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DIPLOMOVÁ PRÁCE

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Modelování kreditního rizika

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MASTER THESIS

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Čestné prohlášení

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 24. dubna, 2009

Michal Bošeľa

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vlastnoruční podpis

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Statement of Honesty

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Abstract

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Abstrakt: Predmetom práce sú oceňovacie modely kreditného rizika s ohľadom na dostupnú informáciu. Z tohto pohľadu je dôležité aká informácia je dostupná tvorcovi modelu a to implikuje aký model má byť použitý, či structural alebo reduced form. Prejednaný je taktiež nový prístup pre modelovanie kreditného rizika, ktorý sa zaoberá otázkou ako modelovať informáciu dostupnú trhu použitím konceptu čiastočnej informácie. Tento prístup sa vyhýba použitiu nedostupného markovského času (inaccesible stopping time). V rámci tohto prístupu sú prejednané otázky oceňovania niektorých kreditných derivátov a taktiež možné aplikácie pre výpočet rezervy na poistné plnenia a straty kreditného portfólia.
Klíčová slova: kreditní riziko, kreditní deriváty, neúplná informace, informace přístupná trhu, informačne založené oceňování aktiv, Brownův most, gamma most, celkový požadavek, zajištění

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Abstract: In this work we study credit risk pricing models from an information based pespective. This perspective implies that to distinguish which model is applicable, structural or reduced form, one needs to understand what information is available to the modeler. We also deal with a new information-based framework for credit risk modelling that is concerned with how to model the market filtration by use of the concept of partial information. This framework avoids the use of inaccesible stopping times. The pricing of several credit risk derivatives is discussed in an information-based framework. Applications of the information-based approach to insurance claims reserves and credit portfolio risk are discussed as well.
Keywords: credit risk, credit derivatives, incomplete information, market filtration, information-based asset pricing, Brownian bridge process, gamma bridge process, aggregate claims, reinsurance

Chapter 1

Introduction

Credit risk management is concerned with the risk of failing to comply with a contracted liability. This research area investigates methodologies to incorporate credit risk in asset prices and pursues the development of hedging instruments that offer protection against losses due to credit risk. Credit-linked securities are also used as a means to transfer credit risk. Our main goal is to present the most important mathematical tools that are used for the risk neutral valuation of defaultable claims, which are also known under the name of credit derivatives. The examples of credit derivatives are defaultable bonds, options on defaultable bonds, credit default swap (CDS), baskets of credit-linked securities, collateralised debt obligation (CDO), etc. Further examples of credit derivatives can be found in J.P. Morgan & The RiskMetrics Group [1999]. Pricing models see the debt as a defaultable zero–coupon bond or as some structure build from it. Hence the main issue is how to price a defaultable zero–coupon bond. There are three main quantitative approaches for credit risk management and pricing of credit derivatives: the structural models, the reduced form models and the incomplete information models.

This thesis has the following structure. Chapter 2 is devoted to the properties of Structural and Reduced form models (in particular intensity-based models) with emphasize to the information set which is assumed to be known by market participants for both models. Structural models use the evolution of the structural variables of a firm, which typically are the value of assets and debts, in order to identify the time of default. Defaults are endogenously generated which carry the information provided by the structural variables. On the other hand, reduced form models use market information regarding the firms' credit structure and do not consider any information provided by the balance sheets. An advantage of such models though is that they are usually more tractable than structural models and easier to calibrate to real data. There is in particular one class of models that has attracted much attention: the so-called intensity-based models. Here default is triggered off by a jump process defined in terms of a default intensity.

There also exist some hybrid models that try to integrate both, the structural and the reduced–form approach. While avoiding their shortcomings, they pick the best features of both models. These models are presented in Chapter 3. The idea here is to convey information carried by the firm's state (structural model) into the default intensity of an intensity-based model.

In Chapter 4 an alternative reduced form model, based on the amount and precision of the information received by market participants about the firm's credit risk, is presented. In this framework the market filtration is modelled explicitly and it is not simply assumed as a given.

Chapter 2

Classical Credit Risk Models

For credit risk modelling, in particular, for the pricing of credit derivatives there are two main approaches: Structural models and Reduced form models. From an information based perspective the difference between these two models is in the information known by the modeler.

Notation and Market Assumptions
Let assume a continuous time model with time period $[0, T]$, where $T > 0$ is a fixed finite date. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ satisfying the usual conditions.[1] Here $\mathbb{P}$ is the statistical (real-world) probability measure and the filtration $\mathbb{F}$ is the information known to the modeler that evaluates the credit risk of a firm. We assume a generic firm which borrows funds in the form of a defaultable zero-coupon bond with the face value 1 and the maturity date T and that is the only liability of the firm. The price of such a bond at time $t \leq T$ is denoted by $D(t, T)$. A default-free zero-coupon bonds of all maturities are traded as well. The price at time t of the unit default-free zero-coupon bond with maturity date T is denoted by $P(t, T)$. The default-free short term interest rate process, denoted by r_t, follows an $\mathbb{F}$-progressively measurable process. Markets for the firm's bond and the default-free bonds are supposed to be arbitrage free. Consequently the existence of an equivalent risk-neutral measure $\mathbb{Q}$ is ensured, in the sense that all discounted bond prices follow martingales under the measure $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$. Here the discount factor at time t is equal to $\exp[-\int_0^t r_s \, \mathrm{d}s]$. Markets need not be complete, so the probability measure $\mathbb{Q}$ may not be unique.

Pricing Building Blocks
The saving account, denoted by B_t, is given by the usual expression

$$B_t = \exp\left[\int_0^t r_s \, \mathrm{d}s\right], \tag{2.0.1}$$

[1] That is, $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets.

for every $t \in \mathbb{R}^+$. From the theory of risk neutral pricing we know that an arbitrage free price $\pi_t(X)$ of a contingent claim paying off X at time $T > t$, where X is an $\mathcal{F}_T$-measurable random variable, is given by formula

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} X | \mathcal{F}_t \right]. \tag{2.0.2}$$

For an introduction to risk neutral pricing we refer to Baxter & Rennie [1996], Shreve [2004], or Musiela & Rutkowski [2005]. Using this formula, the arbitrage free price at time t of the unit default-free zero-coupon bond with maturity date T is given by

$$P(t,T) = B_t \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} | \mathcal{F}_t \right]. \tag{2.0.3}$$

Let us consider a defaultable zero-coupon bond with maturity T and face value 1 which in case of default at time $\tau < T$ generates the recovery payment of $R \in [0,1]$, that is paid at maturity time T. Then, using (2.0.2) the arbitrage free price of the defaultable zero-coupon bond is given by

$$\begin{aligned} D(t,T) &= B_t \, \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} (\mathbb{1}_{\{\tau > T\}} + R \mathbb{1}_{\{\tau \le T\}}) | \mathcal{F}_t \right] \\ &= B_t \, \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} (1 - (1-R) \mathbb{1}_{\{\tau \le T\}}) | \mathcal{F}_t \right] \\ &= P(t,T) - B_t \, \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} (1-R) \mathbb{1}_{\{\tau \le T\}}) | \mathcal{F}_t \right] \end{aligned} \tag{2.0.4}$$

Sometimes, bonds with face value different than 1 will be mentioned. A bond with face value K is exactly the same as K standard bonds with face value 1. The price of such a bond at time $t \le T$ is denoted by $K(t,T) = K\, D(t,T)$.

If we fix t and T we can see that the defaultable zero-coupon bond has a higher yield to maturity. The difference between the yield on a defaultable zero-coupon bond $Y^D(t,T)$ and the yield of an otherwise equivalent default-free zero coupon bond $Y^P(t,T)$ is called the *credit spread*, denoted by $S(t,T)$, and is given by

$$\begin{aligned} S(t,T) = Y^D(t,T) - Y^P(t,T) &= -\frac{\ln(D(t,T))}{T-t} + \frac{\ln(P(t,T))}{T-t} \\ &= -\frac{1}{T-t} \ln\left(\frac{D(t,T)}{P(t,T)}\right). \end{aligned} \tag{2.0.5}$$

2.1 Structural Models

Structural models originated with the paper of Merton [1974] who applied the Black & Scholes [1973] option pricing theory to the modeling of a firm's debt. This was the first credit risk model for a single firm. These models link the credit quality of a firm and the firm's economic and financial situation. In structural models the evolution of the structural variables is used and one makes explicit assumptions about the dynamics of firm's assets, its capital structure and its debt and share holders. The market value of the firm is the central source of uncertainty and the firm defaults if its assets are insufficient according to some measure. In this situation, the firm's liabilities can be seen as an option on the total value of firm's assets.

From an information based perspective structural models assume that the modeler has complete information similar to the information held by the firm's manager. Thus the modeler has continuous-time observations of all the firm's assets and liabilities.

Structural models typically assume that the firm's asset value $\{V_t, t \geq 0\}$ follows a diffusion that stays non-negative, i.e.

$$\frac{dV_t}{V_t} = \mu(t, V_t)\, dt + \sigma(t, V_t)\, dW_t, \tag{2.1.6}$$

where W_t is a standard Brownian motion under the measure $\mathbb{P}$, and μ and σ are Borel-measurable functions on $\mathbb{R}^2$, suitably chosen so that the expression (2.1.6) is well defined (see, e.g., Karatzas & Shreve [1988], Protter [2004], Revuz & Yor [1999]). The function μ represents the mean rate of return on assets and the function σ is the volatility coefficient.

We suppose that modeler's information set contains the natural filtration of the firm's asset value process. Hence $\mathcal{G}_t := \sigma(V_s : s \leq t) \subset \mathcal{F}_t$.

In structural models, the default time τ is ussually defined as the first hitting time of the firm's assets value process $\{V_t, t \geq 0\}$ to a certain prespecified default barrier L_t, determined by the firm's liabilities, i.e.

$$\tau = \inf \{t > 0 : V_t \leq L_t\}. \tag{2.1.7}$$

The default barrier represents some breach of a debt contract. The barrier itself could be a stochastic process. The information set then must be augmented such that it encompasses this stochastic process. So in this case we have $\mathcal{G}_t = \sigma(V_s, L_s : s \leq t)$. Here the default time is a predictable stopping time [2], since firm's value process is assumed to be a diffusion, hence the underlying filtration is generated by a standard Brownian motion. As a result, in the structural models default does not come as a surprise, which makes the models generate very low short-term credit spreads. This is contradicted by the empirical evidence. The default time is determined by the value of the firm process and default triggering barrier, hence it is given endogenously within the model. In these models we do not need to specify recovery rates. They arise from the model as the remaining value of the firm's assets.

2.1.1 Merton's Model

In Merton [1974] the standard conditions for the continuous time Black-Scholes market are assumed. These are the inexistence of transaction costs, taxes, or problems with

[2] A stoping time is a non-negative random variable such that the event $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, T]$. A stopping time is predictable if there exists a sequence of stopping times $\{\tau_n\}_{n \geq 1}$ such that τ_n is increasing, $\tau_n \leq \tau$ on $\{\tau > 0\}$ for all n, and $\lim_{n\to\infty} \tau_n = \tau$ a.s.

indivisibilities of assets; an unrestricted borrowing and lending of funds at the same rate of interest; trading in assets takes place continuously in time; short-sales of all assets, with full use of proceeds, is allowed; the value of the firm is invariant to its capital structure (Modigliani-Miller theorem). The capital structure consists of an equity E and a debt D. Thus the total value of the firm's assets V at time t is given by

$$V_t = E_t + D_t. \tag{2.1.8}$$

In the original paper of Merton it is assumed that the short-term interest rate r is constant. This model assumes that the firm's value process V follows a difussion process under the risk-neutral measure $\mathbb{Q}$ that remains non-negative with constant volatility parameter σ and drift $(r-c)$, i.e.

$$\frac{dV_t}{V_t} = (r-c)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \tag{2.1.9}$$

where the constant c represents the total payout ratio by the firm per unit time to either bondholder or shareholders if positive, and it is an inflow of capital to the firm if negative. The process $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. It can be shown by Itô's formula that

$$V_t = V_0 \exp\left[(r-c-\frac{1}{2}\sigma^2)t + \sigma dW_t^{\mathbb{Q}}\right]. \tag{2.1.10}$$

The debt is represented by a zero-coupon bond with maturity T and the face value K, hence $D_t = K(t,T)$.

In this model default can only happen at the debt's maturity time T. The firm defaults if the firm's assets value process at the time of maturity T is less than the face value K of the firm's debt. In this case the ownership of the firm will be assigned to bondholders. So they receive the remaining value of assets V_T. In total they suffer the loss equal to $K - V_T$. If there is no default, so V_T is enough to redeem the debt, bondholders receive amount K and shareholders receive remainder amount $V_T - K$. Therefore the default time τ is a discrete random variable which can be expressed as

$$\tau = T\ \mathbb{1}_{\{V_T<K\}} + \infty\ \mathbb{1}_{\{V_T\geq K\}}, \tag{2.1.11}$$

where $\mathbb{1}_{\{A\}}$ is the indicator function of the event A and $0\cdot\infty = 0$. It also follows that for the payoff of the defaultable zero-coupon bond at maturity we have

$$\min(V_T, K) = K - (K - V_T)^+, \tag{2.1.12}$$

where $x^+ = \max(x,0)$ for every $x \in \mathbb{R}$, and for the payoff of the equity we have

$$(V_T - K)^+. \tag{2.1.13}$$

Thus the firm's equity can be seen as a call option on firm's assets and the defaultable zero-coupon bond as a difference of the value of a default-free zero-coupon bond with face value K (i.e $KP(t,T)$) and the value of a put option P_t on the firm's assets with strike K and maturity T. Therefore for every $0 \leq t < T$ the price of the defaultable zero-coupon bond $K(t,T)$ is given by Black-Scholes type formula

$$\begin{aligned} K(t,T) &= KP(t,T) - P_t \\ &= V_t \mathrm{e}^{-c(T-t)} \Phi\left(-d_1(t)\right) + KP(t,T)\Phi\left(d_2(t)\right), \end{aligned} \tag{2.1.14}$$

where

$$d_1(t) = \frac{\ln\left(\frac{V_t}{K}\right) \; + \; \left(r - c + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\,, \tag{2.1.15}$$

$$d_2(t) = d_1(t) \; - \; \sigma\sqrt{T-t}\,, \tag{2.1.16}$$

and Φ denotes the cumulative distribution function of a standard normal distribution, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}} \,\mathrm{d}y.$$

The conditional probability of default is the probability that the firm's assets value at maturity V_T will be below K. Using (2.1.10) we have

$$\begin{aligned} p_t &:= \mathbb{Q}\left(V_T < K|\mathcal{F}_t\right) \\ &= \mathbb{Q}\left(\sigma\left(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}\right) < \ln\left(\frac{K}{V_t}\right) \; - \; \left(r - c - \frac{1}{2}\sigma^2\right)(T-t)\,|\mathcal{F}_t\right) \\ &= \Phi\left(-d_2(t)\right), \end{aligned} \tag{2.1.17}$$

since $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}$ is normally distributed with zero mean and variance $(T-t)$.

The expected loss $\mathbb{E}_{\mathbb{Q}}\left[\mathsf{L}\right]$ on the loan computed at time 0 under the risk-neutral probability $\mathbb{Q}$ is equal to the expected pay–off of the put option on the firm's value with strike K, i.e.

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\mathsf{L}\right] &:= \mathbb{E}_{\mathbb{Q}}\left[(K - V_T)^+\right] \\ &= \int_{-\infty}^{\infty} \left(K - V_0 \exp\left[(r - c - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\right]\right)^+ \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x \\ &= K\Phi(-d_2(0)) - V_0 \exp\left[(r-c)T\right] \int_{-\infty}^{-d_2(0)} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x - \sigma\sqrt{T})^2}{2}\right] \mathrm{d}x \\ &= K\Phi(-d_2(0)) - V_0 \exp\left[(r-c)T\right] \Phi(-d_2(0) - \sigma\sqrt{T}). \end{aligned} \tag{2.1.18}$$

An essential feature of Merton's model is that the default time τ is a predictable stopping time with respect to filtration generated by the firm's asset value process V. It is announced by an increasing sequence of $\mathbb{F}^V$ - stopping times, e.g.

$$\tau_n = \inf\left\{t \geq T - \frac{1}{n} : V_t < L\right\}, \tag{2.1.19}$$

with the usual convention that $\inf \emptyset = \infty$.

Distance to Default

To determine the actual probability of default we suppose that the firm's asset value process V under the real-world probability $\mathbb{P}$ satisfies

$$\frac{dV_t}{V_t} = (\mu - c)\, dt + \sigma\, dW_t^{\mathbb{P}}, \tag{2.1.20}$$

where $\mu \in \mathbb{R}$ represents the mean rate of return on assets, $\sigma > 0$ is the constant volatility, c is as above, and $W^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Calculation in the same manner as in (2.1.17) implies that

$$\mathbb{P}\,(\tau \leq T|F_t) = \Phi(-d(t)), \tag{2.1.21}$$

where the distance to default at time t, denoted by $d(t)$, is defined as

$$d(t) = \frac{\ln\left(\frac{V_t}{K}\right) \; + \; \left(\mu - c - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\ . \tag{2.1.22}$$

It measures, in terms of $\sigma\sqrt{T-t}$, the distance of the expected firm's assets total value from the default point K.

Extensions of Merton's Approach

Many extensions to Merton's model have been done. A brief survey of papers devoted to various applications of the original Merton approach and to its extensions can be found in the Section 2.4 of Bielecki & Rutkowski [2002].

2.1.2 First-Passage Model

In Merton's model the default may only occur at maturity. The firm value then can fall to almost nothing and default is not triggered. Hence Black & Cox [1976] extended this approach to allow default prior to maturity if the firm's assets value process V_t falls below some prespecified default barrier L_t. In this situation the firm's bondholders have the right to force the firm to bankruptcy or to reorganize the firm. The default barrier may be endogeneously or exogenously given with respect to model, and it may be a constant, a deterministic, or a random process.

In the original paper of Black & Cox [1976], the firm's assets value process V is assumed to follow a difussion process under the risk-neutral measure $\mathbb{Q}$ that remains non-negative with the constant volatility parameter σ and drift $(r - c)$, i.e.

$$\frac{dV_t}{V_t} = (r - c)\, dt + \sigma\, dW_t^{\mathbb{Q}}, \tag{2.1.23}$$

where the constant $c \geq 0$ is representing the payout ratio. The short-term interest rate is supposed to be a constant r. In the first-passage model the default time is defined as

$$\tau = \min(\tau_1, \tau_2), \tag{2.1.24}$$

where τ_1 is the same as the Merton's default time, i.e.

$$\tau_1 = T\ \mathbb{1}_{\{V_T<K\}} + \infty\ \mathbb{1}_{\{V_T \geq K\}}, \tag{2.1.25}$$

and τ_2 is the first hitting time of the default barrier, i.e.

$$\tau_2 = \inf\left\{t \in (0,T) : V_t < L_t\right\}, \tag{2.1.26}$$

where we assume that the infimum of an empty set is equal to ∞. Hence even if the firm's assets value process does not fall below the barrier and if assets are below the bond's face value at maturity, the firm defaults.
Here we will assume that the default barrier is the face value of the bond discounted at a constant discount factor $\gamma \geq r$. This condition guarantees that the payoff to the bondholders at τ never exceeds the face value of the debt discounted at a risk-free rate. For the default barrier we have then

$$L_t = K\mathrm{e}^{-\gamma(T-t)}. \tag{2.1.27}$$

In this situation for the event $\{V_t < L_t\}$ we have

$$\begin{aligned}\{V_t < L_t\} &= \left\{V_0 \exp\left[(r-c-\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{Q}}\right] < K\mathrm{e}^{-\gamma(T-t)}\right\} \\ &= \left\{\exp\left[(r-c-\frac{1}{2}\sigma^2-\gamma)t + \sigma W_t^{\mathbb{Q}}\right] < \frac{K}{V_0}\mathrm{e}^{-\gamma T}\right\} \\ &= \left\{\nu t + \sigma W_t^{\mathbb{Q}} < \ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)\right\}\ ,\end{aligned} \tag{2.1.28}$$

where $\nu = r - c - \frac{1}{2}\sigma^2 - \gamma$. Let m_t denotes the running minimum of the process $\nu t + \sigma W_t^{\mathbb{Q}}$, i.e.

$$m_t = \min_{0 \leq s \leq t}(\nu t + \sigma W_t^{\mathbb{Q}}). \tag{2.1.29}$$

Using Girsanov's theorem and the reflection principle, one can prove that for every $t > 0$ the joint probability distribution of the Brownian motion $X_t := \nu t + \sigma W_t^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$ and its running minimum m_t is given by the formula

$$\mathbb{Q}\ (X_t \geq x,\ m_t \geq y) = \Phi\left(\frac{-x+\nu t}{\sigma\sqrt{t}}\right) - \exp\left[\frac{2\nu y}{\sigma^2}\right]\Phi\left(\frac{2y-x+\nu t}{\sigma\sqrt{t}}\right). \tag{2.1.30}$$

This result and proof can be found in, e.g. Musiela & Rutkowski [2005] (Corollary B.4.3), or Bielecki & Rutkowski [2002] (Lemma 3.1.3). Differentiating with respect to x leads to

$$\frac{\partial \mathbb{Q}\left(X_T \geq x,\ m_T \geq y\right)}{\partial x} = -\frac{1}{\sigma\sqrt{T}}\varphi\left(\frac{-x+\nu T}{\sigma\sqrt{T}}\right) + \frac{1}{\sigma\sqrt{T}}\exp\left[\frac{2\nu y}{\sigma^2}\right]\varphi\left(\frac{2y-x+\nu t}{\sigma\sqrt{t}}\right). \tag{2.1.31}$$

Consequently, differentiating with respect to x and y, for the joint probability density funtion of (X_T, m_T), one can write

$$f_{X_T,m_T}(x,y) = \frac{-2(2y-x)}{\sigma^3\sqrt{T^3}}\exp\left[\frac{2\nu y}{\sigma^2}\right]\varphi\left(\frac{2y-x+\nu T}{\sigma\sqrt{T}}\right). \tag{2.1.32}$$

Therefore for the default probability, using (2.1.28) and (2.1.30), we have

$$\begin{aligned}
\mathbb{Q}\left(\tau \leq T\right) &= \mathbb{Q}\left(\min(\tau_1,\tau_2) \leq T\right) \\
&= 1 - \mathbb{Q}\left(\tau_1 > T,\ \tau_2 > T\right) \\
&= 1 - \mathbb{Q}\left(V_T \geq K,\ m_T \geq \ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)\right) \\
&= 1 - \mathbb{Q}\left(\nu T + \sigma W_T^{\mathbb{Q}} \geq \ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right), m_T \geq \ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)\right) \\
&= 1 - \Phi\left(\frac{-\ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)+\nu T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)^{\frac{2\nu}{\sigma^2}}\Phi\left(\frac{\ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)+\nu T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)-\nu T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)^{\frac{2\nu}{\sigma^2}}\Phi\left(\frac{\ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)+\nu T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\ln\left(\frac{K}{V_0}\right)-\mu T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right)^{\frac{2(\mu-\gamma)}{\sigma^2}}\Phi\left(\frac{\ln\left(\frac{K}{V_0}\right)+(\mu-2\gamma)T}{\sigma\sqrt{T}}\right),
\end{aligned} \tag{2.1.33}$$

where $\mu = r - c - \frac{1}{2}\sigma^2$. This default probability is higher than the corresponding default probability in the Merton's model (2.1.17), which is obtained if we put $L_t = 0$.

In our case firms defaults iff there exists $t \leq T$ such that $V_t < L_t$. It is equivalent to the situation when $m_t < \ln\left(\frac{K}{V_0}\mathrm{e}^{-\gamma T}\right) =: \ln\bar{K}$. If the default occurs bondholders take control over the firm and they receive the remaining assets V_T. Otherwise they receive the face value K. Hence for the payoff of the defaultable bond at maturity we can write

$$\begin{aligned}
K(T,T) &= K - (K-V_T)^+ + (V_T-K)^+\mathbb{1}_{\left\{m_t<\ln\bar{K}\right\}} \\
&= K - (K-V_T)^+ + V_0\mathrm{e}^{\gamma T}\left(\mathrm{e}^{X_T}-\bar{K}\right)^+\mathbb{1}_{\left\{m_t<\ln\bar{K}\right\}}.
\end{aligned}$$

This is equivalent to a portfolio which consists of a default-free zero-coupon bond with maturity T and face value K, a short European put option on the firm's assets with strike K and maturity T, and a long European down-and-in call option on the firm's assets with strike K and maturity T. Consequently

$$K(0,T) = KP(0,T) - P_0 + V_0 \mathrm{e}^{\gamma T} \mathrm{DIC}_0. \tag{2.1.34}$$

where P_0 is the put option value at time zero and DIC_0 is the price of the European down-and-in call option on the exponential process e^{X_T} with stike $\bar{K}$ at time zero. In the first-passage model bonds are worth at least as much as in the Merton's model. Here bondholders have additionally a barrier option on the firm's assets that becomes active if the firm defaults before the maturity T. Thus for the defaultable bond price at time zero we can write

$$K(0,T) = K^M(0,T) + V_0 \mathrm{e}^{\gamma T} \mathrm{DIC}_0, \tag{2.1.35}$$

where $K^M(0,T)$ is the value of the defaultable bond in Merton's model at time zero (2.1.14). For the valuation of the second term note that the expression

$$\left(\mathrm{e}^{X_T} - \bar{K}\right)^+ \mathbb{1}_{\left\{m_t < \ln \bar{K}\right\}}$$

is non-zero on the set

$$D = \left\{X_T > \ln \bar{K},\ m_t < \ln \bar{K}\right\}. \tag{2.1.36}$$

Hence we can write

$$\begin{aligned} \mathrm{DIC}_0 &= P(0,T)\mathbb{E}_{\mathbb{Q}}\left[\left(\mathrm{e}^{X_T} - \bar{K}\right)^+ \mathbb{1}_{\left\{m_t < \ln \bar{K}\right\}}\right] \\ &= P(0,T)\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{X_T}\mathbb{1}_{\{D\}}\right] - \bar{K}\mathbb{Q}\left(D\right)\right] \\ &= P(0,T)\left[\int_D \mathrm{e}^x f_{X_T,m_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \bar{K}\int_D f_{X_T,m_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right]. \end{aligned} \tag{2.1.37}$$

Using (2.1.31), for the first term in the square brackets in (2.1.37) we can write

$$\begin{aligned} I_1 &:= \int_D \mathrm{e}^x f_{X_T,m_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \exp\left[\frac{2\nu \ln \bar{K}}{\sigma^2}\right] \int_{\ln \bar{K}}^{\infty} \mathrm{e}^x \frac{1}{\sigma\sqrt{T}}\varphi\left(\frac{2\ln \bar{K} - x + \nu T}{\sigma\sqrt{T}}\right)\mathrm{d}x \\ &= \exp\left[\frac{2\nu \ln \bar{K}}{\sigma^2}\right] \int_{\ln \bar{K}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T}}\exp\left[x - \frac{(2\ln \bar{K} - x + \nu T)^2}{2\sigma^2 T}\right]\mathrm{d}x \\ &= \exp\left[\frac{T}{2}(\sigma^2 + 2\nu) + \frac{2(\sigma^2+\nu)}{\sigma^2}\ln \bar{K}\right] \\ &\quad \times \int_{\ln \bar{K}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T}}\exp\left[-\frac{(x - (\sigma^2 T + 2\ln \bar{K} + \nu T))^2}{2\sigma^2 T}\right]\mathrm{d}x \\ &= \bar{K}^{\frac{2\nu}{\sigma^2}+2}\exp\left[\frac{T}{2}(\sigma^2 + 2\nu)\right]\Phi\left(\frac{\ln \bar{K} + (\nu + \sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned} \tag{2.1.38}$$

For the second term in the square brackets in (2.1.37) using (2.1.30) we have

$$
\begin{aligned}
I_2 &:= \bar{K} \int_D f_{X_T, m_T}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \bar{K}\mathbb{Q}\left(X_T > \ln\bar{K},\ m_T < \ln\bar{K}\right) \\
&= \bar{K}\left[\mathbb{Q}\left(X_T > \ln\bar{K}\right) - \mathbb{Q}\left(X_T > \ln\bar{K},\ m_T \geq \ln\bar{K}\right)\right] \\
&= \bar{K}\left[1 - \Phi\left(\frac{\ln\bar{K} - \nu T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{-\ln\bar{K} + \nu T}{\sigma\sqrt{T}}\right) + \exp\left[\frac{2\nu\ln\bar{K}}{\sigma^2}\right]\Phi\left(\frac{\ln\bar{K} + \nu T}{\sigma\sqrt{T}}\right)\right] \\
&= \bar{K}^{\frac{2\nu}{\sigma^2}+1}\Phi\left(\frac{\ln\bar{K} + \nu T}{\sigma\sqrt{T}}\right) \qquad (2.1.39)
\end{aligned}
$$

As a consequence of (2.1.35) and (2.1.37) for the price of the defaultable bond at time zero in this model we have

$$
\begin{aligned}
K(0,T) = {} & V_0 \mathrm{e}^{-cT}\Phi\left(-h_1\right) + KP(0,T)\Phi\left(h_1 - \sigma\sqrt{T}\right) \\
& + KP(0,T)\exp\left[\frac{2\nu}{\sigma^2}\left(\ln\frac{K}{V_0} - \gamma T\right)\right] \\
& \times \left(\frac{K}{V_0}\exp\left[\frac{T}{2}\left(\sigma^2 + 2\nu - 2\gamma\right)\right]\Phi\left(h_2 + \sigma\sqrt{T}\right) - \Phi(h_2)\right)\ , \qquad (2.1.40)
\end{aligned}
$$

where

$$
h_1 = \frac{\ln\left(\frac{V_0}{K}\right) \ + \ (\nu+\gamma)\,T}{\sigma\sqrt{T}} + \sigma\sqrt{T}\ , \qquad (2.1.41)
$$

$$
h_2 = \frac{\ln\left(\frac{K}{V_0}\right) \ + \ (\nu-\gamma)\,T}{\sigma\sqrt{T}}\ , \qquad (2.1.42)
$$

$$
\nu = r - c - \frac{1}{2}\sigma^2 - \gamma\ . \qquad (2.1.43)
$$

For the pricing formulas of barrier options we refer to Section 6.6 in Musiela & Rutkowski [2005] or Hull [2006].

Extensions and Shortcomings

More generally, if the default time is given by (2.1.7), and furthermore we assume the stochastic interest rate and that the value of the barrier process is paid at maturity time T, then the value of the firm's debt can be written as

$$
K(0,T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\int_0^T r_s\,\mathrm{d}s\right]\left(L_\tau \mathbb{1}_{\{\tau\leq T\}} + K\mathbb{1}_{\{\tau > T\}}\right)\right]. \qquad (2.1.44)
$$

As the models for stochastic interest rates in the structural models literature have been used followings

$$\text{the Vasicek model} \qquad dr_t = (\theta - \alpha r_t)\, dt + \sigma_r\, d\widetilde{W}_t, \tag{2.1.45}$$

$$\text{the generalized Vasicek model} \qquad dr_t = (\theta(t) - \alpha(t) r_t)\, dt + \sigma_r(t)\, d\widetilde{W}_t, \tag{2.1.46}$$

$$\text{the Cox-Ingersoll-Ross model} \qquad dr_t = (\theta - \alpha r_t)\, dt + \sigma_r \sqrt{r_t}\, d\widetilde{W}_t, \tag{2.1.47}$$

where $W_t^{\mathbb{Q}}$ and $\widetilde{W}_t$ are correlated Brownian motions, so that $d\widetilde{W}_t dW_t^{\mathbb{Q}} = \rho dt$, and $\rho \in [-1, 1]$.

First-passage models have also been extended to account for debt subordination, strategic default, stochastic default barrier, bankruptcy costs, taxes, jumps in the firm's assets value process, etc. The first passage model supposes that bondholders take control over the firm immediatelly when firm's assets value process falls below the default barrier. In practice, bankruptcy codes let firms reorganize and operate for a period of time. The creditor takes control over the firm's assets if the firm value does not rise. If restructuring is succesful the firm recovers from bankruptcy and continues operating. Thus the firm defaults after its asset's value process spends a given time below the barrier. For these models see Section 2.3 (Excursion approach) in Giesecke [2004a] or Section 2.4 (Liquidation process models) in Elizalde [2005b]. For more structural models and pricing of derivatives we refer to Part I in Bielecki & Rutkowski [2002].

One of the general problems of structural models is that it is difficult to deal effectively with the multiplicity of situations that can lead to default. In particular, default of sovereign state, credit card default, and corporate default would all require different treatments. Thus structural models are viewed as unsatisfactory as a basis of practical modelling. Especially for n^{th}-to-default swaps and collateralised debt obligations. Another problem is the analytical complexity which is increased by involving stochastic interest rates or endogeneous default thresholds. It makes it difficult to get closed form expressions for the value of debt, equity or for the default probability. This forces us to employ numerical methods. The total value of the firm's assets cannot be easily observed and is not a tradeable security.

2.2 Reduced Form Models

Reduced form models originated with the papers of Jarrow & Turnbull [1995], Jarrow et al. [1997], Lando [1998], and Duffie & Singleton [1999]. In this approach firm's assets and its capital structure are not modelled at all. Reduced form models do not address directly why a firm defaults. This approach was developed precisely to avoid modelling unobservable asset value process. An advantage of such models is that they are usually more tractable than structural models and easier to calibrate to real data.

Here the dynamics of default are given exogenously, directly under a pricing probability $\mathbb{Q}$, through a default rate, or default intensity. The default time is characterized as the first jump time of a point process. The most common are used a Poisson process, an inhomogeneous Poisson process or a Cox process. The default time is ussually a totally inaccesible stopping time [3]. This implies the non-zero short-term credit spreads.

The values of credit-sensitive securities can be calculated as if they were default-free, using a credit risk adjusted interest rate, i.e. the risk-free interest rate plus risk-neutral default intensity.

From an information based perspective reduced form models are based on the information set available to the market. This information set typically includes only partial observations of the firm's assets and liabilities.

We can distinguish between the reduced form models that are concerned with the modelling of default time and the reduced form models that are concerned with migration between credit rating classes.

2.2.1 Intensity-Based Models

In the intensity-based models default is triggered off by a jump process defined in terms of a default intensity. Let us assume that default time τ is an $\mathbb{Q}$-a.s. positive random variable, i.e. $\tau : \Omega \to \mathbb{R}^+$ and $\mathbb{Q}(\tau > 0) = 1$. We define the default process by

$$N_t = \mathbb{1}_{\{\tau \leq t\}}. \tag{2.2.48}$$

This is a point process with one jump of size one at the default time. It is obvious that the process N_t is a right-continuous non-negative submartingale with $N_0 = 0$. From Doob-Meyer decomposition we know that there exists an increasing process A^τ such that $N - A^\tau$ becomes a martingale (see, e.g., Karatzas & Shreve [1988], Protter [2004], Revuz & Yor [1999]). The unique process A^τ is often called compensator. Let assume that λ_t is the hazard rate of the random variable τ which has a cumulative distribution function $F(t) = \mathbb{Q}\ (\tau \leq t)$ which is assumed to be differentiable at $t > 0$, i.e.

$$\begin{aligned} \lambda_t &:= \lim_{h \downarrow 0} \frac{\mathbb{Q}\ (t \leq \tau < t + h | \tau > t)}{h} \\ &= \frac{dF(t)}{dt} \frac{1}{1 - F(t)} \\ &= -\frac{d \ln(1 - F(t))}{dt}. \end{aligned} \tag{2.2.49}$$

Then for A^τ we have

$$A_t^\tau = \int_0^{t \wedge \tau} \lambda_s \, \mathrm{d}s = \int_0^t \lambda_s \mathbb{1}_{\{s \leq \tau\}} \, \mathrm{d}s. \tag{2.2.50}$$

[3] A stopping time τ is a totally inaccessible stopping time if for every predictable stopping time ζ it holds that $\mathbb{Q}\ (\tau = \zeta < \infty) = 0$

A non-negative process λ is called the default intensity. If the default intensity λ_t is constant (resp. deterministic, resp. random) then the process N_t is a Poisson process (resp. a time inhomogeneous Poisson process, resp. a Cox process) stopped at its first jump, so at the default time τ. For the default time we can write

$$\tau = \inf\{t > 0 : N_t > 0\}. \tag{2.2.51}$$

1. Poisson Process
In the Poisson process model for N_t, the default intensity is constant, and the default time τ has an exponential distribution with parameter λ. For the risk-neutral probability of default prior to time T in this case we have

$$\mathbb{Q}\,(\tau \leq T) = 1 - \exp\left[-\lambda T\right]. \tag{2.2.52}$$

2. Time Inhomogeneous Poisson Process
In this case, the default intensity is assumed to be a function of time. The time dependency can be estimated from historical market data or can be given exogenously. The default probability we have

$$\mathbb{Q}\,(\tau \leq T) = 1 - \mathbb{Q}\,(N_t = 0) = 1 - \exp\left[-\int_0^T \lambda(s)\,\mathrm{d}s\right]. \tag{2.2.53}$$

3. Cox Process
Here the modeler observes the filtration generated by the defaut time τ and a vector of economy variables X_t, where the default time is a stopping time generated by a Cox process $N_t = \mathbb{1}_{\{\tau \leq t\}}$ with the intensity process $\lambda(X_t)$, i.e. $\mathcal{G}_t = \sigma(\tau, X_s : s \leq t) \subset \mathcal{F}_t$.

In the Cox process setting it is assumed that there exists a d-dimensional background Markov process $\{X_t, t \in [0,T]\}$ that represents economic variables, either state (observable) or latent (unobserved)[4]. These are thought of as risk factors that drive the intensity. Given also is a function $\lambda : \mathbb{R}^d \to \mathbb{R}$ which is assumed to be nonnegative and continuous. The default intensity then is of the form

$$\lambda_t = \lambda(X_t). \tag{2.2.54}$$

This function λ has to be chosen such that $\Lambda(t) := \int_0^t \lambda_s\,\mathrm{d}s < \infty$ a.s. for $t \in [0,T]$. A Cox process is a point process where conditional on the information set generated by

[4]Given the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ together with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, an $\mathbb{F}$-adapted process X is a *Markov process* with respect to $\mathbb{F}$ if

$$\mathbb{E}_{\mathbb{Q}}\left[f(X_t)|\mathcal{F}_s\right] = \mathbb{E}_{\mathbb{Q}}\left[f(X_t)|X_s\right] \;\; a.s.\ w.r.t.\ \mathbb{Q}\,,$$

for all s such that $0 \leq s \leq t$ and for every bounded function $f(x)$. Here f may depend on t as well.

the state variables X_t over the whole time interval, i.e $\sigma(X_s : s \leq T)$, the conditioned process is an inhomogeneous Poisson process with intensity $\lambda(X_t)$. For the conditional survival probability under Cox processes we have

$$\mathbb{Q}\left(\tau > t | (X_s)_{0\leq s\leq t}\right) = \exp\left[-\int_0^t \lambda(X_s)\,\mathrm{d}s\right] = \exp\left[-\int_0^t \lambda_s\,\mathrm{d}s\right], \tag{2.2.55}$$

and for the default probability hence we have

$$\begin{aligned}\mathbb{Q}\left(\tau \leq T\right) &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[N_T = 1|(X_s)_{0\leq s\leq T}\right]\right] \\ &= 1 - \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}|(X_s)_{0\leq s\leq T}\right]\right] \\ &= 1 - \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\int_0^T \lambda_s\,\mathrm{d}s\right]\right].\end{aligned} \tag{2.2.56}$$

Affine Intensity Models
In many financial applicatons that are based on a state process a useful assumption is that the state process is affine.
A Markov process X with some state space $E \subset \mathbb{R}^d$ is called an *affine process* if for any $v \in \mathbb{R}^d$ its conditional characteristic function is of the form

$$\mathbb{E}\left[\mathrm{e}^{iv\cdot X_t}|X_s\right] = \exp\left[\alpha(t-s, iv) + \beta(t-s, iv)\cdot X_s\right], \tag{2.2.57}$$

for some coefficients $\alpha(\cdot, iv)$ and $\beta(\cdot, iv)$. If we take the state space E to be $\mathbb{R}^n_+ \times \mathbb{R}^{d-n}$ for $n \in [0, d]$, then we say that X is *regular* provided the coefficients $\alpha(\cdot, iv)$ and $\beta(\cdot, iv)$ of the characteristic function are differentiable and their derivatives are continuous in zero. The mathematical theory related to affine processes can be found in Duffie et al. [2003] for the time homogeneous case and in Filipovic [2005] for the time inhomogeneous case.

1. **Affine Diffusion Model** An affine diffusion is a solution of the stochastic differential equation of the form

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t^{\mathbb{Q}}, \tag{2.2.58}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion in $\mathbb{R}^d$ under the measure $\mathbb{Q}$ and coefficients are affine functions of the state variables, i.e.

$$\mu(x) = a + Bx \tag{2.2.59}$$

where a is a vector of constants in $\mathbb{R}^d$ and $B \in \mathbb{R}^{d\times d}$ is a matrix of constants, and

$$\left(\sigma(x)\sigma^T(x)\right)_{i,j} = (C)_{i,j} + (D)_{i,j}\cdot x \tag{2.2.60}$$

where $C \in \mathbb{R}^{d\times d}$ and $D \in \mathbb{R}^{d\times d}$ are matrices of constants. Furthermore, let $\lambda(x)$ be also affine in x, i.e.

$$\lambda(x) = \lambda_0 + \lambda_1 \cdot x \tag{2.2.61}$$

for $\lambda_0 \in \mathbb{R}$ and $\lambda_1 \in \mathbb{R}^d$. Then there exists functions $\alpha(t)$ and $\beta(t)$ such that the default probability (2.2.56) is exponentially affine in the initial state X_0, i.e.

$$\mathbb{Q}\,(\tau \le T) = 1 - \exp\left[\alpha(T) + \beta(T)\cdot X_0\right]. \tag{2.2.62}$$

These functions can be in some cases calculated explicitly as the solutions to a system of ordinary differential equations, called a generalized Riccati equation (see, e.g., Duffie [2005], Filipovic & Mayerhofer [2009]).

Example 2.2.1. Assume that $E = \mathbb{R}$, $\lambda(x) = x$, $\mu(x) = c\mu - cx$ for constants $\mu \in \mathbb{R}$ and $c > 0$, and $\sigma^2(x) = \sigma^2$ for a constant $\sigma > 0$ then we obtain the Ornstein-Ulhenbeck (Vasicek) process, i.e.

$$dX_t = c\,(\mu - X_t)\;dt + \sigma\,dW_t^{\mathbb{Q}}\,, \tag{2.2.63}$$

then for the coefficient functions $\alpha(T)$ and $\beta(T)$ in (2.2.62) we have (see, e.g. Shreve [2004])

$$\beta(T) = -\frac{1-\mathrm{e}^{-cT}}{c} \tag{2.2.64}$$

$$\alpha(T) = \mu(\beta(T) - T) + \frac{\sigma^2}{2c^2}\left(T - 2\beta(T) + \frac{1-\mathrm{e}^{-2cT}}{2c}\right). \tag{2.2.65}$$

Example 2.2.2. Let assume that state space $E = \mathbb{R}^+$ and $\sigma^2(x) = \sigma^2 x$ for a constant $\sigma > 0$. Let $\mu(x)$ and $\lambda(x)$ be the same as in the previous example. Then we obtain the square-root diffusion also known as the Feller (Cox-Ingersoll-Ross) process, i.e.

$$dX_t = c\,(\mu - X_t)\;dt + \sigma\sqrt{X_t}\,dW_t^{\mathbb{Q}}\,. \tag{2.2.66}$$

If we assume that $X_0 > 0$ and $2c\mu > \sigma^2$, which is sometimes called the Feller-condition, then the process X stays almost surely strictly positive. In this case, for the coefficient functions $\alpha(T)$ and $\beta(T)$ in (2.2.62) we have

$$\beta(T) = -\frac{2(\mathrm{e}^{\gamma T}-1)}{2\gamma + (\gamma + c)(\mathrm{e}^{\gamma T}-1)} \tag{2.2.67}$$

$$\alpha(T) = \frac{2c\mu}{\sigma^2}\ln\left(\frac{2\gamma\mathrm{e}^{\frac{T}{2}(\gamma+c)}}{(\gamma+c)(\mathrm{e}^{\gamma T}-1)+2\gamma}\right), \tag{2.2.68}$$

where $\gamma := \sqrt{c^2 + 2\sigma^2}$.

2. Affine Jump-Diffusion Model We can extend previous case including unexpected jumps, which model the arrival of news in the economy. Hence we assume that the risk factor X is the solution of the stochastic differential equation of the form

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t^{\mathbb{Q}} + dJ_t, \tag{2.2.69}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion in $\mathbb{R}^d$ under the measure $\mathbb{Q}$, both coefficients μ and $\sigma\sigma^T$ are affine functions of the state variables, and J is a pure jump process with arrival intensity $\{\kappa(X_t) : t \geq 0\}$ which is affine in X_t as well, i.e.

$$\kappa(x) = \kappa_0 + \kappa_1 \cdot x, \tag{2.2.70}$$

for $\kappa_0 \in \mathbb{R}$ and $\kappa_1 \in \mathbb{R}^d$. Conditional on the path of X, the jump times of J are the jump times of a Poisson process with time varying intensity $\{\kappa(X_s) : 0 \leq s \leq t\}$, and the size of the jump of J at time T is independent of $\{X_s : 0 \leq s \leq T\}$ and has the probability distribution j. More details can be found in Duffie et al. [2003] or in Duffie et al. [2000]. If we asume that the default intensity is given by (2.2.61) then the default probability is exponentially affine in the initial state X_0, i.e.

$$\mathbb{Q}\,(\tau \leq T) = 1 - \exp\left[\alpha(T) + \beta(T) \cdot X_0\right], \tag{2.2.71}$$

where the coefficients α and β again solve a system of Riccati ordinary differential equations given in Duffie et al. [2000].

Example 2.2.3. A special example of (2.2.69) is the *basic affine process* with state space $E = \mathbb{R}^+$, $\lambda(x) = x$, $\mu(x) = c\mu - cx$ for constants $\mu \in \mathbb{R}$ and $c > 0$, and $\sigma^2(x) = \sigma^2 x$ for a constant $\sigma > 0$, satisfying

$$dX_t = c\,(\mu - X_t)\; dt + \sigma\sqrt{X_t}\, dW_t^{\mathbb{Q}} + dJ_t, \tag{2.2.72}$$

where J is a compound Poisson process [5], independent of $W^{\mathbb{Q}}$, with iid exponential jumps. The Poisson arrival intensity satisfies $\kappa(x) = \kappa$ and the jump distribution j is exponential. The coefficient functions are provided in Appendix A.5 in Duffie & Singleton [2003].

Valuation of the Defaultable Claims

Firstly, let us assume the case of a Poisson process, so that the default intensity λ is constant. We also suppose that recovery in the case of default is equal to zero and that interest rate r is constant. In this case, for the defaultable zero-coupon bond price at time zero, using (2.2.52), we have

$$D(0,T) = B_T^{-1}(1 - \mathbb{Q}\,(\tau \leq T)) = \exp\left[-(r + \lambda)T\right]. \tag{2.2.73}$$

[5] A compound Poisson process has jumps at iid exponential event times, with iid jump sizes.

Thus, as we mentioned above, the value of the defaultable zero-coupon bond can be calculated as if this bond were default-free using a credit risk adjusted interest rate, i.e. the risk-free interest rate r plus risk-neutral default intensity λ. This analogy extends to more complicated credit derivatives.

A general credit linked security is specified by the amount C_T which is paid at maturity T if no default occurs prior to T, and recovery payment which investors receive precisely at default time τ in the case of default. This recovery payment is modeled as a bounded stochastic process R, with $R_s = 0$ for $s > T$. This specification of recovery payment covers all possible ways of treatment of recovery payments considered in the literature. Various recovery schemes are treated below.

If $C_T = 1$ and R is nontrivial, this security is a defaultable zero-coupon bond. For $C_T = (S_T - K)^+$ and nontrivial R, this security is a vulnerable call option on S with strike K. That is an option contract in which an option writer may default on his obligation. For C_T nontrivial and $R = 0$, this security represents a single fee payment at maturity time T in a default swap, which may be considered as some type of debt insurance contract. The list of credit linked securities can be found in sections 1.1-1.3 of Bielecki & Rutkowski [2002].

Let us assume the case of a Cox process for the intensity. Furthermore we assume that interest rates are stochastic and can be expressed as $r_t = f(X_t)$ for some bounded measurable function $f : \mathbb{R}^d \to [0, \infty)$, and that $C_T = g(X_T)$ for some bounded measurable function $g : \mathbb{R}^d \to \mathbb{R}$, where X is a Markov process as in the case 3. on page 15. Then, using (2.0.2), the price of the defaultable claim at time zero is given by

$$\begin{aligned} C_0 &= \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} C_T \mathbb{1}_{\{\tau > T\}}\right] + \mathbb{E}_{\mathbb{Q}}\left[B_\tau^{-1} R_\tau \mathbb{1}_{\{\tau \leq T\}}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} C_T \mathbb{1}_{\{\tau > T\}} | (X_s)_{0 \leq s \leq T}\right]\right] + \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[B_\tau^{-1} R_\tau \mathbb{1}_{\{\tau \leq T\}} | (X_s)_{0 \leq s \leq T}\right]\right]. \end{aligned} \tag{2.2.74}$$

Taking out what is known (see Williams [1991], 9.7(j)) in the first term, and denoting $p(u)$ the conditional density of τ at u given the path $(X_s)_{0 \leq s \leq T}$ for all $u \in [0, T]$, i.e.

$$p(u) = \frac{\partial}{\partial u} \mathbb{Q}\left(\tau \leq u | (X_s)_{0 \leq s \leq T}\right) = \lambda_u \exp\left[-\int_0^u \lambda_s \, \mathrm{d}s\right], \tag{2.2.75}$$

in the second term (note that in the Cox process framework this density exists), the price of the defaultable claim (2.2.74) can be expressed as

$$\begin{aligned} C_0 &= \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} C_T \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau > T\}} | (X_s)_{0 \leq s \leq T}\right]\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty B_u^{-1} R_u \mathbb{1}_{\{u \leq T\}} p(u) \, \mathrm{d}u\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[C_T \exp\left[-\int_0^T (r_u + \lambda_u) \, \mathrm{d}u\right]\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T R_u \lambda_u \exp\left[-\int_0^u (r_s + \lambda_s) \, \mathrm{d}s\right] \mathrm{d}u\right], \end{aligned} \tag{2.2.76}$$

providing that all technical conditions, which ensure finiteness of the expectations, are satisfied (see Lando [1998], Proposition 3.1).

Recovery Rates

The recovery payment R in the case of default is usually specified by the recovery rate δ. In general, the recovery rate can be a stochastic process with values in [0,1]. This stochastic process δ_t is assumed to be a part of the information set available to the modeler, i.e. $\mathcal{G}_t = \sigma(\tau, X_s, \delta_s : s \leq t)$.

Firstly, to be consistent with the structural model in the previous section, we suppose that the recovery rate δ_τ is paid at time T. Using (2.0.4), the time zero value of the unit defaultable zero-coupon bond with maturiy T and the recovery rate process δ_τ can be written as

$$D(0,T) = \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} (\mathbb{1}_{\{\tau > T\}} + \delta_\tau \mathbb{1}_{\{\tau \leq T\}}) \right] . \tag{2.2.77}$$

For the defaultable zero-coupon with the face value K we have

$$K(0,T) = K\, D(0,T) = \mathbb{E}_{\mathbb{Q}} \left[B_T^{-1} (K \mathbb{1}_{\{\tau > T\}} + K \delta_\tau \mathbb{1}_{\{\tau \leq T\}}) \right] . \tag{2.2.78}$$

A small but crucial difference between this pricing formula in the intensity-based model and the pricing formula (2.1.44) in the structural model is that the recovery process in the structural model is prespecified by a knowledge of the liability structure, whereas here it is given exogenously.

In the credit risk literature there are three main specifications for recoveries.

1. **Recovery of Face Value:** The recovery is assumed to be an exogenously given fraction δ of the face value of the defaultable security. Hence the recovery rate δ is constant and independent of the default time τ. Let us assume a defaultable zero-coupon bond with face value 1 and that a fixed fraction of the bond's face value is paid at time of default τ, then its value can be calculated using (2.2.76) with C_T=1 and $R_t = \delta$.

2. **Recovery of Treasury:** In this case, the recovery payment R is assumed to be an exogenously given fraction δ of the value of an equivalent but default-free version of the security.

3. **Recovery of Market Value:** Here the recovery payment is assumed to be an exogenously given fraction δ of the security market value just before default. In the case of defaultable zero-coupon bond we have $R_\tau = \delta_t D(\tau-, T)$. More generally, if default occurs at time t the recovery process can be written as $R_t = \delta_t C_{t-}$, where $C_{t-} = \lim_{s \uparrow t} C_s$. This convention make only sense if C_τ is different from $C_{\tau-}$. Hence there is a surprise jump at default in the security price. In the structural models it holds that $C_{\tau-} = C_\tau$.

For an extentive review of the treatment of recovery rates we refer to Chapter 6 in Schönbucher [2003].

2.2.2 Credit Rating Migration Models

A firm's credit rating is a measure of the firm's propensity to default. In these models it is assumed that the credit quality of corporate debt is quantified and categorized into a finite number of disjoint *credit rating classes*. The credit quality migrates between various credit classes. The credit rating migration is often modeled using Markov chains with finite state space $\mathcal{S} = \{1, \ldots, K\}$, as was introduced in Jarrow et al. [1997]. Here one should think of 1 as the top rating (AAA, say) and K as default. The default rating class K represents the absorbing state, since multiple defaults are exluded. The *credit migration process* is usually assumed to be either a discrete time or a continuous time Markov chain. The main issue in these models is thus the specification of the matrix of transition probabilities in the discrete time setting or matrix of transition intensities in the continuous time case for the credit rating migration process. There are some problems with using continuous time homogeneous Markov chains:

- **The Markov Property**: Transition probabilities should depend only on the current rating, but empirically there is evidence that if a counterparty downgrade its rating, there will be a higher probability of another downgrade than in the case of a counterparty which has a stable rating or if a current rating was reached by an upgrade. This can be fixed by extending the state space from K ratings $2K-2$. It can be done in the following way $1, 2, 2-, 3, 3-, \ldots, K-1, (K-1)-, K$, where rating i represents the situation when rating i was reached by an upgrade, and rating $i-$ means that rating was reached by a downgrade. The rating K (resp. 1) can be reached only by a downgrade (resp. an upgrade). Thus we have $2K-2$ rating classes.

- **The Aging Effect**: There is dependence of transition probabilities on the time that a firm spends in the same credit rating and also on age. For a homogeneous Markov chain it holds that the distribution of sojourn times (i.e. the time spent by M in some state of K) is exponencial. The exponential distribution is memoryless but there is indeed an apparent momentum in rating transition data. This problem can be solved by means of semi-Markov processes. In semi-Markov processes the transition probabilities are functions of the waiting time spent in some state. The dependence on age can be solved in a general approach by means of a inhomogeneous environment. Both these problems are solved applying an inhomogeneous semi-Markov environment. For the general theory of semi-Markov processes we refer to Janssen & Manca [2006]. Applications for finance and for credit risk migration models can be found in Janssen & Manca [2007] and D'Amico et al. [2005].

- **Constant Rating Intensities**: Real data shows that intensities change over time. Rating based model which include stochastically varying transition intensities was introduced by Lando [1998]. In this model the Cox process framework is proposed to model default time as the first time that the credit migration process M with state space $\mathcal{S} = \{1, \ldots, K\}$ hits the absorbing (default) state, i.e

$$\tau = \inf\{t \in [0, T] : M_t = K\}. \tag{2.2.79}$$

The dynamics of of credit migration process M are characterized by a generator matrix $\mathbf{Q}$:

$$\mathbf{Q}(X_t) = \begin{pmatrix} -\lambda_1(X_t) & \lambda_{1,2}(X_t) & \ldots & \lambda_{1,K-1}(X_t) & \lambda_{1,K}(X_t) \\ \lambda_{2,1}(X_t) & -\lambda_2(X_t) & \ldots & \lambda_{2,K-1}(X_t) & \lambda_{2,K}(X_t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(X_t) & \lambda_{K-1,2}(X_t) & \ldots & -\lambda_{K-1}(X_t) & \lambda_{K-1,K}(X_t) \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}, \tag{2.2.80}$$

where $\lambda_{i,j} : \mathbb{R}^d \to \mathbb{R}^+$, $i, j = 1, \ldots, K$ are non-negative functions which maps the risk factors X into the transition intensity and

$$\lambda_i(X_t) = \sum_{j=1, j\neq i}^{K} \lambda_{i,j}(X_t), \quad i = 1, \ldots, K-1, \tag{2.2.81}$$

for every $t \in [0, \widetilde{T}]$. Intuitively, we can think of the product $\lambda_{i,j}(X_t)\Delta t$, for small Δt, as the probability that the firm currently in rating class i will migrate to class j withhin the time interval Δt, and $\lambda_i(X_t)\Delta t$ as the probability that there will be any rating change for the firm currently in the rating class i within the time interval Δt. The migration process M is determined in such a way that, conditionally on a particular sample path $X_t(\omega)$, $t \in [0, \widetilde{T}]$ of the economy variables process X, the migration process M is a time inhomogeneous Markov chain with finite state space $\mathcal{S} = \{1, ..., K\}$ and time dependent deterministic intensity matrix $\mathbf{Q}(X_t(\omega))$. The corresponding default process N is a Cox process with intensity $\lambda_{M_t,K}(X_t)$ at time t that is represented by the last column in the generator matrix $\mathbf{Q}(X_t)$. This generalises the Jarrow et al. [1997] approach, where the transition intensities are supposed to be constant. Conditionally on the evolution of the economic variables, the transition probabilities of the Markov chain M satisfy

$$\frac{\partial \mathbf{P}_X(s, t)}{\partial s} = -\mathbf{Q}(X_s)\mathbf{P}_X(s, t). \tag{2.2.82}$$

In general we are interested in modeling transition probabilities

$$\mathbf{P}(s, t) = (p_{i,j}(s, t))_{i,j=1}^{K}, \tag{2.2.83}$$

where $p_{i,j}(s,t)$ is the conditional probability that the debtor will be in the rating j at time t given that he is in the rating i at time s. In the time homogeneous case, the matrix of transition probabilities satisfies

$$\mathbf{P}(s,t) = \exp\left[(t-s)\mathbf{Q}\right] := \sum_{k=0}^{\infty} \frac{\mathbf{Q}^k (t-s)^k}{k!}, \tag{2.2.84}$$

where $\mathbf{Q}$ is the $K \times K$ matrix of constant intensities.

If the ratings transition generator $\mathbf{Q}(t)$ is a deterministic matrix function of time, then special methods are required to compute transition probabilities. It is not generally true that one can extend time homogeneous case (2.2.84) to get $\mathbf{P}(s,t) = \exp\left[\int_s^t \mathbf{Q}(u)\,\mathrm{d}u\right]$. But, for this time dependent but deterministic generator, $\mathbf{P}(s,t)$ solves the linear ordinary differential equation

$$\frac{\partial \mathbf{P}(s,t)}{\partial s} = -\mathbf{Q}(s)\mathbf{P}(s,t), \tag{2.2.85}$$

which has to be solved numerically without further assumptions on the generator $\mathbf{Q}(t)$. Computation of $\mathbf{P}(s,t)$ can be hard. One such a case when the linear ordinary differential equation (2.2.85) can be solved more explicitly is when commutativity property

$$\mathbf{Q}(s)\mathbf{Q}(t) = \mathbf{Q}(t)\mathbf{Q}(s) \tag{2.2.86}$$

holds for every s and t. This obviously holds if the transition intensities are constant.

We place ourselves into the Cox process framework with differential equation (2.2.82). Let us assume that for each path of X it holds that the matrix $\mathbf{Q}(X_t)$ can be written in the form

$$\mathbf{Q}(X_t) = \mathbf{C}\,\mathbf{D}(X_t)\,\mathbf{C}^{-1}, \tag{2.2.87}$$

where $\mathbf{C}$ is the $K \times K$ matrix whose columns consist of K eigenvectors of $\mathbf{Q}(X_t)$ and $\mathbf{D}(X_t)$ is the $K \times K$ diagonal matrix with i^{th} diagonal element $d_i(X_t)$ for $i = 1 \ldots K-1$ and $d_K = 0$. Here $d_i : \mathbb{R}^d \to [0,\infty)$, $i = 1, \ldots, K-1$ are non-negative functions defined on the state space of X and we assume that for almost every sample path of X we have $\int_0^T d_i(X_u)\,\mathrm{d}u < \infty$. Also, let us define the diagonal matrix

$$\mathbf{E}_X(s,t) = \begin{pmatrix} \exp\left[\int_s^t d_1(X_u)\,\mathrm{d}u\right] & 0 & \cdots & \cdots & 0 \\ 0 & \exp\left[\int_s^t d_2(X_u)\,\mathrm{d}u\right] & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \exp\left[\int_s^t d_{K-1}(X_u)\,\mathrm{d}u\right] & 0 \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}. \tag{2.2.88}$$

Under the assumption made that $\mathbf{C}$ does not depend on t, while $\mathbf{D}$ may, we have commutativity (2.2.86) and it follows from the theory of linear ordinary differential equations that

$$\mathbf{P}_X(s,t) = \mathbf{C}\,\mathbf{E}_X(s,t)\,\mathbf{C}^{-1} \tag{2.2.89}$$

satisfies Kolmogorov's backward equation (2.2.82) and $\mathbf{P}_X(s,t)$ is the matrix of transition probabilities of an inhomogeneous Markov chain with state space $\mathcal{K}$. The last column of the matrix $\mathbf{P}_X(s,t)$ is a vector of default probabilities from time s to time t for different ratings.

Lando [1998] in the Chapter 5. discusses the valuation of a defaultable European contingent claim whose payoff at time T is $C_T = f(X_T, M_T)$, i.e. the function of background process X and migration process M. Payments are directly linked to the rating class of a certain firm. We give here only the price of defaultable zero-coupon bond with maturity T and zero recovery that is given by

$$D_i(t,T) = B_t \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1}(1 - p_X(t,T)_{i,K}) | \sigma(X_s : 0 \leq s \leq t)\right] \tag{2.2.90}$$

where $p_X(t,T)_{i,K}$ is the $(i,K)^{th}$ element of the matrix $\mathbf{P}_X(t,T)$, i.e. the conditional transition probability that the debtor will default up to time T given that at time t he is in rating class i. Under condition (2.2.87) on $\mathbf{Q}$, with d and $\mathbf{C}$ as discussed above, and if we assume that the short term interest rate satisfies $r_t = R(X_t)$, for some bounded measurable function $R : \mathbb{R}^d \to [0,\infty)$, (2.2.90) can be rewritten as

$$D_i(t,T) = \sum_{j=1}^{K-1} \beta_{ij} \mathbb{E}_{\mathbb{Q}}\left[\exp\left[\int_t^T (d_j(X_u) - R(X_u))\,\mathrm{d}u\right] \mid \sigma(X_s : 0 \leq s \leq t)\right], \tag{2.2.91}$$

where coefficients β_{ij} are defined as $\beta_{ij} = -c_{ij}c_{jK}^{-1}$ with c_{ij} (resp. c_{jK}^{-1}) elements of the matrix $\mathbf{C}$ (resp. $\mathbf{C}^{-1}$). Assuming that d and R are affine functions of diffusions with affine drift and volatility (like in examples in the previous section) we get a class of models whose bond prices are expressed as sums of affine models.

Drawbacks of Intensity Models

Reduced form models lack an endogenous specification of default based on the firm's economic fundaments, i.e. firm's assets and liability structure. These models are not based on any characteristic of the firm's credit quality. The intensity approach is not well adapted to the situation where one wants to model the rise and fall of credit spreads. This can in practise be due in part to changes in the level of investor confidence. Another unsatisfactory feature is that they do not adequately take into account the fact that defaults are typically associated directly with a failure in the delivery of a promised (contractually agreed) cash flow, for example a missed coupon payment.

Chapter 3

Incomplete Information Credit Risk Models

Structural and Reduced form models, treated in the previous chapter, are viewed as competing approaches. However, in recent years, papers by Kusuoka [1999] Duffie & Lando [2001], Çetin et al. [2004], Giesecke [2006], and Guo et al. [2008] have tried to bridge the gap between structural and reduced form models. Jarrow & Protter [2004] provide a survey of the previously mentioned literature with the emphasize on the information set held by the modeler. An introduction to this literature can be also found in Elizalde [2005c]. As shown by the above mentioned authors, a structural model with a predictable default time can be transformed into a reduced form model with a totally inaccessible default time, by introducing *incomplete information*. Relaxing the complete information assumption about the dynamics of the processes which trigger the firm's default they arrive, through different though equivalent routes, to a framework which links both approaches.

Throughout the chapter we work with a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ satisfying the usual conditions. Time T is the final date in the model. The filtration $\mathbb{F}$ represents the evolution of the information. We assume the existence of the probablity measure $\mathbb{Q}$ which is an equivalent risk-neutral measure, i.e. $\mathbb{Q}$ is a martingale measure with respect to the numeraire security with value $B_t = \exp\left[\int_0^t r_s \, \mathrm{d}s\right]$ at time t, where r is an $\mathbb{F}$-progressively measurable process. The no arbitrage assumption guarantees the existence but not the uniqueness of such a probability measure $\mathbb{Q}$.

3.1 Noisy Accounting Report of Assets

The first incomplete information model is introduced by Duffie & Lando [2001]. They retain the first passage time definition of default, but suppose that investors observe the true value of a firm imperfectly. As a model for the firm's assets value process V they assume a geometric Brownian motion

$$V_t = \exp\left[Z_t\right] = \exp\left[Z_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{P}}\right], \tag{3.1.1}$$

where W is a standard Brownian motion, σ is a positive volatility parameter and $\mu \in \mathbb{R}$ represents the expected asset growth rate, like in (2.1.20).

In this model, the firm is operated by equity shareholders who are completely informed about the firm's assets at every time, i.e. they have as information the filtration generated by the firm's asets value process V, $\mathcal{F}_t = \sigma(V_s : s \in [0, t])$. Firm's managers choose the liquidation policy, i.e. an $\mathbb{F}$-stopping time, in order to maximize the value of the equity. The optimal liquidation time is the first time when the assets value process falls below some sufficiently low boundary.

Duffie & Lando point out that in reality bond investors are not fully informed and cannot observe the issuer's asset process V directly. They postulate that the market can only observe the firm's assets value process obscured by adding independent noise and only at selected times t_i, $i = 1, 2, \ldots$ such that $t_i < t_{i+1}$. In particular, they suppose that at each observation time t_i there is a noisy accounting report of assets given by

$$\widehat{V}_{t_i} = \exp\left[N_{t_i}\right] = \exp\left[Z_{t_i} + Y_{t_i}\right], \tag{3.1.2}$$

where Y_{t_i} is the added noise process observed at times t_i and which is normally distributed and independent of Z. This noise generates a market's suprise with respect to default since a firm could nearly be in default and the market is not yet aware of this imminence. An interpretation of this noise is that accounting reports and (or) management press releases either purposefully (e.g. Enron) or inadvertently add extraneous information that obscures the market information about firm's assets value process. The market task is to remove this extraneous noise. This idea was motivated by accounting scandals in the American companies Enron and WorldCom. Both of them had mistakes in their accounting that is by Duffie & Lando modelled by an additional noise.

At each time $t \in [0, \infty)$, the market is also informed whether the equity owners have liquidated the firm. Hence, since one sees $\widehat{V}_t$ and not V_t, the filtration $\mathbb{G}$ available to the market is given by

$$\mathcal{G}_t = \sigma\left(\mathbb{1}_{\{\tau \le s\}}, N_{t_i} : s,\, t_i \in [0, t]\right), \tag{3.1.3}$$

where $N_{t_i} = Z_{t_i} + Y_{t_i}$ and $\tau = \tau(B) = \inf\{t > 0 : V_t \le B\}$ is the first time when the assets value process V falls below the boundary B.

The first objective is to compute the conditional distribution of V_t given $\mathcal{G}_t$. We begin with the case of a single noisy observation at time $t = t_1$. Using properties of the first passage time of a Brownian motion and aplying Bayes' formula and fixing $Z_0 = z_0$, Duffie & Lando derived an explicit formula for the conditional density $g(\cdot|N_t, z_0, t)$ of Z_t given the noisy observation $N_t = Z_t + Y_t$ and $\tau = \inf\{s > 0 : Z_s \leq \underline{v}\} > t$, where $\underline{v} = \ln B$. This explicit formula can be found in the original paper. Given survival to t, this gives us the conditional distribution of assets, since the conditional density of V_t at some level v can be obtained from the conditional density of Z_t at $\ln(v)$.

Using this conditional density, the $\mathcal{G}_t$-conditional probability $p(t,u)$ of survival to some future time $u > t$ can be, for $\tau > t$, expressed as

$$\begin{aligned} p(t,u) &= \mathbb{P}\left(\tau > u | \mathcal{G}_t\right) \\ &= \int_{\underline{v}}^{\infty} \left(1 - \pi\left(u - t, x - \underline{v}\right)\right) g(x|N_t, z_0, t)\, \mathrm{d}x, \end{aligned} \tag{3.1.4}$$

where $\pi(t,x)$ denotes the probability of first passage of a Brownian motion with drift $\mu - \frac{1}{2}\sigma^2$ and volatility σ from an initial condition $x > 0$ to a level below zero before time t. This probability is given by

$$\pi\left(t,x\right) = \Phi\left(\frac{-x - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + \exp\left[-\frac{(2\mu - \sigma^2)x}{\sigma^2}\right]\Phi\left(\frac{-x + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \tag{3.1.5}$$

This result and proof can be found in, e.g. Bielecki & Rutkowski [2002] (Lemma 3.1.2). Duffie & Lando show in a numericall illustration that with pefect information, the conditional probability of default within one year is approximately 2.9%, whereas this conditional probability is approximately 6.7% if accounting assets are reported with a 10% level of accounting noise. As a measure of the degree of accounting noise it is used the standard deviation of Y_t.

The default stopping time τ has an intensity process λ with respect to filtration $\mathcal{G}_t$, if λ is a non-negative progressively measurable process such that

$$\left\{\mathbb{1}_{\{\tau \leq t\}} - \int_0^t \lambda_s\, \mathrm{d}s : t \geq 0\right\} \tag{3.1.6}$$

is a $\mathcal{G}_t$-martingale, provided $\int_0^t \lambda_s\, \mathrm{d}s < \infty$ a.s. for all t.

From above results, at any (ω, t) such that $0 < t < \tau(\omega)$, the $\mathcal{G}_t$-conditional distribution of Z_t has a continuously differentiable conditional density $f(t,\cdot,\omega)$. This density is zero at the boundary $\underline{v}$, and has a derivative $f_x(t,\underline{v},\omega)$ that exists and is non-zero.

Duffie & Lando proved that the process λ defined by

$$\lambda_t(\omega) = \begin{cases} 0, & \text{for } t > \tau \\ \frac{1}{2}\sigma^2 f_x(t,\underline{v},\omega), & \text{for } t \in (0,\tau] \end{cases} \tag{3.1.7}$$

is an $\mathcal{G}_t$-intensity process of τ. Thus the intensity of the default can be expressed in terms of the conditional assets distribution and the default threshold. In structural models, perfect information implies that credit spreads go to zero as maturity tends to zero, regardless of the credit quality of the issuer. However, with imperfect information about the firm's value credit spreads stay bounded away from zero as maturity tends to zero.

Duffie & Lando [2001] also outline extensions of this basic model allowing for inference regarding the distribution of assets from several variables correlated with the asset value or from more than one period of accounting reports. For the latter they do not have explicit solution for the survival probability, but numerical integration can be done. They also discuss how to model re-capitalization when the level of assets reaches some sufficiently high level $\bar{v}$, or decisions by the firm that may be triggered by more than one state variable, such as a stochastic liquidation boundary.

Using the theory of the enlargement of filtrations (see Chapter VI of Protter [2004]), Kusuoka [1999] has proposed a model such that, similar to Duffie & Lando [2001], the information is revealed to the market with an additional noise, but the relevant processes are observed continuously and not discretely. In this approach Kusuoka begins with the modeler's filtration $\mathbb{G} \subset \mathbb{F}$ and a random variable $\tau : \Omega \to \mathbb{R}^+$. Then he expands the filtration so that in the expanded filtration the random variable τ is a stopping time. The filtration expansion is analogous to adding noise to the firm's assets value process.

Duffie & Lando [2001] also characterize the default intensity for cases in which the asset process V satisfies a stochastic differential equation of the form

$$dV_t = \mu(t, V_t)\, dt + \sigma(t, V_t)\, dW_t. \tag{3.1.8}$$

Let us assume an observation scheme as in Kusuoka [1999] for which at time $t < \tau$ there exists a conditional density $f(t, \cdot)$ for V_t. Duffie & Lando then show, under technical assumptions on μ, σ, and f, that the intensity process λ for the first hitting of V at barrier $\underline{v}$ is given by

$$\lambda_t = \frac{1}{2}\sigma^2(\underline{v}, t) f_x(t, \underline{v}), \tag{3.1.9}$$

for $t < \tau$. Here the structural model due to information obscuring is transformed into an intensity-based model.

3.2 Compensators and Pricing Trends

In this section we denote by V_t the issuer's firm value V, and we assume that the issuer defaults when V falls below some random barrier $L < V_0$. We suppose that L is independent of V. Let us assume a random variable $\tau : \Omega \to \mathbb{R}^+$ representing the firm's time of default which is given by

$$\tau = \inf \{t > 0 : V_t \leq L\}. \tag{3.2.10}$$

The related default indicator process N_t generated by τ is given by

$$N_t = \mathbb{1}_{\{\tau \leq t\}}. \tag{3.2.11}$$

This default indicator process is obviously a right-continuous submartingale. The Doob-Meyer decompostion then implies the existence of an unique $\mathbb{F}$-compensator C_t with $C_0 = 0$ for the process N_t such that $N_t - C_t$, called the compensated process, is an $\mathbb{F}$-martingale (see, e.g., Section 4.6 in Bielecki & Rutkowski [2002], or Protter [2004]).

Giesecke [2004b, 2006], and Giesecke & Goldberg [2004a, 2004b] make use of a process Λ_t, referred to as *pricing trend*, related to the $\mathbb{F}$-compensator C_t such that

$$C_t = \Lambda_{t\wedge\tau} = \Lambda_t^{\tau}. \tag{3.2.12}$$

Here $\Lambda_{t\wedge\tau}$ is the stopped process. If the pricing trend is absolutely continuous with respect to Lebesgue measure,i.e.

$$\Lambda_t = \int_0^t \lambda_s \,\mathrm{d}s, \tag{3.2.13}$$

then the density λ is the intensity of arrival of the stopping time τ. The inexistence of an intensity λ_t does not mean that we cannot calculate default probabilities or price of defaultable securities. Using this pricing trend, the conditional default probability can be expressed as

$$\mathbb{Q}\left[\tau \le T|\mathcal{F}_t\right] = 1 - \mathbb{E}_{\mathbb{Q}}\left[\exp\left[\Lambda_t - \Lambda_T\right]|\mathcal{F}_t\right]. \tag{3.2.14}$$

Next we consider the valuation of a defaultable security which pays a bounded amount $X \in \mathcal{F}_T$ at T in the case of default and zero otherwise. If the pricing trend is continuous and we suppose that the process defined by $\mathbb{E}_{\mathbb{Q}}\left[X|\mathcal{F}_t\right]$ is almost surely continuous at τ, then the price of the security at time $t \le T$ is given by

$$\begin{aligned} S_t &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\int_t^T r_s\,\mathrm{d}s\right] X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right] \\ &= \begin{cases} \mathbb{E}_{\mathbb{Q}}\left[X\exp\left[\Lambda_t - \Lambda_T - \int_t^T r_s\,\mathrm{d}s\right]|\mathcal{F}_t\right], & \text{on } \{t<\tau\} \\ 0, & \text{on } \{t \ge \tau\}. \end{cases} \end{aligned} \tag{3.2.15}$$

These two expresions (3.2.14) and (3.2.15) are similar to those observed in an intensity-based model, if (3.2.13) holds.

Let the filtration $\mathbb{F}$ represents the evolution of the information available to investors. Then different specifications of $\mathbb{F}$ will imply different compensator processes and hence different pricing trends. Thus pricing trends are determined by the specification of the information $\mathbb{F}$ and a stopping time τ. Once we have a specification of the default time (3.2.10), assumptions of the information available to the investors regarding the firm's assets value process and default threshold will yield a different pricing trend and thus a different reduced form model. These informational assumptions provide a link between structural and reduced form models.

3.2.1 Complete Information

Firstly, let us assume that the investor can observe both V and L. If we also assume that V has continuous paths then we can find an announcing sequence of stopping times $\tau_n < \tau$ which converges to τ almost surely. Thus default can be anticipated and τ is predictable with respect to the filtration generated by V and L. This is the case of a standard first passage model. In this structural model there is no short-term credit risk that would require compensation. Hence the default indicator process is its own compensator, i.e. $\Lambda_t^\tau = N_t$ and an intensity in the sense of (3.2.13) does not exist.

If we allow for jumps in V as in e.g. Zhou [2001], then assets may continuously diffuse to the default barrier or cross it with a sudden jump. Here the times τ_n just are defined converge to τ only with a probability strictly less than one, hence there is always a chance that the firm jumps below the default barrier, i.e. τ cannot be anticipated anymore. So depending on the state of the world there may or may not be short-term credit risk.

3.2.2 No Information about Default Threshold

Giesecke & Goldberg [2004a] deal with the case of complete information about the asset value, but incomplete information about the default threshold. They claim that the default barrier is random and unobserved, (which they argue is consistent with the recent experiences with Enron, WorldCom, and Tyco) which models the fact investors cannot observe the barrier. Not knowing the default threshold, investors form *a priori* a distribution for its value with distribution function G on $(-\infty, V_0)$. Since the default time depends on this random threshold that cannot be observed by the investor, the default time τ is rendered totally inaccessible. If we also assume that V has continuous paths and G is a continuous function, then it can be proved that the pricing trend is a continuous process and for each $t \geq 0$ it holds that

$$\Lambda_t = -\ln G\left(M_t\right) \quad a.s., \tag{3.2.16}$$

where $M_t = \min_{s \leq t} V_s$ is the historical minimum of the firm assets value at time t. In view of (3.2.13) we need only differentiate the trend to get the intensity. However, under the assumption that G is differentiable, the derivative of $\Lambda(t, \omega)$ with respect to time t is zero for almost all ω. It means that in this specification of the information setting, the pricing trend does not admit an intensity of default. In spite of this fact, there are closed form expressions for probability of default (3.2.14) in some cases.

The reason for the absence of an intensity is that investors *learn* about the default barrier as time goes by. Investors observe the asset process V, hence they are fully informed about the historical minimum of assets. With a time invariant default barrier, if the firm has not defaulted by time t, investors know that the default barrier must lie below M_t at t. Giesecke & Goldberg [2004a] argue that this incomplete information model detects deterioration in the equity market-implied credit quality earlier than the complete information model. This model reacts more quickly to changes in the asset value, since it takes account of the whole history of firm's assets values not just the current value. The rate at which the credit spread converges to zero in case $V_t = M_T$ is much smaller than in the complete information case. As an example of the distribution function of the default barrier D under the pricing probability $\mathbb{Q}$, they give $G(x) = \exp[\alpha x]$, for $x \leq 0$ and some parameter $\alpha \geq 0$.

3.2.3 No Information about Assets

Giesecke [2006] also considers cases where investors cannot observe the firm's assets value process perfectly after the bonds have been issued, and either investor observe the default barrier or not.

Firstly, we consider the case where investors observe the default barrier, but have incomplete information, such as noisy or lagged asset report, for the firm's asset value process after the bonds have been issued at time zero. This is described by the model filtration $\mathbb{G} \subseteq \mathbb{F}$. We denote $H(t, \cdot)$ the $\mathcal{G}_t$-conditional distribution function of the running minimum $M_t = \min_{s \leq t} V_s$. Assume that the variables V_0 and D are $\mathcal{G}_0$-measurable. If the variable $H(t, x) < 1$ a.s. for each $t > 0$ and $x < V_0$, and the process $H(\cdot, x)$ is continuous and monotone for each $x < V_0$, then the default time is totally inaccessible in $\mathbb{F}$. The pricing trend in this case is a continuous process such that for each $t \geq 0$ we have

$$\Lambda_t = -\ln\left(1 - H(t, D)\right) \quad a.s. \tag{3.2.17}$$

If in addition we suppose that for each $x < V_0$ the process $H(\cdot, x)$ is absolutely continuous [1] with a bounded, non-negative, right continuous and $\mathbb{G}$-predictable density process $h(\cdot, x)$, then there exists an intensity and for each $t > 0$ we have

$$\lambda_t = \frac{h(t, D)}{1 - H(t, D)} \quad a.s. \tag{3.2.18}$$

From this expression the intensity derived by Duffie & Lando [2001] can be recovered under the assumptions that default barrier D is equal to the constant $d < V_0$ almost surely and that asset process V follows the stochastic differential equation (3.1.8).

Finally, we consider the case of incomplete information for the firm's assets value after the bonds have been issued, and that the default barrier is unobservable. Let us assume that the variable V_0 is $\mathcal{G}_0$-measurable but the variable D is never $\mathcal{G}_t$-measurable. We also assume that investors form *a priori* a distribution of D with distribution function G on

[1] Non-decreasing process X is called an *absolutely continuous* if the random measure on $\mathbb{R}^+$ associated to X is absolutely continuous w.r.t Lebesgue measure almost surely.

$(-\infty, V_0)$. If the variable $H(t,x) < 1$ a.s. for each $t > 0$ and $x < V_0$ and the process $H(\cdot, x)$ is continuous and monotone for each $x < V_0$, then the default time is totally inaccessible in $\mathbb{F}$. The pricing trend in this case is a continuous process such that for each $t \geq 0$ we have

$$\Lambda_t = -\ln\left(1 - \int_{-\infty}^{V_0} H(t,x)\,\mathrm{d}G(x)\right) \quad a.s. \tag{3.2.19}$$

If in addition we suppose that for each $x < V_0$ the process $H(\cdot, x)$ is absolutely continuous with a bounded, non-negative, right continuous and $\mathbb{G}$-predictable density process $h(\cdot, x)$, then there exists an intensity and for each $t > 0$ we have

$$\lambda_t = \frac{\int_{-\infty}^{V_0} h(t,x)\,\mathrm{d}G(x)}{1 - \int_{-\infty}^{V_0} H(t,x)\,\mathrm{d}G(x)} \quad a.s.\,. \tag{3.2.20}$$

Giesecke [2006] shows that spreads in this case are bounded away from zero for all maturities.

Learning by investors as mentioned in Section 3.2.2 has also important implications in the context of multiple firms. Giesecke [2004b] extends the incomplete information assumption in a structural model to the modeling of default correlation. He provides a structural model of correlated defaults where the firm's default probabilities are connected using a joint distribution for their default thresholds and investors do not have perfect information about neither such thresholds nor their joint distribution. From observing assets and defaults of all firms in the market, investors learn over time about the characteristics of individual firms. Investors form a prior distribution which is updated at any time one of such tresholds is revealed. This happens when one of the firms defaults. In the event of default, the value of the threshold becomes public knowledge. Giesecke [2004b] assumes that investors have complete information about firm's assets.

This framework was extended by Giesecke & Goldberg [2004b] to the case in which investors do not have information about neither firms' default tresholds nor about their assets values. The default correlation in this case is introduced through correlation in the firm value processes. Contagion effects due to counterparty relations are modeled through the dependence of default barrier processes. Investors receive information about the firms asset processes and default thresholds only when they default. Giesecke & Goldberg [2004b] explicitly calculate the pricing trends and the arrival intensity of the n^{th}-to-default in terms of fundamental firm variables.

3.3 Reduced Information

An alternative approach for obtaining a reduced form model from a structural model to the one by Duffie & Lando [2001] is that the market has the similar information as firm's managers but just less of it. Çetin et al. [2004] assume that investors only receive a reduced version of the information set which is available to the firm's management. In their model, the firm's cash flow is the process that triggers default. The firm defaults if the firm's cash flow remains negative for an extended period of time, after exhausting both its lines of credit and easily liquidated assets. The firm's management observes this cash flow process, but the market receives only information about the sign of the cash flows. Thus the market knows only that the firm is in financial distress and the duration of this state. This makes the default time an unpredictable event for the market.

Let us assume a process X which represents the cash balances of the firm normalized by the value of the money market account B_t. This process is observed by the firm's management. Let X follows the stochastic differential equation

$$dX_t = \sigma \, dW_t^{\mathbb{Q}}, \tag{3.3.21}$$

where cash balances are initialized at $X_0 = x > 0$ units of the money market account, $\sigma > 0$, and where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under the probability $\mathbb{Q}$. An interpretation of x should be the target or optimal cash balances for the firm, since the firm gives up attractive investment projects and incurs increased tax liabilities if it holds too many cash, whereas it increases the likelihood of bankruptcy if it has too little cash. The firm makes an effort to maintain cash balances at this optimal level. Çetin et al. [2004] assume, without loss of generality, that $x = 0$ and $\sigma = 1$. Hence cash balances can be positive, zero or negative. The last corresponds to the situations where the firm is in financial distress and the payments owed are not paid. Under the martingale measure cash balances have no drift term.

Let $\mathcal{Z}$ denote the times when the firm's cash balances hit zero, i.e.

$$\mathcal{Z} := \{ t \in [0, T] : X_t = 0 \} . \tag{3.3.22}$$

When a firm has zero or negative cash balances, debt payments can only be made by accessing bank lines of credit or liquidating the firm's assets. The firm can exist with negative cash balances for only a limited period of time. We also define the random time $l(t)$ that represents the last time before t at which cash balances hit zero, i.e.

$$l(t) := \sup \{ s \leq t : \, X_s = 0 \} . \tag{3.3.23}$$

Let τ_α, for some parameter of the default process $\alpha \in \mathbb{R}^+$, be the random time that measures the onset of a possible default situation for the firm, i.e

$$\tau_\alpha := \inf \left\{ t > 0 : \, t - l(t) \geq \frac{\alpha^2}{2}, \text{ where } X_s < 0 \text{ for } s \in (l(t_-), t) \right\} . \tag{3.3.24}$$

Thus τ_α is the first time at which the firm's cash balances have been negative for at least $\frac{\alpha^2}{2}$ units of time. The constant α is a parameter of the default process. It could be estimated from market data. The default time is defined as the first time after τ_α at which the cash balances doubles in magnitude., i.e.

$$\tau := \inf\left\{t > \tau_\alpha :\ X_t = 2X_{\tau_\alpha}\right\}. \tag{3.3.25}$$

Çetin et al. [2004] point out that the doubling in absolute magnitude is only for analytical convenience and it has no economic content. The intuition behind (3.3.25) is that after being below zero for a long time, the firm exhausts all its slacks (e.g. lines of credit) to meet its debt payments. The firm has no slacks left if it ever hits $2X_{\tau_\alpha}$ afterwards, hence it defaults.

The market does not observe the firm's cash balances. Until the random time τ_α, the market only knows when the firm has positive cash balances or when it has negative or zero cash balances. After time τ_α, the market knows whether cash balances are below or above the default threshold $2X_{\tau_\alpha}$. Let us introduce a new process

$$Y_t = \begin{cases} X_t, & \text{for } t < \tau_\alpha, \\ 2X_{\tau_\alpha} - X_t, & \text{for } t \geq \tau_\alpha. \end{cases} \tag{3.3.26}$$

The process Y is also an $\mathbb{F}$-Brownian motion and the definition of the default time (3.3.25) , in terms of this process, can be rewtitten as

$$\tau = \inf\left\{t \geq \tau_\alpha :\ Y_t = 0\right\}. \tag{3.3.27}$$

Let us define

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{for } x \leq 0, \\ 1, & \text{for } x > 0. \end{cases} \tag{3.3.28}$$

Let $\mathbb{G} = \{\mathcal{G}_t :\ t \in [0,T]\}$ denote the $\mathbb{Q}$-complete and right-continuous version of the filtration $\widetilde{\mathbb{G}} = \left\{\widetilde{\mathcal{G}}_t :\ t \in [0,T]\right\}$, where $\widetilde{\mathcal{G}}_t := \sigma\left\{\operatorname{sgn}(Y_s) :\ s \leq t\right\}$. This filtration $\mathbb{G}$ represents the information that is available to market.

Çetin et al. [2004] define

$$M_t := \mathbb{E}_{\mathbb{Q}}\left[\frac{2}{\sqrt{\pi}} Y_t | \mathcal{G}_t\right], \tag{3.3.29}$$

and point out that M_t is Azéma's martingale on the filtered probability space $\left(\Omega, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}\right)$. This martingale M_t satisfies the following claims: quadratic variation of M_t satisfies the structure equation

$$d[M,M]_t = dt - M_{t-}\,dM_t, \tag{3.3.30}$$

Azéma's martingale is a strong Markov process, and M_t can be expressed by

$$M_t = \operatorname{sgn}(Y_t)\sqrt{2}\sqrt{t - \tilde{l}_t}, \tag{3.3.31}$$

where $\tilde{l}_t := \sup\{s \leq t : Y_s = 0\}$. These results on Azema's martingale can be found in Sections IV.5 (p.203-204) and IV.8 of Protter [2004]. The default time τ now can be equivalently written as

$$\tau = \inf\{t > 0 : \Delta M_t \geq \alpha\}. \tag{3.3.32}$$

Therefore, τ is a jump time of Azema's martingale, hence it is totally inaccessible with respect to the filtration $\mathbb{G}$ (see Protter [2004], Theorem 59*(iv)*). The random time can be rewritten as $\tau_\alpha = \inf\{t > 0 : M_{t-} \leq -\alpha\}$. For ΔM_t we have $\Delta M_t = -M_{t-}\mathbb{1}_{\{M_{t-}\neq M_t\}}$, hence $\tau_\alpha \leq \tau$ almost surely. However, τ_α is a predictable stopping time which implies $\mathbb{Q}(\tau = \tau_\alpha) = 0$, hence $\tau_\alpha < \tau$ almost surely. If we define $N_t = \mathbb{1}_{\{\tau \leq t\}}$, then by the Doob-meyer decomposition there exists a continuous, natural increasing process A such that $N - A$ is a $\mathbb{G}$-martingale that has only one jump at τ of the unit size. Çetin et al. [2004] proved that τ has a $\mathbb{G}$-intensity, i.e. process A in the Doob-Meyer decomposition is of the form $A_t = \int_0^{t\wedge\tau} \lambda_s \, \mathrm{d}s$. Furthermore,

$$\lambda_t(\omega) = \begin{cases} 0, & \text{for } t > \tau \\ \frac{1}{2\left(t-\tilde{l}_{t-}\right)}\mathbb{1}_{\{t>\tau_\alpha\}}, & \text{for } t \in [0,\tau]. \end{cases} \tag{3.3.33}$$

Thus under the market's information set, default is given by a totally inaccessible stopping time, and we obtain a reduced form model from the market's perspective. The firm's default intensity is zero until time τ_α, then, after this time, intensity decreases with the length of time that the firm remains in financial distress $(t - \tilde{l}_{t-})$. It can be interpreted as follows: the longer the firm survives in the state of financial distress, the less likely it is to default. With the intensity (3.3.33), the market can value credit linked securities as in the reduced form approach. Valuation of a defaultable zero-coupon bond can be found in the original paper by Çetin et al. [2004].

3.4 Delayed Filtration

Collin-Dufresne et al. [2003], following Duffie & Lando [2001], assume that investors do not observe the actual current firm value. Instead, they assume that investors observe a signal which corresponds to some lagged firm's assets value procces $\widehat{V}_t = V_{t-l}$ where the lag l is not known perfectly. For simplicity Collin-Dufresne et al. assume that l can take on only one of two values, l^H or l^L, where $l^H > l^L$, i.e. firms are either in the high-delay state or the low-delay state. The longer the delay the less is known about how close the current cash flows are to the default boundary. For the firm value process V they use a geometric Brownian motion. Guo et al. [2008] attempt to unify notions of noisy and lagged information into the same framework. They define the notion of a *delayed filtration* for both discrete and continuous case. A continuously delayed filtration allows information to flow in continuously, albeit following a time clock slower than ordinary one. A discretely delayed filtration does not allow new information to flow in between two consecutive observation times. Guo et al. [2008] introduce following two definitions of the delayed filtration.

Definition 3.4.1. (Continuously delayed filtration) Let $\mathbb{H} = \{\mathcal{H}_t,\, t \geq 0\}$ be a filtration that satisfies the usual conditions. Let $\{\alpha_t, t \geq 0\}$ denote an increasing, right-continuous process such that $\alpha_0 = 0$ and α_t is an $\mathbb{H}$-stopping time for every $t \geq 0$. Suppose that $\alpha_t(\omega) \leq t$ a.s. for all $t \geq 0$. The time-changed filtration $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\} = \{\mathcal{H}_{\alpha_t},\, t \geq 0\}$ is then called a *continuously delayed filtration* of $\mathbb{H}$.

Definition 3.4.2. (Discretely delayed filtration) Let $\mathbb{H} = \{\mathcal{H}_t,\, t \geq 0\}$ be a filtration that satisfies the usual conditions. Suppose there are K sequences of $\mathbb{H}$-stopping times $\left\{T_n^k\right\}_{n \geq 0}$, $1 \leq k \leq K$ such that $T_0^k = 0$, $T_n^k \uparrow \infty$ and for all $n \geq 0$, $T_n^k < \infty$ it holds that $T_n^k < T_{n+1}^k$ (i.e each sequence is strictly increasing). Suppose also that $\{\mathcal{G}_{i_1,\ldots,i_K \in \mathbb{N}}\}$ is a family of sub-σ-field of $\mathcal{H}_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{H}_t)$ such that

(i) $\mathcal{G}_{i_1,\ldots,i_K} \subset \mathcal{G}_{j_1,\ldots,j_K}$, if $i_1 \leq j_1, \ldots, i_K \leq j_K$,

(ii) $\mathcal{G}_{i_1,\ldots,i_K} \subset \mathcal{H}_{T_{i_1}^1 \vee \ldots \vee T_{i_K}^K}$, [2]

(iii) For any k, T_n^k is $\mathcal{G}_{i_1,\ldots,i_K}$-measurable whenever $n \leq i_k$. Define

$$\mathcal{F}_t^0 = \bigcup_{i_1 \cdots i_K} \left(\mathcal{G}_{i_1,\ldots,i_K} \cap \left\{ T_{i_k}^k \leq t \leq T_{i_k+1}^k \,:\, 1 \leq k \leq K \right\} \right),$$

and

$$\mathcal{F}_t = \bigcup_{i_1 \cdots i_K} \left(\left(\mathcal{G}_{i_1,\ldots,i_K} \vee \sigma(\mathcal{N}) \right) \cap \left\{ T_{i_k}^k \leq t \leq T_{i_k+1}^k \,:\, 1 \leq k \leq K \right\} \right),$$

where $\mathcal{N}$ is the collection of all negligible sets. Then $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\}$ is a filtration that satisfies the usual conditions, and is called a *discretely delayed filtration* of $\mathbb{H}$.

Under this mathematical framework Guo et al. [2008] generalize both Duffie & Lando [2001] and Collin-Dufresne et al. [2003] to show how a delayed filtration generate the default intensity process for any Markov model, with or without jumps. For the closed-form formulas we refer to original paper by Guo et al. [2008] and for an alternative methodology to obtain closed-form formulas of intensity see Guo & Zeng [2008].

[2] Here $a \vee b = \max\{a, b\}$

Chapter 4

Information-Based Pricing

Another line of research is concerned with how to model the market filtration by use of the concept of partial information. This is constrasted with what is the more common modelling approach in mathematical finance where the market filtration is simply given. Here cash flows, defining a credit-risky asset, are modelled by random variables. The objective is to develop an economic model for the partial information about the value of the cash flows at a later time. This approach to credit risk modelling is followed in Macrina [2006], Brody et al. [2007, 2008a, 2008b], and Rutkowski & Yu [2007]. In this approach there is no attempt to bridge a gap between structural and intensity-based credit risk models, and authors avoid the use of inaccessible stopping times. Brody et al. [2007] present an alternative reduced form model, based on the amount and precision of the information received by market participants about the firm's credit risk. Market participants have only noisy information about forthcoming cash flows. The rate at which true information is provided for each such cash flow is a parameter of the model. This model does not require the use of default intensities. It relies on market prices of defaultable instruments as the only source of information about the firms' credit risk. Default events are associated directly with the failure of obligors to make contractually agreed payments.

Throughout this chapter we work with a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\mathbb{F} = \{\mathcal{F}_t,\, 0 \leq t < \infty\}$, where $\mathbb{Q}$ is the risk-neutral probability measure and $\mathbb{F}$ represents the flow of information to market participants and it will be constructed explicitly. All asset price processes and other information-providing processes accessible to market participants will be adapted to $\mathbb{F}$. The real probability measure does not directly enter into the investigation. We assume absence of arbitrage, hence there exists a risk-neutral measure. Brody et al. also assume the existence of a pricing kernel (see Cochrane [2005]), hence this measure is unique (i.e. the market has chosen a fixed risk-neutral measure $\mathbb{Q}$ for the pricing of all assets and derivatives), even though the market may be be incomplete (i.e. derivative hedging is not always possible). They also assume that the default-free system of interest rates is deterministic. This assumption is relaxed in the paper by Rutkowski & Yu [2007], using a forward measure technique (Musiela & Rutkowski [2005]). If we have a deterministic system of interest rates, then the absence of arbitrage implies that the corresponding default-free system of discount functions, denoted $\{P_{tT}\}_{0\le t\le T<\infty}$, can be written in the form

$$P_{tT} = \frac{P_{0T}}{P_{0t}}, \tag{4.0.1}$$

for $t \le T$, where $\{P_{0t}\}_{0\le t<\infty}$ is the initial discount function. This function is assumed to be differentiable and strictly decreasing, and also that it satisfies $P_{0t} \in (0,1]$ and $\lim_{t\to\infty} P_{0t} = 0$. It is also supposed to be part of the initial data of the model. The expression (4.0.1) is the parallel of the expression (2.0.3) in Chapter 2.

4.1 Cash Flows and Market Factors

We start with the general situation where the asset pays multiple dividends. Each financial asset is defined by a series of random cash flows. We consider an asset that is defined by a set of random cash flows $\{D_{T_k}\}_{k=1,\dots,n}$ which are paid at the pre-specified dates $\{T_k\}_{k=1,\dots,n}$. For simplicity it is assumed that n is finite. Possession of the asset at time t entitles the bearer to the cash flows occuring at times $T_k > t$.

For each date T_k we introduce a set of independent random variables $\left\{X^{\alpha}_{T_k}\right\}^{\alpha=1,\dots,m_k}_{k=1,\dots,n}$ which we call *market factors*. For each value of α it is assumed that the market factor $X^{\alpha}_{T_k}$ is $\mathcal{F}_{T_k}$-measurable, where $\mathbb{F}$ is the market filtration. The market factors $\{X^{\alpha}_{T_j}\}_{j\le k}$ for each value of k represent the independent elements that determine the cash flow occuring at the future time T_k. Thus for each date T_k we introduce a *cash flow function* (or structure function) Δ_{T_k} of $\sum_{i=1}^{k} m_i$ variables such that

$$D_{T_k} = \Delta_{T_k}\left(X^{\alpha}_{T_1},\, X^{\alpha}_{T_2},\, \dots,\, X^{\alpha}_{T_k}\right). \tag{4.1.2}$$

Brody et al. say that for any given asset it is the job of the financial analyst (or actuary) to determine the relevant independent market factors, their *a priori* probabilities, and the form of the cash flow function Δ_{T_k} for each cash flow.

The price of the asset that generates the cash flows $\{D_{T_k}\}_{k=1,\dots,n}$ is given by the risk-neutral valuation formula

$$S_t = \sum_{k=1}^{n} \mathbb{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}_{\mathbb{Q}}\left[D_{T_k} | \mathcal{F}_t\right]. \tag{4.1.3}$$

Example 4.1.1. Defaultable discount bond with random recovery: We consider a defaultable discount bond which at maturity date T pays out the terminal value X_T, i.e. an asset that provides a single cash flow $D_T = X_T$ at time T.

Brody et al. [2007] assume the situation where this defaultable discount bond has a discrete probability spectrum $X_T = x_i$, $i = 0, 1, \ldots, n$ with *a priori* probability $q_i := \mathbb{Q}(X_T = x_i)$, and for convenience they assume that $x_n > x_{n-1} > \cdots > x_1 > x_0$. In this situation we think of $X_T = x_n$ as the case of no default and the other cases as various possible degrees of recovery. If we assume the case $n = 1$, we obtain a simple defaultable *binary* discount bond that matures at time T to pay a principal payment x_1 if there is no default and x_0 in the event of default. Further, if $x_1 = 1$ and $x_0 = 0$ we get a *digital* defaultable discount bond.
Rutkowski & Yu [2007] relax the assumption that X_T has a discrete probability distribution and they postulate that X_T takes values in the interval $[0, 1]$, i.e. the claim X_T can be interpreted as the payoff of a defaultable bond with maturity T and the face value 1.

The price process $\{B_{tT}\}_{0 \le t \le T}$ of the defaultable bond with payoff X_T, using (4.1.3), is given by

$$B_{tT} = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[X_T | \mathcal{F}_t\right], \tag{4.1.4}$$

for all $t \in [0, T]$.

Example 4.1.2. Accumulation process: In finance and insurance there are many important problems that involve the analysis of accumulation processes, i.e. cumulative gains or losses. Let $[0, T]$ be a fixed accounting period, where time zero denotes the present. Brody et al. [2008b] assume contracts for which the payoff, at time T, is given by a random cash flow X_T which is given by the terminal value of a process of accumulation, or, more generally, by a function of X_T. The random variable X_T represents the total accumulation of an irreversible gain process. For instance, in the case of insurance contracts the random variable X_T represents the totality of the payments made at T in settlement of claims arising over the accounting period $[0, T]$. The problem facing the insurance firm is the valuation of the random cash flow. Traditionally, insurance claims reserving has been concerned to a large extent with estimation and prediction. Brody et al. [2008b] instead emphasise the role of valuation and separate the issue of how to manage the reserves from how these should be calculated. The value process $\{S_t\}_{0 \le t \le T}$ of the contract that pays X_T at T is given by

$$S_t = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[X_T | \mathcal{F}_t\right]. \tag{4.1.5}$$

One can interpret S_t as the value of the reserve that needs to be maintained by the insurance firm at t in order to ensure that X_T will be payable at T. Equivalently, it represents the cost of commuting the claim, i.e. one can view S_t as the amount that would have to be paid at t for the insurance firm to relieve itself of the obligation to pay X_T. Likewise, the time t cost C_{tT} of a simple stop-loss reinsurance contract that pays out $(X_T - K)^+$ at time T for some fixed threshold K is given by

$$C_{tT} = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[(X_T - K)^+ | \mathcal{F}_t\right]. \tag{4.1.6}$$

Another example of an accumulation process originate in credit risk management. If we consider a large credit portfolio, X_T may denote the accumulated losses due to defaults over the interval $[0, T]$. For example, at time zero a credit-card firm has a large number of customers, each with an outstanding balance payable in the accounting period. If a customer does not pay the balance by the required date, they will be deemed to be in default, and a loss will be registered. The random variable X_T will denote the totality of such losses. Brody et al. [2008b] assume that once a customer is in default, no further payments are made by that customer (or that any such payments are registered in a separate account). The problem facing the firm is to determine the capital it needs to set aside to cover such losses, and in particular what premium to charge over the base interest rate, to ensure that funds will be in hand to cover the default losses.

There are also many other examples. For instance, X_T may represent the total number of accidents or fatalities of a certain type during the period from 0 to T, or the total number of sales achieved, or the total amount of water, or gas, or electricity consumed, or total GDP, or the total amount of emissions emitted, or the total rainfall, etc.

In the subsequent examples below, X_{T_j}, $j = 1, 2, ..., k$ denote independent binary random variables taking the value 0 in case of default or 1 if there is no default, with *a priori* probabilities $p_0^{(j)}$ (default) and $p_1^{(j)}$ (no default), and the constants $\mathbf{c}$ and $\mathbf{p}$ denote the coupon and the principal.

Example 4.1.3. Simple defaultable coupon bond: We consider a bond with two payments remaining. A coupon D_{T_1} at time T_1, a coupon plus the principal totalling D_{T_2} at time T_2 and no recovery on default. The default can occur at any of the coupon dates. If there is a default at T_1 then no further payment is made at T_2. On the other hand, if the coupon at time T_1 is paid, default may still ocuurs at time T_2. Hence for this defaultable coupon bond we have the following cash flows

$$D_{T_1} = \mathbf{c} X_{T_1} , \tag{4.1.7}$$

$$D_{T_2} = (\mathbf{c} + \mathbf{p}) X_{T_1} X_{T_2} . \tag{4.1.8}$$

Example 4.1.4. Defaultable coupon bond with recovery: We can extend the previous two coupons in Example 4.1.3 by considering that there is partial recovery in the case of default. For instance, by saying that in the case of default on the first coupon we have the recovery rate R_1 as a percentage of coupon plus principal $(\mathbf{c} + \mathbf{p})$, whereas in the case of default on the final payment at time T_2 the recovery rate is R_2. Hence the associated cash flows can be written in the form

$$D_{T_1} = \mathbf{c} X_{T_1} + R_1 (\mathbf{c} + \mathbf{p})(1 - X_{T_1}) , \tag{4.1.9}$$

$$D_{T_2} = (\mathbf{c} + \mathbf{p}) X_{T_1} X_{T_2} + R_2 (\mathbf{c} + \mathbf{p}) X_{T_1} (1 - X_{T_2}) . \tag{4.1.10}$$

Example 4.1.5. Defaultable n-coupon bond with multiple recovery levels: In this example we consider the case of n outstanding payments at the pre-specified payment dates T_k, where $k = 1, \ldots, n$. In the event of default at the date T_k we assume the recovery payment $R_k(\mathbf{c}+\mathbf{p})$, where, as in the Example 4.1.4 above, R_k is a percentage of the owed coupon and principal payment. Here the associated cash flows are given by

$$D_{T_k} = \mathbf{c} \prod_{j=1}^{k} X_{T_j} + R_k(\mathbf{c}+\mathbf{p}) \prod_{j=1}^{k-1} X_{T_j}(1 - X_{T_k}), \qquad \text{for } k = 1, , \ldots, n-1\,, \quad (4.1.11)$$

$$D_{T_n} = (\mathbf{c}+\mathbf{p}) \prod_{j=1}^{n} X_{T_j} + R_n(\mathbf{c}+\mathbf{p}) \prod_{j=1}^{n-1} X_{T_j}(1 - X_{T_n}), \quad \text{for } k = n. \quad (4.1.12)$$

Example 4.1.6. Credit default swap: We consider a CDS written on a defaultable coupon bond. The seller of protection receives a series of premium payments, each of the amount $\mathbf{g}$, at some pre-specified dates. The payments continue until a credit event occurs, which here is modeled as the failure of a coupon payment in the reference bond. In the event of default a lump sum is paid to the buyer of protection at the default time equal, for instance, to the principal minus the effective effective recovery value of the reference bond at that time. If we take as the reference bond the one from Example 4.1.4 with two outstanding coupon payments, and for simplicity we assume that the default-swap premium payments are made immediately after the bond coupon dates, then this CDS has the following cash flows

$$D_{T_1} = \mathbf{g}X_{T_1} - (\mathbf{p} - R_1(\mathbf{c}+\mathbf{p}))\,(1 - X_{T_1})\,, \quad (4.1.13)$$

$$D_{T_2} = \mathbf{g}X_{T_1}X_{T_2} - (\mathbf{p} - R_2(\mathbf{c}+\mathbf{p}))\,X_{T_1}(1 - X_{T_2})\,. \quad (4.1.14)$$

The pricing issues of the credit linked securities from above examples will be treated below in the sections 4.3 and 4.4. Firstly we shall construct the filtration observed by market participants.

4.2 Market Filtration and Market Information Processes

The new conceptual tool underpinning the information-based approach by Brody et al. is the so-called *market information process*, denoted by ξ. For each market factor $X^{\alpha}_{T_k}$, $k = 1, \ldots, n$, $\alpha = 1, \ldots, m_k$ we introduce an information process $\left\{\xi^{\alpha}_{tT_k}\right\}_{0\leq t\leq T_k}$. The information flow available to investors is modelled explicitly and is assumed to be the natural filtration of one or more independendent market information processes. Thus

$$\mathcal{F}_t = \sigma\left(\left\{\xi^{\alpha}_{sT_k}\right\}^{\alpha=1,\ldots,m_k}_{k=1,\ldots,n} : 0 \leq s \leq t\right). \quad (4.2.15)$$

The idea is that the information process reveals the value of $X^{\alpha}_{T_k}$ at time T_k. The information processes $\left\{\xi^{\alpha}_{tT_k}\right\}$ are clearly $\{\mathcal{F}_t\}$-adapted. In the following we will consider two types of information processes: Brownian bridge information and Gamma bridge information.

4.2.1 Brownian Bridge Information Processes

For $t < T_k$, we assume that all information available to market participants about the market factor $X^{\alpha}_{T_k}$ is contained in the information process $\left\{\xi^{\alpha}_{tT_k}\right\}_{0\leq t\leq T_k}$ defined by

$$\xi^{\alpha}_{tT_k} = X^{\alpha}_{T_k}\sigma^{\alpha}_{T_k}t + \beta^{\alpha}_{tT_k}, \tag{4.2.16}$$

where $\sigma^{\alpha}_{T_k}$ is a parameter, and the process $\left\{\beta^{\alpha}_{tT_k}\right\}_{0\leq t\leq T_k}$ is a standard Brownian bridge over the interval $[0, T_k]$. Thus $\left\{\beta^{\alpha}_{tT_k}\right\}_{0\leq t\leq T_k}$ is a Gaussian process defined on $(\Omega, \mathcal{F}, \mathbb{Q})$ satisfying $\beta_{0T_k} = 0$ and $\beta_{T_kT_k} = 0$. It is known (see Karatzas & Shreve [1988], p.358-360) that the mean and the covariance of the Brownian bridge are given by

$$\mathbb{E}_{\mathbb{Q}}\left[\beta_{tT_k}\right] = 0, \tag{4.2.17}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\beta_{sT_k}\beta_{tT_k}\right] = \frac{s\left(T_k - t\right)}{T_k}, \tag{4.2.18}$$

for all s,t satisfying $0 \leq s \leq t \leq T_k$. We assume that market factors $X^{\alpha}_{T_k}$ and the Brownian bridge processes are all independent.

For the explanation of this form of market information processes, we consider for the moment the case in which the asset entails a single payment D_T at time T. Brody et al. make the assumption that some partial information regarding the value of the cash flow D_T is available at earlier times. This information is in general imperfect. The model for such imperfect information is of a simple type that allows for a great deal of analytic tractability. In this model, information about the true value of the cash flow steadily increases, while at the same time the obscuring factors increase in magnitude for the first half of its trajectory, and then eventually die away at the payment day since investors have a perfect information about the value of D_T at time T. Formally, in this case of a single distribution occuring at time T, we assume that the following $\mathbb{F}$-adapted market information process ξ is accessible to market participants

$$\xi_t = D_T\sigma t + \beta_{tT}. \tag{4.2.19}$$

The first part of this process $D_T\sigma t$ contains the true information about the cash flow and grows in magnitude as time t increases. It is assumed that initially all available market information is taken into account in the determination of the price. In the case of a defaultable discount bond, the relevant information is embodied in the *a priori* probabilities (see Example 4.1.1). Since the market filtration $\mathbb{F}$ is generated by the market information process, see (4.2.15)), the cash flow D_T is $\mathcal{F}_T$-measurable but it is not $\mathcal{F}_t$-measurable for $t < T$, i.e. the true value of D_T is not fully accessible until time T. The Brownian bridge process $\{\beta_{tT}\}_{0\le t\le T}$ is assumed to be independent of D_T and thus represents pure noise. The market participants do not have direct access to the bridge process β_{tT}, i.e. β_{tT} is not assumed to be $\mathbb{F}$-adapted. This reflects the fact that until the cash flow is paid the market participants cannot distinguish the true information from the noise in the market. We can thus think of β_{tT} as representing speculation, rumour, overreaction, gossip, misrepresentation, and general disinformation often occuring in connection with financial activity. The parameter σ represents the rate at which information about true value of D_T is revealed as time progresses. Thus low σ indicates that the true vaule of D_T is effectively hidden until very near the payment date of the asset. On the other hand, if σ is high, then D_T is revealed quickly.

More generally, the rate at which the true value of D_T is revealed is not constant. In that case we will have

$$\xi_t = D_T \int_0^t \sigma_u \,\mathrm{d}u + \beta_{tT}, \tag{4.2.20}$$

where σ_t is non-negative and deterministic function. This can be interpreted, for instance, that there is more activity in a market during the day than at night. This consideration is important for short-term investments. Alternatively, more information concerning the future of the firm may be available on the day when the annual report of a firm is going to be published than normal.

When σ is constant, the important feature of the market information process (4.2.19) is that $\{\xi_t\}_{0\le t\le T}$ has the Markov property, i.e.

$$\mathbb{Q}\left(\xi_t \le x | \mathcal{F}_s^{\xi}\right) = \mathbb{Q}\left(\xi_t \le x | \xi_s\right), \tag{4.2.21}$$

for all $x \in \mathbb{R}$, and for all s, t such that $0 \le s \le t \le T$. The proof of this property can be found in Brody et al. [2007] (p.238) or in Rutkowski & Yu [2007]. Along with the fact that D_T is $\mathcal{F}_T$-measurable we thus find that

$$\mathbb{E}_{\mathbb{Q}}\left[D_T | \mathcal{F}_t^{\xi}\right] = \mathbb{E}_{\mathbb{Q}}\left[D_T | \xi_t\right], \tag{4.2.22}$$

which simplifies calculations.

In the case of random interest rate, it is needed to introduce another source of randomness. Let us introduce a Brownian motion $\widetilde{W}^{\mathbb{Q}}$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ independent of ξ and D_T. Then ξ has the Markov property with respect to the joint filtration $\mathbb{G} = \mathbb{F}^{\xi} \vee \mathbb{F}^{\widetilde{W}^{\mathbb{Q}}}$, i.e.

$$\mathbb{Q}\left(\xi_t \le x | \mathcal{G}_s\right) = \mathbb{Q}\left(\xi_t \le x | \xi_s\right), \tag{4.2.23}$$

for all $x \in \mathbb{R}$, and for all s, t such that $0 \le s \le t \le T$.

In the general case of market infomation processes (4.2.16), the Brownian bridges $\beta^{\alpha}_{tT_k}$ represent market noise and only the terms $X^{\alpha}_{T_k}\sigma^{\alpha}_{T_k} t$ contain true market information. The true value of the market factors $X^{\alpha}_{T_k}$ is revealed at time T_k. The parameter $\sigma^{\alpha}_{T_k}$ can be interpreted as the information flow rate for the market factor $X^{\alpha}_{T_k}$. Clearly, the market factor $X^{\alpha}_{T_k}$ is $\mathcal{F}_{T_k}$-measurable. The information process $\xi^{\alpha}_{tT_k}$ is $\mathcal{F}_t$-adapted, but this is not the case for the Brownian bridges $\beta^{\alpha}_{tT_k}$. By calculation it can be also shown that the information processes $\xi^{\alpha}_{tT_k}$ satisfy the Markov property.

Price Process of Single Dividend Paying Asset

Firstly, we shall focus on the derivation of the conditional expectation (4.2.22) if the random variable D_T represents continuously distributed asset payoff. This conditional expectation can take the following form

$$\mathbb{E}_{\mathbb{Q}}\left[D_T|\xi_t\right] = \int_0^{\infty} x f_t(x)\,\mathrm{d}x, \tag{4.2.24}$$

where $f_t(x)$ is the conditional probability density for the random variable D_T given ξ_t, i.e.

$$f_t(x) := f_{D_T|\xi_t}\left(x|\xi_t\right) = \frac{\partial}{\partial x}\mathbb{Q}\left(D_T \le x|\xi_t\right). \tag{4.2.25}$$

It is implicitly assumed that appropriate technical conditions on the distribution of D_T, that will be sufficient to ensure the existence of the expressions under consideration, are satisfied. In what follows, $q(x)$ will denote the *a priori* probability density function of D_T. This density is assumed to be known as an initial condition, and $f_{\xi_t|D_T}(\cdot|x)$ denotes the conditional probability density function of the random variable ξ_t given that $D_T = x$. Properties of market information process, in particular, that the random variable β_{tT} has a normal distribution with mean zero and variance $\frac{t(T-t)}{T}$, imply that the conditional probability density function for ξ_t, evaluated at z, is given by

$$f_{\xi_t|D_T}\left(z|x\right) = \sqrt{\frac{T}{2\pi(T-t)}}\exp\left[-\frac{T\left(z - x\sigma t\right)^2}{2t(T-t)}\right]. \tag{4.2.26}$$

Using a form of Bayes' formula, we can express the *a posteriori* probability density function of D_T given that $\xi_t = z$ as

$$\begin{aligned} f_{D_T|\xi_t}\left(x|z\right) &= \frac{q(x) f_{\xi_t|D_T}\left(z|x\right)}{\int_0^{\infty} q(x) f_{\xi_t|D_T}\left(z|x\right)\,\mathrm{d}x} \\ &= \frac{q(x)\exp\left[\frac{T}{T-t}\left(x\sigma z - \frac{1}{2}x^2\sigma^2 t\right)\right]}{\int_0^{\infty} q(x)\exp\left[\frac{T}{T-t}\left(x\sigma z - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}x}. \end{aligned} \tag{4.2.27}$$

This result and (4.2.25) imply that the conditional expectation (4.2.22) can be expressed as

$$\hat{D}_t := \mathbb{E}_{\mathbb{Q}}\left[D_T|\xi_t\right] = D\left(t, \xi_t\right), \tag{4.2.28}$$

where $D : [0,T] \times \mathbb{R} \rightarrow (0,\infty)$ is a jointly continuous function given by the formula

$$D(t,\xi_t) = \frac{\int_0^\infty xq(x)\exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}x}{\int_0^\infty q(x)\exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}x}. \tag{4.2.29}$$

The process $\hat{D}_t$ is $\mathbb{F}^\xi$-adapted .

If the random variable D_T has an arbitrary cumulative distribution function F then (4.2.29) can be reformulated as

$$D(t,\xi_t) = \frac{\int_0^\infty x\exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}F(x)}{\int_0^\infty \exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}F(x)}. \tag{4.2.30}$$

Furthermore, by (4.1.3), the information-based price process $\{S_t\}_{0\leq t\leq T}$ of a limited-liability asset that pays a single dividend D_T at time T with *a priori* distribution $\mathbb{Q}(D_T \leq u) = \int_0^u q(x)\,\mathrm{d}x$ is given by

$$S_t = \mathbb{1}_{\{t<T\}} P_{tT} \frac{\int_0^\infty xq(x)\exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}x}{\int_0^\infty q(x)\exp\left[\frac{T}{T-t}\left(x\sigma\xi_t - \frac{1}{2}x^2\sigma^2 t\right)\right]\,\mathrm{d}x}. \tag{4.2.31}$$

Asset Price Dynamics

In order to obtain the dynamics of the above asset we need to find the stochastic differential equation to which the process S_t is the solution.

Firstly, we shall investigate the dynamics of the market information process (4.2.19). We know (Karatzas & Shreve [1988], p.358) that a standard Brownian bridge $\{\beta_{tT}\}_{0\leq t\leq T}$ satisfies the stochastic differential equation

$$d\beta_{tT} = -\frac{\beta_{tT}}{T-t}\,dt + d\overline{W}_t^{\mathbb{Q}}, \quad \text{for } 0 \leq t < T, \tag{4.2.32}$$

where $\overline{W}^{\mathbb{Q}}$ is a standard Brownian motion under the measure $\mathbb{Q}$ with respect to its natural filtration $\overline{\mathbb{F}} = \mathbb{F}^{\overline{W}}$. Then for the dynamics of $\xi_t = D_T\sigma t + \beta_{tT}$ we have

$$\begin{aligned} d\xi_t &= D_T\sigma\,dt + d\beta_{tT} \\ &= D_T\sigma\,dt - \frac{\beta_{tT}}{T-t}\,dt + d\overline{W}_t^{\mathbb{Q}} \\ &= \left(D_T\sigma - \frac{\xi_t - D_T\sigma t}{T-t}\right)dt + d\overline{W}_t^{\mathbb{Q}} \\ &= \frac{D_T\sigma T - \xi_t}{T-t}\,dt + d\overline{W}_t^{\mathbb{Q}}. \end{aligned} \tag{4.2.33}$$

Thus the market information process $\{\xi_t\}_{0\leq t\leq T}$ is a continuous semimartingale and for its quadratic variation we have $\langle\xi\rangle_t = \langle\overline{W}^{\mathbb{Q}}\rangle_t = t$ for all $t \in [0,T]$. From this result we see that the market information process can be represented as a solution to a stochastic differential equation driven by some Brownian motion under $\mathbb{Q}$. Let define the process $W^{\mathbb{Q}}$ by the formula

$$W_t^{\mathbb{Q}} = \xi_t - \int_0^t \frac{\hat{D}_s \sigma T - \xi_s}{T-s} \, \mathrm{d}s. \tag{4.2.34}$$

It is not immediately evident, but one can prove (see, e.g. Brody et al. [2007], or Rutkowski & Yu [2007]) that this process $W^{\mathbb{Q}}$ is a continuous martingale with respect to the filtration $\mathbb{F}^{\xi}$. For the quadratic variation of $W^{\mathbb{Q}}$ we have $\langle W^{\mathbb{Q}} \rangle_t = \langle \xi \rangle_t = t$, hence making use of the Lévy's characterization theorem shows that $W^{\mathbb{Q}}$ is a standard Brownian motion with respect to $\mathbb{F}^{\xi}$.

A straightforward application of Itô's formula and (4.2.34) shows that the dynamics of $\hat{D}_t$ under $\mathbb{Q}$ can be written as

$$\begin{aligned} d\hat{D}_t &= \frac{\sigma T}{T-t} \mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right] \left(d\xi_t - \frac{\hat{D}_s \sigma T - \xi_t}{T-t} \, dt \right) \\ &= \frac{\sigma T}{T-t} \mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right] \, dW_t^{\mathbb{Q}}, \end{aligned} \tag{4.2.35}$$

where conditional variance $\mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right]$ can be written as

$$\begin{aligned} \mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right] &= \int_0^\infty \left(x - \hat{D}_t \right)^2 f_t(x) \, \mathrm{d}x \\ &= \int_0^\infty x^2 f_t(x) \, \mathrm{d}x - \left(\int_0^\infty x f_t(x) \, \mathrm{d}x \right)^2. \end{aligned} \tag{4.2.36}$$

Therefore, the dynamics of the asset price process (4.2.31) are given by

$$\begin{aligned} dS_t &= d\left(P_{tT} \hat{D}_t \right) = \hat{D}_t \, dP_{tT} + P_{tT} \, d\hat{D}_t \\ &= r_t P_{tT} \hat{D}_t \, dt + P_{tT} \left(\frac{\sigma T}{T-t} \mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right] \, dW_t^{\mathbb{Q}} \right) \\ &= r_t S_t \, dt + \Sigma_{tT} \, dW_t^{\mathbb{Q}}, \end{aligned} \tag{4.2.37}$$

where the short-term interest rate $r_t = -\frac{d}{dt} \ln P_{0t}$, and the absolute price volatility Σ_{tT} is given by

$$\Sigma_{tT} = P_{tT} \frac{\sigma T}{T-t} \mathsf{Var}_{\mathbb{Q}} \left[D_T | \xi_t \right]. \tag{4.2.38}$$

Brody et al. [2008b] point out that from the point of view of the market it is the process $\{W_t\}$ that drives the asset price dynamics. In this way their framework resolves the paradoxical point of view usually adopted in financial modelling in which $\{W_t\}$ is regarded on the one hand as noise, and yet on the other hand also generates the market information flow. Therefore, instead of hypothesising the existence of a driving process for the dynamics of the markets, they are able to deduce the existence of such a process.

Price Process of Single Dividend Paying Asset with Multiple Market Factors

We asume the more general case when the cash flow D_T is determined by a number of independent market factors, i.e. $D_T = \Delta_T\left(X_T^1, \ldots, X_T^m\right)$. For each market factor we have the vector-valued information process $\xi_{tT}^\alpha = X_T^\alpha \sigma_T^\alpha t + \beta_{tT}^\alpha$, which generates the market filtration. Write $q^\alpha(x)$ for the a priori probability density function of the market factor X_T^α. Calculations in the same manner as above ((4.2.25)-(4.2.27)) lead us to the following expression for the conditional probability density functions $f_t^\alpha(x)$ for $\alpha = 1, \ldots, m$

$$f_t^\alpha(x) = \frac{q^\alpha(x)\mathrm{exp}\left[\frac{T}{T-t}\left(x\sigma^\alpha\xi_{tT} - \frac{1}{2}x^2\left(\sigma^\alpha\right)^2 t\right)\right]}{\int_0^\infty q^\alpha(x)\mathrm{exp}\left[\frac{T}{T-t}\left(x\sigma^\alpha\xi_{tT} - \frac{1}{2}x^2\left(\sigma^\alpha\right)^2 t\right)\right]\,\mathrm{d}x}. \tag{4.2.39}$$

Then owing to the independence of the information processes ξ_{tT}^α associated with the market factors X_T^α, $\alpha = 1, \ldots, m$ we find that

$$\begin{aligned} S_t &= \mathbb{1}_{\{t<T\}} P_{tT} \mathbb{E}_\mathbb{Q}\left[\Delta_T\left(X_T^1, \ldots, X_T^m\right) | \xi_{tT}^1, \ldots, \xi_{tT}^m\right] \\ &= \mathbb{1}_{\{t<T\}} P_{tT} \int_0^\infty \cdots \int_0^\infty \Delta_T\left(x_1, \ldots, x_m\right) f_t^1(x_1) \cdots f_t^m(x_m)\,\mathrm{d}x_1 \cdots \mathrm{d}x_m. \end{aligned} \tag{4.2.40}$$

Black-Scholes Theory from the Information-Based Perspective

It is interesting that in the information-based framework the standard Black-Scholes-Merton theory can be expressed in terms of a normally distributed market factor and an independent Brownian bridge noise process. We consider a limited-liability company that makes a single cash distribution D_T at time T. Let X_T be a standard normally distributed random variable. We assume that D_T has a log-normal distribution under the probability measure $\mathbb{Q}$ and thus can be expressed as

$$D_T = D_0 \mathrm{exp}\left[rT - \frac{1}{2}\rho^2 T + \rho\sqrt{T} X_T\right], \tag{4.2.41}$$

where r and ρ are strictly positive constants. In this setting we assume that the Brownian bridge information process is of the form

$$\xi_t = X_T \sigma t + \beta_{tT}, \tag{4.2.42}$$

and that the information flow rate parameter is of the special form $\sigma = 1/\sqrt{T}$. Using this σ and (4.2.27) where $q(x)$ is the probability density function of a stardard normal distribution, i.e $q(x) = \frac{1}{\sqrt{2\pi}}\mathrm{exp}\left[-\frac{x^2}{2}\right]$, we find that the conditional probability density function of D_T given ξ_t is of the Gaussian form

$$f_t(x) = \sqrt{\frac{T}{2\pi(T-t)}}\,\mathrm{exp}\left[-\frac{(x\sqrt{T} - \xi_t)^2}{2(T-t)}\right]. \tag{4.2.43}$$

Using this conditional density, the value of the asset is given by

$$\begin{aligned} S_t &= \mathbb{1}_{\{t<T\}} P_{tT} \mathbb{E}_{\mathbb{Q}} \left[D_T | \xi_t \right] \\ &= \mathbb{1}_{\{t<T\}} \exp\left[-r\left(T-t\right)\right] \int_{-\infty}^{\infty} D_0 \exp\left[rT - \frac{1}{2}\rho^2 T + \rho\sqrt{T} X_T\right] f_t(x)\,\mathrm{d}x \\ &= \mathbb{1}_{\{t<T\}} D_0 \exp\left[rt - \frac{1}{2}\rho^2 t + \rho\xi_t\right]. \end{aligned} \tag{4.2.44}$$

In our case of normally distributed market factor X_T and information flow parameter $\sigma = 1/\sqrt{T}$, the market information process is of the form

$$\xi_t = X_T \frac{t}{\sqrt{T}} + \beta_{tT}, \tag{4.2.45}$$

and owing to the fact that X_T and $\{\beta_{tT}\}$ are independet it follows that $\mathbb{E}_{\mathbb{Q}}\left[\xi_s \xi_t\right] = s$ for $s \leq t$. This shows that $\{\xi_t\}$ is an $\mathbb{F}$-Brownian motion. Hence setting $W_t = \xi_t$ for $t \in [0, T]$ in (4.2.44) we get the standard geometric Brownian motion model

$$D_t = D_0 \exp\left[(r - \frac{1}{2}\rho^2)t + \rho W_t\right]. \tag{4.2.46}$$

Reversely, starting with (4.2.46) and making use of the orthogonal decomposition of the Brownian motion (see, e.g. Mansuy & Yor [2008])

$$W_t = \frac{t}{T} W_T + \left(W_t - \frac{t}{T} W_T\right), \tag{4.2.47}$$

the second term on the right side of (4.2.47) is a standard representation of the standard Brownian bridge of duration T, and it is independent of the first term of the right side of (4.2.47). Hence if we set $X_T = W_T/\sqrt{T}$ and $\sigma = 1/\sqrt{T}$, we find that the right side of (4.2.47) is indeed the market information process. The special feature of the Black-Scholes theory therefore is the fact that the information flow rate takes the specific form $\sigma = 1/\sqrt{T}$.

4.2.2 Gamma Bridge Information Processes

We assume the situation of example 4.1.2, i.e. the random variable X_T represents the total accumulation of an irreversible gain process. In this setting we will assume that the flow of information available to market participants is generated by an aggregate claims process ξ. For each t the random variable ξ_t stands for the totality of claims known at t to be payable at T. Brody et al. [2008b] assume that this information process ξ is of the form

$$\xi_t = X_T \gamma_{tT}, \tag{4.2.48}$$

where the random variable X_T and the gamma bridge process $\{\gamma_{tT}\}_{0\le t\le T}$ are independent. The motivation for this specific form of the accumulation process arises from the idea that the gamma process can be used as a mathematical basis for describing the aggregate losses associated with insurance claims (see references cited in Brody et al. [2008b]).

For fixed T we define the process $\{\gamma_{tT}\}_{0\le t\le T}$ by

$$\gamma_{tT} = \frac{\gamma_t}{\gamma_T}, \tag{4.2.49}$$

where $\{\gamma_t\}_{0\le t<\infty}$ is a standard gamma process with mean growth rate m. Here by a *standard gamma process* $\{\gamma_t\}_{0\le t<\infty}$ with rate m, defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, we mean a process such that $\gamma_0 = 0$, and with independent increments such that $\gamma_t - \gamma_s$ for $0 \le s \le t$ has a gamma distribution with mean and variance both equal to $m(t-s)$, i.e. the probability density function of the increment $\gamma_t - \gamma_s$ is given by

$$g(x) = \mathbb{1}_{\{x>0\}} \frac{x^{m(t-s)-1}\exp\left[-x\right]}{\Gamma\left[m(t-s)\right]}, \tag{4.2.50}$$

where $\Gamma\left[\alpha\right]$ is the standard gamma function that is for $\alpha > 0$ defined by

$$\Gamma\left[\alpha\right] = \int_0^\infty x^{\alpha-1}\exp\left[-x\right]\,\mathrm{d}x. \tag{4.2.51}$$

From definition (4.2.49) we see that $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$. We refer to $\{\gamma_{tT}\}_{0\le t\le T}$ as the *standard gamma bridge* over the interval $[0, T]$ associated with the gamma process $\{\gamma_t\}$. By computation it can be shown (see Brody et al. [2008b], Proposition 3.1) that the random variable γ_{tT} has a beta distribution with probability density function

$$b(x) = \mathbb{1}_{\{0<x<1\}} \frac{x^{mt-1}(1-x)^{m(T-t)-1}}{B\left[mt, m(T-t)\right]}, \tag{4.2.52}$$

where $B\left[\alpha, \beta\right]$ is the beta function that is for $p, q > 0$ defined by

$$B\left[\alpha, \beta\right] = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x, \tag{4.2.53}$$

and can be also expressed as

$$B\left[\alpha, \beta\right] = \frac{\Gamma\left[\alpha\right]\Gamma\left[\beta\right]}{\Gamma\left[\alpha+\beta\right]}. \tag{4.2.54}$$

Using (4.2.53) we can easily express n^{th} moment of a gamma bridge process, for $n > 0$, as

$$\mathbb{E}_{\mathbb{Q}}\left[\gamma_{tT}^n\right] = \frac{B\left[mt+n, m(T-t)\right]}{B\left[mt, m(T-t)\right]}. \tag{4.2.55}$$

Using the above results on gamma and beta functions, we can show that the mean and the variance of the gamma bridge are given by

$$\mathbb{E}_{\mathbb{Q}}\left[\gamma_{tT}\right] = \frac{t}{T}, \tag{4.2.56}$$

$$\mathsf{Var}_{\mathbb{Q}}\left[\gamma_{tT}\right] = \frac{t(T-t)}{T^2(1+mT)}. \tag{4.2.57}$$

It is interesting to observe that the expectation of γ_{tT} does not depend on the growth rate m, and that the variance of γ_{tT} decreases with increasing m. The gamma bridge has also remarkable property that for all $t \in [0,T]$ the random variables $\frac{\gamma_t}{\gamma_T}$ and γ_T are independent. Some further propeties of gamma bridges can be found in Yor [2007] and paralels between Brownian bridges and gamma bridges are treated in Emery & Yor [2004].

The important feature of the aggregate claims process (4.2.48) is that $\{\xi_t\}_{0\le t\le T}$ has the Markov property, i.e.

$$\mathbb{Q}\left(\xi_t \le x | \mathcal{F}_s^{\xi}\right) = \mathbb{Q}\left(\xi_t \le x | \xi_s\right), \tag{4.2.58}$$

for all $x \in \mathbb{R}$, and for all s, t such that $0 \le s \le t \le T$. This result and its proof can be found in Brody et al. [2008b] (Proposition 4.1, p.1810). Along with the fact that X_T is $\mathcal{F}_T$-measurable we thus find that

$$\mathbb{E}_{\mathbb{Q}}\left[X_T | \mathcal{F}_t^{\xi}\right] = \mathbb{E}_{\mathbb{Q}}\left[X_T | \xi_t\right], \tag{4.2.59}$$

where $\mathcal{F}_t^{\xi} = \sigma\left(\xi_s : 0 \le s \le t\right)$ in accordance with (4.2.15). The value S_t of the claim at time t is then given by

$$S_t = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[X_T | \xi_t\right]. \tag{4.2.60}$$

In (4.2.48) the gamma bridge $\{\gamma_{tT}\}$ is independent of the random variable X_T, and represents in some sense the noise that obscures the true value of X_T , while progressively revealing that value as time passes . In the insurance context, ξ_t represents the totality of claims known already at time t to be payable at time T. In the credit context, ξ_t represents the default losses known at t that will be realisable at T.

4.3 Applications of the Information-Based Pricing to Credit Risk Management

The information-based approach described in the Section 4.2.1 has many applications in modelling of credit risk.

4.3.1 Defaultable Discount Bond with Random Recovery

We assume the situation described in the Example 4.1.1 and that the filtration is generated by market information process $\xi_t = X_T \sigma t + \beta_{tT}$ for all $t \in [0, T]$.

Firstly we consider the simple case of a binary defaultable discount bond which at maturity date pays a principal payment x_1 with *a priori* probability q_1 if there is no default and x_0 with *a priori* probability q_0 in the event of default, where $x_1 > x_0$. Using (4.1.4), for the time zero price of this binary bond we can write

$$B_{0T} = P_{0T}(x_0 q_0 + x_1 q_1). \tag{4.3.61}$$

We assume that the market data B_{0T} and P_{0T} are known. Then the *a priori* probabilities q_0 and q_1 can be worked out as

$$q_0 = \frac{1}{x_1 - x_0}\left(x_1 - \frac{B_{0T}}{P_{0T}}\right), \; q_1 = \frac{1}{x_1 - x_0}\left(\frac{B_{0T}}{P_{0T}} - x_0\right). \tag{4.3.62}$$

Now we consider the more general situation when the defaultable bond has $n+1$ different possible payoffs $x_0 < x_1 < \cdots < x_n$ with *a priori* probabilities $q_i = \mathbb{Q}\left(X_T = x_i\right)$. As we have mentioned above, the case of $X_T = x_n$ corresponds with no default and all the others cases with various possible degrees of recovery. Using the fact that the market information flow $\mathbb{F}$ is generated by the market information process (4.2.19) along with the Markov property of the Brownian bridge information process $\{\xi\}_{0 \le t \le T}$, we can rewrite the price process of this defaultable bond (4.1.4) as

$$B_{tT} = P_{tT} \mathbb{E}_{\mathbb{Q}}\left[X_T | \xi_t\right], \tag{4.3.63}$$

for all $t \in [0, T]$. The bond price can thus be expressed in the form

$$B_{tT} = P_{tT} \sum_{i=0}^{n} x_i p_{it}, \tag{4.3.64}$$

where $p_{it} = \mathbb{Q}\left(X_T = x_i | \xi_t\right)$ is the conditional probability that the defaultable bond pays out x_i at T. This *a posteriori* probability can be calculated following steps parallel to (4.2.24) - (4.2.27) but for the discretely distributed payoff $D_T = X_T$. Thus the conditional probability can be worked out explicitly as a function of ξ_t and t by use of a form of the Bayes formula analogous to (4.2.27), i.e p_{it} can be expressed as

$$p_{it} = \frac{q_i \exp\left[\frac{T}{T-t}\left(x_i \sigma \xi_t - \frac{1}{2} x_i^2 \sigma^2 t\right)\right]}{\sum_{i=0}^{n} q_i \exp\left[\frac{T}{T-t}\left(x_i \sigma \xi_t - \frac{1}{2} x_i^2 \sigma^2 t\right)\right]}. \tag{4.3.65}$$

Consequently on account of (4.3.64), for the price process of the defaultable zero coupon bond we have

$$B_{tT} = P_{tT} \frac{\sum_{i=0}^{n} x_i q_i \exp\left[\frac{T}{T-t}\left(x_i \sigma \xi_t - \frac{1}{2} x_i^2 \sigma^2 t\right)\right]}{\sum_{i=0}^{n} q_i \exp\left[\frac{T}{T-t}\left(x_i \sigma \xi_t - \frac{1}{2} x_i^2 \sigma^2 t\right)\right]}. \tag{4.3.66}$$

In particular, for the above case of the binary bond we have

$$B_{tT} = P_{tT} \frac{x_0 q_0 \exp\left[\frac{T}{T-t}\left(x_0\sigma\xi_t - \frac{1}{2}x_0^2\sigma^2 t\right)\right] + x_1 q_1 \exp\left[\frac{T}{T-t}\left(x_1\sigma\xi_t - \frac{1}{2}x_1^2\sigma^2 t\right)\right]}{q_0 \exp\left[\frac{T}{T-t}\left(x_0\sigma\xi_t - \frac{1}{2}x_0^2\sigma^2 t\right)\right] + q_1 \exp\left[\frac{T}{T-t}\left(x_1\sigma\xi_t - \frac{1}{2}x_1^2\sigma^2 t\right)\right]}. \quad (4.3.67)$$

Analogous to the treatment in the Section 4.2.1 ((4.2.32) - (4.2.38)), we can compute the dynamics of defaultable bond, and it can be written as

$$dB_{tT} = r_t B_{tT}\, dt + \Sigma_{tT}\, dW_t, \quad (4.3.68)$$

where r_t is the short rate and the absolute bond volatility is given by

$$\begin{aligned} \Sigma_{tT} &= P_{tT}\frac{\sigma T}{T-t}\mathsf{Var}_{\mathbb{Q}}\left[X_T|\xi_t\right] \\ &= P_{tT}\frac{\sigma T}{T-t}\sum_{i=0}^{n}(x_i - \hat{X}_t)^2 p_{it} \end{aligned} \quad (4.3.69)$$

Here $\hat{X}_t = \mathbb{E}_{\mathbb{Q}}\left[X_T|\xi_t\right]$. It should be apparent that as the maturity date is approached the absolute discount bond volatility will be high unless the conditional probability has most of its mass concentrated around the "true" outcome. Since $P_{tT}\frac{\partial}{\partial\xi_t}\hat{X}_t = \Sigma_{tT}$, the price process B_{tT} is increasing in ξ_t.

One of the attractive features of formula (4.3.66) is that since B_{tT} is expressed explicitly as a function of the market information process, simulation of the dynamics of the bond price process can be carried out quite efficiently. All we need to do is to simulate the dynamics of $\{\xi_t\}$. We choose at random a value for X_T in accordance with the *a priori* probabilities, and a sample path for the Brownian Bridge. The way how to simulate a Brownian bridge is $\beta_{tT} = W_t - \frac{t}{T}W_T$ as in the decomposition (4.2.47). So the present framework allows for a very simple and natural simulation methodology for the dynamics of defaultable bonds and related structure.

The parameter σ governs the speed at which the bond price converges to its terminal value. In the case of a binary bond with price process (4.3.67) this can be seen as follows. We suppose for instance that in a given run of the simulation the actual value of the payoff is $X_T = x_0$. Hence for market information process in this case we have $\xi_t = x_0\sigma t + \beta_{tT}$ and thus for $\{B_{tT}\}$ we have

$$B_{tT} = P_{tT} \frac{x_0 q_0 \exp\left[\frac{T}{T-t}\left(x_0\sigma\beta_{tT} + \frac{1}{2}x_0^2\sigma^2 t\right)\right] + x_1 q_1 \exp\left[\frac{T}{T-t}\left(x_1\sigma\beta_{tT} + x_0 x_1\sigma^2 t - \frac{1}{2}x_1^2\sigma^2 t\right)\right]}{q_0 \exp\left[\frac{T}{T-t}\left(x_0\sigma\beta_{tT} + \frac{1}{2}x_0^2\sigma^2 t\right)\right] + q_1 \exp\left[\frac{T}{T-t}\left(x_1\sigma\beta_{tT} + x_0 x_1\sigma^2 t - \frac{1}{2}x_1^2\sigma^2 t\right)\right]}. \quad (4.3.70)$$

When we divide the numerator and the denominator by the coefficient of $x_0 q_0$, the result is

$$B_{tT} = P_{tT} \frac{x_0 q_0 + x_1 q_1 \exp\left[-\frac{T}{T-t}\left(\frac{1}{2}\left(x_1 - x_0\right)^2\sigma^2 t - \left(x_1 - x_0\right)\sigma\beta_{tT}\right)\right]}{q_0 + q_1 \exp\left[-\frac{T}{T-t}\left(\frac{1}{2}\left(x_1 - x_0\right)^2\sigma^2 t - \left(x_1 - x_0\right)\sigma\beta_{tT}\right)\right]}. \quad (4.3.71)$$

Hence the convergence of the bond price to the value x_0 is exponential. A similar argument shows that the bond price converges to x_1 provided $X_T = x_1$. The parameter $\sigma^2(x_1 - x_0)$ governs the speed at which the defaultable discount bond converges to its destined terminal value. In particular, if the a priori probability of no default is high (say, $q_1 \approx 1$), and if σ is very small, and if in fact $X_T = x_0$, then it will only be when t is close to T that serious decay in the bond price will set in.

4.3.2 Options on Defaultable Bonds

In this section we shall perform a method which is based on the concept of an Arrow-Debreu security. An alternative method where the price of an European call option written on a defaultable bond with discrete payoff is calculated by use of a change of measure technique to introduce the so-called bridge measure can be found in Section 8 of Brody et al. [2007], and for the same method but with continuous payoff we refer to Rutkowski & Yu [2007]. We shall see that the value of the information flow rate parameter σ can be inferred from option price data.

The value at time 0 of an European call option exercisable at time $t > 0$ on a defaultable bond with maturity date $T > t$ and strike price K is given by risk neutral formula

$$C_0 = P_{0t}\mathbb{E}_{\mathbb{Q}}\left[(B_{tT} - K)^+\right]. \tag{4.3.72}$$

Using formulas (4.3.64) and (4.3.66) for the bond price B_{tT}, the valuation formula (4.3.72) for the option can be rewritten as

$$\begin{aligned} C_0 &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\left(P_{tT}\sum_{i=0}^{n} x_i p_{it} - K\right)^+\right] \\ &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\left(P_{tT}\frac{\sum_{i=0}^{n} x_i q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma\xi_t - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}{\sum_{i=0}^{n} q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma\xi_t - \frac{1}{2}x_i^2\sigma^2 t\right)\right]} - K\right)^+\right]. \end{aligned} \tag{4.3.73}$$

The call option price can thus be viewed as an exotic derivative with payoff $h(\xi_t)$ at maturity t. The underlying of this derivative is the information available to market participants. The price of such an *information derivative* is given by

$$V_0 = P_{0t}\mathbb{E}_{\mathbb{Q}}\left[h(\xi_t)\right]. \tag{4.3.74}$$

We first look at an abstract security expiring at time t with the payoff

$$g(\xi_t) = \delta(\xi_t - y), \tag{4.3.75}$$

where δ is the delta "function", i.e. . To work out the price in this case we use the Fourier representation of the delta function

$$\delta(\xi_t - y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\mathrm{i}(\xi_t - y)u\right]\,\mathrm{d}u. \tag{4.3.76}$$

As we mentioned above, conditional on a specific value of the random variable X_T the information process $\{\xi_t\}$ is normally distibuted. Along with the fact that the random variable X_T can take only a finite number of different states this ensures that the expectation in (4.3.74) and the integral in (4.3.76) can be interchanged. Let us write $A_{0t}(x)$ for the time zero value of the contract with payoff (4.3.75). The value of this elementary information security is given by the discounted risk neutral expectation of the payoff $\delta(\xi_t - y)$, i.e.

$$
\begin{aligned}
A_{0t}(y) &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\delta(\xi_t - y)\right] \\
&= P_{0t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,yu}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,\xi_t u}\right]\mathrm{d}u. \qquad (4.3.77)
\end{aligned}
$$

Conditional on a specific value of the random variable $X_T = x_j$ the information process $\{\xi_t\}$ is normally distibuted with mean $x_j\sigma t$ and variance $t(T-t)/T$. This leads us to the relation

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,\xi_t u}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,\xi_t u}|X_T\right]\right] \qquad (4.3.78) \\
&= \sum_{j=0}^{n} q_j \exp\left[\mathrm{i}\,x_j\sigma t u - \frac{1}{2}\frac{t(T-t)}{T}u^2\right]. \qquad (4.3.79)
\end{aligned}
$$

Here we have used the known expression of characteristic function of a normal distribution. Inserting this result into (4.3.77), swapping the order of integration and sumation, and then working out the integral we obtain

$$
A_{0t}(y) = P_{0t}\sum_{j=0}^{n} q_j\sqrt{\frac{T}{2\pi\,t(T-t)}}\exp\left[-\frac{\left(x_j\sigma t - x\right)^2 T}{2t(T-t)}\right]. \qquad (4.3.80)
$$

The calculation above served the purpose of illustrating an application of the information based approach to the pricing of an Arrow-Debreu security. The value of a general information derivative (4.3.74) can then be expressed as a weighted integral where the elementary information securities with Arrow-Debreu price $A_{0t}(y)$ play the role of the weights, i.e.

$$
\begin{aligned}
V_0 = P_{0t}\mathbb{E}_{\mathbb{Q}}\left[h(\xi_t)\right] &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\int_{-\infty}^{\infty}\delta\left(\xi_t - y\right)h(y)\,\mathrm{d}y\right] \\
&= \int_{-\infty}^{\infty} A_{0t}(y)h(y)\,\mathrm{d}y \qquad (4.3.81)
\end{aligned}
$$

Setting $h(\xi_t) = (B_{tT} - K)^+$ we are in the position to calculate the value C_0 of the call option by rewriting (4.3.73) in terms of $A(y) = A_{0t}(y)/P_{0t}$ which can be called the non-discounted Arrow-Debreu density, i.e.

$$\begin{aligned} C_0 &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\left(P_{tT}\frac{\sum_{i=0}^n x_i q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma\xi_t - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}{\sum_{i=0}^n q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma\xi_t - \frac{1}{2}x_i^2\sigma^2 t\right)\right]} - K\right)^+\right] \\ &= \int_{-\infty}^{\infty} A_{0t}(y)\left(P_{tT}\frac{\sum_{i=0}^n x_i q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}{\sum_{i=0}^n q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]} - K\right)^+ \mathrm{d}y \\ &= P_{0t}\int_{-\infty}^{\infty}\left(P_{tT}\frac{\sum_{i=0}^n x_i q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}{\sum_{i=0}^n q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]} - K\right)^+ A(y)\,\mathrm{d}y. \end{aligned} \tag{4.3.82}$$

Observing that the Arrow-Debreu density $A(y)$ is a positive function, we can rewrite (4.3.82) as

$$C_0 = P_{0t}\int_{-\infty}^{\infty}\left(P_{tT}\frac{\sum_{i=0}^n x_i q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}{\sum_{i=0}^n q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]}A(y) - K\,A(y)\right)^+ \mathrm{d}y. \tag{4.3.83}$$

Using (4.3.80), for the density $A(y)$ we can express the density $A(y)$ as

$$A(y) = \exp\left[-\frac{T}{2t(T-t)}y^2\right]\sqrt{\frac{T}{2\pi\,t(T-t)}}\sum_{i=0}^n q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]. \tag{4.3.84}$$

For sake of brevity we introduce $q_{it} = q_i \exp\left[\frac{T}{T-t}\left(x_i\sigma y - \frac{1}{2}x_i^2\sigma^2 t\right)\right]$ and we plug (4.3.84) into (4.3.83). Hence we obtain

$$\begin{aligned} C_0 &= P_{0t}\sqrt{\frac{T}{2\pi\,t(T-t)}}\int_{-\infty}^{\infty}\left(\exp\left[-\frac{T}{2t(T-t)}y^2\right]\sum_{i=0}^n \left(P_{tT}x_i - K\right)q_{it}\right)^+ \mathrm{d}y \\ &= P_{0t}\sqrt{\frac{T}{2\pi\,t(T-t)}}\int_{-\infty}^{\infty}\left(\sum_{i=0}^n q_i\left(P_{tT}x_i - K\right)\exp\left[-\frac{T}{2t(T-t)}\left(y - x_i\sigma t\right)^2\right]\right)^+ \mathrm{d}y. \end{aligned} \tag{4.3.85}$$

In case that $P_{tT}x_n > \ldots > P_{tT}x_0 > K$ the option is certain to expire in the money and for value of the option we have $C_0 = B_{0T} - P_{0t}K$, and in case that $K > P_{tT}x_n > \ldots > P_{tT}x_0$ the option expires out of money hence $C_0 = 0$. We consider the case where the strike price lies in the range $P_{tT}x_{j+1} > K > P_{tT}x_j$ for some value of $j = 0, 1, \ldots, n$. In this case the option can expire in or out of money and there exists a unique critical value of y, above which the argument of maximum in the expression (4.3.85) is positive. Writing $\bar{y}$ for this critical value, which can be obtain by numerical methods, we define the random variable

$$\bar{Z}(x_i) = \frac{\bar{y} - x_i\sigma t}{\sqrt{\frac{t(T-t)}{T}}}, \tag{4.3.86}$$

and we note that this random variable is normally distributed with zero mean and unit variance. Then the equation (4.3.85) takes the form

$$\begin{aligned} C_0 &= P_{0t} \sum_{i=0}^{n} q_i \left(P_{tT} x_i - K\right) \frac{1}{\sqrt{2\pi}} \int_{\bar{Z}(x_i)}^{\infty} \exp\left[-\frac{z^2}{2}\,\mathrm{d}z\right] \\ &= P_{0t} \sum_{i=0}^{n} q_i \left(P_{tT} x_i - K\right) \Phi\left(-\bar{Z}(x_i)\right), \end{aligned} \tag{4.3.87}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. This semi-analytic option pricing formula can be regarded as fully tractable for practical purposes.

In the case of a binary defaultable bond paying either x_0 or x_1 at maturity we obtain a closed-form expression for the price of the call option. The result for the option price is very similar in form to the Black-Scholes formula. We start with formula (4.3.85) which can be in case of a binary defaultable bond rewritten as

$$\begin{aligned} C_0^b = P_{0t} \sqrt{\frac{T}{2\pi\, t(T-t)}} \int_{-\infty}^{\infty} \Bigg(& q_0 \left(P_{tT} x_0 - K\right) \exp\left[-\frac{T}{2t(T-t)} \left(y - x_0 \sigma t\right)^2\right] \\ & + q_1 \left(P_{tT} x_1 - K\right) \exp\left[-\frac{T}{2t(T-t)} \left(y - x_1 \sigma t\right)^2\right] \Bigg)^{+} \mathrm{d}y. \end{aligned} \tag{4.3.88}$$

As above we write $\bar{y}$ for the critical value which is obtained by setting the argument of maximum function above to zero, i.e.

$$\ln\left(\frac{q_0(P_{tT}x_0 - K)}{q_1(K - P_{tT}x_1)}\right) + \frac{1}{2}\sigma^2\left(x_1^2 - x_0^2\right)\tau = \frac{T}{T-t}\sigma(x_1 - x_0)\bar{y}, \tag{4.3.89}$$

where $\tau = \frac{tT}{T-t}$. Consequently, realizing that $y - x_i \sigma t$, $i = 1, 2$ have a normal distribution with zero mean and variance $t(T-t)/T$, expression (4.3.88) can be rewritten as

$$C_0^b = P_{0t}\left[q_1\left(P_{tT}x_1 - K\right)\Phi(d^+) - q_0\left(K - P_{tT}x_0\right)\Phi\left(d^-\right)\right], \tag{4.3.90}$$

where d^+ and d^- are given by

$$d^{+}_{-} = \frac{\ln\left(\frac{q_1(K - P_{tT}x_1)}{q_0(P_{tT}x_0 - K)}\right) \,{}^{+}_{-}\, \frac{1}{2}(x_1 - x_0)^2\sigma^2\tau}{(x_1 - x_0)\sigma\sqrt{\tau}}. \tag{4.3.91}$$

The information flow rate σ thus plays a role very similar to that of the volatility parameter in the Black-Scholes formula.

Let us define the option vega in this model by $\mathcal{V} = \frac{\partial C_0^b}{\partial \sigma}$. For the vega we then obtain a positive expression

$$\mathcal{V} = \frac{1}{\sqrt{2\pi}} \exp\left[-rt - \frac{A}{2}\right](x_1 - x_0)\sqrt{\tau q_0 q_1 (P_{tT}x_1 - K)(K - P_{tT}x_0)}, \tag{4.3.92}$$

where A is given by

$$A = \frac{1}{(x_1 - x_0)^2\sigma^2\tau}\left(\ln\left(\frac{q_1(K - P_{tT}x_1)}{q_0(P_{tT}x_0 - K)}\right)\right)^2 + \frac{1}{4}(x_1 - x_0)^2\sigma^2\tau. \tag{4.3.93}$$

Thus C_0 is an increasing function of the information flow rate σ. In other words, the more rapidly information regarding the true value of the bond payoff is released, the higher the premium of the call option. Another conclusion is that bond option prices (or, equivalently, the prices of caps and floors) can be used to recover an implied value for the information flow rate σ, and hence to calibrate the model.

Another important feature of the information-based model is that options positions can be hedged with a position in the defaultable bonds. This is because the option price process and the underlying bond price process are one-dimensional diffusions driven by the same Brownian motion. The number of bond unites needed to hedge a short position in a call option is given by the option delta $\Delta = \frac{\partial C_0}{\partial B_{0T}}$. In the case of a binary bond, making use of (4.3.90) and (4.3.62), we can write the option delta as

$$\Delta = \frac{(P_{tT}x_1 - K)\,\Phi(d^+) - (K - P_{tT}x_0)\,\Phi(d^-)}{P_{tT}(x_1 - x_0)}. \tag{4.3.94}$$

4.3.3 Defaultable n-coupon Bond with Multiple Recovery Levels

We consider the case of a defaultable coupon bond where default can occur at any of the n coupon pre-specified payment dates T_k, where $k = 1, \ldots, n$. In this section the market factors X_{T_j}, $j = 1, 2, ..., k$ are modelled as independent binary random variables taking the value 0 in case of default or 1 if there is no default, with *a priori* probabilities $q_0^{(j)}$ (default) and $q_1^{(j)}$ (no default). The constants $\mathbf{c}$ and $\mathbf{p}$ denote the coupon and the principal. In the event of default at the date T_k we assume the recovery payment $R_k(\mathbf{c}+\mathbf{p})$, where R_k is a percentage of the owed coupon and principal payment. At each date T_k there occurs a cash flow H_{T_k} given by

$$D_{T_k} = \mathbf{c}\prod_{j=1}^{k} X_{T_j} + R_k(\mathbf{c}+\mathbf{p})\prod_{j=1}^{k-1} X_{T_j}(1 - X_{T_k}), \qquad \text{for } k = 1, , \ldots, n-1\,, \tag{4.3.95}$$

$$D_{T_n} = (\mathbf{c}+\mathbf{p})\prod_{j=1}^{n} X_{T_j} + R_n(\mathbf{c}+\mathbf{p})\prod_{j=1}^{n-1} X_{T_j}(1 - X_{T_n}), \quad \text{for } k = n. \tag{4.3.96}$$

In this case we introduce a set of market information processes given by

$$\xi_{tT_j} = X_{T_j}\sigma_{T_j}t + \beta_{tT_j}, \tag{4.3.97}$$

and denote by $\mathbb{F}^{\xi}$ the market filtration which is generated collectively by the market information processes. Using (4.1.3), we can write the information-based price process of this defaultable n-coupon bond as

$$S_t = \sum_{k=1}^{n-1} \mathbb{1}_{\{t<T_k\}} P_{tT_k}\mathbb{E}_{\mathbb{Q}}\left[D_{T_k}|\mathcal{F}_t^{\xi}\right] + P_{tT_n}\mathbb{1}_{\{t<T_n\}}\mathbb{E}_{\mathbb{Q}}\left[D_{T_n}|\mathcal{F}_t^{\xi}\right]. \tag{4.3.98}$$

As an ilustration we assume the case of two outstanding payments as in example 4.1.4. In this setting we have two market information process of the form (4.3.97) for $j = 1, 2$, and cash flows (4.1.9), (4.1.10). Then for the price of this bond we have

$$\begin{aligned} S_t = {} & \mathbb{1}_{\{t<T_1\}} P_{tT_1}\left[\mathbf{c}\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right] + R_1(\mathbf{c}+\mathbf{p})\mathbb{E}_{\mathbb{Q}}\left[1 - X_{T_1}|\xi_{tT_1}\right]\right] \\ & + \mathbb{1}_{\{t<T_2\}} P_{tT_2}(\mathbf{c}+\mathbf{p})\left(\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\mathbb{E}_{\mathbb{Q}}\left[X_{T_2}|\xi_{tT_2}\right] + R_2\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\mathbb{E}_{\mathbb{Q}}\left[1 - X_{T_2}|\xi_{tT_2}\right]\right), \end{aligned} \tag{4.3.99}$$

where the two expectations appearing here can be calculated explixitly using an analogy to (4.3.65). Thus we have

$$\mathbb{E}_{\mathbb{Q}}\left[X_{T_j}|\xi_{tT_j}\right] = \sum_{i=0}^{1} x_i p_{it}^{(j)} = \frac{q_1^{(j)}\exp\left[\frac{T}{T_j-t}\left(\sigma_{T_j}\xi_{tT_j} - \frac{1}{2}\sigma_{T_j}^2 t\right)\right]}{q_0^{(j)} + q_1^{(j)}\exp\left[\frac{T}{T_j-t}\left(\sigma\xi_{tT_j} - \frac{1}{2}\sigma_{T_j}^2 t\right)\right]} \tag{4.3.100}$$

In case of two outstanding payments we hence obtain a two-factor model, the factors are the two independent Brownian motions arising in connection with the information processes. In the case of n payments we obtain an n-factor model. A further extension can be done by introducing of random recovery rates.

4.3.4 Credit Default Swaps

In the information-based framework swap-like structures can also readily be treated. We assume the situation outlined in the Example 4.1.6, that is, we assume cash flows (4.1.13) and (4.1.14) from the point of view of the seller of protection. Using risk neutral valuation formula (4.1.3) for the value of the default swap we get

$$\begin{aligned} V_t \;=\; & \mathbf{g}P_{tT_1}\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right] - \left(\mathbf{p} - R_1(\mathbf{c}+\mathbf{p})\right)P_{tT_1}\left(1 - \mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\right) \\ & + \mathbf{g}P_{tT_2}\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\mathbb{E}_{\mathbb{Q}}\left[X_{T_2}|\xi_{tT_2}\right] \\ & - \left(\mathbf{p} - R_2(\mathbf{c}+\mathbf{p})\right)P_{tT_2}\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\left(1 - \mathbb{E}_{\mathbb{Q}}\left[X_{T_2}|\xi_{tT_2}\right]\right), \end{aligned} \tag{4.3.101}$$

where expectations are given in (4.3.100). If we assume a more general case where **g** denotes the premium payment and **n** denotes the payment made to the buyer of the protection in the event of default, then the price of a default swap written on a reference defaultable two coupon bond is given by

$$\begin{aligned} V_t = & - \mathbf{n}P_{tT_1} + ((\mathbf{g}+\mathbf{n})P_{tT_1} - \mathbf{n}P_{tT_2})\,\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right] \\ & + (\mathbf{g}+\mathbf{n})P_{tT_2}\mathbb{E}_{\mathbb{Q}}\left[X_{T_1}|\xi_{tT_1}\right]\mathbb{E}_{\mathbb{Q}}\left[X_{T_2}|\xi_{tT_2}\right]. \end{aligned} \tag{4.3.102}$$

Brody et al. [2007] point out that a similar approach can be used in the multi-name credit situation. They also treat the problem of valuation for a basket of defaultable bonds, when there are correlations in the payoffs. Here the number of independent factors in general grows rapidly with the number of bonds in the portfolio. Consequently, a market which consists of correlated bonds is in general highly incomplete. This fact provides an economic justification for the creation of products such as CDSs and CDOs that enhance the hedgeability of such portfolios.

4.4 Applications of the Information-Based Pricing to Insurance and Credit Portfolio Management

In this section we derive an expression for the value process of a contract that delivers the cash flow X_T at time T, when the market filtration is generated by the accumulation process (4.2.48). We also derive the value of general reinsurance contracts that at some fixed time t gives the contract holder the option to commute the claim X_T by paying a fixed amount K at t.

4.4.1 Aggregate Claims

We assume a contract that pays X_T at T, which is assumed to be positive and integrable random variable. As we have mentioned above, X_T represents the total accumulation of irreversible gains processes. These may be the totality of the payments made at T in settlement of claims arising over the period $[0, T]$, or total losses in a credit portfolio.

Firstly, we also assume that X_T has a continuous distribution. The market filtration is generated by an aggregate claims process

$$\xi_t = X_T \gamma_{tT}, \tag{4.4.103}$$

where process $\{\gamma_{tT}\}$ is a standard gamma bridge under the measure $\mathbb{Q}$ with parameter m. This gamma process represents in some sense the noise that obscures the true value of X_T. As we have mestioned above, In the insurance context, ξ_t represents the totality of claims known already at time t to be payable at time T and in the credit context, ξ_t

represents the default losses known at t that will be realisable at T. It is assumed that X_T and the gamma bridge are independent. Thus for the market filtration we can write $\mathcal{F}_t = \sigma\left(\xi_s : s \in [0,t]\right)$. The value of the contract at $t \leq T$ is given by $S_t = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[X_T|\mathcal{F}_t\right]$. The fact that X_T is $\mathcal{F}_T$-measurable along with the fact that the process $\{\xi_t\}$ has the Markov property leads us to valuation formula

$$S_t = P_{tT}\mathbb{E}_{\mathbb{Q}}\left[X_T|\xi_t\right]. \tag{4.4.104}$$

Next, we shall compute the conditional expectation in (4.4.104) in the same manner as presented above in this chapter. So that the conditional expectation in (4.4.104) takes the following form

$$\mathbb{E}_{\mathbb{Q}}\left[X_T|\xi_t\right] = \int_0^\infty x f_t(x)\,\mathrm{d}x, \tag{4.4.105}$$

where $f_t(x)$ is the conditional probability density for the random variable X_T , i.e.

$$f_t(x) := f_{X_T|\xi_t}\left(x|\xi_t\right) = \frac{\partial}{\partial x}\mathbb{Q}\left(X_T \leq x|\xi_t\right). \tag{4.4.106}$$

Using Bayes' formula, we can write this conditional density in the form

$$f_{X_T|\xi_t}\left(x|y\right) = \frac{q(x) f_{\xi_t|X_T}\left(y|x\right)}{\int_0^\infty q(x) f_{\xi_t|X_T}\left(y|x\right)\,\mathrm{d}x}, \tag{4.4.107}$$

where $q(x)$ denotes the a priori probability density function for X_T, which is assumed to be known, and $f_{\xi_t|X_T}\left(y|x\right)$ denotes conditional density of ξ_t, valued at y, which is given by

$$\begin{aligned} f_{\xi_t|X_T}\left(y|x\right) &= \frac{\partial}{\partial y}\mathbb{Q}\left(\xi_t \leq y|X_T = x\right) = \frac{\partial}{\partial y}\mathbb{Q}\left(X_T\gamma_{tT} \leq y|X_T = x\right) \\ &= \frac{\partial}{\partial y}\mathbb{Q}\left(\gamma_{tT} \leq \frac{y}{x}\right) = \frac{\partial}{\partial y}\int_0^{\frac{y}{x}} b\left(u\right)\,\mathrm{d}u \\ &= \frac{1}{x} b\left(\frac{y}{x}\right). \end{aligned} \tag{4.4.108}$$

We recall that the random variable γ_{tT} has a beta distribution with probability density function b that is given by (4.2.52). Thus (4.4.108) can be rewritten as

$$\begin{aligned} f_{\xi_t|X_T}\left(y|x\right) &= \frac{1}{x}\mathbb{1}_{\{y<x\}}\frac{\left(\frac{y}{x}\right)^{mt-1}\left(1-\frac{y}{x}\right)^{m(T-t)-1}}{B\left[mt, m(T-t)\right]} \\ &= \mathbb{1}_{\{y<x\}} y^{mt-1}\frac{x^{1-mT}(x-y)^{m(T-t)-1}}{B\left[mt, m(T-t)\right]}. \end{aligned} \tag{4.4.109}$$

The conditional density (4.4.107) is thus given by

$$f_t(x) = \mathbb{1}_{\{x>\xi_t\}} \frac{q(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}}{\int_{\xi_t}^{\infty} q(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}. \tag{4.4.110}$$

Hence, inserting (4.4.110) into (4.4.105), we obtain from (4.4.104) that the value S_t at time $t < T$ of the aggregate claim that pays the continuous variable $X_T > 0$ at time T is given by

$$S_t = P_{tT} \frac{\int_{\xi_t}^{\infty} q(x)x^{2-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}{\int_{\xi_t}^{\infty} q(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}. \tag{4.4.111}$$

When X_T is a discrete random variable taking values x_i, $i = 1, \ldots, n$ with a priori probabilities p_i, a similar calculation gives

$$S_t = P_{tT} \frac{\sum_{i=0}^{n} \mathbb{1}_{\{x_i>\xi_t\}} q_i x_i^{2-mT}(x_i-\xi_t)^{m(T-t)-1}}{\sum_{i=0}^{n} \mathbb{1}_{\{x_i>\xi_t\}} q_i x_i^{1-mT}(x_i-\xi_t)^{m(T-t)-1}}. \tag{4.4.112}$$

It is straightforward to see that if t tends to T the expression (4.4.112) converges to the correct terminal value.

Since the value S_t of the reserve at time t is given explicitly as a function of the accumulation ξ_t (the cumulative gain so far achieved), it becomes a straightforward matter to simulate trajectories of the reserve process, and hence also to value financial products that depend in a general way on the value of the reserve. The parameter m can be interpreted as the information flow rate for the market factor X_T.

4.4.2 Valuation of Reinsurance Products

Firstly we assume the case of a simple reinsurance contract that pays out $(X_T - K)^+$ at T for some fixed threshold K. The value process of this simple stop-loss reinsurance policy is given by $C_{tT} = \mathbb{E}_{\mathbb{Q}}\left[(X_T - K)^+ | \xi_t\right]$ and using the results of previous section it can be calculated as follows

$$\begin{aligned} C_{tT} &= P_{tT} \int_0^{\infty} (x-K)^+ \, f_t(x)\,\mathrm{d}x \\ &= P_{tT} \frac{\int_{\xi_t}^{\infty} q(x)\,(x-K)^+ \; x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}{\int_{\xi_t}^{\infty} q(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}, \end{aligned} \tag{4.4.113}$$

for $t < T$. Once a time t has been reached such that $\xi_t \geq K$, then $C_{sT} = P_{st}\,(S_t - K)$ for all s such that $s \in [t, T]$, i.e. once a sufficient number of claims have accumulated the option is sure to expire in the money. In the case of a large credit portfolio, C_{tT} has the interpretation of being the value at t of a contract that pays at T an amount equal to the total loss incurred by the portfolio, in excess of some threshold K.

A more general situation can be assumed, when we consider a contract that at a fixed time $t < T$ allows the firm (policy holder) the option of commuting the claim X_T in exchange for a pre-fixed settlement K. For the time zero value of such a option we have

$$C_{0t} = P_{0t}\mathbb{E}_{\mathbb{Q}}\left[(S_t - K)^+\right], \tag{4.4.114}$$

where S_t is time t value of the claim paying C_T at T. In the context of a credit portfolio, S_t represents the value at t of a contract that pays an amount equal to the accumulated losses in the portfolio at time T. Then C_{0t} is the time zero price of a contract that pays at time t the excess of S_t over K. Since the payout of the option is function of ξ_t, we can use the concept of Arrow-Debreu security mentioned in the Section 4.3.2. Thus the value at time zero of a simple option on the reserve (4.4.114) takes the form

$$\begin{aligned} C_{0t} &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[(S(t,\xi_t) - K)^+\right] \\ &= P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \delta\left(\xi_t - y\right)\left(S(t,y) - K\right)^+ \mathrm{d}y\right] \\ &= \int_0^\infty A_{0t}(y)\left(S(t,y) - K\right)^+ \mathrm{d}y. \end{aligned} \tag{4.4.115}$$

Here we have written $A_{0t}(y) = P_{0t}\mathbb{E}_{\mathbb{Q}}\left[\delta(\xi_t - y)\right]$ for the price of an Arrow-Debreu security on the aggregate gains process, with delta function payoff. Using Fourier transformation this price can be written as

$$\begin{aligned} A_{0t}(y) &= P_{0t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,yu}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,\xi_t u}\right]\mathrm{d}u \\ &= P_{0t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,yu}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,X_T\gamma_{tT}u}|X_T\right]\right]\mathrm{d}u \\ &= P_{0t}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,yu}\left(\int_0^\infty q(x)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\,x\gamma_{tT}u}\right]\mathrm{d}x\right)\mathrm{d}u. \end{aligned}$$

Swapping the order of integration, using the inverse Fourier transform of the characteristic function and using the fact that conditional on a value of the random variable X_T the gamma bridge process γ_{tT} has density function specified in (4.4.109) we derive the following expression for the price of the Arrow-Debreu security

$$A_{0t}(y) = P_{0t}\frac{y^{mt-1}}{B\left[mt, m(T-t)\right]}\int_y^\infty q(x)x^{1-mT}(x-y)^{m(T-t)-1}\,\mathrm{d}x. \tag{4.4.116}$$

Inserting this result and expression (4.4.111) into (4.4.115) we therefore have the following expression for the option price

$$C_{0t} = \frac{P_{0t}}{B\left[mt, m(T-t)\right]}\int_0^\infty y^{mt-1}\left(\int_y^\infty q(x)(xP_{tTt} - K)x^{1-mT}(x-y)^{m(T-t)-1}\,\mathrm{d}x\right)^+ \mathrm{d}y. \tag{4.4.117}$$

We let $\bar{y}$ denote the critical value of y such that the argument of maximum function vanishes. After some arrangements we are able to obtain the following option price

$$C_{0t} = P_{0t} \int_{\bar{y}}^{\infty} q(x)(xP_{tT} - K)\,\mathrm{B}\left(\frac{\bar{y}}{x}\right)\,\mathrm{d}x, \tag{4.4.118}$$

where

$$\mathrm{B}\,(a) = \frac{\int_a^1 z^{mt-1}(1-z)^{m(T-t)-1}\,\mathrm{d}z}{\int_0^1 z^{mt-1}(1-z)^{m(T-t)-1}\,\mathrm{d}z}, \tag{4.4.119}$$

is the complementary beta distribution function.

The option price process C_{st} for $0 \le s \le t \le T$ can be calculated by a similar method. The result is as follows

$$C_{st} = P_{st} \int_{\bar{y}}^{\infty} f_s(x)(xP_{tT} - K)\,\mathrm{B}\left(\frac{\bar{y}-\xi_s}{x-\xi_s}\right)\,\mathrm{d}x. \tag{4.4.120}$$

The derivation of this result can be found in the original paper by Brody et al. [2008b]. This expression lets us simulate trajectories of the value of the option. Note in particular that C_{st} is given as a function of ξ_s.

4.5 Conclusions

Analytic solutions to valuation problems are a rarity in finance, but the gamma process has, like Brownian motion, a set of special mathematical properties that make this possible. In the case of Brownian motion it is the additive decomposition of the Brownian motion into orthogonal components consisting of its terminal value and the associated bridge process. In the case of the gamma process there is a corresponding multiplicative decomposition.

The information-based asset pricing methodology makes it clear that the modelling of the filtration itself should be regarded as an essential component of pricing theory. This can be achieved by the introduction of the ideas of market factors and information processes. We have shown how closed-form solutions for the price processes of assets can be obtained in a number of different examples, and that complex cash-flow structures can be modelled efficiently with a good deal of flexibility. The price process of a defaultable zero-coupon bond is given by a closed-form expression that leads to an efficient simulation methodology. In the case where a binary defaultable bond is the underlying, the price of a call option can be exactly computed, and turns out to be of the Black-Scholes form. The information-based models for binary defaultable bonds can be calibrated by use of bond options.

Bibliography

[1] Martin BAXTER & Andrew RENNIE [1996]. *Financial Calculus: An Introduction to Derivative Pricing.* Cambridge University Press, Cambridge.

[2] Tomasz R. BIELECKI & Marek RUTKOWSKI [2002]. *Credit Risk : Modeling, Valuation and Hedging.* Springer Finance. Springer, Berlin, 1st edition.

[3] Fischer BLACK & John C. COX [1976]. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2): pp. 351–367.

[4] Fischer BLACK & Myron SCHOLES [1973]. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3): pp. 637–654.

[5] Dorje C. BRODY, Lane P. HUGHSTON, & Andrea MACRINA [2007]. Beyond hazard rates: a new framework for credit-risk modelling. In *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan*, edited by R. ELLIOTT, M. FU, R. JARROW, & J.-Y. YEN, Applied and Numerical Harmonic Analysis, pp. 231–258. Birkhäuser, Boston.

[6] Dorje C. BRODY, Lane P. HUGHSTON, & Andrea MACRINA [2008a]. Information-based asset pricing. *International Journal of Theoretical and Applied Finance*, 11(01): pp. 107–142.

[7] Dorje C. BRODY, Lane P. HUGHSTON, & Andrea MACRINA [2008b]. Dam rain and cumulative gain. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 464(2095): pp. 1801–1822.

[8] Umut ÇETIN, Robert JARROW, Philip PROTTER, & Yildiray YILDIRIM [2004]. Modeling credit risk with partial infromation. *The Annals of Applied Probability*, 14(3): pp. 1167–1178.

[9] John H. COCHRANE [2005]. *Asset Pricing.* Princeton Univeristy Press, Princeton.

[10] Pierre COLLIN-DUFRESNE, Robert S. GOLDSTEIN, & Jean HELWEGE [2003]. Is credit event risk priced? Modeling contagion via the updating of beliefs. Working paper.

[11] Guglielmo D'AMICO, Jacques JANSSEN, & Raimondo MANCA [2005]. Credit risk migration semi-markov models: a reliability approach. In *Proceedings of the XI Applied Stochastic*

Models and Data Analysis Symposium, edited by J. Janssen & P. Lenca, pp. 950–957. ENST Bretagne Technopole Brest.

[12] Darrell Duffie [2005]. Credit risk modeling with affine processes. *Journal of Banking and Finance*, 29(11): pp. 2751–2802.

[13] Darrell Duffie, Damir Filipovic, & Walter Schachermayer [2003]. Affine processes and applications in finance. *Annals of Applied Probability*, 13(3): pp. 984–1053.

[14] Darrell Duffie & David Lando [2001]. Term structures of credit spreads with incomplete accounting information. *Econometrica*, 69(3): pp. 633–64.

[15] Darrell Duffie, Jun Pan, & Kenneth Singleton [2000]. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6): pp. 1343–1376.

[16] Darrell Duffie & Kenneth J. Singleton [1999]. Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12(4): pp. 687–720.

[17] Darrell Duffie & Kenneth J. Singleton [2003]. *Credit Risk: Pricing, Measurement, and Management*. Princeton Series in Finance. Princeton University Press, Princeton (NJ).

[18] Abel Elizalde [2005a]. Credit risk models I: Default correlation in intensity models. Downloadable at: http://abelelizalde.com/pdf/survey1%20-%20intensity.pdf.

[19] Abel Elizalde [2005b]. Credit risk models II: Structural models. Downloadable at: http://abelelizalde.com/pdf/survey2%20-%20structural.pdf.

[20] Abel Elizalde [2005c]. Credit risk models III: Reconciliation reduced-structural models. Downloadable at: http://abelelizalde.com/pdf/survey3%20-%20reconciliation.pdf.

[21] Michel Emery & Marc Yor [2004]. A parallel between Brownian bridges and gamma bridges. *Publications of the Research Institute for Mathematical Sciences*, 40(3): pp. 669–688.

[22] Damir Filipovic [2005]. Time-inhomogeneous affine processes. *Stochastic Processes and their Applications*, 115(4): pp. 639–659.

[23] Damir Filipovic & Eberhard Mayerhofer [2009]. Affine diffusion processes: Theory and applications. Vienna Institute of Finance working paper.

[24] Kay Giesecke [2004a]. Credit risk modeling and valuation: An introduction. In *Credit Risk: Models and Management*, edited by David Shimko, chapter 16, pp. 487–526. Risk Books, London, 2nd edition.

[25] Kay Giesecke [2004b]. Correlated default with incomplete information. *Journal of Banking and Finance*, 28(7): pp. 1521–1545.

[26] Kay GIESECKE [2006]. Default and information. *Journal of Economic Dynamics & Control*, 30(11): pp. 2281–2303.

[27] Kay GIESECKE & Lisa R. GOLDBERG [2004a]. Forecasting default in the face of uncertainty. *Journal of Derivatives*, 12(1): pp. 11–25.

[28] Kay GIESECKE & Lisa R. GOLDBERG [2004b]. Sequential defaults and incomplete information. *Journal of Risk*, 7(1): pp. 1–26.

[29] Xin GUO, Robert A. JARROW, & Yan ZENG [2008]. Credit risk models with incomplete information. Cornell University working paper, To appear in Mathematics of Operations Research. Downloadable at: http://www.ieor.berkeley.edu/˜xinguo/papers/II.C.2.pdf.

[30] Xin GUO & Yan ZENG [2008]. Intensity process and compensator: A new filtration expansion approach and the Jeulin–Yor theorem. *The Annals of Applied Probability*, 18(1): pp. 120–142.

[31] John C. HULL [2006]. *Options, Futures and Other Derivatives*. Prentice Hall, New Jersey, 6 edition.

[32] Jacques JANSSEN & Raimondo MANCA [2006]. *Applied Semi–Markov Processes*. Springer, New York.

[33] Jacques JANSSEN & Raimondo MANCA [2007]. *Semi-Markov Risk Models for Finance, Insurance and Reliability*. Springer.

[34] Robert A. JARROW, David LANDO, & Stuart M. TURNBULL [1997]. A Markov model for the term structure of credit risk spreads. *Review of Financial Studies*, 10(2): pp. 481–523.

[35] Robert A. JARROW & Philip PROTTER [2004]. Structural versus reduced form models: a new information based perspective. *Journal of Investment Management*, 2(2): pp. 1–10.

[36] Robert A. JARROW & Stuart M. TURNBULL [1995]. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50(1): pp. 53–85.

[37] J.P. MORGAN & THE RISKMETRICS GROUP [1999]. *The J.P. Morgan Guide To Credit Derivatives*. Downloadable at:
http://www.defaultrisk.com/_pdf6j4/The_JP_Morgan_Guide_2_Crdt_Drvtvs.pdf.

[38] Ioannis KARATZAS & Steven E. SHREVE [1988]. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1st edition.

[39] Shigeo KUSUOKA [1999]. A remark on default risk models. *Advances in Mathematical Economics*, 1: pp. 69–82.

[40] David LANDO [1998]. On Cox processes and credit risky securities. *Review of Derivatives Research*, 2(2–3): pp. 99–120.

[41] Andrea Macrina [2006]. An information-based framework for asset pricing: X-factor theory and its applications. Ph.D. thesis, Department of Mathematics King's College London.

[42] Roger Mansuy & Marc Yor [2008]. *Aspects of Brownian motion*. Universitext. Springer-Verlag, Berlin Heidelberg.

[43] Robert C. Merton [1974]. On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29(2): pp. 449–470.

[44] Marek Musiela & Marek Rutkowski [2005]. *Martingale Methods in Financial Modelling*. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, Heidelberg, 2nd edition.

[45] Philip E. Protter [2004]. *Stochastic Integration and Differential Equations*. Stochastic Modelling and Applied Probability. Springer, Berlin, 2nd edition.

[46] Daniel Revuz & Marc Yor [1999]. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Heidelberg, 3rd edition.

[47] Marek Rutkowski & Nannan Yu [2007]. An extension of the Brody-Hughston-Macrina approach to modeling of defaultable bonds. *International Journal of Theoretical and Applied Finance*, 10(3): pp. 557–589.

[48] Philipp J. Schönbucher [2003]. *Credit Derivatives Pricing Models: Models, Pricing and Implementation*. John Wiley & Sons Ltd, Chichester.

[49] Steven E. Shreve [2004]. *Stochastic Calculus for Finance II: Continuous–Time Models*. Springer Finance. Springer, New York, 1st edition.

[50] David Williams [1991]. *Probability with Martingales*. Cambridge University Press.

[51] Marc Yor [2007]. Some remarkable properties of gamma processes. In *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan*, edited by R. Elliott, M. Fu, R. Jarrow, & J.-Y. Yen, Applied and Numerical Harmonic Analysis, pp. 37–47. Birkhäuser, Boston.

[52] Chunsheng Zhou [2001]. The term structure of credit spreads with jump risk. *Journal of Banking & Finance*, 25(11): pp. 2015–2040.