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Park's Conjecture

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Abstrakt: Konečná algebra konečného typu (t.j. v konečném jazyce) je *konečně bázovaná* právě tehdy, když lze varietu, již generuje, definovat konečným počtem rovnic. Slavná *Parkova domněnka* tvrdí, že jestliže se ve varietě generované konečnou algebrou **A** nachází pouze konečně mnoho subdirektně ireducibilních algeber a všechny jsou konečné, pak je **A** konečně bázovaná. V této práci představím nejdůležitější výsledky tohoto milénia a drobnou ochutnávku výsledků starších. Nejdůležitější věty tohoto textu lze rozdělit do dvou skupin: mnohé důkazy se odvolávají na Jónssonovu větu z roku 1979 (například důkazy Bakerovy věty pro kongruenčně distributivní variety, a obecnější Willardovy věty pro průsekově polodistributivní variety), zatímco jiné důkazy jsou spíše syntaktické povahy (Lyndonova věta o dvouprvkových algebrách, Ježkova věta o chudých signaturách, věta o regularizaci a Perkinsova věta o komutativních pologrupách). Ačkoli se předpokládá znalost základních pojmů univerzální algebry a matematické logiky, nejdůležitější výsledky těchto oborů jsou zmíněny bez důkazu.

Klíčová slova: konečná báze, varieta, konečná reziduální mez

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Abstract: A finite algebra of finite type (i.e. in a finite language) is *finitely based* iff the variety it generates can be axiomatized by finitely many equations. *Park's conjecture* states that if a finite algebra of finite type generates a variety in which all subdirectly irreducible members are finite and of bounded size, then the algebra is finitely based. In this thesis, I reproduce some of the finite basis results of this millennium, and give a taster of older ones. The main results fall into two categories: applications of Jónsson's theorem from 1979 (Baker's theorem in the congruence distributive setting, and its extension by Willard to congruence meet-semidistributive varieties), whilst other proofs are syntactical in nature (Lyndon's theorem on two element algebras, Ježek's on poor signatures, Perkins's on commutative semigroups and the theorem on regularisation). The text is self-contained, assuming only basic knowledge of logic and universal algebra, and stating the results we build upon without proof.

Keywords: finite basis, variety, finite residual bound

CHAPTER I

Preliminaries

I expect the reader to be familiar with Chapter II and Sections V.1 and V.2 of [**BuSa81**], or some other introduction to universal algebra and mathematical logic. All undefined terms can be found in [**BuSa81**] or [**Jež08**].

Unless otherwise stated, we always assume that the signature (type, language) σ is finite. The reader should be aware that the vast majority of results presented in the text is not valid without this assumption.

We make the usual distinction between the equality sign $\approx$ in the signature and the equality $=$ of elements in an algebra.[1] However, we do not always make the distinction between an algebra $\mathbf{A}$ and its underlying set A.

In order to simplify notation, we write $\forall \overline{x}\ \varphi(\overline{x})$ or alternatively $\forall x_1 x_2 \dots x_n\ \varphi(x_1, \dots, x_n)$ instead of $\forall x_1 \dots \forall x_n\ (\varphi(x_1, \dots, x_n))$ and similarly $\exists xy\ \varphi(x, y)$ etc.

We denote the set of natural numbers by $\mathbb{N}$. When n is a natural number, then $\widehat{n} = \{1, 2, \dots, n\}$ and (by abuse of notation) $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ are sets of natural numbers. For two sets S and T, we write $S \subseteq_{FIN} T$ iff S is a finite subset of T.

The symbols $\leadsto$ and $\lightning$ mark the beginning and end of a proof by contradiction. We mark the parts of a proof by induction by $1°$ (the induction hypothesis) and $2°$ (the induction step). We use the symbol $:=$ in defining equations; for example, if we define c to be equal to $a+b$, we write $c := a + b$.

I.1. Elements of logic

THEOREM I.1.1 (Compactness theorem).

(1) Let Σ be a set of first-order sentences such that for every $\Sigma_0 \subseteq_{FIN} \Sigma$ there exists a model of Σ_0. Then there exists a model of Σ.

(2) Let Σ be a set of sentences and φ another sentence. If $\Sigma \models \varphi$, then for some $\Sigma_0 \subseteq_{FIN} \Sigma$, $\Sigma_0 \models \varphi$.

[1] In other words, $\approx$ is part of the syntax, whilst $=$ is a meta-equality.

DEFINITION I.1.2. These are some types of first-order formulae:

open: φ does not include any quantifiers
sentence: every occurrence of a variable in φ is quantified
universal: φ is of the form $\forall\overline{x}\varphi'(\overline{x})$, where φ' is an open formula
existential: φ is of the form $\exists\overline{x}\varphi'(\overline{x'})$, where φ' is an open formula and every variable from $\overline{x'}$ appears in $\overline{x}$
positive: φ is a formula in the prenex normal form, which only includes the connectives $\wedge$ and $\vee$
quasi-equation: φ is an implication[2] such that the premise is a (possibly empty) conjunction of equations and the conclusion is an equation: $(t_{a_1} \approx t_{b_1} \ \wedge\ \dots\ \wedge\ t_{a_n} \approx t_{b_n}) \ \Rightarrow\ t_a \approx t_b, \quad n \geq 0$
equation: a formula of the form $p \approx t$, where p, t are terms: sometimes considered simply as a pair $(p, q) \in \mathrm{Term}^2_\sigma$
regular equation: exactly the same variables appear on both sides
linear equation: each side has at most one occurrence of every variable

DEFINITION I.1.3. The *sup-algebra* in the language σ is the algebra $(\{0,1\}, \sigma)$, where for each n-ary operation $f \in \sigma$,

$$f(a_1, \dots, a_n) = \begin{cases} 0, & \text{if } a_1 = \dots = a_n = 0 \\ 1, & \text{else.} \end{cases}$$

This algebra is unique up to isomorphism.

If $\mathbf{A} = (A, \sigma)$ is any algebra, the *complex algebra of* $\mathbf{A}$ is $\mathbf{B} = (B, \sigma)$, where $B = \{C \subseteq A;\ C \neq \emptyset\} = \mathscr{P}(A) \setminus \{\emptyset\}$ and for any n-ary operation $f \in \sigma$,

$$f(C_1, \dots, C_n) = \{f(a_1, \dots, a_n);\ \forall i \in \widehat{n}\ a_i \in C_i\}.$$

DEFINITION I.1.4. These are the most common operators on classes of algebras:

I($\mathcal{K}$)**:** isomorphic copies of algebras from $\mathcal{K}$
S($\mathcal{K}$)**:** subalgebras of algebras from $\mathcal{K}$
H($\mathcal{K}$)**:** homomorphic images of algebras from $\mathcal{K}$
P($\mathcal{K}$)**:** direct products of non-empty families of algebras from $\mathcal{K}$
$\mathrm{P_R}(\mathcal{K})$**:** reduced products of non-empty families of algebras from $\mathcal{K}$

[2] A *Horn clause* is a disjunction of literals (i.e. atomic formulae or their negations) with at most one positive literal. In a language without relational symbols, a Horn clause is either a *strictly Horn clause*, also called *definite clause*, i.e. $t_{a_1} \not\approx t_{b_1} \ \vee\ \dots\ \vee\ t_{a_n} \not\approx t_{b_n} \vee\ t_a \approx t_b$, where $n \geq 0$, or a *goal clause*, i.e. disjunction of negations of equalities $t_{a_1} \not\approx t_{b_1} \ \vee\ \dots\ \vee\ t_{a_n} \not\approx t_{b_n}$, where $n > 0$. In this case, each quasi-equation is equivalent to some strictly Horn clause and vice versa. Note that the trivial (one-element) algebra is a model of a Horn clause φ iff φ is a strictly Horn clause.

$P_U(\mathcal{K})$**:** ultraproducts of non-empty families of algebras from $\mathcal{K}$

$\equiv(\mathcal{K})$**:** algebras that are elementarily equivalent with some algebra from $\mathcal{K}$, i.e. algebras in which the same sentences are true

DEFINITION I.1.5. Let $\mathcal{K}$ be a class of first-order structures in the same signature σ. We say that $\mathcal{K}$ is *closed under the operator* O iff $O(\mathcal{K}) \subseteq \mathcal{K}$. We call a class $\mathcal{K}$ of algebras *algebraic* iff it is closed under isomorphism.

By the class $\mathcal{K}'$ we mean the class of all algebras in the signature σ that are not in $\mathcal{K}$.

THEOREM I.1.6. *Let $\mathcal{K}$ be a class of first-order structures in the same signature. Let $\mathcal{K}$ be closed under* I. *The table summarises the "iff" relationships between axomatizability by a certain kind of formulae and closure under certain operators:*[3]

type of formulae	$\mathcal{K}$ is closed under	name of the class	reference
first-order	$\equiv, P_U$	axiomatic, elementary	[**FMS62, Koc61**]
universal	S, P_U	universal	
quasi-equations	S, P, P_U	quasivariety	[**Mal71, Mal73**]
quasi-equations	S, P_R	quasivariety	[**Mal71, Mal73**]
equations	H, S, P	variety	[**Bir35**]
regular equations	H, S, P, sup-algebras		[**JóNe74, Pło75**]
linear equations	H, S, P, complex algebras		[**Gau57, BSW73**]

Some such conditions involve closure of $\mathcal{K}'$ as well:

axiomatizable by ... formulae	$\mathcal{K}$ is closed under	$\mathcal{K}'$ is closed under	name of the class
first-order	P_U	ultrapowers	axiomatic, elementary
finitely many first-order	P_U	P_U	strictly elementary

Other such conditions involve more assumption on $\mathcal{K}$:

axiomatizable by ... formulae	$\mathcal{K}$ is closed under	$\mathcal{K}$ is	reference
positive	H	strictly elementary	[**Lyn59**]

OBSERVATION I.1.7. *A class $\mathcal{K}$ is strictly elementary iff it is axiomatizable by a single first-order formula.*

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[3] For some more theorems of this kind see for example [**Tay79**, Section 3]=[**Grä79**, §63] and [**Grä79**, §45]. The terminology varies from author to author:

	axiomatizable class	strictly elementary class
Burris, Sankappanavar [**BuSa81**]	elementary	strictly elementary
Grätzer [**Grä79**]	axiomatic	elementary
Burris [**Bur79**]		basic elementary

See also [**Grä79**, p. 256] for more references. To avoid confusion, we use the terms "axiomatizable" and "strictly elementary"; however, we say that a class is *finitely axiomatizable by a formula* φ to emphasise the fact that it's a strictly elementary class.

COROLLARY I.1.8. *If a sentence φ is preserved under formation of homomorphic images, subalgebras and products, then φ is equivalent to a (finite) conjunction of equations.*

PROOF. Use Compactness Theorem I.1.1. □

DEFINITION I.1.9. We say that a variety $\mathcal{V}$ is *finitely generated* iff there exists a finite algebra $\mathbf{A}$ such that $\mathcal{V} = \mathrm{HSP}(\mathbf{A})$.

PROPOSITION I.1.10. *If $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$ for a finite set $\mathcal{K}$ of finite algebras, then $\mathcal{V}$ is finitely generated.*

PROOF. $\mathcal{V}$ is generated by the algebra $\prod_{\mathbf{A}\in\mathcal{K}} \mathbf{A}$. □

PROPOSITION I.1.11. *An equation $s \approx t$ can be proved from a set E of equations iff there exists a sequence $s = s_0, s_1, \ldots, s_j = t$ such that for each $i \in \mathbb{Z}_j$, one side of an equation $e \approx e'$ from E (say e) is matched with a subterm of s_i; then s_{i+1} is the same as s_i with this subterm replaced by e'.*

In other words, $s_i \approx s_j$ follows from some equation $e \approx e' \in E$ by a substitution of the following kind: first we substitute some terms $t_1, \ldots t_n$ for variables occurring in e and e' and thus obtain an equation $e^* \approx e'^*$; then we substitute e^* and e'^* for some variable into the same term f, and thus obtain the equation

$$s_i = f(e^*, y_1, \ldots, y_n) \approx f(e'^*, y_1, \ldots, y_n) = s_{i+1}.$$

PROOF. The "if" direction is clear. On the other hand, $E \vdash s \approx t$ means by definition that there exists a sequence of equations

$$p_1 \approx q_1, \ \ldots, \ p_k \approx q_k$$

such that the last equation is actually $s \approx t$ and for each $i \leq k$, $p_i \approx q_i$ belongs to E, or is of the form $p_i \approx p_i$, or can be obtained from some equation with index $j < i$ by symmetry, substitution or replacement of p_j by q_j in some term f, or can be obtained from two equations with smaller indices by transitivity. By induction on k we immediately see that the proposition holds. □

DEFINITION I.1.12. We say that a variety $\mathcal{V}$ is *finitely based* iff there exists a finite set of equations E_0 such that $\mathcal{V} = \mathrm{Mod}\, E_0$. We say that a set E of equations is *finitely based* iff $\mathrm{Mod}\, E$ is finitely based.

OBSERVATION I.1.13.

(1) *A variety is finitely based iff it is finitely axiomatizable.*

(2) The intersection of two finitely based varieties is finitely based.

EXAMPLE I.1.14 (J. Karnofsky). We show that the join of two finitely based varieties need not be finitely based. First, we need to realise that there is a Galois correspondence between varieties and equational theories: the larger the variety, the fewer equations are satisfied in all of its algebras. Thus the join of two varieties corresponds to the meet of the corresponding equational theories.

Let us have equational theories E_1 and E_2 with the following bases:

$$\begin{array}{lrcl} E_1: & x(yz) & \approx & (xy)z \\ & (xyz)^2 & \approx & x^2y^2z^2 \\ & x^3y^3z^2w^3 & \approx & y^3x^3z^2w^3 \end{array}$$

$$\begin{array}{lrcl} E_2: & x(yz) & \approx & (xy)z \\ & x^3y^3 & \approx & y^3x^3 \end{array}$$

The theory $E_1 \cap E_2$ contains the equations

$$s := x^3y^3v_0^2 \dots v_{2k}^2w^3 \approx y^3x^3v_0^2 \dots v_{2k}^2w^3 =: t$$

for all $k \geq 0$. We show that any base B of $E_1 \cap E_2$ must contain all of these equations (up to a substitution of different variables).

First note that modulo E_2 the left-hand side s is equal to nothing but itself and the right-hand side t. Thus also modulo the smaller theory $E_1 \cap E_2$,

$$[x^3y^3v_0^2 \dots v_{2k}^2w^3]_{E_1 \cap E_2} = \{x^3y^3v_0^2 \dots v_{2k}^2w^3,\ y^3x^3v_0^2 \dots v_{2k}^2w^3\}.$$

Now from Proposition I.1.11 we see that if $B \models s \approx t$, then there must be an equation $e \approx e' \in B$ such that $s = f(e^*, y_1, \dots, y_n)$ and $t = f(e'^*, y_1, \dots, y_n)$ for some term f and substitution *. But a proper subterm e^* of s is modulo $E_1 \cap E_2$ not equal to anything but itself. Thus $s \approx t \in B$.

PROPOSITION I.1.15. *A set E of equations is finitely based iff there exists an $E_0 \subseteq_{FIN} E$ such that* $\operatorname{Mod} E_0 = \operatorname{Mod} E$.

PROOF. ($\Leftarrow$) is clear.
($\Rightarrow$) Let $E_1 \subseteq_{FIN} \operatorname{Eq}(\operatorname{Mod} E)$ be such that $\operatorname{Mod} E_1 = \operatorname{Mod} E$. Then

$$E \models \bigwedge_{\varphi \in E_1} \varphi,$$

and hence (by Compactness) there exists a finite $E_0 \subseteq_{FIN} E$ such that

$$E_0 \models \bigwedge_{\varphi \in E_1} \varphi.$$

Clearly, $\operatorname{Mod} E_0 = \operatorname{Mod} E$. ☐

Definition I.1.16. Let k be a natural number.

Let $\mathrm{Term}_\sigma(k) = \mathrm{Term}_\sigma(\{x_1, \dots, x_k\})$; it is the set of all terms in k variables.

Let $\mathrm{Eq}_k\mathcal{V}$ be the set of equations $s \approx t$ such that $s, t \in \mathrm{Term}_\sigma(k)$ and $\mathcal{V} \models s \approx t$. Let $\mathcal{V}^{(k)} := \mathrm{Mod}(\mathrm{Eq}_k\,\mathcal{V})$.

We define the *length* of a term as the number of occurrences of fundamental operations appearing in it. Let $\mathrm{Term}_{(k)}(X) \subseteq \mathrm{Term}_\sigma(X)$ denote the set of σ-terms with length at most k and variables from X.

Let $\mathrm{Eq}_{(k)}(X, \mathcal{V})$ be the set of equations $p \approx q \in (\mathrm{Term}_{(k)}(X))^2$ true in $\mathcal{V}$; if X is a countably infinite set of variables, we write simply $\mathrm{Eq}_{(k)}\mathcal{V}$.

Note the difference between $\mathrm{Term}_\sigma(k)$ and $\mathrm{Term}_{(k)}(X)$, and the difference between $\mathrm{Eq}_k\mathcal{V}$ and $\mathrm{Eq}_{(k)}\mathcal{V}$.

Example I.1.17. If $\mathcal{V}$ is finitely based, then it has a base in k variables for some $k \in \mathbb{N}$. We show that the converse is not true even for $k = 1$. Indeed, the following equational theory is not finitely based:

$$E = \{fg^k f(x) \approx fgf^{n-2}gf(x); \quad k \geq 2\},$$

where for example $fg^3f(x)$ is the term $f(g(g(g(f(x)))))$.

$\leadsto$ To show that E is not finitely based, assume $E_0 \subseteq_{FIN} E$ is a basis of E. Then there would exists largest n such that

$$fg^n f(x) \approx fgf^{n-2}gf(x) \in E_0.$$

According to Proposition I.1.11, $s \approx t$ may be derived from E_0 iff there exists a sequence $s = s_0, \dots, s_j = t$ such that for each $i \in \mathbb{Z}_j$, one side of an equation $e \approx e' \in E_0$ (say e) is matched with a subterm of s_i; then s_{i+1} is the same as s_i with this subterm replaced by e'.

However, subterms of the left-hand side of the equation

$$(\boxdot) \qquad fg^{n+1}f(x) \approx fgf^{n-1}gf$$

do not match any equation appearing in E_0; hence $(\boxdot)$ does not follow from E_0, which is a contradiction with the assumption that E_0 is a basis of E. $\lightning$

Thus we see that even if a variety V has a basis in finitely many variables, it does not need to be finitely based. In the following proposition we show that a bound on the length of terms appearing in a basis is a much stronger condition than a bound on the number of variables.

Proposition I.1.18.

(1) $\mathcal{V}$ *is finitely based iff there exists a* k *such that* $\mathcal{V}$ *is axiomatized by* $\mathrm{Eq}_{(k)}(\mathcal{V})$.

*(2) A variety $\mathcal{V}$ is **not** finitely based iff for all $k \in \mathbb{N}$, there exists a subdirectly irreducible algebra $\mathbf{A}_k \notin \mathcal{V}$ which satisfies all identities true in* $\mathrm{Eq}_{(k)}(\mathcal{V})$.

PROOF. (1⇒) and (2⇐) If $\mathcal{V}$ is finitely based, there is a maximal length of terms k among all terms appearing in the finite basis. Hence $\mathbf{A}_k$ cannot exist.

(1⇐) If n is the maximum arity of an operation symbol in σ, let $m = 2n^k$. Any identity in $(\mathrm{Term}_{(k)}(X))^2$ has at most m variables occurring in it, which implies that all equations in $\mathrm{Eq}_{(k)}(\mathcal{V})$ are deductive consequences of $\mathrm{Eq}_{(k)}(\{x_1, \ldots, x_m\}, \mathcal{V})$. $\mathrm{Term}_{(k)}(\{x_1, \ldots, x_m\})$ is a finite set of terms, showing that $\mathrm{Eq}_{(k)}(\{x_1, \ldots, x_m\}, \mathcal{V})$ is finite.

(2⇒) We write out the proof, although the claim is a clear consequence of (1) and I.2.4.

Assume that $\mathcal{V}$ is not finitely based. According to part (1), $\mathrm{Eq}_{(k)}(\mathcal{V})$ does not axiomatize $\mathcal{V}$ for any k, so for each k there exists an algebra $\mathbf{B}_k \notin \mathcal{V}$ such that $\mathbf{B} \models \mathrm{Eq}_{(k)}(\mathcal{V})$. According to Theorem I.2.3, $\mathbf{B}_k$ is subdirect product of its subdirectly irreducible factors; if all these factors would lie in $\mathcal{V}$, $\mathbf{B}_k \in \mathrm{SP}(\mathcal{V})$ would be in $\mathcal{V}$ as well; hence at least one of these factors ($\mathbf{A}_k$) is not in $\mathcal{V}$. Since $\mathbf{A}_k$ satisfies all the identities $\mathbf{B}_k$ does, we are finished. □

For the sake of finite basis proofs, the following proposition allows us to assume that (finitely many) important terms in the variety are already part of the signature. This assumption simplifies the proofs in which we use certain terms to write formulae with specific meaning.

PROPOSITION I.1.19. *Let $\mathcal{V}$ be a variety in the language σ and let $t(x_1, \ldots, x_n) \in \mathrm{Term}_\sigma(n)$. Let $\sigma^* = \sigma \cup \{f\}$ be a language enriched by a new n-ary operation symbol, and let $\mathcal{V}^*$ be the variety defined by $\mathrm{Eq}(\mathcal{V}) \cup \{f(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)\}$. Then $\mathcal{V}$ is finitely based iff $\mathcal{V}^*$ is finitely based.*

A SEMANTIC PROOF. Let $E = \mathrm{Eq}(\mathcal{V})$, $E' = E \cup \{f \approx t\}$ and let E_0^* be a finite basis of $E^* = \mathrm{Eq}(\mathcal{V}^*)$. Then

$$E' \models E_0^*,$$

so by Theorem I.1.1 there exists a finite $E_0' \subseteq E'$ such that

$$E_0' \models E_0^*.$$

So E_0' is a finite set of equations for $\mathcal{V}^*$. Then

$$E_0' \subseteq E_0 \cup \{f \approx t\}$$

for some finite $E_0 \subseteq E$. To any $\mathbf{A}$ satisfying E_0 one may add a new operation f to obtain $\mathbf{A}^*$ satisfying E_0'; hence $E_0 \models E$. □

A SYNTACTIC PROOF. Let E_0^* be a finite basis of $\mathcal{V}^*$. Let

$$r \approx s \in \mathrm{Eq}(\mathcal{V});$$

then

$$E_0^* \vdash r \approx s,$$

so there is a proof $r_1 \approx s_1, \ldots, r_n \approx s_n$ such that each $r_i \approx s_i$ is either in E_0^* or it is a consequence of the equations with smaller indices. We turn E_0^* into E_0 in the following way: in each equation use the term t instead of each appearance of the operation symbol f; we also apply this transformation to all equations in the proof, showing that

$$E_0 \vdash r \approx s. \qquad \square$$

DEFINITION I.1.20. A *clone* of the algebra $\mathbf{A}$ is a set C of operations on the set A such that

(1) for any n, C contains all the projections

$$\begin{array}{rccll} \pi_k : & \mathbf{A}^n & \to & \mathbf{A}, & k \in \widehat{n} \\ & (x_1, \ldots, x_n) & \mapsto & x_k. & \end{array}$$

(2) C is closed under composition: if $f, g_1, \ldots, g_m \in C$ such that f is m-ary, and all g_j are n-ary, then C contains the n-ary operation

$$h(x_1, \ldots, x_n) := f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

In particular, the clone generated by the fundamental operations is called the *clone of (term) operations*; it is the set of the various operations $f : \mathbf{A}^n \to \mathbf{A}$, $n \in \mathbb{N}$, which can be defined by some term.

COROLLARY I.1.21. *If two algebras $\mathbf{A} = (A, \sigma)$ and $\mathbf{A}' = (A, \sigma')$ in finite languages have the same elements and the same clone of operations, then $\mathbf{A}$ is finitely based iff $\mathbf{A}'$ is finitely based.*

PROOF. Each basic operation $f_i \in \sigma'$ of $\mathbf{A}'$ may be expressed by some term $t_i \in \mathrm{Term}_\sigma$. By Proposition I.1.19, (A, σ) is strictly elementary iff $(A, \sigma \cup \sigma')$ is strictly elementary iff (A, σ') is strictly elementary. In other words, if $\mathcal{V} = \mathrm{Mod}(\mathrm{Eq}\,\mathbf{A})$ and

$$\mathcal{V}^* = \mathrm{Mod}\left(\mathrm{Eq}\,\mathbf{A} \cup \{f_i(x_1, \ldots, x_n) \approx t_i(x_1, \ldots, x_n);\ f_i \in \sigma'\}\right),$$

then $\mathcal{V}$ is finitely axiomatizable iff $\mathcal{V}^*$ is finitely axiomatizable. But the equational theory generated by $\mathrm{Eq}\,\mathbf{A} \cup \{f_i \approx t_i;\ f_i \in \sigma'\}$ is equal to $\mathrm{Eq}(A, \sigma \cup \sigma')$. $\square$

I.2. Subdirectly irreducible algebras

DEFINITION I.2.1. An algebra $\mathbf{A}$ is called *subdirectly irreducible* iff $0_{\mathbf{A}}$ is a completely meet-irreducible element of the congruence lattice of $\mathbf{A}$. An algebra $\mathbf{A}$ is called *finitely subdirectly irreducible* iff $0_{\mathbf{A}}$ is a meet-irreducible member of the congruence lattice of $\mathbf{A}$. Notation: For a variety $\mathcal{V}$, let $\mathcal{V}_{SI}$ (resp. $\mathcal{V}_{FSI}$) be the class of all subdirectly irreducible (resp. finitely subdirectly irreducible) members of $\mathcal{V}$.

OBSERVATION I.2.2.

(1) The following conditions are equivalent:
- $\mathbf{A}$ *is* subdirectly irreducible, *i.e.* $0_{\mathbf{A}}$ *is a completely meet-irreducible element of* $\operatorname{Con}\mathbf{A}$. *In other words, if* $0_{\mathbf{A}}$ *is the intersection of a non-empty family of congruences of* $\mathbf{A}$, *then at least one of the congruences equals* $0_{\mathbf{A}}$.
- $\operatorname{Con}\mathbf{A}$ *contains a least non-zero congruence, the so called* monolith.
- *Any family of homomorphisms separating points of* $\mathbf{A}$ *must contain some injection.*
- *If* $\mathbf{A}$ *is a subdirect product of algebras* $\mathbf{A}_i, i \in I$, *then* $\mathbf{A} \simeq \mathbf{A}_i$ *for some* i.

(2) The following conditions are equivalent:
- $\mathbf{A}$ *is* finitely subdirectly irreducible, *i.e.* $0_{\mathbf{A}}$ *is a meet-irreducible element of* $\operatorname{Con}\mathbf{A}$.
- *For any two non-injective homomorphisms* h_1, h_2 *with domain* $\mathbf{A}$ *there exists* $a \neq b \in \mathbf{A}$ *which are separated by neither* h_1 *nor* h_2.
- *For any* $a, b, c, d \in \mathbf{A}$, *if* $\operatorname{Cg}(a,b) \cap \operatorname{Cg}(c,d) = 0_{\mathbf{A}}$ *then* $a = b$ *or* $c = d$.

THEOREM I.2.3 (Birkhoff's representation theorem [**Bir44**]). *Every algebra is isomorphic to a subdirect product of its subdirectly irreducible factors.*

PROOF. See the proof of Theorem I.5.11. □

COROLLARY I.2.4. *Each variety* $\mathcal{V}$ *is uniquely determined by the class of its (finitely) subdirectly irreducible members. In other words, the following implications hold:*

$$\text{if} \quad \mathcal{V}_{FSI} = \mathcal{W}_{FSI} \qquad \text{then} \quad \mathcal{V}_{SI} = \mathcal{W}_{SI};$$
$$\text{if} \quad \mathcal{V}_{SI} = \mathcal{W}_{SI} \qquad \text{then} \quad \mathcal{V} = \mathcal{W}.$$

THEOREM I.2.5 (Taylor [**Tay72**]; McKenzie, Shellah [**McSh74**]; McKenzie [**McK96a**]). *If $\mathcal{V}$ is a variety in a countable language, then one of the following is true: the cardinalities of the irreducible members of $\mathcal{V}$*

(1) are bounded by a finite cardinal;
(2) include arbitrarily large finite cardinals but no infinite cardinal;
(3) include $\aleph_0$ but no cardinal larger than $\aleph_0$;
(4) include all infinite cardinals $\lambda \leq 2^{\aleph_0}$ but no larger cardinal;
(5) are not bounded by any cardinal at all.

For each one of these types, there exists a finite algebra generating a variety of the given type.

DEFINITION I.2.6. In case (1) of the previous theorem, we say that $\mathcal{V}$ has *a finite residual bound* or that $\mathcal{V}$ is *residually very finite*; in cases (1) and (2), the variety is called *residually finite*; in cases (1)–(4), it is called *residually small*; in the last case, it is *residually large*.

In the case of a finite language, $\mathcal{V}$ is residually small iff there exists a set $\mathcal{K}$ such that for each $\mathbf{A} \in \mathcal{V}_{SI}$ there exists $\mathbf{A}' \in \mathcal{K}$ isomorphic to $\mathbf{A}$, and $\mathcal{V}$ has a finite residual bound iff there are only finitely many non-isomorphic subdirectly irreducibles in $\mathcal{V}$ and all of them are finite. Except in case (2), the algebras witnessing the above theorem have a finite language.

PROPOSITION I.2.7. *Let $\mathcal{V}$ be a locally finite variety.*

(1) If $\mathbf{A} \in \mathcal{V}_{SI}$, $\mathbf{B} \leq \mathbf{A}$ and $\mathbf{B}$ is finite, then there is some finite $\mathbf{A}' \in \mathcal{V}_{SI}$ such that $\mathbf{B} \leq \mathbf{A}'$.
(2) (Quackenbush [**Qua71**])
   (a) If $\mathcal{V}$ contains an infinite subdirectly irreducible algebra $\mathbf{A}$, then for any $n \in \mathbb{N}$, $\mathcal{V}$ contains a finite subdirectly irreducible algebra $\mathbf{A}_n$ such that $|\mathbf{A}_n| \geq n$.
   (b) If $\mathcal{V}$ has, up to isomorphism, only finitely many finite subdirectly irreducible members, then $\mathcal{V}$ has no infinite subdirectly irreducible members (and hence has a finite residual bound).

PROOF. (1⇒2a) Any n different elements of $\mathbf{A}$ generate a finite subalgebra $\mathbf{B} \leq \mathbf{A}$ of size at least n; thus if $\mathbf{B} \leq \mathbf{A}_n$ for some finite $\mathbf{A}_n \in \mathcal{V}_{SI}$, then also $|\mathbf{A}_n| \geq n$.

(2a⇔2b) can be seen easily. However, we give separate proofs of (1) and (2b) to show different strategies for proving the Proposition.

THE PROOF OF (1) uses Proposition I.3.1. Let

$$(a, a') \in \mu,$$

where μ is the monolith of $\mathbf{A}$, and let $\mathbf{B} = \{b_1, \ldots, b_k\}$. Then

$$(a, a') \in \mathrm{Cg}^{\mathbf{A}}(b_i, b_j)$$

for any $i \neq j$. Let $\{c_1, \ldots, c_m\}$ be the set of all links in the Mal'cev chains used to show that $(a, a') \in \mathrm{Cg}^{\mathbf{A}}(b_i, b_j)$ and all constants used to construct the unary polynomials in these chains for all $i \neq j$. Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}$ generated by the set $\{a, a', b_1, \ldots, b_k, c_1, \ldots, c_m\}$:

$$\mathbf{C} \; := \; \big\langle \{a, a', b_1, \ldots, b_k, c_1, \ldots, c_m\} \big\rangle_{\mathbf{A}}.$$

$\mathbf{C}$ is finite since $\mathcal{V}$ is locally finite.

For any $i \neq j$, the elements $c_1, \ldots, c_m$ ensure that the Mal'cev chain certifying that $(a, a') \in \mathrm{Cg}^{\mathbf{C}}(b_i, b_j)$ can be constructed in $\mathbf{C}$. Let θ be a maximal congruence in $\mathrm{Con}\,\mathbf{C}$ such that $(a, a') \notin \theta$. If $\alpha > \theta$, then $(a, a') \in \alpha$, so

$$(a, a') \;\in \bigwedge_{\substack{\alpha \in \mathrm{Con}\,\mathbf{C}, \\ \alpha > \theta}} \alpha,$$

hence θ is a strictly meet-irreducible element of $\mathrm{Con}\,\mathbf{C}$ and

$$\mathbf{A}' := \mathbf{C}/\theta$$

is subdirectly irreducible.

From $(a, a') \in \mathrm{Cg}(b_i, b_j)$ we get $[b_i]_\theta \neq [b_j]_\theta$ for all $i \neq j$, so there is an injection from $\mathbf{B}$ to $\mathbf{A}'$. As the property of being a closed under operations is carried over to homomorphic images,

$$\mathbf{B} \simeq \mathbf{B}/\theta \leq \mathbf{A}'.$$

THE PROOF OF (2b) is based on an ultraproduct construction typical to model theory:

FACT. *Every first-order structure can be embedded in an ultraproduct of its finitely generated substructures.*

Let $\mathcal{V}^*$ be the class of the finite algebras $\mathbf{F} \in \mathcal{V}_{SI}$; by the assumption, $\mathcal{V}^*$ is (up to isomorphism) a finite set of finite algebras.

Let $\mathbf{A} \in \mathcal{V}$ and let $\mathcal{K}$ be the class of the finitely generated subalgebras of $\mathbf{A}$. Because of local finiteness, if $\mathbf{B} \in \mathcal{K}$, then $\mathbf{B}$ is finite, and by Birkhoff's representation theorem I.2.3, $\mathbf{B}$ is a subdirect product of algebras from $\mathcal{V}^*$. By the fact mentioned earlier we have

$$\mathbf{A} \in \mathrm{SP_U}(\mathcal{K}) \subseteq \mathrm{SP_U}(\mathrm{SP}(\mathcal{V}^*)).$$

Therefore

$$\mathbf{A} \in \mathrm{SPP_U}(\mathcal{V}^*)$$

because an algebraic class $\mathcal{K}'$ closed under $\mathrm{S}, \mathrm{P_U}$ and P is equal to $\mathrm{SPP_U}\,\mathcal{K}'$. (See I.1.6 and I.5.1.)

As an ultraproduct of finitely many finite algebras is isomorphic to one of the algebras, we have

$$\mathbf{A} \in \mathrm{SP}(\mathcal{V}^*),$$

and hence $\mathbf{A}$ is a subdirect product of algebras from $\mathcal{V}^*$. However, this means that $\mathbf{A}$ cannot be both infinite and subdirectly irreducible. □

Definition I.2.8. Let $\mathbf{A}$ be an algebra; we define

$$\sigma_{\mathbf{A}} \;:=\; \sigma \cup \{c_a;\, a \in \mathbf{A}\}$$

as the language enriched by the names of all elements of $\mathbf{A}$. Then *the diagram of* $\mathbf{A}$, denoted $\Delta_{\mathbf{A}}$, is the following set of equations and their negations:[4]

$$\begin{aligned}\{c_a \not\approx c_b;\quad & a \neq b \in \mathbf{A}\}\\ \cup\quad \{f(c_{a_1},\dots,c_{a_n}) = c_b;\quad & f \in \sigma,\ \ a_1,\dots,a_n, b \in \mathbf{A},\\ & \mathbf{A} \models f(a_1,\dots,a_n) = b\}.\end{aligned}$$

In [**Bur79**], Burris asked whether the following Proposition is true; the positive answer is folklore, as is the lemma:

Proposition I.2.9. *If a variety $\mathcal{V}$ has a finite residual bound, then $\mathcal{V}_{SI}$ is strictly elementary and $\mathcal{V}_{FSI} = \mathcal{V}_{SI}$.*

Proof. $\mathcal{V}_{SI}$ is a finite set of finite algebras; therefore it can be axiomatized by the disjunction ψ of formulae that describe each $\mathbf{A} \in \mathcal{V}_{SI}$ up to isomorphism.[5]

For all varieties, $\mathcal{V}_{FSI} \supseteq \mathcal{V}_{SI}$. In the following lemma we show that if $\mathbf{B} \in \mathcal{V}_{FSI}$, then $\mathbf{B}$ embeds into some $\mathbf{A} \in \mathcal{V}_{SI}$ and hence $\mathbf{B}$ is finite. But finite finitely subdirectly irreducible algebras are subdirectly irreducible, which means that $\mathcal{V}_{FSI} \subseteq \mathcal{V}_{SI}$. □

Lemma I.2.10. *Let $\mathcal{V}$ be a variety such that $\mathcal{V}_{SI}$ is axiomatizable by a set Ψ of elementary sentences. Then every algebra $\mathbf{B} \in \mathcal{V}_{FSI}$ is embeddable into some $\mathbf{A} \in \mathcal{V}_{SI}$.*

[4]The concept of a diagram comes from model theory. Under the usual definition, it is a set of all atomic sentences in signature $\sigma_{\mathbf{A}}$ and their negations which are true in the given structure; the usual definition does not involve the restriction to terms with at most one operation symbol.

[5]For a given $\mathbf{A} \in \mathcal{V}_{SI}$, the corresponding formula is of this form:
$\exists x_1 \exists x_2 \dots \exists x_n$
$[x_1 \not\approx x_2 \wedge \dots \wedge x_1 \not\approx x_n \wedge \dots \wedge x_{n-1} \not\approx x_n$ — $\mathbf{A}$ has at least n elements
$\wedge\ \forall x (x \approx x_1 \vee x \approx x_2 \vee \dots \vee x \approx x_n)$ — $\mathbf{A}$ has at most n elements
$\wedge\ f(x_1, \dots x_1) \approx x_j\ \wedge\ \dots]$ — equations that say how basic operations act on $\mathbf{A}$

Proof ([**BMW04**]). Let Δ be the diagram of $\mathbf{B}$. We show that $\Delta \cup \Psi$ is a consistent set of formulae; by the Compactness Theorem I.1.1 we only need to show the consistency of Γ for all $\Gamma \subseteq_{FIN} \Delta \cup \Psi$.

Let $S = \{b \in \mathbf{B};\ c_b \text{ appears in } \Gamma\}$. S is a finite subset of $\mathbf{B} \in \mathcal{V}_{FSI}$, therefore there exist p, q such that

$$(p,q) \in \bigcap_{\substack{r,s \in S,\\ r \neq s}} \mathrm{Cg}^{\mathbf{B}}(r,s)$$

Let $\alpha \in \mathrm{Con}\,\mathbf{B}$ be a maximal congruence such that $(p,q) \notin \alpha$ (it exists by Zorn's lemma). Then $\mathbf{B}/\alpha$ is subdirectly irreducible, hence it satisfies Ψ. True atomic sentences are carried over to homomorphic images, and $\mathbf{B}/\alpha \models r \not\approx s$ for all $r, s \in S$, so $\mathbf{B}/\alpha$ satisfies Γ.

We have shown the consistency of $\Delta \cup \Psi$. Take any $\mathbf{A}$ such that $\mathbf{A} \models \Delta \cup \Psi$. Then $\mathbf{B}$ embeds into $\mathbf{A}$ (because $\mathbf{A} \models \Delta$) and[6] $\mathbf{A} \in \mathcal{V}_{SI}$ (because $\mathbf{A} \models \Psi$). □

I.3. Congruences

Proposition I.3.1 (Mal'cev chains). *Let* $\mathbf{A}$ *be an algebra and* $X \subseteq A^2$. *Then* $\mathrm{Cg}^{\mathbf{A}}(X)$ *is equal to*

$$\big\{(x,y) \in A^2;\ \exists x_1 \dots x_n \exists y_1 \dots y_n \exists z_0 \dots z_n \in A\ \exists p_1 \dots p_n \in \mathrm{Pol}_1 \mathbf{A}$$
$$(x_i, y_i) \in X \ \&\ x = z_0 \ \&\ y = z_n \ \&\ \{z_{i-1}, z_i\} = \{p_i(x_i), p_i(y_i)\}\big\}.$$

Proof. Since terms preserve congruences, polynomials do as well. Hence the set C given by the above definition is subset of $\mathrm{Cg}^{\mathbf{A}}(X)$ and we only need to prove that it is a congruence. Obviously, it is a reflexive, symmetric and transitive set of pairs. Let f be any k-ary operation, and for $i \in \widehat{k}$, let $z_{i0}, z_{i1}, \dots, z_{in_i}$ be a chain for $(a_i, b_i) \in C$. Then there exists a chain from $f^{\mathbf{A}}(a_1, \dots, a_k)$ to $f^{\mathbf{A}}(b_1, \dots, b_k)$: it is the chain

$$f^{\mathbf{A}}(a_1, a_2, \dots, a_k) = f^{\mathbf{A}}(z_{10}, a_2, \dots, a_k),$$
$$\dots, f^{\mathbf{A}}(z_{1n_1}, a_2, \dots, a_k) = f^{\mathbf{A}}(b_1, z_{20}, \dots, a_k), f^{\mathbf{A}}(b_1, z_{21}, \dots, a_k),$$
$$\dots, f^{\mathbf{A}}(b_1, b_2, \dots, z_{kn_k}) = f^{\mathbf{A}}(b_1, b_2, \dots, b_k).$$

□

[6]This is not quite exact: $\mathbf{A}$ is an algebra in the language $\sigma_{\mathbf{B}}$, so it cannot lie in $\mathcal{V}$; however, it is true for the σ-reduct of $\mathbf{A}$.

DEFINITION I.3.2. A term t is *linear* iff no variable occurs more than once in t. Let Sl be the set of all linear terms $p(x, \overline{y})$ such that the variable x occurs in every non-variable subterm of p.[7]

For $t \in \mathrm{Sl}$ we define the *depth*[8] $d(t)$ to be the number of occurrences of fundamental operation symbols in t; let $\mathrm{Sl}_n = \{t \in \mathrm{Sl};\ d(t) \leq n\}$.

DEFINITION I.3.3. Let $p(x) \in \mathrm{Pol}_1\, \mathbf{A}$ be a unary polynomial. We say that $p(x)$ is a *translation* iff there exists a term $t \in \mathrm{Sl}$ and elements $a_1, \dots a_k \in \mathbf{A}$ such that $p(x) = t(x, \overline{a})$. The set of all translations will be denoted by $\mathrm{Tr}\, \mathbf{A}$, and the set of all translations obtained from terms $t \in \mathrm{Sl}_n$ by $\mathrm{Tr}_n\, \mathbf{A}$; in particular, a function $p \in \mathrm{Tr}_1\, \mathbf{A}$ is called a *basic translation*, and the identity map is the only translation of depth 0.

EXAMPLE I.3.4. A basic translation is of the form

$$f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n),$$

where f is an n-ary operation symbol and all $a_i \in \mathbf{A}$.

Figure 1 gives a picture of what a translation might look like. Note that any translation is a composition of some finite sequence of basic translations, and $p(x) \in \mathrm{Tr}_n$ iff the length of the sequence is at most n.

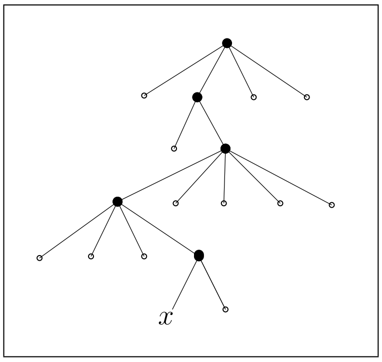


FIGURE 1. An element of Tr_5. The full dots represent fundamental operations, while the small dots represent constants.

The term $x \cdot y$ is not a translation because it is not a unary polynomial, the term $x \cdot x$ is not a translation because it is not obtained from a linear term and the term $x \cdot (a \cdot b)$ is not a translation because x does not appear in every non-variable subterm. (The latter two terms

[7]We define a *slender term* inductively:

1° each variable is a slender term;

2° for a k-ary operation symbol f, a slender term p and variables $y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k$, the term $t = f(y_1, \dots, y_{i-1}, p, y_{i+1}, \dots, y_k)$ is slender.

It can be seen easily that Sl is the set of slender linear terms $p(x, \overline{y})$ with the occurence of x at maximal depth.

[8][**BMW04**] call this property the *complexity* of t.

are unary polynomials.) However, the term $a \cdot (x \cdot b)$ is a translation, as is the term $a \cdot (a \cdot x)$.

Observation I.3.5. *If a polynomial* $p(x) = t^{\mathbf{A}}(x, a_1, \dots, a_n)$ *for some linear term* t*, we can find a translation* t^* *such that* $p(x) = t^*(x)$*.*

Proof. Evaluate all maximal subterms that do not include x. □

Corollary I.3.6 (Refined Mal'cev chains). $\mathrm{Cg}^{\mathbf{A}}(X)$ *is equal to the equivalence relation on* $\mathbf{A}$ *generated by the set of pairs*

$$\{(p(a), p(b)) : \ p \in \mathrm{Tr}\,\mathbf{A}, (a, b) \in X\}.$$

In other words, we may replace $\mathrm{Pol}_1\,\mathbf{A}$ *in Proposition I.3.1 by* $\mathrm{Tr}\,\mathbf{A}$*.*

Proof. The idea is to refine the Mal'cev chain from Proposition I.3.1: for $p \in \mathrm{Pol}_1\,\mathbf{A}$, we connect $p(a)$ and $p(b)$ with a chain of images of $\{a, b\}$ under translations. By the definition of a polynomial, there exists a term $t(x, y_1, \dots, y_n)$ and elements $a_1, \dots, a_n$ such that $p(x) = t(x, a_1, \dots, a_n)$. Now turn t into a linear term t': if there are k occurrences of the variable x in t, replace them by variables $x_1, \dots, x_k$ and similarly for all y's:

$$t(x, y_1, \dots, y_n) \ \rightsquigarrow \ t'(x_1, \dots, x_k, y_{11}, \dots, y_{1k_1}, y_{21}, \dots, y_{nk_n}).$$

We have that

$$p(x) = t'(\underbrace{x, \dots, x}_{k}, \underbrace{a_1, \dots, a_1}_{k_1}, \underbrace{a_2, \dots, a_2}_{k_2}, \underbrace{\dots}_{\dots}, \underbrace{a_n, \dots, a_n}_{k_n}).$$

For $i \in \mathbb{Z}_{k+1}$ let

$$t'_i(x) := t'(\underbrace{a, \dots, a}_{k-i}, x, \underbrace{b, \dots, b}_{i-1}, \underbrace{a_1, \dots, a_1}_{k_1}, \underbrace{a_2, \dots, a_2}_{k_2}, \underbrace{\dots}_{\dots}, \underbrace{a_n, \dots, a_n}_{k_n}).$$

According to the previous observation, there exist translations t_i^* such that $t_i^*(x) = t'_i(x)$. Then

$$t_0^*(a) = p(a), \quad t_k^*(b) = p(b) \quad \text{and } t_i^*(b) = t_{i+1}^*(a).$$

□

I.4. Congruence lattices

I.4.1. Congruence permutable varieties.

THEOREM I.4.1 (Mal'cev [**Mal54**]). *Let $\mathcal{V}$ be a variety. Then the following conditions are equivalent:*

(1) $\mathcal{V}$ is congruence permutable, i.e. for each $\mathbf{A} \in \mathcal{V}$ and each $\theta, \mu \in \operatorname{Con} \mathbf{A}$,

$$\theta \circ \mu = \mu \circ \theta.$$

(2) There exists a Mal'cev term, i.e. ternary term p such that

$$\mathcal{V} \models p(x, x, y) \approx y \approx p(y, x, x).$$

EXAMPLE I.4.2. For example, groups, rings and modules have Mal'cev terms:
in rings and modules, it is the term $x - y + z$;
in groups, it is the term $x * y^{-1} * z$.

I.4.2. Congruence distributive varieties.

THEOREM I.4.3. *Let $\mathcal{V}$ be a variety. Then the following conditions are equivalent:*

(1) $\mathcal{V}$ is congruence-distributive, i.e. for $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, the following equalities hold:

$$\alpha \cap (\beta \vee \gamma) = (\alpha \cap \beta) \vee (\alpha \cap \gamma)$$

$$\alpha \vee (\beta \cap \gamma) = (\alpha \vee \beta) \cap (\alpha \vee \gamma).$$

(2) Neither[9] *M_3 nor N_5 is a sublattice of $\operatorname{Con} \mathbf{A}$ for some $\mathbf{A} \in \mathcal{V}$.*

(3) (Jónsson [**Jón67**]) *$\mathcal{V}$ has Jónsson terms, i.e. there exist ternary terms $t_0, t_1, \dots, t_k$ such that $\mathcal{V}$ satisfies*
- *$t_0(x, y, z) \approx x$*
- *for every i, $t_i(x, y, x) \approx x$*
- *for i even, $t_i(x, x, y) \approx t_{i+1}(x, x, y)$*
- *for i odd, $t_i(x, y, y) \approx t_{i+1}(x, y, y)$*
- *$t_k(x, y, z) \approx z$.*

(4) (Burris [**Bur79**], Baker[10]) *There exist terms $p_1, \dots, p_{k-1}$ such that*
- *for every i, $\mathcal{V} \models p_i(x, u, x) \approx p_i(x, v, x)$*
- *if $x \neq y$ then there exists an i so that $p_i(x, x, y) \neq p_i(x, y, y)$.*

[9][**BuSa81**] use the name M_5 for the lattice that is now generally called M_3 (see, e.g., [**HoMc88**]).

[10]According to Burris, Baker noticed that $3 \Rightarrow 4$ is actually an equivalence.

COROLLARY I.4.4 (Pixley [**Pix63**]). *If $\mathcal{V}$ has a ternary polynomial $p(x, y, z)$ which is a majority function, i.e.*

$$p(x, x, y) \approx p(x, y, x) \approx p(y, x, x) \approx x,$$

then $\mathcal{V}$ is congruence distributive.

EXAMPLE I.4.5. Every lattice, possibly with other operations added, generates a congruence distributive variety. Indeed, the term

$$(x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

is a majority function.

THEOREM I.4.6 (Jónsson's lemma [**Jón67**]). *If $\mathcal{V}$ is a congruence distributive variety generated by a finite algebra $\mathbf{A}$, then*

$$\mathcal{V}_{SI} = \mathcal{V}_{FSI} \subseteq \mathrm{HS}(\mathbf{A}).$$

Hence $\mathcal{V}$ has a finite residual bound and $\mathcal{V}_{SI} = \mathcal{V}_{FSI}$ is strictly elementary.

PROPOSITION I.4.7. *Let $\mathbf{A}_i, \mathbf{C}$ be algebras in the same signature, where $\mathbf{C}$ is finite and congruence distributive. Let $\overline{p}, \overline{q}, \overline{r}, \overline{s} \in \mathbf{C}$. If there is an embedding $\mathbf{C} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$, then*

$$(\overline{r}, \overline{s}) \in \mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}) \quad \textit{iff} \quad \mathbf{A}_i \models (r_i, s_i) \in \mathrm{Cg}^{\mathbf{A}_i}(p_i, q_i) \textit{ for all } i$$

PROOF. ($\Rightarrow$) is clear.
($\Leftarrow$) By the right hand side,

$$\mathrm{Cg}^{\mathbf{C}}(\overline{r}, \overline{s}) \leq \mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}) \vee \ker \pi_i$$

for any i. Since $\mathbf{C}$ is finite there are only finitely many possible kernels, so the distributive law applies:

$$\mathrm{Cg}^{\mathbf{C}}(\overline{r}, \overline{s}) \leq \bigcap_{i \in I} \left(\mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}) \vee \ker \pi_i\right) =$$
$$= \mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}) \vee \bigcap_{i \in I} \ker \pi_i = \mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}) \vee 0_{\mathbf{A}} = \mathrm{Cg}^{\mathbf{C}}(\overline{p}, \overline{q}). \quad \square$$

I.4.3. Congruence modular varieties.

PROPOSITION I.4.8. *If $\mathbf{A}$ is either congruence permutable or congruence distributive, then $\mathbf{A}$ is congruence modular.*

THEOREM I.4.9 ([**FrMc81**, Theorem 8]). *If $\mathbf{A}$ is finite and $\mathrm{HSP}\,\mathbf{A}$ is congruence modular, then $\mathrm{HSP}\,\mathbf{A}$ is residually small iff it has a finite residual bound.*

I.4.4. Congruence meet-semidistributive varieties.

THEOREM I.4.10. *Let $\mathcal{V}$ be a variety. Then the following conditions are equivalent:*

(1) *$\mathcal{V}$ is congruence meet-semidistributive (*CSD($\wedge$) *in short), i.e. for any $\mathbf{A} \in \mathcal{V}$ and any $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, the following quasi-equality holds:*

$$\alpha \cap \beta = \alpha \cap \gamma \;\Rightarrow\; \alpha \cap \beta = \alpha \cap (\beta \vee \gamma).$$

(2) *For any $\mathbf{A} \in \mathcal{V}$ and any $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, the following quasi-equation holds:*

$$\alpha \cap \beta = \alpha \cap \gamma = 0_{\mathbf{A}} \;\Rightarrow\; \alpha \cap (\beta \vee \gamma) = 0_{\mathbf{A}}.$$

(3) *In the $\mathcal{V}$-free algebra over $\{x, y, z\}$ we have $(x, z) \in \beta_n$ for some n, where*

$$\begin{aligned} &\alpha = \mathrm{Cg}(x, z) & & \\ &\beta = \mathrm{Cg}(x, y) = \beta_0 & &\beta_{n+1} = \beta \vee (\alpha \cap \gamma_n) \\ &\gamma = \mathrm{Cg}(y, z) = \gamma_0 & &\gamma_{n+1} = \gamma \vee (\alpha \cap \beta_n) \end{aligned}$$

(4) [**Wil00**], formulation from [**Jež08**] *$\mathcal{V}$ has Willard terms, i.e. there exist finitely many ternary terms s_e, t_e such that*
- *for every e, $\mathcal{V} \models s_e(x, y, x) \approx t_e(x, y, x)$*
- *$x = y$ iff for all e,*
$$s_e(x, x, y) = t_e(x, x, y) \;\Leftrightarrow\; s_e(x, y, y) = t_e(x, y, y).$$

(5) [**Wil00**], formulation from [**BMW04**] *$\mathcal{V}$ has Willard terms, i.e. there exists a bracket expression*[11] *β and terms $p_0, \ldots, p_n$ such that*

$$\begin{aligned} &p_0(x, y, z) \approx x, & & \\ &p_n(x, y, z) \approx z, & & \\ &p_i(x, x, y) \approx p_{i+1}(x, x, y) & &\textit{even } i < n, \\ &p_i(x, y, y) \approx p_{i+1}(x, , y) & &\textit{odd } i < n, \\ &p_i(x, y, x) \approx p_j(x, y, x) & &\textit{if the brackets at the } i\textit{-th and the } j\textit{-th} \\ & & &\quad \textit{position of } \beta \textit{ bound a bracket expression.} \end{aligned}$$

(6) [**Wil00**] *For all $\mathbf{A} \in \mathcal{V}$ and $a_0, \ldots, a_n \in \mathbf{A}$, if $a_0 \neq a_n$ then there exists $i \in \mathbb{Z}_n$ such that* $\mathrm{Cg}(a_0, a_n) \cap \mathrm{Cg}(a_i, a_{i+a}) \neq 0_{\mathbf{A}}$.

---

[11] By a *bracket expression* we mean a string of bracket symbols, e.g. $\langle\langle\langle\rangle\langle\rangle\rangle\langle\rangle\rangle$, constructed recursively by the rules
1° $\langle\rangle$ is a bracket expression
2° for $k \geq 1$ and bracket expressions $\beta_1, \ldots, \beta_k$, $\langle\beta_1 \ldots \beta_k\rangle$ is a bracket expression.

(7) [**HoMc88**] *The 3-generated $\mathcal{V}$-free algebra is congruence meet-semidistributive.*

(8) [**KeSz98**], [**Lip98**] *The lattice M_3 is not a sublattice of* $\operatorname{Con}\mathbf{A}$ *for any* $\mathbf{A} \in \mathcal{V}$.

Proof. $(1 \Rightarrow 2)$ is clear, while $(2 \Rightarrow 1)$ follows from the fact that we can factor $\mathbf{A}$ by the congruence $\alpha \cap \beta$ to obtain an algebra $\mathbf{A}' \in \mathcal{V}$ to which (2) may be applied.

$(1 \Rightarrow 3)$ We define $\beta_\infty := \bigcup_{n\in\mathbb{N}} \beta_n$ and $\gamma_\infty := \bigcup_{n\in\mathbb{N}} \gamma_n$.

Here are a few facts about the congruences α_n and β_n:

Claim 1. *For all i, $\beta_i \subseteq \beta_{i+1}$ and $\gamma_i \subseteq \gamma_{i+1}$.*

Simple exercise: show that $\beta_0 \subseteq \beta_1$ and $\gamma_0 \subseteq \gamma_1$, and then proceed by induction.

Claim 2. *β_∞ and γ_∞ are congruences; moreover, $\alpha \cap \beta_\infty = \alpha \cap \gamma_\infty$.*

β_∞ and γ_∞ are unions of chains of congruences, and hence $\beta_\infty, \gamma_\infty \in \operatorname{Con}\mathbf{A}$. If $(a,b) \in \alpha \cap \beta_\infty$, then $(a,b) \in \beta_n$ for some n. Therefore, $(a,b) \in \alpha \cap \beta_n \subseteq \gamma \vee (\alpha \cap \beta_n) = \gamma_{n+1} \subseteq \gamma_\infty$. This shows that $\alpha \cap \beta_\infty \subseteq \alpha \cap \gamma_\infty$ and analogically we prove the other inclusion.

Now we are ready to prove (3) from (1):

$$(x,z) \in \alpha \cap (\beta \vee \gamma) \subseteq \alpha \cap (\beta_\infty \vee \gamma_\infty) = \alpha \cap \beta_\infty$$

and thus $(x,z) \in \beta_n$ for some n.

$(3 \Rightarrow 4)$ In the following construction, the ternary terms s_e, t_e are indexed by finite sequences of natural numbers. We construct the indexing set E as a union of inductively defined sets $E_k, k = 0, \dots, n$, where n is such that $(x,z) \in \beta_n$, and each E_k contains sequences of length at most k. Moreover, we construct pairs

$$(s_e, t_e), e \in E_k \text{ such that } \begin{cases} (s_e, t_e) \in \alpha \cap \beta_{n-k} & \text{if } k \text{ is even} \\ (s_e, t_e) \in \alpha \cap \gamma_{n-k} & \text{if } k \text{ is odd.} \end{cases}$$

Let E_0 contain a single element, the empty sequence, and put $s_\emptyset = x$ and $t_\emptyset = z$. Clearly, $(s_\emptyset, t_\emptyset) \in \alpha \cap \beta_n$.

Assume that E_k and s_e, t_e $(e \in E_k)$ are already defined for some $k < n$, and take any $e \in E_k$. If k is even, then

$$(s_e, t_e) \in \beta_{n-k} = \beta \vee (\alpha \cap \gamma_{n-k-1}),$$

so there exists a finite sequence of ternary terms[12] $s_{e,1}, t_{e,1}, \dots, s_{e,m}, t_{e,m}$ such that the consecutive pairs are members of the congruences as

[12] Notice that the elements of the sequence are indexed by sequences of length $k+1$ that extend the sequence e by a natural number $i \in \widehat{m}$.

suggested in the diagram:

$$s_e \ \sim_\beta \ s_{e,1} \ \sim_{\alpha\cap\gamma_{n-k}} \ t_{e,1} \ \dots \ s_{e,m} \ \sim_{\alpha\cap\gamma_{n-k}} \ t_{e,m} \ \sim_\beta \ t_e.$$

In other words,

$$(\bowtie) \qquad \begin{aligned} &(s_e, s_{e,1}) \in \beta \\ &(t_{e,m}, t_e) \in \beta \\ &(s_{e,i}, t_{e,i}) \in \alpha \cap \gamma_{n-k} && \text{for each } i \in \widehat{m} \\ &(t_{e,i}, s_{e,i+1}) \in \beta && \text{for each } i \in \mathbb{Z}_m. \end{aligned}$$

If k is odd, proceed analogically.

Let $E_{k+1} := \{s_{e,1}, t_{e,1}, \dots, s_{e,m}, t_{e,m}\}$. Put $E = E_0 \cup \dots \cup E_n$. The set E can be imagined as a rooted tree, with the root $\emptyset$ and leaves the sequences from E that cannot be extended to longer sequences in E (e.g. the sequences from E_n). From ($\bowtie$) we see that $(s_e, t_e) \in \alpha$ for any $e \in E$; hence $\mathcal{V} \models s_e(x, y, x) \approx t_e(x, y, x)$.

Let u, v be any ternary terms. Then

$$\begin{aligned} &\text{if } (u, v) \in \beta && \text{then } u(x, x, y) \approx v(x, x, y), \\ &\text{if } (u, v) \in \gamma && \text{then } u(x, y, y) \approx v(x, y, y). \end{aligned}$$

Claim 3. *If for all $e \in E$*

$$s_e(a, a, b) = t_e(a, a, b) \Leftrightarrow s_e(a, b, b) = t_e(a, b, b),$$

then in all cases

$$s_e(a, a, b) = t_e(a, a, b) \qquad \textit{and} \qquad s_e(a, b, b) = t_e(a, b, b).$$

Proof by induction on $n-k$. If e is a leaf then (s_e, t_e) belongs either to β or to γ, so one of the equalities holds and by the assumption the other one holds too. If $e \in E_k$ can be continued to sequences $(e, 1)$, $(e, 2)$, …, (e, m) as described above and k is even, then each two neighbours in the sequence

$$s_e(a, a, b) \ \sim_\beta \ s_{e,1}(a, a, b) \ \dots \ \sim_{\alpha\cap\gamma_{n-k}} \ t_{e,m}(a, a, b) \ \sim_\beta \ t_e(a, a, b)$$

are equal either because they are congruent modulo β or because of the induction hypothesis. Hence $s_e(a, a, b) = t_e(a, a, b)$ and $s_e(a, b, b) = t_e(a, b, b)$ follows from the equivalence. For odd k proceed similarly.

The claim shows that if

$$s_e(a, a, b) = t_e(a, a, b) \Leftrightarrow s_e(a, b, b) = t_e(a, b, b)$$

for all $e \in E$, then $a = s_\emptyset$ is equal to $b = t_\emptyset$.

($3 \Rightarrow 5$) This is the same as the proof of ($3 \Rightarrow 4$): let the sequence $p_0, \ldots, p_n$ be equal to the sequence $s_\emptyset, s_1, s_{11}, \ldots t_{1m'}, t_1, s_2, \ldots, t_m, t_\emptyset$ and in the corresponding bracket expression put '$\langle$' for any s_e and '$\rangle$' for any t_e.

($5 \Rightarrow 4$) We show that the Willard terms in the sense of (5) satisfy the conditions of (4). For $i < n$, let i^* be the index such that the brackets at the positions i and i^* bound a bracket expression. Let E be the set of $i < n$ such that $i < i^*$; for $i \in E$, let $s_i := p_i$ and $t_i := p_{i^*}$. The condition $\mathcal{V} \models s_e(x, y, x) \approx t_e(x, y, x)$ is clearly satisfied.

We need to show that if for all $i \in E$,

$$(\div) \qquad p_i(a, a, b) = p_{i^*}(a, a, b) \Leftrightarrow p_i(a, b, b) = p_{i^*}(a, b, b),$$

then $a = b$. The proof is the same as the proof of Claim 3: start with the shortest bracket subexpressions of β, that is $\langle\rangle$. For the corresponding indices i and $i^* = i + 1$, one of the equalities in $(\div)$ holds by

$$(\oplus) \qquad \begin{array}{ll} p_i(x, x, y) \approx p_{i+1}(x, x, y) & \text{for even } i < n, \text{ or} \\ p_i(x, y, y) \approx p_{i+1}(x, y, y) & \text{for odd } i < n, \end{array}$$

and the other one follows from $(\div)$. By induction, the interplay between $\oplus$ and $(\div)$ leads to $a = p_0 = p_n = b$.

($4 \Rightarrow 6$) We prove this in Lemma III.8.5.

($6 \Rightarrow 2$) $\rightsquigarrow$ For contradiction suppose that $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$ are such that $\alpha \cap \beta = \alpha \cap \gamma = 0_{\mathbf{A}}$ but $\alpha \cap (\beta \vee \gamma) \neq 0_{\mathbf{A}}$. There are elements $a = a_0, a_1, \ldots, a_n = b$ with $a \neq b$, $(a, b) \in \alpha$ and $(a_i, a_{i+1}) \in \beta \cup \gamma$ for $i \in \mathbb{Z}_n$. By (6) there is an $i \in \mathbb{Z}_n$ with $\operatorname{Cg}(a, b) \cap \operatorname{Cg}(a_i, a_{i+1}) \neq 0_{\mathbf{A}}$. But then either $\alpha \cap \beta \neq 0_{\mathbf{A}}$ or $\alpha \cap \gamma \neq 0_{\mathbf{A}}$. ↯

($1 \Rightarrow 7$) is clear.

($7 \Rightarrow 3$)

$$(x, z) \in \alpha \cap (\beta \circ \gamma) \subseteq \alpha \cap (\beta \vee \gamma) \subseteq \alpha \cap (\beta_\infty \vee \gamma_\infty) = \alpha \cap \beta_\infty$$

(we have used meet semi-distributivity in the last equation) and hence $(x, z) \in \beta_n$ for some n.

($2 \Rightarrow 8$) The three "middle" elements of M_3 do not satisfy the required quasi-equation.

We leave the fact that (8) implies the other conditions without proof. □

Definition I.4.11. We say that $\mathbf{A}$ is a *semilattice* iff there exists a binary term operation $\wedge$ on $\mathbf{A}$ such that

$$x \wedge x = x, \qquad x \wedge y = y \wedge x, \qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$$

EXAMPLE I.4.12. We show that if a variety has a semilattice operation, then it is congruence meet-semidistributive. In particular, if $\mathbf{A}$ is a semilattice, then $\operatorname{Con}\mathbf{A}$ is meet-semidistributive. We claim that the terms

$$\begin{aligned} s_1(x,y,z) &:= x \wedge y, & t_1(x,y,z) &:= y \wedge z, \\ s_2(x,y,z) &:= x \wedge y \wedge z, & t_2(x,y,z) &:= y \wedge z, \end{aligned}$$

are Willard terms. Indeed, if $x = y$ then

$$s_i(x,x,y) = t_i(x,x,y) \;\Leftrightarrow\; s_i(x,y,y) = t_i(x,y,y)$$

for $i \in \widehat{2}$. On the other hand, if $x < y$ or $y < x$, then the equivalence does not hold for $i = 1$; and finally if x and y are incomparable, then the equivalence does not hold for $i = 2$.

PROPOSITION I.4.13 ([**KeWi99**]). *Every residually finite, congruence meet-semidistributive variety has a finite residual bound.*

I.5. Quasivarieties

See definition I.1.2 for the definition of a quasi-equation, and Theorem I.1.6 for the definition of a quasivariety.

THEOREM I.5.1. *Let $\mathcal{Q}$ and $\mathcal{K}$ be algebraic classes. $\mathcal{Q}$ is a quasivariety iff $\mathrm{SPP_U}\,\mathcal{Q} = \mathcal{Q}$. In the case that $\mathcal{K}$ is up to isomorphism a finite set of finite algebras, the quasivariety generated by $\mathcal{K}$ is equal to $\mathrm{SP}(\mathcal{K})$.*

DEFINITION I.5.2. Let $\mathcal{Q}$ be a quasivariety, $\mathbf{A} \in \mathcal{Q}$ and $\alpha \in \operatorname{Con}\mathbf{A}$. We say that α is a *Q-congruence* iff $\mathbf{A}/\alpha \in \mathcal{Q}$. We denote the set of $\mathcal{Q}$-congruences of $\mathbf{A}$ by $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}$. We denote the join and meet in $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ by $\vee_{\mathcal{Q}}$ and $\wedge_{\mathcal{Q}}$.

PROPOSITION I.5.3.

(1) $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ *is closed under finite and infinite meets.*
(2) For any $\alpha \in \operatorname{Con}\mathbf{A}$, there exists the smallest $\overline{\alpha} \supseteq \alpha$ such that $\overline{\alpha} \in \operatorname{Con}_{\mathcal{Q}}\mathbf{A}$.
(3) If $\mathbf{A} \in \mathcal{Q}$, then $\overline{} : \operatorname{Con}\mathbf{A} \to \operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ preserves $0_{\mathbf{A}}$, $1_{\mathbf{A}}$ and $\vee$.
(4) $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ *is a complete algebraic lattice.*

PROOF. (1) This is due to the fact that intersection of congruences always corresponds to a subdirect product of factors. More precisely,

$$\begin{array}{cccc} f: & \mathbf{A}/\alpha\wedge\beta & \to & \mathbf{A}/\alpha \times \mathbf{A}/\beta \\ & [x]_{\alpha\wedge\beta} & \mapsto & ([x]_\alpha, [x]_\beta) \end{array}$$

is an injective homomorphism which is onto each of the coordinates, and the same may be done for infinite meets.

(2) is a direct consequence of (1): Let

$$\overline{\alpha} \quad := \bigwedge_{\substack{\alpha\subseteq\alpha',\\ \alpha'\in\operatorname{Con}_{\mathcal{Q}}\mathbf{A}}} \alpha'.$$

(3) Obviously, $\overline{} : \operatorname{Con}\mathbf{A} \to \operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ is monotone and preserves $1_{\mathbf{A}}$. $0_{\mathbf{A}} \in \operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ because $\mathbf{A} \in \mathcal{Q}$. We show that

$$\overline{\alpha \vee \beta} = \overline{\alpha} \vee_{\mathcal{Q}} \overline{\beta}.$$

$\alpha \vee \beta \subseteq \overline{\alpha} \vee_{\mathcal{Q}} \overline{\beta}$ and the right hand side is a $\mathcal{Q}$-congruence. Hence

$$\overline{\alpha \vee \beta} \subseteq \overline{\alpha} \vee_{\mathcal{Q}} \overline{\beta}.$$

The opposite inclusion is a consequence of $\overline{\alpha \vee \beta} \supseteq \overline{\alpha}$ and $\overline{\alpha \vee \beta} \supseteq \overline{\beta}$.

(4) Completeness follows directly from (1). $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ is algebraic, because each $\mathcal{Q}$-congruence is the join of its finitely generated subcongruences (due to (3)). □

Definition I.5.4. We say that a quasivariety $\mathcal{Q}$ has the *extension property* iff $\overline{} : \operatorname{Con}\mathbf{A} \to \operatorname{Con}_{\mathcal{Q}}\mathbf{A}$ is a lattice homomorphism, i.e., iff $\overline{}$ preserves meets. We say that $\mathcal{Q}$ has the *weak extension property*, *WEP* in short, iff for any $\mathbf{A} \in \mathcal{Q}$ and any $\alpha, \beta \in \operatorname{Con}\mathbf{A}$

$$\alpha \cap \beta = 0_{\mathbf{A}} \quad \Rightarrow \quad \overline{\alpha} \cap \overline{\beta} = 0_{\mathbf{A}}.$$

Definition I.5.5. Let $\mathcal{Q}$ be a quasivariety. We use terms such as *congruence distributive (CD)*, *congruence modular (CM)* or *congruence meet-semidistributive (CSD(∧))* referring to $\operatorname{Con}\mathbf{A}, \mathbf{A} \in \mathcal{Q}$, and *relatively congruence distributive ($\mathcal{Q}$-CD)* etc. referring to $\operatorname{Con}_{\mathcal{Q}}\mathbf{A}, \mathbf{A} \in \mathcal{Q}$.

Theorem I.5.6 ([**KeMc92, MaMc04, DMMN09**]). *For any quasivariety, the following implications hold:*

CD ⇔ CM + CSD(∧)
$\mathcal{Q}$-CD ⇔ $\mathcal{Q}$-CM + $\mathcal{Q}$-CSD(∧)
CSD(∧) ⇒ $\mathcal{Q}$ has Willard terms
$\mathcal{Q}$-CSD(∧) ⇔ WEP + $\mathcal{Q}$ has Willard terms
$\mathcal{Q}$-CD ⇒ $\mathcal{Q}$-CM ⇒ EP ⇒ WEP.

However, CD does not imply $\mathcal{Q}$-CD, and $\mathcal{Q}$-CD does not imply CD. Also, some quasivarieties have Willard terms but are not CSD(∧).

Definition I.5.7. Let $\mathcal{Q}$ be a quasivariety. We say that an algebra $\mathbf{A} \in \mathcal{Q}$ is *$\mathcal{Q}$-subdirectly irreducible* iff $0_{\mathbf{A}}$ is a completely meet-irreducible element of $\operatorname{Con}_{\mathcal{Q}} \mathbf{A}$.

Observation I.5.8. *$\mathbf{A}$ is $\mathcal{Q}$-subdirectly irreducible iff $\operatorname{Con}_{\mathcal{Q}} \mathbf{A}$ contains a monolith, in other words, iff*

$$\bigwedge_{\substack{\theta \in \operatorname{Con}_{\mathcal{Q}} \mathbf{A}, \\ \theta \neq 0_{\mathbf{A}}}} \theta \neq 0_{\mathbf{A}}.$$

Proposition I.5.9. *Let $\mathcal{Q}$ be a quasivariety. Then every subdirectly irreducible algebra $\mathbf{A} \in \mathcal{Q}$ is $\mathcal{Q}$-subdirectly irreducible.*

Proof. $\operatorname{Con}_{\mathcal{Q}} \mathbf{A} \subseteq \operatorname{Con} \mathbf{A}$; hence if $0_{\mathbf{A}}$ is completely meet-irreducible in $\operatorname{Con} \mathbf{A}$, it is also completely meet-irreducible in $\operatorname{Con}_{\mathcal{Q}} \mathbf{A}$. □

Example I.5.10. However, note that a $\mathcal{Q}$-subdirectly irreducible algebra need not be subdirectly irreducible. As an example, take the algebra $(\mathbb{Z}, +)$ in the quasivariety defined by infinitely many quasi-equations of the form

$$nx = 0 \;\Rightarrow\; x = 0,$$

where n is any integer and nx is an abbreviation for $x + x + \cdots + x$. Any homomorphic image of $\mathbb{Z}$ is of the form $\mathbb{Z}_n$ for some natural number n. Because

$$\mathbb{Z}_n \not\models nx = 0 \;\Rightarrow\; x = 0,$$

$\mathbb{Z}$ does not have any $\mathcal{Q}$-congruences other than $0_{\mathbb{Z}}$ and $1_{\mathbb{Z}}$, and hence is $\mathcal{Q}$-subdirectly irreducible. However, $\mathbb{Z}$ is not subdirectly irreducible, because

$$\mathbb{Z} \leq \prod_{p \text{ is prime}} \mathbb{Z}_p.$$

Indeed, each factor morphism $\pi_p : \mathbb{Z} \to \mathbb{Z}_p$ is onto, and any two elements $n, k \in \mathbb{Z}$ may be distinguished by the morphism π_p for any $p > n, k$.

More generally, if $\mathcal{Q}$ and $\mathcal{Q}'$ are two different quasivarieties, an algebra $\mathbf{A}$ may be $\mathcal{Q}$-subdirectly irreducible, but not $\mathcal{Q}'$-subdirectly irreducible. This is not the case in varieties, where being subdirectly irreducible is not relative to any particular variety.

Theorem I.5.11 (Mal'cev; Burris [**Bur76**]). *Any $\mathbf{A} \in \mathcal{Q}$ is isomorphic to a subdirect product of $\mathcal{Q}$-subdirectly irreducible algebras.*

Proof. We prove the following claim later:

Claim 1. *Any element of an algebraic lattice L may be represented as the meet of completely meet-irreducible elements.*

By the claim, $0_{\mathbf{A}}$ is the meet of all completely meet-irreducible elements in $\operatorname{Con}_{\mathcal{Q}} \mathbf{A}$. Similarly as in varieties,

$$\operatorname{Con}_{\mathcal{Q}} {}^{\mathbf{A}}\!/_{\theta} \quad \simeq \quad [\theta, 1_{\mathbf{A}}] \subseteq \operatorname{Con}_{\mathcal{Q}} \mathbf{A};$$

thus if $\theta \in \operatorname{Con}_{\mathcal{Q}} \mathbf{A}$ is completely meet-irreducible in $\operatorname{Con}_{\mathcal{Q}} \mathbf{A}$, then ${}^{\mathbf{A}}\!/_{\theta}$ is $\mathcal{Q}$-subdirectly irreducible. Hence

$$\mathbf{A} \hookrightarrow \prod_{\substack{\theta \in \operatorname{Con}_{\mathcal{Q}} \mathbf{A}, \\ \theta \text{ is completely} \\ \wedge\text{-irreducible}}} {}^{\mathbf{A}}\!/_{\theta}$$

is a subdirect representation of $\mathbf{A}$.

PROOF OF THE CLAIM. BY CONTRADICTION.

Let M be the set of the completely meet-irreducible elements of L. By definition,

$$a \leq a' := \bigwedge_{\substack{x \in M \\ a \leq x}} x$$

for every $a \in \mathbf{A}$. ⇝ Suppose the equality does not hold, i.e.

$$a < a'.$$

In an algebraic lattice, each element is uniquely determined by the set of compact elements below it. Hence there is a compact element c,

$$c \leq a' \quad \text{but} \quad c \not\leq a.$$

Let

$$Z := \{x \in \mathbf{A}; a \leq x, c \not\leq x\}.$$

Clearly, $a \in Z$. We show that we may apply Zorn's lemma to Z: indeed, if C is a chain in Z, then $a \leq \bigvee C$. Moreover, $c \leq \bigvee C$ means that there is some $z \in C$ such that $c \leq z$, which immediately leads to a contradiction. Thus $\bigvee C \in Z$.

Hence Z includes a maximal element, say m. If $m \in M$, then $m \geq a' \geq c$, which is a contradiction. Hence

$$m = \bigwedge B,$$

where $B \subseteq \mathbf{A}$ and for every $b \in B$, $b \neq m$. Since m is maximal in Z and $a \leq b$ for every $b \in B$, we have $c \leq b$ for every b. However, then $c \leq m$, a contradiction.↯ □

CHAPTER II

An overview of known finite basis results

Probably the first result concerning the finite basis of equations is due to G. Birkhoff [**Bir35**]: *the set of equations in n variables true in a finite algebra is finitely based.* Already two years later, B. H. Neumann [**Neu37**] posed the following question: Does every finite group have a finite basis of equations?

The first finite non-finitely based algebra was discovered by R. C. Lyndon in [**Lyn54**]. Since then, the finite basis question has become one of the most researched topics in universal algebra, both in the stricter sense (is a finite algebra finitely based?) and in the broader sense (is a variety finitely based?).

The difference between the two questions is well illustrated by the case of lattices: every finite lattice is finitely based according to [**McK70**]; however, not all varieties of lattices are finitely based: an example of a variety of modular lattices that is not finitely based is given in [**Bak69**].

Concerning the stricter question, the following result is of interest:

FACT II.0.12 (Murskiĭ [**Mur75**]). *"Almost all" finite algebras are finitely based. More exactly, for a fixed type σ*

$$\lim_{k\to\infty} \frac{|\{\mathbf{A}; |\mathbf{A}| = k, \mathbf{A} \textit{ is finitely based}\}|}{|\{\mathbf{A}; |\mathbf{A}| = k\}|} = 1;$$

and for a fixed natural number k

$$\lim_{|\sigma|\to\infty} \frac{|\{\mathbf{A}; |\mathbf{A}| = k, \mathbf{A} \textit{ is finitely based}\}|}{|\{\mathbf{A}; |\mathbf{A}| = k\}|} = 1,$$

where $|\sigma|$ is the number of operations in σ, $\mathbf{A} = (A, \sigma)$, and the number of algebras is taken up to isomorphism.

McKenzie showed in [**McK96c**] that the answer to the question "is the given finite algebra finitely based?" is not recursive. In other words, *there is no algorithm which determines whether a given finite algebra is finitely based.* This only confirms an earlier finding that finitely and non-finitely based algebras would be hard to distinguish: all finite groups are finitely based [**OaPo64**], but Bryant [**Bry82**] gives an example of a non-finitely based finite group with just one extra constant operation. As the congruence lattice of a pointed group is the same as the congruence lattice of its underlying group, this example destroys all hope for a simple characterisation of finitely based algebras in the setting of universal algebra.

Yet, McKenzie also showed in [**McK96b**] that *there is no algorithm which would determine the residual size of a finite algebra.* Thus, there still remains enough space for the famous Park's conjecture:

CONJECTURE II.0.13 (Park [**Par76**]). *Every finitely generated variety of finite type and with a finite residual bound is finitely based.*

The most important finite basis results confirm Park's conjecture under additional assumptions on the congruence lattices of the algebras in the variety:

– any *variety with definable principal congruences and a finite residual bound* [**McK78**]

– any *finite algebra generating a congruence distributive variety* (the finite residual bound is a consequence of the other conditions) [**Bak77**]. The theorem has been reproved many times, and extended in the following directions:

- *congruence distributive varieties such that $\mathcal{V}_{FSI}$ is strictly elementary* [**Jón79a**];
- *finite algebras generating a residually small, congruence modular variety* [**McK87**]. The assumption that the generated variety is residually small cannot be omitted, as is shown by an example of a non-finitely based finite non-associative ring [**Pol76**], and an example of a non-finitely based finite pointed group [**Bry82**][1]
- *varieties with definable principal subcongruences and strictly elementary $\mathcal{V}_{SI}$* [**BaWa02**];
- *congruence meet-semidistributive varieties* with a finite residual bound [**Wil00**]. By [**KeWi99**], *residually finite* is enough. Willard's theorem has been extended in several directions:
  - *locally finite, congruence meet-semidistributive varieties with bounded critical diameter such that $\mathcal{V}_{SI}$ is axiomatizable and $\mathcal{V}_{FSI}$ is strictly elementary* [**BMW04**]: an algebra to which this theorem applies but Willard's theorem does not is given in the article;
  - [**MaMc04**] extend Willard's theorem to the setting of quasivarieties.

[1]Both rings and groups have Mal'cev terms and thus generate congruence permutable varieties. See also Theorem I.4.9.

The finite basis question has been extensively studied in well known classes of algebras. Finitely based are:

– any *primal algebra*, i.e. a nontrivial finite algebra such that every function on **A** is a polynomial [**Ros42, Yaq57**].

– concerning rings, positive answers have been obtained for:
- *finite rings* [**Lvo73, Kru73**],
- *nilpotent rings* and *commutative rings* [**BaMa75**],
- *finite non-associative rings without zero divisors* [**Lvo75**].

However, [**Pol76**] gives an example of a finite non-associative ring which is not finitely based.

– *finite groups* [**OaPo64**]. Not all varieties of groups are finitely based [**Ols70, Vau70**]. There is not much hope that this theorem could be extended in the setting of universal algebra, due to an example of a non-finitely based finite pointed group [**Bry82, Ali91**]. Also note that the case of finite groups does not fall under Park's conjecture, as either of the two eight-element non-commutative groups generates a residually large variety.

– in the case of semigroups, positive results have been obtained for:
- *varieties of commutative semigroups* [**Per69**],
- any *uniformly periodic permutative semigroup*, i.e. a semigroup satisfying $x^n \approx x^{n+k}$ for some $k \geq 1$ and a permutation identity $x_1x_2 \dots x_n \approx x_{\pi(1)}x_{\pi(2)} \dots x_{\pi(n)}$ [**Per69**],
- *bands*, i.e. idempotent semigroups [**Bir70, Fen71, Ger70**].

Negative results in the case of semigroups include
- a 6-element monoid (semigroup with identity) [**Per69**] (this is the smallest possible non-finitely based semigroup),
- an infinite permutative semigroup [**Per69**].

See [**Vol01**] for an overview of other finite basis results for semigroups.

– any finite simple 2-generated quasigroup [**McK76**].

– *finite lattices* possibly with other operations added [**McK70**].

However, not all varieties of lattices are finitely based:
- an example of a variety of modular lattices that is not finitely based is given in [**Bak69**];
- an example of a non-finitely based infinite lattice can be found in [**McK70**];
- examples that the join of two finitely based lattice varieties need not be finitely based can be found in [**Bak82**] and [**Jón76**].

– any finite algebra **A** such that every subalgebra of **A** is simple and **A** generates a congruence permutable variety [**McK78**].

– any *finite Lie algebra over a finite ring* [**BaOl75**]. A Lie algebra over non-noetherian ring is not finitely based. For more information on finite basis question in Lie algebras, see [**Bah87**].

Some finite basis results have been obtained for what we could call "small cases". This includes

– *all 2-element algebras* [**Lyn51**]; this cannot be extended, which is shown by Murskiĭ's three element grupoid which is not finitely based [**Mur65**].

– *3, 4 and 5-element semigroups* [**Per69, Bol71, Tis80, Tra83**]; an example of a 6-element non-finitely based semigroup is given in [**Per69**].

– *all varieties in finite poor signatures*, i.e. signatures with finitely many constants and at most one at most unary operation symbol [**Jež69**]. Murskiĭ's grupoid shows that this cannot be extended to binary operations.

– *locally finite varieties in signatures consisting only of constants and unary symbols* - this is a consequence of Birkhoff's theorem. The assumption of local finiteness is necessary: [**Jež69**] describes uncountably many varieties in the signature with two unary symbols, each one of them generated by a single countably infinite algebra; however, for any finite signature, there are only countably many finite bases, hence there exists a non-finitely based variety in the signature with two unary symbols.

– *locally finite varieties with polynomial free spectrum*, i.e. varieties where the cardinality of the n-generated free algebra can be bound by some polynomial.

The following table sums up the connections between some of the results listed above; it also gives resources where alternative proofs can be found:

original result of	has also been proved in	and has been extended in
[**Lyn51**]	[**Ber80**]	
[**Per69**]	[**Eva71**]	[**Bol71, Tis80, Tra83**]
[**McK70**]		[**Bak77**]
[**Bak77**]	[**Mak73, Tay78, Bur79**]	[**Jón79a, Wil00, BaWa02**]
[**McK78**]		[**BaWa02**]
[**Wil00**]		[**KeWi99, BMW04, MaMc04**]

Finitely based varieties include:

reference	special condition	residual condition	congruence condition	section
[**Bir35**]	$\lvert\mathbf{F}_{\mathcal{V}}(n)\rvert < \omega$, n variables			III.1.1
[**Sha87**]	polynomial free spectrum			III.1.2
[**Jež69**]	poor signature			III.1.3
[**Per69**]	commutative semigroups			III.3
[**McK78**]		fin. res. bound	def. principal congr.	III.5.1
[**Jón79a**]		$\mathcal{V}_{FSI}$ fin. axiom.	distributive	III.7.1
[**Wil00**]		res. finite	$\wedge$-semidistributive	III.8
[**BaWa02**]		$\mathcal{V}_{FSI}$ fin. axiom.	def. principal subcongr.	III.5.3

The following finite algebras are also finitely based:

reference	type	residual condition	congruence condition	section
[**Bak77**]			distributive	III.7
[**McK78**]	$\mathbf{B} \leq \mathbf{A} \Rightarrow \mathbf{B}$ is simple		permutable	
[**McK87**]		res. small	modular	
[**Ros42, Yaq57**]	primal			
[**OaPo64**]	group			
[**Kru73, Lvo73**]	ring			
[**McK70**]	has lattice operations			
[**Lyn51**]	2-element			III.1.4
[**Per69, Bol71, Tra83**]	1–5-element semigroup			
[**Bir70, Fen71, Ger70**]	possibly infinite idempotent semigroup			
[**Per69**]	possibly infinite uniformly periodic semigroup satisfying a permutation identity			

CHAPTER III

Methods for proving finite basis results

III.1. Small cases

III.1.1. Birkhoff's Theorem. Probably the oldest finite basis theorem is due to Birkhoff and dates as far back as 1935:

PROPOSITION III.1.1 (Birkhoff [**Bir35**]). *Let $\mathcal{V}$ be a variety which has an equational base in n variables. If the n-generated free algebra in $\mathcal{V}$ is finite, then $\mathcal{V}$ is finitely based.*

PROOF. Let $\mathbf{T}_n = \mathrm{Term}_\sigma(n) = \mathrm{Term}_\sigma(\{x_1, \dots, x_n\})$ and let $\mathbf{F}_n$ be the free algebra over $\{x_1, \dots, x_n\}$ in $\mathcal{V}$. There is a unique homomorphism

$$\begin{aligned} h :\ \mathbf{T}_n &\to \mathbf{F}_n \\ x_i &\mapsto x_i \qquad i \in \widehat{n}. \end{aligned}$$

For every $a \in \mathbf{F}_n$, take one $a^* \in \mathbf{T}_n$ such that $h(a^*) = a$; specifically, take $x_i^* = x_i$. This means that the set $Q := \{a^*,\ a \in \mathbf{F}_n\}$ is a set of representatives of the partition $\mathbf{T}_n/_{\ker h}$. Let E be the set of equations

$$\begin{aligned} E = \{f(a_1^*, \dots, a_k^*) \approx a^*;\quad & f \in \sigma \text{ is an operation symbol of arity } k, \\ & a_1, \dots, a_k, a \in \mathbf{F}_n, \quad f(a_1, \dots, a_k) = a\}. \end{aligned}$$

Since both σ and $\mathbf{F}_n$ are finite, E is also finite. Moreover, $E \subseteq \mathrm{Eq}\,\mathcal{V}$. We show that E is a basis for $\mathrm{Eq}\,\mathcal{V}$. By the assumption, $\mathcal{V}$ has a basis in n variables and therefore $\mathcal{V} = \mathrm{Mod}\,\mathrm{Eq}_n\,\mathcal{V}$. Thus it is enough to show that if $s \approx t \in \mathrm{Eq}_n\,\mathcal{V}$, then $E \models e \approx t$.

Let $\mathbf{A}$ be any model of E. We show that if $(s,t) \in \mathrm{Eq}_n\mathcal{V}$ and $e :\ \{x_1, \dots, x_n\} \to \mathbf{A}$ is any evaluation of variables, then $\overline{e}(s) = \overline{e}(t)$, where $\overline{e} :\ \mathbf{T}_n \to \mathbf{A}$ is the unique extension of e.

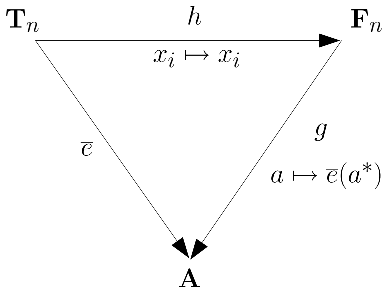
Define a mapping

$$\begin{aligned} g :\quad \mathbf{F}_n &\to \mathbf{A} \\ a &\mapsto \overline{e}(a^*). \end{aligned}$$

g is a homomorphism. Indeed, for any $f \in \sigma$ and $a_1, \ldots, a_k, a \in \mathbf{F}_n$ such that $f(a_1, \ldots, a_k) = a$, we have

$$\begin{aligned} f_{\mathbf{A}}(g(a_1), \ldots, g(a_k)) &= f_{\mathbf{A}}(\overline{e}(a_1^*), \ldots, \overline{e}(a_k^*)) = \\ &= \overline{e}(f(a_1^*, \ldots, a_k^*)) = \\ &= \overline{e}(a^*) = g(a) = \\ &= g(f(a_1, \ldots, a_k)). \end{aligned}$$

For any i we have $gh(x_i) = g(x_i) = \overline{e}(x_i^*) = e(x_i)$, so the diagram



commutes. Consequently, $\overline{e}(s) = gh(s) = gh(t) = \overline{e}(t)$.

This finishes the proof that any model of E is a model of $\mathrm{Eq}_n\mathcal{V}$. □

This theorem says that if we place a finite bound on the number of variables, then every locally finite variety is finitely based. In particular it means that in a non-finitely based finite algebra, there must be equations with arbitrarily large number of variables which cannot be derived from equations with fewer variables.

It is also useful to compare Birkhoff's theorem with Proposition I.1.18. In I.1.18 (1) we assume an upper bound on the length of terms, which is a much stricter assumption than the upper bound on the number of variables in Birkhoff's Theorem.

III.1.2. Varieties with polynomial free spectrum.

Definition III.1.2. We define the *free spectrum* of a variety as

$$f_{\mathcal{V}}(n) := |\mathbf{F}_{\mathcal{V}}(n)|.$$

The free spectrum may be thought of as the cardinality of the clone of operations of that variety.

Example III.1.3. For Boolean algebras, $f_{BA}(n) = 2^{2^n}$. Indeed, the free Boolean algebra generated by $\{x_1, \ldots, x_n\}$ contains the elements $x_1^{\perp}, \ldots, x_n^{\perp}$, which together give rise to 2^n elements of the form

$$x_1^{\epsilon_1} \wedge x_2^{\epsilon_2} \wedge \cdots \wedge x_n^{\epsilon_n},$$

where $\epsilon_i \in \{\emptyset, \perp\}$. Finally, it can be proved that each element of the n-generated Boolean algebra may be expressed as a join of some of these 2^n elements (this corresponds to the disjunctive normal form in logic); as there are 2^{2^n} such joins (including the empty join, which corresponds to the least element of the Boolean algebra),

$$|\mathbf{F}_{BA}(n)| = 2^{2^n}.$$

Observation III.1.4.

- *$\mathcal{V}$ is locally finite iff $f_\mathcal{V}(n)$ has a finite value for every n.*
- *If $\mathcal{V} = \mathrm{HSP}\,\mathbf{A}$ and $|\mathbf{A}| = k$, then $f_\mathcal{V}(n) \leq k^{k^n}$, so the free spectrum is always at most doubly exponential.*

The variety of Boolean algebras, generated by the two-element Boolean algebra, certifies that the upper bound in the observation can not be improved.

Proposition III.1.5 (Berman). *The following conditions are equivalent for any variety $\mathcal{V}$ and any integer $k \geq 1$:*

(1) $f_\mathcal{V}(n) \leq c \cdot n^k$ for some c and every n.
(2) $\mathcal{V}$ is locally finite and has no term operations depending on more than k variables.
(3) There is a polynomial $p(x)$ with rational coefficients and of degree at most k such that for large n, $|\mathbf{F}_\mathcal{V}(n)| = p(n)$.

Proof. $(1 \Rightarrow 2)$ Let e_n be the number of functions in the clone of $\mathcal{V}$ that depend exactly on n variables. Because we can identify the free algebra with the partition of the term algebra by the equality of terms, we see that

$$(\triangleright) \qquad f_\mathcal{V}(n) = |\mathbf{F}_\mathcal{V}(n)| = \sum_{i=0}^{n} e_i \cdot \binom{n}{i}.$$

Thus for any $n > i > k$,

$$c \cdot n^k \geq f_\mathcal{V}(n) \geq e_i \cdot \binom{n}{i}.$$

The inequality

$$\binom{n}{i} \geq \frac{n^i}{i^i}$$

implies that

$$c\, i^i \cdot \frac{n^k}{n^i} \geq e_i$$

for arbitrarily large n; the left hand side tends to zero, hence $e_i = 0$. Thus (1) implies (2).

(2 ⇒ 3) By the assumption, $e_n = 0$ for all $n > k$. But then the sum on the right hand side of (▷), viewed as a function of n, defines a polynomial.
(3 ⇒ 1) is clear. □

Definition III.1.6. If the conditions of the previous theorem are satisfied, then we say that $\mathcal{V}$ has *polynomial free spectrum.*

Proposition III.1.7 (Shapiro [**Sha87**]). *Let $\mathcal{V}$ be a locally finite variety with polynomial free spectrum. Then $\mathcal{V}$ is finitely based.*

Proof. Let k be such that each term depends on at most k variables; we assume that k is greater than any arity of a basic operation. By assumption $\mathbf{F}_{\mathcal{V}}(n)$ is finite for any n, and by Birkhoff's theorem $\mathcal{V}^{(n)} = \operatorname{Mod}\operatorname{Eq}_n\mathcal{V}$ is finitely based. Let E be the finite basis of $\mathcal{V}^{(2k)}$.

We proceed in a similar way as in the proof of Birkhoff's theorem: let Q be a finite set of terms representing elements in $\mathbf{F}_{\mathcal{V}}(k)$. Then for each term t, there exists a term $t^* \in Q$ such that

$$\mathcal{V} \models t \approx t^*.$$

If $\mathcal{V} \models t \approx s$, then also $E \models t^* \approx s^*$; the only problem is to find a finite set of identities I such that $I \models t \approx t^*$.

For a basic operation ω and terms $q_{i_0}, \dots, q_{i_{l-1}} \in Q$ there is a term $q_{i_l} \in Q$ such that

$$\mathcal{V} \models \omega(q_{i_0}(x_1, \dots, x_k), \dots, q_{i_{l-1}}(x_{(l-1)k+1}, \dots, x_{lk})) \approx q_{i_l}(y_1, \dots, y_k),$$

where $\{y_1, \dots, y_k\} \subseteq \{x_1, \dots, x_{lk}\}$. Let I be the set of all of these equations. I is finite, because both Q and the signature are finite. It follows by induction on complexity of t that indeed $I \models t \approx t^*$. □

III.1.3. Poor signatures. The following Proposition is an immediate consequence of Birkhoff's theorem:

Proposition III.1.8. *A locally finite variety in a language σ which contains only constants and unary symbols is finitely based.*

Proof. Any equation in such a language contains at most two different variables (one on each side of the equation). □

If we drop the assumption of local finiteness, we have at least the following weaker version of the Proposition:

Definition III.1.9. We say that the signature σ is *poor* iff it consists only of constants and at most one unary operation symbol.

Proposition III.1.10 (Ježek [**Jež69**]). *Every variety of a finite poor signature is finitely based.*

Proof. In the case that the signature σ is just a finite set C of constants, any non-trivial equational theory is an equivalence on C, and hence it is finite. Therefore we assume that there is a unary operation symbol $f \in \sigma$. We define $f^n, n \geq 0$ inductively:

- $f^0(x) = x$
- $f^{n+1}(x) = f(f^n(x))$

Let E be a non-trivial equational theory.

The following table summarises the proof: the set of equations of the form given in the first column, with the scope of variables n, m and k given in the second column, is equivalent to the equation(s) in the third column (where "the least (n, m)" means "the equation of the given form with the pair (n, m) the least in the lexicographical order", and similarly with "the least k"). The last column gives the number of equations of the given form that appear in the final (finite) basis. Let x, y be two different variables, and let $c, d \in C$ represent any given pair of two different constants. $k, n, m, l \geq 0$ denote natural numbers.

$f^n(x) \approx f^m(c)$	$0 \leq n, m$	$f^n(x) \approx f^n(y)$ and $f^n(c) \approx f^m(c)$	
$f^n(x) \approx f^m(y)$	$0 \leq n < m$	$f^n(x) \approx f^n(y)$ and $f^n(y) \approx f^m(y)$	
$f^k(x) \approx f^k(y)$	$0 \leq k$	the least k	1
$f^n(x) \approx f^m(x)$	$0 \leq n < m$	the least (n, m)	1
$f^n(c) \approx f^m(d)$	$0 \leq n, m$	the least (n, m) and equations of the form $f^k(c) = f^l(c)$	$\lvert C\rvert^2 - \lvert C\rvert$
$f^n(c) \approx f^m(c)$	$0 \leq n < m$	the least (n, m)	$\lvert C\rvert$

This means that there is a basis with at most $|C|^2+2$ equations. $\square$

III.1.4. Two-element algebras.

Whilst the previous proposition gives a positive answer to Park's conjecture for varieties of particularly small signatures, the following one solves it for varieties generated by particularly small algebras.

Proposition III.1.11 (Lyndon [**Lyn51**]). *Any two-element algebra is finitely based.*

Lyndon's result lies in the scope of Park's conjecture:

Fact III.1.12 (Taylor [**Tay76**]). *The variety generated by a two-element algebra in any (possibly infinite) language has at most three subdirectly irreducible algebras, each of cardinality at most three.*

Lyndon's original proof relies on Post's classification of all clones on a two-element algebra [**Pos41**]. Corollary I.1.21 is crucial, as it means that only one algebra from each class of algebras with the same clone needs to be studied.[1] Furthermore, each two-element algebra

[1]Post's monograph showed that any two element algebra of arbitrary similarity type generates an equational class polynomially equivalent to an equational class

is isomorphic to its *dual algebra* (i.e. the algebra obtained under the interchange of the two elements 0 and 1), so only one of each pair of dual algebras needs to be investigated. Nonetheless, four countably infinite families of algebras are treated in the paper, as well as over twenty specific algebras.

One of the four infinite families consists of algebras whose clone is generated by the classical implication function[2] together with the function

$$\begin{aligned} d_n(x_1,\dots,x_n) &= \bigvee_{i\in\hat{n}}(x_1\wedge\ldots\wedge x_{i-1}\wedge x_{i+1}\wedge\ldots\wedge x_n) \\ &= x_2x_3\cdots x_n\vee x_1x_3\cdots x_n\vee\cdots\vee x_1x_2\cdots x_{n-1}. \end{aligned}$$

Lyndon uses the following earlier result:

Fact III.1.13 (Henkin [**Hen49**]). *Every two-valued logic containing the classical implication and closed under modus ponens is deductively axiomatizable.*

By deductive axiomatizability we mean that there exists a finite set S of formulae such that all other true formulae of the given logic can be derived from S by the use of substitution and modus ponens. Lyndon extends this result into the following lemma:

Lemma III.1.14. *If the clone of* **A** *contains the classical implication function, then* **A** *is finitely based.*

Proof. 1 is in the clone of **A**, because $1 = x\Rightarrow x$. By III.1.13, **A** seen as a deductive system has some deductive axioms $\varphi_1,\dots,\varphi_n$. The equational basis B consists of

$$\begin{aligned} \varphi_1 &\approx 1, \\ &\vdots \\ \varphi_n &\approx 1, \\ (\circ)\qquad x\Rightarrow x &\approx 1, \\ (\star)\qquad 1\Rightarrow x &\approx x, \\ (\diamond)\qquad (x\Rightarrow y)\Rightarrow y &\approx (y\Rightarrow x)\Rightarrow x. \end{aligned}$$

of finite similarity type. Lyndon's result implies that any two-element algebra is polynomially equivalent to a finitely based equational class. However, this does not mean that any two element algebra in an infinite signature is finitely based: any polynomial on such an algebra may be expressed in terms of operations appearing in the given finite basis, but the behaviour of infinitely many basic operations can not be determined by a finite basis.

[2] f is the classical implication function iff $f(1,0)=0$ and otherwise $f(x,y)=1$.

All of these equations are true in $\mathbf{A}$, so we only need to show that any equation which is true in $\mathbf{A}$ may be derived from B.

Claim 1. *If φ is a theorem of the logic with the connectives corresponding to the operations of $\mathbf{A}$, then $\varphi \approx 1$ can be derived from B.*

The claim is true for the axioms. Theorems of a deductive system are obtained recursively by substitution and modus ponens, so we need to prove that these two ways of making a step in a proof may be reproduced in B. That consequences of B are closed under substitution is clear from the definition of an equational theory.

Suppose that $\psi_1 \Rightarrow \psi_2$ and ψ_1 are theorems and that $\psi_1 \Rightarrow \psi_2 \approx 1$ and $\psi_1 \approx 1$ are derivable from B. We substitute 1 for ψ_1 (substituting an equal term is allowed in a derivation from a given basis) and get that $1 \Rightarrow \psi_2 \approx 1$, which by ($\star$) means that $\psi_2 \approx 1$ can be derived from B.

Claim 2. *If $\mathbf{A} \models s \approx t$, then $s \approx t$ may be derived from B.*

Let $\mathbf{A} \models s \approx t$ for two terms t and s. Then $t \Rightarrow s \approx 1$ and $s \Rightarrow t \approx 1$ because of ($\circ$). But then $t \Rightarrow s$ and $s \Rightarrow t$ must be theorems of the corresponding logic. From Claim 1 we see that $t \Rightarrow s \approx 1$ and $s \Rightarrow t \approx 1$ can be derived from B. ($\diamond$) gives $(t \Rightarrow s) \Rightarrow s \approx (s \Rightarrow t) \Rightarrow t$, whence by substitution $1 \Rightarrow s \approx 1 \Rightarrow t$. Now we use ($\star$) to finish the derivation of $s \approx t$ from B. □

Lyndon gives an explicit basis of equations for all other possible clones on a two-element algebra. Some of these cases are reducts of Boolean algebras. For the other cases, completeness is proved in the following way: first it is shown that a free countably generated algebra $\mathbf{F}$ in the variety determined by the given basis $B \subset \operatorname{Eq} \mathbf{A}$ is isomorphic to an algebra $\mathbf{F}'$ of sets. Because $\mathbf{F}'$ is a subalgebra of a direct product of the two-element algebras in the same signature, all equations true in $\mathbf{A}$ are also true in $\mathbf{F}'$, and hence are consequences of the given basis. In short,

$$\mathbf{F}_\infty(\operatorname{Mod} B) \simeq \mathbf{F}' \leq \prod_{i \in I} \mathbf{A} \quad \text{and hence} \quad \operatorname{Eq}(\mathbf{F}_\infty(\operatorname{Mod} B)) = \operatorname{Eq} \mathbf{A}.$$

Instead of going into further details of Lyndon's proof, we look at a later proof by J. Berman. It is based on the following lemma, which effectively reduces the number of clones for which we need to give an explicit basis:

Lemma III.1.15 (Berman [**Ber80**]). *Let* $\mathbf{A} = (\{0,1\}, f)$*, where* f *is an* n*-ary operation depending on all* n *variables,* $n \geq 2$*. Then the variety generated by* $\mathbf{A}$ *is congruence distributive (CD), congruence permutable (CP), or* f *is a semilattice operation, i.e.* $f(x_1, \dots, x_n) = x_1 \wedge \cdots \wedge x_n$ *or* $f(x_1, \dots, x_n) = x_1 \vee \cdots \vee x_n$*.*

Sketch of the proof. The proof is based on Mal'cev's characterisation of congruence permutability I.4.1 and Pixley's sufficient condition for congruence distributivity I.4.4. For $\mathbf{u} \in \{0,1\}^n$, let $\mathbf{u}' = (u_1', u_2', \dots, u_n')$ where $u_i' = 1 - u_i$. In particular, let $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := \mathbf{0}' = (1, \dots, 1)$. Four possible cases may occur:

(1) for all $\mathbf{u}$, $f(\mathbf{u}) = f(\mathbf{u}')$: HSP $\mathbf{A}$ is CP;
(2) $f(\mathbf{0}) = f(\mathbf{1})$, but for some $\mathbf{u}$, $f(\mathbf{u}) \neq f(\mathbf{u}')$: HSP $\mathbf{A}$ is CD;
(3) $f(\mathbf{0}) \neq f(\mathbf{1})$, but for some $\mathbf{u}$, $f(\mathbf{u}) = f(\mathbf{u}')$: HSP $\mathbf{A}$ is both CD and CP, only CD, or f is a semilattice operation;
(4) for all $\mathbf{u}$, $f(\mathbf{u}) \neq f(\mathbf{u}')$: HSP $\mathbf{A}$ is both CD and CP, only CD, or only CP.

For example, let us investigate case (1) that for all $\mathbf{u}$, $f(\mathbf{u}) = f(\mathbf{u}')$. Since f depends on all variables, there is some $\mathbf{u}$ for which $f(\mathbf{0}) \neq f(\mathbf{u})$. Let $g(x,y)$ be defined in the following way:

$$g(x,y) = f(x_1, x_2, \dots, x_n) \quad \text{where} \quad x_i = \begin{cases} x & u_i = 1, \\ y & u_i = 0. \end{cases}$$

Thus $g(0,0) = g(1,1) \neq g(1,0) = g(0,1)$. It is easy to check that the term $g(g(x,y),z)$ is a Mal'cev term, and it lies in the clone of $\mathbf{A}$, because it is derived from f by substitution. Hence $\mathbf{A}$ is congruence permutable.

The analysis of the other cases is similar, only more technical. □

Berman's proof of Lyndon's theorem. By Lemma III.1.15, the following cases appear among varieties generated by two-element algebras:

(i) $\mathbf{A}$ has only constants and unary operations - we may use Proposition III.1.8
(ii) some reduct of HSP $\mathbf{A}$, and hence also HSP $\mathbf{A}$, is congruence permutable - use McKenzies result from [**McK78**]: every finite algebra $\mathbf{A}$ such that every subalgebra of $\mathbf{A}$ is simple and $\mathbf{A}$ generates a congruence permutable variety is finitely based
(iii) some reduct of HSP $\mathbf{A}$, and hence also HSP $\mathbf{A}$, is congruence distributive - use Baker's theorem III.6.1
(iv) the clone of $\mathbf{A}$ contains a semilattice operation. In that case, HSP $\mathbf{A}$ is congruence meet-semidistributive. Moreover, HSP $\mathbf{A}$

has a finite residual bound by III.1.12. Thus we may use Willard's theorem III.8.1.[3] □

III.2. Regular and non-regular equations

Definition III.2.1. We say that an equation is *regular* iff the sets of variables that occur on each side are equal; in the case that there exists a variable which occurs on one side of the equation but not on the other, we say that the equation is *non-regular*.[4] An equational theory is *regular* iff it consists only of regular equations; otherwise, it is *non-regular*.

Proposition III.2.2. *Let E be a non-regular equational theory, E_0 the set of all regular equations in E and $u \approx v$ any non-regular equation in E. Then $E = \mathrm{Eq}(E_0 \cup \{u \approx v\})$.*[5]

Proof. Without loss of generality, let x be a variable occurring in u but not in v. Let $s \approx t$ be a non-regular equation in $E \setminus E_0$. We show that $s \approx t$ is a consequence of $E_0 \cup \{u \approx v\}$.

Case 1. σ contains no operation symbol of arity greater than 1. Hence,[6]

$$(\heartsuit) \qquad u(x) \approx v \approx u[p] \quad \text{for any term } p,$$

because the variable x does not appear in v.

Case 1a. s contains some variable y. Therefore $s(y) \approx s[p]$ lies in E for any term p, and we obtain the following proof of $s \approx t$:

$$\begin{array}{ll} s(y) \approx s[y : u[x : y]] & \text{is in } E_0 \\ s[y : u[x : y]] \approx s[y : u[x : t]] & \text{is a consequence of } u \approx v \text{ by } (\heartsuit) \\ s[y : u[x : t]] \approx t & \text{is in } E_0 \end{array}$$

[3]At the time when Berman published his proof, Willard's theorem was not yet available. Berman argues that either (a) **A** has both a meet and a join semilattice operation; thus **A** has a lattice reduct and HSP **A** is congruence distributive, or (b) the clone of **A** is generated by a binary semilattice operation possibly with nullary and unary operations and may be easily seen to be finitely based by the inspection of all such cases.

[4]In semigroup theory, regular identities are called *homotypical* and non-regular identities are called *heterotypical*.

[5]According to Ježek [**Jež08**], this theorem belongs to a Russian mathematician whose name is forgotten and the reference is lost. It was proved around 1950.

[6]We write $s(y)$ to show that s contains variable y and no others; we use $s[y : p]$ for the term that we get from s by simultaneous substitution of p for all occurrences of y; if there is only one variable in s, we write $s[p]$ instead of $s[y : p]$.

CASE 1b. s contains no variable. Then t contains some variable, and we may apply Case 1a to obtain $t \approx s$; the desired equation $s \approx t$ follows by symmetry.

CASE 2. σ contains an operation symbol f of arity at least 2. Assume that the variables $x_1, \ldots, x_m$ appear in s but not in t and the variables $y_1, \ldots, y_k$ appear in t but not in s. Without loss of generality, $m \geq 1$ (otherwise interchange the role of s and t).

If $u = u(x, z_1, \ldots, z_n)$, where x is a variable not occurring in v, let

$$a := f(u(y, z, \ldots, z), z, \ldots, z),$$

so that $a = a(z, y)$ and the non-regular equation $a(z, y) \approx a(z, y')$ is a consequence of $E_0 \cup \{u \approx v\}$. Also,

$$\begin{aligned} s(x_1, \ldots, x_m, z_1, \ldots, z_l) \approx t(y_1, \ldots, y_k, z_1, \ldots, z_l) \approx \\ \approx s(x'_1, \ldots, x'_m, z_1, \ldots, z_l), \end{aligned}$$

where $z_1, \ldots, z_l$ are the common variables of s and t. Hence[7]

$$\begin{array}{rll} s \approx s[\forall i\ \ x_i : a(x_i, x_i)] & \text{is in } E_0 \\ s[\forall i\ \ x_i : a(x_i, x_i)] \approx s[\forall i\ \ x_i : a(x_i, t)] & \text{due to } a(z, y) \approx a(z, y') \\ s[\forall i\ \ x_i : a(x_i, t)] \approx s[\forall i\ \ x_i : a(t, x_i)] & \text{is in } E_0 \\ s[\forall i\ \ x_i : a(t, x_i)] \approx s[\forall i\ \ x_i : a(t, t)] & \text{due to } a(z, y) \approx a(z, y') \\ s[\forall i\ \ x_i : a(t, t)] \approx t. & \text{is in } E_0 & \square \end{array}$$

DEFINITION III.2.3. A variety $\mathcal{V}$ is called *regular* iff all identities in $\operatorname{Eq}\mathcal{V}$ are regular; otherwise $\mathcal{V}$ is called *irregular*. The variety $\widetilde{\mathcal{V}}$ defined by the regular identities in $\operatorname{Eq}\mathcal{V}$ is called the *regularization of* $\mathcal{V}$.

COROLLARY III.2.4. *A variety $\mathcal{V}$ is finitely based iff its regularization $\widetilde{\mathcal{V}}$ is finitely based.*

[7] We introduce the abbreviation $s[\forall i\ \ x_i : a(x_i, x_i)]$ for
$s[x_1 : a(x_1, x_1), \ldots, x_m : a(x_m, x_m)]$.

III.3. Perkins's result: commutative semigroups

Let $(S, *)$ be a non-empty set with a binary operation. We often write

xy	instead of $x * y$
$x * yz$	instead of $x * (y * z)$
x^k	instead of $\underbrace{x * x * \cdots * x}_{k \times \text{times}}$
x^0	for the empty sequence of symbols

Definition III.3.1. $(S, *)$ is a *semigroup* iff

$$\mathbf{S} \models x * yz \approx xy * z.$$

In other words, $*$ is associative.

A semigroup $\mathbf{S}$ is *commutative* iff $\mathbf{S} \models x * y \approx y * x$.

Definition III.3.2. By a *normal form* of a term t we mean a term $t' = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$, where each $\epsilon_i \geq 0$, $\epsilon_n > 0$, all brackets are pushed as far to the right as possible[8] and $t \approx t'$ is provable from the associative and commutative law. Note that the order of the variables is fixed.[9]

By a *normal form* of an equation $s \approx t$ we mean the equation $s' \approx t'$, where s' and t' are the normal forms of s and t.

Observation III.3.3. *Each term has a uniquely determined normal form.*

Definition III.3.4. A semigroup $\mathbf{S}$ is called *uniformly periodic* iff there exist integers $m, k > 0$ such that $\mathbf{S} \models x^m \approx x^{m+k}$. Let m_0 be the least such m and k_0 the least such k for m_0; we then say that $\mathbf{S}$ is (m_0, k_0)-*uniformly periodic.*

Proposition III.3.5. $\mathbf{S}$ *is uniformly periodic iff there exists an equation $s \approx t$ valid in $\mathbf{S}$ and a variable v such that the number of occurrences of v is not the same in s as in t.*

Proof. $(\Rightarrow)$ is clear. We prove $(\Leftarrow)$: if s and t have different lengths, then we may plot in x for all variables and we obtain an equation of the form $x^k \approx x^l$, $k \neq l$. So let s and t have the same lengths. Without loss of generality,[10] we assume that $s \approx t$ is of the form

$$x^k y^l \approx x^{k+m} y^{l-m}$$

[8] By pushing the brackets to the right we mean that

$$t' := x_1 * (x_1 * (\cdots * (x_1 * (x_2 * (\cdots * (x_n * x_n) \ldots))) \ldots)).$$

[9] We assume that only variables from the infinite countable set $\{x_1, x_2, \ldots\}$ can be used, and that they are ordered by the natural order of their indices.

[10] If there are more than two variables in $s \approx t$, let one of them be x and plot in y for all others. If $k = l$, then $k + m \neq l - m$, so we may assume $k \neq l$.

for some $k, l, m \in \mathbb{N}$, $k \neq l$, $m \neq 0$. We show that there is a deductive consequence $s' \approx t'$ of $s \approx t$ such that the length of s' is not equal to the length of t'. Indeed, the following equalities hold:

$$x^{kl(k+m)}y^{kl(l-m)} = (x^{kl})^{k+m}(y^{kl})^{l-m} \approx$$
$$\approx (x^{kl})^k(y^{kl})^l = (x^{k^2})^l(y^{l^2})^k \approx$$
$$\approx (y^{l^2})^{k+m}(x^{k^2})^{l-m} = y^{l^2(k+m)}x^{k^2(l-m)}.$$

(Here, we use "$=$" for equality following from associativity and "$\approx$" for equality following from $s \approx t$.) The length of the leftmost term is $kl(k+m) + kl(l-m) = kl(k+l)$, whilst the length of the rightmost term is $l^2(k+m)+k^2(l-m) = kl(k+l)+m(l^2-k^2)$; by the assumption that $k \neq l$ we see that these are not equal. □

Definition III.3.6. By a *non-trivial equation* in a commutative semigroup we mean any semigroup equation which is not true in all commutative semigroups, i.e. an equation $s \approx t$ such that

$$a * (b * c) \approx (a * b) * c, a * b \approx b * a \not\vdash s \approx t.$$

Corollary III.3.7. *If some non-trivial equation is true in a commutative semigroup* $\mathbf{S}$*, then* $\mathbf{S}$ *is uniformly periodic.*

Definition III.3.8. By a *reduced normal form* of a term t we mean a term $t' = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ in normal form such that each $\epsilon_i < m_0 + k_0$ and $t \approx t'$ is provable from the associative and commutative law and $x^{m_0} \approx x^{m_0+k_0}$. By a *reduced normal form* of an equation $s \approx t$ we mean the equation $s' \approx t'$, where s' and t' are the reduced normal forms of s and t. If no confusion arises, we talk simply of a *reduced term* or a *reduced equation*.

Let $0 \leq p, q < m_0 + k_0$ and let s, t be two reduced terms. A set $B = \{x_{i_1}, \ldots, x_{i_n}\}$ is called a *p-block of* s iff each $x_i \in B$ has exponent p in s. B is a (p, q)*-block of* $s \approx t$ iff it is a p-block of s and a q-block of t.

Observation III.3.9. *For a given* p *and* q*, if* $s \approx t$ *has a* (p, q)*-block, then it has a maximal* (p, q)*-block (called* the (p, q)-block of $s \approx t$*).*

Example III.3.10. Let $S \models x^7 \approx x^4$. The reduced form of the equation

$$x_1^2x_2x_4^8x_2x_5^2x_2 \approx x_1^6x_5^{12}x_6^7$$

is the equation

$$x_1^2x_2^3x_4^5x_5^2 \approx x_1^6x_5^6x_6^4,$$

which is reduced iff $m_0 + k_0 > 6$. In that case, $\{x_1\}, \{x_5\}$ and $\{x_1, x_5\}$ are $(2, 6)$-blocks, $\{x_1, x_5\}$ is the $(2, 6)$ block and $\{x_6\}$ is the $(0, 6)$-block.

Definition III.3.11. For an equation $e = s \approx t$, let $\lambda_{pq}(e)$ be the length of the (p,q)-block of e if such exists and 0 otherwise. We define a vector of dimension $(m_0 + k_0)^2 - 1$:

$$\begin{aligned} \lambda^e = (\quad & \lambda^e_1, & \lambda^e_2, & \quad \ldots, & \lambda^e_{m_0+k_0}, & \quad \ldots, & \lambda^e_{(m_0+k_0)^2-1} & \quad) \\ := (\quad & \lambda_{01}(e), & \lambda_{02}(e), & \quad \ldots, & \lambda_{10}(e), & \quad \ldots, & \lambda_{m_0+k_0-1,m_0+k_0-1}(e) & \quad). \end{aligned}$$

The choice of the dimension of the vector λ^e is motivated by it's later use in the setting of (m_0, k_0)-uniformly periodic semigroups. The following lemma will later be applied to vectors λ^e for an infinite set of equations.

Lemma III.3.12. *If S is an infinite set of vectors of dimension d with components in $\mathbb{N}$, and $\prec$ is the partial order induced by the natural order of the components, then there is an infinite linearly ordered chain $C \subseteq (S, \prec)$.*

Proof. We proceed by induction od d:

1° The case $d = 1$ is clear.

2° If S is infinite, then either the set

$$S' := \{u \in \mathbb{N}^{d-1};\ \exists n \in \mathbb{N}, (u,n) \in S\}$$

or the set

$$'S := \{u \in \mathbb{N}^{d-1};\ \exists n \in \mathbb{N}, (n,u) \in S\}$$

is infinite.

Without loss of generality we assume that S' is infinite. By the induction hypothesis, it contains an infinite chain $\{u_i,\ i \in \mathbb{N}\}$.

For each u_i, choose one n_i such that $(u_i, n_i) \in S$. We show by contradiction that there has to exist the smallest i such that the set

$$N_i := \{n_j;\ j > i, n_j \geq n_i\}$$

is infinite.

$\leadsto$ If N_i is finite for every i, then the set

$$N'_i := \{n_j;\ j > i, n_j < n_i\}$$

is infinite for every i. Therefore, we could construct an infinite decreasing sequence of natural numbers

$$n_1,\ n_{j_1} \in N'_1,\ n_{j_2} \in N'_{j_1},\ n_{j_3} \in N'_{j_2}, \ldots \qquad ↯$$

We are ready to construct an infinite chain in S. Let

$$v_1 := (u_{i_1}, n_{i_1}),$$

where i_1 is the smallest i such that N_i is infinite. Let

$$v_{j+1} := (u_{i_{j+1}}, n_{i_{j+1}}),$$

where i_{j+1} is the smallest $i \in N_{i_j}$ such that N_i is infinite. Then $n_i \leq n_j$ whenever $i < j$, and hence $(u_i, n_i) \prec (u_j, n_j)$. □

Theorem III.3.13 (Perkins [**Per69**]). *Every (finite or infinite) commutative semigroup* $\mathbf{S}$ *is finitely based.*
Proof.
Case 1. If $\mathbf{S}$ is not uniformly periodic, then by Corollary III.3.7, the normal forms of all equations true in $\mathbf{S}$ are $t \approx t$. Thus this case is trivial.

Case 2. ↯ Let $\mathbf{S}$ be uniformly periodic, but not finitely based. There must exist an infinite sequence of equations $e_i \in \operatorname{Eq} \mathbf{S}$, $i \in \mathbb{N}$, such that for any i, e_{i+1} cannot be proved from $e_1, \dots, e_i$.

By Lemma III.3.12, there exists an infinite chain $C \subseteq \{\lambda^{e_i};\, i \in \mathbb{N}\}$. Moreover, we may assume that also the indices of the equations are increasing in this chain (we could add i as the last coordinate to λ^{e_i}).

However, if $n < m$ and $\lambda_k^{e_n} \leq \lambda_k^{e_m}$ for any $k = 1, \dots, (m_0 + k_0)^2 - 1$, we could substitute a linear term involving unused variables for some variable in any k-block where $\lambda_k^{e_n} < \lambda_k^{e_m}$, thus deducing an equation e'_n for which $\lambda_k^{e'_n} = \lambda_k^{e_m}$ for all k. By changing the variables, we could deduce e_m from e'_n, contradicting the original assumption on the sequence e_i.↯ □

Example III.3.14. Consider the equations

$$e : x^2 y^3 = x^4$$

and

$$e' : x^2 y^2 z^3 = x^4 y^4.$$

Both equations contain a $(2,4)$-block and a $(3,0)$-block, and the lengths of the blocks are increasing from e to e'. Thus $\lambda^e \prec \lambda^{e'}$. And indeed, we may deduce e' from e by substituting xu for x in e, and then changing the variable y to z and u to y.

Corollary III.3.15. *Every variety of commutative semigroups is finitely based.*

Proof. Indeed, we have not made any use of finite generation of the variety $\operatorname{HSP} \mathbf{S}$ in the proof of Theorem III.3.13: the only assumption which we needed was that the equational theory under consideration contains the associative and commutative law. □

In his paper, Perkins also gives an upper bound on the number of variables appearing in a basis for a semigroup $\mathbf{S}$.

Definition III.3.16. An element $a \in \mathbf{S}$ is called *prime* iff for all $b, c \in \mathbf{S}$, $a \neq bc$.

For example, if **S** has an identity element or is idempotent, then it has no primes at all. If **S** is finitely generated, then it has only finitely many primes.

FACT III.3.17. *Let* **S** *be* (m_0, k_0)*-uniformly periodic. As a consequence,* $\mathbf{S} \models x^{m_0} \approx x^{m_0+k_0}$.

(1) If p is the number of primes in **S** *and p is finite then* Eq **S** *has a finite basis involving no more than* $(pm_0 + 2)[(m_0 + k_0)^2 - 1]$ *variables.*
(2) If **S** *has a finite set of n generators, then* Eq **S** *has a basis involving no more than* $2nm_0 + 1$ *variables.*
(3) For each integer n, a commutative semigroup $\mathbf{S}_n$ *can be constructed such that* Eq $\mathbf{S}_n$ *has no basis with fewer than n variables.*

COROLLARY III.3.18. *If* **S** *is an* (m_0, k_0)*-uniformly periodic semigroup and* **S** *has a finite set of n generators, then* **S** *has a basis B with no more than* $\frac{(m_0+k_0)^{4nm_0+1}}{2} + 3$ *equations.*

PROOF. Let B contain the associativity and commutativity law and the equation $x^{m_0} \approx x^{m_0+k_0}$. We may assume that all other equations in B are in the reduced form. As there are only $(m_0 + k_0)^v$ different reduced terms in v variables, there are at most $\frac{((m_0+k_0)^v)^2-(m_0+k_0)^v}{2} = \frac{(m_0+k_0)^{2v-1}}{2}$ different non-trivial equations (here we consider $s \approx t$ and $t \approx s$ to be the same equation). Hence

$$|B| \leq \frac{(m_0 + k_0)^{2(2nm_0+1)-1}}{2} + 3 = \frac{(m_0 + k_0)^{4nm_0+1}}{2} + 3. \qquad \square$$

III.4. Jónsson's Finite Basis Theorem

THEOREM III.4.1 (Jónsson [**Jón79b**]). *Let $\mathcal{V}$ be a variety. If there exist classes $\mathcal{S}$ and $\mathcal{K}$ satisfying*

- $\mathcal{V} \subseteq \mathcal{K}$
- *$\mathcal{K}$ is finitely axiomatizable by a first-order formula ψ_1*
- *$\mathcal{S}$ is axiomatizable by a set of first-order formulae*
- $\mathcal{K}_{SI} \subseteq \mathcal{S}$
- *$\mathcal{V} \cap \mathcal{S}$ is finitely axiomatizable by a first-order formula ψ_2*

then $\mathcal{V}$ is finitely based.

PROOF. $\leadsto$ For contradiction, assume that $\mathcal{V}$ is not finitely based. According to Proposition I.1.18, for each $k \geq 1$ there exists a subdirectly irreducible algebra

$$\mathbf{A}_k \in \mathrm{Mod}(\mathrm{Eq}_{(k)}\mathcal{V}) \setminus \mathcal{V}.$$

Let $\mathbf{A}$ be the ultraproduct of $\mathbf{A}_k, k \geq 1$ over some ultrafilter U containing all complements of finite subsets of $\mathbb{N} \setminus \{0\}$.

Since $\mathcal{V} \subseteq \mathcal{K}$, we get that $\mathrm{Eq}\,\mathcal{V} \models \psi_1$, and by the Compactness Theorem I.1.1 there exists a finite set $\Sigma_0 \subseteq \mathrm{Eq}\,\mathcal{V}$ such that $\Sigma_0 \models \psi_1$. Let n be the maximal depth of terms used in Σ_0. For all $k \geq n$, $\mathbf{A}_k \in \mathcal{K}_{SI} \subseteq \mathcal{S}$. Hence each of the sentences axiomatizing $\mathcal{S}$ is true in all but finitely many $\mathbf{A}_k$, so $\mathbf{A} \in \mathcal{S}$.

Each identity in $\mathrm{Eq}\,\mathcal{V}$ is a member of all but finitely many $\mathrm{Eq}_{(k)}\mathcal{V}$, and hence almost all $\mathbf{A}_k$ satisfy it. Therefore $\mathbf{A} \models \mathrm{Eq}\,\mathcal{V}$, or in other words, $\mathbf{A} \in \mathcal{V}$.

We have shown that $\mathbf{A} \in \mathcal{S} \cap \mathcal{V} = \mathrm{Mod}\,\psi_2$; however, this is impossible, because $\mathbf{A}_k \not\models \psi_2$ for all $k \geq 1$.↯ □

COROLLARY III.4.2 ([**BaWa02**]). *Let $\mathcal{V}$ be a variety and $\mathcal{K}$ a strictly elementary class of algebras such that $\mathcal{V} \subseteq \mathcal{K}$.*

(1) If $\mathcal{K}_{SI}$ is strictly elementary, then $\mathcal{V}$ and $\mathcal{V}_{SI}$ either both are strictly elementary or both are not strictly elementary.
(2) If $\mathcal{K}_{FSI}$ is strictly elementary, then $\mathcal{V}$ and $\mathcal{V}_{FSI}$ either both are strictly elementary, or both are not strictly elementary.

PROOF. (1) Let $\mathcal{K}$ and $\mathcal{K}_{SI}$ be both strictly elementary. If $\mathcal{V}$ is strictly elementary, then so is $\mathcal{V}_{SI} = \mathcal{K}_{SI} \cap \mathcal{V}$. On the other hand, if $\mathcal{V}_{SI}$ is strictly elementary, then $\mathcal{V}$ is finitely based by Jónsson's theorem III.4.1 with the choice $\mathcal{S} := \mathcal{K}_{SI}$.
(2) is similar. □

Many known finite basis proofs can be reduced to the following special case of B. Jónsson's theorem:

Corollary III.4.3 ([**Wil04**]). *Suppose a variety $\mathcal{V}$ has a finite residual bound. If there exist classes $\mathcal{S} \subseteq \mathcal{K}$ satisfying*

- $\mathcal{V} \subseteq \mathcal{K}$
- *$\mathcal{K}$ is finitely axiomatizable by a first-order formula ψ_1*
- *$\mathcal{S}$ is finitely axiomatizable by a first-order formula ψ_3*
- $\mathcal{K}_{SI} \subseteq \mathcal{S}$
- $\mathcal{V}_{SI} = \mathcal{V} \cap \mathcal{S}$,

then $\mathcal{V}$ is finitely based.

A proof without reference to Jónsson's Theorem.

According to Proposition I.2.9, $\mathcal{V}_{SI}$ is axiomatized by a single formula ψ_2. Every member of $\mathcal{V}$ satisfies $\psi_1 \wedge (\psi_3 \Rightarrow \psi_2)$. Let E be the equational base of the variety $\mathcal{V}$. Due to the Compactness Theorem I.1.1, there exists a finite set $E_0 \subseteq E$ such that $E_0 \models \psi_1 \wedge (\psi_3 \Rightarrow \psi_2)$. Let $\mathcal{W} = \operatorname{Mod} E_0$, so that the following (in)equalities hold:

$\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{K}$	since $\mathcal{W} \models \psi_1$
$\mathcal{W} \cap \mathcal{S} \subseteq \mathcal{V}_{SI} = \mathcal{V} \cap \mathcal{S}$	since $\mathcal{W} \models \psi_3 \Rightarrow \psi_2$
$\mathcal{W} \cap \mathcal{S} = \mathcal{V} \cap \mathcal{S}$	since $\mathcal{V} \subseteq \mathcal{W}$
$\mathcal{W}_{SI} \subseteq \mathcal{K}_{SI} \subseteq \mathcal{S}$	since $\mathcal{W} \subseteq \mathcal{K}$
$\mathcal{W}_{SI} = (\mathcal{W} \cap \mathcal{S})_{SI} = (\mathcal{V} \cap \mathcal{S})_{SI} = \mathcal{V}_{SI}$	

Recall that a variety is uniquely determined by its subdirectly irreducible members; hence $\mathcal{W} = \mathcal{V}$, which means that $\mathcal{V}$ is finitely based. □

A typical use of Jónsson's theorem is as follows: define formulae ψ_1 and ψ_3 such that

- $\mathcal{V} \models \psi_1 \wedge (\psi_3 \Rightarrow \psi_2)$
- ψ_1 gives some important characteristics of symbols from ψ_3
- ψ_3 gives a characterization of irreducibility (assuming that the symbols obey ψ_1)

Now simply let $\mathcal{K} = \operatorname{Mod} \psi_1$ and $\mathcal{S} = \operatorname{Mod} \psi_3$.

III.5. Principal congruences

III.5.1. McKenzie's Theorem: Definable principal congruences.

Definition III.5.1. A variety $\mathcal{V}$ has *definable principal congruences* iff there exists a first order formula $\varphi(x, y, u, v)$ such that for any $\mathbf{A} \in \mathcal{V}$ and any elements $a, b, c, d \in \mathbf{A}$ the following equivalence holds:

$$\mathbf{A} \models \varphi(a, b, c, d) \qquad \text{iff} \qquad (a, b) \in \operatorname{Cg}^{\mathbf{A}}(c, d).$$

DEFINITION III.5.2. A *principal congruence formula* is any formula $\pi(x, y, u, v)$ of the form

$$\exists \overline{w} \quad \left(x = p_1(z_1, \overline{w}) \;\wedge\; \bigwedge_{i \in \widehat{n-1}} p_i(z_i', \overline{w}) = p_{i+1}(z_{i+1}, \overline{w}) \;\wedge\; p_n(z_n', \overline{w}) = y \right)$$

where p_i are some terms and $\{z_i, z_i'\} = \{u, v\}$ for all $i \in \widehat{n}$. In other words, it is a formula which describes a Mal'cev chain implying that $(x, y) \in \mathrm{Cg}^{\mathbf{A}}(u, v)$.

Let Π be the set of principal congruence formulae.

PROPOSITION III.5.3.

(1) Let $\mathbf{A}$ *be a* σ*-algebra and let* $a, b, c, d \in \mathbf{A}$*. Then* $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ *iff there exist* $\pi \in \Pi$ *such that* $\mathbf{A} \models \pi(a, b, c, d)$*.*

(2) $\mathcal{V}$ *has principal congruences definable by a first order formula* $\varphi(x, y, u, v)$ *iff there exists* $\Pi_0 \subseteq_{FIN} \Pi$ *such that for all* $\mathbf{A} \in \mathcal{V}$ *and* $a, b, c, d \in \mathbf{A}$*,*

$$(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d) \text{ iff there exists } \pi \in \Pi_0 \text{ such that } \mathbf{A} \models \pi(a, b, c, d).$$

PROOF. (1) This is just a rewording of I.3.1.

($2 \Leftarrow$) Just take

$$\varphi = \bigvee_{\pi \in \Pi_0} \pi(x, y, u, v).$$

($2 \Rightarrow$) The idea of this proof is quite common in model theory: we add some constants in order to force the same evaluation of variables occurring in all principal congruence formulae; then we use the Compactness Theorem, and go back to the original language.

Let $\sigma' = \sigma \cup \{c_a, c_b, c_c, c_d\}$ be an enriched language and let $\mathcal{V}'$ be the variety of type σ' defined by the same identities Σ as $\mathcal{V}$. Let

$$\neg\Pi = \{\neg\pi(c_a, c_b, c_c, c_d);\ \pi(x, y, u, v) \in \Pi\}.$$

Then for any $\mathbf{A} \in \mathcal{V}'$, if $\mathbf{A} \models \neg\Pi$ then there is no Mal'cev chain certifying that $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ and hence $\mathbf{A} \models \neg\varphi(c_a, c_b, c_c, c_d)$. In other words, $\Sigma \cup \neg\Pi \models \neg\varphi(c_a, c_b, c_c, c_d)$, and according to the Compactness Theorem I.1.1, there exists a finite $\Pi_0 \subseteq \Pi$ such that

$$\Sigma \cup \neg\Pi_0 \models \neg\varphi(c_a, c_b, c_c, c_d).$$

This means that[11]

$$\Sigma \models \varphi(c_a, c_b, c_c, c_d) \Rightarrow \bigvee_{\pi \in \Pi_0} \pi(c_a, c_b, c_c, c_d).$$

[11] We use the Deduction Theorem of first order logic: if $\Sigma, \psi \models \phi$, then $\Sigma \models \psi \Rightarrow \phi$; we also use the fact that $\neg\phi \Rightarrow \neg\psi$ is equivalent to $\psi \Rightarrow \phi$, and $\neg \bigwedge_{\pi \in \Pi_0} \neg\pi$ is equivalent to $\bigvee_{\pi \in \Pi_0} \pi$.

Now it is easy to verify that for an algebra $\mathbf{A} \in \mathcal{V}$ and elements $a, b, c, d \in \mathbf{A}$, $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ iff there exists $\pi \in \Pi_0$ such that $\mathbf{A} \models \pi(a, b, c, d)$: simply interpret $\mathbf{A}$ as an algebra in $\mathcal{V}'$, with the evaluation of the constants c_a, c_b, c_c and c_d by the elements a, b, c, d. $\square$

Example III.5.4. An algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, e, /)$ is a *commutative idempotent residuated lattice* iff

- $(A, \vee, \wedge)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, i.e. $\cdot$ is associative and e is a unit,
- $(A, \cdot)$ is a semilattice, i.e. $\cdot$ is associative, commutative and idempotent,
- for every $a, b, c \in A$, $\quad ab \le c \Leftrightarrow a \le c/b$.

Thus for every $a, b \in A$ there is a greatest c such that $cb \le a$; then $a/b = c$. For example, every Heyting algebra is a commutative idempotent residuated lattice, where $ab := a \wedge b$ and $a/b := b \to a$.

It has been shown in [**Sta07**] that if $\mathbf{A}$ is a commutative idempotent residuated lattice, then $\mathrm{Con}\,\mathbf{A}$ is isomorphic to the lattice of filters on $\mathbf{A}^- := \{a \in \mathbf{A}; a \le e\}$. Principal congruences correspond to principal filters, which are first-order definable. This means that commutative idempotent residuated lattices have definable principal congruences.

Theorem III.5.5 (McKenzie [**McK78**]). *Let $\mathcal{V}$ be a variety satisfying one of the following two conditions:*

(1) either $\mathcal{V}$ has a finite residual bound,
(2) or V is locally finite and has only finitely many finite subdirectly irreducible algebras.

If $\mathcal{V}$ has definable principal congruences, then $\mathcal{V}$ is finitely based.

Proof. First note that (2) implies (1) by Proposition I.2.7.

We define formulae ψ_1, ψ_2 and ψ_3 so that we can use Theorem III.4.3:

Let ψ_2 be the formula describing $\mathcal{V}_{SI}$ and let φ be the formula which defines principal congruences in $\mathcal{V}$.

We want ψ_1 and ψ_3 to have the following meaning:

ψ_1**:** for any u, v, $\varphi(\_, \_, u, v)$ generates a congruence containing (u, v)
ψ_3**:** there exist x, y such that (x, y) lies in the monolith of $\mathbf{A}$, i.e. $\mathbf{A} \in \mathcal{V}_{SI}$

ψ_3 is the following formula:
$$\exists xy\big(x \not\approx y \ \wedge\ \forall uv\ (u \not\approx v \Rightarrow \varphi(x, y, u, v))\big).$$

ψ_1 is the closure of the conjunction of the following formulae:

$\varphi(u, v, u, v)$ — $(u, v) \in \mathrm{Cg}(u, v)$

$\varphi(x, x, u, v)$ — $\mathrm{Cg}(u, v)$ is a reflexive relation

$\varphi(x, y, u, v) \Rightarrow \varphi(y, x, u, v)$ — $\mathrm{Cg}(u, v)$ is a symmetric relation

$\varphi(x, y, u, v) \wedge \varphi(y, z, u, v) \Rightarrow \varphi(x, z, u, v)$ — $\mathrm{Cg}(u, v)$ is transitive

$\big(\varphi(x_1, y_1, u, v) \wedge \cdots \wedge \varphi(x_n, y_n, u, v)\big) \Rightarrow$
$\varphi(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), u, v)$
— $\mathrm{Cg}(u, v)$ is compatible with all operations f in the signature

With these definitions, it is easy to verify the conditions of Theorem III.4.3. □

By investigation of the proof we see that the following generalisation is also valid:

Corollary III.5.6 ([**Jón79a**]). *If $\mathcal{V}$ is a variety with definable principal congruences and such that either $\mathcal{V}_{SI}$ or $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely based.*

Proof. The case when $\mathcal{V}_{SI}$ is strictly elementary is clear. The case when $\mathcal{V}_{FSI}$ is strictly elementary requires the use of Theorem III.4.1 and the following definition of ψ_3:

$$\forall uvwz \exists xy \big((u \neq v \wedge w \neq z) \Rightarrow (x \not\approx y \wedge \varphi(x, y, u, v) \wedge \varphi(x, y, w, z))\big).$$

The claim also follows from Theorem III.5.18. □

III.5.2. Definability of the disjointness property of principal congruences. Already in the article with the previous theorem, McKenzie showed that the lattice M_3 does not have definable principal congruences. Hence it was clear that his technique could not be used to obtain a finite basis theorem for congruence distributive (or even more general) varieties. However, we shall see later that many general results rely on the definability of another characteristics concerning principal congruences—namely the disjointness property.

Definition III.5.7. A class of algebras $\mathcal{K}$ has *definable disjointness property of principal congruences* (in short: *DDPC*) iff there exists a first-order formula $\varphi(x, y, u, v)$ such that for any $\mathbf{A} \in \mathcal{K}$ and any elements $a, b, c, d \in \mathbf{A}$ the following equivalence holds:

$$\mathbf{A} \models \varphi(a, b, c, d) \qquad \text{iff} \qquad \mathrm{Cg}^{\mathbf{A}}(a, b) \cap \mathrm{Cg}^{\mathbf{A}}(c, d) = 0_{\mathbf{A}}.$$

Observation III.5.8. *If $\mathcal{V}$ has principal congruences definable by $\varphi(x, y, u, v)$, then $\forall xy[(\varphi(x, y, a, b) \wedge \varphi(x, y, c, d)) \Rightarrow x \approx y]$ defines disjointness of principal congruences in $\mathcal{V}$.*

Burris conjectured in [**Bur79**] that any variety with a finite residual bound and DDPC is finitely based. The claim may be proved via Jónsson's theorem III.4.3:

THEOREM III.5.9. *Let $\mathcal{V}$ be a variety such that*

either (1) $\mathcal{V}$ has a finite residual bound,
or (2) $\mathcal{V}_{FSI}$ is finitely axiomatizable by a formula ψ_2.

If there exists a strictly elementary class $\mathcal{K} \supseteq \mathcal{V}$ with DDPC, then $\mathcal{V}$ is finitely based.

PROOF. Let ψ_1 be the formula that defines class $\mathcal{K}$.

Let $\mathcal{S} := \mathcal{K}_{FSI}$ be the class of finitely subdirectly irreducible algebras from $\mathcal{K}$. $\mathcal{S}$ is definable by the first-order formula

$$\psi_3 = \psi_1 \ \wedge \ \forall xyuv[\varphi(x, y, u, v) \Rightarrow (x = y \vee u = v)].$$

(1) It is easy to verify the conditions of Corollary III.4.3. For the last condition, use Proposition I.2.9.

(2) The conditions of Jónsson's theorem III.4.1 are satisfied. □

III.5.3. Definable principal subcongruences.

DEFINITION III.5.10. A *congruence formula* is any positive existential formula $\Gamma(u, v, x, y)$ such that $\Gamma(u, v, x, x) \Rightarrow u \approx v$ in any algebra.

Let $\Gamma(\_, \_, c, d) := \{(e, f); \Gamma(e, f, c, d)\}$.

For example, any principal congruence formula is a congruence formula.

PROPOSITION III.5.11. *If $\Gamma(u, v, x, y)$ is a congruence formula and $\mathbf{A} \models \Gamma(e, f, c, d)$, then $(e, f) \in \mathrm{Cg}(c, d)$. In other words,*

$$\Gamma(\_, \_, c, d) \subseteq \mathrm{Cg}(c, d).$$

PROOF. Γ is a positive formula and hence its satisfaction is carried over to homomorphic images (see Theorem I.1.6): if $\mathbf{A} \models \Gamma(e, f, c, d)$, then

$$\mathbf{A}/_{\mathrm{Cg}(c,d)} \models \Gamma([e]_{\mathrm{Cg}(c,d)}, [f]_{\mathrm{Cg}(c,d)}, [c]_{\mathrm{Cg}(c,d)}, [d]_{\mathrm{Cg}(c,d)}).$$

But $\Gamma(u, v, x, x) \Rightarrow u \approx v$, and hence

$$\mathbf{A}/_{\mathrm{Cg}(c,d)} \models [e]_{\mathrm{Cg}(c,d)} = [f]_{\mathrm{Cg}(c,d)},$$

which is the claim. □

PROPOSITION III.5.12. *For a finite family F of finite algebras of the same type, there is a congruence formula $\Gamma(u,v,x,y)$ such that for any $a,b \in \mathbf{A} \in F$,* $\mathrm{Cg}^{\mathbf{A}}(a,b) = \Gamma(\_,\_,a,b)$.

PROOF. For given elements a,b,c,d in a given algebra $\mathbf{A} \in F$, $(c,d) \in \mathrm{Cg}(a,b)$ is certified by some principal congruence formula $\Gamma^{\mathbf{A}}_{cdab}(c,d,a,b)$ which describes the corresponding Mal'cev chain. Moreover, the disjunction of congruence formulae is again a congruence formula. Hence we may define

$$\Gamma(u,v,x,y) \;:=\; \bigvee_{\substack{\mathbf{A}\in F,\\ a,b,c,d\in\mathbf{A}}} \Gamma^{\mathbf{A}}_{cdab}(u,v,x,y). \qquad \square$$

DEFINITION III.5.13. A variety $\mathcal{V}$ has *definable principal subcongruences* (in short: *DPSC*) iff there is a congruence formula Γ such that given any algebra $\mathbf{A} \in \mathcal{V}$ and any elements $a \neq b \in \mathbf{A}$, there exist elements $c \neq d \in \mathbf{A}$ such that $\mathbf{A} \models \Gamma(c,d,a,b)$ and $\mathrm{Cg}(c,d) = \Gamma(\_,\_,c,d)$.

LEMMA III.5.14. *For a congruence formula Γ, there exists a first-order formula $\Pi_\Gamma(x,y)$ such that $\mathbf{A} \models \Pi_\Gamma(c,d)$ iff* $\mathrm{Cg}(c,d) = \Gamma^{\mathbf{A}}(\_,\_,c,d)$.

PROOF. $\Pi_\Gamma(x,y)$ asserts that $\Gamma(\_,\_,x,y)$ is an equivalence relation compatible with the fundamental operations and that $\Gamma(x,y,x,y)$ holds.[12]

If $\mathbf{A} \models \Pi_\Gamma(c,d)$, then $\Gamma(\_,\_,c,d)$ is an equivalence containing c and d and hence $\Gamma(\_,\_,c,d) \supseteq \mathrm{Cg}(c,d)$. The other inclusion follows from the previous Proposition. On the other hand, if $\mathrm{Cg}(c,d) = \Gamma^{\mathbf{A}}(\_,\_,c,d)$, then the conditions for $\Gamma(\_,\_,c,d)$ stipulated by $\Pi_\Gamma(c,d)$ are clearly satisfied. $\square$

PROPOSITION III.5.15. *$\mathcal{V}$ has definable principal subcongruences iff there exist congruence formulae Γ_1 and Γ_2 such that given any algebra $\mathbf{A} \in \mathcal{V}$ and any elements $a \neq b \in \mathbf{A}$, there exist elements $c \neq d \in \mathbf{A}$ such that $\mathbf{A} \models \Gamma_1(c,d,a,b) \wedge \Pi_{\Gamma_2}(c,d)$.*

PROOF. If Γ satisfies the conditions of Definition III.5.13, we take $\Gamma_1 = \Gamma_2 = \Gamma$. On the other hand, if Γ_1 and Γ_2 are as in the statement, then we define $\Gamma(u,v,x,y) := \Gamma_1(u,v,x,y) \vee \Gamma_2(u,v,x,y)$. Then for every $a \neq b$, there exists $c \neq d$ such that $\Gamma_1(c,d,a,b)$ and hence also $\Gamma(c,d,a,b)$. Moreover, as $\Gamma_1(\_,\_,c,d) \subseteq \mathrm{Cg}(c,d)$ and $\Gamma_2(\_,\_,c,d) = \mathrm{Cg}(c,d)$, we get

$$\Gamma(\_,\_,c,d) = \Gamma_1(\_,\_,c,d) \cup \Gamma_2(\_,\_,c,d) = \mathrm{Cg}(c,d). \qquad \square$$

[12]Compare this with the definition of (i)–(iv) of Theorem III.9.2. If $\varphi := \Gamma$, then $\Pi_\Gamma :=$ i' $\wedge$ ii' $\wedge$ iii' $\wedge$ iv' $\wedge\, \Gamma(x,y,x,y)$, where i' – iv' are obtained from (i)–(iv) by erasing the quantification for x and y.

This means that $\mathcal{V}$ has definable principal subcongruences iff for each $\mathbf{A} \in \mathcal{V}$ and each $a \neq b$ there exist $c \neq d$ such that

- $(c, d) \in \mathrm{Cg}(a, b)$ is certified by Γ_1, and
- $\mathrm{Cg}(c, d)$ is defined by Γ_2.

We have seen in Proposition III.5.3 that $\mathcal{V}$ has definable principal **congruences** (DPC) iff there exists a finite set of principal congruence formulae sufficient to compute any principal congruence. If $\mathcal{V}$ has definable principal **subcongruences** (DPSC), then Γ_1 is sufficient to reach a principal congruence which can be fully computed using Γ_2.

Observation III.5.16. *If $\mathcal{V}$ has principal congruences definable by $\varphi(x, y, u, v)$, then $\mathcal{V}$ has principal subcongruences defined by the same formula.*

Example III.5.17. The variety generated by the five-element lattice M_3 has DPSC, but does not have DPC. The fact that $\mathcal{V}(M_3)$ has DPSC follows from Theorem III.6.2 and the fact that every lattice generates a congruence distributive variety (Example I.4.5). On the other hand, [**McK78**] shows that $\mathcal{V}$ does not have DPC.

Theorem III.5.18. *Let $\mathcal{V}$ be a variety with definable principal subcongruences. The following conditions are equivalent:*

(1) $\mathcal{V}$ is finitely based,
(2) $\mathcal{V}_{SI}$ is strictly elementary,
(3) $\mathcal{V}_{FSI}$ is strictly elementary.

Proof. (1⇔2) Let $\mathcal{K}$ be the class of algebras for which Γ witnesses DPSC: $\mathcal{K}$ is the class of models of the formula

$$\Phi \;:=\; \forall ab\big[a \not\approx b \Rightarrow \exists cd\big(c \not\approx d \wedge \Gamma(c, d, a, b) \wedge \Pi_\Gamma(c, d)\big)\big].$$

Then $\mathcal{K}_{SI}$ is the class of models of $\Phi \wedge \Psi$, where

$$\Psi \;:=\; \exists rs\big[r \not\approx s \wedge \forall ab\big[a \not\approx b \Rightarrow \exists cd\big(\Gamma(c, d, a, b) \wedge \Gamma(r, s, c, d)\big)\big]\big].$$

Since $\mathcal{V} \subseteq \mathcal{K}$, Corollary III.4.2(1) applies.
(1⇔3) The proof is the same, only take

$$\begin{aligned}\Psi \;:=\; \forall abcd\big[\big(a \not\approx b \wedge c \not\approx d\big) \Rightarrow &\\ \exists efghrs\big(\Gamma(e, f, a, b) \wedge \Gamma(g, h, c, d) \wedge &\\ r \not\approx s \wedge \Gamma(r, s, e, f) \wedge \Gamma(r, s, g, h)\big)\big]&\end{aligned}$$

and use part (2) of the Corollary III.4.2. □

III.6. Baker's Theorem: Congruence-distributivity

THEOREM III.6.1 (Baker [**Bak77**]). *Every finitely generated congruence distributive variety is finitely based.*

Baker announced this theorem in [**Bak72**]. Before his own proof was officially published five years later, several other proofs appeared: whilst Baker's proof employs a Ramsey argument to obtain a finite basis from an infinite one, Makkai [**Mak73**] used compactness; Taylor [**Tay78**] also used compactness, but found a recursive procedure to find the finite basis.

The shortest known proof of Baker's theorem was given by Baker and Wang in [**BaWa02**], that is thirty years after Baker announced the result. On the space of mere seven pages, Baker and Wang develop the concept of definable principal subcongruences, apply it to prove Baker's theorem III.6.1 (but not its extension III.7.1), and also show that the method cannot be used to give the famous result of Oates and Powell for finite groups – the group S_3 does not have definable principal subcongruences.

PROOF. In the view of Theorem III.5.18 on definable principal subcongruences and Jónsson's theorem I.4.6 on subdirectly irreducible algebras in a congruence distributive variety, we only need to show that if $\mathbf{A}$ is a finite algebra generating a congruence-distributive variety $\mathcal{V}$, then $\mathcal{V}$ has definable principal subcongruences. This is done in the next theorem. □

THEOREM III.6.2. *Let* $\mathbf{A}$ *be a finite algebra such that* $\mathrm{HSP}\,\mathbf{A}$ *is congruence distributive. Then* $\mathrm{HSP}\,\mathbf{A}$ *has definable principal subcongruences.*

STRUCTURE OF THE PROOF. By Jónsson's lemma I.4.6, $\mathcal{V}$ has a finite residual bound N. Given any algebra $\mathbf{B} \in \mathrm{HSP}\,\mathbf{A}$ and $a \neq b \in \mathbf{B}$, we shall

(1) construct $\mathbf{D} \leq \mathbf{B}$ with at most N generators (including a and b);
(2) select $c \neq d \in \mathbf{D}$ so that $\mathrm{Cg}^{\mathbf{D}}(c,d) \leq \mathrm{Cg}^{\mathbf{D}}(a,b)$;
(3) for any given $r, s \in \mathbf{B}$ such that $\mathrm{Cg}^{\mathbf{B}}(r,s) \leq \mathrm{Cg}^{\mathbf{B}}(c,d)$, define $\mathbf{C}_{rs}$ as the subalgebra of $\mathbf{B}$ generated by $\mathbf{D}$ and r, s;
(4) show that $(r,s) \in \mathrm{Cg}^{\mathbf{C}_{rs}}(c,d)$.

By local finiteness, $|\mathbf{D}|$ and $|\mathbf{C}_{rs}|$ have finite bounds depending only on $\mathbf{A}$, and hence the algebras $\mathbf{D}$ and $\mathbf{C}_{rs}$ can take only finitely many different shapes. By Proposition III.5.12, there are congruence formulae $\Gamma_1(u,v,x,y)$ and $\Gamma_2(u,v,x,y)$ such that $\mathbf{D} \models \Gamma_1(c,d,a,b)$ and $\mathbf{C} \models \Gamma_2(r,s,c,d)$. But then $\mathbf{B} \models \Gamma_1(c,d,a,b) \wedge \Gamma_2(r,s,c,d)$ because positive existential formulae carry over to extensions, which shows that $\mathrm{HSP}\,\mathbf{A}$ has DPSC.

Details of the proof.

(1) construct $\mathbf{D} \le \mathbf{B}$: Let $\mathbf{B} \hookrightarrow \prod_{i\in I} \mathbf{S}_i$ be a subdirect representation of $\mathbf{B}$. Choose $j \in I$ so that $\pi_j(a) \ne \pi_j(b)$ and $n_j := |\mathbf{S}_j|$ is as large as possible. Choose $e_1 := a, e_2 := b$ and $e_3, \dots e_{n_j} \in \mathbf{B}$ to be some pre-images of the elements of $\mathbf{S}_j$ under π_j. Let $\mathbf{D}$ be the subalgebra of $\mathbf{B}$ generated by $e_1, \dots, e_{n_j}$. Thus $\pi_j(\mathbf{D}) = \mathbf{S}_j$.

(2) select $c, d \in \mathbf{D}$: For convenience, we write $\pi_i^{\mathbf{D}}$ for $\pi_i|_{\mathbf{D}}$. Since $\mathbf{S}_j$ is subdirectly irreducible, $\ker \pi_j^{\mathbf{D}}$ is completely meet irreducible in $\operatorname{Con} \mathbf{D}$. By the congruence distributivity of HSP $\mathbf{A}$, the interval $[0_{\mathbf{D}}, \ker \pi_j^{\mathbf{D}}] \subseteq \operatorname{Con} \mathbf{D}$ is a prime ideal[13].

A set-theoretic complement of a prime ideal is a prime filter[14]. This means that $\operatorname{Con} \mathbf{D} \setminus [0_{\mathbf{D}}, \ker \pi_j^{\mathbf{D}}]$ has a least element α which is join-irreducible. Since $\mathbf{D}$ is finite, α is a finite join of principal congruences. Thus α must be a principal congruence, $\alpha = \operatorname{Cg}^{\mathbf{D}}(c, d)$. Because α is the least congruence in $\operatorname{Con} \mathbf{D}$ not under $\ker \pi_j$ and because $\operatorname{Cg}^{\mathbf{D}}(a, b) \not\le \ker \pi_j$, we have

$$\operatorname{Cg}^{\mathbf{D}}(c, d) \le \operatorname{Cg}^{\mathbf{D}}(a, b).$$

(3) define $\mathbf{C}_{rs}$: Let $r, s \in \mathbf{B}$ be given so that $\operatorname{Cg}^{\mathbf{B}}(r, s) \le \operatorname{Cg}^{\mathbf{B}}(c, d)$. We define $\mathbf{C}_{rs}$ to be the subalgebra of $\mathbf{B}$ generated by $\mathbf{D}$ and r, s.

(4) show that $(r, s) \in \operatorname{Cg}^{\mathbf{C}_{rs}}(c, d)$: $\mathbf{C}_{rs}$ is finite by local finiteness of HSP $\mathbf{A}$, so we can apply Proposition I.4.7 to c, d, r, s and

$$\mathbf{C}_{rs} \hookrightarrow \prod_{i\in I} \mathbf{S}_i,$$

as follows:

Let us say that $i \in I$ *separates* $u, v \in \mathbf{B}$ iff $\pi_i(u) \ne \pi_i(v)$. It is equivalent to saying that π_i does not glue u and v together, i.e. $\operatorname{Cg}^{\mathbf{B}}(u, v) \not\le \ker \pi_i$.

Let i separate c, d. Then i also separates a, b. Again because $\alpha = \operatorname{Cg}^{\mathbf{D}}(c, d)$ is the least congruence not under $\ker \pi_j$, we have

$$\ker \pi_i^{\mathbf{D}} \le \ker \pi_j^{\mathbf{D}}.$$

[13]A *prime ideal* in a lattice L is a subset $I \subseteq L$ such that

- for every $x \in I$ and any $y \in L$, if $y \le x$, then $y \in I$: I is a *lower set*;
- For every $x, y \in I$, there is some element $z \in I$, such that $z \ge x$ and $z \ge y$: I is *upward directed*;
- for every elements $x, y \in L$, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$ (*primality*).

[14]A *prime filter* is an upper, downward directed set which is prime in the dual sense: if the filter contains a join of some elements, then it contains at least one of them.

Then there is an induced surjective map

$$\mathbf{D}/_{\ker \pi_i^{\mathbf{D}}} \simeq \pi_i(\mathbf{D}) \quad \twoheadrightarrow \quad \mathbf{D}/_{\ker \pi_j^{\mathbf{D}}} \simeq \pi_j(\mathbf{D}) \simeq \mathbf{S}_j.$$

By the choice of j so that $|\mathbf{S}_j|$ is greatest possible among those $\mathbf{S}_i$ for which π_i separates a, b,

$$\pi_i : \mathbf{D} \twoheadrightarrow \mathbf{S}_i.$$

Looking at $\mathbf{C}_{rs} \hookrightarrow \prod_{i\in I} \mathbf{S}_i$, we see that if i separates c, d, then $\mathbf{S}_i \supseteq \pi_i(\mathbf{C}_{rs}) \supseteq \pi_i(\mathbf{D}) = \mathbf{S}_i = \pi_i(\mathbf{B})$, so $\pi_i(\mathbf{C}_{rs}) = \pi_i(\mathbf{B})$ and

$$(r_i, s_i) \in \mathrm{Cg}^{\pi_i(\mathbf{B})}(c_i, d_i) = \mathrm{Cg}^{\pi_i(\mathbf{C}_{rs})}(c_i, d_i),$$

where r_i, s_i, c_i and d_i are images in $\mathbf{S}_i$.

On the other hand, if i does not separate c, d, then neither does it separate r, s, so again $(r_i, s_i) \in \mathrm{Cg}^{\pi_i(\mathbf{C}_{rs})}(c_i, d_i) = 0_{\pi_i(\mathbf{C}_{rs})}$. Then the Proposition I.4.7 applies to show that

$$(r, s) \in \mathrm{Cg}^{\mathbf{C}_{rs}}(c, d). \qquad \square$$

Several comments can be made about the proof:

- One could economise on generators of $\mathbf{D}$ by using only enough elements to generate $\mathbf{S}_j$. Then $\mathbf{D}$ and $\mathbf{C}_{rs}$ have respectively at most $g + 2$ and $g + 4$ generators, where g is the maximum of the minimum numbers of generators needed for the various subdirectly irreducible members of HSP $\mathbf{A}$.
- The proof is valid for any locally finite, congruence distributive variety with a finite residual bound.

III.7. Burris's proof of Baker's theorem

In 1979, Jónsson proved the following extension of Baker's theorem:

Theorem III.7.1 (Jónsson [**Jón79a**]). *If $\mathcal{V}$ is a congruence distributive variety such that $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely based.*

It is clear from Jónsson's lemma I.4.6 that Jónsson's theorem covers Baker's. Here we give a proof from Burris [**Bur79**].

We assume that terms $p_1, \ldots, p_{k-1}$ from Theorem I.4.3 are part of the signature σ (we may do this by Proposition I.1.19). Let ϕ be the formula

$$\forall xuv \left[\bigwedge_{i\in\widehat{k-1}} p_i(x, u, x) = p_i(x, v, x) \right]$$

$$\wedge \ \forall xy \left[x \not\approx y \Rightarrow \bigvee_{i\in\widehat{k-1}} p_i(x, x, y) \not\approx p_i(x, y, y) \right]$$

Observation III.7.2. *$\mathcal{V}$ is congruence distributive iff there exists a formula ϕ of the form given above such that $\mathcal{V} \models \phi$.*

Burris's proof of Theorem III.7.1. The idea of the proof is to find a formula δ_N certifying that $\mathcal{V}$ has DDPC, and then use Theorem III.5.9. For clarity, we do not use this theorem but rather write out the full structure of the proof:

(1) Define $\delta_n, n \in \mathbb{N}$ so that if $\mathbf{A} \in \operatorname{Mod} \phi$ and $a, b, c, d \in \mathbf{A}$ then $\operatorname{Cg}(a, b) \cap \operatorname{Cg}(c, d) \neq 0_{\mathbf{A}}$ iff $\mathbf{A} \models \delta_n(a, b, c, d)$ for some n.
(2) Define formula γ_N so that it ensures that if $\mathbf{A} \models \delta_n(a, b, c, d)$ for some n, then $\mathbf{A} \models \delta_N(a, b, c, d)$.
(3) Let $\psi_1 := \phi \wedge \gamma_N$,
    ψ_2 be the sentence axiomatizing $\mathcal{V}_{FSI}$,
    $\psi_3 := \forall xyuv[\neg\delta_N(x, y, u, v) \Rightarrow (x \approx y \vee u \approx v)]$.
    Show that $\mathcal{V} \models \psi_1 \wedge (\psi_3 \Rightarrow \psi_2)$.
(4) Use compactness to get a $E_0 \subseteq_{FIN} \operatorname{Eq} \mathcal{V}$ such that

$$E_0 \models \psi_1 \wedge (\psi_3 \Rightarrow \psi_2).$$

Let $\mathbf{A} \in \operatorname{Mod} E_0$ be subdirectly irreducible and let $a, b, c, d \in \mathbf{A}$ be such that $a \neq b, c \neq d$.
    $\mathbf{A} \models \phi$ implies that for some n, $\mathbf{A} \models \delta_n(a, b, c, d)$.
    $\mathbf{A} \models \gamma_N$ and $\mathbf{A} \models \delta_n(a, b, c, d)$ imply that $\mathbf{A} \models \delta_N(a, b, c, d)$.
    Now $\mathbf{A} \models \psi_3$, therefore $\mathbf{A} \models \psi_2$, so $\mathbf{A} \in \mathcal{V}_{FSI}$.
We see that any subdirectly irreducible algebra $\mathbf{A} \in \operatorname{Mod} E_0$ lies in $\mathcal{V}$, therefore $\operatorname{Mod} E_0 \subseteq \mathcal{V}$. But the opposite inclusion is clear.

To complete the proof, we need to fill in steps (1), (2) and (3) with sufficient detail. We'll do this in the subsequent sections. □

Burris's paper also includes an explicit upper bound on the number of variables in the finite set of equations. It is

$$\left[2\binom{S+1}{2} - 1\right]\left[2S^S(R-1) + 4\right],$$

where S is the maximum size of $\mathbf{A} \in \mathcal{V}_{FSI}$ and R is the maximum arity of an operation symbol in σ (due to the assumption that $p_i \in \sigma$, $R \geq 3$).

III.7.1. Definition of δ_n.

DEFINITION III.7.3. For $n \in \mathbb{N}$ let $\delta_n(x, y, u, v)$ be the first-order formula[15]

$$\bigvee_{\substack{i \in \widehat{k-1},\\ r,s \in \mathrm{Sl}_n}} \exists \overline{z} \exists \overline{w}\ p_i(r(x, \overline{z}), s(u, \overline{w}), r(y, \overline{z})) \neq p_i(r(x, \overline{z}), s(v, \overline{w}), r(y, \overline{z})).$$

OBSERVATION III.7.4. $\mathbf{A} \models \delta_n(a, b, c, d) \Rightarrow \delta_{n+1}(a, b, c, d)$

LEMMA III.7.5. *Let $\mathbf{A}$ be any algebra of type σ satisfying ϕ and let $a, b, c, d \in \mathbf{A}$. Then the following conditions are equivalent*

(1) $\mathrm{Cg}^{\mathbf{A}}(a, b) \cap \mathrm{Cg}^{\mathbf{A}}(c, d) \neq 0_{\mathbf{A}}$.

(2) $p_i^{\mathbf{A}}(r(a), s(c), r(b)) \neq p_i^{\mathbf{A}}(r(a), s(d), r(b))$
for some $r, s \in \mathrm{Tr}\,\mathbf{A}$ *and* $i \in \widehat{k-1}$.

(3) $\mathbf{A} \models \delta_n(a, b, c, d)$ *for some* $n \in \mathbb{N}$.

PROOF. $(1 \Rightarrow 2)$ Let $(e, f) \in \mathrm{Cg}(a, b) \cap \mathrm{Cg}(c, d)$ for some $e \neq f$.

CLAIM 1. *For some* $t \in \mathrm{Tr}\,\mathbf{A}$ *and some* $p_j, j \in \widehat{k-1}$, *we have*

$$p_j(e, t(a), f) \neq p_j(e, t(b), f).$$

⇝ Assume we always have $p_j(e, t(a), f) = p_j(e, t(b), f)$. We can use Lemma I.3.6 and the fact that $(e, f) \in \mathrm{Cg}(a, b)$ to see that $p_j(e, e, f) = p_j(e, f, f)$ for all j. But this is a contradiction with the existence of j such that $\mathcal{V} \models p_j(e, e, f) \not\approx p_j(e, f, f)$. ↯

Choose t and p_j as in Claim 1 and let $r(x) := p_j(e, t(x), f)$.[16]

CLAIM 2. *For some* $s \in \mathrm{Tr}\,\mathbf{A}$ *and some* p_i, *we have*

$$p_i(r(a), s(c), r(b)) \neq p_i(r(a), s(d), r(b)).$$

By definition, $r(a) \neq r(b)$. Furthermore,

$$p_i(e, t(a), f) \sim_{\mathrm{Cg}(e,f)} p_i(e, t(a), e) = p_i(e, t(b), e) \sim_{\mathrm{Cg}(e,f)} p_i(e, t(b), f),$$

where the equality follows from ϕ. Thus

$$(r(a), r(b)) \in \mathrm{Cg}(e, f) \subseteq \mathrm{Cg}(c, d),$$

so we may use Claim 1 with $e = r(a), f = r(b)$ and c, d in place of a, b.

$(2 \Rightarrow 1)$ Let $t_i(x, y, z) = p_i(r(x), s(y), r(z))$.

Clearly, the pair

$$(t_i(a, c, b), t_i(a, d, b)) \in \mathrm{Cg}(c, d).$$

[15] To make the definition fully precise, we have to limit the scope of the variables: if r is the maximum arity of a function symbol from σ, then the w's come from $\{w_1, \ldots, w_{n(r-1)}\}$ and the z's come from $\{z_1, \ldots, z_{n(r-1)}\}$. Sl_n is defined in I.3.2.

[16] Here we use the assumption that $p_j \in \sigma$, otherwise r would not be a translation.

Since $\quad (r(a), r(b)) \in \mathrm{Cg}(a,b) \quad$ and $\quad \mathcal{V} \models p_i(x,u,x) \approx p_i(x,v,x),$
$$(t_i(a,c,b), t_i(a,d,b)) \in \mathrm{Cg}(a,b).$$

($2 \Leftrightarrow 3$) This is clear from the definitions. □

III.7.2. Definition of γ_N.

DEFINITION III.7.6. For $n \in \mathbb{N}$, let γ_n be the first-order sentence
$$\forall xyuv \quad \delta_{n+1}(x,y,u,v) \Rightarrow \delta_n(x,y,u,v).$$

LEMMA III.7.7. *Let* $\mathbf{A}$ *be any algebra in signature* σ *such that* $\mathbf{A} \models \gamma_n$. *Then*

(1) $\mathbf{A} \models \gamma_{n+1}$.
(2) If $\mathbf{A} \models \delta_k(a,b,c,d)$ *for some* $a,b,c,d \in \mathbf{A}$ *and* $k \in \mathbb{N}$, *then* $\mathbf{A} \models \delta_n(a,b,c,d)$.

PROOF. (1) Assume that
$$\mathbf{A} \models \gamma_n \wedge \delta_{n+2}(a,b,c,d).$$
We want to show that $\delta_{n+1}(a,b,c,d)$.
Choose $p, q \in \mathrm{Sl}_{n+2}$ and $\overline{e}, \overline{\epsilon} \in \mathbf{A}$ witnessing δ_{n+2}:
$$p_i(p(a,\overline{e}), q(c,\overline{\epsilon}), p(b,\overline{e})) \neq p_i(p(a,\overline{e}), q(e,\overline{\epsilon}), p(b,\overline{e})).$$
We know that there exist some function symbols $f, g \in \sigma$ and some $p', q' \in \mathrm{Sl}_{n+1}$ such that
$$p(x,\overline{y}) = p'(f(x,\overline{y_1}), \overline{y_2})$$
$$q(x,\overline{z}) = q(g(x,\overline{z_1}), \overline{z_2}).$$
Let
$$a' = f(a,\overline{e_1}), \quad b' = f(b,\overline{e_1}), \quad c' = g(c,\overline{\epsilon_1}), \quad d' = g(d,\overline{\epsilon_1}).$$
Then
$$p_i(p'(a',\overline{e_2}), q'(c',\overline{\epsilon_2}), p'(b',\overline{e_2})) \neq p_i(p'(a',\overline{e_2}), q'(d',\overline{\epsilon_2}), p'(b',\overline{e_2})).$$
This means that
$$\mathbf{A} \models \delta_{n+1}(a',b',c',d').$$
Now we use γ_n to get
$$\delta_n(a',b',c',d').$$
To finish the proof, we need to show that $\delta_{n+1}(a,b,c,d)$. But this is clear: we substitute $f(a,\overline{e_1})$ etc. for their corresponding elements a',b',c',d', and the depth of terms certifying that $\delta_n(a',b',c',d')$ increases by one.

(2) For $k \leq n$ use Observation III.7.4. For $k > n$, use part (1) to get $\gamma_{n+1}, \gamma_{n+2}, \ldots, \gamma_{k-1}$ and then $\delta_{k-1}, \delta_{k-2}, \ldots, \delta_n$. □

III.7.3. The formula $\psi_1 \wedge (\psi_3 \Rightarrow \psi_2)$ holds in $\mathcal{V}$.

As we have mentioned before, formulae ψ_1 and ψ_3 are defined thus:

$$\psi_1 := \phi \wedge \gamma_N.$$
$$\psi_3 := \forall xyuv[\neg\delta_N(x,y,u,v) \Rightarrow (x \approx y \vee u \approx v)].$$

Equivalently,

$$\psi_3 = \forall xyuv[(x \not\approx y \wedge u \not\approx v) \Rightarrow \delta_N(x,y,u,v)].$$

LEMMA III.7.8. *If $\mathcal{V}_{FSI}$ is axiomatized by a first-order formula ψ_2, then there exists $N \in \mathbb{N}$ such that*

(1) $\mathcal{V}_{FSI} \models \psi_3$;
(2) $\mathcal{V} \models \gamma_N$ (and hence also $\mathcal{V} \models \psi_1$);
(3) $\mathcal{V} \models \psi_3 \Rightarrow \psi_2$.

PROOF. (1) Let $\sigma^* = \sigma \cup \{a, b, c, d\}$. Let

$$\varphi_m = \psi_2 \;\wedge\; a \not\approx b \;\wedge\; c \not\approx d \;\wedge\; \neg\delta_m(a,b,c,d).$$

CLAIM 1. *$\{\varphi_m : m \in \mathbb{N}\}$ has no model.*

If $\mathbf{A}^*$ is a σ^*-algebra and $\mathbf{A}$ is its σ-reduct, then the algebras $\mathbf{A}^*$ and $\mathbf{A}$ share two important characteristics: the same σ-sentences are true in them and they have the same congruences.

Let

$$\mathbf{A}^* \models \psi_2 \;\wedge\; a \not\approx b \;\wedge\; c \not\approx d.$$

$\mathbf{A}^* \models \psi_2$ means that $\mathbf{A} \in \mathcal{V}_{FSI}$. This has two consequences: firstly, as $\mathbf{A} \in \mathcal{V}$,

$$\mathbf{A}^* \models \phi.$$

Secondly,[17]

$$\mathrm{Cg}^{\mathbf{A}^*}(a,b) \cap \mathrm{Cg}^{\mathbf{A}^*}(c,d) \neq 0_{\mathbf{A}^*}.$$

Lemma III.7.5 guarantees the existence of some n such that

$$\mathbf{A}^* \models \delta_n(a,b,c,d).$$

But this means that $\mathbf{A}^* \not\models \varphi_n$, showing that no σ^*-algebra can be a model of all $\varphi_m, m \in \mathbb{N}$ simultaneously.

Due to the Compactness theorem I.1.1, there is an $N \in \mathbb{N}$ such that $\{\varphi_m : m \leq N\}$ has no model. In other words, any σ^*-algebra $\mathbf{A}^*$ satisfies

$$\psi_2 \Rightarrow \left[(a \not\approx b \;\wedge\; c \not\approx d) \Rightarrow \bigvee_{m \leq N} \delta_m(a,b,c,d) \right].$$

[17] We write a for a^{A^*} etc., in other words we use the same symbol for the constant (element of the language) and for the element of the algebra (in the extended signature) which is denoted by this constant.

However, due to Observation III.7.4, this is equivalent to

$$\psi_2 \Rightarrow [(a \not\approx b \ \wedge\ c \not\approx d) \Rightarrow \delta_N(a,b,c,d)]\,.$$

This is true for any evaluation of the constants $a, b, c, d \in \mathbf{A}^*$, so we may go back to σ-reducts: for any σ-algebra $\mathbf{A}$,

$$\mathbf{A} \models \psi_2 \Rightarrow \forall xyuv\,[(x \not\approx y \ \wedge\ u \not\approx v) \Rightarrow \delta_N(x,y,u,v)]\,,$$

showing that

$$\mathcal{V}_{FSI} \models (x \not\approx y \ \wedge\ u \not\approx v) \Rightarrow \delta_N(x,y,u,v),$$

which is the claim.

(2) If $\mathbf{A} \in \mathcal{V}_{FSI}$, $a, b, c, d \in \mathbf{A}$ and $\mathbf{A} \models \delta_{N+1}(a,b,c,d)$, then by Lemma III.7.5 we see that $\mathrm{Cg}(a,b) \cap \mathrm{Cg}(c,d) \neq 0_{\mathbf{A}}$, so $a \neq b, c \neq d$ and from part (1) we have $\mathbf{A} \models \delta_N(a,b,c,d)$. We see that

$$\mathbf{A} \in \mathcal{V}_{FSI} \qquad \Rightarrow \qquad \mathbf{A} \models \gamma_N.$$

Let $\mathbf{A} \in \mathcal{V}$. According to Birkhoff's Theorem I.2.3,

$$\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$$

for some $\mathbf{A}_i \in \mathcal{V}_{SI} \subseteq \mathcal{V}_{FSI}$. For an element $a \in \mathbf{A}$, let $a^i \in \mathbf{A}_i$ denote the i-th projection of a. We show that a sentence of the form γ_N is preserved by subdirect products, hence $\mathcal{V} \models \gamma_N$.

Claim 2. *$\mathbf{A} \models \delta_n(a,b,c,d)$ iff for some $i \in I$, $\mathbf{A}_i \models \delta_n(a^i,b^i,c^i,d^i)$.*

If $a, b, c, d \in \mathbf{A}$ and

$$\mathbf{A} \models \delta_n(a,b,c,d),$$

then for some $j \in \widehat{k-1}$, terms $p, q \in \mathrm{Sl}_n$ and elements $\overline{e}, \overline{\epsilon}$

(*) $$p_j^{\mathbf{A}}(p(a,\overline{e}), q(c,\overline{\epsilon}), p(b,\overline{e})) \neq p_j^{\mathbf{A}}(p(a,\overline{e}), q(e,\overline{\epsilon}), p(b,\overline{e})).$$

This means that for some $i \in I$

(†) $$p_j^{\mathbf{A}_i}(p(a^i,\overline{e^i}), q(c^i,\overline{\epsilon^i}), p(b^i,\overline{e^i})) \neq p_j^{\mathbf{A}_i}(p(a^i,\overline{e^i}), q(e^i,\overline{\epsilon^i}), p(b,\overline{e^i})).$$

Therefore $\mathbf{A}_i \models \delta_n(a^i,b^i,c^i,d^i)$.

The opposite direction in the proof of the claim is done similarly: if $\mathbf{A}_i \models \delta_n(a^i,b^i,c^i,d^i)$, just pick any tuples $\overline{e}, \overline{\epsilon}$ such that their i-th projections satisfy (†). The left-hand side and the right-hand side of (*) evaluate differently in at least one coordinate, proving that $\mathbf{A} \models \delta_n(a,b,c,d)$.

Since $\mathbf{A}_i \in \mathcal{V}_{FSI}$,

$$\mathbf{A}_i \models \delta_{N+1}(a^i,b^i,c^i,c^i) \quad \text{implies} \quad \mathbf{A}_i \models \delta_N(a^i,b^i,c^i,d^i).$$

According to the claim, this means that

$$\mathbf{A} \models \gamma_N.$$

(3) Let $\mathbf{A} \in \mathcal{V}$ be such that $\mathbf{A} \models \psi_3$. This means that for any 4-tuple $a, b, c, d \in \mathbf{A}$ such that $a \neq b, c \neq d$, we have $\delta_N(a, b, c, d)$ and hence (according to Lemma III.7.5), $\mathrm{Cg}(a, b) \cap \mathrm{Cg}(c, d) \neq 0_{\mathbf{A}}$. However, this is equal to saying that $\mathbf{A} \in \mathcal{V}_{FSI}$ and therefore $\mathbf{A} \models \psi_2$. □

III.8. Willard's Theorem: Congruence meet-semidistributivity

Theorem III.8.1 (Willard [**Wil00**]). *Let $\mathcal{V}$ be a congruence meet-semidistributive variety with a finite residual bound. Then $\mathcal{V}$ is finitely based.*

Proof. By Theorem I.4.10 (4) there exists a finite collection s_e, t_e of Willard terms for $\mathcal{V}$. By I.1.19 we may assume that the Willard terms are already part of the signature. Let $\mathcal{V}^*$ be the class of algebras that satisfy conditions for Willard terms as expressed by I.4.10 (4); this class is finitely axiomatizable by the conjunction

$$\left(x = y \quad \Leftrightarrow \quad \bigwedge_{e \in E} \left[s_e(x, x, y) = t_e(x, x, y) \ \Leftrightarrow \ s_e(x, y, y) = t_e(x, y, y)\right]\right) \\ \wedge \ \bigwedge_{e \in E} s_e(x, y, x) \approx t_e(x, y, x).$$

By definition, $\mathcal{V} \subseteq \mathcal{V}^*$.

There exists an integer m such that $\mathbf{A} \in \mathcal{V}_{SI} \Rightarrow |\mathbf{A}| \leq m$. We define a formula ϕ_m such that any model of ϕ_m has a subdirectly irreducible homomorphic image with more than m elements; then $\mathcal{V} \models \neg\phi_m$. Let $\mathcal{K}$ be the class of algebras defined by $\neg\phi_m$ and the formula defining $\mathcal{V}^*$; then $\mathcal{K}$ is a strictly elementary class such that $\mathcal{V} \subseteq \mathcal{K} \subseteq \mathcal{V}^*$.

We also define a formula μ_m which certifies that $\mathcal{K}$ has DDPC. Then we can apply Theorem III.5.9.

To finish the proof, we only need to define formulae ϕ_m and μ_m; we do this in the subsequent sections. □

III.8.1. Approximating Mal'cev chains. According to Corollary I.3.6, the property "$(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$" is equivalent to the property "there exists a Mal'cev chain from c to d consisting of images of a and b under translations"; however, it is not expressible by a first-order formula for two reasons:

- there is no upper bound on the length of the Mal'cev chain from c to d, and
- there are infinitely many different translations.

If for any congruence meet-semidistributive variety $\mathcal{V}$ we could find an upper bound

- on the length of the Mal'cev chain, and
- on the depth of the translations used in the Mal'cev chain,

then the relation "$(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$" would be definable by a first-order formula in $\mathcal{V}$. However, that would lead to an effective proof that congruence meet-semidistributive varieties have definable principal congruences. This is not possible: McKenzie [**McK78**] gives an example of a lattice that does not have definable principal congruences—and any algebra with a semilattice operation generates a congruence meet-semidistributive variety.

Similarly as in the proof of Baker's Theorem, the main trick of the proof is to show that although $\mathcal{V}$ does not have definable principal congruences, the disjointness of principal congruences is definable. Willard's proof is similar to original Baker's proof and does not use compactness in order to find the bounds on the length and depth of the Mal'cev chains in the formulae ϕ_m and μ_m; the argument is based on Ramsey's theorem, the existence of Willard terms and the existence of a finite residual bound. We use Compactness only after we have shown that $\mathcal{V}$ has DDPC (in the proof sketched above, it is hidden behind the use of Theorem III.5.9).

The following notation turns out useful when handling Mal'cev chains:

Definition III.8.2. Let $A^{(2)}$ denote the set of 2-element subsets of A. Assume that $\{a,b\},\{c,d\} \in A^{(2)}$ and that

$$(\odot) \qquad c = c_0, c_1, \ldots, c_n = d$$

is a Mal'cev chain certifying that

$$(c,d) \in \mathrm{Cg}(a,b)$$

such that for each $i \in \mathbb{Z}_n$, either there exists $p_i \in \mathrm{Tr}_k\,\mathbf{A}$ such that $\{p_i(a), p_i(b)\} = \{c_i, c_{i+1}\}$ or else $c_i = c_{i+1}$. In that case we say that there exists a *Mal'cev chain of length at most n and depth at most k* and write[18]

$$\{a,b\} \xrightarrow[k]{n} \{c,d\}.$$

Occasionally, we write $\xrightarrow[k]{?}$ if we want to leave n unknown.

[18] Willard's notation is $\Rightarrow_{k,n}$; then $\Rightarrow_k$ may be defined as "there exists n such that $\Rightarrow_{k,n}$", and $\rightarrow_k$ as an abbreviation for $\Rightarrow_{k,1}$. [**BMW04**] use $\looparrowright^n_k$.

Proposition III.8.3.

(1) $(c,d) \in \mathrm{Cg}(a,b)$ *iff* $\{a,b\} \xrightarrow[k]{n} \{c,d\}$ *for some* k *and* n.

(2) For each k *and* n*, there exists a finite set* $\Pi_{k,n}$ *of principal congruence formulae such that*

$$(*) \qquad u \neq v \qquad \wedge \bigvee_{\pi \in \Pi_{k,n}} \pi(u,v,x,y)$$

defines the relation $\{x,y\} \xrightarrow[k]{n} \{u,v\}$.

Note: for brevity, we often write $\{x,y\} \xrightarrow[k]{n} \{u,v\}$ instead of $(*)$.

(3) A class $\mathcal{K}$ *has DDPC iff there exist* $k,n \in \mathbb{N}$ *so that for all* $\mathbf{A} \in \mathcal{K}$ *and all* $\{a_1,b_2\},\{a_2,b_2\} \in A^{(2)}$,

if $\mathrm{Cg}(a_1,b_1) \cap \mathrm{Cg}(a_2,b_2) \neq 0_{\mathbf{A}}$,

then there exists $c \neq d \in \mathbf{A}$ *such that*

$$\{a_1,b_1\} \xrightarrow[k]{n} \{c,d\} \quad \text{and} \quad \{a_2,b_2\} \xrightarrow[k]{n} \{c,d\}.$$

(4) $\{a,b\} \xrightarrow[k]{n} \{c,d\} \xrightarrow[l]{m} \{e,f\}$ *implies* $\{a,b\} \xrightarrow[k+l]{nm} \{e,f\}$*. In other words, compositions of* $\xrightarrow[k]{n}$ *are additive in* k *and multiplicative in* n.

(5) If $\{a,b\} \xrightarrow[k+l]{1} \{e,f\}$ *then* $\{a,b\} \xrightarrow[k]{1} \{c,d\} \xrightarrow[l]{1} \{e,f\}$ *for some* c,d.

Proof. (1) is just a restatement of I.3.6.

(2) There are only finitely many translations of depth k, and hence there are only finitely many combinations in which we can construct a principal congruence formula describing a Mal'cev chain of length n and depth k.

($3 \Rightarrow$) Let $\varphi(x,y,u,v)$ be the formula which defines disjointness of principal congruences in $\mathcal{K}$. Let Σ be the infinite set of formulae

$$\left\{ \neg\exists wz \ \left(\{x,y\} \xrightarrow[k]{n} \{w,z\} \ \wedge \ \{u,v\} \xrightarrow[k]{n} \{w,z\} \right); \quad k,n \in \mathbb{N} \right\}.$$

Then $\Sigma \models \varphi$. To finish the proof, use the Compactness Theorem and the fact that $\{a,b\} \xrightarrow[n]{k} \{c,d\}$ implies $\{a,b\} \xrightarrow[n']{k'} \{c,d\}$ whenever $n \geq n'$ and $k \geq k'$.

($3 \Leftarrow$) If such n and k exist, then the formula

$$\neg\exists wz \ \left(\{x,y\} \xrightarrow[k]{n} \{w,z\} \ \wedge \ \{u,v\} \xrightarrow[k]{n} \{w,z\} \right)$$

defines disjointness of principal congruences in $\mathcal{K}$.

(4) Assume that

$$e = e_0, e_1, \ldots, e_m = f$$

is a Mal'cev chain certifying that

$$(e,f) \in \mathrm{Cg}(c,d)$$

such that for each $j \in \mathbb{Z}_m$, either there exists $t_j \in \mathrm{Tr}_l\,\mathbf{A}$ such that $\{t_j(c), t_j(d)\} = \{e_j, e_{j+1}\}$ or else $e_j = e_{j+1}$. Moreover, we have the sequence ($\odot$) of Definition III.8.2 at hand. We construct a sequence

$$e_j = e_{j0}, e_{j1}, \ldots, e_{jn} = e_{j+1}$$

from e_j to e_{j+1} in the following way:
if $e_j = t_j(c)$ and $c_i = p_i(a)$, then $e_{ji} = t_j(p_i(a))$
if $e_j = t_j(c)$ and $c_i = p_i(b)$, then $e_{ji} = t_j(p_i(b))$
if $e_j = t_j(d)$ and $c_{n-i+1} = p_{n-i}(a)$, then $e_{ji} = t_j(p_{n-i}(a))$
if $e_j = t_j(d)$ and $c_{n-i+1} = p_{n-i}(b)$, then $e_{ji} = t_j(p_{n-i}(b))$

$t_j(p_i(x))$ is a translation of depth $k + l$ for any j and i. Hence the sequence

$$e = e_0 = e_{00}, e_{01}, \ldots, e_{0(n-1)}, e_1 = e_{10}, \ldots, e_{m(n-1)} = e_m = f$$

certifies that $\{a, b\} \xrightarrow[k+l]{nm} \{e, f\}$.

(5) If $\{a, b\} \xrightarrow[k+l]{1} \{e, f\}$, then there exists some $t \in \mathrm{Tr}_{k+l}$ such that $\{e, f\} = t(\{a, b\})$. By the definition of Tr, we see that there exist terms $p_k \in \mathrm{Tr}_k$ and $p_l \in \mathrm{Tr}_l$ such that $t(x) = p_l(p_k(x))$. Then $\{c, d\} := p_k(\{a, b\})$ satisfies $\{a, b\} \xrightarrow[k]{1} \{c, d\} \xrightarrow[l]{1} \{e, f\}$. □

III.8.2. The length of the Mal'cev chains. In the last lemma of this section, we show that for proving the fact that n principal congruences are not disjoint, we only need Mal'cev chains of length at most 2^n. This is the length we use in the definitions of ϕ_m and μ_m.

Throughout this section, we use the assumption that $\mathcal{V}$ is congruence meet-semidistributive and the terms s_e, t_e are part of the signature. We also assume that $\mathbf{A} \in \mathcal{V}$ and $a \neq b$ are elements of $\mathbf{A}$.

Definition III.8.4. By a *sequence from a to b* we mean a finite sequence $S = (a_0, a_1, \ldots, a_n)$ of elements from $\mathbf{A}$ such that $a_0 = a$ and $a_n = b$. By a *link* we mean any pair $\{a_i, a_{i+1}\}$ with $i \in \mathbb{Z}_n$ and $a_i \neq a_{i+1}$.

Lemma III.8.5 (Single-sequence lemma). *Let S be a sequence of elements of $\mathbf{A}$ from a to b. Then there exist elements $c \neq d \in \mathbf{A}$ and a link $\{a_i, a_{i+1}\}$ in S such that*

$$\{a, b\} \xrightarrow[1]{2} \{c, d\} \quad \textit{and} \quad \{a_i, a_{i+1}\} \xrightarrow[1]{2} \{c, d\}.$$

Proof. Because $a \neq b$, there exists $e \in E$ such that for the corresponding Willard terms

$$s_e(a, a, b) = t_e(a, a, b) \nLeftrightarrow s_e(a, b, b) = t_e(a, b, b).$$

Without loss of generality, we assume that $s_e(a,a,b) = t_e(a,a,b)$ but $s_e(a,b,b) \neq t_e(a,b,b)$. Let a_{i+1} be the first element in the sequence S for which $s_e(a,a_{i+1},b) \neq t_e(a,a_{i+1},b)$. Then $s_e(a,a_i,b) = t_e(a,a_i,b)$ and $\{a_i,a_{i+1}\}$ is a link in S. Let

$$\begin{aligned} c &:= s_e(a,a_{i+1},b) \\ d &:= t_e(a,a_{i+1},b) \\ u &:= s_e(a,a_i,b) = t_e(a,a_i,b) \\ v &:= s_e(a,a_{i+1},a). \end{aligned}$$

Then define

$$\begin{aligned} f_1(x) &:= s_e(a,x,b) && : \{a_i,a_{i+1}\} \mapsto \{c,u\} \\ f_2(x) &:= t_e(a,x,b) && : \{a_i,a_{i+1}\} \mapsto \{u,d\} \\ g_1(x) &:= s_e(a,a_{i+1},x) && : \{a,b\} \mapsto \{c,v\} \\ g_2(x) &:= t_e(a,a_{i+1},x) && : \{a,b\} \mapsto \{v,d\}. \end{aligned}$$

Because we assume that Willard terms are part of the signature, these functions are basic translations, and hence certify the claim. □

LEMMA III.8.6 (Multi-sequence lemma). *Let $S_1,\ldots,S_n$ be sequences of elements of $\mathbf{A}$ from a to b. Then there exist elements $c \neq d \in \mathbf{A}$ and links $\{x_i,y_i\}$ in S_i such that*

$$\{a,b\} \overset{2^n}{\underset{n}{\to}} \{c,d\} \quad \text{and for all } i,\ \{x_i,y_i\} \overset{2^n}{\underset{n}{\to}} \{c,d\}.$$

PROOF BY INDUCTION. The case $n = 1$ is the Single-sequence lemma. Let $n > 1$; by the induction hypothesis applied to $S_1,\ldots,S_{n-1}$ there exist $u \neq v$ and links $\{x_i,y_i\}$ in S_i such that

$$\{a,b\} \overset{2^{n-1}}{\underset{n-1}{\longrightarrow}} \{u,v\} \quad \text{and for all } i \in \mathbb{Z}_n,\quad \{x_i,y_i\} \overset{2^{n-1}}{\underset{n-1}{\longrightarrow}} \{u,v\}.$$

Let $m \leq 2^{n-1}$ be such that there exist distinct elements u_i, $i \in \mathbb{Z}_{m+1}$, so that

$$u = u_0, u_1, \ldots, u_m = v \text{ certifies that } \{a,b\} \overset{2^{n-1}}{\underset{n-1}{\longrightarrow}} \{u,v\}.$$

Each $\{u_i,u_{i+1}\}$ is an image of $\{a,b\}$ under some $f_i \in \operatorname{Tr}_{n-1}\mathbf{A}$. Let S be a sequence from u to v consisting of images of the sequence S_n (possibly in the reversed order) under these translations: if

$$S_n \text{ is the sequence } a = a_0, a_1, \ldots, a_l = b$$

let

$$u_{i0} = f_i(a_0) = u_i, u_{i1} = f_i(a_1), \ldots, u_{il} = f_i(a_l) = u_{i+1} \qquad \text{if } f_i(a) = u_i \text{ and } f_i(b) = u_{i+1}$$

$$u_{i0} = f_i(a_l) = u_i, u_{i1} = f_i(a_{l-1}), \ldots, u_{il} = f_i(a_0) = u_{i+1}$$
$$\text{if } f_i(a) = u_{i+1} \text{ and } f_i(b) = u_i$$

S is the sequence

$$\overbrace{u = \underbrace{u_{00}, \ldots, u_{0l}}_{f_0(S_n)}}^{u_0} = \overbrace{\underbrace{u_{10}, \ldots, u_{1l}}_{f_1(S_n)}}^{u_1} = \overbrace{u_{20}, \ldots,}^{u_2} \overbrace{u_{ml} = v}^{u_m}.$$

By the preceding lemma, there exists a link $\{x, y\}$ in S and $c \neq d \in A$ such that

$$\{x, y\} \xrightarrow[1]{2} \{c, d\} \quad \text{and} \quad \{u, v\} \xrightarrow[1]{2} \{c, d\}.$$

Because of the construction, $\{x, y\} = \{f_j(a_k), f_j(a_{k+1})\}$ for some $j \leq m$ and $k \leq l$; let $\{x_n, y_n\} = \{a_k, a_{k+1}\}$ be the corresponding link in S_n. The following three lines yield the desired properties of $\{c, d\}$ and $\{x_i, y_i\}$, $i \in \widehat{n}$:

$$\{x_i, y_i\} \xrightarrow[n-1]{2^{n-1}} \{u, v\} \xrightarrow[1]{2} \{c, d\}$$

$$\{x_n, y_n\} \xrightarrow[n-1]{1} \{x, y\} \xrightarrow[1]{2} \{c, d\}$$

$$\{a, b\} \xrightarrow[n-1]{2^{n-1}} \{u, v\} \xrightarrow[1]{2} \{c, d\}$$

□

LEMMA III.8.7. *If* $\{a_1, b_1\}, \ldots, \{a_n, b_n\}, \{u, v\} \in A^{(2)}$ *and if for each* $i \in \widehat{n}$

$$\{a_i, b_i\} \xrightarrow[k]{?} \{u, v\},$$

then there exist $\{x_1, y_1\}, \ldots, \{x_n, y_n\}, \{u', v'\} \in A^{(2)}$ *such that*

$$\{u, v\} \xrightarrow[n]{2^n} \{u', v'\} \quad \text{and for all } i,\ \{a_i, b_i\} \xrightarrow[k]{1} \{x_i, y_i\} \xrightarrow[n]{2^n} \{u', v'\}.$$

In particular,

$$\{a_i, b_i\} \xrightarrow[k+n]{2^n} \{u', v'\}.$$

PROOF. For each $i \in \widehat{n}$, there exists a Mal'cev chain S_i from u to v certifying that $\{a_i, b_i\} \xrightarrow[k]{?} \{u, v\}$. Apply the Multi-sequence lemma to $\{u, v\}$ and $S_1, \ldots, S_n$. □

III.8.3. Definition of μ_m and ϕ_m.

From now on, let $m \geq 2$ be any natural number. In this section, we define formulae ϕ_m and μ_m such that:

- any model of ϕ_m has a subdirectly irreducible homomorphic image with more than m elements
- μ_m witnesses that the class of algebras defined by $\neg\phi_m$ has DDPC

Note that Lemma III.8.7 gives more information than just a bound on the length of the Mal'cev chains: it says that if the depth of the original Mal'cev chains was bounded by k, then the depth of the chains with bounded length is bounded by $k+n$. This is reflected in the definitions of ϕ_m and μ_m, in which $k = DM$, where[19]

$M\colon = R(m+1, m+1)$.

M is the Ramsey number: in any graph with at least M vertices, there is either an $m+1$ element clique (an induced subgraph which is a complete graph) or an $m+1$ element anticlique (an induced graph with no edges).[20]

$D\colon = 2\binom{M+1}{2} + 3 = (M+1)M + 3$.

We also define

$L\colon = \binom{2m}{m}$.

ϕ_m should force the algebra to contain $m+1$ different elements: thus we obtain L pairs of elements, each one of them generating a nontrivial congruence.

Lemma III.8.8. *Let*

$$\phi_m := \exists x_0 x_1 \ldots x_m y z \left(y \not\approx z \ \wedge \bigwedge_{\substack{i<j,\\ i,j \in \mathbb{Z}_{m+1}}} \{x_i, x_j\} \xrightarrow[DM+L]{2^L} \{y, z\} \right).$$

Then any model of ϕ_m has a subdirectly irreducible homomorphic image with more than m elements.

Proof. If $\mathbf{A} \models \phi_m$, then there are $m+1$ different elements $a_0, \ldots, a_m$ and elements $b \neq c$ such that $(b,c) \in \operatorname{Cg}(a_i, a_j)$ for each $i \neq j$. With the use of Zorn's lemma it is easy to show that there is a congruence $\theta \in \operatorname{Con}\mathbf{A}$ maximal among all congruences which do not

[19] This particular choice of k is suited to the proof of Lemma III.8.10.

[20] We could define $M := \binom{2m}{m}$, because $\binom{2m}{m}$ is a known upper bound on $R(m+1, m+1)$; the value is not known exactly for any $m > 3$. We could also take $M = R - 1$ because we actually apply the Ramsey theorem to a graph with the underlying set $\mathbb{Z}_{M+1}$, which has $M+1$ elements. However, the exact definition of M does not have any effect on the validity of the proof.

contain (b,c) and that θ is strictly meet-irreducible. This means that ${}^{\mathbf{A}}\!/\!_{\theta}$ is a subdirectly irreducible algebra, and because for $i \neq j$ we have $(a_i, a_j) \notin \theta$, ${}^{\mathbf{A}}\!/\!_{\theta}$ has at least $m+1$ elements. We have shown that if $\mathbf{A} \models \phi_m$, then $\mathbf{A}$ has a subdirectly irreducible image with at least $m+1$ elements.[21] □

μ_m says that two principal congruences are not disjoint:

$$\mu_m(x_1, y_1, x_2, y_2) := \ \exists wz \left(w \not\approx z \ \wedge \ \bigwedge_{i \in \widehat{2}} \{x_i, y_i\} \xrightarrow[DM+2]{4} \{w, z\} \right).$$

In the next section, we show that μ_m witnesses that $\{\mathbf{A}; \mathbf{A} \not\models \phi_m\}$ has DDPC.

III.8.4. The depth of the Mal'cev chains.

LEMMA III.8.10. *Let $\mathbf{A} \in \mathcal{V}$. Then one of the following two statements must hold in $\mathbf{A}$:*

either $\mathbf{A} \models \psi_m$.

or for all $a, b, c, d \in \mathbf{A}$ we have

$$\mathbf{A} \models \mu_m(a,b,c,d) \quad \textit{iff} \quad \mathrm{Cg}^{\mathbf{A}}(a,b) \cap \mathrm{Cg}^{\mathbf{A}}(c,d) \neq 0_{\mathbf{A}}.$$

PROOF. Let $\mathbf{A} \in \mathcal{V}$ be such that $\mathbf{A} \not\models \psi_m$. Clearly, whenever $\mathbf{A} \models \mu_m(a,b,c,d)$, then $\mathrm{Cg}^{\mathbf{A}}(a,b) \cap \mathrm{Cg}^{\mathbf{A}}(c,d) \neq 0_{\mathbf{A}}$, so we only need to prove the other implication. We proceed BY CONTRADICTION:

$\leadsto$ Assume that $\mathrm{Cg}^{\mathbf{A}}(a,b) \cap \mathrm{Cg}^{\mathbf{A}}(c,d) \neq 0_{\mathbf{A}}$, but $\mathbf{A} \not\models \mu_m(a,b,c,d)$. This means that there must exist some $e \neq f \in \mathbf{A}$ and some k such

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[21] With a slightly different bound on the depth of Mal'cev chains, the opposite implication is valid at least for subdirectly irreducible algebras:

FACT III.8.9. ([**Wil00**, Corollary 3.5], [**Jež08**, 10.4.5]) *Let $\mathbf{A}$ be any subdirectly irreducible algebra with at least $m+1$ elements. Then $\mathbf{A} \models \phi_m^*$, where*

$$\phi_m^* = \exists x_0 x_1 \dots x_m yz \left(y \neq z \ \wedge \ \bigwedge_{\substack{i<j,\\ i,j \in \mathbb{Z}_m}} \{x_i, x_j\} \xrightarrow[2^{6m}]{2^L} \{y, z\} \right)$$

or

$$\phi_m^* = \exists x_0 x_1 \dots x_m yz \left(y \neq z \ \wedge \ \bigwedge_{\substack{i<j,\\ i,j \in \mathbb{Z}_m}} \{x_i, x_j\} \xrightarrow[(3+DM)L]{2^L} \{y, z\} \right).$$

Moreover, ϕ_m may be replaced by ϕ_m^* in the rest of the proof.

It can be shown that $DM + L \leq 2^{6m}$.

that $\{a,b\} \xrightarrow[k]{?} \{e,f\}$ and $\{c,d\} \xrightarrow[k]{?} \{e,f\}$; according to Lemma III.8.7, there exist $\{r,s\},\{r',s'\},\{u,v\} \in A^{(2)}$ such that

$$(\circ) \qquad \{a,b\} \xrightarrow[k]{1} \{r,s\} \xrightarrow[2]{4} \{u,v\} \quad \text{and} \quad \{c,d\} \xrightarrow[k]{1} \{r',s'\} \xrightarrow[2]{4} \{u,v\}.$$

Because we assume that $\mu_m(a,b,c,d)$ fails, $k > DM$ for any choice of r,s,r',s',u and v. Let r,s,r',s',u,v be chosen so that $u \neq v$, $(\circ)$ holds and k is minimal.

Let $t = k - DM$. According to Observation III.8.3 (5), we can select $\{a_i,b_i\} \in A^{(2)}$ and $f_i \in \mathrm{Tr}_{D(M-i)}\,\mathbf{A}$, where $i \in \mathbb{Z}_{M+1}$, such that

$$(\bullet) \qquad \{a,b\} \xrightarrow[t]{1} \{a_0,b_0\} \xrightarrow[D]{1} \{a_1,b_1\} \xrightarrow[D]{1} \cdots \xrightarrow[D]{1} \{a_M,b_M\} = \{r,s\}$$

and such that $f_i(a_i) = r$ and $f_i(b_i) = s$.

Analogically, $\{c_i,d_i\} \in A^{(2)}$ and $g_i \in \mathrm{Tr}_{D(M-i)}\,\mathbf{A}$ can be selected so that $\{c,d\} \xrightarrow[t]{1} \{c_0,d_0\}$, $g_i(c_i) = r'$ and $g_i(d_i) = s'$.

For any $i \in \mathbb{Z}_{M+1}$, we may rewrite $(\circ)$ and $(\bullet)$ as

$$\{a,b\} \xrightarrow[t]{1} \{a_0,b_0\} \xrightarrow[Di]{1} \{a_i,b_i\} \xrightarrow[D(M-i)]{1} \{r,s\} \xrightarrow[2]{4} \{u,v\}$$

and

$$\{c,d\} \xrightarrow[t]{1} \{c_0,d_0\} \xrightarrow[Di]{1} \{c_i,d_i\} \xrightarrow[D(M-i)]{1} \{r',s'\} \xrightarrow[2]{4} \{u,v\},$$

where the long arrows are certified by f_i and g_i respectively. Fix a Mal'cev chain

$$(\dagger) \qquad u = u_0, u_1, u_2, u_3, u_4 = v$$

such that for $i \in \mathbb{Z}_4$ there exists some $p_i \in \mathrm{Tr}_2\,\mathbf{A}$ such that

$$p_i(\{r,s\}) = \{u_i, u_{i+1}\}.$$

For any $0 \le i < j \le M$ we have a sequence

$$(\ddagger) \qquad r = f_j(a_j), f_j(a_i), f_j(b_i), f_j(b_j) = s.$$

We define a sequence $S_{ij}\colon\ v_0^{ij}, v_1^{ij}, \ldots, v_{12}^{ij}$ from u to v as a combination of $(\dagger)$ and $(\ddagger)$:

if $p_h(r) = u_h$ and $p_h(s) = u_{h+1}$ then
$$\begin{aligned} v_{3h}^{ij} &= p_h(f_j(a_j)) = p_h(r) = u_h \\ v_{3h+1}^{ij} &= p_h(f_i(a_i)) \\ v_{3h+2}^{ij} &= p_h(f_i(b_i)) \\ v_{3h+3}^{ij} &= p_h(f_j(b_j)) = p_h(s) = u_{h+1} \end{aligned}$$

if $p_h(r) = u_{h+1}$ and $p_h(s) = u_h$ then
$$\begin{aligned} v_{3h}^{ij} &= p_h(f_j(b_j)) = p_h(s) = u_h \\ v_{3h+1}^{ij} &= p_h(f_i(b_i)) \\ v_{3h+2}^{ij} &= p_h(f_i(a_i)) \\ v_{3h+3}^{ij} &= p_h(f_j(a_j)) = p_h(r) = u_{h+1} \end{aligned}$$

The most important fact about the sequences S_{ij} is that for any $h \in \mathbb{Z}_{12}$, one of the following three cases occurs:

(1) $\{a_i, a_j\} \xrightarrow[D(M-j)+2]{1} \{v_h^{ij}, v_{h+1}^{ij}\}$

(2) $\{b_i, b_j\} \xrightarrow[D(M-j)+2]{1} \{v_h^{ij}, v_{h+1}^{ij}\}$

(3) $\{a_i, b_i\} \xrightarrow[D(M-j)+2]{1} \{v_h^{ij}, v_{h+1}^{ij}\}$

In the same way, we obtain chains $S'_{ij}:\ w_0^{ij}, w_1^{ij}, \ldots, w_{12}^{ij}$ from u to v such that for any $h \in \mathbb{Z}_{12}$ one of the following three occurs:

(1') $\{c_i, c_j\} \xrightarrow[D(M-j)+2]{1} \{w_h^{ij}, w_{h+1}^{ij}\}$

(2') $\{d_i, d_j\} \xrightarrow[D(M-j)+2]{1} \{w_h^{ij}, w_{h+1}^{ij}\}$

(3') $\{c_i, d_i\} \xrightarrow[D(M-j)+2]{1} \{w_h^{ij}, w_{h+1}^{ij}\}$

We have obtained $N := 2\binom{M+1}{2} = M(M+1)$ sequences from u to v.[22] From the Multi-sequence lemma III.8.6 we know that there exist $u' \neq v' \in \mathbf{A}$ and for all $0 \leq i < j \leq M$ there exist x_{ij}, y_{ij}, x'_{ij} and y'_{ij} such that

$$\{x_{ij}, y_{ij}\} \xrightarrow[N]{2^N} \{u', v'\} \quad \text{and} \quad \{x'_{ij}, y'_{ij}\} \xrightarrow[N]{2^N} \{u', v'\}$$

where $\{x_{ij}, y_{ij}\}$ are consecutive members of S_{ij} and $\{x'_{ij}, y'_{ij}\}$ are consecutive members of S'_{ij}. Now there are two possible cases:

Case 1: $\rightsquigarrow$ If there exist $i < j$ and $i' < j'$ such that case (3) occurs for $\{x_{ij}, y_{ij}\}$ and case (3') occurs for $\{x'_{i'j'}, y'_{i'j'}\}$, then we derive a contradiction with the minimality of k: case (3) implies that

$$\{a, b\} \xrightarrow[t+Di]{1} \{a_i, b_i\} \xrightarrow[D(M-j)+2]{1} \{x_{ij}, y_{ij}\} \xrightarrow[N]{2^N} \{u', v'\},$$

so

$$\{a, b\} \xrightarrow[k-1]{?} \{u', v'\}$$

since[23]

$$\begin{aligned} t + Di + D(M-j) + 2 + N \ &= \ t + DM + N + 2 + D(i-j) \ \leq \\ &\leq \ k + N + 2 - D \ = \ k - 1. \end{aligned}$$

Similarly, we get that

$$\{c, d\} \xrightarrow[k-1]{?} \{u', v'\}.$$

Corollary III.8.7 implies that there exist elements $u'' \neq v''$ and $r_1, s_1, r_2, s_2 \in \mathbf{A}$ such that

$$\{a, b\} \xrightarrow[k-1]{1} \{r_1, s_1\} \xrightarrow[2]{4} \{u'', v''\} \quad \text{and} \quad \{c, d\} \xrightarrow[k-1]{1} \{r_2, s_2\} \xrightarrow[2]{4} \{u'', v''\}.$$

---

[22]Half of them are S_{ij} and the other half S'_{ij}.

[23]Actually, this is the calculation that determines the choice of D.

↯ This is the desired contradiction with the minimality of k, so either case (3) or case (3') never occurs.

Case 2: Without loss of generality we assume that case (3) never occurs. We may define an associated undirected graph on the vertex set $\mathbb{Z}_{M+1}$ such that ij is an edge iff case (1) occurs for $\{x_{ij}, y_{ij}\}$. By the choice of M to be the Ramsey number $R(m+1, m+1)$, there either exists an $m+1$ element clique or an $m+1$ element anticlique in the graph.

In the first case, there exists a set $S = \{i_0, \ldots, i_m\}$ of indices such that case (1) occurs for every $\{x_{i_g i_h}, y_{i_g i_h}\}$ with $i_g < i_h$ and $i_g, i_h \in S$:

$$\{a_{i_g}, a_{i_h}\} \xrightarrow[D(M-i_h)+2]{1} \{x_{i_g i_h}, y_{i_g i_h}\} \overset{2^N}{\underset{N}{\rightarrow}} \{u', v'\}.$$

Hence

$$\{a_{i_g}, a_{i_h}\} \xrightarrow[DM]{?} \{u', v'\},$$

and from Corollary III.8.7 we see that ϕ_m holds for $x_g = a_{i_g}$, $y = u'$ and $z = v'$.

The case of an anticlique is analogical, just replace (1) with (2) and a_{i_g} with b_{i_g}. In both cases, we get a contradiction with $\mathbf{A} \not\models \phi_m$. ↯ □

III.9. Dziobiak's proof of Willard's Theorem via quasivarieties

The proof of Willard's Theorem which we give in Section III.8 relies heavily on the use of Willard terms: they are used in the Single-sequence lemma III.8.5, without which the whole calculation would be impossible. If one would like to generalize the theorem, one would have to find a Mal'cev condition for some more general class of algebras, and then find results similar to the Single-sequence and Multi-sequence lemmas (this is exactly how Willard generalized the original proof of Baker's Theorem).

Here we would like to show another approach to the problem, one which does not use any Mal'cev condition. Although Park's conjecture is stated for varieties, in this proof, due to Dziobiak, we work with quasi-varieties. One might be led to considering quasivarieties for example by the following fact: although in general, quasivariety generation involves ultraproducts, in the case that the generating class of algebras is a finite set of finite algebras, the quasivariety generated by $\mathcal{K}$ is equal to $\mathrm{SP}(\mathcal{K})$. Moreover, for any variety $\mathcal{V}$, $\mathcal{V} = \mathrm{SP}(\mathcal{V}_{SI})$. As a corollary we get the following fact:

Fact III.9.1. *If $\mathcal{V}_{SI}$ is a finite set of finite algebras, then $\mathcal{V}$ is the quasivariety generated by $\mathcal{V}_{SI}$.*

Plan for the proof of Willard's Theorem.

As usually, the strategy of the proof is to use DDPC. In the next subsection, we use Theorem III.4.1, assuming that $\mathcal{V}$ is congruence meet-semidistributive and has DDPC. Next, we work with two quasivarieties $\mathcal{Q}_4 \subseteq \mathcal{V}_4$ closely related to $\mathcal{V}$; it turns out that $\mathcal{V}$ has DDPC iff there exists a finite set of quasi-equations Σ such that $\mathcal{Q}_4 = \{\mathbf{A} \in \mathcal{V}_4, \mathbf{A} \models \Sigma\}$. The last step of the proof thus requires a "relative finite basis" theorem for quasivarieties. Notice, however, that this theorem does not directly give a finite basis for $\mathcal{V}$; it is only used for showing that $\mathcal{V}$ has DDPC. □

III.9.1. CSD(∧) varieties with DDPC.

Theorem III.9.2. *Let $\mathcal{V}$ be a congruence meet-semidistributive variety with DDPC. If $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely based.*

Proof. Let ψ_2 be the sentence axiomatizing $\mathcal{V}_{FSI}$ and φ the sentence which defines disjointness of principal congruences in $\mathcal{V}$. Then

$$\begin{aligned}\mathcal{V}_{FSI} &= \operatorname{Mod}\psi_2 = \\ &= \operatorname{Mod}\big(\operatorname{Eq}\mathcal{V} \cup \{\forall xyuv[\varphi(v,w,x,y) \Rightarrow (v=w \vee x=y)]\}\big).\end{aligned}$$

By the Compactness Theorem I.1.1, there exists $\Sigma \subseteq_{FIN} \operatorname{Eq}\mathcal{V}$ such that

$$\Sigma \cup \{\forall xyuv[\varphi(v,w,x,y) \Rightarrow (v=w \vee x=y)]\} \models \psi_2.$$

We shall see that, with x and y fixed, $\varphi(\_,\_,x,y)$ defines a congruence:

(i) $\forall vwxy \quad [v=w \Rightarrow \varphi(v,w,x,y)]$ reflexivity

(ii) $\forall vwxy \quad [\varphi(v,w,x,y) \Rightarrow \varphi(w,v,x,y)]$ symmetry

(iii) $\forall uvwxy \quad [(\varphi(u,v,x,y) \wedge \varphi(v,w,x,y)) \Rightarrow \varphi(u,w,x,y)]$ transitivity

(iv) $\forall \overline{x}\,\overline{y} \quad [\wedge_{i\in\widehat{n}}\varphi(x_i,y_i,x,y) \Rightarrow \varphi(f(\overline{x}),f(\overline{y}),x,y)]$
closure under any basic operation f of arity n

The role of x,y and v,w in $\varphi(v,w,x,y)$ is symmetrical:

(v) $\forall vwxy \quad [\varphi(v,w,x,y) \Rightarrow \varphi(x,y,v,w)]$

Finally, we claim that if $x \neq y$ then $\operatorname{Cg}(x,y) \cap \operatorname{Cg}(x,y) \neq 0_{\mathbf{A}}$:

(vi) $\forall xy \quad [\varphi(x,y,x,y) \Rightarrow x=y]$

Let us define Γ to be the finite set of sentences consisting of (i)–(vi).

Let $\mathcal{K} = \operatorname{Mod}(\Sigma \cup \Gamma)$; thus $\mathcal{K}$ is strictly elementary. We claim that $\mathcal{V} \subseteq \mathcal{K}$. Indeed, $\mathcal{V} \models \Sigma$ and (i), (ii), (v) and (vi) are satisfied in $\mathcal{V}$ trivially. From congruence meet-semidistributivity, it follows that if $\operatorname{Cg}(x_i,y_i) \cap \operatorname{Cg}(x,y) = 0_{\mathbf{A}}$ for each $i \in \overline{n}$, then

$$\operatorname{Cg}(f(\overline{x}),f(\overline{y})) \cap \operatorname{Cg}(x,y) \subseteq \operatorname{Cg}(\{x_1,x_2,\dots,y_1,y_2,\dots\}) \cap \operatorname{Cg}(x,y) = 0_{\mathbf{A}},$$

which gives (iv). Transitivity may be proved in the same way.

To finish the proof, we use Jónsson's Theorem III.4.1, with $\mathcal{S} = \mathcal{V}_{FSI}$. The last thing we need to show is that $\mathcal{K}_{SI} \subseteq \mathcal{V}_{FSI}$. In order to do this, we show that $\mathcal{K}_{SI} \models \forall xyuv[\varphi(v, w, x, y) \Rightarrow (v = w \vee x = y)]$, and hence $\mathcal{K}_{SI} \models \psi_2$.

So take any $\mathbf{A} \in \mathcal{K}_{SI}$ such that $\mathbf{A} \models \varphi(a, b, c, d)$. Define

$$\begin{array}{lll} \theta & := & \{(e, f) \in \mathbf{A}^2,\ \ \mathbf{A} \models \varphi(e, f, c, d)\}, \\ \psi & := & \{(e, f) \in \mathbf{A}^2,\ \ \forall gh \in \theta\ \ \mathbf{A} \models \varphi(g, h, e, f)\}. \end{array}$$

$\theta \in \operatorname{Con} \mathbf{A}$ because of axioms (i)–(iv); $\psi \in \operatorname{Con} \mathbf{A}$ due to (i)–(v) and the fact that an intersection of congruences is a congruence.

Moreover, if $(e, f) \in \theta \cap \psi$, then $\varphi(e, f, e, f)$, and by (vi) we get that $e = f$. Hence $\theta \cap \psi = 0_{\mathbf{A}}$. But the algebra $\mathbf{A}$ is subdirectly irreducible, implying that either $\theta = 0_{\mathbf{A}}$ or $\psi = 0_{\mathbf{A}}$. From $(a, b) \in \theta$, $(c, d) \in \psi$ we get that

$$\mathbf{A} \models \varphi(a, b, c, d) \Rightarrow (a = b \vee c = d). \qquad \square$$

III.9.2. Turning to quasivarieties: a characterisation of DDPC. According to Theorem III.9.2, in order to prove Willard's theorem, we only need to show that a congruence meet-semidistributive variety has DDPC. In this section, we turn it into a problem concerning quasi-varieties.

Let $\sigma^* = \sigma \cup \{a, b, c, d\}$. Let

$$\mathcal{V}_4\ :=\ \{(A, \sigma, a, b, c, d);\ (A, \sigma) \in \mathcal{V}\}$$

be the variety of σ^*-algebras that is defined by the same identities as $\mathcal{V}$.

Let

$$\mathcal{Q}_4\ :=\ \{\mathbf{A} \in \mathcal{V}_4;\ \operatorname{Cg}(a, b) \cap \operatorname{Cg}(c, d) = 0_{\mathbf{A}}\}.$$

According to I.3.1, $(u, v) \in \operatorname{Cg}(a, b)$ iff there exists a formula of the form $\exists \overline{w} \Gamma_1(u, v, a, b, \overline{w})$, where Γ_1 is just a conjunction of equations which describes a Mal'cev chain. Relative to $\mathcal{V}_4$, $\mathcal{Q}_4$ has a basis consisting of all formulae of the form

$$[\exists \overline{w} \overline{z}\ \Gamma_1(u, v, a, b, \overline{w}) \wedge \Gamma_2(u, v, c, d, \overline{z})]\ \ \Rightarrow\ \ u = v,$$

where Γ_1 and Γ_2 describe the Mal'cev chains testifying that $(u, v) \in \operatorname{Cg}(a, b)$ and $(u, v) \in \operatorname{Cg}(c, d)$.

Proposition III.9.3. *$\mathcal{V}$ has DDPC iff $\mathcal{Q}_4$ is finitely based relative to $\mathcal{V}_4$, i.e. iff there exists a finite set Σ of quasi-equations such that $\mathcal{Q}_4 = \{\mathbf{A} \in \mathcal{V}_4;\ \mathbf{A} \models \Sigma\}$.*

Proof. ($\Rightarrow$) Let φ define disjointness of principal congruences: for any $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in \mathbf{A}$,

$$\mathbf{A} \models \varphi(a, b, c, d) \quad \text{iff} \quad \mathrm{Cg}(a, b) \cap \mathrm{Cg}(c, d) = 0_{\mathbf{A}}.$$

We use the Compactness theorem I.1.1: we know that $\mathcal{Q}_4$ is a quasivariety defined by $\mathrm{Eq}\,\mathcal{V} \cup \{\varphi(a, b, c, d)\}$; hence there exists a finite set Σ of quasi-equations such that $\Sigma \models \varphi(a, b, c, d)$ and $\mathcal{Q}_4 = \mathrm{Mod}(\mathrm{Eq}\,\mathcal{V} \cup \Sigma)$.

($\Leftarrow$) Let $\mathcal{Q}_4 = \mathrm{Mod}(\mathrm{Eq}\,\mathcal{V} \cup \Sigma)$, where Σ is a finite set of quasi-equations.[24] Define

$$\varphi(x, y, x', y') := \bigwedge_{\psi \in \Sigma} \psi.$$

It follows from the definitions of $\mathcal{V}_4$ and $\mathcal{Q}_4$ that φ defines the disjointness of principal congruences in $\mathcal{V}$. $\square$

We prove the following lemma for later use:

Lemma III.9.4. *If $\mathcal{V}$ is congruence meet-semidistributive, then $\mathcal{Q}_4$ has the weak extension property. That is, if $\mathbf{A} \in \mathcal{Q}_4$ and $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ are such that $\alpha \cap \beta = 0_{\mathbf{A}}$, then $\overline{\alpha} \cap \overline{\beta} = 0_{\mathbf{A}}$.*

Proof. It suffices to prove that $\alpha \cap \beta = 0_{\mathbf{A}}$ implies $\overline{\alpha} \cap \beta = 0_{\mathbf{A}}$. We introduce the following notation: $\theta_1 := \mathrm{Cg}(a, b)$, $\theta_2 := \mathrm{Cg}(c, d)$.

Claim 1. *γ is a $\mathcal{Q}_4$-congruence iff $\gamma = (\gamma \vee \theta_1) \cap (\gamma \vee \theta_2)$.*

In the factor-algebra $\mathbf{A}/\gamma$, $\theta_1 \cap \theta_2 = 0_{\mathbf{A}}$.

Claim 2. $\overline{\alpha} = (\alpha \vee \theta_1) \cap (\alpha \vee \theta_2)$.

Let $\alpha' := (\alpha \vee \theta_1) \cap (\alpha \vee \theta_2)$. Then $\overline{\alpha} \geq \alpha' \geq \alpha$ because

$$\overline{\alpha} = \bigcap_{\substack{\gamma \geq \alpha, \\ \gamma \in \mathrm{Con}_{\mathcal{Q}}\,\mathbf{A}}} (\gamma \vee \theta_1) \cap (\gamma \vee \theta_2) \;\geq\; (\alpha \vee \theta_1) \cap (\alpha \vee \theta_2) \;=\; \alpha'.$$

We show that $\alpha'' = \alpha'$ and use Claim 1.

$$\begin{aligned} \alpha' \vee \theta_1 &= [(\alpha \vee \theta_1) \cap (\alpha \vee \theta_2)] \vee \theta_1 \leq (\alpha \vee \theta_1) \vee \theta_1 = \alpha \vee \theta_1 \\ \alpha'' &= (\alpha' \vee \theta_1) \cap (\alpha' \vee \theta_2) \leq (\alpha \vee \theta_1) \cap (\alpha \vee \theta_2) = \alpha' \end{aligned}$$

But $\alpha' \leq \alpha''$ is obvious, and hence $\alpha' = \alpha''$. This proves Claim 2.

[24] Due to the Compactness theorem, we may assume that Σ is a subset of

$$\Sigma'(x, y, x', y') = \{\forall uv\overline{w}\overline{z} \quad [(\Gamma_1(u, v, x, y, \overline{w}) \wedge \Gamma_2(u, v, x', y', \overline{z})) \Rightarrow u = w]\,; \; \Gamma_1 \text{ and } \Gamma_2 \text{ are Mal'cev schemes}\}.$$

Claim 3. $\alpha \cap \beta = 0_{\mathbf{A}} \Rightarrow \overline{\alpha} \cap \beta = 0_{\mathbf{A}}$.

Under the assumption of congruence meet-semidistributivity,

$$(\alpha \vee \theta_1) \cap (\alpha \vee \theta_2) \cap \beta = 0_{\mathbf{A}}$$
$$\text{iff}$$
$$\alpha \cap (\alpha \vee \theta_2) \cap \beta = 0_{\mathbf{A}} \text{ and } \theta_1 \cap (\alpha \vee \theta_2) \cap \beta = 0_{\mathbf{A}}$$

(The direct implication follows from monotonicity. The reverse implication follows from congruence meet-semidistributivity.)

$\alpha \cap (\alpha \vee \theta_2) \cap \beta = 0_{\mathbf{A}}$ holds by the assumption that $\alpha \cap \beta = 0_{\mathbf{A}}$. $\theta_1 \cap (\alpha \vee \theta_2) \cap \beta = 0_{\mathbf{A}}$ holds iff $\theta_1 \cap \alpha \cap \beta = 0_{\mathbf{A}}$ and $\theta_1 \cap \theta_2 \cap \beta = 0_{\mathbf{A}}$. Both are trivial. □

III.9.3. A finite basis theorem for quasivarieties. To finish the proof of Willard's theorem, we only need to show that $\mathcal{Q}_4$ finitely based relative to $\mathcal{V}_4$. This is a simple consequence of the following theorem (together with the lemma just proved), with the choice $\mathcal{K} = \mathcal{Q}_4$, $\mathcal{L} = \mathcal{L}' = \mathcal{V}_4$.

Theorem III.9.5 ([**DMMN09**]). *Suppose that $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{L}'$ are quasivarieties such that $\mathcal{K}$ satisfies WEP and $\mathcal{L}'$ is finitely generated. Then $\mathcal{K}$ is finitely based relative to $\mathcal{L}$.*

Proof. We may assume $\mathcal{K} \neq \mathcal{L}$, because otherwise the statement is trivial. We start with some Introductory definitions:

We say that a finite algebra $\mathbf{A} \in \mathcal{L} \setminus \mathcal{K}$ is *$(\mathcal{K}, \mathcal{L})$-minimal* iff every proper subalgebra of $\mathbf{A}$ belongs to $\mathcal{K}$; in that case, we let $\theta_{\mathcal{K}}^{\mathbf{A}}$ denote the smallest congruence on $\mathbf{A}$ such that $\mathbf{A}/\theta_{\mathcal{K}}^{\mathbf{A}} \in \mathcal{K}$. The relation of isomorphism partitions the family of all $(\mathcal{K}, \mathcal{L})$-minimal algebras; let S be a selector of that partition. Thus S is a minimal set such that if $\mathbf{A}$ is $(\mathcal{K}, \mathcal{L})$-minimal, then there exists exactly one $\mathbf{A}' \in S$ which is isomorphic to $\mathbf{A}$.

For $\mathbf{A} \in S$, let $a \mapsto x_a$ be a fixed one-to-one assignment of variables to elements of $\mathbf{A}$. Let

$$D_{\mathbf{A}} := \{f(x_{a_1}, \ldots, x_{a_n}) = x_b; \quad f \in \sigma, \ \ a_1, \ldots, a_n, b \in \mathbf{A}, \\ \mathbf{A} \models f(a_1, \ldots, a_n) = b\}.$$

(The concept is similar to Definition I.2.8, with two differences: here we work with variables rather than constants, and we do not include inequalities.) Finally, let

$$\Sigma_{\mathbf{A}} := \left\{ \bigwedge D_{\mathbf{A}} \Rightarrow x_a \approx x_b; \quad a \neq b \in \mathbf{A}, \ (a, b) \in \theta_{\mathcal{K}}^{\mathbf{A}} \right\}.$$

Plan for the proof. Claim 1 says that $\mathcal{K} = \mathcal{L} \cap \operatorname{Mod}\left(\bigcup_{\mathbf{A}\in S}\Sigma_{\mathbf{A}}\right)$. Claim 3 shows that there are only finitely many $(\mathcal{K},\mathcal{L})$-minimal algebras, and hence S is finite. This is all we need.

Claim 1. $\mathcal{K} = \mathcal{L} \cap \operatorname{Mod}\left(\bigcup_{\mathbf{A}\in S}\Sigma_{\mathbf{A}}\right)$.

($\subseteq$) Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \in S$. Let $a \neq b$ be such that $(a,b) \in \theta_{\mathcal{K}}^{\mathbf{A}}$. Let $v : \{x_a; a \in \mathbf{A}\} \to \mathbf{B}$ be an assignment of values under which the equations of $D_{\mathbf{A}}$ are satisfied. Let

$$\begin{aligned} \varphi :\ & \mathbf{A} \to \mathbf{B} \\ & a \mapsto v(x_a). \end{aligned}$$

φ is a homomorphism from $\mathbf{A}$ to $\mathbf{B} \in \mathcal{K}$, hence $\mathbf{A}/_{\ker\varphi} \in \mathcal{K}$. So

$$\theta_{\mathcal{K}}^{\mathbf{A}} \leq \ker\varphi.$$

This gives $\varphi(a) = \varphi(b)$, which means that $v(x_a) = v(x_b)$, proving that

$$\mathbf{B} \models \bigwedge D_{\mathbf{A}} \Rightarrow x_a \approx x_b.$$

Because this is true for any $\mathbf{A} \in S$, we have that

$$\mathbf{B} \models \bigcup_{\mathbf{A}\in S} \Sigma_{\mathbf{A}};$$

moreover, $\mathbf{B} \in \mathcal{K} \subseteq \mathcal{L}$ by the assumptions.

($\supseteq$) Let $\mathbf{B} \in \mathcal{L} \setminus \mathcal{K}$ be finite. Then there is a $(\mathcal{K},\mathcal{L})$-minimal subalgebra of $\mathbf{B}$; we denote it by $\mathbf{A}$ and without loss of generality assume that $\mathbf{A} \in S$. From $\mathbf{A} \not\models \Sigma_{\mathbf{A}}$ we see that $\mathbf{B} \not\models \Sigma_{\mathbf{A}}$. Hence $\mathbf{B} \notin \bigcup_{\mathbf{A}\in S}\Sigma_{\mathbf{A}}$.

A couple more definitions

If $\mathcal{L}' = \mathrm{SP}(\{\mathbf{G}_i, i \in I\})$, where all $\mathbf{G}_i$ are finite algebras and I is a finite set, let $n \in \mathbb{N}$ be an upper bound on the size of $\mathbf{G}_i, i \in I$.

Let m be the maximum size of an n-generated subalgebra of any algebra in $\mathcal{L}'$. $m \in \mathbb{N}$ because L' is locally finite.

For $X, Y \subseteq \operatorname{Con}\mathbf{A}$, we define $X \preceq Y$ iff for every $\alpha \in Y$ there exists $\beta \in X$ such that $\beta \leq \alpha$. Note that $\preceq$ restricted to antichains is a partial order.

Claim 2. *For any $\mathbf{A} \in \mathcal{L} \setminus \mathcal{K}$, there exists a maximal antichain $Z \subseteq \operatorname{Con}\mathbf{A}$ such that $\bigcap Z = 0_{\mathbf{A}}$ and for any $\gamma \in Z$, $|\mathbf{A}/_{\gamma}| \leq n$. It consists of meet-irreducible congruences, some of which are not $\mathcal{K}$-congruences.*

If $\mathbf{A} \in \mathcal{L}$, then

$$\mathbf{A} \leq \prod_{j \in J} \mathbf{G}_j,$$

where each $\mathbf{G}_j$ is one of $\mathbf{G}_i, i \in I$. Because $\bigcap_{j \in J} \ker \pi_j = 0_{\mathbf{A}}$, there exists an antichain

$$X \subseteq \{\ker \pi_j;\ j \in J\} \subseteq \operatorname{Con} \mathbf{A}$$

whose intersection is $0_{\mathbf{A}}$ and such that for $\alpha \in X$, the quotient algebra $\mathbf{A}/_{\alpha}$ has at most n elements.

Let Z be a $\preceq$-maximal antichain such that $X \preceq Z$ and $\bigcap Z = 0_{\mathbf{A}}$. Thus for any $\gamma \in Z$ there exists $\alpha \in X$ such that $\alpha \leq \gamma$, and hence $|\mathbf{A}/_{\gamma}| \leq n$ (partition by a bigger congruence yields a smaller algebra).

$\leadsto$ Assume for contradiction that for some $\gamma \in Z$, $\gamma = \gamma_1 \cap \gamma_2$.

If $Z' := Z \cup \{\gamma_1, \gamma_2\} \setminus \{\gamma\}$ is an antichain, we get a contradiction with the maximality of Z.

If there exists $\alpha_1 \in Z \setminus \{\gamma\}$, $\alpha_1 \leq \gamma_1$, but no $\alpha_2 \in Z \setminus \{\gamma\}$ such that $\alpha_2 \leq \gamma_2$, then $Z' := Z \cup \{\gamma_2\} \setminus \{\gamma\}$ is an antichain, $Z \preceq Z'$ and

$$\begin{aligned}\bigcap Z' = \bigcap (Z \cup \{\gamma_2\} \setminus \{\gamma\}) &= \\ = \bigcap (Z \cup \{\alpha_1, \gamma_2\} \setminus \{\gamma\}) &\leq \bigcap (Z \cup \{\gamma_1, \gamma_2\} \setminus \{\gamma\}) = \\ &= \bigcap Z = 0_{\mathbf{A}}.\end{aligned}$$

Finally, if there exist $\alpha_1, \alpha_2 \in Z \setminus \{\gamma\}$ such that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$, then let $Z' := Z \setminus \{\gamma\}$. We get that

$$\begin{aligned}\bigcap Z' = \bigcap (Z \setminus \{\gamma\}) &= \\ = \bigcap (Z \cup \{\alpha_1, \alpha_2\} \setminus \{\gamma\}) &\leq \bigcap (Z \cup \{\gamma_1, \gamma_2\} \setminus \{\gamma\}) = \\ &= \bigcap Z = 0_{\mathbf{A}}.\end{aligned}$$

Hence Z' is an antichain such that $\bigcap Z' = 0_{\mathbf{A}}$; it is easy to check that $Z \preceq Z'$, giving a contradiction with the maximality of Z. ↯

Since $\bigcap Z = 0_{\mathbf{A}}$,

$$\mathbf{A} \leq \prod_{\gamma \in Z} \mathbf{A}/_{\gamma}.$$

If all $\gamma \in Z$ were $\mathcal{K}$-congruences, we would get $\mathbf{A} \in \mathcal{K}$; however, because $\mathbf{A} \notin \mathcal{K}$, there is some $\gamma^* \in Z$ such that $\mathbf{A}/_{\gamma^*} \notin \mathcal{K}$.

Let δ be the unique cover of γ^* in $\operatorname{Con} \mathbf{A}$.

Claim 3. *Any finite algebra $\mathbf{A} \in \mathcal{L} \setminus \mathcal{K}$ has a subalgebra $\mathbf{B} \notin \mathcal{K}$ such that $|\mathbf{B}| \leq m$.*

By the maximality of Z, there exist $a \neq b \in \mathbf{A}$ such that

$$(a,b) \;\in\; \delta \cap \bigcap (Z \setminus \{\gamma^*\}).$$

Choose $a_0 := a,\ a_1 := b$ and $a_2, \ldots, a_{k-1}$ to be a selector on ${}^{\mathbf{A}}\!/_{\gamma^*}$. In other words, choose one $a_i, i \in \mathbb{Z}_k$, from each γ^*-class. Then $2 \leq k \leq n$ because $[a]^{\gamma^*} \neq [b]^{\gamma^*}$ and due to Claim 2. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $a_0, \ldots, a_k$. From the definition of m, $|\mathbf{B}| \leq m$.

Let $\gamma' := \gamma^*|_{\mathbf{B}}$ and $\beta := \bigcap (Z \setminus \{\gamma^*\})|_{\mathbf{B}}$ be the restrictions to $\mathbf{B}$. Obviously, $\gamma' \cap \beta = 0_{\mathbf{B}}$.

$\leadsto$ Assume that $\mathbf{B} \in \mathcal{K}$. By WEP of $\mathcal{K}$,

$$\overline{\gamma'} \cap \overline{\beta} = 0_{\mathbf{B}}.$$

From $a, b \in \mathbf{B}$ we see that $(a,b) \in \beta \subseteq \overline{\beta}$; this implies that

$$(a,b) \notin \overline{\gamma'}.$$

The definition of $\mathbf{B}$ and γ' implies that ${}^{\mathbf{B}}\!/_{\gamma'}$ is isomorphic to ${}^{\mathbf{A}}\!/_{\gamma^*}$. Thus γ' has a unique cover in $\operatorname{Con} \mathbf{B}$ to which (a,b) belongs: it is $\delta' := \delta|_{\mathbf{B}}$. This means that

$$\overline{\gamma'} = \gamma'$$

because otherwise $\overline{\gamma'} \geq \delta'$ which would mean that $(a,b) \in \overline{\gamma'}$.

But

$${}^{\mathbf{B}}\!/_{\gamma'} \cong {}^{\mathbf{A}}\!/_{\gamma^*} \notin \mathcal{K}.$$

Hence

$${}^{\mathbf{B}}\!/_{\overline{\gamma'}} = {}^{\mathbf{B}}\!/_{\gamma'} \notin \mathcal{K}. \qquad ↯$$

□

CHAPTER IV

Open problems

We only consider varieties of finite type.

Conjecture IV.0.6 (Park [**Par76**]). *Every finitely generated residually finite variety is finitely based.*

Conjecture IV.0.7 (Jónsson [**Gum76**, Problem 9, p. 28]). *Every variety such that $\mathcal{V}_{FSI}$ is strictly elementary is finitely based.*

In search of a common generalisation of Willard's theorem and McKenzie's theorem, one should consider the following problems:

Problem IV.0.8 ([**Wil01**, Problem 5.6]). *What is the smallest Mal'cev class of varieties containing both the congruence modular and the congruence meet-semidistributive varieties?*

In the language of tame congruence theory, congruence modularity is equivalent to omitting types **1** and **5**, whilst congruence meet-semidistributivity is characterised by omitting types **1** and **2**. Thus we are led to the following question:

Problem IV.0.9 ([**Wil01**, Problem 5.4]). *Suppose $\mathcal{V}$ is a variety with a finite residual bound which omits type* **1**. *Must $\mathcal{V}$ be finitely based?*

Another possible class generalising both congruence modularity and congruence meet-semidistributivity is the class of varieties with a difference term. The commutator operation $[\_,\_]$ is well behaved in these varieties, promising that important ideas of McKenzie's proof from [**McK87**] could be extended.

Definition IV.0.10. Let $\mathcal{K}$ be a class of algebras. A term $p(x,y,z)$ is a *difference term for K* iff for all $\mathbf{A} \in \mathcal{K}$ and all $\theta \in \operatorname{Con}\mathbf{A}$ and all $(a,b) \in \theta$,

$$p(a,b,b) = a, \qquad \text{and}$$
$$(p(a,a,b),b) \in [\theta,\theta].$$

Problem IV.0.11 ([**Wil04**]). *Prove Park's conjecture for finite algebras belonging to varieties having a difference term.*

In an algebra **A** let $C(x, y, z, w)$ denote the 4-ary relation

$$[\mathrm{Cg}(x, y), \mathrm{Cg}(z, w)] \neq 0_{\mathbf{A}}$$

where $[\_, \_]$ denotes the TC commutator (see [**HoMc88**, Ex. 3.8(3)]).

$C(x, y, z, w)$ defines disjointness of principal congruences in any congruence meet-semidistributive variety (since $[\alpha, \beta] = \alpha \cap \beta$ in such a variety, by [**KeSz98**, Corollary 4.7] or [**Lip98**, Theorem 4.1]), but this is no longer true in varieties that admit types **1** and **2**. A key step in McKenzie's proof of his finite basis theorem was the demonstration that $C(x, y, z, w)$ is definable in any congruence modular variety in a finite language which has a finite residual bound.

Problem IV.0.12 ([**Wil00**, Problem 3]). *Suppose $\mathcal{V}$ is a variety which omits type* **1** *and has a finite residual bound. Is the relation $C(x, y, u, v)$ definable in $\mathcal{V}$ by a first-order formula?*

Although Park's conjecture has been affirmed in the case of congruence meet-semidistributive and congruence modular varieties, Jónssons conjecture has not been extended to these cases. Hence the following questions arise:

Problem IV.0.13 ([**Wil01**, Problem 4.9]). *Suppose $\mathcal{V}$ is a CSD($\wedge$) variety such that $\mathcal{V}_{FSI}$ is elementary. Does it follow that $\mathcal{V}$ is finitely based?*

Problem IV.0.14 ([**Wil01**, Problem 5.5]). *Suppose $\mathcal{V}$ is a congruence modular variety. If $\mathcal{V}_{FSI}$ is elementary, must $\mathcal{V}$ be finitely based?*[1]

We could ask for simplifications of the proof of Willard's theorem:

Problem IV.0.15 ([**Wil01**, Problem 4.6]). *Suppose $\mathcal{V}$ is a CSD($\wedge$) variety. If $\mathcal{V}_{FSI}$ is axiomatizable, does it follow that $\mathcal{V}$ has DDPC?*

Problem IV.0.16 ([**Wil01**, Problem 4.7]). *Suppose $\mathcal{V}$ is a CSD($\wedge$) variety. If $\mathcal{V}_{SI}$ has DDPC, does it follow that $\mathcal{V}$ has DDPC?*

[**BMW04**] have solved this problem under the latter additional stipulation that $\mathcal{V}$ has bounded critical depth.

A positive answer to either of these two questions provides an elementary proof of Willard's theorem III.8.1 and its extension to varieties $\mathcal{V}$ for which $\mathcal{V}_{FSI}$ is elementary, giving a positive answer to Jónsson's conjecture in the case of CSD($\wedge$) varieties.

[1] Willard suggests that this problem is more complex than IV.0.13.

Any CSD(∧) variety with bounded Mal'cev depth has DDPC, so the following questions arise:

Problem IV.0.17 ([**Wil01**, Problem 5.2]). *Suppose $\mathcal{V}$ is a CSD(∧) variety residually bounded by some m. Does it follow that $\mathcal{V}$ has a bounded Mal'cev depth?*

Problem IV.0.18 ([**Wil01**, Problem 5.3]). *Suppose $\mathcal{V}$ is a CSD(∧) (or even CD) such that $\mathcal{V}_{FSI}$ is elementary. Does it follow that $\mathcal{V}$ has a bounded Mal'cev depth?*

[**BMW04**] offer some progress on IV.0.17 and IV.0.18, but with the notion of bounded critical depth replacing bounded Mal'cev depth.

Another possible simplification of Willard's theorem would be through the method of Baker and Wang, showing that $\mathcal{V}$ has definable principal subcongruences:

Problem IV.0.19 ([**Wil04**]). *If $\mathcal{V}$ is finitely generated congruence meet-semidistributive variety with a finite residual bound, does it follow that $\mathcal{V}$ has definable principal subcongruences?*

APPENDIX A

Resources

I have worked with the following resources:

textbooks [**BuSa81, Grä79, Jež08, Mar08**],

the book [**HoMc88**],

articles [**Ber80, Bur79, BuLa81, DMMN09, Jež69, Jón77, Jón79b, Lyn51, MaMc04, NuSt09, Per69, Sha87, Tay79, Wil00, Wil01, Wil04, Wil08**],

slide presentation [**NuSt09**].

Unless otherwise stated, the numbers in the following list of resources refer to numbers of theorems and definitions (rather than sections).

I. Preliminaries

With the following exceptions, this chapter is based on [**BuSa81**] and [**Mar08**].

In Theorem I.1.6, I tried to gather various results from various resources, including [**Jež08**], [**Grä79**] and [**Tay79**]. Similarly, Observation I.2.2 gives a list of equivalent definitions as used by different authors.

Example I.1.14 is taken from [**Tay79**, Section 13] = [**Grä79**, §70].

Theorem I.2.3 is quoted from [**Wil08**, 6.3].

The two proofs of Proposition I.2.7 come from [**Mar08**, III.3.9] and [**BuSa81**, V.3.8].

Proposition I.2.9 can be found in [**Jež08**, 10.3.2] and in [**Mar08**, V.1.3]; both authors give a proof by contradiction. Our proof, together with the proof of Lemma I.2.10 comes from [**BMW04**].

Proposition I.4.7 comes from [**BaWa02**], Theorem I.4.9 from [**McK87**, p. 233-234].

The proof of Theorem I.4.10 comes from [**Jež08**, 7.8.1] and from a lecture by Michał Stronkowski. I prefer the definition of Willard terms from [**Jež08**] because it avoids the somewhat complicated concepts of parenthesis terms and coloured ordered trees (used in [**Wil00**, 2.1]). See [**HoMc88**, 9.10] for a proof of equivalence of (1), (3) and

(8) in locally finite varieties, as well as a tame congruence theoretic characterization of meet semi-distributive varieties.

The chapter on quasi-varieties is based on a lecture by Michał Stronkowski, with the exception of Theorem I.5.6, which is taken from [**MaMc04**] and [**DMMN09**].

II. An overview of known finite basis results

I used the surveys [**Tay79**] and [**Wil08**] to get a perspective on the historical evolution of universal algebra; the article [**Wil04**] mentions the most important general results, whilst a more detailed list of finite and non-finite basis results comes from [**Tay79**, Sections 9 and 10][1] and [**Mar08**, Chapter 2]. I also consulted [**Jón77**, Section 8], [**Bah87**] and [**Art00**, Section 1.4].

III. Methods for proving finite basis results

Sections III.1.1 and III.1.3 include proofs from [**Jež08**] and [**Jež69**].

Proposition III.1.5 in Section III.1.2 is actually [**HoMc88**, 12.1.]; there it is attributed to a preprint by Joel Berman.

Section III.1.4 is based on [**Lyn51**] and [**Ber80**]. I only included the interesting results, leaving out the technical parts of both papers, so I am actually not giving a complete proof of Lyndon's theorem.

Section III.2 is based on [**Jež08**] and Section III.3 on [**Per69**].

The notation in Section III.4 and some of the following sections is consistent with [**Mar08**, Section IV.3]. The proof of Theorem III.4.1 in this text is based on [**Mar08**, V.1.2]; similar proofs may be found in [**Jež08**, 10.3.1] and in Jónsson's original text. This particular formulation of Corrolary III.4.3 appears in Willard's paper [**Wil04**].

The same proof of McKenzie's Theorem III.5.5 appears in [**Mar08**, Section IV.3], [**Jež08**, 10.2.3] and [**BuSa81**, V.4.3]. The term DDPC (Definition III.5.7) comes from [**BuLa81**], but has been coined by Baker; the proof of III.5.9 can be found in [**Mar08**] and [**Jež08**].

Section III.5.3 and section III.6 on DPSC are based on [**BaWa02**], but I also used [**NuSt09**]. In particular, note that Definition III.5.13 of definable principal subcongruences is taken from the latter source, and Proposition III.5.15 shows that it is equivalent to the definition of Baker and Wang. Proposition III.5.12 also comes from [**NuSt09**].

I like the proof of Baker's Theorem (Section III.7) as written by Burris in [**Bur79**], but in the article quite a few details are left to the reader, so I also used the textbooks [**BuSa81**] and [**Mar08**]. The basic structure of the proof is the same in all three resources.

[1]This appears as Paragraphs 67 and 68 in [**Grä79**].

When writing up the proof of Willard's Theorem (Section III.8), [**Mar08**] and [**Wil00**] were helpful when I was trying to understand the structure of the proof, whilst [**Jež08**] was appealing because it is the most concise.

Section III.9 is based on a lecture delivered by Michał Stronkowski to a couple of students in the winter semester of 2008/2009. The proof of III.9.5 comes from [**DMMN09**] - in this article, which is a follow-up to [**MaMc04**], an extension of Willard's Theorem to quasivarieties is proved.

IV. Open problems

The formulations of the open problems are drawn from [**Wil01**] and [**Wil04**].

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