Charles University
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Doctoral Thesis

Structural and algorithmic properties of graph coloring

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Preface

Graph coloring problems can be traced to the Four Color Problem from the end of the 19th century. Aside from the classical graph colorings, numerous generalizations found its use in various industries. As an example, let us mention the radio frequency assignment problem whose importance has grown with cellular network development. In a different direction, colorings of embedded graphs dualize the notion of nowhere-zero flows, closely related to the cycle covers of graphs studied extensively in structural graph theory. This area is full of important open problems with deep connection to other parts of graph theory, such as Tutte's 3-flow Conjecture, 4-flow Conjecture, 5-flow Conjecture, and the Cycle Double Cover Conjecture.

In my thesis, I am not giving an overview of the field of graph coloring and related problems, as the area is huge; instead, I have selected a few particular problems that I have worked on. The thesis consists of four parts. The first part is focused on coloring of squares of planar graphs with no short cycles, the second part deals with similar problems for K_4-minor free graphs, the third part contains results on extending 5-colorings of cylinder graphs, and the fourth part is devoted to cycle covers of graphs. Each part consists of several chapters and is equipped with a brief introductory chapter to the particular problems addressed in the part. The results contained in the thesis originate in my research papers, some of them coauthored by my friends and colleagues, as listed on page v.

Most of the notation used in the thesis is standard and we do not provide a complete introduction to graph theory and the definitions of the terms used; instead, we just provide the extra definitions needed and those definitions where a confusion may arise. For the notation not covered here, we refer the reader to and of the standard graph theory textbooks, such as [20, 66, 84], if necessary.

I would also like to thank Department of Applied Mathematics for generous support during my doctoral studies. My special thanks belong to my advisor, Daniel Král', for leading me through my doctoral studies, presenting interesting problems to me, giving me advises regarding my research and for such a friendly atmosphere during the studies. I am grateful to my other coauthors, Zdeněk Dvořák, Tomáš Kaiser, Bernard Lidický, Robert Šámal, Riste Škrekovski, and Xuding Zhu for a great time during our joint research and numerous discussions on the topics. Last but not least, I would like to thank all the members and the students of Department of Applied Mathematics and Institute for Theoretical Computer Science for the wonderful environment, support, and excellent background without which I would hardly write this thesis.

As required by Charles University, I hereby declare that I wrote the thesis on my own and that I have included all the sources of information which I used to the references. I also authorize Charles University to lend this document to other institutions or individuals for academic and research purposes.

Prague, September 2008 Pavel Nejedlý

List of My Research Papers

The results contained in the thesis appear in the following papers:

1. Z. Dvořák, D. Král', P. Nejedlý, R. Škrekovski: *Coloring squares of planar graphs with girth six*, European Journal of Combinatorics **29(4)** (2008), 838–849.

2. Z. Dvořák, D. Král', P. Nejedlý, R. Škrekovski: *Distance constrained labelings of planar graphs with no short cycles*, accepted to a special issue of Discrete Applied Mathematics.

3. T. Kaiser, D. Král', B. Lidický, P. Nejedlý: *Short cycle covers of graphs with minimum degree three*, submitted. A preliminary version is available as ITI report 2008-375.

4. D. Král', P. Nejedlý: *Distance constrained labelings of K_4-minor free graphs*, accepted to a special issue of Discrete Mathematics.

5. D. Král', P. Nejedlý, R. Šámal: *Short cycle covers of cubic graphs*, submitted. A preliminary version is available as ITI report 2008-374.

6. D. Král', P. Nejedlý, X. Zhu: *Choosability of Squares of K_4-minor Free Graphs*, unpublished manuscript available as ITI report 2008-378. The same results were independently obtained by Hetherington and Woodall [42] and therefore we decided not to publish the paper.

7. P. Nejedlý: *Extending 5-colorings of Cylinder Graphs*, in preparation.

Other publications not included in the thesis:

8. D. Král', P. Nejedlý: *Group coloring and list group coloring are Π_2^P-complete*, Proceedings 29th International Symposium Mathematical Foundations of Computer Science 2004 (MFCS'04), Lecture Notes in Computer Science vol. 3153, Springer-Verlag, 2004, pp. 274–287.

9. P. Nejedlý: *Choosability of graphs with infinite sets of forbidden differences*, Discrete Mathematics **307(23)** (2007), 3040–3047.

Contents

Part I

Coloring Squares of Planar Graphs with no Short Cycles

Chapter 1

Introduction

Graph coloring is a very active area of graph theory. A particular attention is focused on colorings of planar graphs. In this part, we focus on coloring squares of planar graphs.

A (distance) *square* of a graph G is the graph on the same vertex set as G whose edges join precisely the pairs of vertices at distance at most two in G. One may view the problem as assigning colors to the vertices of the original graph in such a way that no two vertices at distance at most two will get the same color.

Speaking of coloring of squares of planar graphs, one has to mention Wegner's conjecture that asserts that the square of every planar graph with maximum degree Δ has a coloring with approximately $3\Delta/2$ colors.

Conjecture 1.1 (Wegner 1977, [83]) *Let G be a planar graph with maximum degree Δ. The chromatic number of G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \le \Delta \le 7$, and at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$, otherwise.*

Conjecture 1.1 has been recently verified by Thomassen [78] for graphs with maximum degree three and it remains open for $\Delta \ge 4$. For larger values of Δ, there is a series $[1, 2, 13, 43, 50, 51, 85]$ of improvements of the upper bound with the currently best known upper bound $\lfloor 5\Delta/3 \rfloor + 78$ due to Molloy and Salavatipour $[68, 69]$. A significant progress on the conjecture has been recently achieved by Havet, van den Heuvel, McDiarmid and Reed [40] who established the upper bound $(3/2+o(1))\Delta$, i.e., they verified the conjecture to be asymptotically true.

Another concept closely related to colorings of squares of graphs is $L(p, q)$-labeling. An $L(p, q)$-*labeling* of a graph G for integers $p \ge q \ge 1$ is a labeling of its vertices by non-negative integers such that the labels of adjacent vertices differ by at least p and those at distance two by at least q. The smallest K for which there exists an $L(p, q)$-labeling with labels $0, \ldots, K$ is called the $L(p, q)$-*span* of G and denoted by $\lambda_{p,q}(G)$. Clearly, the $L(1, 1)$-span of a graph G is equal to the chromatic number of G^2 decreased by one.

The research in the area of $L(p, q)$-labelings has been focused mainly on $L(2, 1)$-labelings because of their practical applications. Another reason is the conjecture of Griggs and Yeh [39] which asserts that $\lambda_{2,1}(G) \leq \Delta^2$ for every graph G with maximum degree $\Delta \geq 2$. The conjecture was verified for several special classes of graphs, including graphs of maximum degree two, outer planar graphs [15], planar graphs with maximum degree $\Delta \neq 3$ [9], chordal graphs [74] (see also [17, 59]), hamiltonian cubic graphs [54, 55], direct and strong products of graphs [56], etc. For general graphs, the original bound $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$ of [39] was improved to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$ in [18]. A more general result contained in [58] yields $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$ and the best known bound of $\Delta^2 + \Delta - 2$ was given by Gonçalves [37]. Recently, Havet, Reed and Sereni [40] proved that if the maximum degree of a graph is large enough, then $\lambda_{2,1}(G) \leq \Delta^2$, which settles the conjecture for graphs with large maximum degrees. Algorithmic aspects of $L(2, 1)$-labelings as well as $L(p, q)$-labelings are also well investigated [3, 12, 28, 31, 57, 67].

$L(p, q)$-labelings of planar graphs were intensively studied as well: van den Heuvel et al. [43] showed that $\lambda_{p,q}(G) \leq (4q - 2)\Delta + 10p + 38q - 24$, and Borodin et al. [13] provided the bound of $\lambda_{p,q}(G) \leq (2q - 1)\lceil 9\Delta/5 \rceil + 8p - 8q + 1$ for $\Delta \geq 47$. The best asymptotic result $\lambda_{p,q}(G) \leq q\lceil 5\Delta/3 \rceil + 18p + 77q - 18$ is due to Molloy and Salavatipour [68, 69]. Better bounds are known for planar graphs without short cycles [82]:

- $\lambda_{p,q}(G) \leq (2q - 1)\Delta + 4p + 4q - 4$ if G is a planar graph of girth at least seven,
- $\lambda_{p,q}(G) \leq (2q - 1)\Delta + 6p + 12q - 9$ if G is a planar graph of girth at least six, and
- $\lambda_{p,q}(G) \leq (2q - 1)\Delta + 6p + 24q - 15$ if G is a planar graph of girth at least five.

In this part of the thesis, we focus on a conjecture of Wang and Lih [82] asserting that whenever Δ is large enough, then for every planar graph G of girth at least six and maximum degree at most Δ, both the chromatic number of G^2 and $L(2, 1)$-span are bounded by $\Delta + 1$. In [14], Borodin at al. proved the conjecture to be true for the chromatic number of G^2 and girth at least seven, they also disproved it for girth six. We aim to complete their results by proving that for girth six, the chromatic number is bounded by $\Delta + 2$ (in Chapter 2) and showing the original conjecture is true also for $L(2, 1)$-labeling if the girth is at least seven (in Chapter 3).

1.1 Preliminaries

We now introduce particular notation used throughout this part of the thesis. All graphs considered in the following two chapters are simple, i.e., without parallel edges and loops. A *d-vertex* is a vertex of degree exactly d. An *$(\leq d)$-vertex* is a vertex of degree at most d. Similarly, an *$(\geq d)$-vertex* is a vertex of degree at least d. A *k-thread* is an induced path comprised of k 2-vertices. The set of all the neighbors of a vertex v is called the *neighborhood* of v and the neighborhood enhanced by v is called the *closed neighborhood* of v.

An *ℓ-face* is a face of length ℓ (counting multiple incidences, i.e., bridges incident to the face are counted twice). If the boundary of a face f forms a connected subgraph, then the subgraph formed by the boundary (implicitly equipped with the orientation determined by the embedding) is called the *facial walk*. A face f is said to be *biconnected* if its boundary is formed by a single simple cycle. The neighbors of a vertex v on the facial walk are called *f-neighbors* of v. Note that if f is biconnected, then each vertex incident with f has exactly two f-neighbors.

Let us consider a biconnected face f, and let $v_1, \ldots, v_k$ be ($\geq$ 3)-vertices incident to f listed in the order on the facial walk of f. The *type* of f is a k-tuple $(\ell_1, \ldots, \ell_k)$ if the part of the facial walk between v_i and v_{i+1} is an ℓ_i-thread. In particular, if v_i and v_{i+1} are f-neighbors, then ℓ_i is zero. Two face types are considered to be the same if they can be types of the same face, i.e., they differ only by a cyclic rotation and/or a reflection.

If the face f is biconnected and v is a vertex incident to f, then the neighbors of v that are not its neighbors on the facial walk are said to be *opposite* to the face f. Similarly, if both the faces f_1 and f_2 incident to an edge uv are biconnected, then the faces incident to v distinct from f_1 and f_2 are *opposite* to the vertex u (with respect to the vertex v).

Like many other results in this area, the proofs of our main results are based on the discharging method. Some of our arguments also use elementary facts on list colorings (choosability of graphs). List colorings were introduced independently by Erdős, Rubin and Taylor [24] and Vizing [81]. A graph G is said to be ℓ-choosable if for any assignment of lists $L(v)$ of sizes ℓ to the vertices of G, there exists a proper coloring c of G such that $c(v) \in L(v)$ for every vertex v. The gap between the list chromatic number (the smallest ℓ for which the graph is ℓ-choosable) and the usual chromatic number can be arbitrary large: for every integer ℓ, there exists a bipartite graph that is not ℓ-choosable. However, the only simple fact that we need in our consideration is the following: any cycle of even length is 2-choosable. The reader can figure out details of a simple proof of this statement him/herself or can consult [24].

Chapter 2

Planar Graphs with Girth Six

In this chapter, we prove that if Δ is large enough, the chromatic number of the square of every planar graph with maximum degree Δ is bounded by $\Delta + 2$. Together with the result of Borodin et al. [14], showing that $\Delta + 1$ is not enough, this gives the optimal upper bound.

The key notion of our proof is the notion of D-minimal graph. For an integer $D \geq 8821$, a graph G is called *D-good* if its maximum degree is at most D and the chromatic number of G^2 is at most $D + 2$. A planar graph G of girth at least 6 and maximum degree at most D is *D-minimal* if G is not D-good but every proper subgraph of G is D-good. If G is a D-minimal graph, then G is connected. Observe that G is also 2-connected: otherwise, color the blocks of G separately and afterwards permute the colors so that the colors of the cut-vertices match and the colors of their neighbors are pairwise distinct. In particular, the minimum degree of a D-minimal graph is at least two.

A vertex is said to be *small* if its degree is at most 1763, and it is said to be *big* otherwise.

In Sections 2.1–2.5, we show, using the discharging method, that there is no D-minimal graph. We assume that there is a D-minimal graph and assign charge to its vertices and its faces. The total amount of initial charge will be negative. We then redistribute charge in two phases as determined by the rules presented in Sections 2.3 and 2.4. We eventually obtain contradiction with our assumption that there exists a D-minimal graph by showing that the total final amount of charge is non-negative.

2.1 Reducible Configurations

Let us first describe several configurations that cannot appear in a D-minimal graph. Such a configuration is called *reducible*.

Lemma 2.1 *The following configurations are reducible:*

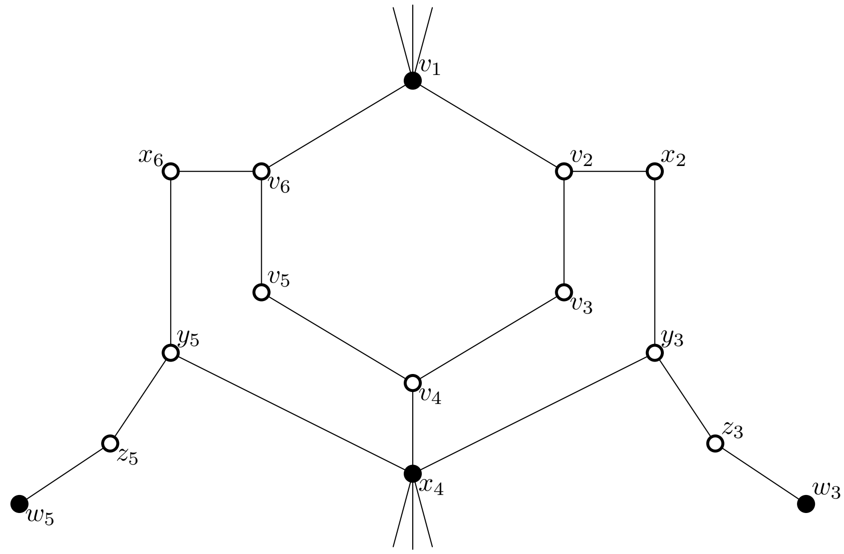

Figure 2.1: The reducible configuration from Lemma 2.1(5). The vertices that are not removed in the proof are represented by full circles.

1. *A small vertex u and a vertex v joined by a 2-thread.*

2. *Vertices u and v joined by two 2-threads.*

3. *A small vertex v joined by a 1-thread to a vertex u of degree at most six, such that all the neighbors of u are small.*

4. *Two adjacent 3-vertices u and v such that all the neighbors of u and v are small and at least one of the neighbors of u is a 2-vertex.*

5. *The configuration in Figure 2.1, where v_2, v_4, v_6, y_3 and y_5 are 3-vertices, v_3, v_5, x_2, x_6, z_3 and z_5 are 2-vertices, and w_3 and w_5 are small vertices (there is no restriction on the degrees of v_1 and x_4).*

Proof: Let G be a D-minimal graph, in particular, $\chi(G^2) > D + 2$. We deal with the configurations separately. In each of the cases, we first assume that G contains the configuration described in the statement of the lemma and we obtain contradiction by showing that G is not D-minimal.

1. Let x and y be the vertices of the 2-thread, where x is the vertex adjacent to u. Consider the graph $G' = G \setminus \{x, y\}$. Since G is D-minimal, the square of G' is $(D + 2)$-colorable. Since the degree of v in G' is at most $D - 1$, there are at least two colors distinct from the colors of v and its neighbors. At least one of them (call it γ) is distinct from the color of u. Assign the color γ to the vertex y. Since u is small, the degree of x in G^2 is at most

$1763 + 3 < D$. Therefore, we can choose a color distinct from colors of u, its neighbors in G', v and y for x. We obtained a proper coloring of G^2 by $(D + 2)$ colors. This contradicts the D-minimality of G.

2. Let the vertices of the 2-threads be x_1, x_2, y_1 and y_2 where x_i is adjacent to y_i and u for $i = 1, 2$. The square of the graph $G' = G \setminus \{x_1, x_2, y_1, y_2\}$ is $(D+2)$-colorable by the D-minimality of G. Fix a coloring of G' with $D+2$ colors. Let C_u and C_v be the sets of the colors which are assigned to no vertex in the closed neighborhood of u and v, respectively. Since the degrees of u and v in G' are at most $D-2$, both C_u and C_v have sizes at least three. Let c_u and c_v be the colors of u and v, respectively. Let $C'_u = C_u \setminus \{c_v\}$ and $C'_v = C_v \setminus \{c_u\}$. Assign the list C'_u to the vertices x_1 and x_2 and the list C'_v to the vertices y_1 and y_2. The subgraph of G^2 induced by $\{x_1, x_2, y_1, y_2\}$ is a 4-cycle. This graph is 2-choosable. Therefore, its vertices can be colored from the assigned lists. The coloring obtained by extending the coloring of G' to G in this way is a proper coloring of G^2 with $D + 2$ colors that contradicts our assumption that G is D-minimal.

3. Let x be the 2-vertex of the 1-thread. The square of the graph $G' = G \setminus \{x\}$ is $(D + 2)$-colorable. Fix such a coloring. The degree of u in G'^2 is at most $5 \cdot 1763 + 5 < D$. Therefore, we can modify the coloring by changing the color of u so that it is distinct from the color of v as well as from the colors of the neighbors if u in G'^2. The degree of x in G^2 is at most $1763 + 7 < D$. Hence, we can extend this coloring to x. This contradicts the D-minimality of G.

4. Let x be a 2-vertex adjacent to u. Let y be the vertex adjacent to x distinct from u. Let w be the neighbor of u distinct from x and v. By the D-minimality of G, the square of the graph $G' = G \setminus \{x, u\}$ is $(D + 2)$-colorable. Fix such a coloring. The vertex y has degree at most $D - 1$ in G', therefore at least two colors are unused on closed neighborhood of y in G'. Choose a color for x from the unused colors so that it is distinct from the color of w. The degree of v in G'^2 is at most $2 \cdot 1763 + 2 < D$. Therefore, it is possible to change the color of v so that it is distinct from the colors of x and w. Finally choose a color for u: its degree is at most $1763 + 6 < D$ in G^2. Therefore, it is always possible. This contradicts the D-minimality of G.

5. The square of the graph $G' = G \setminus \{v_2, v_3, v_4, v_5, v_6, x_2, x_6, y_3, y_5, z_3, z_5\}$ is $(D + 2)$-colorable (the removed vertices are marked by empty circles in Figure 2.1). Fix a coloring of G' with $D + 2$ colors. Since the degree of x_4 in G' is at most $D - 3$, there are at least four colors which are not assigned to a vertex of the closed neighborhood of x_4 in G'. Let L_4 be the set of the unused colors. The degree of v_1 in G' is at most $D - 2$, therefore the

set L_1 of colors that do not appear on closed neighborhood of v_1 has size at least three. Let c_5 be the color of w_5 and c_3 the color of w_3. Assign the list L_1 to vertices v_2 and v_6, the list L_4 to the vertex v_4, the list $L_4 \setminus \{c_5\}$ to the vertex y_5 and the list $L_4 \setminus \{c_3\}$ to the vertex y_3. All 2-vertices of the configuration are adjacent only to small vertices. Therefore, if we were able to color the subgraph G'' of G^2 induced by $\{v_2, v_4, v_6, y_3, y_5\}$ from the lists, we could choose colors for the 2-vertices of the configuration carefully and extend the coloring to the coloring of the whole graph G^2. This would eventually contradict the D-minimality of G.

However, such a coloring of G'' always exists. Choose a color for v_4 from L_4 arbitrarily, and remove this color from the lists of the remaining four vertices. The graph $G'' \setminus \{v_4\}$ is a 4-cycle. Since it is 2-choosable, the remaining vertices of G'' can be colored from the assigned lists.

∎

2.2 Initial Charge

We now describe the amounts of initial charge of vertices. The initial charge of a d-vertex v is set to

$$\mathrm{ch}(v) = d - 3,$$

and the initial charge of an ℓ-face f to

$$\mathrm{ch}(f) = \ell/2 - 3.$$

It is easy to verify that the sum of initial charges is negative:

Proposition 2.2 *If G is a connected planar graph, then the sum of all initial charges of the vertices and faces of G is -6.*

Proof: Since G is connected, Euler's formula yields that $n + f = m + 2$ where n is the number of the vertices of G, m is the number of its edges and f is the number of its faces. The sum of initial charges of the vertices of G is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = 2m - 3n.$$

The sum of initial charges of the faces of G is equal to

$$\sum_{f \in F(G)} \left(\frac{\ell(f)}{2} - 3 \right) = m - 3f.$$

Therefore, the sum of initial charges of all the vertices and faces is $3m-3n-3f = -6$.

■

Note that the amounts of initial charge were chosen such that each face of size at least 6 (consequently, each face of a D-minimal graph) has non-negative charge, the charge of 6-faces is zero and only 2-vertices have negative charge of -1 unit.

2.3 First Discharging Phase

The goal of the first phase is that each 2-vertex receives 2ε units of charge and the amount of charge of other vertices and faces is not decreased too much where $\varepsilon = 1/588$.

If u is a 2-vertex, an edge $e = uv$ is *void* if either $d(v) \in \{2, 4, 5, 6\}$, or v is a 3-vertex and all its neighbors are small. Intuitively, the void edges are those through which it may be impossible to send any charge to u.

In order to simplify the analysis of final charge of big vertices, we send all charge transfered from a big vertex through the edges incident to it. Each rule that deals with big vertices specifies through which edge the charge is (considered to be) sent. The value of ε and the bound on the degree of big vertices was chosen in such a way that a big vertex is able to send $1-\varepsilon$ units of charge through each edge incident to it, and its final charge is still non-negative.

If v is a big vertex, we call an edge uv *red* if one of the following conditions holds:

- the vertex u is a 2-vertex, $e \neq uv$ is the other edge incident to u, and e is void, or
- the vertex u is a 3-vertex, x_1 and x_2 are the neighbors of u distinct from v, both x_1 and x_2 are 2-vertices, and all the neighbors of x_1 and x_2 are small.

The edges incident to big vertices which are not red are called *green*. Intuitively, the green edges are those through which the big vertex does not need to send "too much" charge and the red ones are those through which almost one unit of charge has to be sent.

In order to simplify the description of the rules, we define the following operation: if f is a 6-face and F is the set containing f and all the 6-faces sharing an edge with f, a 6-face f is *boosted* from a vertex or face z when 3ε units of charge are transferred from z to each face of F. Note that the charge of z decreases by at most 21ε.

The discharging rules of the first phase are the following:

F1 Each (≥ 7)-face boosts all the 6-faces sharing an edge with it.

F2 If v is a big vertex, e is a green edge incident to it and f is a 6-face incident to e, then the vertex v boosts f. The charge is sent through the edge e.

F3 If v is a small vertex of degree at least 4, then it boosts all the incident 6-faces.

F4 If v is a 2-vertex and f is a face incident to v, then f sends ε units of charge to v.

Note that no charge is sent through a red edge in the first phase. We now analyze the amounts of charge after the first phase:

Lemma 2.3 *Let G be a D-minimal graph. After the first phase of discharging, the following claims hold:*

1. *at most $1/8$ units of charge was sent through each green edge,*
2. *the charge of a small vertex of degree $d \geq 4$ has decreased by at most $d/16$,*
3. *the charge of each 2-vertex is $2\varepsilon - 1$, and*
4. *the charge of each face is non-negative.*

Proof: We prove each claim separately:

1. Charge is sent through green edges only by Rule F2. Each green edge e is incident to at most two 6-faces and thus the total amount of charge sent through e is at most $42\varepsilon \leq 1/8$.
2. Charge is sent from small vertices only by Rule F3. A d-vertex is incident to at most d 6-faces. Therefore, the total amount of sent charge is at most $21\varepsilon d \leq d/16$.
3. Each 2-vertex receives ε units of charge from both the incident faces by Rule F4. Therefore, its charge becomes $2\varepsilon - 1$.
4. Charge is sent from faces by Rules F1 and F4. A d-face f shares an edge with at most d 6-faces. Therefore, the total amount of charge sent from f by Rule F1 is at most $21\varepsilon d$. Since at most d 2-vertices are incident to f, at most εd units of charge are sent by Rule F4. In total, at most $22\varepsilon d$ units of charge are sent from f.

   The charge of a d-face with $d \geq 7$ after the first phase is at least

   $$\frac{d}{2} - 3 - 22\varepsilon d = \left(\frac{1}{2} - 22\varepsilon\right) d - 3 \geq \frac{3}{7}d - 3 \geq 0.$$

   Hence, if f is a (≥ 7)-face, its final charge is non-negative.

It remains to consider the case when f is a 6-face. Let k be the number of 2-vertices incident to f. Observe that k does not exceed 3: otherwise f contains at least four 2-vertices and it thus contains either a 3-thread or two vertices connected by two 2-threads. Both configurations are reducible by Lemma 2.1.

Initial charge of f is zero and f sends out charge of $k\varepsilon$ by Rule F4. If $k = 0$, the final charge of f is non-negative. Assume that $k > 0$. It is sufficient to prove that f receives at least 3ε units of charge by Rules F1, F2 and F3. We show that f or one of the 6-faces incident to f is boosted during the first phase.

If f shares an edge with a (≥ 7)-face, f is incident to a small vertex of degree at least 4, or f is incident to a green edge, then f itself is boosted. Therefore, we may assume that no edge incident to f is green, all the vertices incident to f are either big or have degree 2 or 3, and all the faces sharing an edge with f are 6-faces.

Let $v_1, \ldots, v_6$ be the vertices of f in a cyclic order around the face.

Suppose first that f is incident to at least two big vertices. Assume that v_1 is a big vertex. The second big vertex of f is v_4: otherwise, the two big vertices are either f-neighbors or share an f-neighbor and at least one of the edges of f is green. If all the f-neighbors of v_1 and v_4 were 2-vertices, then v_1 and v_4 would be joined by two 2-threads, which is impossible by Lemma 2.1. Therefore at least one of the big vertices is adjacent to a 3-vertex. Assume that v_2 is a 3-vertex. But since v_4 is big, the edge v_1v_2 is green regardless of the degree of v_3. Therefore, the face f is boosted.

If f is incident to no big vertex, then no two 2-vertices of f are adjacent by Lemma 2.1(1). Assume that v_2 is a 2-vertex. Therefore, v_1 and v_3 are 3-vertices. Let x_1 and x_3 be the neighbors of v_1 and v_3 not incident to f. Since v_6 and v_4 are small, both x_1 and x_3 are big by Lemma 2.1(3). Let f' be the 6-face incident to v_2 distinct from f. Note that both x_1 and x_3 belong to the 6-face f' and share a common f'-neighbor. Hence, at least one of the edges incident to f' is green. Consequently, f' is boosted and f receives the charge of 3ε units.

It remains to consider the case when f contains exactly one big vertex, say v_1. If v_4 were a 2-vertex, we could use a similar argument as in the previous paragraph to show that the other face incident to v_4 is boosted. Therefore, we can assume that v_4 is a 3-vertex. In addition, either v_2 or v_3 is a 2-vertex, since the edge v_1v_2 is not green.

First suppose that v_2 is a 2-vertex. Hence v_3 is a 3-vertex. Let x_3 and x_4 be the neighbors of v_3 and v_4 not incident to f. If x_3 is big, then the edge v_1v_2 is green. And, if x_4 is big, then the edge x_4v_4 is green. In both the cases, f

receives the required charge. If both x_3 and x_4 are small, the configuration is reducible by Lemma 2.1(4). The case that v_6 is a 2-vertex is symmetrical.

Suppose now that both v_2 and v_6 are 3-vertices and v_3 is a 2-vertex. We may assume that the neighbors of v_2 and v_6 (including v_5) distinct from v_1 are 2-vertices: otherwise, one of the edges v_1v_2 and v_1v_6 would be green. Let x_2, x_4 and x_6 be the vertices adjacent to v_2, v_4 and v_6 and not incident to f. By Lemma 2.1(3), the vertex x_4 is big. Let f_3 and f_5 be the faces incident to v_3 and v_5 and distinct from f. Let y_5 be the remaining vertex of f_5 distinct from x_6, x_4, v_4, v_5 and v_6. Let y_3 be the remaining vertex of f_3 distinct from x_2, x_4, v_2, v_3 and v_4. The degrees of both y_3 and y_5 must be 3: they cannot be two by Lemma 2.1(1) and if one of them were greater than 3, then one of the edges y_3x_4 and y_5x_4 would be green and f would receive charge because of boosting from f_3 or f_5. Let z_3 and z_5 be the neighbors of y_3 and y_5 distinct from x_6, x_4 and x_2. Both z_3 and z_5 must be 2-vertices and all their neighbors must be small, since otherwise one of edges y_3x_4 or y_5x_4 is green. However, the resulting configuration is reducible by Lemma 2.1(5). This finishes the proof of the claim.

■

2.4 Second Phase of Discharging

In this phase we redistribute the charge so that the final charge of all vertices is non-negative. The following rules are used during this phase:

S1 If v is a big vertex adjacent to a 2-vertex u, then v sends $1-\varepsilon$ units of charge to u if uv is red and it sends $3/4$ units of charge to u if uv is green. The charge is sent through the edge uv.

S2 If v is a big vertex adjacent to a 3-vertex u and the edge uv is red, then v sends $(1-\varepsilon)/2$ units of charge to both the 2-vertices adjacent to u. The charge is sent through the edge uv.

S3 Suppose that v is a big vertex adjacent to a 3-vertex u, the edge uv is green, and x is a 2-vertex adjacent to u. If x has a big neighbor, then v sends charge of $1/4$ to x. Otherwise, v sends charge of $1/2$ to x. The charge is sent through the edge uv.

S4 If v is a big vertex adjacent to a d-vertex u, $4 \le d \le 6$, then the vertex v sends $3/4$ units of charge to u. The charge is sent through the edge uv.

S5 If v is a d-vertex, $4 \le d \le 6$, adjacent to a 2-vertex u, and if v has at least one big neighbor, then v sends $1/2$ units of charge to u.

S6 If v is a small vertex of degree $d > 6$ adjacent to a 2-vertex u, then v sends $1/2$ units of charge to u.

We now analyze the amounts of charge sent during the second phase:

Lemma 2.4 *Let G be a D-minimal graph. The following claims hold:*

1. *at most $3/4$ units of charge was sent through each green edge during the second phase,*
2. *at most $1-\varepsilon$ units of charge was sent through each red edge during the second phase, and*
3. *the charge of each vertex is non-negative after performing the first and the second phase.*

Proof: We prove each claim separately:

1. At most one of Rules S1, S3 and S4 applies to each green edge. At most $3/4$ units of charge is sent through such an edge by any of the rules. The only case in which this is not obvious is the case of Rule S3. However, there can be at most one vertex x without a big neighbor that satisfies the assumptions of the rule: otherwise the edge uv is red.
2. At most one of Rules S1 and S2 applies to each red edge and the charge sent through such an edge is exactly $1-\varepsilon$ by any of the rules.
3. Let v be a d-vertex of G. We consider several cases regarding the degree of the vertex v:

   $d = 2$**:** Let x and y be the neighbors of v. It suffices to show that v received at least $1-\varepsilon$ units of charge during the second phase because charge of v was at least 2ε after the first phase by Lemma 2.3.

   Suppose first that x is big. If the edge vy is void, then the edge xv is red and v received charge of $1-\varepsilon$ from x by Rule S1. Assume that the edge vy is not void and that the edge xv is green. Consequently, v received $3/4$ units of charge by Rule S1. Additionally, since vy is not void, then either y is a 3-vertex and has a big neighbor w, or y is a (≥ 7)-vertex. In the former case, v receives $1/4$ units of charge from w by Rule S3. In the latter case, y sends $1/2$ units of charge to v by Rules S1 or S6. In both the cases, the total charge received by v is at least 1.

   The final case is that both x and y are small. By Lemma 2.1(1), neither x nor y has degree 2. We show that v receives at least $(1-\varepsilon)/2$ units of charge through x. Note that by symmetry v also receives at least

$(1-\varepsilon)/2$ units of charge through y, i.e., v receives $1-\varepsilon$ units of charge in total. Let d' be the degree of x. If $3 \le d' \le 6$, at least one neighbor of x must be big by Lemma 2.1(3). Consequently, v receives at least $(1-\varepsilon)/2$ by one of Rules S2, S3 and S5. If $d' \ge 7$, then v receives $1/2$ from x by Rule S6.

$d = 3$**:** None of the discharging rules changes the charge of a vertex of degree three. Therefore, the final charge of v is zero.

$4 \le d \le 6$**:** The d-vertex v sent charge of at most $d/16$ units during the first phase by Lemma 2.3(2). If v is not adjacent to a big vertex, then it does not send anything during the second phase. Otherwise, it sends at most $(d-1)/2$ units of charge by Rule S5 and receives charge of at least $3/4$ units by Rule S4. Therefore, the final charge of v is

$$d - 3 - \frac{d}{16} - \frac{d-1}{2} + \frac{3}{4} = \frac{7d}{16} - \frac{7}{4} \ge 0.$$

$d \ge 6$ **and** v **is small:** The vertex v sends at most $d/16$ units of charge during the first phase by Lemma 2.3(2) and at most $d/2$ units of charge during the second phase by Rule S6. Therefore, the final charge of v is at least

$$d - 3 - \frac{d}{16} - \frac{d}{2} = \frac{7d}{16} - 3 > 0.$$

v **is big:** All the charge sent out from the big vertex v was sent through some of the edges incident to it. Charge is sent through a red edge e only in the second phase and the total amount of such charge is at most $1-\varepsilon$ by the previous claim of this lemma. At most $1/8$ units of charge is sent through a green edge e in the first phase by Lemma 2.3 and at most $3/4$ units in the second phase, thus in total $7/8 < 1-\varepsilon$. Therefore, v has the final charge of at least $d-3-(1-\varepsilon)d = \varepsilon d - 3 \ge 0$ (recall that v is a d-vertex with $d > 1763$).

■

2.5 Final Step

We now combine our arguments from the previous sections:

Theorem 2.5 *If G is a planar graph of maximum degree $\Delta \ge 8821$ and girth at least six, then G has a proper $L(1,1)$-labeling with span $\Delta+1$, i.e., $\chi(G^2) \le \Delta+2$.*

Proof: If the statement of the theorem is false, then there exists a D-minimal graph. Consider such a D-minimal graph G. Assign charge to the vertices and

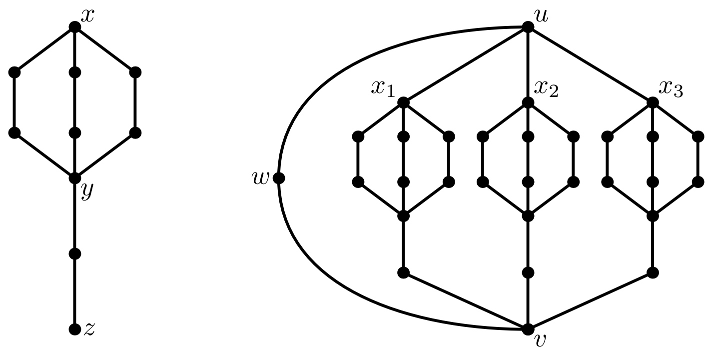

Figure 2.2: The graphs G'_4 and G_4.

the faces of G as described in Section 2.2. By Proposition 2.2, the sum of all the charges is negative. Apply the discharging rules of the two phases described in Sections 2.3 and 2.4. The final amount of charge of each face is non-negative after the first phase by Lemma 2.3 and it is preserved during the second phase, i.e., it is non-negative after the second phase. The final amount of charge of each vertex is non-negative after the second phase by Lemma 2.4. Therefore, the total final amount of charge is non-negative. We conclude that there is no D-minimal graph.

■

2.6 Lower Bound

For the sake of completeness, we also present a construction of planar graphs G with $\chi(G^2) = \Delta+2$ and girth six. This shows that our bound is the best possible. A different construction of such graphs can be found in [14]. One of the reasons that also led us to include our construction to the thesis is that our construction yields graphs with fewer vertices than that of [14].

Let G'_Δ be a graph of order $2\Delta + 2$ formed by two vertices x and y joined by $(\Delta - 1)$ 2-threads and a vertex z joined to y by a 1-thread. Let G_Δ be a graph obtained by taking $\Delta - 1$ copies of G'_Δ, identifying all the vertices z of the copies into a single vertex v, and adding a vertex u joined to v by a 1-thread and by an edge to the vertex x of each copy of G'_Δ (see Figure 2.2). Clearly, the girth of G_Δ is six and the maximum degree of G_Δ is Δ. The chromatic number of G_Δ is determined in the next proposition:

Proposition 2.6 *The chromatic number of the square of the graph G_Δ is $\Delta + 2$ for every $\Delta \geq 2$.*

Proof: It is easy to construct a coloring of G^2_Δ by $\Delta + 2$ colors. We focus on showing that it cannot be colored by $\Delta + 1$ colors.

We first show that in any proper coloring of the square of G'_Δ, the colors assigned to x and z are distinct. Suppose for contradiction that there exists a proper coloring of G'^2_Δ by the colors $0, \ldots, \Delta$ such that the colors of both x and z are the same, say 0. Since the vertex y has degree Δ, either y or one of its neighbors must have color 0. This is impossible because each of these vertices is at distance at most two from x or z.

Suppose now that the graph G_Δ can be colored by the colors $0, \ldots, \Delta$. Let $x_1, \ldots, x_{\Delta-1}$ be the vertices of the copies of G'_Δ adjacent to the vertex u. Let w be the vertex adjacent to u and distinct from all x_i, $1 \leq i < \Delta$. We may assume that the color of v is 0. By the observation from the previous paragraph, the color of each vertex x_i is distinct from 0. The vertex u has degree Δ. Therefore, either u or one of its neighbors has color 0. This is impossible since the colors of vertices x_i are distinct from 0 and both u and w are at distance at most two from the vertex v. We conclude that there is no proper coloring of G^2_Δ with $\Delta+1$ colors.

■

Chapter 3

Planar Graphs with Girth Seven

In this chapter, we prove that $L(2,1)$-span of a planar graph of girth at least seven and sufficiently large maximum degree Δ is bounded by $(\Delta + 1)$. Since we prove the result in a more general setting of $L(p,1)$-labelings, we obtain an alternative proof of the result of Borodin et al. [14] (by setting $p = 1$) as well.

For the purpose of the proof, we use the following notation (note that similar notation was used in the previous chapter, however, the constants were different). For an integer $D \geq 192$, a graph G is *D-good* if its maximum degree is at most D and it has an $L(p,1)$-labeling of span at most $D+2p-2$ for every $p \leq (D-190)/2$. A planar graph G of girth at least 7 and maximum degree at most D is said to be *D-minimal* if it is not D-good but every proper subgraph of G is D-good. Clearly, if G is D-minimal, then it is connected. A vertex of G is said to be *small* if its degree is at most 95, and *big* otherwise.

In the proof, we again use discharging method to show that there is no D-minimal graph, i.e., all planar graphs of girth at least seven and maximum degree at most D are D-good. In order to show this, we first describe configurations that cannot appear in a D-minimal graph (reducible configurations). In the proof, we consider a potential D-minimal graph and assign each vertex and each face a certain amount of charge. The amounts are assigned in such a way that their sum is negative. The charge is then redistributed among the vertices and faces according to the rules described in Section 3.3. It is shown that if the considered graph is D-minimal, then the final charge of every vertex and every face is non-negative after the redistribution. Since the sum of the initial charges is negative, we obtain a contradiction and conclude that there is no D-minimal graph.

3.1 Structure of D-minimal Graphs

We now identify configurations that cannot appear in D-minimal graphs. The following argument is often used in our considerations: we first assume that there exists a D-minimal graph G that contains a certain configuration. We remove some vertices of G and find a proper $L(p,1)$-labeling of the new graph (the labeling exists because G is D-minimal). We then recolor some of the vertices: at this stage, we state the properties that the new colors of the recolored vertices should have, and recolor the vertices such that the properties are met (and show that it is possible). If the original colors of such vertices already have the desired properties, then the vertices just keep their original colors. Finally, the labeling is extended to the removed vertices.

We have already seen that every D-minimal graph is connected. Similarly, it is not hard to see that the minimum degree of a D-minimal graph is at least two:

Lemma 3.1 *If G is a D-minimal graph, then its minimum degree is at least two.*

Proof: Assume that G contains a vertex v of degree one (since G is connected, it has no vertices of degree zero). Fix an integer $p \leq (D-190)/2$ such that G has no proper $L(p,1)$-labeling of span $D+2p-2$. Let v' be the neighbor of v in G. Remove v from G. Since G is D-minimal, the obtained graph has a proper $L(p,1)$-labeling c of span $D+2p-2$. We extend the labeling c to v: the vertex v cannot be assigned at most $2p-1$ colors whose difference from the color of v' is less than p and it cannot be assigned at most $D-1$ colors which are assigned to the other neighbors of v'. Therefore, there are at most $D+2p-2$ forbidden colors for v. In particular, there exists a color that can be assigned to v, and thus c can be extended to v. This contradicts our assumption that G is D-minimal.

■

Observe that Lemma 3.1 implies that every ℓ-face of a D-minimal graph G for $\ell \leq 13$ is biconnected because of the girth assumption and that the facial walk of every ℓ-face with $\ell \leq 11$ induces a chordless cycle of G.

Next, we focus on 2-, 3- and 4-threads contained in D-minimal graphs:

Lemma 3.2 *If vertices v and w of a D-minimal graph G are joined by a 2-thread, then at least one of the vertices v and w is big.*

Proof: Fix an integer $p \leq (D-190)/2$ such that G has no proper $L(p,1)$-labeling with span $D+2p-2$. Let $v'w'$ be the 2-thread between v and w in G (where v' is the neighbor of v). Assume for the sake of contradiction that neither v nor w is big. Remove the vertices v' and w' from G. Since G is D-minimal, there exists a proper $L(p,1)$-labeling c of the obtained graph whose span does not exceed $D+2p-2$. We extend the labeling c to the vertices v' and w'.

Let A_v be the set of the colors that differ by at least p from the color of v and are different from the colors of all the neighbors of v and from the color of w. Similarly, let A_w be the set of the colors that differ by at least p from the color of w and are different from the colors of all the neighbors of w and from the color of v. Since w is not a big vertex, the number of these colors is at least

$(D + 2p - 1) - (2p - 1) - 94 - 1 \geq 2p$, since $D - 95 \geq 2p$. Similarly, we have $|A_v| \geq 2p$.

Color now the vertices v' and w' by colors from A_v and A_w that differ by at least p (observe that such colors always exist). The obtained labeling c is a proper $L(p, 1)$-labeling of G with span at most $D + 2p - 2$.

■

The following two statements readily follow:

Lemma 3.3 *No D-minimal graph G contains a 4-thread.*

Proof: Assume that a D-minimal graph G contains a 4-thread $vv'v''v'''$. By Lemma 3.2, v or v''' is big and $vv'v''v'''$ is not a 4-thread.

■

Lemma 3.4 *If vertices v and w of a D-minimal graph G are joined by a 3-thread, then both v and w are big.*

Proof: Let $v'v''v'''$ be the 3-thread joining v and w. By Lemma 3.2, v or v''' is big. Since v''' is a 2-vertex, v is big. Similarly, we infer that w is big.

■

Next, we focus on cycles of lengths seven and eight contained in D-minimal graphs. Note that the boundary of every 7-face and 8-face is biconnected (because of the girth assumption and Lemma 3.1), i.e., its boundary is a simple cycle of length seven or eight, and thus the following lemma can always be applied in such cases.

Lemma 3.5 *Let $v_1v_2v_3v_4v_5v_6v_7$ be a part of a 7-cycle or an 8-cycle contained in a D-minimal graph G. If v_2, v_3, v_5 and v_6 are 2-vertices, then v_1 or v_7 is a big vertex.*

Proof: Fix an integer $p \leq (D-190)/2$ such that G has no proper $L(p, 1)$-labeling with span $D + 2p - 2$. Note that the distance between the vertices v_1 and v_7 is at most two. Assume that neither v_1 nor v_7 is big. Remove the vertices v_2, v_3, v_5 and v_6 from G. Since G is D-minimal, the new graph has an $L(p, 1)$-labeling c of span at most $2p + D - 2$. Let A be the set of colors γ that differ from the color of v_4 by at least p and such that no neighbor of v_4 is colored with γ. Since there are $2p + D - 1$ colors available and the degree of v_4 in the new graph does not exceed $D - 2$, we infer that $|A| \geq 2$.

We extend the labeling c to the removed vertices. Color the vertices v_5 and v_3 by distinct colors from A in such a way that the colors of v_5 and v_7 are different, and the colors of v_3 and v_1 are also different. Since the colors of v_7 and v_1 are different (the distance of v_7 and v_1 in G is at most two), this is always possible.

Color now the vertex v_6 by a color that differs by at least p from the colors of v_5 and v_7 and that differ from the colors of v_4 and (at most 94) neighbors of v_7. Since there are at most $95 + 4p - 2 \leq 2p + D - 2$ forbidden colors for v_6, the vertex v_6 can be colored. Similarly, it is possible to color the vertex v_2. Since the obtained labeling is a proper $L(p, 1)$-labeling with span at most $2p + D - 2$, the graph G is not D-minimal.

■

The following result is an easy consequence of Lemma 3.5:

Lemma 3.6 *No D-minimal graph G contains a pair of vertices joined by two 3-threads.*

Proof: Assume for the sake of contradiction that G contains two vertices v and w joined by two 3-threads. The vertices v, w and the two 3-threads joining them comprise an 8-cycle in G. By Lemma 3.5, at least one of the neighbors of w in the 3-threads is big, but both the neighbors are 2-vertices.

■

We now focus on 3-vertices in D-minimal graphs:

Lemma 3.7 *Let $v_1v_2v_3v_4$ be a path of a D-minimal graph G where v_2 is a 3-vertex. If neither v_1 nor v_4 is big and v_3 is a 2-vertex, then the remaining neighbor w of v_2 is big.*

Proof: Fix an integer $p \leq (D - 190)/2$ such that G has no $L(p, 1)$-labeling of span $2p + D - 2$. Assume that w is not big. Remove the vertex v_3 from G. Since G is D-minimal, there exists a proper $L(p, 1)$-labeling of the obtained graph with span at most $2p + D - 2$. We first change the color of v_2 and then we extend the labeling c to the vertex v_3.

Recolor the vertex v_2 by a color that differs from the colors of v_1 and w by at least p, and that is different from the colors of all the neighbors of v_1 and w and from the color of v_4. Since neither v_1 nor w is big, there are at most $2(2p - 1) + 2 \cdot 94 + 1 \leq 2p + D - 2$ forbidden colors for v_2. Hence, the vertex v_2 can be recolored.

Finally, color the vertex v_3 by a color that differs from the colors of v_2 and v_4 by at least p, and that is different from the colors of all the neighbors of v_2 and v_4. Since v_2 is a 3-vertex and v_4 is not big, there are at most $2(2p-1)+94+2 \leq 2p + D - 2$ forbidden colors and v_3 can be colored.

■

We finish this section by establishing a lemma on the structure of faces of type $(2, 1, 1)$:

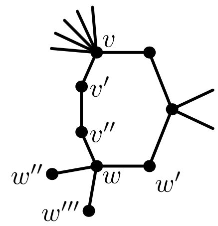

Figure 3.1: Notation used in the proof of Lemma 3.8.

Lemma 3.8 *The following configuration does not appear in a D-minimal graph G: a 7-face f of type $(2, 1, 1)$ with one big and two 4-vertices such that both the 4-vertices of f are adjacent only to small vertices.*

Proof: By Lemma 3.2, the big vertex incident to f delimits the 2-thread. Let v be the big vertex and w the other vertex delimiting the 2-thread and let $v'v''$ be the 2-thread (the 2-vertex v' is an f-neighbor of v). Let w', w'' and w''' be the neighbors of w different from v'' (see Figure 3.1) and assume that w' is an f-neighbor of w.

Fix an integer $p \leq (D - 190)/2$ such that G has no proper $L(p, 1)$-labeling with span $2p + D - 2$. Remove the vertices v'' and w' from G. Since G is D-minimal, there exists a proper $L(p, 1)$-labeling c of the new graph whose span is at most $2p + D - 2$. Next, we change the color of w and we extend the labeling c to the vertices v'' and w'.

Recolor the vertex w by a color that differs by at least p from the colors w'' and w''', and that is different from the colors of all the neighbors of w'' and w''' and that is also different from the color of v' and the other 4-vertex incident to f. Since none of the vertices w'' and w''' is big, the number of colors forbidden for w does not exceed $2(2p - 1) + 2 \cdot 94 + 2 \leq D + 2p - 2$. Hence, the vertex w can be recolored.

Next, color the vertex w' by a color that differs from the colors of both the 4-vertices incident with f by at least p and that is also different from the colors of all the six neighbors of the 4-vertices. Since the number of such forbidden colors does not exceed $2(2p - 1) + 6 \leq D + 2p - 2$, the vertex w' can be colored.

Finally, we color the vertex v'' by a color that differs from the colors of v' and w by at least p and that is different from the colors of the vertices v, w', w'' and w'''. Since there are at most $4p + 2 \leq D + 2p - 2$ forbidden colors, the labeling c can be also extended to the vertex v''.

$\blacksquare$

3.2 Initial Charge

We now describe the amounts of initial charge of vertices. The initial charge of a d-vertex v is set to

$$\mathrm{ch}(v) = d - 3,$$

and the initial charge of an ℓ-face f to

$$\mathrm{ch}(f) = \ell/2 - 3.$$

It is easy to verify that the sum of initial charges is negative:

Proposition 3.9 *If G is a connected planar graph, then the sum of all initial charges of the vertices and faces of G is -6.*

Proof: Since G is connected, Euler's formula yields that $n + f = m + 2$ where n is the number of the vertices of G, m is the number of its edges and f is the number of its faces. The sum of initial charges of the vertices of G is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = 2m - 3n.$$

The sum of initial charges of the faces of G is equal to

$$\sum_{f \in F(G)} \left(\frac{\ell(f)}{2} - 3 \right) = m - 3f.$$

Therefore, the sum of initial charges of all the vertices and faces is $3m - 3n - 3f = -6$.

■

Note that the amounts of initial charge were chosen such that each face of size at least 6 (consequently, each face of a D-minimal graph) has non-negative charge, the charge of 6-faces is zero and only 2-vertices have negative charge of -1 unit.

3.3 Discharging Rules

Next, the charge is redistributed among the vertices and faces of a (potential) D-minimal graph by the following rules:

R1 Each face f sends a charge of $1/2$ to every incident 2-vertex.

R2 Each 4-vertex sends a charge of $1/4$ to every incident face.

R3 Each small (≥ 5)-vertex sends a charge of $5/16$ to every incident face.

R4 Each big vertex adjacent to a 3-vertex w sends a charge of $5/16$ to the opposite face through w.

R5 Each big vertex adjacent to a 4-vertex w sends a charge of $1/16$ to each of the two opposite faces through w.

R6 If v is a big vertex incident to a face f and v_1 and v_2 are its f-neighbors, then v sends the following charge to f:

$1/2$	if $k = 0$,
$3/4$	if $k = 1$,
$15/16$	if $k = 2$ and the type of f is not $(3, 2)$, and
1	if the type of f is $(3, 2)$,

where k is the number of 2-vertices in set $\{v_1, v_2\}$.

If there are multiple incidences, the charge is sent according to the appropriate rule(s) several times, e.g., if a 2-vertex v is incident to a bridge, then it is incident to a single face f and f sends a charge of $1/2$ to v twice by Rule R1.

3.4 Final Charge of Vertices

In this section, we analyze the final amounts of charge of vertices.

Lemma 3.10 *If a graph G is D-minimal, then the final charge of every (≤ 4)-vertex is zero.*

Proof: The initial charge of a 2-vertex v is -1 and it receives a charge of $1/2$ from each of the two incident faces by Rule R1. Therefore, its final charge is zero. Since a 3-vertex does not receive or send out any charge, its final charge is zero. Similarly, a 4-vertex sends a charge of $1/4$ to each of the four incident faces by Rule R2. Since its initial charge is 1, its final charge is also zero.

■

Lemma 3.11 *If a graph G is D-minimal, then the final charge of every small (≥ 5)-vertex is non-negative.*

Proof: Consider a small vertex v of degree $d \geq 5$. The vertex v sends a charge of $5/16$ to each of the d incident faces by Rule R3. Hence, it sends out a charge of at most $5d/16$. Since the initial charge of v is $d - 3 \geq 5d/16$, the final charge of v is non-negative.

■

The analysis of final charge of big vertices needs finer arguments:

Lemma 3.12 *If a graph G is D-minimal, then the final charge of every big vertex is non-negative.*

Proof: Let v be a big vertex of degree d. Let $v_1, \ldots, v_d$ be the neighbors of v in a cyclic order around the vertex v and let $f_1, \ldots, f_d$ be the faces incident to v in the order such that the f_i-neighbors of v are the vertices v_i and v_{i+1}. Note that some of the faces f_i can coincide. Let $\varphi(v_i)$ be the amount of charge sent from v through a vertex v_i. Similarly, $\varphi(f_i)$ is the amount of charge sent to f_i. Note that this is a slight abuse of our notation since the faces f_i are not necessarily mutually distinct—in such case, $\varphi(f_i)$ is the amount of charge sent from v because of this particular incidence to f_i.

We show that the following holds for every $i = 1, \ldots, d$ (indices are modulo d):

$$\frac{\varphi(v_i)}{2} + \varphi(f_i) + \varphi(v_{i+1}) + \varphi(f_{i+1}) + \frac{\varphi(v_{i+2})}{2} \le \frac{31}{16}. \tag{3.1}$$

Summing (3.1) over all $i = 1, \ldots, d$ yields the following:

$$\sum_{i=1}^{d} (2\varphi(v_i) + 2\varphi(f_i)) \le \left(2 - \frac{1}{16}\right) d. \tag{3.2}$$

Recall now that the initial charge of v is $d - 3$. Because v is big, its degree d is at least 96. Since the charge sent out by v is at most $d - d/32$ by (3.2), the final charge of v is non-negative. Therefore, in order to establish the statement of the lemma, it is enough to show that the inequality (3.1) holds.

Let us fix an integer i between 1 and d. We distinguish several cases according to which of the vertices v_i, v_{i+1} and v_{i+2} are of degree 2:

None of the vertices v_i, v_{i+1} and v_{i+2} is a 2-vertex. The vertex v sends through each of the vertices v_i, v_{i+1} and v_{i+2} a charge at most $5/16$ by Rules R4 and R5, i.e., $\varphi(v_i), \varphi(v_{i+1}), \varphi(v_{i+2}) \le 5/16$. By Rule R6, both the faces f_i and f_{i+1} receive charge of $1/2$ from v, i.e., $\varphi(f_i), \varphi(f_{i+1}) \le 1/2$. Hence, the sum (3.1) of charges is at most $13/8 < 31/16$.

The vertex v_{i+1} is not a 2-vertex and one of v_i and v_{i+2} is a 2-vertex. By symmetry, we can assume that v_i is a 2-vertex and v_{i+2} is a (≥ 3)-vertex. Since v_i is a 2-vertex, v sends no charge through it, i.e., $\varphi(v_i) = 0$. By Rule R6, $\varphi(f_i) = 3/4$ and $\varphi(f_{i+1}) = 1/2$. By Rules R4 and R5, the amounts of charge sent from v through v_{i+1} and v_{i+2} do not exceed $5/16$, i.e., $\varphi(v_{i+1}), \varphi(v_{i+2}) \le 5/16$. Therefore, the sum (3.1) is bounded by $3/4 + 1/2 + 3/2 \cdot 5/16 < 31/16$.

The vertex v_{i+1} is not a 2-vertex and both v_i and v_{i+2} are 2-vertices. The vertex v sends a charge of $3/4$ to both the faces f_i and f_{i+1} by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. No charge is sent through the vertices

v_i and v_{i+2}, i.e., $\varphi(v_i) = \varphi(v_{i+2}) = 0$. The amount of charge sent through v_{i+1} is at most $5/16$ (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_{i+1}) \leq 5/16$. We conclude that the sum (3.1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

The vertex v_{i+1} is a 2-vertex and neither v_i nor v_{i+2} is a 2-vertex. The vertex v sends a charge of $3/4$ to both the faces f_i and f_{i+1} by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. The amount of charge sent through each of v_i or v_{i+2} is at most $5/16$ (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_i), \varphi(v_{i+2}) \leq 5/16$. Since no charge is sent through v_{i+1}, i.e., $\varphi(v_{i+1}) = 0$, the sum (3.1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

The vertex v_{i+1} is a 2-vertex and one of v_i and v_{i+2} is a 2-vertex. By symmetry, we can assume that v_i is a 2-vertex and v_{i+2} is a (≥ 3)-vertex. Since v_i and v_{i+1} are 2-vertices, v sends no charge through v_i or v_{i+1}, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = 0$. By Rule R6, the face f_i receives a charge of at most 1 and the face f_{i+1} a charge of at most $3/4$, i.e., $\varphi(f_i) \leq 1$ and $\varphi(f_{i+1}) \leq 3/4$. Finally, the charge sent from v through v_{i+2} is at most $5/16$, i.e., $\varphi(v_{i+2}) \leq 5/16$. We infer that the sum (3.1) is bounded by $1 + 3/4 + 5/32 < 31/16$.

All the vertices v_i, v_{i+1} and v_{i+2} are 2-vertices. There is no charge sent from v through any of the vertices v_i, v_{i+1} and v_{i+2}, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = \varphi(v_{i+2}) = 0$. If at least one of the faces f_i and f_{i+1} is not a $(3,2)$-face, then the total amount of charge sent to both of them by Rule R6 is at most $15/16 + 1 = 31/16$ as desired. In the rest, we consider the case when both the faces f_i and f_{i+1} are $(3,2)$-faces. Let v' be the other big vertex incident to f_i and f_{i+1} (v' is big by Lemma 3.4). The vertex v_{i+1} lies in a 2-thread or a 3-thread shared by the faces f_i and f_{i+1}. If the faces f_i and f_{i+1} share a 2-thread, then the vertices v and v' are joined by two 3-threads—this is impossible by Lemma 3.6. On the other hand, if they share a 3-thread, then the vertices v and v' together with the two 2-threads form a 6-cycle contradicting the girth assumption.

■

3.5 Final Charge of Faces

In this section, we analyze the final amounts of charge of faces. First, we start with faces that are not biconnected. Recall that a maximal 2-connected subgraph of a graph is called a *block*. Blocks form a tree-like structure. The blocks that contain (at most) one vertex in common with other blocks are referred to as *end-blocks*.

Lemma 3.13 *Let f be a face of a D-minimal graph G. If f is not biconnected, then its final charge is non-negative.*

Proof: Let P be the facial walk of f. Since f is not biconnected, P consists of two or more blocks. In particular, it contains at least one cut-vertex. Each end-block of P is a cycle by Lemma 3.1. In addition, observe that the end-blocks of P are cycles of length at least seven. Let C_1 and C_2 be two different end-blocks of P and w_1 and w_2 be their cut-vertices (note that w_1 may be equal to w_2), respectively.

Let k be the number of incidences of f with (≥ 3)-vertices, counting multiplicities. If $w_1 \neq w_2$, then each of w_1 and w_2 contributes at least two to k, thus w_1 and w_2 together contribute by at least 4 to k. Otherwise, the vertex $w_1 = w_2$ contributes at least two to k (it contributes two if P is comprised of two blocks).

Since the length of C_1 is at least seven, it has at least one (≥ 3)-vertex different from w_1 by Lemma 3.3. If C_1 contains exactly one such (≥ 3)-vertex, then it has a 3-thread (it cannot have a 4-thread by Lemma 3.3), and the vertex w_1 is big by Lemma 3.4. Similar statements hold for C_2. Therefore, there are at least two (≥ 3)-vertices incident with f that are distinct from w_1 and w_2. We conclude that $k \geq 4$. Moreover, if $w_1 \neq w_2$ or $w_1 = w_2$ is small, then $k \geq 6$. Note that in the latter case, there are at least four (≥ 3)-vertices incident with f that are distinct from $w_1 = w_2$.

If f is an ℓ-face, its initial charge is $\ell/2 - 3$. The face f sends out a charge of $(\ell - k)/2$ by Rule R1. If $k \geq 6$, then this is at most $\ell/2 - 3$ and thus the final charge of the face is non-negative.

If $k < 6$, then $w_1 = w_2$ is a big vertex (this follows from our previous discussion) and it has two incidences with f. Therefore f receives a charge of at least one unit from w_1 by Rule R6 and its final charge is $\ell/2 - 3 - (\ell - k)/2 + 1 \geq 0$.

■

Next, we analyze biconnected faces starting with 7-faces:

Lemma 3.14 *The final charge of each 7-face f of a D-minimal graph G is non-negative.*

Proof: The initial charge of the face f is $1/2$. By Lemma 3.3, f does not contain a 4-thread, and thus the face f is incident to at least two (≥ 3)-vertices. We distinguish five cases according to the number of (≥ 3)-vertices incident to f:

The face f is incident to two (≥ 3)-vertices. In this case, the type of f is $(3, 2)$. By Lemma 3.4, both the (≥ 3)-vertices are big and each of them sends a charge of 1 unit to f by Rule R6. Since f sends out a charge of $5/2$ to the five incident 2-vertices, its final charge is zero.

The face f is incident to three (≥ 3)-vertices. Since f sends a charge of two units to the incident 2-vertices, it is enough to show that it receives a charge of at least $3/2$ from the incident (≥ 3)-vertices. Since G does not contain a 4-thread by Lemma 3.3, the type of f is $(3,1,0)$, $(2,2,0)$ or $(2,1,1)$.

If f is incident to two big vertices, then each of them sends a charge of at least $3/4$ to f by Rule R6, and the final charge of f is non-negative. In the rest, we assume that f is incident to at most one big vertex. Consequently, the type of f is $(2,2,0)$ or $(2,1,1)$ by Lemma 3.4 and f is incident to exactly one big vertex by Lemma 3.2.

Assume that the type of f is $(2,2,0)$. By our assumption, f is incident to a single big vertex and, by Lemma 3.2, this vertex delimits both the 2-threads of f. However, Lemma 3.5 yields that one of the other two (≥ 3)-vertices is also big (contrary to our assumption).

The final case to consider is that the type of f is $(2,1,1)$. Let v be the big vertex incident to f. By Lemma 3.2, v delimits the 2-thread. Since both f-neighbors of v are 2-vertices, v sends a charge of $15/16$ to f. Let v' be any of the other two (≥ 3)-vertices incident to f. If v' is a 3-vertex, its neighbor opposite to f is big by Lemma 3.7 and it sends (through v') a charge of $5/16$ to f by Rule R4. If v' is a 4-vertex, it sends a charge of $1/4$ to f, and if v' has a big neighbor opposite to f, then the big neighbor sends f an additional charge of $1/16$ by Rule R5. Finally, if v' is a small (≥ 5)-vertex, it sends a charge of $5/16$ to f by Rule R3. We conclude that if f receives a total charge of less than $3/2$, then both the (≥ 3)-vertices incident to f are 4-vertices with no big neighbors. However, this is impossible by Lemma 3.8.

The face f is incident to four (≥ 3)-vertices. Since f is incident to three 2-vertices, it sends out a charge of $3/2$. We show that, on the other hand, it receives a charge of at least one unit from the incident (≥ 3)-vertices. This will imply that the final charge of f is non-negative. If f is incident to two big vertices, then it receives a charge of at least $1/2$ from each of them, i.e., a charge of at least one unit in total. Hence, we can assume in the rest that f is incident to at most one big vertex. In particular, by Lemma 3.4, f has no 3-thread. Therefore, the type of f is one of the following: $(2,1,0,0)$, $(2,0,1,0)$ or $(1,1,1,0)$.

Assume first that f is incident to no big vertex. By Lemma 3.2, the type of f is $(1,1,1,0)$. Let v be any of the four (≥ 3)-vertices incident to f. Note that v has an f-neighbor that is a 2-vertex. If v is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ units from v by Rules R2 and R3. If v is a 3-vertex, then its neighbor opposite to f is big by Lemma 3.7 and it sends a charge of $5/16$ through v to f by Rule R4. Since the choice of v was arbitrary, the amount of charge sent from (or through) each incident

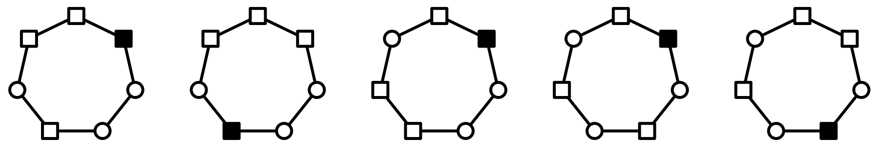

Figure 3.2: All configurations (up to symmetry) of a 7-face of types $(2,1,0,0)$, $(2,0,1,0)$ and $(1,1,1,0)$ when the face is incident to a single big vertex. The big vertices are represented by full squares, the small (≥ 3)-vertices by empty squares and the 2-vertices by circles. Note that a 2-thread must be bounded by at least one big vertex by Lemma 3.2.

(≥ 3)-vertex is at least $1/4$ and f receives a charge of at least 1 unit in total.

We now consider the case that exactly one vertex incident to f is big. We say that a vertex x incident to f has Property S if the following conditions are satisfied:

1. x is small,
2. both f-neighbors of x are small, and
3. one of the f-neighbors of x is a 2-vertex with no big f-neighbor.

It is routine to check that the following claim holds (consult Figure 3.2): unless the type of f is $(2,1,0,0)$ and the big vertex delimits both the 2-thread and the 1-thread of f, the face f is incident to two different (≥ 3)-vertices w_1 and w_2 that have Property S.

Under the assumption that the type of f is not $(2,1,0,0)$, we show that the face f receives a charge of at least $1/4$ from (or through) each of w_1 and w_2: if w_i is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ from it. Otherwise, w_i is a 3-vertex and, by Lemma 3.7, its neighbor opposite to f is big. Consequently, it sends through w_i a charge of $5/16$ to f. Since f receives in addition the charge of at least $1/2$ from the big vertex, its final charge is non-negative as desired.

It remains to consider the case when the type of f is $(2,1,0,0)$ and the big vertex delimits both the 2-thread and the 1-thread of f. In this case, f receives a charge of $15/16$ from the incident big vertex by Rule R6. Moreover, there exists a vertex w that has Property S (consult Figure 3.2). Similarly as in the previous paragraph, the charge sent from w to f is at least $1/4$. Altogether, f receives a charge of at least 1 and the final charge is thus non-negative.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of 1 unit to the two incident 2-vertices. Thus it is enough to show that the face

f receives a charge of at least $1/2$ from incident vertices. If f is incident to a big vertex, then f receives a charge of at least $1/2$ from it by Rule R6. We assume in the rest that f is only incident to small vertices. In particular, f has no 2-thread (by Lemma 3.2).

Let v be a 2-vertex incident to f and let v^- and v^+ be the two f-neighbors of v. Note that both v^- and v^+ are (≥ 3)-vertices. If v^- is a (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. If v^- is a 3-vertex, then its neighbor opposite to f is big by Lemma 3.7, and it sends a charge of $5/16$ through v^- to f. Similarly, f receives a charge of at least $1/4$ from (or through) v^+. Hence, the total charge received by f from the vertices v^- and v^+ is at least $1/2$ and the final charge of f is non-negative.

The face f is incident to six or seven (≥ 3)-vertices. Since the face f is incident to at most one 2-vertex, it sends out a charge of at most $1/2$ and its final charge is non-negative.

■

Next, we analyze the final charge of 8-faces.

Lemma 3.15 *The final charge of each biconnected 8-face f of a D-minimal graph G is non-negative.*

Proof: First note that the initial charge of the face f is one. By Lemma 3.3, the face f does not contain a 4-thread. Therefore, the face f is incident to at least two (≥ 3)-vertices. We distinguish five cases based on the number of (≥ 3)-vertices incident to the face f:

The face f is incident to two (≥ 3)-vertices. Since f does not contain a 4-thread, the type of f is $(3,3)$. However, this is impossible by Lemma 3.6.

The face f is incident to three (≥ 3)-vertices. Since f sends a charge of $5/2$ to the incident 2-vertices, it is enough to show that it receives a charge of at least $3/2$ from the incident (≥ 3)-vertices. Since f does not contain a 4-thread, the type of f is $(3,2,0)$, $(3,1,1)$ or $(2,2,1)$.

If the type of f is $(3,2,0)$ or $(3,1,1)$, then the 3-thread is delimited by two big vertices (by Lemma 3.4) and f receives from each of them a charge of at least $3/4$ by Rule R6. Hence, the final charge of f is non-negative.

Assume that the type of f is $(2,2,1)$. It is enough to show that f is incident to at least two big vertices because each of them would send a charge of $3/4$ to f by Rule R6. If this is not the case, then f is incident to exactly one big vertex that is common to the two 2-threads by Lemma 3.2. However, by Lemma 3.5, at least one of the other two (≥ 3)-vertices is also big. We conclude that f is incident to at least two big vertices.

The face f is incident to four (≥ 3)-vertices. Since f is incident to four 2-vertices, f sends out a charge of two units. We claim that it also receives a charge of at least one unit from the incident vertices. This will imply that the final charge of f is non-negative. If f is incident to two big vertices, then it receives a charge of at least $1/2$ from each of them and the claim holds. We assume in the rest that f is incident to at most one big vertex. In particular, by Lemma 3.4, f does not have a 3-thread.

Assume that f contains a 2-thread. Let v and v' be the vertices delimiting the 2-thread. By Lemma 3.2, v or v' is big, say v. Since v is incident to a 2-vertex, it sends a charge of at least $3/4$ to f by Rule R6. If v' is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ from v' and the final charge of f is non-negative. Otherwise, v' is a 3-vertex incident to a 2-thread and its f-neighbor not contained in the 2-thread is a small vertex. By Lemma 3.7, the neighbor of v' opposite to f is a big vertex. Hence, the face f receives a charge of $5/16$ from the big neighbor of v' and thus its final charge is non-negative.

In the rest, we assume that f has neither a 3-thread nor a 2-thread. Consequently, the type of f must be $(1, 1, 1, 1)$. Let v_1, v_2, v_3 and v_4 be the (≥ 3)-vertices incident to f in the order as they appear on the facial walk of f. We have already established that f is incident with at most one big vertex. First assume that f is incident to a single big vertex, say v_1. Note that f receives a charge of $15/16$ from v_1 by Rule R6. If v_3 is a (≥ 4)-vertex, it sends a charge of $1/4$ to f and the final charge of f is non-negative. If v_3 is a 3-vertex, then its neighbor opposite to f is big (by Lemma 3.7) and sends a charge of $5/16$ to f, and thus the final charge of f is non-negative.

It remains to consider the case when the type of f is $(1, 1, 1, 1)$ and f is not incident to a big vertex. Let us consider a vertex v_1. If v_1 is (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. If v_1 is 3-vertex, then its neighbor opposite to f is big, and it sends a charge of $5/16$ to f through v_1. Similarly, we can infer that f receives a charge of at least $1/4$ from (or through) the vertices v_2, v_3 and v_4. Hence, f receives a charge of at least one unit from the incident vertices and its final charge is non-negative.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of $3/2$ units to the incident 2-vertices. Thus it is enough to show that the face f receives a charge of at least $1/2$ from incident (≥ 3)-vertices. If f is incident to a big vertex, then f receives a charge of at least $1/2$ from it and the final charge is non-negative. We assume in the rest that f is only incident to small vertices.

Let v be a 2-vertex incident to f. Since f is incident to no big vertex, both the neighbors v^- and v^+ of v are (≥ 3)-vertices by Lemma 3.2. If v^- is a (≥ 4)-vertex, it sends a charge of at least $1/4$ to f. And if v^- is a 3-vertex, then its neighbor opposite to f is big by Lemma 3.7 and it sends through v^- to f a charge of 5/16. Similarly, f receives a charge of at least 1/4 from (or through) v^+. Hence, f receives a charge of at least 1/2 in total from the two neighbors of v and the final charge of f is non-negative.

The face f is incident to six or more (≥ 3)-vertices. Since the face f is incident to at most two 2-vertices, it sends out a charge of at most one unit and the final charge of f is non-negative.

■

Finally, we analyze the case of (≥ 9)-faces:

Lemma 3.16 *The final charge of each biconnected (≥ 9)-face f of a D-minimal graph is non-negative.*

Proof: Since f does not contain a 4-thread by Lemma 3.3, the face f is incident to at least three (≥ 3)-vertices. The initial charge of f is $\ell/2 - 3$ where ℓ is the length of f. We distinguish four cases according to the number of (≥ 3)-vertices incident to f:

The face f is incident to three (≥ 3)-vertices. The face f sends out a charge of $(\ell - 3)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least 3/2 from the incident vertices. If f has a 3-thread, then the 3-thread is delimited by two big vertices. Both of them send a charge of at least 3/4 to f by Rule R6. Therefore, if the total charge received by f is less than 3/2, then f has no 3-thread. Consequently, the length of f is nine and its type is $(2, 2, 2)$. By Lemma 3.2, at least two of the (≥ 3)-vertices are big and f receives a charge of at least 3/2 from them by Rule R6 in this case.

The face f is incident to four (≥ 3)-vertices. The face f sends a charge of $(\ell - 4)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least 1 from the incident vertices. If f has a 3-thread, then the 3-thread is delimited by two big vertices (by Lemma 3.4) and each of them sends a charge of at least 1/2 to f by Rule R6. If f has at least three 2-threads, then these threads are delimited by at least two different big vertices by Lemma 3.2, and f receives a charge of at least 1/2 from each of them by Rule R6. If none of the above cases holds, i.e., f has no 3-thread and at most two 2-threads, then its type must be one of the following: $(2, 2, 1, 0)$, $(2, 1, 2, 0)$, $(2, 1, 1, 1)$, $(2, 2, 1, 1)$, and $(2, 1, 2, 1)$—see Figure 3.3.

Assume that the type of f is one of those five types. Since f has a 2-thread, it must be incident to a big vertex v by Lemma 3.2. Let v', v'' and v''' be

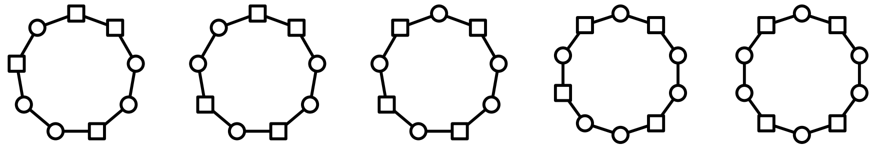

Figure 3.3: Possible types of a 9-face or a 10-face with no 3-thread and at most two 2-threads. The (≥ 3)-vertices are represented by squares and the 2-vertices by circles.

the remaining (≥ 3)-vertices incident to f. The face f receives a charge of at least $1/2$ from the vertex v by Rule R6. If at least one of v', v'' and v''' is big, then it sends an additional charge of at least $1/2$ to f by Rule R6, and the total amount of charge received by f is at least one. Let us assume in the rest that all the vertices v', v'' and v''' are small.

Observe that in this case the type of f is $(2,2,1,0)$, $(2,1,1,1)$ or $(2,2,1,1)$. If v' is a (≥ 4)-vertex, f receives a charge of at least $1/4$ from v' by Rule R2 or Rule R3. If v' is a 3-vertex, its neighbor opposite to f is big by Lemma 3.7 and it sends through v' to f a charge $5/16$ by Rule R4. Similarly, f receives a charge of at least $1/4$ from v'' and v'''. We conclude that the total charge received by f is at least one.

The face f is incident to five (≥ 3)-vertices. The face f sends a charge of $(\ell-5)/2$ to the incident 2-vertices. It is enough to show that f receives a charge of at least $1/2$ from the incident vertices. If f is incident to a big vertex, then it receives a charge of at least $1/2$ by Rule R6 from this vertex. Assume in the rest that f is only incident to small vertices. In particular, the length of every 2-thread of f is one by Lemma 3.2. Let v be a 2-vertex incident to f and v^- and v^+ the f-neighbors of v. Note that both v^- and v^+ are (≥ 3)-vertices. If v^- is a (≥ 4)-vertex, then f receives a charge of at least $1/4$ from v^- by Rule R2 or Rule R3. If v^- is a 3-vertex, then its neighbor opposite to v is big by Lemma 3.7 and the face f receives a charge of $5/16$ from it through v. Similarly, f receives a charge of at least $1/4$ from (or through) v^+. Altogether, f receives a charge of at least $1/2$ as required.

The face f is incident to six or more (≥ 3)-vertices. The face f sends out a charge of at most $(\ell-6)/2$ by Rule R1. Since the initial charge of f is $\ell/2-3$ and $\ell \geq 9$, the final charge is non-negative.

∎

3.6 Final Step

We now combine our observations from the previous sections together:

Theorem 3.17 *If G is a planar graph of maximum degree $\Delta \geq 190 + 2p$, $p \geq 1$, and the girth of G is at least seven, then G has a proper $L(p, 1)$-labeling with span $2p + \Delta - 2$.*

Proof: Consider a possible counterexample G and set $D = \Delta$. Since G is not D-good, there exists a D-minimal graph G'. Assign charge to the vertices and faces of G' as described in Section 3.2. Apply the rules given in Section 3.3 to G'. By Proposition 3.9, the sum of the amounts of initial charge assigned to the vertices and edges of G' is -6. On the other hand, the final amounts of charge of every vertex (Lemmas 3.10–3.12) and every face (Lemmas 3.13–3.16) are non-negative. However, this is impossible since the total amount of charge is preserved by the rules.

■

We use an argument applied in [69] to derive the following result for $L(p, q)$-labelings:

Corollary 3.18 *If G is a planar graph of maximum degree $\Delta \geq 190 + 2\lceil p/q \rceil$, $p, q \geq 1$, and girth at least seven, then G has a proper $L(p, q)$-labeling with span $2p + q\Delta - 2$.*

Proof: Let $p' = \lceil p/q \rceil$. By Theorem 3.17, the graph G has a proper $L(p', 1)$-labeling c' with span $2p' + \Delta - 2$. Define a labeling c by setting $c(v) = qc'(v)$ for each vertex v. The labeling c is a proper $L(p'q, q)$-labeling. Therefore, it is also a proper $L(p, q)$-labeling of G. The span of c is at most the following:

$$q(2p' + \Delta - 2) = 2\left(p' - \frac{q-1}{q}\right)q + q\Delta - 2 \leq 2p + q\Delta - 2.$$

■

3.7 Conclusion

One may ask whether the bound proven in Theorem 3.17 cannot be further improved, e.g., to $2p + \Delta - 3$. However, the bound is tight for all considered pairs of Δ and p as shown in the following proposition (though the next proposition follows from results of [35], see Proposition 3.20, we include its short proof for the sake of completeness):

Proposition 3.19 *Let p and $\Delta \geq 2p$ be arbitrary integers. There exists a tree T with maximum degree Δ such that the span of an optimal $L(p,1)$-labeling of T is $2p+\Delta-2$.*

Proof: It can be easily proven by induction on the order of a tree that the span of an optimal labeling of any tree with maximum degree Δ is at most $2p+\Delta-2$. Therefore, it is enough to construct a tree with no $L(p,1)$-labeling with span less than $2p+\Delta-2$. Let us consider the following tree T: a vertex v_0 is adjacent to Δ vertices $v_1,\ldots,v_\Delta$ and each of the vertices $v_1,\ldots,v_\Delta$ is adjacent to $\Delta-1$ leaves. Clearly, the maximum degree of T is Δ.

Assume that T has a proper $L(p,1)$-labeling c of span at most $2p+\Delta-3$. Since $\Delta \geq 2p$, the color of at least one of the vertices $v_0,\ldots,v_\Delta$ is between $p-1$ and $p+\Delta-2$, i.e., $c(v_i) \in \{p-1,\ldots,p+\Delta-2\}$ for some i. The color of each neighbor of v_i is either at most $c(v_i)-p$ or at least $c(v_i)+p$. Since there are only $\Delta-1$ such colors, two of the neighbors of v_i have the same color and the labeling c is not proper.

■

One may also ask whether the condition $\Delta \geq 190+2p$ in Theorem 3.17 cannot be further weakened. The answer is positive (we strongly believe that the bound for $p=2$ can be lowered to approximately 50) but we decided not to try to refine the discharging phase and the analysis in order to avoid adding more pages to the thesis. It is also natural to consider $L(p,q)$-labelings of planar graphs with no short cycles for $q>2$. In such case, the following result of Georges and Mauro [35, Theorems 3.2–3.5] comes to use:

Proposition 3.20 *Let p and q, $p \geq q$, be two positive integers. There exists a Δ_0 such that the span of an optimal $L(p,q)$-labeling of the infinite Δ-regular tree T_Δ, $\Delta \geq \Delta_0$ (Δ_0 depends on p and q), is the following:*

$$\lambda_{p,q}(T_\Delta) = \begin{cases} q\Delta+2p-2q & \text{if } p/q \text{ is an integer, i.e., } q|p, \\ q\Delta+p & \text{if } 1<\frac{p}{q}\leq\frac{3}{2}, \\ q\Delta+\left\lfloor\frac{p}{q}\right\rfloor q+p-q & \text{if } 2\leq\left\lfloor\frac{p}{q}\right\rfloor<\frac{p}{q}\leq\left\lfloor\frac{p}{q}\right\rfloor+\frac{1}{2}, \\ q\Delta+2\left\lfloor\frac{p}{q}\right\rfloor q & \text{otherwise.} \end{cases}$$

Proposition 3.20 provides lower bounds on optimum spans of $L(p,q)$-labelings of planar graphs with large girth as every infinite tree T_Δ contains a finite subtree T with $\lambda_{p,q}(T)=\lambda_{p,q}(T_\Delta)$. The lower bounds can be complemented by the following (rather straightforward) upper bound which is tight if $q=1$:

Proposition 3.21 *Let p and q, $p \geq q$, be two positive integers. There exists an integer Δ_0, which depends on p and q, such that every planar graph G of maximum degree Δ and of girth at least 18 has an $L(p,q)$-labeling of span at most $qD+2p+q-3$ where $D=\max\{\Delta_0,\Delta\}$.*

Proof: Fix p, q and Δ and let $\Lambda = qD + 2p + q - 3$. We prove the proposition for $\Delta_0 = (2p-1)/q + 3$. Let G be a planar graph of the smallest order such that the maximum degree of G is at most Δ, G contains no cycle of length less than 18 and $\lambda_{p,q}(G) > \Lambda$. Clearly, G is connected. We partition the vertices of G into three classes and refer the vertices in the classes as to red, green and blue vertices: the vertices of degree one will be red, the vertices adjacent to at most two vertices that are not red will be green and the remaining vertices will be blue.

Assume first that there is a red vertex adjacent to a green vertex. Let v be that green vertex and W all red vertices adjacent to v. By the choice of G, $G \setminus W$ has an $L(p,q)$-labeling of span at most Λ. Since v is green, it is adjacent to at most two vertices that are green or blue. We consider the case that v is adjacent to two such vertices, say v_1 and v_2, and leave the other cases to be verified by the reader since our arguments readily translate to those cases. Note that the vertices v, v_1 and v_2 are the only vertices at distance at most two from the vertices of W in G.

Our aim now is to find $\Delta - 2$ numbers $a_1, \ldots, a_{\Delta-2}$ such that the difference between any two numbers a_i and a_j, $i \neq j$, is at least q, the difference between any number a_i and the label of v is at least p and the difference between a_i and the label of v_1 or v_2 is at least q. The numbers are constructed inductively as follows. Set $a_1 = 0$ and $i = 1$ and apply the following three rules:

Rule 1 If the difference between a_i and the label of v is smaller than p, increase a_i by $2p - 1$.

Rule 2 If the difference between a_i and the label of v_1 or v_2 is smaller than q, increase a_i by $2q - 1$.

Rule 3 Suppose that neither Rule 1 nor Rule 2 applies. If $i = \Delta - 2$, stop. Otherwise, set $a_{i+1} = a_i + q$ and increase i by one.

Observe that Rule 1 can apply at most once and Rule 2 at most twice during the entire process. This yields that the value of $a_{\Delta-2}$ does not exceed $(\Delta-3)q + 2p - 1 + 4q - 2 \leq \Lambda$. Hence, the labeling of $G \setminus W$ can be extended to G by assigning the vertices of W the labels $a_1, \ldots, a_{\Delta-2}$ to an $L(p,q)$-labeling of span Λ which contradicts our choice of G. We conclude that green vertices are adjacent to green and blue vertices only. In particular, every green vertex has degree two.

Let G' be the subgraph of G induced by all green and blue vertices. Observe that the degree of each green vertex in G' is two and the degree of each blue vertex is at least three. Hence, the minimum degree of G' is two. As the girth of G' is at least 18, G' contains a 3-thread comprised of (green) vertices v_1, v_2 and v_3. Since green vertices are adjacent to green and blue vertices in G only, the vertices v_1, v_2 and v_3 also form a 3-thread in G.

By the choice of G, the graph $G \setminus v_2$ has an $L(p,q)$-labeling of span at most Λ. We aim to extend the labeling of $G \setminus v_2$ to v_2. Let us count the number of labels that cannot be assigned to v_2. There are at most $2p-1$ labels that cannot be assigned to v_2 because of the label assigned to v_1 and there are at most $2q-1$ additional labels that cannot be assigned to v_2 because of the label assigned to the neighbor of v_1 different from v_2. Similarly, there are at most $2p+2q-2$ labels that cannot be assigned to v_2 because of the labels of v_3 and the other neighbor of v_3. In total, there at most $4p+4q-4$ labels that cannot be assigned to v_2. Since there are at least $\Lambda+1 = qD+2p+q-3+1 \geq q\Delta_0+2p+q-2 = (2p-1)+3q+2p+q-2 = 4p+4q-3$ labels, the labeling can be extended to v_2 contradicting our choice of G. The proof of the proposition is now finished.

■

Note that Proposition 3.21 can be generalized to minor-closed classes of graphs (with the bound on the girth depending on the considered class of graphs). However, we think that the assumption on the girth in the proposition is not optimal and can be weakened to seven:

Conjecture 3.22 *Let p and q, $p \geq q$, be two positive integers. There exists an integer Δ_0, which depends on p and q, such that every planar graph G of maximum degree Δ and of girth at least seven has an $L(p,q)$-labeling of span at most $qD+2p+q-3$ where $D = \max\{\Delta_0, \Delta\}$.*

The lower and upper bounds given in Propositions 3.20 and 3.21 do match for $q = 1$ but they differ for $q \neq 1$. We leave as an open problem to determine the optimal values of spans of $L(p,q)$-labelings of planar graph with large maximum degree and no short cycles for $q \neq 1$.

Part II

K_4-minor Free Graphs

Chapter 4

Introduction

In the first part of the thesis, we have investigated $L(p, q)$-labelings of planar graphs. In this part, we again study $L(p, q)$-labelings, but we focus on graphs that do not contain a complete graph on four vertices as a minor. As planar graphs are precisely those that contain neither K_5 or $K_{3,3}$ as a minor and outerplanar graphs contain no K_4 or $K_{2,3}$ minor, K_4-minor free graphs form a subclass of planar graphs that contains all outerplanar graphs. In addition, K_4-minor free graphs can be obtained from single edges by two simple operations which we introduce later.

In [62], Lih, Wang and Zhu proved Wegner's conjecture (Conjecture 1.1) for K_4-minor free graphs, extending a previous result [61] for outerplanar graphs.

Theorem 4.1 (Lih, Wang and Zhu 2003) *The chromatic number of the square of a K_4-minor free graph G of maximum degree Δ is at most $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$ and at most $\Delta + 3$ if $2 \leq \Delta \leq 3$.*

In the following two chapters, we establish two extensions of Theorem 4.1. In Chapter 5, we extend the result to $L(p, q)$-labelings for general p and q (recall that coloring the square of a graph can be viewed as an $L(1, 1)$-labeling), and in Chapter 6, we prove a list-coloring counterpart of Theorem 4.1.

4.1 Structure of Series-parallel Graphs

In this section, we describe the structure of K_4-minor free graphs and introduce a useful notation for their description.

Before introducing series-parallel graphs, let us remark that the definition of series-parallel graphs slightly varies throughout the literature and thus the reader can find using this term in a slightly different meaning elsewhere. *Series-parallel graphs* can be obtained by the following recursive construction based on graphs with two distinguished vertices called *poles*. The simplest series-parallel graph is an edge uv and the two poles of it are its end-vertices. If G_1 and G_2 are series-parallel graphs with poles u_1 and v_1, and u_2 and v_2, respectively, then the graph G obtained by identifying the vertices v_1 and u_2 is also a series-parallel graph and its two poles are the vertices u_1 and v_2. The graph G obtained in this way is called the *serial join* of G_1 and G_2. The *parallel join* of G_1 and G_2 is the graph obtained by identifying the pairs of vertices u_1 and u_2 and v_1 and v_2 with the poles being the identified vertices. The series-parallel graphs are precisely those that can be obtained from edges by a series of serial and parallel joins.

It is well-known that every 2-edge-connected K_4-minor free graph is a series-parallel graph. Let us now state this fact as a separate lemma:

Lemma 4.2 *Every block of a K_4-minor free graph is a series-parallel graph.*

The construction of a particular series-parallel graph G can be encoded by a rooted tree which is called the *SP-decomposition tree* of G. Each node of the tree corresponds to a subgraph of G obtained at a step of the recursive construction of G. The leaves correspond to simple paths with their end-vertices being poles (such graphs are obtained by successive serial joins from edges) and each inner node of the tree corresponds to either a serial or a parallel join. Based on this, there are two types of inner nodes: *S-nodes* and *P-nodes*. The inner nodes have at least two children: the subgraphs corresponding to their children were joined together by a sequence of serial or parallel joins depending on the type of the node. Since the result of a sequence of serial joins depends on the order in which the serial joins are applied, the children of each inner node are ordered. Without loss of generality, we can assume that the children of a P-node are S-nodes and leaves only, and the children of an S-node are P-nodes and leaves only. We can also assume that no two consecutive children of an S-node are leaves.

An SP-decomposition tree corresponding to a series-parallel graph G is not unique. In fact, there is a lot of freedom in its choice as can be seen in the following well-known result:

Lemma 4.3 *Let G be a series-parallel graph and v a vertex of G. There exists an SP-decomposition tree such that v is one of the poles of the graph corresponding to the root of the SP-decomposition tree.*

In the proof of our results, we show that a minimal possible counter-examples do not contain certain subgraphs. Their structure is based on the subtrees corresponding to them in the SP-decomposition tree. A subgraph of G corresponding to a leaf of the tree, i.e., a path consisting of vertices of degree two, is called an *ℓ-subgraph* of G (ℓ stands for leaf). A subgraph obtained by a parallel join of A_1-subgraph, A_2-subgraph, $\ldots$, A_k-subgraph, is a $P(A_1, \ldots, A_k)$-subgraph and a subgraph obtained by a serial join of such subgraphs is an $S(A_1, \ldots, A_k)$-subgraph. For instance, a $P(\ell, \ell, \ell)$-subgraph is a subgraph of G that corresponds

Figure 4.1: A $S(P,\ell)$-subgraph and the corresponding subtree. The subgraph is obtained by a serial join (using the vertex v) of an edge vw and a P-subgraph A with poles u and v. The P-subgraph A itself is a parallel join of several paths with endvertices u and v.

to a P-node with three leaves. Since the result of a serial join depends on the order in which the subgraphs are joined, we require the sequence $A_1,\dots,A_k$ to respect this order. Subgraphs obtained by a parallel join of several A-subgraphs are called $P(A*)$-subgraphs and those obtained by a serial join $S(A*)$-subgraphs. $P(\ell*)$-subgraphs are called P-subgraphs for short. An example of this notation can be found in Figure 4.1.

Finally, we introduce a special name for particular P-subgraphs of a series-parallel graph G. A P-subgraph of G obtained by a parallel join of several two-edge paths and possibly an edge is called a *crystal*. Its vertices distinct from its poles are said to be its *inner* vertices. A crystal whose poles are not adjacent is a *diamond*. The *size* of a crystal is the number of edges incident with each of its poles, i.e., if the poles are adjacent, the size of a crystal is the number of the paths forming it increased by one. Notice that size of any crystal is at least 2. If A is a crystal, Inner(A) denotes its set of inner vertices and size(A) denotes its size.

Chapter 5

Distance Constrained Labelings of K_4-minor Free Graphs

In this chapter, we state and prove our results on $L(p, q)$-labelings of K_4-minor free graphs. In particular, we prove a generalization of Theorem 4.1 for $L(p, q)$-labeling.

Theorem 5.1 *For every positive integers $p \geq q$, there exists Δ_0 such that every K_4-minor free graph with maximum degree $\Delta \geq \Delta_0$ has an $L(p, q)$-labeling with span at most $q \lfloor 3\Delta/2 \rfloor$.*

In the proof of Theorem 5.1, we use the following notation. For integers $p \geq 1$ and $D \geq 1$, a graph G is said to be (D, p)-*bad* if G is K_4-minor free, has maximum degree at most D and has no $L(p, 1)$-labeling with span at most $\lfloor 3D/2 \rfloor$. A graph G is (D, p)-*minimal* if it is (D, p)-bad and there is no (D, p)-bad graph of smaller order. Finally, a function $c : V(G) \to \{0, 1, \ldots, k\}$ is an $L_D(p, 1)$-labeling of G if it is an $L(p, 1)$-*labeling* of G and its span is at most $\lfloor 3D/2 \rfloor$. Clearly a graph G with maximum degree at most D is (D, p)-bad if it is K_4-minor free and there is no $L_D(p, 1)$-labeling of G.

The following theorem shows that (for a fixed positive integer p) there are only finitely many (D, p)-minimal graphs.

Theorem 5.2 *For every positive integer p, there exists an integer D_0 such that there is no (D, p)-bad graph for any $D \geq D_0$.*

Before we present the proof of Theorem 5.2, we show that Theorem 5.2 implies Theorem 5.1.

Proof of Theorem 5.1 using Theorem 5.2. Fix integers p and q, and set Δ_0 to be the constant D_0 from Theorem 5.2 for $p' = \lceil p/q \rceil$. To see that Δ_0 has the required properties, fix a K_4-minor free graph G such that $\Delta(G) \geq \Delta_0$. By Theorem 5.2 (for $D = \Delta(G)$), there exists an $L(\lceil p/q \rceil, 1)$-labeling c of G of span at most $\lfloor 3\Delta(G)/2\rfloor$. Set $c'(v) = qc(v)$ for each $v \in V(G)$. Since the differences of the labels assigned to neighboring vertices by c' are at least q and the differences of the labels of vertices at distance two are at least $q\lceil p/q\rceil \geq p$, c' is an $L(p,q)$-labeling of G and since its span is at most $q\lfloor 3\Delta(G)/2\rfloor$, the statement of the theorem follows.

∎

The rest of this chapter contains the proof of Theorem 5.2. In a series of lemmas, we show that if D is sufficiently large (in terms of p), then certain subgraphs cannot appear in a (D,p)-minimal graph and we eventually conclude that there is no (D,p)-minimal graph. The main idea of each of the proofs is to modify the given (D,p)-minimal graph G to a smaller one which has an $L_D(p,1)$-labeling by the (D,p)-minimality of G and use that labeling to obtain an $L_D(p,1)$-labeling of G contradicting the assumption that G is (D,p)-bad.

5.1 Overture

Clearly, every (D,p)-minimal graph is connected. In the following lemma, we show that it cannot contain vertices of degree one or two adjacent vertices of degree two.

Lemma 5.3 *For every positive integer p, there exists a constant $D_{5.3}$ such that no (D,p)-minimal graph, $D \geq D_{5.3}$, contains a vertex of degree at most one or two adjacent vertices of degree two.*

Proof: We prove the lemma for $D_{5.3} = 8p-4$. Let us fix a (D,p)-minimal graph G, $D \geq D_{5.3}$. First, consider the case that there is a vertex v in G of degree one. Remove the vertex and find an $L_D(p,1)$-labeling c of $G \setminus v$ (such a labeling exists by the minimality of G). We now aim to extend c to an $L_D(p,1)$-labeling of the entire graph G. To show that there is a suitable label for v, we count the number of labels in the set $\{0,\dots,\lfloor 3D/2\rfloor\}$ which cannot be used on v without violating the constraints of $L(p,1)$-labelings. We say that those labels are *forbidden* for v. In particular, we show that the number of labels forbidden for v is at most $\lfloor 3D/2\rfloor$, and thus there is at least one label available for v. The label of the only neighbor w of v forbids at most $2p-1$ labels to be assigned to v and the neighbors of w forbid additional $D-1$ labels or less. Hence, the total number of labels which cannot be assigned to v is at most $2p-1+D-1 = D+2p-2 \leq D+4p-2 = D+\lfloor D_{5.3}(p)/2\rfloor \leq \lfloor 3D/2\rfloor$. In particular, c can be extended to an $L_D(p,1)$-labeling of G, i.e., G is not (D,p)-bad—a contradiction.

Next, we show that there are no two adjacent vertices u and v of degree two. Remove u and v from G and find an $L_D(p,1)$-labeling c of $G \setminus \{u,v\}$. Let x be the neighbor of u different from v and y the neighbor of v different from u, i.e., G

contains a path $xuvy$. We first find a label for u: there are at most $2p-1$ labels forbidden by x, at most $D-1$ forbidden by the neighbors of x and at most one label forbidden by y. Together, there are at most $2p-1+D-1+1 = D+2p-1 \leq \lfloor 3D/2 \rfloor$ labels forbidden for u and therefore, we can label u properly. The case of v is analogous, except that there are at most $2p-1$ additional labels forbidden by u. We conclude that c can be extended to the entire graph G which contradicts the (D,p)-minimality of G.

■

In the rest of the proof, we choose one of the end-blocks of the block-decomposition of a K_4-minor free graph G (we choose the entire G if G is 2-connected) and show that it cannot contain certain types of subgraphs. We refer to the chosen end-block as to the *final block*, and write $G^\star$ for it. By Lemma 4.2, the final block is a series-parallel graph and, by Lemma 4.3, we may assume that one of the poles of the graph corresponding to the root of its SP-decomposition is its cut-vertex. In case that G is 2-vertex-connected, we consider an arbitrary SP-decomposition of G. One such (fixed) decomposition of $G^\star$ will be denoted by $T^\star$. Notice that since $G^\star$ is 2-connected, the root of $T^\star$ is a P-node.

We adopt the notation of A-subgraphs introduced in Section 4.1, and we say that an A-subgraph is *contained* in $G^\star$, if there is a subtree T_A of the form described by A with root r in $T^\star$ such that there is no descendant w of r in $T^\star$ whose depth (measured from r) is greater than depth of every descendant of r in T_A (in other words, we allow the subtree of the node r to be more complex than just A-subgraph, but we do not want it to be significantly more complex).

An immediate consequence of Lemma 5.3 is that all P-subgraphs contained in the final block are crystals. In fact, crystals are the "building blocks" of many of the reducible subgraphs we deal with later in the proof. The following lemma gives two useful estimates on the size of crystals in (D,p)-minimal graphs.

Lemma 5.4 *For every positive integer p, there exist a constant K such that no (D,p)-minimal graph G, contains a crystal of size greater than $\lceil D/2 \rceil + K$. Moreover, if C_1 and C_2 are two crystals in G sharing a vertex v such that v is incident to no vertex of G except for the vertices of C_1 and C_2 and C_2 contains at least one inner vertex, then the size of C_1 is at least $\lfloor D/2 \rfloor - K$.*

Proof: We prove the lemma for $K = 4p-4$. Let us fix a (D,p)-minimal graph G. To see the first claim, suppose that there is a crystal C with poles u and v of size $k \geq \lceil D/2 \rceil + 4p - 3$ and let w be an inner vertex of C. Remove w and find a proper $L_D(p,1)$-labeling c of $G \setminus w$. It is now possible to extend c to an $L_D(p,1)$-labeling of G because there are at most $\lfloor 3D/2 \rfloor$ labels which cannot be assigned to w: the labels of vertices u and v forbid at most $2p-1$ labels each, at most $k-1$ labels are forbidden by the labels of the remaining inner vertices of C, at most $D-k$ labels are forbidden by neighbors of u outside C and other at most $D-k$ labels are forbidden by the labels of the neighbors of v outside C. Altogether, there are at most $2D+4p-3-k \leq \lfloor 3D/2 \rfloor$ forbidden labels, hence there is still at least one label available for w, thus G is not (D,p)-bad—a contradiction.

To prove the second claim, suppose there are two crystals C_1 and C_2 with the common pole v which is connected only to the vertices of C_1 and C_2, and C_2 contains an inner vertex w. Let u be the pole of C_2 different from v, k be the number of inner vertices of C_2 and $r \leq \lfloor D/2 \rfloor - 4p + 3$ be the size of C_1. As in the first part of the proof, we remove w from G and find an $L_D(p,1)$-labeling of $G \setminus w$ which we extend to an $L_D(p,1)$-labeling of G. The number of labels forbidden for w is again at most $\lfloor 3D/2 \rfloor$: vertices u and v forbid at most $2p-1$ labels each, at most $k-1$ labels are forbidden by the inner vertices of C_2, at most r labels are forbidden by the vertices in C_1 neighboring with v, and at most $D-k$ labels are forbidden by neighbors of u outside of C_2; altogether $4p-3+D+r \leq \lfloor 3D/2 \rfloor$ forbidden labels, thus there is again at least one label available for w. Hence, there exists an $L_D(p,1)$-labeling of G which contradicts the (D,p)-minimality of G.

∎

Let us turn our attention back to the SP-decomposition $T^\star$. We already know that the deepest inner nodes are P-nodes and that they correspond to crystals; to proceed with the proof, we investigate the neighborhood of those crystals. There are two possibilities—either the P-node is the entire decomposition $T^\star$ or it has an S-node parent S. Let us start with the former case:

Lemma 5.5 *For every positive integer p, there exists a constant $D_{5.5}$ such that the SP-decomposition of the final block of every (D,p)-minimal graph with $D \geq D_{5.5}$ has at least two inner nodes.*

Proof: We prove the lemma for $D_{5.5} = \max\{8p-6, D_{5.3}\}$, where $D_{5.3}$ is the constant from Lemma 5.3. For the sake of contradiction, assume that there is a (D,p)-minimal graph G (for some $D \geq D_{5.5}$) whose final block $G^\star$ violates the statement. By Lemma 5.3, the entire decomposition cannot be just a leaf. Hence, we may assume that the decomposition consists of a single P-node with several leaves. In other words, $G^\star$ is a single crystal with poles u and v, which is possibly connected to the rest of G through the pole v. Let w be one of the inner vertices of the crystal. Remove w, find an $L_D(p,1)$-labeling c of $G \setminus w$ and then extend the labeling to w. The number of labels forbidden for w is at most $4p-3+D \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most $k-1$ because of the inner vertices of $G^\star$, and at most $D-k$ because of the neighbors of v outside $G^\star$. Hence, we can extend c to an $L_D(p,1)$-labeling of G. Therefore, G is not (D,p)-bad.

∎

Figure 5.1: A $S(P, \ell)$-subgraph and the corresponding subtree.

5.2 Allegro

By Lemma 5.5, if D is large enough, we know that every bottommost P-node P_0 in $G^\star$ has an S-node parent S_0. Let us investigate the other children of S_0: in the next two lemmas, we show that S_0 must have exactly two children, both being P-subgraphs.

Lemma 5.6 *For every positive integer p, there exists a constant $D_{5.6}$ such that there is no (D, p)-minimal graph, $D \geq D_{5.6}$, whose final block contains an $S(P, \ell)$-subgraph.*

Proof: We prove the lemma for $D_{5.6} = 8p - 4$. Let us fix a (D, p)-minimal G (for some $D \geq D_{5.6}$). Suppose that $G^\star$ contains an $S(P, \ell)$-subgraph. In other words, there is a crystal A of size $k \geq 2$ with poles u and v connected to an edge vx (see Figure 5.1). Since the size of A is at least two, it contains an inner vertex w. Remove w and find an $L_D(p, 1)$-labeling c of $G \setminus w$. Then, extend the labeling to w. The number of labels forbidden for w is at most $D + 4p - 2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p - 1)$ because of u and v, at most $k - 1$ because of the inner vertices of A, at most $D - k$ because of neighbors of u outside A, and 1 because of the vertex x. Hence, there is at least one label available for w, and therefore c can be extended to G. This implies that G is not (D, p)-bad—a contradiction.

■

The lemma we just proved shows that every bottommost P-subtree has an S-node parent S_0 whose children are only P-subgraphs. The following lemma yields that there exactly two such children, i.e., the subtree of S_0 is an $S(P, P)$-subgraph as claimed before.

Lemma 5.7 *For every positive integer p, there exists a constant $D_{5.7}$ such there is no (D, p)-minimal graph, $D \geq D_{5.7}$, whose final block contains an $S(P, P, P)$-subgraph.*

Proof: We prove the lemma for $D_{5.7} = 4K + 16p - 4$, where K is the constant from Lemma 5.4. Let G be a (D, p)-minimal graph for some $D \geq D_{5.7}$ whose final block contains an $S(P, P, P)$-subgraph. In other words, there exist three crystals A, B, and C with poles u and v, v and w, and w and x, respectively.

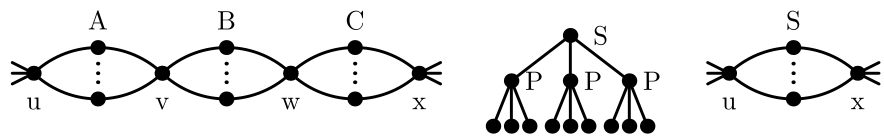

Figure 5.2: An $S(P, P, P)$-subgraph, the subtree corresponding to it and its reduction.

The configuration is depicted in Figure 5.2. By Lemma 5.4, we know that the size of each of the crystals is between $\lfloor D/2 \rfloor - K$ and $\lceil D/2 \rceil + K$. By symmetry, we may assume that the size of A is smaller than or equal to the size of C.

Construct an auxiliary graph G' as follows: remove the crystals A, B and C from G and connect u and x by r paths of length two, where r is the size of A. This newly created crystal is denoted by S. Since the order of G' is smaller than the order of G, there exists an $L_D(p, 1)$-labeling c of G'. We now extend c to the original graph G. First, the vertices u, x, and all the vertices outside S get the same label as they are assigned by c. We use the labels assigned by c to inner vertices of S to label all vertices in the crystal A and size(A) vertices in the crystal C (note that since the distance of an inner vertex of A from an inner vertex of C is at least three, the labels of those vertices are not in a conflict). After this operation, all inner vertices of A are properly labeled and there are at most $2K + 1$ vertices in C without a label.

Next, we find a label for the vertex v avoiding the conflicting labels except for the labels of inner vertices of A. The number of forbidden labels for v is at most $D + 2p - 1 \leq \lfloor 3D/2 \rfloor$: at most $2p - 1$ because of u, at most r because of the inner vertices of C (which have already been labelled), and at most $D - r$ because of neighbors of u outside A. To resolve possible conflicts with the labels in A, unlabel the inner vertices of A in conflict. Notice that at most $2p - 1$ vertices can be unlabeled. Use a similar approach to label w—but this time, the roles of A and C are interchanged, i.e., we avoid the labels of inner vertices of A and if there is a conflict with an inner vertex of C, we unlabel the conflicting vertex. The number of labels forbidden for w is at most $D + 4p - 2 \leq \lfloor 3D/2 \rfloor$, since we have to avoid the label of v as well.

When v and w are labeled, we can finish labeling the inner vertices of A and C (those which did not get label yet or have been unlabeled). Let k be the number of inner vertices of A. The number of forbidden labels of an inner vertex of A (similarly for C) is at most $D + 4p - 2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p - 1)$ because of u and v, at most 1 because of w, at most $k - 1$ because of the inner vertices of A, and at most $D - k$ because of the neighbors of u outside A. The final step is labeling of the inner vertices of B. Notice that the inner vertices of A and C use at most size(A) $+ 2K + 4p - 1$ distinct labels: at most size(A) for the labels taken from S, at most $2K + 1$ for vertices which did not get the initial labels, and

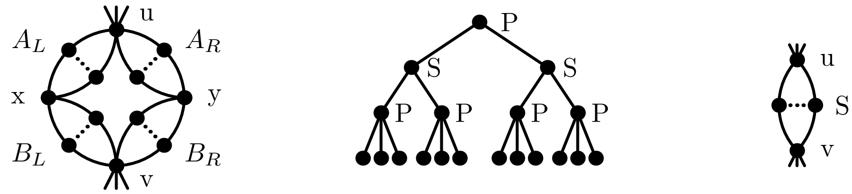

Figure 5.3: A $P(S(P,P),S(P,P))$-subgraph, the subtree corresponding to it and its reduction.

at most $2(2p-1)$ new labels of the unlabeled vertices. Therefore, the number of labels forbidden for an inner vertex of B is at most $D+2K+8p-2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of v and w, at most $D - \text{size}(A) - 1$ because of the other inner vertices of B, at most size(A) $+ 2K + 4p - 1$ because of the inner vertices of A and C, and at most 2 because of u and x. We infer from the preceding calculations that c can be extended to G. Hence, G is not (D,p)-bad.

■

Lemma 5.7 provides a nice characterization of possible configurations of S-nodes of the largest depth. It shows that those nodes are roots of $S(P,P)$-subgraphs in $G^\star$. Since the root of the decomposition $T^\star$ must be a P-node, every $S(P,P)$-subgraph must have a P-node parent. The following lemma shows that no such P-node has two or more $S(P,P)$-children. In particular, this shows that every $S(P,P)$-subgraph of the largest depth in $T^\star$ is contained in a $P(S(P,P),l*)$-subgraph.

Lemma 5.8 *For every positive integer p, there exists a constant $D_{5.8}$ such that there is no (D,p)-minimal graph, $D \geq D_{5.8}$, with a final block containing an $P(S(P,P),\ S(P,P))$-subgraph.*

Proof: We prove the lemma for $D_{5.8} = 16p + 8K - 4$, where K is the constant from Lemma 5.4. Fix a (D,p)-minimal graph G, $D \geq D_{5.8}$, such that its final block $G^\star$ contains a $P(S(P,P),S(P,P))$-subgraph. In particular, there are two vertices u and v connected by two crystals A_L and B_L with the common pole x and by another two crystals A_R and B_R with the common pole y (see Figure 5.3). The entire subgraph (the four vertices and four crystals) is denoted by R. By Lemma 5.4, the size of each of the crystals is at least $\lfloor D/2 \rfloor - K$, so the number neighbors of u (or v) outside R is small (at most $2K+1$). We construct an auxiliary graph G' as follows: remove the interior of R (leave only u and v) and join u and v by r paths where $r = \min\{\text{size}(A_L)+\text{size}(A_R), \text{size}(B_L)+\text{size}(B_R)\}$. The newly added crystal is denoted by S. Since G' has less vertices than G and its maximum degree is at most D, there exists a proper $L_D(p,1)$-labeling c of G'. We extend c to a proper $L_D(p,1)$-labeling of G in the following way. First, split the labels of the inner vertices of S into two sets X and Y, such that each set contains at least $\lfloor D/2 \rfloor - K$ labels. The elements of X are used to label as many inner vertices of A_L and B_R as possible. The elements of Y are used to label the inner vertices of A_R and B_L in a similar way. Note that after this step, at most $4K+2$ inner vertices of crystals in R are not labeled. Next, label vertices x and y. As in Lemma 5.7, unlabel some neighboring inner vertices if there is a conflict. The number of forbidden labels for x (analogously for y) is at most $D + 4p + 2K \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most r because of the inner vertices of A_R and B_R, at most $2K+1$ because of the neighbors of u outside R, at most $D-r$ because of neighbors of v outside R, and at most 1 because of the vertex y. Since the number of forbidden labels is at most $\lfloor 3D/2 \rfloor$, it is possible to label both the vertices x and y. Note that the number of unlabeled inner vertices is bounded by $2(2p-1)$. Finally, we label the remaining inner vertices (those which were unlabeled or were not labeled yet). Since there are at most $2(2p-1)+4K+2$ such vertices, the number of labels forbidden for an inner vertex of A_L is bounded by $D+8p+4K-2 \leq \lfloor 3D/2 \rfloor$: $2(2p-1)$ because of x and v, $D-r$ because of the neighbors of v outside R, 1 because of u, and $r+2(2p-1)+4K+1$ because of the labels of the inner vertices of A_L, B_L, and B_R. The cases of the inner vertices of the remaining three crystals are analogous. Therefore, G can be properly $L_D(p,1)$-labeled—a contradiction.

■

5.3 Intermezzo

Before we continue with the proof, let us establish the following technical lemma. Before stating it, we need some additional notation: if p, t, and K are non-negative integers, then $B_K(t,p)$ denotes the set of integers x such that $0 \leq x \leq K$ and $|t-x| < p$. Notice that if u and v are two adjacent vertices of G and u is labeled with t, then $B_K(t,p)$ is precisely the set of labels which cannot be used to label v in any proper $L(p,1)$-labeling of G with span K.

Lemma 5.9 *Let p be a non-negative integer, G a graph with no adjacent vertices of degree two, c a partial $L(p,1)$-labeling of span at most $K \geq |V(G)|-1$ such that every vertex which is not labeled by c has degree two, and every label is used at most once in c. Further, let $P = \{v_1, \ldots, v_k\}$ be the set of all vertices whose degree is different from 2. If every label in the set $\bigcup_{i=1}^{k} B_K(c(v_i), p)$ is used on some vertex v in $V(G)$, then c can be extended to an $L(p,1)$-labeling of the entire graph G with span at most K.*

Proof: In order to extend c to the entire G, we assign the unused labels (from the set $\{0, \ldots, K\}$) arbitrarily to the vertices of degree two which are not labeled by c, in such a way that each label is used at most once. It is now routine to check that this extension of c is an $L(p,1)$-labeling. The condition for vertices

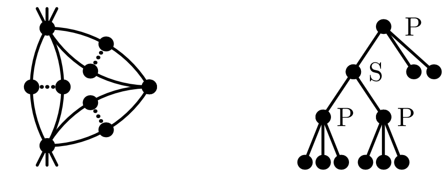

Figure 5.4: A $P(S(P,P),\ell*)$-subgraph and the corresponding subtree.

at distance two is clearly satisfied as no two vertices get the same label. If u and v are neighboring vertices, we know that at least one of them, say u, has degree different from 2 and therefore, is labeled by c. If v is not labeled by c, the distance of the labels of u and v must be at least p because all the labels conflicting with $c(u)$ are used somewhere else in the prelabeling c. If v is labeled by c, then the proper difference of labels is guaranteed by the fact that c is a partial $L(p,1)$-labeling.

■

The main benefit of the lemma is that we do not have to specify the assignment of all the labels, but only the labels of vertices with degrees different from two and the labels which are "close" (in terms of p) to those labels. This will be quite useful in the proofs of the next few lemmas which involve constructions of $L_D(p,1)$-labelings of potentially large graphs with only a few vertices which do not have degree two.

5.4 Largo

By Lemmas 5.5–5.8, if D is large enough, the final block of a (D,p)-minimal graph contains a $P(S(P,P),\ell*)$-subgraph (see Figure 5.4). This subgraph is either the entire final block or the root of the subtree corresponding to it has a parent (which is an S-node and must have another parent which is a P-node). First, we deal with the former case.

Lemma 5.10 *For every positive integer p, there exists a constant $D_{5.10}$ such that there is no (D,p)-minimal graph, $D \geq D_{5.10}$, whose entire final block is a $P(S(P,P),\ell*)$-subgraph.*

Proof: We prove the lemma for

$$D_{5.10} = \max\left\{\frac{2}{3}(10p+4Kp-2K-3), 6K+4p+4\right\}$$

where K is the constant from Lemma 5.4. For the sake of contradiction, assume that there is a (D,p)-minimal graph G, $D \geq D_{5.10}$, whose final block $G^\star$ is a

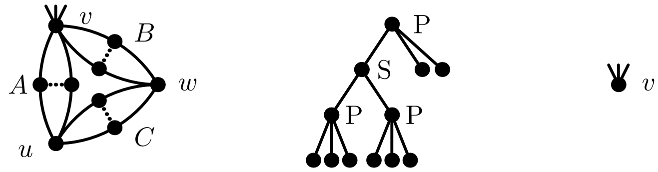

Figure 5.5: A $P(S(P,P),\ell*)$-subgraph being the entire final block, the corresponding subtree, and its reduction.

$P(S(P,P),\ell*)$-subgraph. In particular, $G^\star$ consists of three vertices u, v, w and three crystals A, B, and C such that the poles of the crystal A are u and v, the poles of the crystal B are v and w, and the poles of the crystal C are u and w. If $G^\star$ is not the entire graph G, then v is the cut-vertex separating $G^\star$ from the rest of G. Let N_v be the set of neighbors of v outside $G^\star$. The configuration is depicted in Figure 5.5. In order to produce an $L_D(p,1)$-labeling of G, we construct an auxiliary graph G' from G by replacing $G^\star$ by a single vertex v and eventually find an $L_D(p,1)$-labeling c' of G'.

In the rest of the proof, we aim to extend c' to an $L_D(p,1)$-labeling c of the entire graph G. By Lemma 5.4, the sizes of A, B, and C are at least $\lfloor D/2\rfloor - K$, and thus $|N_v| \leq 2K+1$. We start with the vertices u, v and w. Their labels should satisfy the following: they differ by at least $2p$ from each other and and each of them differs by at least p from all the labels of the vertices in N_v. Since this is satisfied for v, the vertex v can keep its original label and we only have to label u and w. Calculating the number of labels forbidden for u (similarly for w), we get that there are at most $2(4p-1)+(2K+1)(2p-1) \leq \lfloor 3D/2\rfloor$ such labels. In particular, there exist suitable labels for u and w.

To finish the prelabeling, we assign the labels of vertices in N_v to some inner vertices of C and we assign any unused labels in $B_{\lfloor 3D/2\rfloor}(c(v),p)$ to some inner vertices in the crystal C. Since $\lfloor D/2\rfloor - K > (2K+1)+2p-2+1$, the crystal C always contains enough inner vertices for the assignment and moreover, there will remain at least one inner vertex of C without an assigned label. The existence of such a vertex will be important in the final part of the proof. Similarly, we assign any unused elements of $B_{\lfloor 3D/2\rfloor}(c(u),p)$ to some inner vertices of B and the unused elements of $B_{\lfloor 3D/2\rfloor}(c(w),p)$ to some inner vertices of A. Notice that the resulting labeling is a valid partial $L_D(p,1)$-labeling of $G^\star$.

Next, we would like to estimate the size of $G^\star$. By the degree condition for vertices u, v, and w, we obtain the following inequalities:

$$\begin{aligned}
\mathrm{size}(A)+\mathrm{size}(C) &\leq D\\
\mathrm{size}(A)+\mathrm{size}(B)+|N_v| &\leq D\\
\mathrm{size}(B)+\mathrm{size}(C) &\leq D
\end{aligned}$$

Summing these values up, we get that

$$\begin{aligned}|V(G^\star)| &= |\mathrm{Inner}(A)| + |\mathrm{Inner}(B)| + |\mathrm{Inner}(C)| + 3\\ &\le \mathrm{size}(A) + \mathrm{size}(B) + \mathrm{size}(C) + 3 \le \lfloor 3D/2 \rfloor + 3.\end{aligned}$$

However, the statement of Lemma 5.9 requires $|V(G^\star)| \le \lfloor 3D/2 \rfloor + 1$. To overcome this problem, we consider the following cases.

Case 1: *None of A, B, and C contains an edge joining the poles of the crystal.* In this case, we can remove one unlabeled inner vertex w' from A and one unlabeled inner vertex u' from B. Let G' be the resulting graph. Since two vertices are removed in the construction of G', $|V(G')| \le \lfloor 3D/2 \rfloor + 1$ as required. By Lemma 5.9, we obtain an $L_D(p,1)$-labeling $c^\star$ of G'. The labeling $c^\star$ is eventually extended to the entire $G^\star$ by setting $c^\star(w') = c^\star(w)$ and $c^\star(u') = c^\star(u)$.

Case 2: *Exactly one of A, B, and C contains an edge joining the poles of the crystal.* First assume that the crystal containing the edge joining the poles is A (or, by symmetry, B). In particular, $\mathrm{size}(A) = |\mathrm{Inner}(A)| + 1$. Remove an unlabeled inner vertex w' from A and let G' be the resulting graph. Again, $|V(G')| \le \lfloor 3D/2 \rfloor + 1$ as required. By Lemma 5.9, there exists an $L_D(p,1)$-labeling $c^\star$ of G' which can be extended to an $L_D(p,1)$-labeling of $G^\star$ by setting $c^\star(w') = c^\star(w)$.

The case when the edge joining the poles is contained in C is similar. We choose an unlabeled inner vertex v' from C, remove it, and find a suitable labeling $c^\star$ of the new graph. Finally, we set $c^\star(v') = c^\star(v)$.

Case 3: *At least two crystals contain an edge connecting the poles.* Then,

$$\begin{aligned}|\mathrm{Inner}(A)| + |\mathrm{Inner}(C)| + 1 \le D, |\mathrm{Inner}(A)| + |\mathrm{Inner}(B)| + |N_v| + 1 \le D,\\ |\mathrm{Inner}(B)| + |\mathrm{Inner}(C)| + 1 \le D,\end{aligned}$$

and at least one of those inequalities is strict. Therefore, we get $|V(G^\star)| \le \lfloor 3D/2 \rfloor + 1$ and Lemma 5.9 can be applied to $G^\star$ directly, yielding an $L_D(p,1)$-labeling $c^\star$ of $G^\star$.

Based on the discussion above, we can find an $L_D(p,1)$-labeling $c^\star$ of $G^\star$ consistent with the prelabeling. It is routine to check that(because of the construction of the prelabeling) $c^\star$ combined with c is a proper $L_D(p,1)$-labeling of the G, hence G is not (D,p)-bad—a contradiction.

■

By Lemmas 5.5–5.10, the final block of a (D,p)-minimal graph has to contain an $S(P(S(P,P),\ell*),\ell)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, or an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph. In the next three lemmas, we show that none of these cases actually applies if D is large enough.

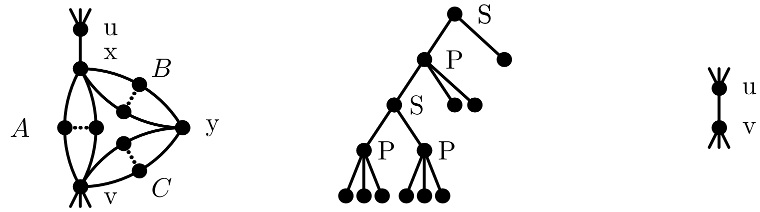

Figure 5.6: An $S(P(S(P,P),\ell*),\ell)$-subgraph, the subtree corresponding to it and its reduction.

Lemma 5.11 *For every positive integer p, there exists a constant $D_{5.11}$ such that there is no (D,p)-minimal graph, $D \geq D_{5.11}$, whose final block contains an $S(P(S(P,P),\ell*),\ell)$-subgraph.*

Proof: We prove the statement of the lemma with

$$D_{5.11} = \max\{4K + 2L + 8p, (4p-2)(K+L+1) + 24p - 8\}$$

where K is the constant from Lemma 5.4 and $L = 4p - 3$. Let G be a (D,p)-minimal graph for some $D \geq D_{5.11}$ such that its final block $G^\star$ contains an $S(P(S(P,P),\ell*),\ell)$-subgraph. In particular, $G^\star$ contains four vertices u, v, x and y, three crystals A, B, and C such that the poles of crystal A are x and v, the poles of crystal B are x and y, the poles of crystal C are y and v, and there is an edge joining u with x. The entire subgraph (the four vertices and the three crystals) is denoted by R and is depicted in Figure 5.6. Finally, let N_u and N_v be the set of the neighbors of u and v that are not contained in R. Moreover, if there is an edge uv in $G^\star$ which is not contained in R, then we also set $v \in N_u$ and $u \in N_v$. Both N_u and N_v are nonempty, otherwise x is a cut-vertex and $G^\star$ is not 2-connected.

By Lemma 5.4, the sizes of both B and C are at least $\lfloor D/2 \rfloor - K$. Next, we show that the size of A is at least $\lfloor D/2 \rfloor - L$. Assume the contrary, i.e., $\mathrm{size}(A) \leq \lfloor D/2 \rfloor - L - 1$. Let us remove an inner vertex w from B and find an $L_D(p,1)$-labeling c of $G \setminus w$. Since the number of labels forbidden for w is at most $D - 1 + \lfloor D/2 \rfloor - L - 1 + 1 + 2(2p-1) = \lfloor 3D/2 \rfloor - L + 4p - 3 = \lfloor 3D/2 \rfloor$, c can be extended to G. However, this is impossible by the (D,p)-minimality of G. Hence, the size of A must be at least $\lfloor D/2 \rfloor - L$. Combined with the lower bound on the size of C, we obtain that $|N_v| \leq K + L + 1$.

In order to prove the statement of the lemma, we construct a new graph G' from G by replacing R with an edge uv and find an $L_D(p,1)$-labeling c of G' which we eventually extend to an $L_D(p,1)$-labeling of G. The proof proceeds similarly to the proof of Lemma 5.10. First, we find the prelabeling: the labels of the vertices x and y are chosen in such a way that they differ by at least $2p$ from the labels of both u and v, by at least p from the labels of the vertices in

N_v, by at least one from the labels of the vertices in N_u, and by at least $2p$ from each other. The number of forbidden labels is bounded by $2(4p-1)+(K+L+1)(2p-1)+D-1+4p-1 \leq \lfloor 3D/2 \rfloor$.

Next, the labels of vertices in $N_v \setminus u$ are used to label some inner vertices of B, and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(v),p) \cup B_{\lfloor 3D/2 \rfloor}(c(u),p)$ are used to label some inner vertices of B. Since $\lfloor D/2 \rfloor - K > (K+L+1)+2(2p-2)+1$, the number of inner vertices in the crystal B is sufficient so that all the labels described above can be used on some vertices of B. Moreover, there always remains at least one inner vertex of B without a label. Finally, the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(y),p)$ are used to label some inner vertices of A and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(x),p)$ are used to label some inner vertices of C.

It is straightforward to verify that this partial labeling satisfies the conditions on the prelabeling given in Lemma 5.9 for the subgraph R and span $\lfloor 3D/2 \rfloor$ with a possible exception for the condition that $|V(R)| \leq \lfloor 3D/2 \rfloor$. Since the degree of each of v, x, and y is at most D, we obtain the following inequalities:

$$\begin{aligned} \mathrm{size}(A) + \mathrm{size}(B) + 1 &\leq D, \\ \mathrm{size}(B) + \mathrm{size}(C) &\leq D, \\ \mathrm{size}(A) + \mathrm{size}(C) + |N_v| &\leq D. \end{aligned}$$

Summing these inequalities up and using $|N_v| \geq 1$, we conclude that

$$2(\mathrm{size}(A) + \mathrm{size}(B) + \mathrm{size}(C)) \leq 3D - 2.$$

Thus,

$$\begin{aligned} |V(R)| &= 4 + |\mathrm{Inner}(A)| + |\mathrm{Inner}(B)| + |\mathrm{Inner}(C)| \\ &\leq 4 + \lfloor (3D-2)/2 \rfloor = \lfloor 3D/2 \rfloor + 3. \end{aligned}$$

As in Lemma 5.10, we cannot apply Lemma 5.9 directly to R in general, because the number of vertices could be greater than $\lfloor 3D/2 \rfloor + 1$. However, by considering the same three cases as in Lemma 5.10, we conclude that the prelabeling can always be extended to an $L_D(p,1)$-labeling c_R of R. By the construction of the prelabeling, c_R can be combined with c to yield an $L_D(p,1)$-labeling of the entire graph G. Hence, G is not (D,p)-bad, a contradiction.

■

Lemma 5.12 *For every positive integer p, there exists a constant $D_{5.12}$ such that there is no (D,p)-minimal graph, $D \geq D_{5.12}$, whose final block contains an $S(P(S(P,P),\ell*),P)$-subgraph.*

Proof: We prove this lemma for $D_{5.12} = 10p(K+6p+6)$ where K is the constant from Lemma 5.4. For the sake of contradiction, assume that there

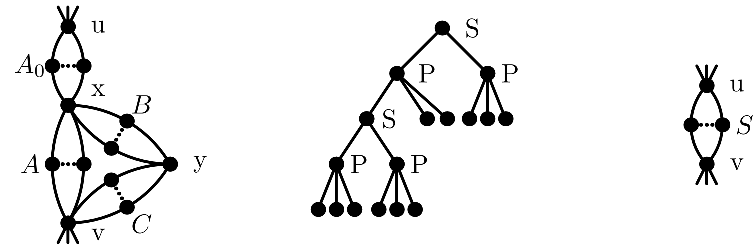

Figure 5.7: An $S(P(S(P,P),\ell*),P)$-subgraph, the subtree corresponding to it and its reduction.

exists a (D,p)-minimal graph G, $D \geq D_{5.11}$, whose final block $G^\star$ contains an $S(P(S(P,P),\ell*),P)$-subgraph. In particular, $G^\star$ contains four vertices u, v, x, and y and four crystals A_0, A, B, and C, such that the poles of A_0 are u and x, the poles of A are x and v, the poles of B are x and y, and the poles of C are v and y. The entire subgraph (the four vertices and the four crystals) is denoted by R and is depicted in Figure 5.7. Let N_u and N_v be the sets of neighbors of u and v outside R and set $M = 5p$.

By Lemma 5.4, the sizes of the crystals B and C are at least $\lfloor D/2 \rfloor - K$. Consequently, the sizes of A_0 and A sum to at most $\lceil D/2 \rceil + K$. By an argument analogous to that used to prove Lemma 5.4, we show that the sum of the sizes of A and A_0 is at least $\lfloor D/2 \rfloor - L$, where $L = 4p - 4$. Assume the contrary, i.e., $\mathrm{size}(A) + \mathrm{size}(A_0) \leq \lfloor D/2 \rfloor - L - 1$. Then, remove an inner vertex w from B and find an $L_D(p,1)$-labeling c of $G \setminus w$. Next, we label the vertex w. The number of forbidden labels for it is at most $D - 1 + \lfloor D/2 \rfloor - L - 1 + 2(2p-1) = \lfloor 3D/2 \rfloor - L + 4p - 4 = \lfloor 3D/2 \rfloor$: at most $D-1$ labels are forbidden by the inner vertices of B and C, at most $\lfloor D/2 \rfloor - L - 1$ labels are forbidden by the remaining neighbors of the vertex x, and at most additional $2(2p-1)$ labels can be forbidden by the vertices x and y. Consequently, c can be extended to the entire graph G, contradicting the (D,p)-minimality of G.

Let $M = 5p$. We distinguish two cases: $\mathrm{size}(A_0) \geq \frac{D}{2M}$ and $\mathrm{size}(A_0) < \frac{D}{2M}$.

Case $\mathrm{size}(A_0) \geq \frac{D}{2M}$**:** Consider the graph G' obtained from G by contracting the subgraph induced by A, B, and C into the vertex v. In particular, R is transformed to a crystal S with poles u and v. Next, we find an $L_D(p,1)$-labeling c of G' and extend it to an $L_D(p,1)$-labeling of G as described in the following. The labels assigned to the inner vertices of S are used to label the inner vertices of A_0 and as many inner vertices of C as possible. Then, suitable labels for x and y are found (possibly by unlabeling some vertices in A_0 and C). The number of forbidden labels for x and y is at most $D - \frac{D}{2M} + \lceil D/2 \rceil + K + 2(2p-1) + 1 \leq \lfloor 3D/2 \rfloor$. The inner vertices of A and the remaining vertices of A_0 are labeled next. The number of forbidden labels is at most $D-1+2p-1+1+2(2p-1) \leq \lfloor 3D/2 \rfloor$ for the inner vertices of A and at most $D-1+\lceil \frac{D}{2} \rceil + K + 1 - \frac{1}{2M}D + 1 + 2(2p-1) \leq \lfloor 3D/2 \rfloor$

for the inner vertices of A_0.

The next step is labeling of the remaining inner vertices of C. The number of forbidden labels for those vertices is bounded by $D-1+2(2p-1)+1 \leq \lfloor 3D/2 \rfloor$. Finally, label the inner vertices of B (the number of forbidden labels for these vertices is at most $D-1+\lceil D/2 \rceil + K - \frac{D}{2M} + 2p - 1 + 2(2p-1) \leq \lfloor 3D/2 \rfloor$). We conclude that c can be extended to an $L_D(p,1)$-labeling of the entire G—a contradiction.

Case $\text{size}(A_0) < \frac{D}{2M}$**:** Since $\text{size}(A) + \text{size}(A_0) \geq \lfloor D/2 \rfloor - L$, $\text{size}(A) \geq \frac{(M-1)D}{2M} - L$. Therefore, $|N_v| \leq \frac{D}{2M} + K + L \leq \frac{D}{M}$. By the 2-connectivity of $G^\star$, $|N_v| \geq 1$ (otherwise, x would be a cut-vertex). We proceed analogously to the proof of Lemma 5.11. Transform G to G' by replacing R with a single edge uv, and find an $L_D(p,1)$-labeling c of G'. In the rest of the proof, we demonstrate how to extend c to an $L_D(p,1)$-labeling of the entire graph G.

First, we find labels for x and y that differ from the labels of u and v by at least $2p$, from the labels of the vertices in N_v by at least p, from the labels of the vertices in N_u by at least one, and from each other by at least $2p$. This is always possible since the number of labels forbidden for x and y is at most $D + \frac{D}{M}(2p-1) + 3(4p-1) \leq \lfloor 3D/2 \rfloor$. Next, labels for the inner vertices of A_0 are found in such a way that the difference of these labels from the labels of u, v, x, and y is at least p and they are different from the labels of all the vertices in $N_u \cup N_v$. Note that the number of forbidden labels for each vertex of A_0 is bounded by $D + 4(2p-1) + \frac{D}{2M} + K + L \leq \lfloor 3D/2 \rfloor$.

Now, assign all the labels of the inner vertices of A_0 to some inner vertices of C and assign the labels of the vertices in N_v to some inner vertices of B (omit the label $c(u)$ if ux is an edge). Next, construct an auxiliary graph G_0 by taking the subgraph of G induced by the set $\{v,x,y\} \cup A \cup B \cup C$. If ux is an edge in G, add u and the edge ux to G_0 as well. Let c_0 be the obtained prelabeling of G_0. To meet the conditions of Lemma 5.9, we extend c_0 as follows: the unused labels in $B_{\lfloor 3D/2 \rfloor}(c_0(u),p)$ and $B_{\lfloor 3D/2 \rfloor}(c_0(v),p)$ are assigned to some vertices in B and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c_0(x),p)$ and $B_{\lfloor 3D/2 \rfloor}(c_0(y),p)$ are assigned to some inner vertices of C and A, respectively. Notice that since $\lfloor D/2 \rfloor - K > \frac{D}{M} + 2(2p-2) + 1$ and $\frac{(M-1)D}{2M} - L > 2(2p-2)+1$, every crystal contains enough inner vertices for the assignment of the labels and every crystal will always contain at least one inner vertex without a label.

Finally, we have to show that $|V(G_0)| \leq \lfloor 3D/2 \rfloor + 1$. Since the degree of each of the vertices x, y, and v is bounded by D, we obtain that

$$\begin{aligned} |\text{size}(A)| + |\text{size}(B)| + \text{size}(A_0) &\leq D \\ |\text{size}(B)| + |\text{size}(C)| &\leq D \\ |\text{size}(A)| + |\text{size}(C)| + |N_v| &\leq D \end{aligned}$$

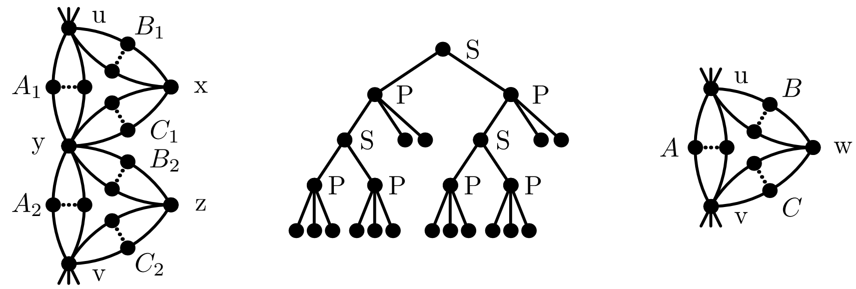

Figure 5.8: An $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph, the subtree corresponding to it and its reduction.

Summing these inequalities up and using $|N_v| \geq 1$ and $\text{size}(A_0) \geq 2$, we conclude that

$$2(\text{size}(A) + \text{size}(B) + \text{size}(C)) \leq 3D - 3.$$

Thus,

$$\begin{aligned}|V(G_0)| &= 4 + |\text{Inner}(A)| + |\text{Inner}(B)| + |\text{Inner}(C)| \\ &\leq 4 + \lfloor (3D-3)/2 \rfloor = \lfloor (3D-1)/2 \rfloor + 3.\end{aligned}$$

As in the previous two proofs, Lemma 5.9 cannot be applied to G_0 directly. However, considering the same cases and analyzing them as in Lemma 5.10, we conclude that G is not (D,p)-bad.

■

Lemma 5.13 *For every positive integer p, there exists a constant $D_{5.13}$ such that there is no (D,p)-minimal graph, $D \geq D_{5.13}$, whose final block contains an $S(P(S(P,P),\ell*),\ P(S(P,P),\ell*))$-subgraph.*

Proof: We prove the lemma for $D_{5.13} = 24p + 28K + 4$ where K is the constant from Lemma 5.4. For the sake of contradiction, fix G to be a (D,p)-minimal graph for some $D \geq D_{5.13}$ whose final block $G^\star$ contains an $S(P(S(P,P),\ell*),$ $P(S(P,P),\ell*))$-subgraph. In particular, $G^\star$ contains five vertices u, v, x, y, and z and six crystals A_1, B_1, C_1, A_2, B_2 and C_2 such that the poles of the crystal A_1 are u and y, the poles of the crystal B_1 are u and x, the poles of the crystal C_1 are x and y, the poles of the crystal A_2 are y and v, the poles of the crystal B_2 are y and z, and the poles of the crystal C_2 are v and z. The entire subgraph (the five vertices and the six crystals) is denoted by R and is depicted in Figure 5.8. Let N_u and N_v be the sets of the neighbors of u and v outside R.

By Lemma 5.4, the sizes of the crystals B_1, C_1, B_2, and C_2 are at least $\lfloor D/2 \rfloor - K$. Hence, the sizes of A_1 and A_2 sum to at most $2K+1$. An $L_D(p,1)$-labeling of G is obtained as follows: construct a new graph G' from G by contracting the subgraph induced by A_1, B_1, and C_1 into the vertex u. Since the degree of the vertex u could be greater than D after the contraction, it might be necessary to remove several (at most $2K+1$) vertices from the crystal corresponding to B_2. Let A, B and C be the "new" crystals and let w be the common pole of B and C. Find an $L_D(p,1)$-labeling c of G'. We will extend c to an $L_D(p,1)$-labeling of G in what follows.

First, use the labels assigned to the inner vertices of B to label as many inner vertices of B_1 as possible and use the labels assigned to the inner vertices of C to label as many inner vertices of C_2 as possible. The vertex y is assigned the label of an arbitrarily chosen inner vertex of A. Next, find suitable labels for x and z and unlabel some vertices in B_1 or C_2 if required. The number of forbidden labels for each of the vertices x and z is bounded by $D+2(2p-1)+2$. The inner vertices of A_1 and A_2 are labeled next. The number of forbidden labels for those vertices is at most $D+2(2p-1)+2K \leq \lfloor 3D/2 \rfloor$.

It remains to label the inner vertices of B_2 and C_1, and the remaining inner vertices of B_1 and C_2. We start with the remaining vertices of B_1 and C_2. The number of forbidden labels is bounded by $D-1+2(2p-1)+1 \leq \lfloor 3D/2 \rfloor$. Next, label as many inner vertices of C_1 as possible using labels of inner vertices of C and as many inner vertices of B_2 using labels of inner vertices of B. Notice that there are at most $2(2K+1+2(2p-1))$ labels which are used on inner vertices of B_1 but not on inner vertices of B_2, and the same relation holds between C_2 and C_1. Finally, label the remaining vertices of C_1 and B_2. The number of forbidden labels for those vertices is at most

$$D-1+2(2K+1+2(2p-1))+2(2p-1) \leq \lfloor 3D/2 \rfloor.$$

We infer from the above that c can be extended to the entire graph G, which contradicts its (D,p)-minimality.

■

5.5 Finale

Proof of Theorem 5.2. Fix p and set $D_0 = \max\{D_{5.5}, \ldots, D_{5.13}\}$ where the values $D_{5.5}, \ldots, D_{5.13}$ are the constants from Lemmas 5.5–5.13. For the sake of contradiction, let us assume that there exists a (D,p)-bad graph G, $D \geq D_0$. Since the empty graph is clearly not (D,p)-bad, there must exist a (D,p)-minimal graph G'. Further, let $G^\star$ be the final block of G' and $T^\star$ be its SP-decomposition tree such that if $G^\star$ contains a cut-vertex v of G', then v is one of the poles of the root node of $T^\star$.

By Lemmas 5.5–5.7, the final block $G^\star$ contains an $S(P,P)$-subgraph. Consider an $S(P,P)$-subgraph G_0 whose depth in $T^\star$ is the largest among all the

$S(P,P)$-subgraphs. By Lemma 5.8, there is no $P(S(P,P),S(P,P))$-subgraph. So, G_0 must be contained in a $P(S(P,P),l*)$-subgraph G_1. Lemma 5.10 yields that G_1 cannot be the entire subgraph $G^\star$. In particular, the P-node corresponding to G_1 must have an S-node parent in $T^\star$ which corresponds to an $S(\ldots)$-subgraph G_2. However, Lemmas 5.11–5.13 imply that no such G_2 exist. We infer from the above arguments that no (D,p)-bad graph exists.

■

Corollary 5.1 yields that the upper bound on the $L(p,1)$-span of K_4-minor free graphs of the maximum degree Δ matches the corresponding upper bound on the chromatic number of the square if Δ is large enough. Analogous results are known for some other graph classes as well. For instance, the bounds on the $L(p,1)$-span and the $L(1,1)$-span of planar graphs of maximum degree Δ obtained by Molloy and Salavatipour [68, 69] differ only by an additive term which (linearly) depends on p. We suspect that this is not a mere coincicence and believe that the following more general statement actually holds.

Conjecture 5.14 *Let H be a graph and let $f_p^H(\Delta)$ be the maximum $L(p,1)$-span of an H-minor free graph of the maximum degree Δ. For every positive integer p, there exist two constants Δ_0 and K such that $f_p^H(\Delta) \leq f_1^H(\Delta) + K$ for every $\Delta \geq \Delta_0$.*

For the case of K_4-minor free graphs, i.e., $H = K_4$, we have established the above conjecture with $K = 0$ (Corollary 5.1). It could turn out that this is just an exposure of a more general fact, i.e., Conjecture 5.14 is true with $K = 0$ for all graphs H.

Chapter 6

Choosability of Squares of K_4-minor Free Graphs

In the previous chapter, we have shown how Theorem 4.1 extends to $L(p,q)$-labelings. In this chapter, we focus on list coloring instead, and prove an analogous result. Unlike Theorem 5.1, this extension applies for all values of Δ.

Theorem 6.1 *The list chromatic umber of the square of a K_4-minor free graph G of maximum degree Δ is at most $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$ and at most $\Delta + 3$ if $\Delta \in \{2, 3\}$.*

Before presenting the proof of Theorem 6.1, let us mention that the same result was obtained independently by Hetherington and Woodall [42].

The main part of the proof lies in examining properties of a possible counterexample. For an integer $D \geq 1$, a graph is said to be D-*bad* if it is K_4-minor free, its maximum degree is at most D, and its list chromatic number is greater than $\lfloor 3D/2 \rfloor + 1$. Further, a graph is said to be D-*minimal* if it is D-bad and there is no D-bad graph of smaller order.

Clearly, our main result (see Theorem 6.1) for $D \geq 4$ is equivalent to stating that there are no D-bad graphs and no D-minimal graphs in particular for $D \geq 4$. In what follows, we exhibit a (long) series of lemmas that yield the proof of the statement.

6.1 Even Maximum Degree

Let us start with observations on the structure of D-minimal graphs which lead straightforwardly to a proof of the desired bound for graphs with maximum degree that is even (but they are also useful for graphs with odd maximum degree). Clearly, every D-minimal graph is connected. In the next lemma, we show that D-minimal graphs do not contain vertices of degree one and neighboring vertices of degree two.

Lemma 6.2 *No D-minimal graph G, for $D \geq 4$, contains a vertex of degree one or two adjacent vertices of degree two.*

Since proofs of most of the lemmas use the same technique, we explain the notation and principle in more detail here. This will allow us to make the other proofs more compact and to concentrate on their main aspects.
Proof: Fix a list assignment L giving each vertex of G a list of at least $\lfloor 3D/2 \rfloor + 1$ colors.

We first consider the case that G contains a vertex v of degree one. Remove the vertex v, and find a proper list-coloring c of the square of the resulting graph using the lists given by L which exists by the D-minimality of G. We now aim to extend c to v. To show that this is always possible, we count the number of colors in $L(v)$ that cannot be used to color the vertex v since the color is already used to color a vertex at distance at most two from v. We say that the colors that cannot be used to color v because of the above reason are *forbidden* for v; the remaining colors in $L(v)$ are said to be *available* for v. In particular, if we show that the number of available colors for v is at least one (i.e., the number of forbidden colors is at most $\lfloor 3D/2 \rfloor$), we can conclude that the coloring c can be extended to v. Let us proceed with counting. The only neighbor of v forbids at most one color, and its neighbors excluding v forbid at most additional $D - 1$ colors. In total, there are at most $D \leq \lfloor 3D/2 \rfloor$ forbidden colors. Hence, c can be extended to v.

Next, we show that there are no two vertices u and v of degree two joined by an edge, i.e., there is no path $xuvy$ where u and v have degree two. As before, we find a list-coloring c of the square of $G \setminus \{u, v\}$. Now, we find a color for u: the neighbors of x forbid at most $D - 1$ colors, and two more colors can be forbidden by x a y. Hence, there are at most $D + 1$ forbidden colors for the vertex u and therefore, u can be properly colored. The case of the vertex v is similar: the number of forbidden colors will be at most $D + 2$, since the vertex u has already been colored. We conclude that the coloring c can be extended to a proper list-coloring of the square of G, which contradicts the D-minimality of G.

■

For the rest of the section, we consider a block-decomposition of a D-minimal graph G, and focus on one of its endblocks (we consider the entire graph G if it is itself 2-connected). The chosen block will be referred to as to the *final block* of G, and will be denoted by $G^\star$.

By Lemma 4.2, $G^\star$ is a series-parallel graph. If $G \neq G^\star$, $G^\star$ is connected to the rest of G through a vertex $v^\star$. If $G = G^\star$ (i.e., G itself is already 2-connected), $v^\star$ will be an arbitrarily chosen vertex of $G^\star$. By Lemma 4.3, there is an SP-decomposition $T^\star$ of $G^\star$ such that $v^\star$ is one of the poles corresponding to the root of of the decomposition $T^\star$. Note that the root of $T^\star$ always corresponds to a P-node since $G^\star$ is 2-connected.

We adopt the notation of A-subgraphs introduced in Section 4.1, and we say that an A-subgraph is *contained* in $G^\star$, if there is a subtree T_A of the form described by A with root r in $T^\star$ such that there is no descendant w of r in $T^\star$ whose depth is greater than the maximum depth of a descendant of r in T_A (in other words, we allow the subtree of the node r to be more complex than just a subtree corresponding to an A-subgraph, but we do not want it to be higher).

An immediate consequence of Lemma 6.2 is that all P-subgraphs contained in the final block are crystals. The following lemma shows that sizes and types of crystals in D-minimal graphs are quite restricted.

Lemma 6.3 *In a D-minimal graph, $D \geq 2$, the size of each crystal is at most $\lceil D/2 \rceil$, with the equality holding only for diamonds.*

Proof: Fix a D-minimal graph G, $D \geq 2$, that contains a crystal C with poles u and v whose size is S, and a list-assignment L of G giving each vertex a list of at least $\lfloor 3D/2 \rfloor + 1$ colors. Further, let w be an arbitrary inner vertex of C. Let G' be a graph obtained from G by deleting w. Since G is D-minimal, there exists a proper list-coloring c of the square of G' using the list-assignment L. We now show that if the statement of the lemma is violated, then c can be extended to the entire graph G.

First assume that C is a diamond and $S \geq \lceil D/2 \rceil + 1$. As in Lemma 6.2, we count the number of forbidden colors for w. The neighbors of u and v outside C forbid at most $2(D - S)$ colors, the inner vertices of $C \setminus w$ forbid at most another $S - 1$ colors, and 2 more colors may be forbidden by the vertices u and v themselves. Summing up, there are at most $2D - S + 1 \leq \lfloor 3D/2 \rfloor$ forbidden colors. Hence, the list $L(w)$ contains at least one available color.

Next, assume that C is not a diamond and $S \geq \lceil D/2 \rceil$. This time, the number of forbidden colors for w is at most $2D - S$: the neighbors of u and v outside C forbid at most $2(D - S)$ and the inner vertices of $C \setminus w$ forbid at most another $S-2$ colors (recall that there are only $S-1$ inner vertices if C is not a diamond). In particular, there are at most $\lfloor 3D/2 \rfloor$ forbidden colors, and therefore there exists a color that can be used to color w.

■

The next lemma complements Lemma 6.3 by showing that the size of certain crystals can be bounded from below as well.

Lemma 6.4 *Let C_1 and C_2 be two crystals in a D-minimal graph G sharing a common pole v and v has no other neighbors in G except for the vertices in C_1 and C_2. If $D \geq 4$, then the sizes of C_1 and C_2 belong to the set $\{\lfloor D/2 \rfloor, \lceil D/2 \rceil\}$. Moreover, one of the following properties must hold:*

- *both C_1 and C_2 are diamonds, or*

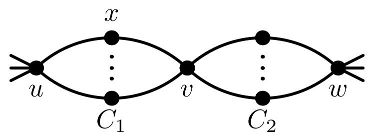

Figure 6.1: The proof of Lemma 6.4.

- *D is odd, one crystal is a diamond of size $\lceil D/2 \rceil$, and the other crystal is of size $\lfloor D/2 \rfloor$ and is not a diamond.*

Proof: First, we fix a D-minimal graph G and crystals C_1 and C_2 as specified above. By Lemma 6.3, the size of each of the crystals is bounded by $\lceil D/2 \rceil$ from above. Hence, it suffices to show that the size is also bounded by $\lfloor D/2 \rfloor$ from below and to discuss the possibilities whether a certain crystal is a diamond or not. In particular, we show that C_2 has the required property, depending on properties of C_1; the rest will follow by symmetry.

Before we start distinguishing the cases, let S_1 and S_2 be the sizes of crystals C_1 and C_2, respectively, u and v the poles of C_1, v and w the poles of C_2, and x an arbitrary inner vertex of C_1 (see Figure 6.1). We now fix an arbitrary list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors and find a proper list-coloring c of the square $G \setminus x$ using those lists. Next, the coloring c is extended to the square of the entire graph G as follows.

Case 1. *C_1 is not a diamond and S_2 is at most $\lfloor D/2 \rfloor$.* Let us calculate the number of forbidden colors of x. At most S_2 colors are forbidden by the inner vertices of C_2 and possibly by w, $S_1 - 2$ by the inner vertices of C_1 except for x, $D - S_1$ by neighbors of u outside of C_1, and finally 2 more colors may be forbidden by the vertices u and v. Altogether, there are at most $D + S_2 \leq \lfloor 3D/2 \rfloor$ forbidden colors. Hence, there is at least one color available for x.

Case 2. *C_1 is a diamond and S_2 is at most $\lfloor D/2 \rfloor$ - 1.* The number of forbidden colors for x is again bounded by $\lfloor 3D/2 \rfloor$: at most S_2 colors are forbidden by the inner vertices of C_2 and possibly by w, $S_1 - 1$ by the inner vertices of C_1 except for x, $D - S_1$ by neighbors of u outside of C_1, and finally 2 more colors may be forbidden by the vertices u and v. Therefore, there is always at least one color available for x, i.e., the coloring can be extended to the entire graph G.

Finally, let us show the two possibilities mentioned at the end of the statement of the lemma are the only possible cases. First assume that both C_1 and C_2 are not diamonds. Case 1 immediately yields that the sizes of both C_1 and C_2 are at least $\lfloor D/2 \rfloor + 1$, hence the degree of v is at least $D + 1$ which is not possible. Next assume that D is even, C_1 is a diamond, and C_2 is not. By Case 1, $S_1 \geq \lfloor D/2 \rfloor + 1$, and by Case 2, $S_2 \geq \lfloor D/2 \rfloor$. In particular, this yields that the degree of the vertex v exceeds D, which is, again, not possible.

■

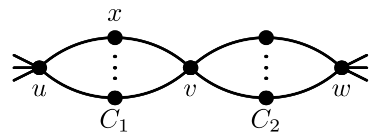

Figure 6.2: The proof of Lemma 6.6.

While the preceding two lemmas focused on the interior of the crystals, the next lemma describes something about their neighborhood.

Lemma 6.5 *If $D \geq 4$, then each pole of a crystal C in a D-minimal graph has at least 2 neighbors outside of C.*

Proof: Fix a D-minimal graph G, $D \geq 4$, and a crystal C with a pole v that has at most one neighbor outside of C. Further fix a list-assignment L giving each vertex a list of at least $\lfloor 3D/2 \rfloor + 1$ colors. Next, remove an arbitrary inner vertex w from the crystal C and find a list-coloring c of the square of the new graph with respect to the list-assignment L. To show that c can be extended to w, we again calculate the number of forbidden colors for w: at most $D-1$ colors are forbidden by the neighbors of the pole of C different from v, at most two colors are forbidden by the two poles, and at most one more color are forbidden by the neighbor (if any) of the vertex v outside of C. Again, this gives at most $D+2 \leq \lfloor 3D/2 \rfloor$, hence a suitable color for w exists.

■

We now turn our attention back to the final block $G^\star$ and its SP-decomposition $T^\star$. A corollary of Lemma 6.5 is that $G^\star$ cannot be just a crystal (i.e., just a P-subgraph) since at least one of the poles of the crystal would have no neighbors outside of the crystal.

Let us further examine the properties of the deepest P-nodes in $T^\star$. Since they do not form the entire $T^\star$, they must have an S-node parent. By Lemma 6.5, this S-node can only have P-subgraphs as its children. In the next lemma, we further examine properties of such P-subgraphs.

Lemma 6.6 *If $D \geq 4$, then no final block of a D-minimal graph contains an $S(P,P)$-subgraph whose both P-subgraphs are diamonds.*

Proof: Fix a D-minimal graph G, $D \geq 4$, its final block $G^\star$ containing an $S(P,P)$-subgraph whose both P-subgraphs are diamonds, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. Let the two diamonds be C_1 and C_2, their sizes S_1 and S_2, and their poles u and v, and v and w, respectively (see Figure 6.2). Without loss of generality, we may assume that $S_2 \leq \lfloor D/2 \rfloor$.

Choose x arbitrarily among the inner vertices of C_1 and find a proper list-coloring c of the square of the graph $G \setminus x$ with respect to L. Next, uncolor the

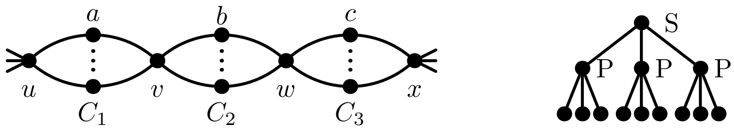

Figure 6.3: An $S(P, P, P)$-subgraph and the subtree corresponding to it.

vertex v, obtaining a partial coloring c' of G. We claim that c' can be extended to the entire graph G. The number of forbidden colors for x is bounded by $D + S_2 - 1 \leq \lfloor 3D/2 \rfloor$: there are $S_1 - 1$ colors forbidden by other inner vertices of C_1, at most $D - S_1$ forbidden by the neighbors of u outside of S_1, at most S_2 forbidden by the inner vertices of C_2, and at most one more color forbidden by the vertex u.

Next, we find a suitable color for the vertex v. The number of forbidden colors is bounded by $D + 2 \leq \lfloor 3D/2 \rfloor$: at most D colors are forbidden by the inner vertices of the two crystals, and additional two colors may be forbidden by the vertices u and w.

■

Lemmas 6.4 and 6.6 imply that there there is no D-minimal graph for even D. The case when D is odd, requires additional work, and is dealt with in the following section.

6.2 Odd Maximum Degree

In this section, we state and prove the lemmas necessary for proving that there are no D-minimal graphs for odd $D \geq 5$.

The first lemma excludes a possibility that the deepest S-node in $T^\star$ corresponds to an $S(P, P, P)$-subgraph.

Lemma 6.7 *If $D \geq 5$, then no final block of a D-minimal graph contains an $S(P, P, P)$-subgraph.*

Proof: Let G be a D-minimal graph for $D \geq 5$ whose final block contains an $S(P, P, P)$-subgraph. In particular, there exist three crystals C_1, C_2, and C_3 with poles u, v, w, and x, as depicted in Figure 6.3. Further, choose vertices a, b, and c among the inner vertices of C_1, C_2, and C_3, respectively. Next, fix an arbitrary list-assignment L assigning each vertex a list of at least $\lfloor 3D/2 \rfloor + 1$ colors. We show that G can be properly list-colored with respect to L.

First, by Lemma 6.6, we know that crystals and non-crystals must alternate among C_is, i.e., both C_1 and C_3 have the same properties (size and being diamond) which differ from those of C_2. We now delete all the three crystals except for the vertices u and x from G and find a list-coloring c' of the square of the new graph using the lists given by L. We aim to extend c' to the entire graph G.

Two cases have to be distinguished, based on which crystals are diamonds.

Case 1. *C_1 and C_3 are diamonds, C_2 is not.* By Lemma 6.4, the sizes of C_1 and C_2 are $\lceil D/2 \rceil$, and the size of C_2 is $\lfloor D/2 \rfloor$. Consequently, each of the vertices u and x has at most $\lfloor D/2 \rfloor$ neighbors outside the three crystals.

First, we aim to find colors for a and c in such a way that the number of colors available for b will remain at least $\lfloor 3D/2 \rfloor$. There are at least $\lfloor 3D/2 \rfloor + 1 - 1 - \lfloor D/2 \rfloor = D$ colors available for a (and, by symmetry, for c), as the are at most $\lfloor D/2 \rfloor$ colors forbidden by the neighbors of the vertex u and one more color may be forbidden by the vertex u itself. As there are no colors forbidden for the vertex b so far, there are $\lfloor 3D/2 \rfloor + 1$ colors available for it. If there is a color α available for both a and c, we color both the vertices by such a color α. Otherwise, as $2D > \lfloor 3D/2 \rfloor + 1$, there is a color β available for one of those vertices, say a, which is not available for b. In that case, color a by β and color c using any color available for it. Let this partial coloring be $c^\star$.

Next, we color the vertices v and w—the number of colors forbidden for each of them, say w, is at most 4: those forbidden by the vertices v, a, c, and x. We continue by coloring of the remaining inner vertices of C_1. The number of forbidden colors for those vertices is at most $D+2 \leq \lfloor 3D/2 \rfloor$: at most $D-1$ colors are forbidden by the neighbors of u and additional three colors may be forbidden by the vertices u, v, and w. Next, we color the inner vertices of C_2 except for b. The number of forbidden colors is bounded by $D \leq \lfloor 3D/2 \rfloor$: there are at most $\lfloor D/2 \rfloor - 3$ colors forbidden by the inner vertices of C_2, at most $\lceil D/2 \rceil$ by the inner vertices of C_1, and three more colors may be forbidden by the vertices c, v, and w. Then, we color the remaining inner vertices of C_3. The number of forbidden colors is bounded by $\lfloor 3D/2 \rfloor$: at most $D-1$ colors are forbidden by the neighbors of x, at most $\lfloor D/2 \rfloor - 2$ by the inner vertices of C_2 and at most three other colors are forbidden by the vertices v, w, and x.

As the last step, we have to find the color for b. To show that we can color the vertex b, we calculate the number of vertices we colored since the coloring $c^\star$ was made; each of those vertices can forbid at most one color that was available for vertex b at the time $c^\star$ was made. In particular, we show that number of such vertices is less than the number of available colors, hence it follows that there must be at least one color that can be used for coloring the vertex b. Let us calculate the vertices that were colored since $c^\star$ was made: there are $\lceil D/2 \rceil - 1$ such inner vertices in both C_1 and C_3, $\lfloor D/2 \rfloor - 2$ inner vertices in D_2, and the remaining two colored vertices are v and w; we conclude that exactly $\lfloor 3D/2 \rfloor - 1$ vertices have been colored. Since the number of colors available for b in $c^\star$ was at least $\lfloor 3D/2 \rfloor$, we infer that b can be properly colored.

Case 2. *C_2 is a diamond, C_1 and C_3 are not.* By Lemma 6.4, the sizes of C_1 and C_2 are $\lfloor D/2 \rfloor$, the size of C_2 is $\lceil D/2 \rceil$, and each of the vertices u and x has at most $\lceil D/2 \rceil$ neighbors outside the three crystals.

As in the previous case, we calculate the numbers of colors available to the vertices a, b, and c. In particular, there are at least $D-1$ colors available for a (the same holds for c), as the only colors forbidden for a could be those assigned to at most $\lceil D/2 \rceil$ neighbors of the vertex u and the color of the vertex u itself. Similarly, the vertex b has at least $\lfloor 3D/2 \rfloor - 1$ available colors since at most two colors can be forbidden—the colors of the vertices u and x. Hence, we can use the same approach to color the vertices a and c in such a way that the vertex b has at least $\lfloor 3D/2 \rfloor - 2$ available colors after the coloring. This partial coloring will be denoted by $c^\star$.

Continue by choosing the colors for the vertices v and w. The number of forbidden colors for each of them, say w, will be at most $\lceil D/2 \rceil + 3$: at most $\lceil D/2 \rceil$ colors are forbidden by the neighbors of the vertex x outside C_3, and at most three other colors may be forbidden by the vertices c, v, and x.

The remaining inner vertices of C_1 are colored next; the number of forbidden colors is bounded by D: at most $\lfloor D/2 \rfloor - 2$ colors are forbidden by the inner vertices of C_1, at most $\lceil D/2 \rceil$ colors are forbidden by the neighbors of the vertex u outside C_1, and at most two more colors are forbidden by the vertices u and v. Next, color the remaining inner vertices of C_3, the situation is symmetrical. Then, color the inner vertices of C_2 except for b. The number of forbidden colors is at most $\lfloor 3D/2 \rfloor$: at most $\lceil D/2 \rceil - 2$ colors are forbidden by the inner vertices of C_2, at most $2(\lfloor D/2 \rfloor - 1)$ colors are forbidden by the inner vertices of C_1 and C_4, and at most four colors are forbidden by the vertices u, v, w, and x. Now, the only vertex missing a color is b. Let us calculate the number of vertices that are now colored now did not have a color in $c^\star$. There are $\lfloor 3D/2 \rfloor - 3$ such vertices: $\lfloor D/2 \rfloor - 2$ vertices in both C_1 and C_3, $\lceil D/2 \rceil - 1$ vertices in C_2, and the vertices v and w. Since the vertex b had at least $\lfloor 3D/2 \rfloor - 2$ available colors in $c^\star$, there is still at least one available color which can be used to color it.

In both cases, we proved that c' can be extended to the entire graph G, hence the statement of lemma follows.

■

Lemmas 6.5 and 6.7 imply that the deepest S-nodes in $T^\star$ must correspond to roots of $S(P, P)$-subgraphs having exactly two P-subgraphs as their children. Since the root of $T^\star$ is a P-node, those S-nodes must have a P-node parent. This node corresponds either to a $P(S(P, P), S(P, P))$-subgraph (Figure 6.4) or to a $P(S(P, P), \ell*)$-subgraph (Figure 6.5). The next lemma excludes the former case.

Lemma 6.8 *If $D \geq 5$, then no final block of a D-minimal graph contains an $P(S(P, P)S(P, P))$-subgraph.*

Proof: Consider four vertices x, y, u, and v and four crystals C_{UL}, C_{UR}, C_{LL}, and C_{LR} as depicted in Figure 6.4. Finally, the entire structure may be connected to the rest of the graph through the vertices u and v.

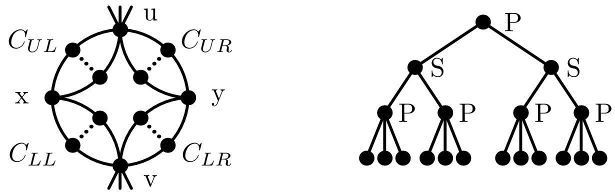

Figure 6.4: A $P(S(P,P),S(P,P))$-subgraph and the subtree corresponding to it.

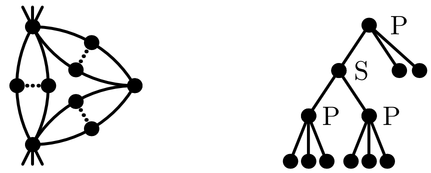

Figure 6.5: A $P(S(P,P),\ell*)$-subgraph and the subtree corresponding to it.

Applying Lemma 6.6 to C_{UL} and C_{LL}, we get that one of the crystals is of size $\lfloor D/2 \rfloor$ and the size of other one is $\lceil D/2 \rceil$; the same holds for C_{UR} and C_{LR}. Since both C_{UL} and C_{UR} cannot be of size $\lceil D/2 \rceil$ at the same time, we infer that one of C_{LL} and C_{LR} must be of size $\lceil D/2 \rceil$; the same again for C_{UL} and C_{UR}. Hence, the four crystals comprise the entire graph G as degrees of the vertices u and v cannot exceed D. However, we may find another decomposition of G as a parallel join of an $S(P,P,P)$-subgraph (corresponding to the crystals C_{UL}, C_{UR}, and C_{LR}) with several paths (corresponding to the crystal C_{LL}). Since the existence of an $S(P,P,P)$-subgraph was already excluded in Lemma 6.7, the statement of the lemma follow.

■

We now consider $P(S(P,P),\ell*)$-subgraphs.

Lemma 6.9 *If $D \geq 5$, then no final block of a D-minimal graph can be a $P(S(P,P),\ell*)$-subgraph.*

Proof: Let G be a D-minimal graph for $D \geq 5$ whose final block $G^\star$ consists of three vertices u, v, and w joined by three crystals C_1, C_2, and C_3 as in Figure 6.6. $G^\star$ may be connected to the rest of G through the vertex u. As in Lemma 6.8, one can quickly infer that C_2 is a diamond of size $\lceil D/2 \rceil$, while C_1 and C_3 are not, and their sizes are $\lfloor D/2 \rfloor$. In particular, u may have at most one neighbor outside of $G^\star$.

In the rest of the proof, we show that G^2 can be properly colored using lists containing at least $\lfloor 3D/2 \rfloor + 1$ colors. First, fix a list-assignment L assigning each vertex a list of $\lfloor 3D/2 \rfloor + 1$ colors. Next, remove all the vertices of $G^\star$ except for

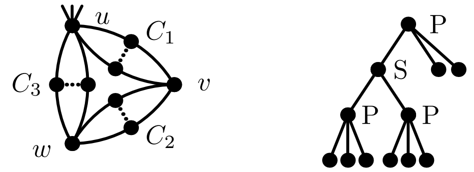

Figure 6.6: A $P(S(P,P),\ell*)$-subgraph with one of the poles being the cut-vertex in G.

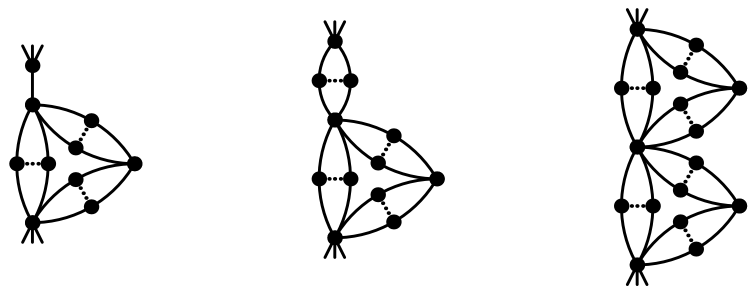

Figure 6.7: An $S(P(S(P,P),\ell*),\ell*)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, and an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph.

u and find a proper list-coloring c of the square of the new graph with respect to L. We extend the coloring c to the rest of the graph G. First, color the vertices v and w, which have at most 3 forbidden colors. Next, we color the inner vertices of C_1 and C_3. These vertices have at most $D-1+2 \leq \lfloor 3D/2 \rfloor$ forbidden colors: at most $D-1$ because of the colors of the neighbors of the vertex u and at most 2 because of the two poles of the crystal they belong to. Finally, we finish with coloring of the inner vertices of C_2. The number of forbidden color in this case is at most $D-1+\lfloor D/2 \rfloor -1+2 = \lfloor 3D/2 \rfloor$: at most $D-1$ colors are forbidden by the neighbors of the vertex v, at most $\lfloor D/2 \rfloor - 1$ colors are forbidden by the inner vertices of the crystal C_3, and additional 2 colors can be forbidden by the vertices v and w. Hence, c can be extended to the entire G, contradicting the minimality of G.

■

Since the $P(S(P,P),\ell*)$-subgraph is not the entire final block $G^\star$, the P-node corresponding to it must have an S-node parent in $T^\star$. In particular, any $P(S(P,P),\ell*)$-subgraph of the maximum depth must be contained either in an $S(P(S(P,P),\ell*),\ell*)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, or an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph. The structures are depicted in Figure 6.7

It is not hard to see that the final block of a D-minimal graph cannot contain an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph, as the degree of the pole connect-

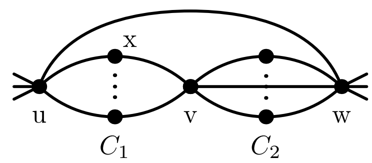

Figure 6.8: The proof of Lemma 6.10.

ing the two $P(S(P,P),\ell*)$-subgraphs would have degree at least $D+1$. In the following two lemmas, we consider the remaining two possibilities.

Lemma 6.10 *If $D \geq 5$, then no final block of a D-minimal graph contains an $S(P,P)$-subgraph whose poles are joined by an edge.*

Proof: Fix a D-minimal graph G, $D \geq 5$, its final block $G^\star$ containing an $S(P,P)$-subgraph whose poles are joined by an edge, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. Let the two P-subgraphs (crystals) of the $S(P,P)$-subgraph mentioned above be C_1 and C_2, and their poles u and v, and w, as depicted in Figure 6.8. By Lemma 6.4, we can assume that C_1 is a diamond of size $\lceil D/2 \rceil$ and C_2 is a crystal of size $\lfloor D/2 \rfloor$ that is not a diamond.

We pick an arbitrary inner vertex x in diamond C_1. By minimality, there exists a proper list-coloring c of the square of the graph $G \setminus x$ with respect to L; we fix one such coloring and extend it to x. The number of forbidden colors for x can be bounded as follows: there are at most $D-1$ colors forbidden by neighbors of u, additional $\lfloor D/2 \rfloor - 1$ colors may be forbidden by the inner vertices of C_2, and two more colors may be forbidden by the vertices u and v. This gives at most $\lfloor 3D/2 \rfloor$ colors forbidden altogether, i.e., c can be extended to vertex x.

■

Having established Lemma 6.10, we are ready to prove the final lemma of this section.

Lemma 6.11 *If $D \geq 5$, then the final block of a D-minimal graph contains neither an $S(P(S(P,P),\ell*),\ell*)$-subgraph nor an $S(P(S(P,P),\ell*),P)$-subgraph.*

Proof: Fix a D-minimal graph G, $D \geq 5$, its final block $G^\star$ containing one of the subgraphs from the statement of the lemma, and a list-assignment L giving each vertex at least $\lfloor 3D/2 \rfloor + 1$ colors. In particular, $G^\star$ contains four vertices u, v, w, and x; a crystal C_1 with poles u and v, a crystal C_2 with poles v and w, S_3 vertex-disjoint paths of length at most 2 connecting the vertices u and w, and S_4 vertex-disjoint paths of length at most 2 connecting the vertices w and x (see Figure 6.9). The described subgraph is connected to the rest of G through the vertices u and x. By Lemma 6.10, none of the S_3 paths connecting u and w is an edge. On the other hand, the S_4 paths between w and x may or may

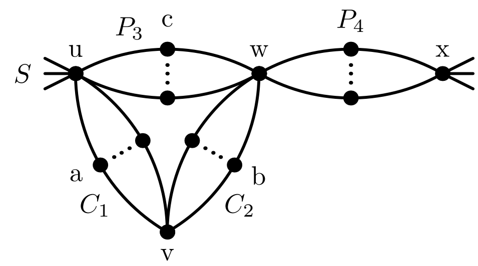

Figure 6.9: The proof of Lemma 6.11.

not be crystals. In order to simplify the notation used in the proof, P_3 and P_4 denote the two sets of the paths between u and w and the paths between w and x, respectively. Finally, let S be the number of neighbors of u except for the vertices in C_1 and the paths in P_3.

We now choose an arbitrary inner vertex a of the crystal C_1, an arbitrary inner vertex b of the crystal C_2, and an arbitrary internal vertex c of one of the paths in P_3. We find a list-coloring c' of the square of $G \setminus b$ with respect to L. In each of the following three cases, we show how to use c' to obtain a proper list-coloring of the square of the entire graph G.

Case 1. *C_2 is a diamond, C_1 is not.* In this case, $S + S_3 \leq \lceil D/2 \rceil$ and $S_4 \leq \lfloor D/2 \rfloor - 1$. First, we uncolor the inner vertices of both C_1 and C_2, together with the vertices v, w, and c. At this point, vertex a has at least $\lfloor 3D/2 \rfloor + 1 - S - (S_3 - 1) - 1 = D$ available colors, as the only colors forbidden for it are the colors used on the S neighbors of the vertex u outside the structure, on the $S_3 - 1$ colored internal vertices of the paths in P_3, and on the vertex u. Similarly, the vertex b has at least $\lfloor 3D/2 \rfloor + 1 - (\lfloor D/2 \rfloor - 1) - 1 = D + 1$ available colors as the only forbidden colors are used on the $\lfloor D/2 \rfloor - 1$ neighbors of the vertex w in P_3 and P_4 and one more color can be forbidden by the vertex u. Finally, the vertex w has at most $\lfloor 3D/2 \rfloor - 1$ forbidden colors (and thus at least 2 available colors): at most D colors are forbidden by the vertex x and its neighbors (notice that if there is no edge wx then at most $\lfloor D/2 \rfloor - 1$ neighbors of x are at distance at most 2 from the vertex w), at most $S_3 - 1$ colors are forbidden by the internal vertices of the paths in P_3, and one more color can be forbidden by the vertex u. In particular, we can choose the colors for w and a in such a way that there will remain at least D colors available for b: either the previously stated bound on available colors for b is not sharp, or there is a color α available for both a and w, or, as $D + 2 > D + 1$, there is a color α available for a or w which is not available for b.

We now choose suitable colors for the vertices c and v. The number of colors forbidden for the vertex c is bounded by $\lceil D/2 \rceil - 1 + \lfloor D/2 \rfloor - 1 + 3 = D + 1$: there are at most $S + S_3 - 1$ colors forbidden by the neighbors of the vertex u, at most S_4 colors forbidden by the remaining neighbors of the vertex w, and three more colors may be forbidden by the vertices u, w and a. Similarly, the number of colors forbidden for the vertex v is bounded by $\lceil D/2\rceil + 3 \leq \lfloor 3D/2\rfloor$: there are at most $S+S_3$ colors forbidden by the neighbors of the vertex u and at most three other colors are forbidden by the vertices u, w and a.

The remaining inner vertices of C_1 are colored next—there are at most $D-2+2$ forbidden colors: at most $S+S_3+\lfloor D/2\rfloor-2$ colors are forbidden by the neighbors of the vertex u and two more colors may be forbidden by the vertices u and v. We continue with the inner vertices of C_2 except for b. This time, the number of forbidden colors is at most $D-2+\lfloor D/2\rfloor-1+3 = \lfloor 3D/2\rfloor$: at most $\lceil D/2\rceil-2$ colors are forbidden by the inner vertices of C_2, at most additional $S_3+S_4 \leq \lfloor D/2\rfloor$ colors are forbidden by the remaining neighbors of the vertex w, at most $\lfloor D/2\rfloor-1$ colors are forbidden by the inner vertices of C_1, and three more colors may be forbidden by the vertices u, v, and w. Let us now calculate the number of vertices we have colored after coloring the vertices a and w: $\lfloor D/2\rfloor-2$ inner vertices of C_1, $\lceil D/2\rceil-1$ inner vertices of C_2, and the vertices c and v. In particular, at most $D-1$ additional colors might have been forbidden for b. Since b had at least D available colors after coloring the vertices a and w, there is still an available color for b.

Case 2a. *C_1 is a diamond, C_2 is not, and there is at least one path of length two in P_4.* In this case, $S+S_3 \leq \lfloor D/2\rfloor$. In particular, $S_3 \leq \lfloor D/2\rfloor-1$ as $S \geq 1$. Further, we choose an arbitrary vertex d among the inner vertices of the paths of length 2 in P_4.

We proceed similarly to the previous case. We uncolor v, w, c, d, and all the inner vertices of C_1 and C_2. Next, we would like to choose colors for the vertices a and d such that there will remain at least $D+1$ colors available for b. We proceed as in Case 1: there are at least $D+2$ colors available of the vertex b (the only forbidden colors are those used to color the $\lceil D/2\rceil-2$ neighbors of w in P_3 and P_4, excluding the vertices c and d), at least $D+1$ colors are available for a (there are at most $S+S_3-1$ colors forbidden by the neighbors of the vertex u and one more color can be forbidden by the vertex u itself), and at least 3 colors are available for d (there are at most $D-1$ colors forbidden by the neighbors of the vertex x, at most S_3-1 colors are forbidden by the internal vertices of the paths in P_3, and one more color can be forbidden by the vertex x). We follow the discussion in Case 1 and conclude that either we can color a and d with the same color, or, as $(D+1)+3 > D+2$, there exists a color β that is not available for b, but is available for one of the vertices a and d.

Vertex w is colored next—there are at most $D-1+\lfloor D/2\rfloor-2+3 = \lfloor 3D/2\rfloor$ colors forbidden for it: at most $D-1$ colors are forbidden by the neighbors of the vertex x (if vx is an edge) or the internal vertices of the paths in P_4, at most $\lfloor D/2\rfloor-2$ colors are forbidden the remaining neighbors of the vertex w, and three more colors may be forbidden by the vertices a, u, and x.

Then, we continue exactly as in Case 1: we start with c and v, continue with the rest of the inner vertices of C_1, and the inner vertices of C_2 except for b. In particular, we color $3+\lceil D/2\rceil-1+\lfloor D/2\rfloor-2=D$ vertices (including the vertex w discussed above). The only vertex without a color is the vertex b, and because it had at least $D+1$ available colors after coloring the vertices a and d, there must be at least one color that is still available for b.

Case 2b. *C_1 is a diamond, C_2 is not, and there is no path of length two in P_4 (i.e., P_4 consists of just a single path of length one).* As in the previous case, we observe that $S_3 \leq \lfloor D/2\rfloor - 1$.

In this case, the coloring c' can directly be extended to the vertex b. Let us count the forbidden colors for b: there at most $D-1$ colors forbidden by the neighbors of v, at most two more colors are forbidden by the vertices v and x, and at most $\lfloor D/2\rfloor$ colors are forbidden by the inner vertices of the paths of length two connecting u and w. In particular, there is at least one color available for b.

■

6.3 Final Step

Lemmas 6.2–6.11 exclude the existence of a D-minimal graph for $D \geq 4$ (see the discussion before Lemma 6.10). In order to complete the proof of Theorem 6.1, it is necessary to consider K_4-minor free graphs with maximum degree two and three. Such graphs are considered in the next two propositions.

Proposition 6.12 *The list chromatic number of the square of a graph of maximum degree 2 is at most 5.*

Proof: The statement follows easily from the fact that if G has maximum degree 2, then G^2 has maximum degree at most 4.

■

Proposition 6.13 *The list chromatic number of the square of a K_4-minor free graph of maximum degree 3 is at most 6.*

Proof: Fix a vertex-minimal K_4-minor free graph G with maximum degree three for that there exists a list-assignment L giving each vertex a list of 6 colors such that G^2 cannot be properly colored.

By considering the last level of an SP-decomposition tree of a final block of G, we obtain that G contains a vertex of degree one, two adjacent vertices of degree two or a crystal. If G contains a vertex v of degree one or a vertex v of degree two adjacent to another vertex of degree two, contract an edge incident with v and find a proper list-coloring c of the square of the resulting graph. The vertices of G preserve their colors and the vertex v can be assigned a color from its list since there are at most $3+2=5 \leq 6$ colors forbidden for the vertex v.

If G contains a crystal C of size $S \geq 2$, remove an arbitrarily chosen inner vertex w of the crystal C and find a list-coloring c of the square of $G \setminus w$. Let us calculate the number of colors forbidden for w: there are at most $S-1$ colors forbidden by the inner vertices of C, at most $2 \cdot (3-S)$ colors are forbidden by the neighbors of the poles of C that are outside the crystal, and two more colors may be forbidden by the poles. In particular, there are $7-S \leq 5$ forbidden colors, therefore c can be extended to the entire graph G.

■

Combining Lemmas 6.2–6.11 and Propositions 6.12 and 6.13, we obtain the proof of Theorem 6.1.

Part III

Extending Colorings of Cylinder Graphs

Chapter 7

Introduction

Colorings of graphs on surfaces has attracted attention of researchers for very long time. A classical result of Heawood [41] asserts that the chromatic number of a graph embedded on a surface of Euler genus g, $g \geq 1$, is bounded by the Heawood number $H(g) = \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$. The case $g = 0$ was proved many years later [8, 73] and became known as the Four Color Theorem. Further, Dirac's Map Color Theorem [21, 22] asserts that a graph G embedded on a surface of Euler genus $g \neq 0, 2$ is $(H(g) - 1)$-colorable unless G contains a complete graph of order $H(g)$ as a subgraph. This characterization of not-$(H(g) - 1)$-colorable graphs leads to study of color-critical graphs embeddable on surfaces. A graph G is *k-color-critical* if G cannot be colored with $(k - 1)$ colors, but every proper subgraph of G can be.

Fisk [29] has proved that for any surface different from the plane, the list of 5-color-critical graphs is infinite. On the other hand, a simple application of Euler's formula yields that there are only finitely many k-critical graphs, $k \geq 8$, that can be embedded on a fixed surface. The number of 7-critical graphs that can be embedded on a fixed surface is also finite by classical results of Gallai [33, 34] as pointed out by Thomassen in [79]. Finally, Thomassen [80] completed the results by showing that the number of 6-color-critical subgraphs is finite for any fixed surface.

Unfortunately, obtaining the actual list of 6-color-critical graphs for a given surface seems to be quite challenging. The only surfaces for which the precise lists are known are the the plane (with no 6-color-critical graphs), the projective plane (with K_6 being the only graph), the torus [79], and the Klein bottle [19, 52]. Let us state the last two results as separate theorems.

Theorem 7.1 (Thomassen [80]) *The only non-isomorphic 6-color-critical graphs that can be embedded on the torus are the four graphs in Figure 7.1.*

Theorem 7.2 (Chenette et al. [19] and Kawarabayashi et al. [52]) *The*

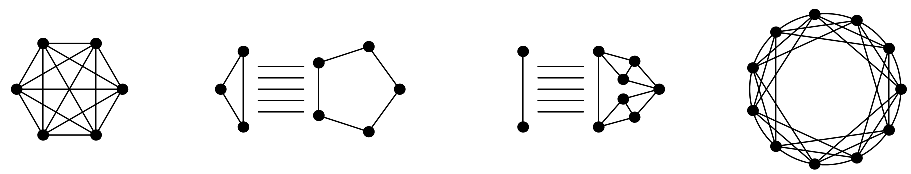

Figure 7.1: The list of 6-color-critical graphs on the torus. The straight edges between two parts of a graph represent that the graph is obtained as the join of the two parts.

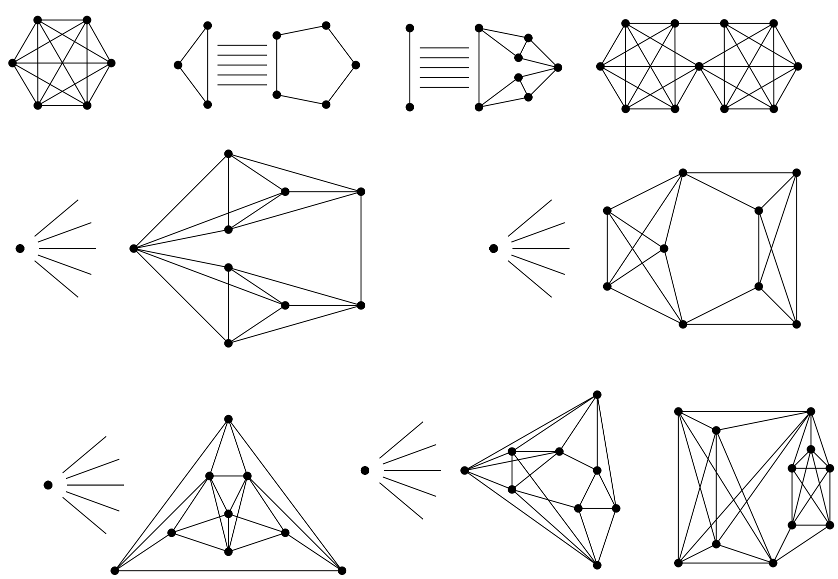

Figure 7.2: The list of 6-color-critical graphs on the Klein bottle. The straight edges between two parts of a graph represent that the graph is obtained as the join of the two parts.

only non-isomorphic 6*-color-critical graphs that can be embedded on the torus are the nine graphs in Figure 7.2.*

A coloring problem related to and motivated by the study of colorings of graphs embedded on surfaces is extending precolorings of embedded graphs [4–6]. A *cylinder graph* is a plane graph G with two distinguished faces f_1 and f_2 bounded by cycles C_1 and C_2. The two cycles are called the *inner cycle* and the *outer cycle*. G can be drawn in the plane in such a way that f_2 is the outer face, and hence f_1 is called the *inner face*. The union of the inner and the outer cycle is called the *boundary* of the cylinder graph. The rest of the graph (i.e., $G \setminus (C_1 \cup C_2)$) is the *interior* of the graph. Similarly, any vertex in the interior of G is an *internal vertex* and any edge that belongs neither to C_1 or C_2 is an *internal edge*. A (proper) subgraph G' of a cylinder graph G is said to be *(proper) cylinder subgraph* if it preserves the boundary. Note that the roles of the inner and outer cycles can freely be interchanged.

A *cylinder triangulation* is a cylinder graph whose every face except for the inner and outer faces is a triangle. In the next chapter, we solve a problem of Thomassen on extending 5-colorings of cylinder triangulations [80, Problem 4]. In particular, we characterize the cylinder triangulations with both inner and outer cycles being triangles such that there is a precoloring of the boundary that cannot be extended to a 5-coloring of the entire graph.

Chapter 8

Solution of a Problem of Thomassen

In [80], Thomassen posed the following problem:

Problem 8.1 [80, Problem 4] *Let G be a cylinder triangulation with precolored inner triangle C_1 and precolored outer triangle C_2. Suppose that the coloring $C_1 \cup C_2$ can be extended to a 5-coloring of every subgraph of G with at most 14 vertices. Can it then be extended to G?*

In the same paper, Thomassen has shown that 14 is the best possible value. One of the examples showing that the bound cannot be improved can be found in Figure 8.1. In this chapter, we answer Thomassen's question in the affirmative, i.e., we show that the value 14 in Problem 8.1 is sufficient. Our proof of this result is computer-assisted.

8.1 Critical Cylinder Graphs

In this section, we introduce the notion of critical cylinder graphs which is crucial to our proof.

If G is a cylinder graph with the inner cycle C_1 and the outer cycle C_2, a precoloring $\varphi : V(C_1) \cup V(C_2) \to \{1, \ldots, 5\}$ of its boundary with five colors is said to be *bad* if it induces a proper coloring of both C_1 and C_2, but it cannot be extended to the entire graph G. A cylinder graph G is said to be *φ-bad* if both its inner and outer cycles are triangles and φ is a bad precoloring of its boundary. A graph G is said to be *φ-critical* if G is φ-bad and its every proper cylinder subgraph G' is not φ-bad. Note that a φ-critical cylinder graph may contain a proper cylinder subgraph that is φ'-critical; however, precolorings φ and φ' must be different in such a case. Finally, G is said to be *critical* if it is φ-critical for some precoloring φ.

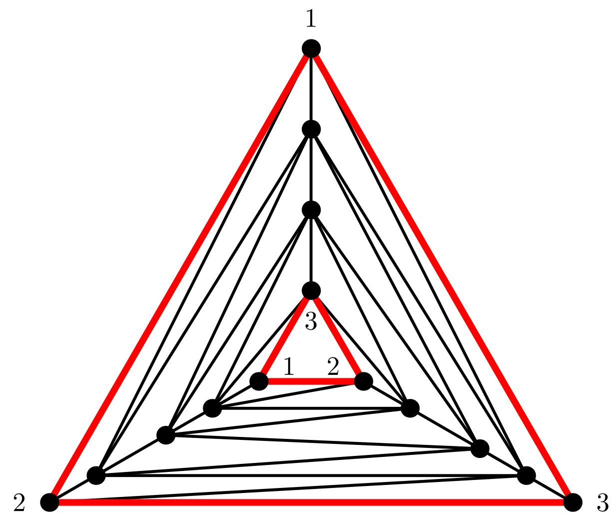

Figure 8.1: A graph showing that the value 14 in Problem 8.1 cannot be replaced by 13. The inner and outer cycles are drawn with thick red lines and the numbers near the vertices of the boundary represent a precoloring that can be extended to every cylinder subgraph with at most 13 vertices but cannot be extended to the whole graph.

In particular, Problem 8.1 is equivalent to stating that every critical cylinder graph has at most 14 vertices. The following is a straightforward observation on the structure of critical cylinder graphs.

Proposition 8.2 *Every internal vertex of a critical cylinder graph has degree at least* 5*.*

Proof: Let G be a φ-critical cylinder graph and v an internal vertex of G violating the statement of the proposition. By the criticality, φ can be extended to a proper 5-coloring c of $G \setminus v$. Consequently, as the degree of v is at most four, there is a color not used on its neighbors. Therefore, c (and thus φ) can be extended to a 5-coloring of the entire graph G, a contradiction.

■

Clearly, a cylinder graph consisting of just a single edge connecting the inner and outer cycle is critical (consider a precoloring assigning both endvertices of the internal edge the same color). We call such a cylinder graph a *trivial critical cylinder* graph. There are three such graphs (see Figure 8.2), depending on the size of the intersection of the inner and outer cycles. The remaining critical cylinder graphs are referred to as *non-trivial*. The trivial critical cylinder graphs are the only ones that contain an edge connecting two vertices on the boundary.

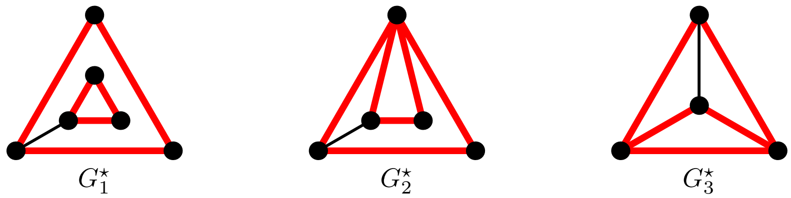

Figure 8.2: The list of all trivial critical cylinder graphs. The inner and outer cycles are drawn with thick red lines.

Proposition 8.3 *Every non-trivial critical cylinder graph does not contain an edge connecting the inner and outer cycles.*

Proof: Consider a φ-critical graph G containing an edge uv connecting the inner cycle with the outer one and at least one other internal edge e'. If $\varphi(u) \neq \varphi(v)$, then φ must be a bad precoloring for $G \setminus uv$ as well, hence it is not φ-critical. On the other hand, if $\varphi(u) = \varphi(v)$, $G \setminus e'$ is φ-bad because of the edge uv, contradicting φ-criticality of G.

■

The following observation shows that if there is an internal vertex adjacent to at least two vertices on the boundary of a φ-critical cylinder graph, then no two neighbors on the boundary may get the same color in φ.

Proposition 8.4 *If v is an internal vertex of a φ-critical cylinder graph G, then all its neighbors on the boundary of G must be assigned different colors by φ.*

Proof: Let G is a φ-critical graph, u its internal vertex, and v and w two neighbors of u on the boundary of G such that $\varphi(v) = \varphi(w)$. Consider a graph $G' = G \setminus uv$. By the criticality, G' is not φ-bad, therefore there exists a 5-coloring c of G'. However, this coloring is consistent even with edge uv, so c is a proper 5-coloring extension of φ to G.

■

In our proof, we use the notion of k-minimal graphs from [52]. A plane graph G with an outer face bounded by a cycle C of length k is said to be k-minimal if for every edge $e \in E(G) \setminus E(C)$ there exists a proper precoloring φ_e of C with five colors that cannot be extended to a proper 5-coloring of G and can be extended to a proper 5-coloring of $G \setminus e$. Precolorings φ_e may be different for different choices of e. Also note that a cycle C_k of length k is vacuously k-minimal. C_k is said to be the *trivial* k-minimal graph; the other k-minimal graphs are *non-trivial*. An algorithm for computing k-minimal graphs was given in [52] and the list of k-minimal graphs for $k \leq 10$ can be found at `http://kam.mff.cuni.cz/~bernard/klein/`. The list of k-minimal graphs for

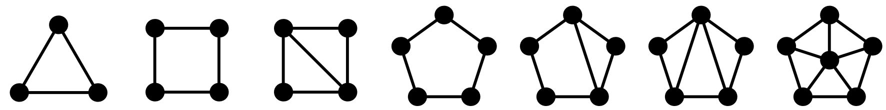

Figure 8.3: The list of k-minimal graphs for $k \in \{3,4,5\}$.

$k \in \{3,4,5\}$ is given in Figure 8.3 and the list of 6-critical graphs can be found in Figure 8.6.

The connection between critical cylinder graphs and k-minimal graphs is explored in the following lemma.

Lemma 8.5 *Let G be a critical cylinder graph and let C be a closed walk of length k that does not cross itself and whose interior does not contain the inner cycle of G. Then, C and its interior can be obtained from a k-minimal graph H by identifying some of the vertices on its outer cycle of H.*

Proof: Let G be a φ-critical graph and let C be a closed walk as described in the statement of the lemma. Let $H^\star$ be the union of C and its interior. We construct H from $H^\star$ by expanding the walk C to a cycle of the same length. If C itself is a cycle, set $H = H^\star$. Otherwise, there are several vertices v_i which occur ℓ_i times ($\ell_i \geq 2$) on C. We replace ℓ_i occurrences of each v_i by ℓ_i new vertices in such a way that C becomes a cycle and the face structure of the interior of C is preserved.

Now, we have to show that H is k-minimal. Let e be an edge of H not contained in C. Since G is φ-critical, $G \setminus e$ is not φ-bad and therefore there is a 5-coloring c_e extending φ to $G \setminus e$ for every edge e in the interior of $H^\star$. The coloring c_e induces a precoloring φ'_e of C. Clearly, φ'_e can be extended to $H \setminus e$ (the coloring c_e yields the extension), but it cannot be extended to a 5-coloring of H—if it could, combining such an extension and c_e outside $H^\star$ would give an extension of φ to a 5-coloring of G.

■

8.2 Touching Triangles

In this section, we analyze critical cylinder graphs whose inner and outer cycles share a common vertex or an edge.

Lemma 8.6 *Every non-trivial critical cylinder graph whose inner and outer cycles share a common vertex or an edge is isomorphic to one of the three graphs $G_1, \ldots, G_3$ from Figure 8.4.*

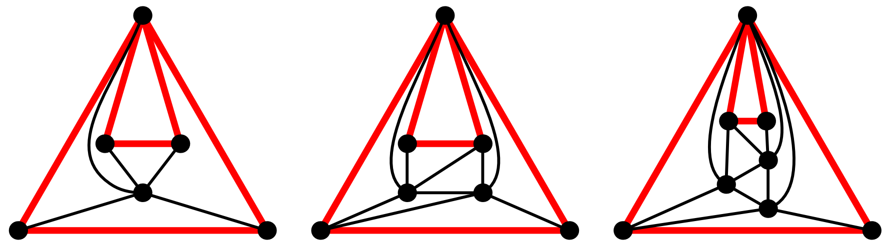

Figure 8.4: The list of all non-isomorphic non-trivial critical graphs whose inner and outer cycles intersect. The inner and outer cycles are drawn with thick red lines.

Figure 8.5: The proof of Lemma 8.6. The 6- and 4-minimal graphs will be glued into the faces colored with gray.

Proof: Our main tool is Lemma 8.5. We start with the case that triangles share a single vertex. Let G be one such critical cylinder graph whose inner and outer cycles intersect at a vertex a. Further, let b and c be the remaining vertices of the inner triangle and d and e the remaining vertices of the outer triangle in such a way that $C = abcade$ is a closed walk which does not cross itself (see Figure 8.5).

The interior of the closed walk C, by Lemma 8.5, forms a 6-minimal graph. This simple but important observation gives us a way to generate all critical cylinder graphs whose cycles share a common vertex. We simply try to "glue" 6-minimal graphs into the face bounded by C and check whether the resulting cylinder graph is critical.

We now consider all 6-minimal graphs and identify the vertices of the outer face with the vertices of the facial walk of C; we consider all rotations of the k-minimal graphs and their mirror images. The list of all 6-minimal graphs can be found in Figure 8.6. We claim that critical graphs that can be obtained in this way are precisely the graphs $G_1, \ldots, G_3$ from Figure 8.4 and the trivial graph $G_2^\star$ from Figure 8.2.

We first verify that each G_i is critical. The bad precolorings φ_i for G_i, $i = 1, \ldots, 3$, such that G_i is φ_i-critical, are given in Figure 8.7. Each of the graphs G_i, $1 \le i \le 3$, contains i vertices in the interior. The vertices form a clique and each of them is connected to $6 - i$ vertices on the boundary. In particular, each vertex can be colored with one of $i - 1$ available colors $7 - i, \ldots, 5$, which are not

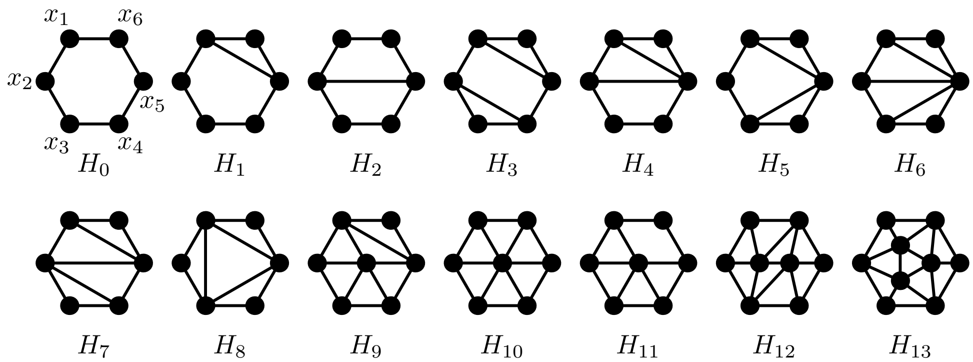

Figure 8.6: The list of all 6-minimal graphs. The vertices on the outer face of each of the graphs are labeled $x_1, \ldots, x_6$ in the same way as shown on H_0.

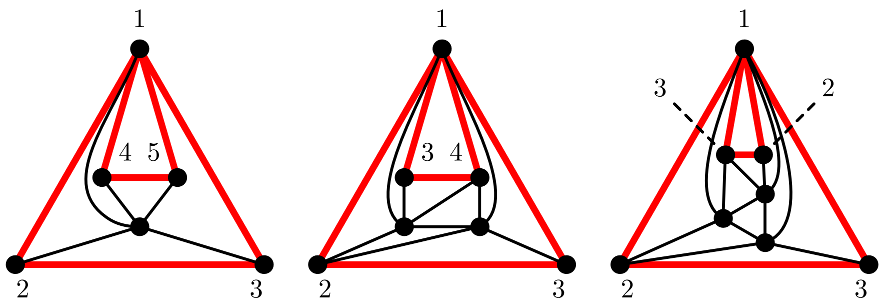

Figure 8.7: The bad precolorings for the graphs in Lemma 8.6.

used on its neighbors. Hence, the i internal vertices cannot be colored as only $i-1$ colors are available. If any edge e in the interior of G is removed, then either one vertex can be colored with a color $c \leq 6-i$, or the internal vertices do not form a clique anymore. In both cases, the internal vertices can be colored from their lists, producing a proper extension of φ_i.

Finally, we show that the described process cannot produce any other non-trivial critical graphs. First, observe that if the graph after gluing contains parallel edges or loops, it cannot be critical. Let us now examine gluing of each of the graphs $H_0, \ldots, H_{13}$ separately. The trivial 6-minimal graph H_0 does not add any new vertices or edges and therefore produces no bad cylinder graph. H_1 and H_2 contain a single chord, i.e., the result of gluing will be isomorphic to graph $G_2^\star$ Figure 8.2. The graphs $H_3, \ldots, H_9$ contain a chord and other vertices/edges; after gluing, the chord becomes an edge connecting the two triangles. However, the resulting graph is not critical by Proposition 8.3. We are now left with the graphs $H_{10}, \ldots, H_{13}$. By gluing them into the face bounded by C, we identify, for some i, the vertex x_i with the vertex x_{i+3}; these vertices will be glued to the two occurrences of vertex a in the facial walk. Not all choices of i are feasible since they may produce parallel edges. In particular, no i is feasible for H_{10}, the only feasible choice for H_{11} is $i = 3$ (producing a G_1), feasible choices for H_{12} are $i = 2, 3$ (producing a G_2) and all values of i are feasible for H_{13} (producing a G_3).

An analogous argument can be used to list all critical cylinder graphs where the inner and outer cycles share an edge. As the only two 4-minimal graphs are C_4 and C_4 with a chord, the only such critical graph is $G_3^\star$ from Figure 8.2.

■

The above lemma settles the case when the inner and outer cycles have a common vertex. Therefore, in the next two sections, we assume that the inner and outer cycles are disjoint.

8.3 Precolorings with Three or Four Colors

As noted in [80], the characterization of 6-color-critical graphs on the torus yields a characterization of φ-critical graphs for the case when the inner and outer cycles are precolored $1, 2,$ and 3 in the same clockwise order by φ. Similarly, the characterization of 6-color-critical graphs on the Klein bottle yields a characterization of φ-critical graphs for the case when the order of $1, 2,$ and 3 on the cycles differ in the orientation.

Indeed, if G is a non-trivial φ-critical cylinder graph with the inner cycle abc and the outer cycle def such that $\varphi(a) = \varphi(d) = 1, \varphi(b) = \varphi(e) = 2, \varphi(c) = \varphi(f) = 3$, we can identify the vertex a with d, b with e, and c with f (and deleting one edge from each pair of parallel edges ab, bc and ac) to obtain a graph G'. Notice that G' contains neither loops nor parallel edges by Propositions 8.3 and 8.4. The graph G' is not 5-colorable: if it were, the vertices $a, b,$ and c would get, by symmetry, the colors $1, 2,$ and 3 and the colors of the remaining vertices would induce a 5-coloring of G that is an extension of φ. Moreover, removal of any edge e of G' except for those in the triangle abc results in a 5-colorable graph (extension of φ to $G \setminus e$ yields a 5-coloring of $G' \setminus e$). In other words, there is a 6-color-critical graph $G^\star$ that can be obtained form G' by deletion of some (or none) of the edges $ab, bc,$ and ac. If the order of the vertices a, b, c induces the same orientation of the inner cycle as the order d, e, f on the outer cycle, the resulting graph G' can be embedded on the torus, and if the orientations differ, G' can be embedded on the Klein bottle. As $G^\star$ is a subgraph of G' and the list of 6-color-critical graphs on both torus and the Klein bottle are known, the list of non-trivial critical cylinder graphs with a bad precoloring using the colors $1, 2,$ and 3 can be obtained by reversing the process of identifying the vertices of the inner and outer cycles.

The method for dealing with bad precolorings with exactly four colors is similar. Consider a non-trivial φ-critical cylinder graph G with the inner cycle

abc, the outer cycle def and $\varphi(a) = \varphi(d) = 1, \varphi(b) = \varphi(e) = 2, \varphi(c) = 3$, and $\varphi(f) = 4$. In this case, we construct graph G' by identifying a with d, b with e, adding an edge between c and f, and deleting one of the two parallel edges ab. Again, G' cannot be colored with 5 colors, but for every edge e' of G' except for those in the clique $abcf$, $G' \setminus e'$ is 5-colorable. We conclude that a 6-color-critical graph $G^\star$ can be obtained from G' by deleting some of the edges in the clique $abcf$ and $G^\star$ is embeddable on the torus or the Klein bottle, depending on the orientation of the inner and outer face cycles induced by φ.

The above observations imply the main result of this section.

Lemma 8.7 *If G is a φ-critical cylinder graph and φ uses at most four different colors, then G has at most 14 vertices.*

Proof: The trivial critical cylinder graph containing a single edge connecting disjoint inner and outer cycles contains six vertices (see Figure 8.2). The remaining φ-critical cylinder graphs with φ using at most four colors can be reduced, by identifying vertices and deleting edges, to 6-color-critical graphs embeddable on the torus or the Klein bottle as argued before the lemma. If φ uses three colors, then three pairs of vertices are identified; if φ uses four colors, two pairs are identified. In particular, as the maximum number of vertices of a 6-color-critical graph embeddable on the torus is 11 (by Theorem 7.1) and it is also 11 for the Klein bottle (by Theorem 7.2), the maximum number of vertices of a φ-critical cylinder graphs with φ using at most four colors is at most 14.

■

We have also written a program that sequentially examines the 6-color-critical graphs embeddable on the torus and the Klein bottle, and reverses the process of identifying vertices to obtain the list of critical cylinder graphs with a bad precoloring using at most four colors. The source code and the results of the program (i.e., the list of the graphs) are available both on the CD accompanying the thesis and at the web page `http://kam.mff.cuni.cz/~bim/cylinder`.

8.4 Precolorings with Five Colors

We are now approaching the main part of the proof. It remains to analyze precolorings with five colors. By symmetry, we may assume that the inner cycle is colored with colors $1, 2$, and 3 and the outer cycle is colored with $1, 4$, and 5.

In this the section, we use the fact that a cylinder graph is planar and therefore 4-colorable [8, 73] to obtain a strategy for extending a precoloring to a proper coloring of the entire graph. The coloring tables, which we now define, relate the 4-coloring, the precoloring of the boundary and the sought 5-coloring of the entire graph.

1	1	1	1	1
2	4	4	4	4
3	3	3	5	5
5	5	2	2	2

(1)

1	1	1	1	1
2	4	4	4	4
3	3	3	3	3
5	5	5	5	5

(2)

1	1	1	1	1
2	2	2	2	2
3	3	4	4	4
5	5	5	5	5

(3)

1	1	1	4	4
2	2	2	2	1
3	5	5	5	5
4	4	3	3	3

(4)

1	1	4	4	4
2	2	2	1	1
3	3	3	3	3
5	5	5	5	5

(5)

1	1	1	4	4
2	5	5	5	5
3	3	3	3	1
4	4	2	2	2

(6)

1	1	4	4	4
2	2	2	2	5
3	3	3	3	3
5	5	5	1	1

(7)

1	1	4	4	4
2	2	2	2	2
3	3	3	1	1
5	5	5	5	5

(8)

1	1	4	4	4
2	2	2	2	2
3	3	3	3	5
5	5	5	1	1

(9)

1	1	1	3	3
2	2	2	2	1
3	3	4	4	4
5	5	5	5	5

(10)

1	1	2	2	2
2	4	4	4	4
3	3	3	1	1
5	5	5	5	5

(11)

1	1	2	2	2
2	4	4	4	4
3	3	3	3	5
5	5	5	1	1

(12)

Figure 8.8: The proof of Lemma 8.8 (a) and (b). The non-blank entries of the original partial coloring tables are colored with gray.

A *partial coloring table* with m columns is a matrix $A = (a_{i,j})_{4\times m}$ with four rows and m columns; each of the entries of the table is either an integer in the range from 1 to 5 or blank (denoted by a star, $\star$). In addition, any integer may occur at most once in each column and for any two adjacent columns, say j and $j+1$, and any i, if $a_{i,j}$ is not a blank, its value may occur in the $(j+1)$-th column only as $a_{i,j+1}$, and vice versa for the columns $j+1$ and j. A *coloring table* is a partial coloring table with no blanks. We say that a partial coloring table A can be *completed* if there is a way to assign integers 1 to 5 to the blank entries of A in such a way that the resulting matrix is a coloring table.

Lemma 8.8 *Let A be partial coloring table with 5 columns satisfying one of the following:*

(a) The first column is $(1, 2, 3, \star)$.

(b) The first column is $(1, 2, \star, \star)$ and the second column is $(\star, \star, 3, \star)$.

(c) The first column is $(\star, 2, 3, \star)$ and the second column is $(1, \star, \star, \star)$.

If the fifth column contains integers 1, 4, 5 and a single blank and all the remaining entries of A are blank, then A can be completed.

1	1	1	1	1
2	4	4	4	4
3	3	3	5	5
5	5	2	2	2

(1)

1	1	1	1	1
2	4	4	4	4
3	3	3	3	3
5	5	5	5	5

(2)

1	1	1	1	1
2	2	2	2	2
3	4	4	4	4
5	5	5	5	5

(3)

1	1	1	4	4
2	2	2	2	1
3	5	5	5	5
4	4	3	3	3

(4)

1	1	4	4	4
2	2	2	1	1
3	3	3	3	3
5	5	5	5	5

(5)

1	1	1	4	4
2	5	5	5	5
3	3	3	3	1
4	4	2	2	2

(6)

1	1	4	4	4
2	2	2	2	5
3	3	3	3	3
5	5	5	1	1

(7)

1	1	4	4	4
2	2	2	2	2
3	3	3	1	1
5	5	5	5	5

(8)

1	1	4	4	4
2	2	2	2	2
3	3	3	3	5
5	5	5	1	1

(9)

1	1	3	3	3
2	2	2	1	1
3	4	4	4	4
5	5	5	5	5

(10)

1	1	2	2	2
2	4	4	4	4
3	3	3	1	1
5	5	5	5	5

(11)

1	1	2	2	2
2	4	4	4	4
3	3	3	3	5
5	5	5	1	1

(12)

Figure 8.9: The proof of Lemma 8.8 (c). The non-blank entries of the original partial coloring tables are colored with gray.

Proof: We prove the lemma by exhibiting the completed tables for all combinations of values in the fifth column. There are $4! = 24$ possibilities, however, because of the symmetry between the values 4 and 5, we may assume that the value 4 appears in the fifth column *before* the value 5, reducing the number of cases to 12. In Figure 8.8, we present the completed tables for table with the first column being $(1, 2, 3, \star)$ and the second one $(\star, \star, 3, \star)$, thus proving the parts (a) and (b) of the lemma at the same time. The completed tables proving the part (c) are given in Figure 8.9.

■

The lemma that we have just proved allows us to prove that in a φ-critical graph, the length of the shortest path connecting a vertex of the inner cycle and a vertex of the outer cycle is at most three.

Lemma 8.9 *If G is a φ-critical cylinder graph and φ uses all five colors, then the distance between the inner and outer cycles of G is at most three.*

Proof: Let G be a φ-critical cylinder graph with the distance between the inner and outer cycles at least four and φ uses all five colors. Our aim is to find a proper 5-coloring π of G extending φ, which would contradict φ-criticality of G. Without loss of generality, we may assume that the inner cycle abc is

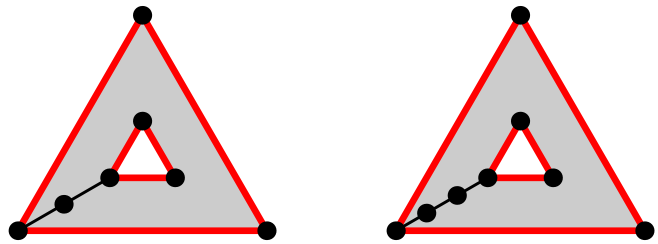

Figure 8.10: The graphs where the inner and outer cycles are connected by a path of length two or three. Gluing suitable 10- and 12-minimal graphs into the gray faces will produce all the φ-critical graphs with φ using all five colors and the distance between the inner and outer cycles being at least two.

precolored with $1, 2$, and 3 (in this order) and the outer cycle def by $1, 4$, and 5. Let ψ be a 4-coloring of G such that $\psi(a) = 1, \psi(b) = 2, \psi(c) = 3$. Let A be a partial coloring table with 5 columns, the first column being $(1, 2, 3, \star)$ and $a_{\psi(d),5} = \varphi(d), a_{\psi(e),5} = \varphi(e), a_{\psi(f),5} = \varphi(f)$. The rows of the table correspond to the colors given by ψ, the entries correspond to the precoloring φ, and the first and last column correspond to the inner and outer cycle of G. Later, the numbers in the table will be used to obtain the 5-coloring π.

By Lemma 8.8 (a), the table A can be completed to a coloring table $B = (b_{i,j})_{4\times 5}$. We now construct a 5-coloring π of G that extends precoloring φ. First, we assign each vertex v a *column index* $\gamma(v)$ based on the distances of v from the inner and outer cycles. A column index for the vertices of the inner cycle is 1 and the column index of their neighbors outside the cycle is 2. Similarly, the column index of the vertices of the outer cycle is 5 and the vertices at distance one from the outer cycle get 4. The column index of the remaining vertices is 3. Since the distance between the inner and outer cycle is at least four, if uv is an edge, then $|\gamma(u) - \gamma(v)| \leq 1$.

The color of a vertex v in the coloring π is the value in the $\psi(v)$-th row and $\gamma(v)$-th column, i.e., $\pi(v) = b_{\psi(v),\gamma(v)}$. By the construction of the partial coloring table A, π agrees with the precoloring φ on vertices a, b, c, d, e, and f.

We next show that π is a proper coloring. Consider an edge uv such that $\pi(u) = \pi(v)$. By symmetry, either $\gamma(u) = \gamma(v)$ or $\gamma(u) = \gamma(v)+1$. If $\gamma(u) = \gamma(v)$, it must hold that $\psi(u) = \psi(v)$ (since each number occurs at most once in a column). If $\gamma(u) = \gamma(v) + 1$, then again $\pi(u) = \pi(v)$ implies $\psi(u) = \psi(v)$ because of the requirement for the neighboring columns in the table. However, $\psi(u) \neq \psi(v)$ since ψ is a proper coloring. Hence, there is no monochromatic edge uv, and therefore, π is a proper 5-coloring of G extending φ.

■

At this point, we know that the inner and outer cycles of a φ-critical graph are connected by a path of length at most three. The case that is length is zero

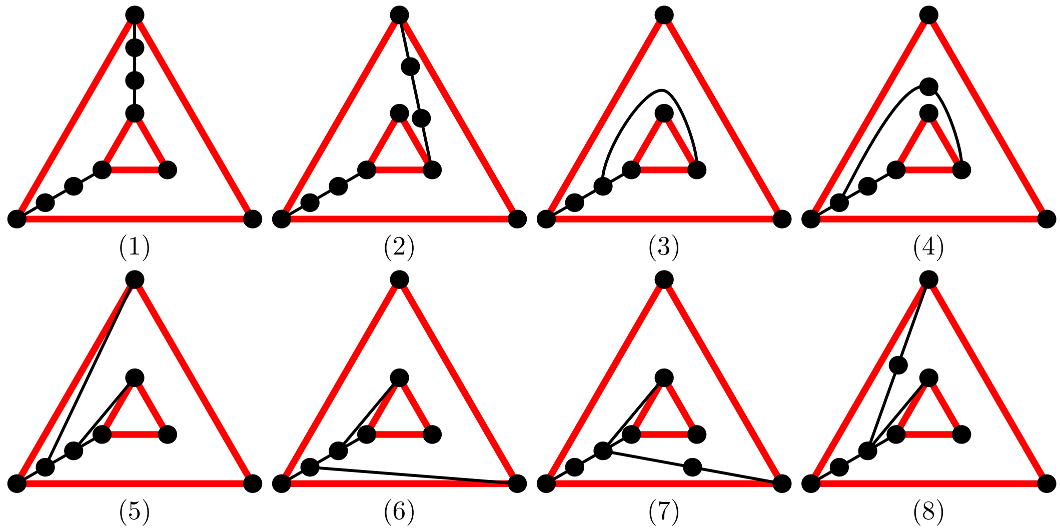

Figure 8.11: The list of subgraphs (including their drawings) that can be used to obtain the critical cylinder graphs with a bad precoloring using five colors and distance between the inner and outer cycle exactly three.

(i.e., the cycles share a vertex) is covered by Lemma 8.6. The only critical graph with the distance between the inner and outer cycle one is the trivial critical graph $G_1^\star$ in Figure 8.2 by Proposition 8.3.

It remains to analyze the cases where the distance of the two cycles is two or three. The situation depicted in Figure 8.10 suggests that we could use the same approach as in Lemma 8.6 which would consist of gluing 10- and 12-minimal graphs inside the gray faces. Unfortunately, listing all 12-minimal graphs would take years to complete (at least using the algorithm from [52] and our current hardware). The following lemma helps us to split the large face into smaller ones, thus allow us to use gluing in this case as well.

Lemma 8.10 *Let G be a φ-critical cylinder graph such that the precoloring φ uses all five colors. If the distance between the inner and outer cycles of G is three, than G contains one of the graphs in Figure 8.11 as a subgraph.*

In the proof of Lemma 8.10, the following auxiliary lemma is needed.

Lemma 8.11 *If a cylinder graph G with an inner cycle C_1 and an outer cycle C_2 satisfies the assumptions of Lemma 8.10, one of the following cases applies:*

(a) all paths of length three connecting C_1 with C_2 share a single end-vertex in C_1, or

(b) all paths of length three connecting C_1 with C_2 share a single end-vertex in C_2, or

(c) G contains a subgraph isomorphic to one of the graphs in Figure 8.11.

Proof: Let G be a cylinder graph with inner cycles $C_1 = abc$ and outer cycle $C_2 = def$ such that the distance of C_1 and C_2 is three. Let $\mathcal{P}$ be the set of all paths of length three connecting C_1 and C_2. In the rest of the proof, we assume that neither (a) nor (b) applies, and infer that (c) does. We show this by examining the following three cases.

Case 1. *$\mathcal{P}$ contains two vertex-disjoint paths.* In this case, the two mentioned paths between C_1 and C_2 can be drawn in two non-isomorphic ways—(1) and (2) in Figure 8.11.

Case 2. *$\mathcal{P}$ contains two intersecting paths with distinct endvertices in both C_1 and C_2.* Without loss of generality, we may assume that $P_1 = auvd$ and $P_2 = bxye$ are the two paths. Let w be the common point of P_1 and P_2 closest to C_1 and z be the common point closest to C_2. In particular, either $w = z$ or wz is an edge. Let G' be subgraph of G containing only C_1, C_2, P_1, and P_2. First, let us examine the cycle $C = a \ldots w \ldots b$ where $a \ldots w$ is the subpath of P_1 from the vertex a to w and $w \ldots b$ is the subpath of P_2 from w to b. If C does not bound a face of G', then G' (and thus G) contains one of the graphs (3) and (4) from Figure 8.11 as a subgraph. Similarly, one of the graphs (3) and (4) with the inner and outer cycles swapped will be a subgraph of G if $C' = d \ldots z \ldots e$ is not a facial cycle of G'. We may now assume that both C and C' are facial cycles of G'. If wz is an edge, then G' is isomorphic to either (5) or (6) in Figure 8.11 and if $w = z$, then G' is isomorphic to either (7) or (8) in Figure 8.11. In all the cases the roles of the inner and outer cycles can be swapped.

Case 3. *Neither Cases 1 nor Case 2 apply.* In this case, $\mathcal{P}$ contains two paths P_1 and P_2 such that P_1 and P_2 share a common endvertex on C_1 but differ at endvertices on C_2. Moreover, $\mathcal{P}$ contains a path P_3 such that P_1 and P_3 share a common endvertex on C_2 and differ at endvertices on C_1. We may assume that $P_1 = au_1v_1d$, $P_2 = au_2v_2e$, and $P_3 = bu_3v_3d$. However, paths P_2 and P_3 witness that either Case 1 or Case 2 should have applied.

■

We are now ready to prove Lemma 8.10.

Proof of Lemma 8.10. Let G be a φ-critical cylinder graph with the distance between the inner and outer cycles being three, and a precoloring φ using all five colors. Further, let $C_1 = abc$ be the inner cycle and $C_2 = def$ be the outer one. We may assume that $\varphi(a) = 1, \varphi(b) = 2, \varphi(c) = 3$, and the colors assigned to d, e, and f by φ are $1, 4$, and 5 (in any order).

We use Lemma 8.11. In order to prove the statement of Lemma 8.10, it is enough to show that neither case (a) nor (b) in Lemma 8.11 applies. As the cases (a) and (b) are symmetric, it is enough to prove that (a) does not apply. We do so by showing that if (a) applies, then there is a proper 5-coloring π that extends the precoloring φ.

Because of the symmetry between vertices b and c, we may assume that the common endvertex of all paths of length three connecting C_1 and C_2 is either a or c. We use an approach similar to the one in Lemma 8.9. Let us start with the case that the common vertex is the vertex a. In this case, the column index γ of the vertices b and c will be 1, the column index of their neighbors (including the vertex a) will be 2, the column index of the vertices of C_2 will be 5, and their neighbors will get 4. The remaining vertices will be assigned the column index 3. Again, if uv is an edge, it holds that $|\gamma(u)-\gamma(v)| \leq 1$. Next, let ψ be a 4-coloring of G such that $\psi(a) = 1, \psi(b) = 2, \psi(c) = 3$. Using the coloring ψ, we construct a partial coloring table A with first column being $(\star, 2, 3, \star)$, the second column being $(1, \star, \star, \star)$, and $a_{\psi(d),5} = \varphi(d), a_{\psi(e),5} = \varphi(e), a_{\psi(f),5} = \varphi(f)$.

The construction of the coloring table for the case that the common vertex is c is similar, except that the roles of the vertices a and c are swapped. In particular, the vertices with the column index 1 are a and b, and the first two columns of the coloring table are $(1, 2, \star, \star)$ and $(\star, \star, 3, \star)$.

The rest of the proof is the same as that of Lemma 8.9: by Lemma 8.8, (b) or (c), the table A can be completed to a coloring table B. The coloring π of G is obtained by setting $\pi(v) = b_{\psi(v),\gamma(v)}$. The analysis that π is a proper coloring extending φ is the same as in Lemma 8.9. In particular, the fact that π agrees with φ on the boundary of G follows from the construction of the partial coloring table A.

■

Finally, we can find all φ-critical cylinder graphs with precoloring φ using all five colors.

Lemma 8.12 *If G is a φ-critical cylinder graph and the precoloring φ uses all five colors, then G has at most 13 vertices.*

Proof: The proof of this lemma is computer-assisted. Since trivial critical cylinder graphs have at most six vertices, we may focus on non-trivial graphs only. The program starts with a set S of initial graphs—those in Figure 8.11 and the first graph from Figure 8.10. For each of these graphs, all faces of size at least 4 are found. The program then sequentially tries to glue minimal graphs into each of the faces; all possible combinations of minimal graphs, rotations, and mirror images are considered. Each of the resulting graphs is checked for criticality, the critical graphs are printed to the output. Notice that there is no need to glue minimal graphs to faces of size three as the only 3-critical graph is C_3. The list of k-minimal graphs is taken from `http://kam.mff.cuni.cz/~bernard/klein/`.

The following pseudocode illustrates the algorithm.

```
S := { the nine initial graphs }
M_k := { the k-minimal graphs } for k <= 10
```

```
X := {}
forall G in S do
  F = { f_i : f_i is a face of G of size k_i >= 4 }
  forall possible choices of g_i from M_(k_i) for each i do
    forall possible rotations and mirrorings of graphs g_i do
      G' = G with g_i glued into face f_i forall i
      if G' contains no parallel edges and/or loops then
        if G' is critical and not yet in X then
          add G' to X
        endif
      endif
    endfor
  endfor
endfor
output X
```

The source code and output produced by the program just described can be found both at the web page `http://kam.mff.cuni.cz/~bim/cylinder` and on the CD accompanying the thesis. Lemmas 8.5, 8.9, and 8.10 imply that the program outputs the list of all non-trivial φ-critical cylinder graphs with φ using all five colors. The program has produced 96 non-isomorphic graphs, the largest of them has 13 vertices.

■

8.5 Conclusion

Combining Lemmas 8.6, 8.7, and 8.12, we obtain the main result of this chapter.

Theorem 8.13 *Every critical cylinder graph has at most* 14 *vertices.*

Proof: Let G be a φ-critical cylinder graph with the inner cycle C_1 and the outer cycle C_2. If $V(C_1) \cap V(C_2) \neq \emptyset$, Lemma 8.6 yields $|V(G)| \leq 8$. If $V(C_1) \cup V(C_2) = \emptyset$, either φ uses all five colors and Lemma 8.12 then yields $|V(G)| \leq 13$, or φ uses at most four colors and Lemma 8.7 yields $|V(G)| \leq 14$.

■

An immediate consequence of Theorem 8.13 is a positive answer to Problem 8.1.

Corollary 8.14 *Let G be a cylinder triangulation with precolored inner and outer triangles. If a precoloring of the boundary of G can be extended to a 5-coloring of every subgraph of G with at most 14 vertices, then it can be extended to the entire graph G.*

Proof: Let G be a cylinder triangulation with the inner cycle C_1 and the outer cycle C_2, both C_1 and C_2 being triangles. Further, let φ be a precoloring of the boundary of G that can be extended to a 5-coloring of every subgraph of G with at most 14 vertices, but not to G itself. As G is φ-bad, there exists a φ-critical cylinder subgraph G' of G. Theorem 8.13 yields that $|V(G')| \leq 14$, however this contradicts the fact that φ can be extended to any subgraph of G with at most 14 vertices.

■

Part IV

Short Cycle Covers

Chapter 9

Introduction

Cycle covers of graphs are closely related to several deep and open problems on graph colorings. A *cycle* in a graph is a subgraph with all degrees even. A *cycle cover* is a collection of cycles such that each edge is contained in at least one of the cycles; we also say that each edge is *covered*. The Cycle Double Cover Conjecture of Seymour [76] and Szekeres [77] asserts that every bridgeless graph G has a collection of cycles containing each edge of G exactly twice which is called a *cycle double cover*. In fact, it was conjectured by Celmins [16] and Preissmann [71] that every graph has such a collection of five cycles.

The Cycle Double Cover Conjecture is known to be implied by several other conjectures. One of the conjectures implying the Cycle Double Cover Conjecture is the Shortest Cycle Cover Conjecture of Alon and Tarsi [7] asserting that every bridgeless graph with m edges has a cycle cover of total length at most $7m/5$. Recall that the *length* of a cycle is the number of edges contained in it and the length of the cycle cover is the sum of the lengths of its cycles. The reduction of the Cycle Double Cover Conjecture to the Shortest Cycle Cover Conjecture can be found in the paper of Jamshy and Tarsi [48].

The best known general result on short cycle covers is due to Alon and Tarsi [7] and Bermond, Jackson and Jaeger [10]: every bridgeless graph with m edges has a cycle cover of total length at most $5m/3 \approx 1.667m$. As it is the case with most conjectures in this area, there are numerous results on short cycle covers for special classes of graphs, e.g., graphs with no short cycles, well connected graphs or graphs admitting a nowhere-zero 4-/5-flow, see e.g. [25, 26, 45, 46, 49, 72]. The reader is referred to the monograph of Zhang [86] for further exposition of such results where an entire chapter is devoted to results on the Shortest Cycle Cover Conjecture.

The least restrictive of such refinements of the general bound of Alon and Tarsi [7] and Bermond, Jackson and Jaeger [10] is the result of Fan [25] that every *cubic* bridgeless with m edges has a cycle cover of total length at most $44m/27 \approx 1.630m$.

In this part of the thesis, we strengthen the result of Fan [25] in two ways.

First, we improve the bound for bridgeless cubic graphs with m edges to $34m/21 \approx 1.619m$ and then we show that every m-edge bridgeless graph with minimum degree three has a cycle cover of total length at most $44m/27 \approx 1.630m$, i.e., we extend the result from [25] on cubic graphs to all graphs with minimum degree three. As in [25], the cycle covers that we construct consist of at most three cycles.

Though the improvements of the original bound of $5m/3 = 1.667m$ on the length of a shortest cycle cover of an m-edge bridgeless graph can seem to be rather minor, obtaining a bound below $8m/5 = 1.600m$ for a significant class of graphs might be quite challenging since the bound of $8m/5$ is implied by Tutte's 5-Flow Conjecture [49].

9.1 Notation

Graphs considered in this part of the thesis can have loops and multiple edges. If E is a set of edges of a graph G, then $G \setminus E$ denotes the graph with the same vertex set and with the edges of E removed. If $E = \{e\}$, we simply write $G \setminus e$ instead of $G \setminus \{e\}$. For an edge e of G, G/e is the graph obtained by contracting the edge e, i.e., G/e is the graph with the end-vertices of e identified, the edge e removed and all the other edges preserved. In particular, the edges parallel to e become loops in G/e. Also note that if e is a loop, then $G/e = G \setminus e$. Finally, for a set E of edges of a graph G, G/E denotes the graph obtained by contracting all edges contained in E. If G is a graph and v a vertex of G of degree two, then the graph obtained from G by *suppressing* the vertex v is the graph obtained from G by contracting one of the edges incident with v, i.e., the graph obtained by replacing the two-edge path with the inner vertex v by a single edge.

An *edge-cut* in a graph G is a set E of edges such that the vertices of G can be partitioned into two sets A and B such that E contains precisely the edges with one end-vertex in A and the other in B. Such an edge-cut is also denoted by $E(A, B)$ and its size by $e(A, B)$. We abuse this notation a little bit and also use $e(A, B)$ for the number of edges between any two disjoint sets A and B which do not necessarily form a partition of the vertex set of G. An edge forming an edge-cut of size one is called a *bridge* and graphs with no edge-cuts of size one are said to be *bridgeless*. Note that we do not require edge-cuts to be minimal sets E such that $G \setminus E$ has more components than G. A graph G with no edge-cuts of odd size less than k is said to be *k-odd-connected*. For every set F of edges of G, cuts in G/F correspond to cuts (of the same size) in G. Therefore, if G has no edge-cuts of size k, then also G/F has no edge-cuts of size k.

As said before, a *cycle* of a graph G is a subgraph of G with all vertices of even degree. A *circuit* is a connected subgraph with all vertices of degree two and a 2-*factor* is a spanning subgraph with all vertices of degree two.

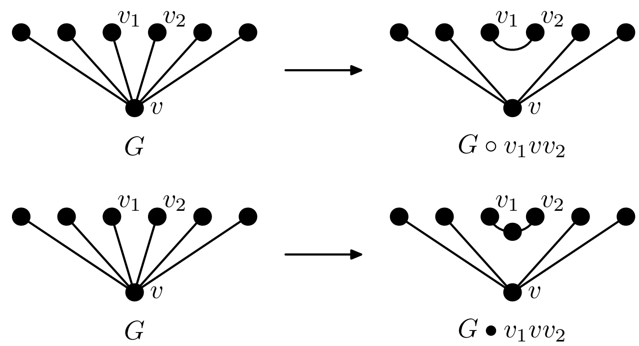

Figure 9.1: Splitting the vertices v_1 and v_2 from the vertex v.

9.2 Splitting and Expanding Vertices

In the proof of our results, we will need to construct a nowhere-zero 4-flow of a special type. In order to exclude some "bad" nowhere-zero flows, we will first modify a graph in such a way that some of its edges must get the same flow value. This will be achieved by splitting some of the vertices of the considered graph. Let G be a graph, v a vertex of G and v_1 and v_2 some of the neighbors of v in G. Let $G \circ v_1 v v_2$ be the graph obtained by removing the edges vv_1 and vv_2 from G and adding the edge $v_1 v_2$ and $G \bullet v_1 v v_2$ be the graph obtained by removing the edges vv_1 and vv_2 from G and connecting the vertices v_1 and v_2 with a path of length two (see Figure 9.1). The graphs $G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$ are said to be obtained by *splitting the vertices* v_1 *and* v_2 *from the vertex* v. Notice that $G \bullet v_1 v v_2$ can be obtained from $G \circ v_1 v v_2$ by subdividing the edge $v_1 v_2$ and that both $G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$ preserve the degrees of all vertices in G. Also observe that the odd-connectivity of $G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$ is the same.

Classical (and deep) results of Fleischner [30], Mader [65] and Lovász [63] assert that it is possible to split vertices without creating new small edge-cuts. Let us now formulate two of the corollaries of their results.

Lemma 9.1 *Let G be a bridgeless graph. For every vertex v of G of degree four or more, there exist two neighbors v_1 and v_2 of the vertex v such that both the graphs $G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$ are bridgeless as well.*

Lemma 9.2 *Let G be a 5-odd-connected graph. For every vertex v of G of degree four, six or more, there exist two neighbors v_1 and v_2 of the vertex v such that both the graphs $G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$ are 5-odd-connected as well.*

Zhang [87] proved a version of Lemma 9.2 where only some pairs of vertices are allowed to be split off.

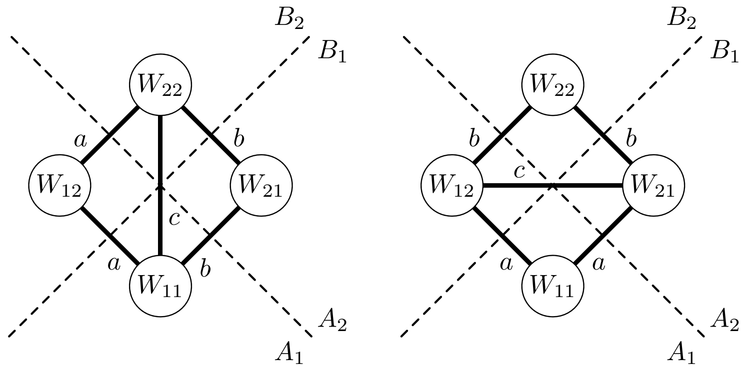

Figure 9.2: The two configurations described in the statement of Lemma 9.4.

Lemma 9.3 *Let G be an ℓ-odd-connected graph for an odd integer ℓ. For every vertex v of G with neighbors $v_1, \ldots, v_k$, $k \notin \{2, \ell\}$, there exist two neighbors v_i and v_{i+1} such that the graphs $G \circ v_i v v_{i+1}$ and $G \bullet v_i v v_{i+1}$ are also ℓ-odd-connected (indices are modulo k).*

However, any of these results is not sufficient for our purposes since we need to specify more precisely which pair of the neighbors of v should be split from v. This is guaranteed by the lemmas we establish in the rest of this section. Let us remark that Lemma 9.3 can be obtained as a consequence of our results. We start with a modification of the well-known fact that "minimal odd cuts do not cross" for the situation where small even cuts can exist.

Lemma 9.4 *Let G be an ℓ-odd-connected graph (for some odd integer ℓ) and let $E(A_1, A_2)$ and $E(B_1, B_2)$ be two cuts of G of size ℓ. Further, let $W_{ij} = A_i \cap B_j$ for $i, j \in \{1, 2\}$. If the sets W_{ij} are non-empty for all $i, j \in \{1, 2\}$, then there exist integers a, b and c such that $a + b + c = \ell$ and one of the following holds:*

- *$e(W_{11}, W_{12}) = e(W_{12}, W_{22}) = a$, $e(W_{11}, W_{21}) = e(W_{21}, W_{22}) = b$, $e(W_{11}, W_{22}) = c$ and $e(W_{12}, W_{21}) = 0$, or*
- *$e(W_{12}, W_{11}) = e(W_{11}, W_{21}) = a$, $e(W_{12}, W_{22}) = e(W_{22}, W_{21}) = b$, $e(W_{12}, W_{21}) = c$ and $e(W_{11}, W_{22}) = 0$.*

See Figure 9.2 for an illustration of the two possibilities.

Proof: Let w_{ij} be the number of edges with exactly one end-vertex in W_{ij}. Observe that

$$w_{11} + w_{12} = e(A_1, A_2) + 2e(W_{11}, W_{12}) = \ell + 2e(W_{11}, W_{12}) \quad \text{and} \qquad (9.1)$$

$$w_{12} + w_{22} = e(B_1, B_2) + 2e(W_{12}, W_{22}) = \ell + 2e(W_{12}, W_{22}) \; .$$

In particular, one of the numbers w_{11} and w_{12} is even and the other is odd. Assume that w_{11} is odd. Hence, w_{22} is also odd. Since G is ℓ-odd-connected, both w_{11} and w_{22} are at least ℓ.

If $e(W_{11}, W_{12}) \leq e(W_{12}, W_{22})$, then

$$\begin{aligned} w_{12} &= e(W_{11}, W_{12}) + e(W_{12}, W_{21}) + e(W_{12}, W_{22}) \\ &\geq e(W_{11}, W_{12}) + e(W_{12}, W_{22}) \geq 2e(W_{11}, W_{12})\,. \end{aligned} \tag{9.2}$$

The equation (9.1), the inequality (9.2) and the inequality $w_{11} \geq \ell$ imply that $w_{11} = \ell$. Hence, the inequality (9.2) is an equality; in particular, $e(W_{11}, W_{12}) = e(W_{12}, W_{22})$ and $e(W_{12}, W_{21}) = 0$. If $e(W_{11}, W_{12}) \geq e(W_{12}, W_{22})$, we obtain the same conclusion. Since the sizes of the cuts $E(A_1, A_2)$ and $E(B_1, B_2)$ are the same, it follows that $e(W_{11}, W_{21}) = e(W_{21}, W_{22})$. We conclude that the graph G and the cuts have the structure as described in the first part of the lemma.

The case that w_{11} is even (and thus w_{12} is odd) leads to the other configuration described in the statement of the lemma.

∎

Next, we use Lemma 9.4 to characterize graphs where some splittings of neighbors of a given vertex decrease the odd-connectivity. In the statement of Lemma 9.5, the graph G is assumed to be simple just to avoid unnecessary technical complications in its proof; the lemma also holds for graphs with loops and parallel edges (with a suitable definition of vertex splitting).

Lemma 9.5 *Let G be a simple ℓ-odd-connected graph for an odd integer $\ell \geq 3$, v a vertex of G and $v_1, \ldots, v_k$ some neighbors of v. If every graph $G \circ v_i v v_{i+1}$, $i = 1, \ldots, k-1$, contains an edge-cut of odd size smaller than ℓ, the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $v_i \in V_2$ for $i = 1, \ldots, k$ and the size of the edge-cut $E(V_1, V_2)$ is ℓ.*

Proof: The proof proceeds by induction on k. The base case of the induction is that $k = 2$. Let $E(V_1, V_2)$ be an edge-cut of $G \circ v_1 v v_2$ of odd size less than ℓ. By symmetry, we can assume that $v \in V_1$. If both $v_1 \in V_1$ and $v_2 \in V_1$, then $E(V_1, V_2)$ as an edge-cut of G has the same size as in $G \circ v_1 v v_2$ which contradicts the assumption that G is ℓ-odd-connected. If $v_1 \in V_1$ and $v_2 \in V_2$, then $E(V_1, V_2)$ is also an edge-cut of G of the same size as in $G \circ v_1 v v_2$ which is again impossible.

Hence, both v_1 and v_2 must be contained in V_2, and the size of the edge-cut $E(V_1, V_2)$ in G is larger by two compared to its size in $G \circ v_1 v v_2$. Since G has no edge-cuts of size $\ell - 2$, the size of the edge-cut $E(V_1, V_2)$ in $G \circ v_1 v v_2$ is $\ell - 2$ and its size in G is ℓ. Hence, V_1 and V_2 form the partition of the vertices as in the statement of the lemma.

We now consider the case that $k > 2$. By the induction assumption, G contains a cut $E(V_1', V_2')$ of size ℓ such that $v \in V_1'$ and $v_1, \ldots, v_{k-1} \in V_2'$. Similarly, there is a cut $E(V_1'', V_2'')$ of size ℓ such that $v \in V_1''$ and $v_2, \ldots, v_k \in V_2''$. Let

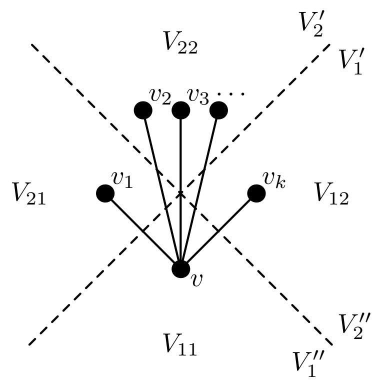

Figure 9.3: Notation used in the proof of Lemma 9.5.

$V_{ij} = V_i' \cap V_j''$ for $i, j \in \{1, 2\}$ (see Figure 9.3). Apply Lemma 9.4 for the graph G with $A_i = V_i'$ and $B_i = V_i''$. Since $e(V_{11}, V_{22}) = e(V_1' \cap V_1'', V_2' \cap V_2'') \geq k-2 > 0$, the first case described in Lemma 9.4 applies and the size of the cut $E(V_1' \cap V_1'', V_2' \cup V_2'')$ is ℓ. Hence, the cut $E(V_1' \cap V_1'', V_2' \cup V_2'')$ is a cut of size ℓ

■

We finish this section with a series of lemmas that we need in Section 10.1. All these lemmas are simple corollaries of Lemma 9.5. Note that Lemma 9.6 is also implied by Lemma 9.3.

Lemma 9.6 *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree four and v_1, v_2, v_3 and v_4 its four neighbors. Then, the graph $G \circ v_1 v v_2$ or the graph $G \circ v_2 v v_3$ is also 5-odd-connected graph.*

Proof: Observe that the graph $G \circ v_1 v v_2$ is 5-odd-connected if and only if the graph $G \circ v_3 v v_4$ is 5-odd-connected. Lemma 9.5 applied for $\ell = 5$, the vertex v, $k = 4$ and the vertices v_1, v_2, v_3 and v_4 yields that the vertices of G can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $\{v_1, v_2, v_3, v_4\} \subseteq V_2$ and $e(V_1, V_2) = 5$. Hence,

$$e(V_1 \setminus \{v\}, V_2 \cup \{v\}) = e(V_1, V_2) - 4 = 5 - 4 = 1 \ .$$

This contradicts our assumption that G has no edge-cuts of size one.

■

Lemma 9.7 *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree six and $v_1, \ldots, v_6$ its neighbors. At least one of the graphs $G \circ v_1 v v_2$, $G \circ v_2 v v_3$ and $G \circ v_3 v v_4$ is also 5-odd-connected.*

Proof: Lemma 9.5 applied for $\ell = 5$, the vertex v, $k = 4$ and v_1, v_2, v_3 and v_4 yields that the vertices of G can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $\{v_1, v_2, v_3, v_4\} \subseteq V_2$ and $e(V_1, V_2) = 5$. If V_2 contains σ neighbors of v (note that $\sigma \geq 4$), then

$$e(V_1 \setminus \{v\}, V_2 \cup \{v\}) = e(V_1, V_2) - \sigma + (6 - \sigma) = 11 - 2\sigma$$

which is equal to 1 or 3 contradicting the fact that G is 5-odd-connected.

■

Lemma 9.8 *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree k, $k \geq 6$ and $v_1, \ldots, v_k$ its neighbors. At least one of the graphs $G \circ v_1 v v_2$, $G \circ v_2 v v_3$, $G \circ v_3 v v_4$, $G \circ v_4 v v_5$ and $G \circ v_5 v v_6$ is also 5-odd-connected.*

Proof: Since there is no partition of the vertices of G into two parts V_1 and V_2 such that $v \in V_1$, $v_i \in V_2$ for $i = 1, \ldots, 6$, and $e(V_1, V_2) = 5$ (because $e(V_1, V_2) \geq 6$), Lemma 9.5 applied for $\ell = 5$, the vertex v and the vertices $v_1, \ldots, v_6$ yields the statement of the lemma.

■

As the two variants of vertex splitting ($G \circ v_1 v v_2$ and $G \bullet v_1 v v_2$) differ only in subdividing edges, which cannot introduce any edge-cuts of size one or three if they did not exist before, Lemmas 9.6–9.8 hold for the other variant of vertex splitting as well. Moreover, since every graph can be made simple by subdividing edges, the requirement for the graphs being simple is not essential. We do not formulate Lemmas 9.6–9.8 for graphs G with parallel edges, as a graph $G \circ v_1 v v_2$ could contain loops which we forbid. However, this complication does not arise for $G \bullet v_1 v v_2$. We now state the modified variants of Lemmas 9.6–9.8.

Lemma 9.9 *Let G be a graph with no edge-cuts of size one or three, and let v be a vertex of degree four and v_1, v_2, v_3 and v_4 its four neighbors. The graph $G \bullet v_1 v v_2$ or the graph $G \bullet v_2 v v_3$ contains no edge-cuts of size one or three.*

Lemma 9.10 *Let G be a 5-odd-connected graph, v a vertex of degree six, and $v_1, \ldots, v_6$ its neighbors. At least one of the graphs $G \bullet v_1 v v_2$, $G \bullet v_2 v v_3$ and $G \bullet v_3 v v_4$ is also 5-odd-connected.*

Lemma 9.11 *Let G be a 5-odd-connected graph, v a vertex of degree six or more, and $v_1, \ldots, v_k$ its neighbors ($k \geq 6$). At least one of the graphs $G \bullet v_1 v v_2$, $G \bullet v_2 v v_3$, $G \bullet v_3 v v_4$, $G \bullet v_4 v v_5$ and $G \bullet v_5 v v_6$ is also 5-odd-connected.*

Finally, we will also need in Section 10.1 the following corollary of Lemma 9.5.

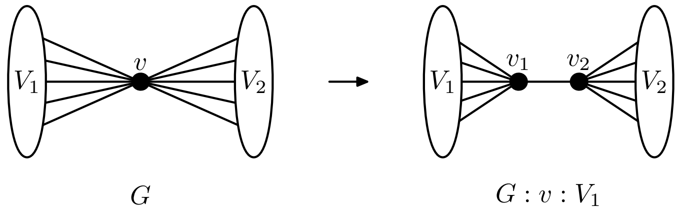

Figure 9.4: An example of the expansion a vertex v with respect to set V_1.

Lemma 9.12 *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree eight and $v_1, \ldots, v_8$ its neighbors. Suppose that $G \circ v_1 v v_2$ is 5-odd-connected. At least one of the following graphs is also 5-odd-connected: $G \circ v_1 v v_2 \circ v_3 v v_4$, $G \circ v_1 v v_2 \circ v_7 v v_8$, $G \circ v_1 v v_2 \circ v_8 v v_3 \circ v_4 v v_5$ and $G \circ v_1 v v_2 \circ v_8 v v_3 \circ v_4 v v_6$.*

Proof: The degree of the vertex v in $G \circ v_1 v v_2$ is six. By Lemma 9.7, at least one of the graphs $G \circ v_1 v v_2 \circ v_3 v v_4$, $G \circ v_1 v v_2 \circ v_3 v v_8$ and $G \circ v_1 v v_2 \circ v_7 v v_8$ is 5-odd-connected.

If $G \circ v_1 v v_2 \circ v_3 v v_8$ is 5-odd-connected, we apply Lemma 9.6 for the vertex v and its neighbors v_5, v_4, v_6 and v_7 (in this order). Hence, the graph $G \circ v_1 v v_2 \circ v_8 v v_3 \circ v_4 v v_5$ (which is homeomorphic to $G \circ v_1 v v_2 \circ v_8 v v_3 \circ v_6 v v_7$) or the graph $G \circ v_1 v v_2 \circ v_8 v v_3 \circ v_4 v v_6$ is 5-odd-connected.

■

We need one more vertex operation in our arguments in Section 11.6—vertex expansions. If G is a graph, v a vertex of G and V_1 and V_2 a partition of the neighbors of v into two sets, then the graph $G : v : V_1$ is the graph obtained from G by removing the vertex v and introducing two new vertices v_1 and v_2, joining v_1 to the vertices of V_1, v_2 to the neighbors of v not contained in V_1, and adding an edge $v_1 v_2$. We say that $G : v : V_1$ is obtained by *expanding the vertex v with respect to the set V_1*. See Figure 9.4 for an example. Let us remark that this operation will be applied only to vertices v incident with no parallel edges.

In Section 11.6, we use the following auxiliary lemma which directly follows from results of Fleischner [30]:

Lemma 9.13 *Let G be a bridgeless graph and v a vertex of degree four in G incident with no parallel edges. Further, let v_1, v_2, v_3 and v_4 be the four neighbors of v. The graph $G : v : \{v_1, v_2\}$ or the graph $G : v : \{v_2, v_3\}$ is also bridgeless.*

Chapter 10

Short Cycle Covers of Cubic Graphs

In this chapter, we show that a bridgeless cubic graph with m edges has a cycle cover of total length at most $34m/21 \approx 1.619m$.

10.1 Rainbow Lemma

Before we start with proving our result, we state and prove a generalization of a lemma usually referred to as the Rainbow Lemma.

Lemma 10.1 (Rainbow Lemma) *Every cubic bridgeless graph G contains a 2-factor F such that the edges of G not contained in F can be colored with three colors, red, green and blue in the following way:*

- *every even circuit of F contains an even number of vertices incident with red edges, an even number of vertices incident with green edges and an even number of vertices incident with blue edges, and*
- *every odd circuit of F contains an odd number of vertices incident with red edges, an odd number of vertices incident with green edges and an odd number of vertices incident with blue edges.*

In the rest, a 2-factor F with an edge-coloring satisfying the constraints given in Lemma 10.1 will be called a *rainbow 2-factor*. Rainbow 2-factors implicitly appear in, e.g., [25, 60, 64], and are related to the notion of parity 3-edge-colorings from the Ph.D. thesis of Goddyn [36].

A key ingredient in the proof of Lemma 10.1 is the following classical result of Jaeger:

Theorem 10.2 (Jaeger [47]) *If G is a 5-odd-connected graph, then G has a nowhere-zero 4-flow.*

A classical result of Petersen [70] asserts that every cubic bridgeless graph has a perfect matching. The following strengthening of this result, which appears, e.g., in [53, 86], is another ingredient for the proof of Rainbow Lemma.

Theorem 10.3 *Every cubic bridgeless graph G has a 2-factor such that the graph G/F is 5-odd-connected.*

We are now ready to provide a modification of the Rainbow Lemma that is used in the proof of our main theorem. In addition to the constraints on the edge-coloring of a perfect matching given in Lemma 10.1, we exclude certain color patterns from appearing on the edges incident with cycles of specific lengths. Let us be more precise. The *pattern* of a circuit $C = v_1 \dots v_k$ of a rainbow 2-factor F is $X_1 \dots X_k$ where X_i is the color of the edge incident with the vertex v_i; we use R to represent the red color, G the green color and B the blue color. Two patterns are said to be *symmetric* if one of them can be obtained from the other by a rotation, a reflection and/or a permutation of the red, green and blue colors. For example, the patterns RRGBGB and RBRBGG are symmetric but the patterns RRGBBG and RRGBGB are not.

Let us now state and prove a generalization of the Rainbow Lemma.

Lemma 10.4 *Every cubic bridgeless graph G contains a rainbow 2-factor F such that*

- *no circuit of the 2-factor F has length three,*
- *every circuit of length four has a pattern symmetric to RRRR or RRGG, and*
- *every circuit of length eight has a pattern symmetric to one of the following 16 patterns:*
  RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB,
  RRGGRRGG, RRGGRRBB, RRRRGRRG, RRRRGBBG,
  RRGGRGGR, RRGGRBBR, RRGGBRRB, RRRRGRGR,
  RRRGBGBR, RRGRGRGG, RRGRBRBG and RRGGBGBG.

Proof: By Theorem 10.3, there exists a 2-factor F such that the graph G/F is 5-odd-connected. Since G is cubic and G/F 5-odd-connected, the 2-factor F contains no circuits of length three. Let H_0 be the graph obtained from the graph G/F by subdividing each edge three times. Clearly, H_0 is also 5-odd-connected. This modification of G/F to H_0 is needed only to simplify our arguments later in the proof since it guarantees that the graph is simple and thus we can easily apply Lemmas 9.6–9.8 and 9.12.

In a series of steps, we iteratively modify the graph H_0 to graphs H_1, H_2, etc. During this process, the degree of each vertex of H_i is the same as in H_0 though

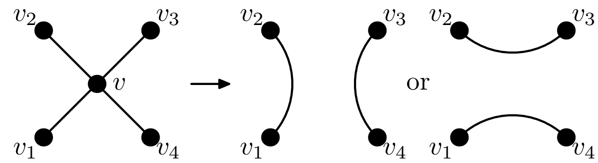

Figure 10.1: Reduction of a vertex of degree four in a graph H_i in the proof of Lemma 10.4.

some vertices may be removed (and thus not present in H_i). All the graphs H_1, H_2, ... will be simple and 5-odd-connected.

If the graph H_i contains a vertex v of degree four, then the vertex v corresponds to a circuit C of length four in F. Let v_1, v_2, v_3 and v_4 be the neighbors of v in the order in which the edges vv_1, vv_2, vv_3 and vv_4 correspond to edges incident with the circuit C. To obtain H_{i+1}, we consider among the graphs $H_i \circ v_1vv_2$ and $H_i \circ v_2vv_3$ one that is 5-odd-connected (by Lemma 9.6 at least one of them is). The graph H_{i+1} is then obtained by suppressing the vertex v in this graph; see Figure 10.1. Clearly, H_{i+1} is 5-odd-connected and all the vertices of H_{i+1} have the same degree as in H_i.

If the graph H_i contains a vertex v of degree eight, we proceed as follows. The vertex v corresponds to a circuit C of F of length eight; let $v_1, \ldots, v_8$ be the neighbors of v in H_i in the order in that they correspond to the edges incident with the circuit C. By Lemma 9.8, we can assume that the graph $H_i \circ v_1vv_2$ is 5-odd-connected (for a suitable choice of the cyclic rotation of the neighbors of v).

By Lemma 9.12, at least one of the following graphs is 5-odd-connected: $H_i \circ v_1vv_2 \circ v_3vv_4$, $H_i \circ v_1vv_2 \circ v_7vv_8$, $H_i \circ v_1vv_2 \circ v_8vv_3 \circ v_4vv_5$ and $H_i \circ v_1vv_2 \circ v_8vv_3 \circ v_4vv_6$. If the graph $H_i \circ v_1vv_2 \circ v_3vv_4$ is 5-odd-connected, we then apply Lemma 9.6 to the graph $H_i \circ v_1vv_2 \circ v_3vv_4$ and conclude that the graph $H_i \circ v_1vv_2 \circ v_3vv_4 \circ v_5vv_6$ or the graph $H_i \circ v_1vv_2 \circ v_3vv_4 \circ v_6vv_7$ is 5-odd-connected. Since the case that the graph $H_i \circ v_1vv_2 \circ v_7vv_8$ is 5-odd-connected is symmetric to the case of the graph $H_i \circ v_1vv_2 \circ v_3vv_4$, it can be assumed that (at least) one of the following four graphs is 5-odd-connected: $H_i \circ v_1vv_2 \circ v_3vv_4 \circ v_5vv_6$, $H_i \circ v_1vv_2 \circ v_3vv_4 \circ v_6vv_7$, $H_i \circ v_1vv_2 \circ v_8vv_3 \circ v_4vv_5$ and $H_i \circ v_1vv_2 \circ v_8vv_3 \circ v_4vv_6$. Let H_{i+1} be the graph obtained by suppressing the vertex v in a 5-odd-connected graph in this list (see Figure 10.2).

We eventually reach a 5-odd-connected graph H_k with no vertices of degree four or eight. By Theorem 10.2, the graph H_k has a nowhere-zero 4-flow and, equivalently, it has a nowhere-zero $\mathbb{Z}_2^2$-flow. This flow yields a nowhere-zero $\mathbb{Z}_2^2$-flow in $H_{k-1}, \ldots, H_0$ and eventually in G/F: in each step, we subdivide some edges (and give them the flow-value of the original edge), and identify some vertices. In particular, the pairs of edges split away from a vertex are assigned the same flow value.

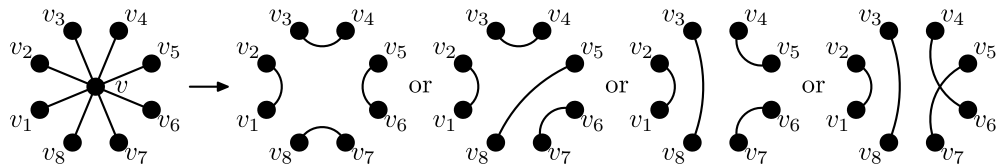

Figure 10.2: Reduction of a vertex of degree eight in a graph H_i in the proof of Lemma 10.4.

The nowhere-zero $\mathbb{Z}_2^2$-flow of G/F gives the coloring of the edges: the edges with the flow value $(0,1)$ are colored red, those with the flow value $(1,0)$ green and those with the value $(1,1)$ blue. Since the colors of the edges of G/F correspond to a nowhere-zero 4-flow of G/F, F is a rainbow 2-factor with respect to this edge-coloring.

We now verify that the edge-coloring of G/F also satisfies the additional two constraints given in the statement. Let us start with the first constraint, and let C be a circuit of F of length four, v the vertex of G/F corresponding to C and e_1, e_2, e_3 and e_4 the four (not necessarily distinct) edges leaving C in G. In H_0, the edge e_i corresponds to an edge vv_i for a neighbor v_i of v. During the construction of H_k, either the vertices v_1 and v_2 or the vertices v_2 and v_3 are split away from v. In the former case, the colors of the edges e_1 and e_2 are the same and the colors of the edges e_3 and e_4 are the same; in the latter case, the colors of the edges e_1 and e_4 and the colors of the edges e_2 and e_3 are the same. In both cases, the pattern of C is symmetric to RRRR or RRGG.

Let C be a circuit of F of length eight, v the vertex of G/F corresponding to C and $e_1, \ldots, e_8$ the eight edges leaving C in G (note that some of the edges $e_1, \ldots, e_8$ can be the same). Let c_i be the color of the edge e_i. Based on the splitting, one of the following four cases (up to symmetry) applies:

1. $c_1 = c_2$, $c_3 = c_4$, $c_5 = c_6$ and $c_7 = c_8$,
2. $c_1 = c_2$, $c_3 = c_4$, $c_5 = c_8$ and $c_6 = c_7$,
3. $c_1 = c_2$, $c_3 = c_8$, $c_4 = c_5$ and $c_6 = c_7$, and
4. $c_1 = c_2$, $c_3 = c_8$, $c_4 = c_6$ and $c_5 = c_7$.

In the first case, the pattern of the circuit C is symmetric to RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB, RRGGRRGG or RRGGRRBB. In the second case, the pattern of the circuit C is symmetric to one of the patterns RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB, RRRRGRRG, RRRRGBBG, RRGGRGGR, RRGGRBBR or RRGGBRRB. The third case is symmetric to the second one (see Figure 10.2). In the last case, the pattern of C is symmetric to one of RRRRRRRR, RRRRRRGG, RRRRGRGR, RRRRGRRG,

RRRRGGGG, RRGRGRGG, RRRGBGBR, RRGRBRBG, RRGGBGBG or RRRRGBBG. In all the four cases, the pattern of C is one of the patterns listed in statement of the lemma.

■

10.2 Intermezzo

In order to help the reader to follow the arguments presented in the next section, we reprove a restricted version of the classical result of Alon and Tarsi [7] and Bermond, Jackson and Jaeger [10]: every cubic bridgeless graph with m edges has a cycle cover of length at most $5m/3$. We restrict our attention to cubic graphs only and use a technique similar to that used by Fan in [25]. In the next section, we refine the presented proof to improve the bound.

Let us now introduce additional notation used in the proof of Theorem 10.5. Let G be a cubic graph and F a 2-factor of G. For a circuit C contained in F and for a set of edges of E such that $C \cap E = \emptyset$, we define $C(E)$ to be the set of vertices of C incident with the edges of E. Our goal in the proof will be to extend a certain set E of edges of G to a cycle by adding edges of F. This is impossible if $|C(E)|$ is odd for any circuit C of F. If $|C(E)|$ is even, we partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that each of them induces paths with end-vertices being the vertices of $C(E)$. If $C(E) = \emptyset$, then $C(E)^A$ contains no edges of C and $C(E)^B$ contains all the edges of C (or vice versa). Observe that for every set E of edges not contained in F, adding one of the sets $C(E)^A$ and $C(E)^B$ for each circuit C of F yields a cycle of G. In the rest, we will always assume that the number of edges of $C(E)^A$ does not exceed the number of edges of $C(E)^B$.

Theorem 10.5 *Every cubic bridgeless graph G with m edges has a cycle cover of length at most* $5m/3$.

Proof: We first apply Lemma 10.1 to G and obtain a rainbow 2-factor F. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the sets of red, green and blue edges and r, g and b their numbers. By symmetry, we can assume that $r \le g \le b$. Also observe that $r + g + b = m/3$.

The desired cycle cover of G, which is comprised of three cycles, is defined as follows. The first cycle $\mathcal{C}_1$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^A$ for all circuits C of the 2-factor F. The second cycle $\mathcal{C}_2$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^B$ for all circuits C of F. Finally, the third cycle $\mathcal{C}_3$ contains all the red and blue edges and the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ for all circuits C of F.

It remains to verify that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ cover all edges and to estimate their total length. Each edge of F is covered once by either the cycle $\mathcal{C}_1$ or $\mathcal{C}_2$; since $|C(E)^A| \le |C(E)^B|$ for every circuit C of F, at most half of the edges of F

is also covered by the cycle $\mathcal{C}_3$. Since each red edge is covered three times, each green edge twice and each blue edge once, the total length of the constructed cycle cover is at most:

$$3r + 2g + b + |F| + |F|/2 \leq 2(r + g + b) + 3|F|/2 = 2m/3 + m = 5m/3 .$$

The proof of the theorem is now finished.

■

10.3 Main Result

We are now ready to prove the main result of this chapter.

Theorem 10.6 *Every cubic bridgeless graph G with m edges has a cycle cover comprised of three cycles of total length at most $34m/21$.*

Proof: We present two bounds on the length of a cycle cover of G and the bound claimed in the statement of the theorem is eventually obtained by combining the two presented bounds. In both bounds, the constructed cycle cover will consist of three cycles. Fix a rainbow 2-factor F and an edge-coloring of the edges not contained in F with the red, green and blue colors as described in Lemma 10.4. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the sets of the red, green and blue edges, respectively, and let r, g and b be their numbers. Finally, let d_ℓ be the number of circuits of lengths ℓ contained in F. By Lemma 10.4, $d_3 = 0$.

The first cycle cover. Before we proceed with constructing the first cycle cover, recall the notation of $C(E)^A$ and $C(E)^B$ introduced before Theorem 10.5. The cycle cover is comprised of three cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycle $\mathcal{C}_1$ contains all red and green edges, the cycle $\mathcal{C}_2$ contains all red and blue edges and the cycles $\mathcal{C}_3$ contains all green and blue edges. In addition, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain some edges of the 2-factor F as described further.

Let C be a circuit of the 2-factor F. Lemma 10.4 allows us to assume that if the length of C is four, then the pattern of C is either BBBB or GGBB (otherwise, we can—for the purpose of extending the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ to C—switch the roles of the red, green and blue colors and the roles of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ in the remaining analysis; note that we are not recoloring the edges, just apply the arguments presented in the next paragraphs with respect to a different permutation of colors). Similarly, we can assume that the pattern of the circuit C of length eight is one of the following 16 patterns:

BBBBBBBB, BBBBBBRR, BBBBRRRR, BBBBRRGG,
BBRRBBRR, BBRRBBGG, BBBBRBBR, BBBBRGGR,
BBRRBRRB, BBRRBGGB, RRGGBRRB, BBBBRBRB,
BBBRGRGB, RRBRBRBB, BBRBGBGR and BBRRGRGR.

Let us now choose edges of the circuit C that are included in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycle $\mathcal{C}_1$ contains the edges of $C^1 = C(\mathcal{R}\cup\mathcal{G})^A$. The cycle $\mathcal{C}_2$ contains the edges C^2 of either $C(\mathcal{R}\cup\mathcal{B})^A$ or $C(\mathcal{R}\cup\mathcal{B})^B$—we choose the set with smaller intersection with $C(\mathcal{R}\cup\mathcal{G})^A$. Finally, the edges included to $\mathcal{C}_3$ are chosen so that every edge of C is covered odd number of times; explicitly, the edges $C^3 = C^1 \triangle C^2 \triangle C$ are included to $\mathcal{C}_3$. Note that C^3 is either $C(\mathcal{G}\cup\mathcal{B})^A$ or $C(\mathcal{G}\cup\mathcal{B})^B$. In particular, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form cycles.

We now estimate the number of the edges of C contained in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The sum of the numbers edges contained in each of the cycles is:

$$\begin{aligned}|C^1| &+ |C^2| + |C^1 \triangle C^2 \triangle C| \\ &= |C^1\cup C^2| + |C^1\cap C^2| + |C\setminus(C^1\cup C^2)| + |C^1\cap C^2| \\ &= |C| + 2|C^1\cap C^2|\end{aligned}$$

Since $|C^1| = |C(\mathcal{R}\cup\mathcal{G})^A| \le |C(\mathcal{R}\cup\mathcal{G})^B|$, the number of edges of C^1 is at most $\ell/2$. By the choice of C^2, $|C^1\cap C^2| \le |C^1|/2 \le \ell/4$. Hence, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most $\ell + 2\lfloor \ell/4\rfloor$ edges of the circuit C.

The estimate on the number of edges of C contained in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ can further be improved if the length of the circuit C is four: if the pattern of C is BBBB, then $C^1 = C(\mathcal{R}\cup\mathcal{G})^A = \emptyset$ and thus $C^1\cap C^2 = \emptyset$. If the pattern is GGBB, then $C^1\cap C^2 = C(\mathcal{R}\cup\mathcal{G})^A \cap C(\mathcal{R}\cup\mathcal{B})^A = \emptyset$. In both the cases, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain (at most) $|C| + 2|C^1\cap C^2| = \ell = 4$ edges of C.

Similarly, the estimate on the number of edges of C contained in the cycles can be improved if the length of C is eight. As indicated in Figure 10.3, it holds that $|C^1| = |C(\mathcal{R}\cup\mathcal{G})^A| \le 3$. Hence, $|C^1\cap C^2| \le |C^1|/2 \le 3/2$. Consequently, $|C^1\cap C^2| \le 1$ and the number of edges of C included in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is at most $|C| + 2|C^1\cap C^2| \le 8+2 = 10$.

Based on the analysis above, we can conclude that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most the following number of edges of the 2-factor F in total:

$$\begin{aligned}&2d_2 + 4d_4 + 7d_5 + 8d_6 + 9d_7 + 10d_8 + 13d_9 + 14d_{10} + 15d_{11} + \sum_{\ell=12}^{\infty}\frac{3\ell}{2}d_\ell \\ = \ &\frac{3}{2}\sum_{\ell=2}^{\infty}\ell d_\ell - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11}\ . \qquad (10.1)\end{aligned}$$

Since the 2-factor F contains $2m/3$ edges, the estimate (10.1) translates to:

$$m - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11}\ . \qquad (10.2)$$

Since each red, green or blue edge is contained in exactly two of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ and there are $m/3$ such edges, the total length of the cycle cover of G formed by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ does not exceed:

$$\frac{5m}{3} - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11}\ . \qquad (10.3)$$

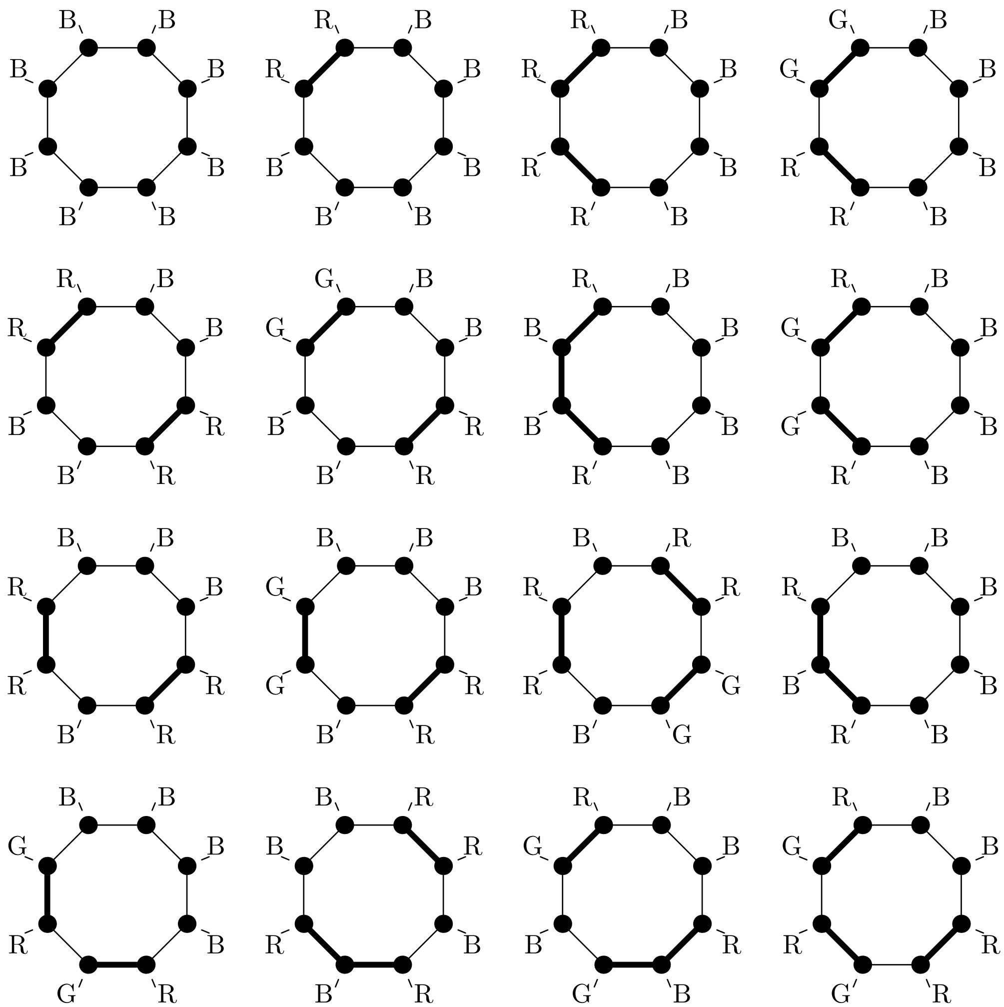

Figure 10.3: The sets $C^1 = C(\mathcal{R} \cup \mathcal{G})^A$ for circuits C with length eight; the edges contained in the set are drawn bold.

This finishes the construction and the analysis of the first cycle cover of G.

The second cycle cover. We keep the 2-factor F and the coloring of the edges of G by red, green and blue colors fixed. As long as the graph $H = G/F$ contains a red circuit, choose a red circuit of $H = G/F$ and recolor its edges with blue. Similarly, recolor edges of green circuits with blue. The modified edge-coloring still gives a rainbow 2-factor but the two additional constraints given in Lemma 10.4 need not be met anymore. Let $\mathcal{R}'$, $\mathcal{G}'$ and $\mathcal{B}'$ be the sets of red, green and blue in the modified edge-coloring and r', g' and b' their cardinalities.

The construction of the cycle cover now follows the lines of the proof of Theorem 10.5. The first cycle $\mathcal{C}_1$ is formed by the red and green edges and the edges of $C(\mathcal{R}' \cup \mathcal{G}')^A$ for every circuit C of the 2-factor F. The cycle $\mathcal{C}_2$ is also formed by the red and green edges and it contains the edges of $C(\mathcal{R}' \cup \mathcal{G}')^B$ for every circuit C of F. Finally, the cycle $\mathcal{C}_3$ is formed by the red and blue edges and the edges of $C(\mathcal{R}' \cup \mathcal{B}')^A$ for every circuit C of F. Clearly, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are cycles of G and they cover all the edges of G.

Let us now estimate the lengths of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each red edge is contained in all the three cycles, each green edge in two cycles and each blue edge in one cycle. Each edge of a circuit C of length ℓ of the 2-factor F is contained either in $\mathcal{C}_1$ or in $\mathcal{C}_2$ and at most half of the edges of C is also contained in the cycle $\mathcal{C}_3$. Hence, the total length of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is at most:

$$3r' + 2g' + b' + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell \, . \tag{10.4}$$

Since the red edges form an acyclic subgraph of G/F, the number of red edges is at most the number of the cycles of F, i.e., $d_2 + d_3 + d_4 + d_5 + \ldots$ Similarly, the number of green edges does not exceed the number of the cycles of F. Since $r' + g' + b' = m/3$, the expression (10.4) can be estimated from above by

$$\frac{m}{3} + 2r' + g' + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell \le \frac{m}{3} + \sum_{\ell=2}^{\infty} 3d_\ell + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell = \frac{m}{3} + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} + 3 \right\rfloor d_\ell \, . \tag{10.5}$$

Since the 2-factor F contains $2m/3 = 2d_2 + 3d_3 + 4d_4 + \ldots$ edges, the bound (10.5) on the number of edges contained in the constructed cycle cover can be rewritten to

$$\frac{m}{3} + \frac{7 \cdot 2 \cdot m}{4 \cdot 3} + \sum_{\ell=2}^{\infty} \left(\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} \right) d_\ell \le \frac{3m}{2} + \sum_{\ell=2}^{10} \left(\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} \right) d_\ell \, . \tag{10.6}$$

Note that the last inequality follows from the fact that $\lfloor \frac{3\ell}{2} + 3 \rfloor - \frac{7\ell}{4} \le \frac{3\ell}{2} - \frac{7\ell}{4} + 3 = 3 - \frac{\ell}{4} \le 0$ for $\ell \ge 12$ and the expression $\lfloor \frac{3\ell}{2} + 3 \rfloor - \frac{7\ell}{4} = -1/4$ is also non-positive for $\ell = 11$. The estimate (10.6) can be expanded to the following form (recall that $d_3 = 0$):

$$\frac{3m}{2} + \frac{5}{2}d_2 + 2d_4 + \frac{5}{4}d_5 + \frac{3}{2}d_6 + \frac{3}{4}d_7 + d_8 + \frac{1}{4}d_9 + \frac{1}{2}d_{10} \, . \tag{10.7}$$

The length of the shortest cycle cover of G with three cycles exceeds neither the bound given in (10.3) nor the bound given in (10.7). Hence, the length of such cycle cover of G is bounded by any convex combination of the two bounds, in particular, by the following:

$$\frac{5}{7} \cdot \left(\frac{5m}{3} - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11} \right) +$$

$$\frac{2}{7} \cdot \left(\frac{3m}{2} + \frac{5}{2}d_2 + 2d_4 + \frac{5}{4}d_5 + \frac{3}{2}d_6 + \frac{3}{4}d_7 + d_8 + \frac{1}{4}d_9 + \frac{1}{2}d_{10} \right) =$$

$$\frac{34m}{21} - \frac{6}{7}d_4 - \frac{2}{7}d_6 - \frac{6}{7}d_7 - \frac{8}{7}d_8 - \frac{2}{7}d_9 - \frac{4}{7}d_{10} - \frac{15}{14}d_{11} \le \frac{34m}{21} \, .$$

The proof of Theorem 10.6 is now completed.

■

Chapter 11

Short Cycle Covers of Graphs with Minimum Degree Three

In this chapter, we present the proof of the second result on short cycle covers of graphs. In particular, we prove that every bridgeless graph with minimum degree at least three has a cycle cover of total length at most $44m/27 \approx 1.630m$.

11.1 Weighted Variant of the Rainbow Lemma

In Section 10.1 we have proved a generalization of so-called Rainbow Lemma which was useful for the proof of Theorem 10.6. The proof of the main result of this chapter will need a weighted version of the lemma. We present the statement of the weighted variant and its proof in this section. Later, in Section 11.4, we further generalize the argument to exclude certain edge-colorings of the edges not contained in the 2-factor F. However, we think that presenting a less general version of the lemma first will help the reader to follow our arguments later.

As in Chapter 10, a 2-factor F with an edge-coloring satisfying the constraints given in Lemma 10.1 will be called a *rainbow 2-factor*.

The key ingredient for the proof of our modifications of the Rainbow Lemma is the notion of fractional perfect matchings. Let us briefly survey some classical results from this area. The reader is referred to a recent monograph of Schrijver [75] for a more detailed exposition.

A *perfect matching* M of a graph G is the set of edges such that every vertex of G is incident with exactly one edge of M. A perfect matching M can also be viewed as a zero-one vector $u_M \in \{0,1\}^{E(G)}$ such that for each vertex v, the entries of u corresponding to the edges incident v sum to one. A fractional perfect matching is a generalization of this notion: a non-negative vector $u \in \mathbb{R}^{E(G)}$ is said to be a *fractional perfect matching* of the graph G if it can be expressed as a convex combination of vectors u_M corresponding to perfect matchings M of G. The convex polytope formed by all vectors corresponding to fractional perfect matching is called the *perfect matching polytope* of the graph G.

A natural question is whether it is possible to explicitly find the inequalities describing the perfect matching polytope for a graph G. Clearly, all vectors u of the perfect matching polytope have non-negative entries between 0 and 1 (inclusively) and satisfy that the sum of the entries of u corresponding to the edges incident a vertex v sum to one for every vertex v. These two constraints turn out to fully describe the perfect matching polytope if the graph is bipartite [11], however, they are not sufficient for a full description of the perfect matching polytope of non-bipartite graphs. In the general case, the description of the perfect matching polytope is given as follows:

Theorem 11.1 (Edmonds [23]) *Let G be a graph. A vector $u \in \mathbb{R}^{E(G)}$ is contained in the perfect matching polytope of G if and only if:*

- *all the entries of u are between 0 and 1 (inclusively),*
- *the sum of the entries corresponding to the edges incident with a vertex v is equal to one for every vertex v of G, and*
- *the sum of the entries corresponding to the edges with one end-vertex in a subset $V' \subseteq V(G)$ and with the other end-vertex not in V' is at least one for every subset $V' \subseteq V(G)$ of odd cardinality.*

Note that the last condition of Theorem 11.1 applied for $V' = V(G)$ implies that the perfect matching polytope is empty if the number of the vertices of G is odd.

We are now ready to prove a weighted variant of the Rainbow Lemma.

Lemma 11.2 *Let G be a bridgeless cubic graph with edges assigned weights and let w_0 be the total weight of all the edges of G. The graph G contains a rainbow 2-factor F such that the total weight of the edges of F is at least $2w_0/3$ and the 2-factor F contains no circuits of length three.*

Proof: Observe first that Theorem 11.1 implies that the vector $u \in \mathbb{R}^{E(G)}$ with all entries equal to $1/3$ is contained in the perfect matching polytope of G. Hence, there exist perfect matchings $M_1, \ldots, M_k$ of G and coefficients $\alpha_i \in (0, 1]$, $i = 1, \ldots, k$, such that

$$u = \sum_{i=1}^{k} \alpha_i u_{M_i} \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = 1 \ .$$

Let w_i be the sum of the weights of the edges contained in the perfect matching M_i. Since $u = \sum_{i=1}^{k} \alpha_i u_{M_i}$, we conclude that

$$w_0/3 = \sum_{i=1}^{k} \alpha_i w_i \ .$$

Since $w_0/3$ is a convex combination of the weights w_i, there exists an index $i_0 \in \{1, \ldots, k\}$ such that $w_{i_0} \leq w_0/3$. Let F be the complement of M_{i_0}.

Let us now focus on the graph $H = G/F$. Every edge-cut of H corresponds to an edge-cut of G of the same size. In particular, H has no edge-cuts of size one. Assume that H has an edge-cut of size three and let V_1 and V_2 be the vertices of G corresponding to the two parts of H. Since the graph G is cubic and the size of the edge-cut $E(V_1, V_2)$ is odd, both the parts V_1 and V_2 must contain an odd number of vertices of G.

Let $E(V_1, V_2) = \{e_1, e_2, e_3\}$. The sum of the entries of each of the vectors $u_{M_1}, \ldots, u_{M_k}$ corresponding to the edges e_1, e_2 and e_3 is at least one since V_1 contains an odd number of vertices. On the other hand, the sum of the entries of the vector u, which is a convex combination of the vectors $u_{M_1}, \ldots, u_{M_k}$, is equal to one. Hence, the sum of the three entries of each of the vectors $u_{M_1}, \ldots, u_{M_k}$ corresponding to the edges e_1, e_2 and e_3 must also be equal to one. In particular, M_{i_0} contains exactly one of the edges e_1, e_2 and e_3 which is impossible since $\{e_1, e_2, e_3\} \subseteq M_{i_0}$. We conclude that H has no edge-cuts of size one or three. This also implies that F has no circuits of length three.

Theorem 10.2 yields that H has a nowhere-zero 4-flow. Fix a nowhere-zero flow $\varphi : E(H) \to \mathbb{Z}_2^2$. The edges of $\varphi^{-1}(01)$ are colored with red, the edges of $\varphi^{-1}(10)$ with green and the edges of $\varphi^{-1}(11)$ with blue. Since φ is a $\mathbb{Z}_2^2$-flow of H, a vertex of H of odd degree is incident with an odd number of red edges, an odd number of green edges and an odd number of blue edges (counting loops twice). Similarly, the vertices of H of even degree are incident with an even number of red edges, green edges and blue edges. Since the weight of the edges of M_{i_0} is at most $w_0/3$, the statement of the lemma follows.

■

11.2 Intermezzo

In Section 10.2, we presented another proof of the result of Alon and Tarsi [7] and Bermond, Jackson, and Jaeger [10] that every bridgeless cubic graph with m edges has a cycle cover of length at most $5m/3$. In this section, we show an alternative proof of the same result for general graphs. The reason for that is to demonstrate the technique we use later in the proof of the main result of this chapter. Actually, the proof of Theorem 10.5 is a simplified version of the proof we are about to present. In the rest of the chapter, we refine the arguments presented in this section to obtain an improved bound for graphs with minimum degree three.

Theorem 11.3 *Let G be a bridgeless graph with m edges. G has a cycle cover of length at most $5m/3$.*

Proof: If G has a vertex v of degree four or more, then, by Lemma 9.1, v has two neighbors v_1 and v_2 such that the graph $G \bullet v_1vv_2$ is also bridgeless. Let G' be the graph $G \bullet v_1vv_2$. The number of edges of G' is the same as the number of edges of G and every cycle of G' corresponds to a cycle of G of the same length. Hence, a cycle cover of G' corresponds to a cycle cover of G of the same length. Through this process we can reduce any bridgeless graph to a bridgeless graph with maximum degree three. In particular, we can assume without loss of generality that the graph G has maximum degree three and G is connected (otherwise, cover each component separately).

If G is a circuit, the statement is trivial. Otherwise, we proceed as described in the rest. First, we suppress all vertices of degree two in G. Let G_0 be the resulting cubic (bridgeless) graph. We next assign each edge e of G_0 the weight equal to the number of edges in the path corresponding to e in G. In particular, the total weight of the edges of G_0 is equal to m. Let F_0 be a rainbow 2-factor with the properties described in Lemma 11.2.

The 2-factor F_0 corresponds to a set F of disjoint circuits of the graph G which do not necessarily cover all the vertices of G. Let w_F be the weight of the edges contained in the 2-factor F_0, and r, g and b the weight of red, green and blue edges, respectively. By symmetry, we can assume that $r \leq g \leq b$. Since the weight w_F of the edges contained in the 2-factor F_0 is at least $2m/3$, the sum $r + g + b$ is at most $m/3$. Finally, let $\mathcal{R}$ be the set of edges of G corresponding to red edges of G_0, $\mathcal{G}$ the set of edges corresponding to green edges, and $\mathcal{B}$ the set of edges corresponding to blue edges. By the choice of edge-weights, the cardinality of $\mathcal{R}$ is r, the cardinality of $\mathcal{G}$ is g and the cardinality of $\mathcal{B}$ is b.

For a circuit C contained in F and for a set of edges of E such that $C \cap E = \emptyset$, we define $C(E)$ to be the set of vertices of C incident with the edges of E. If $C(E)$ has even cardinality, it is possible to partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that

- each vertex of $C(E)$ is incident with one edge of $C(E)^A$ and one edge of $C(E)^B$, and
- each vertex of C not contained in $C(E)$ is incident with either two edges of $C(E)^A$ or two edges of $C(E)^B$.

Note that if $C(E) = \emptyset$, then $C(E)^A$ contains no edges of C and $C(E)^B$ contains all the edges of C (or vice versa). We will always assume that the number of edges of $C(E)^A$ does not exceed the number of edges of $C(E)^B$, i.e., $|C(E)^A| \leq |C(E)^B|$.

The desired cycle cover of G which is comprised of three cycles can now be defined. The first cycle $\mathcal{C}_1$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^A$ for all circuits C of the 2-factor F. The second cycle $\mathcal{C}_2$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^B$ for all circuits C of F. Finally, the third cycle $\mathcal{C}_3$ contains all the red and blue edges and the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ for all circuits C of F.

Let us first verify that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ cover the edges of G. Clearly, every edge not contained in F, i.e., a red, green or blue edge, is covered by at least one of the cycles. On the other hand, every edge of F is contained either in the cycle $\mathcal{C}_1$ or the cycle $\mathcal{C}_2$. Hence, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form a cycle cover of G.

It remains to estimate the lengths of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each edge of F is covered once by the cycles $\mathcal{C}_1$ and $\mathcal{C}_2$; since $|C(E)^A| \le |C(E)^B|$ for every circuit C of F, at most half of the edges of F is also covered by the cycle $\mathcal{C}_3$. We conclude that the total length of the constructed cycle cover is at most:

$$3r + 2g + b + |F| + |F|/2 \le 2(r+g+b) + 3w_F/2 =$$

$$3(r+g+b+w_F)/2 + (r+g+b)/2 \le 3m/2 + m/6 = 5m/3 \ .$$

This finishes the proof of the theorem.

■

11.3 Special Types of $\mathbb{Z}_2^2$-flows

As mentioned before, we need a modification of the Rainbow Lemma excluding certain edge-colorings of the graph $H = G/F_0$. Some of the "bad" edge-colorings will be excluded by vertex splitting introduced in Section 9.2. However, vertex splitting itself is not sufficient to exclude all bad edge-colorings. In this section, we establish an auxiliary lemma that guarantees the existence of a special nowhere-zero $\mathbb{Z}_2^2$-flow.

Lemma 11.4 *Let G be a bridgeless graph admitting a nowhere-zero $\mathbb{Z}_2^2$-flow. Assume that*

- *for every vertex v of degree five, there are given two multisets A_v and B_v of three edges incident with v such that $|A_v \cap B_v| = 2$ (loops can appear twice in the same set), and*
- *for every vertex v of degree six, the incident edges are partitioned into three multisets A_v, B_v and C_v of size two each (loops appear twice in these sets).*

The graph G has a nowhere-zero $\mathbb{Z}_2^2$-flow φ such that

- *for every vertex v of degree five, the flow φ is constant on neither of the sets A_v and B_v, and*
- *for every vertex v of degree six, the edges incident with v have all the three possible flow values, or the flow φ is constant on A_v, or it is constant on B_v, or it is not constant on C_v.*

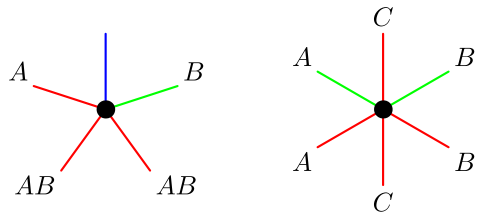

Figure 11.1: Bad vertices of degree five and six (symmetric cases are omitted). The letters indicate edges contained in the sets A, B and C.

Proof: By Theorem 10.2, G has a nowhere-zero $\mathbb{Z}_2^2$-flow φ. For simplicity, we refer to edges with the flow value 01 red, 10 green and 11 blue. Note that each vertex of odd degree is incident with odd numbers of red, green and blue edges and each vertex of even degree is incident with even numbers of red, green and blue edges (counting loops twice). We say that a vertex v of degree five is *bad* if φ is constant on A_v or on B_v, and it is *good*, otherwise. Similarly, a vertex v of degree six is *bad* if φ has only two possible flow values at v and it is not constant on A_v and on B_v and is constant on C_v; otherwise, v is good. Choose a $\mathbb{Z}_2^2$-flow φ of H with the least number of bad vertices. If there are no bad vertices, then there is nothing to prove. Assume that there is a bad vertex v.

Let us first analyze the case that the degree of v is five. Let $e_1, \ldots, e_5$ be the edges incident with v. By symmetry, we can assume that $A_v = \{e_1, e_2, e_3\}$, $B_v = \{e_2, e_3, e_4\}$, the edges e_1, e_2 and e_3 are red, the edge e_4 is green and the edge e_5 is blue (see Figure 11.1). We now define a closed trail W in H formed by red and blue edges. The first edge of W is e_1.

Let $f = ww'$ be the last edge of W defined so far. If $w' = v$, then f is one of the edges e_2, e_3 and e_5 and the definition of W is finished. Assume that $w' \neq v$. If w' is not a vertex of degree five or six or w' is a bad vertex, add to the trail W any red or blue edge incident with w' that is not already contained in W.

If w' is a good vertex of degree five, let $f_1, \ldots, f_5$ be the edges incident with w', $A_{w'} = \{f_1, f_2, f_3\}$ and $B_{w'} = \{f_2, f_3, f_4\}$. If w' is incident with a single red and a single blue edge, leave w' through the other edge that is red or blue. Otherwise, there are three red edges and one blue edge or vice versa. The next edge f' of the trail W is determined as follows (note that the role of red and blue can be swapped):

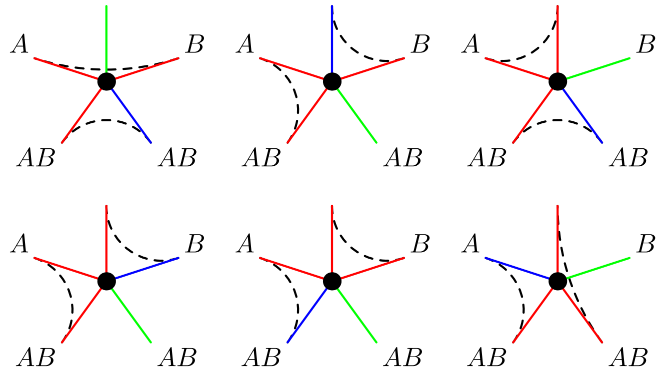

Figure 11.2: Routing the trail W (indicated by dashed edges) through a good vertex of degree five with three red edges. The letters indicate edges contained in the sets A and B. Symmetric cases are omitted.

Red edges	Blue edge	$f = f_1$	$f = f_2$	$f = f_3$	$f = f_4$	$f = f_5$
f_1, f_2, f_4	f_3	$f' = f_4$	$f' = f_3$	$f' = f_2$	$f' = f_1$	N/A
f_1, f_2, f_4	f_5	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_1, f_2, f_5	f_3	$f' = f_5$	$f' = f_3$	$f' = f_2$	N/A	$f' = f_1$
f_1, f_2, f_5	f_4	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_1, f_4, f_5	f_2	$f' = f_2$	$f' = f_1$	N/A	$f' = f_5$	$f' = f_4$
f_2, f_3, f_5	f_1	$f' = f_2$	$f' = f_1$	$f' = f_5$	N/A	$f' = f_3$

See Figure 11.2 for an illustration of these rules.

If w' is a good vertex of degree six, proceed as follows. If φ is constant on $A_{w'}$ and $f \in A_{w'}$, let the next edge f' of W be the other edge contained in $A_{w'}$; if φ is constant on $A_{w'}$ and $f \notin A_{w'}$, let f' be any red or blue edge not contained in $A_{w'}$ or in W. A symmetric rule applies if φ is constant on $B_{w'}$, i.e., f' is the other edge of $B_{w'}$ if $f \in B_{w'}$ and f' is a red or blue edge not contained in $B_{w'}$ or W, otherwise.

If φ is not constant on $C_{w'}$ and $f \in C_{w'}$ and the other edge of $C_{w'}$ is red or blue, set f' to be the other edge of $C_{w'}$; if $f \in C_{w'}$ and the other edge of $C_{w'}$ is green, choose f' to be any red or blue edge incident with w' that is not contained in W. If $f \notin C_{w'}$ (and φ is not constant on $C_{w'}$), choose f' to be a red or blue edge incident with w' not contained in W that is also not contained in $C_{w'}$. If such an edge does not exist, choose f' to be the red or blue edge contained in $C_{w'}$ (note that the other edge of $C_{w'}$ is green since w' is incident with an even number of red, green and blue edges).

It remains to consider the case that w' is incident with two edges of each color and is constant on $C_{w'}$ and neither of $A_{w'}$ and $B_{w'}$. If f is blue, set f' to be any red edge incident with w' not contained in W and if f is red, set f to be any such blue edge. See Figure 11.3 for an illustration of these rules.

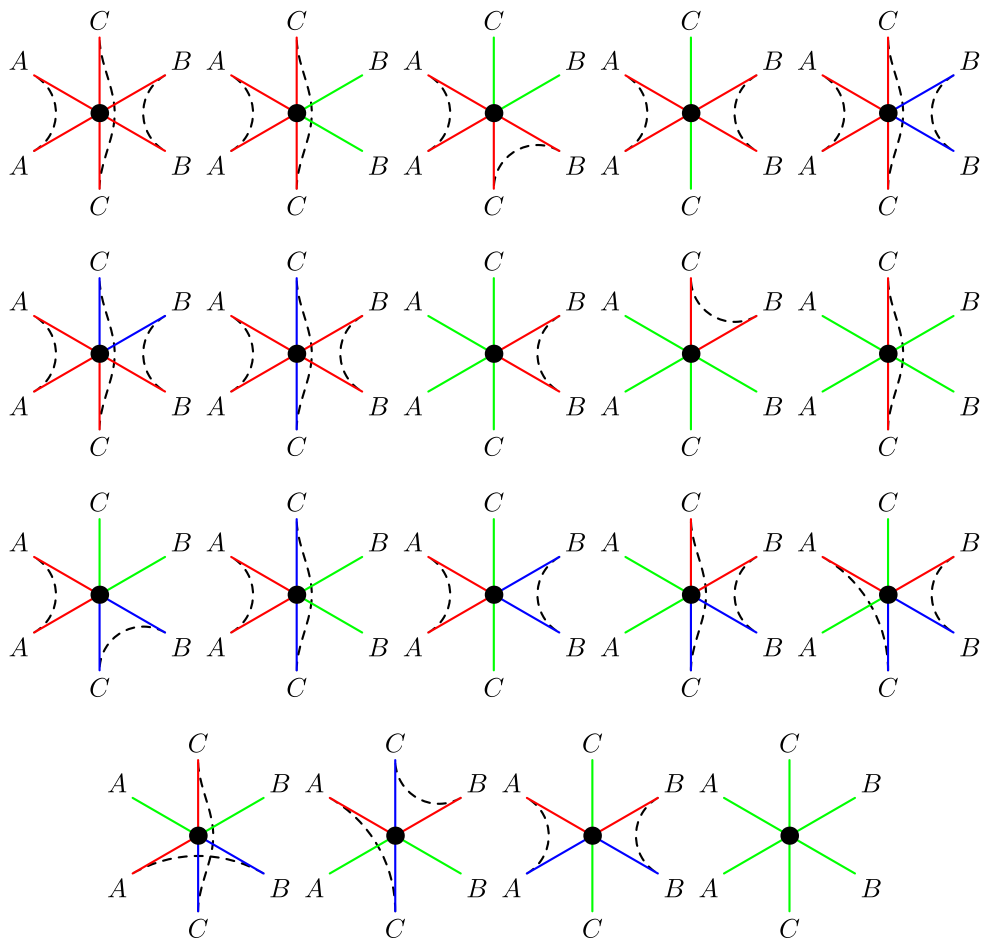

Figure 11.3: Routing the trail W (indicated by dashed edges) through a good vertex of degree six. The letters indicate edges contained in the sets A, B and C. Symmetric cases are omitted.

The definition of the trail W is now finished. Let us swap the red and blue colors on W. It is straightforward to verify that all good vertices remain good and the vertex v become good (see Figures 11.1–11.3). In particular, the number of bad vertices is decreased which contradicts the choice of φ.

Let us now analyze the case that there is a bad vertex v of degree six, i.e., the colors of the edges of A_v are distinct, the colors of the edges of B_v are distinct and the colors of the edges of C_v are the same and not all the flow values are present at the vertex v (see Figure 11.1). By symmetry, we can assume that the two edges of A_v are red and green, the two edges of B_v are also red and green, and the two edges of C_v are both red (recall that the vertex v is incident with even numbers of red, green and blue edges). As in the case of vertices of degree five, we find a trail formed by red and blue edges and swap the colors of the edges on the trail. The first edge of the trail is any red edge incident with v and the trail W is finished when it reaches again the vertex v. After swapping red and blue colors on the trail W, the vertex v is incident with two edges of each of the three colors. Again, the number of bad vertices has been decreased which contradicts our choice of the flow φ.

■

11.4 Rainbow Lemma Revisited

In this section, we establish a modification of the Rainbow Lemma from Section 11.1. In addition to the statement of Lemma 11.2, we exclude certain edge-colorings of edges incident with short circuits of the chosen 2-factor. Let us be more precise. If $C = v_1 \dots v_k$ is a circuit of a cubic graph and e_i the edge incident with v_i not contained in C, then the pattern of C is a k-tuple $X_1 \dots X_k$ where X_i is R if the color of e_i is red, G if it is green, and B if it is blue. A pattern P is *compatible* with a pattern P' if P' can be obtained from P by a permutation of the red, green and blue colors followed by replacement of some of the colors with the letter x (which represents a wild-card). For example, the pattern RGRGBBGG is compatible with RBRxxxBx.

We can now state and prove the modification of the Rainbow Lemma.

Lemma 11.5 *Let G be a bridgeless cubic graph with edges assigned non-negative integer weights and w_0 be the total weight of the edges. In addition, suppose that no two edges with weight zero have a vertex in common. The graph G contains a rainbow 2-factor F such that the total weight of the edges of F is at most $2w_0/3$. Moreover, the patterns of circuits with four edges of weight one are restricted as follows. Every circuit $C = v_1 \dots v_k$ of F that consists of four edges of weight one and at most four edges of weight zero (and no other edges) has a pattern:*

- *compatible with RRxx or xRRx if C has no edges of weight zero (and thus $k = 4$),*

- *compatible with RxGxx or RRRGB if the only edge of C of weight zero is v_4v_5 (and thus $k = 5$),*

- *compatible with xxRRxx, xxxxRR, xxRGGR or xRxGGR if the only edges of C of weight zero are v_3v_4 and v_5v_6 (and thus $k = 6$),*

- *not compatible with RRGRRG, RRGRGR, RGRRRG or RGRRGR if the only edges of C of weight zero are v_2v_3 and v_5v_6 (and thus $k = 6$),*

- *compatible with xRRxxxx, xxxRRxx, xxxxxRR, xRGxxRB, xRGxxBR, xRGxxGB, xRGxxBG, xxxRGRG or xxxRGGR if the only edges of C of weight zero are v_2v_3, v_4v_5 and v_6v_7 (and thus $k = 7$), and*

- *compatible with RRxxxxxx, xxRRxxxx, xxxxRRxx, xxxxxxRR, RGGRxxxx, xxRGGRxx, xxxxRGGR or GRxxxxRG if the edges v_1v_2, v_3v_4, v_5v_6 and v_7v_8 of C have weight zero (and thus $k = 8$).*

Proof: As in the proof of Lemma 11.2, we first find a perfect matching M with weight at least $w_0/3$ such that the graph $H = G/F$ has no edge-cuts of size one or three where F is the 2-factor of G complementary to M. Note that in the proof of Lemma 11.2, we found a matching M with weight at most $w_0/3$ but the same argument also yields the existence of a matching M with weight at least $w_0/3$.

Next, we modify the graph $H = G/F$ in such a way that an application of Lemma 11.4 will yield a $\mathbb{Z}_2^2$-flow that yields an edge-coloring satisfying the conditions from the statement of the lemma. Let w be a vertex of H corresponding to a circuit $v_1 \dots v_k$ of F consisting of four edges with weight one and some edges with weight zero, and let e_i be the edge of M incident with v_i. Finally, let w_i be the neighbor of w in H that corresponds to the circuit containing the other end-vertex of the edge e_i. The graph H is modified as follows (see Figure 11.4):

- if $k = 4$, split the pair w_1 and w_2 or the pair w_2 and w_3 from w in such a way that the resulting graph has no edge-cuts of size one or three (at least one of the two splittings works by Lemma 9.9).

- if $k = 5$ and the weight of the edge v_4v_5 is zero, set $A_w = \{e_1, e_3, e_5\}$ and $B_w = \{e_1, e_3, e_4\}$.

- if $k = 6$ and the weights of the edges v_3v_4 and v_5v_6 are zero, split the pair w_3 and w_4, w_4 and w_5, or w_5 and w_6 from w without creating edge-cuts of size one or three (one of the splitting works by Lemma 9.10). If the pair w_4 and w_5 is split off, split further the pair w_2 and w_6, or the pair w_3 and w_6 from w again without creating edge-cuts of size one or three (one of the splitting works by Lemma 9.9).

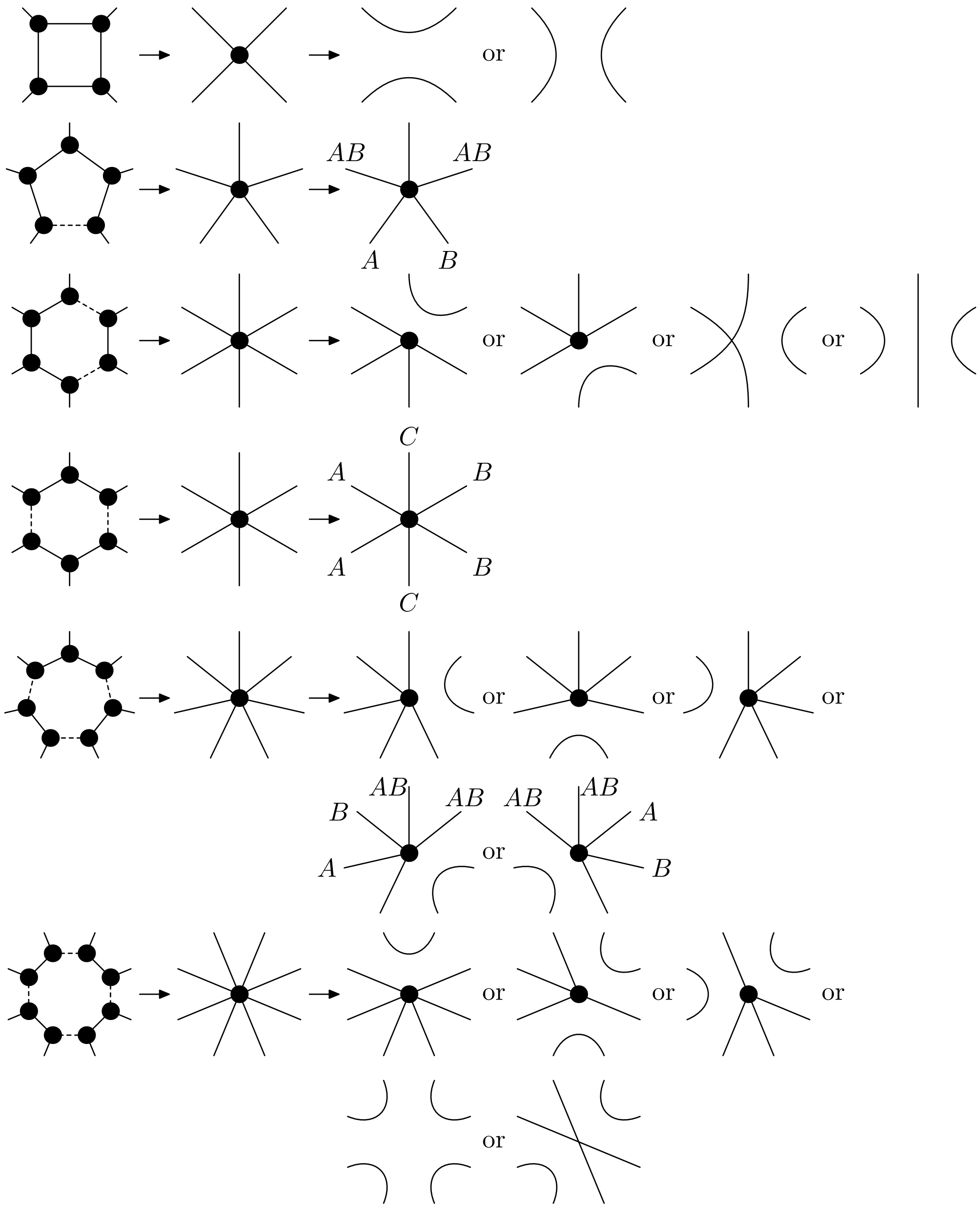

Figure 11.4: Modifications of the graph H performed in the proof of Lemma 11.4. The edges of weight one are solid and the edges of weight zero are dashed. The sets A_w, B_w and C_w are indicated by letters near the edges. Vertices of degree two obtained through splittings are not depicted and some symmetric cases are omitted in the case of a circuit of length eight.

- if $k = 6$ and the weights of the edges v_2v_3 and v_5v_6 are zero, set $A_w = \{e_2, e_3\}$, $B_w = \{e_5, e_6\}$ and $C_w = \{e_1, e_4\}$.

- if $k = 7$ and the weights of the edges v_2v_3, v_4v_5 and v_6v_7 are zero, split one of the pairs w_i and w_{i+1} from w for $i \in \{2, 3, 4, 5, 6\}$ (the existence of such a splitting is guaranteed by Lemma 9.11). If w_3 and w_4 is split off, set $A_w = \{e_1, e_2, e_6\}$ and $B_w = \{e_1, e_2, e_7\}$. If w_5 and w_6 is split off, set $A_w = \{e_1, e_2, e_7\}$ and $B_w = \{e_1, e_3, e_7\}$.

- if $k = 8$ and the weights of the edges v_1v_2, v_3v_4, v_5v_6 and v_7v_8 are equal to zero, split one of the pairs w_i and w_{i+1} from w for some $i \in \{1, \ldots, 8\}$ (indices taken modulo eight) without creating edge-cuts of size one or three. This is possible by Lemma 9.11. If i is odd, then there are no further modifications to be performed. If i is even, one of the pairs w_{i+3} and w_{i+4}, w_{i+4} and w_{i+5}, and w_{i+5} and w_{i+6} is further split off from the vertex w in such a way that no edge-cuts of size one or three are created (one of the splittings has this property by Lemma 9.10). In case that the vertices w_{i+4} and w_{i+5} are split off, split further the pair of vertices w_{i+2} and w_{i+3} or the pair of vertices w_{i+3} and w_{i+6}, again, without creating edge-cuts of size one or three (and do not split off other pairs of vertices in the other cases). Lemma 9.9 guarantees that one of the two splittings work.

Fix a nowhere-zero $\mathbb{Z}_2^2$-flow φ with the properties described in Lemma 11.4 with respect to the sets A_w, B_w and C_w as defined before (and where the sets A_w, B_w and C_w are undefined, choose them arbitrarily). The edges of $\varphi^{-1}(01)$ are colored with red, the edges of $\varphi^{-1}(10)$ with green and the edges of $\varphi^{-1}(11)$ with blue as in the proof of Lemma 11.2. This defines the coloring of the edges of G not contained in F.

Clearly, F is a rainbow 2-factor. It remains to verify that the patterns of circuits with four edges of weight one are as described in the statement of the lemma. Let $C = v_1 \ldots v_k$ be a circuit of F consisting of four edges with weight one and some edges with weight zero, and let c_i be the color of the edge of M incident with v_i. We distinguish six cases based on the value of k and the position of zero-weight edges (symmetric cases are omitted):

- if $k = 4$, then all the edges of C have weight one. By the modification of H, it holds that $c_1 = c_2$ or $c_2 = c_3$. Hence, the pattern of C is compatible with RRxx or xRRx.

- if $k = 5$ and the weight of v_4v_5 is zero, then either $c_1 \neq c_3$, or $c_1 = c_3 \notin \{c_4, c_5\}$. Since C is incident with an odd number of edges of each color, its pattern is compatible with RxGxx or RRRGB.

- if $k = 6$ and the weights of v_3v_4 and v_5v_6 are zero, then $c_3 = c_4$, or $c_4 = c_5$ and $c_2 = c_6$, or $c_4 = c_5$ and $c_3 = c_6$, or $c_5 = c_6$. Hence, the pattern of C

is compatible with xxRRxx, xRxRRR or xRxGGR, xxRRRR or xxRGGR, or xxxxRR.

- if $k = 6$ and the weights of v_2v_3 and v_5v_6 are zero, then the pattern of C contains all three possible colors or it is compatible with xRRxxx, xxxxRR or RxxGxx. In particular, it is not compatible with any of the patterns listed in the statement of the lemma.

- if $k = 7$ and the weights of v_2v_3, v_4v_5 and v_6v_7 are zero, then $c_i = c_{i+1}$ for some $i \in \{2, 3, 4, 5, 6\}$ by the modification of H. If i is even, then the pattern of C is compatible with xRRxxxx, xxxRRxx or xxxxxRR. If $i = 3$, then $c_1 \neq c_2$ or $c_1 = c_2 \notin \{c_6, c_7\}$. Hence, the pattern of C is compatible with RGRRxxx, RGBBxxx, RRGGxGB or RRGGxBG (unless $c_2 = c_3$). Since C is incident with an odd number of edges of each colors, its pattern is compatible with one of the patterns listed in the statement of the lemma. A symmetric argument applies if $i = 5$ and either $c_1 \neq c_7$ or $c_1 = c_7 \notin \{c_2, c_3\}$.

- if $k = 8$ and the weights v_iv_{i+1}, $i = 1, 3, 5, 7$, then $c_i = c_{i+1}$ for $i \in \{1, \ldots, 8\}$ by the modification of H. If there is such odd i, the pattern of C is compatible with RRxxxxxx, xxRRxxxx, xxxxRRxx or xxxxxxRR. Otherwise, at least one of the following holds for some even i: $c_{i+3} = c_{i+4}$, $c_{i+4} = c_{i+5}$ or $c_{i+5} = c_{i+6}$. In the first and the last case, the pattern is again compatible with RRxxxxxx, xxRRxxxx, xxxxRRxx or xxxxxxRR. If $c_{i+4} = c_{i+5}$, then $c_{i+2} = c_{i+3}$ or $c_{i+3} = c_{i+6}$. Hence, the pattern of C is compatible with xR-RGGRRx, xRRGGBBx, xRRxRGGR, xRRxGRRG, xRRxGBBG or one of the patterns rotated by two, four or six positions. All these patterns are listed in the statement of the lemma.

■

11.5 Reducing Parallel Edges

In this section, we show that it is enough to prove our main theorem for graphs that do not contain parallel edges of certain type. We state and prove four auxiliary lemmas that simplify our arguments presented in Section 11.6. The first two lemmas deal with the cases when there is a vertex incident only with parallel edges leading to the same vertex.

Lemma 11.6 *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 3$ parallel edges. If the degree of v_1 is k, the degree of v_2 is at least $k + 3$ and the graph $G' = G \setminus v_1$ has a cycle cover with three cycles of length at most $44(m - k)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof: Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k)/27$ covering the edges of G' and $e_1,\ldots,e_k$ the k parallel edges between the vertices v_1 and v_2. If k is even, add the edges $e_1,\ldots,e_k$ to $\mathcal{C}_1$. If k is odd, add the edges $e_1,\ldots,e_{k-1}$ to $\mathcal{C}_1$ and the edges e_{k-1} and e_k to $\mathcal{C}_2$. Clearly, we have obtained a cycle cover of G with three cycles. The length of the cycles is increased at most by $k+1$ and thus it is at most

$$\frac{44m-44k}{27}+k+1=\frac{44m-17k+27}{27}\leq\frac{44m}{27}\ .$$

■

Lemma 11.7 *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k\geq 4$ parallel edges. If the degree of v_1 is k, the degree of v_2 is $k+2$ and the graph G' obtained from G by removing all the edges between v_1 and v_2 and suppressing the vertex v_2 has a cycle cover with three cycles of length at most $44(m-k-1)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof: Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k-1)/27$ covering the edges of G' and $e_1,\ldots,e_k$ the k parallel edges between the vertices v_1 and v_2. Let v' and v'' be the two neighbors of v_2 distinct from v_1. Note that it can hold that $v'=v''$. The edges v_2v' and v_2v'' are included in those cycles $\mathcal{C}_i$ that contain the edge $v'v''$. The edges $e_1,\ldots,e_{k-1}$ are included to $\mathcal{C}_1$. In addition, the edge e_k is included to $\mathcal{C}_1$ if k is even. Otherwise, the edges e_{k-1} and e_k are included to $\mathcal{C}_2$.

The length of all the cycles is increased by at most $3+k+1=k+4$. Hence, the total length of the cycle cover is at most

$$\frac{44m-44k-44}{27}+k+4=\frac{44m-17k+64}{27}\leq\frac{44m}{27}\ .$$

■

In the next two lemmas, we deal with the case that each of the two vertices joined by several parallel edges is also incident with another vertex.

Lemma 11.8 *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k\geq 2$ parallel edges. If the degree of v_1 is at least $k+1$, the degree of v_2 is at least $k+2$ and the graph G' obtained by contracting all the edges between v_1 and v_2 has a cycle cover with three cycles of length at most $44(m-k)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof: Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k)/27$ covering the edges of G' and $e_1, \ldots, e_k$ the k parallel edges between the vertices v_1 and v_2. By symmetry, we can assume that the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_{i_0}$ contain an odd number of edges incident with v_1 and the cycles $\mathcal{C}_{i_0+1}, \ldots, \mathcal{C}_3$ contain an even number of such edges for some $i_0 \in \{0, 1, 2, 3\}$. Since $\mathcal{C}_1, \ldots, \mathcal{C}_3$ form a cycle cover of G', if v_1 is incident with an odd number of edges of $\mathcal{C}_i$, $i = 1, 2, 3$, then v_2 is incident with an odd number of edges of $\mathcal{C}_i$ and vice versa.

The edges are added to the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ as follows based on the value of i_0 and the parity of k:

i_0	k	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
0	odd	$e_1, \ldots, e_{k-1}$	e_{k-1}, e_k	
0	even	$e_1, \ldots, e_k$		
1	odd	$e_1, \ldots, e_k$		
1	even	$e_1, \ldots, e_{k-1}$	e_{k-1}, e_k	
2	odd	$e_1, \ldots, e_k$	e_k	
2	even	$e_1, \ldots, e_{k-1}$	e_k	
3	odd	$e_1, \ldots, e_{k-2}$	e_{k-1}	e_k
3	even	$e_1, \ldots, e_{k-1}$	e_{k-1}	e_k

Clearly, we have obtained a cycle cover of G with three cycles. The length of the cycles is increased at most by $k+1$ and thus it is at most

$$\frac{44m - 44k}{27} + k + 1 = \frac{44m - 17k + 27}{27} \leq \frac{44m}{27} .$$

■

Lemma 11.9 *Let G be an m-edge bridgeless graph with vertices v_1 and v_2 joined by $k \geq 3$ parallel edges. If the degrees of v_1 and v_2 are $k+1$ and the graph G' obtained by contracting all the edges between v_1 and v_2 and suppressing the resulting vertex of degree two has a cycle cover with three cycles of length at most $44(m-k-1)/27$, then G has a cycle cover with three cycles of length at most $44m/27$.*

Proof: Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the cycles of total length at most $44(m-k-1)/27$ covering the edges of G', let $e_1, \ldots, e_k$ be the k parallel edges between the vertices v_1 and v_2, and let v'_i be the other neighbor of v_i, $i = 1, 2$. Add the edges incident with $v_1v'_1$ and $v_2v'_2$ to those cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ that contain the edge $v'_1v'_2$ and then proceed as in the proof of Lemma 11.8. The length of the cycles is increased by at most $3 + k + 1 = k + 4$ and thus it is at most

$$\frac{44m - 44k - 44}{27} + k + 4 = \frac{44m - 17k + 64}{27} \leq \frac{44m}{27}$$

where the last inequality holds unless $k = 3$. If $k = 3$ and the edge $v_1'v_2'$ is contained in at most two of the cycles, the length is increased by at most $2+k+1 = k+3 = 6$. If $k = 3$ and the edge $v_1'v_2'$ is contained in three of the cycles, each of the parallel edges is added to exactly one of the cycles and thus the length is increased by at most $3+k = 6$. In both cases, the length of the new cycle cover can be estimated as follows:

$$\frac{44m - 44 \cdot 3 - 44}{27} + 6 = \frac{44m - 14}{27} \le \frac{44m}{27} \, .$$

■

11.6 Main Result

We are now ready to prove the main result of this chapter.

Theorem 11.10 *Let G be a bridgeless graph with m edges and with minimum degree three or more. The graph G has a cycle cover of total length at most $44m/27$ that is comprised of at most three cycles.*

Proof: By Lemmas 11.6–11.9, we can assume without loss of generality that if vertices v_1 and v_2 of G are joined by k parallel edges, then either $k = 2$ and the degrees of both v_1 and v_2 are equal to $k+1 = 3$, or $k = 3$, the degree of v_1 is $k = 3$ and the degree of v_2 is $k+2 = 5$ (in particular, both v_1 and v_2 have odd degrees). Note that the graphs G' from the statement of Lemmas 11.6–11.9 are also bridgeless graphs with minimum degree three and have fewer edges than G which implies that the reduction process described in Lemmas 11.6–11.9 eventually finishes.

Let us now proceed with the proof under the assumption that the only parallel edges contained in G are pairs of edges between two vertices of degree three and triples of edges between a vertex of degree three and a vertex of degree five. As the first step, we modify the graph G into bridgeless graphs $G_1, G_2, \ldots$ eventually obtaining a bridgeless graph G' with vertices of degree two, three and four. Set $G_1 = G$. If G_i has no vertices of degree five or more, let $G' = G_i$. If G_i has a vertex v of degree five or more, then Lemma 9.1 yields that there are two neighbors v_1 and v_2 of v such that the graph $G_i \bullet v_1vv_2$ is also bridgeless. We set G_{i+1} to be the graph $G_i \bullet v_1vv_2$. We continue while the graph G_i has vertices of degree five or more. Clearly, the final graph G' has the same number of edges as the graph G and every cycle of G' corresponds to a cycle of G.

Next, each edge of G' is assigned weight one, each vertex of degree four is expanded to two vertices of degree three as described in Lemma 9.13 and the edge between the two new vertices of degree three is assigned weight zero (note that the vertex splitting preserves the parity of the degree of the split vertex and thus no vertex of degree four is incident with parallel edges). The resulting graph is denoted by G_0. Note that every cycle C of G_0 corresponds to a cycle C' of G and the length of C' in G is equal to the sum of the weights of the edges of C. Next, the vertices of degree two in G_0 are suppressed and each edge e is assigned the weight equal to the sum of the weights of edges of the path of G_0 corresponding to e. The resulting graph is denoted by G_0'. Clearly, G_0' is a cubic bridgeless graph. Also note that all the edges of weight zero in G_0 are also contained in G_0' and no vertex of G_0' is incident with two edges of weight zero. Finally, observe that the total weight of the edges of G_0' is equal to m.

We apply Lemma 11.5 to the cubic graph G_0'. Let F_0' be the rainbow 2-factor of G_0' and let F_0 be the cycle of G_0 corresponding to the 2-factor of F_0'. Note that F_0 is a union of disjoint circuits. Let $\mathcal{R}_0$, $\mathcal{G}_0$ and $\mathcal{B}_0$ be the sets of edges of G_0 contained in paths corresponding to red, green and blue edges in G_0'. Let r_0 be the weight of the red edges in G_0, g_0 the weight of green edges and b_0 the weight of blue edges. Lemma 11.5 yields $r_0 + g_0 + b_0 \geq m/3$.

We construct two different cycle covers, each comprised of three cycles, and eventually combine the bounds on their lengths to obtain the bound claimed in the statement of the theorem.

The first cycle cover. The first cycle cover that we construct is a cycle cover of the graph G_0 (which yields a cycle cover of G of the same length as explained earlier). Let d_ℓ be the number of circuits of F_0 of weight ℓ. Note that d_3 can be non-zero since a circuit of weight three need not have length three in G_0. The cycle $\mathcal{C}_1$ contains all the red and green edges, i.e., the edges contained in $\mathcal{R}_0 \cup \mathcal{G}_0$, the cycle $\mathcal{C}_2$ contains the red and blue edges and the cycle $\mathcal{C}_3$ contains the green and blue edges. Recall now the notation $C(E)^A$ and $C(E)^B$ used in the proof of Theorem 11.3 for circuits C and set E of edges that are incident with even number of vertices of C. In addition, $C(E)^A_*$ denotes the edges of $C(E)^A$ with weight one and $C(E)^B_*$ denotes such edges of $C(E)^B$. In the rest of the construction of the first cycle cover, we always assume that $|C(E)^A_*| \leq |C(E)^B_*|$. The sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are completed to cycles in a way similar to that used in the proof of Theorem 11.3.

For a circuit C of F_0, the edges of $C^1 = C(\mathcal{R}_0 \cup \mathcal{G}_0)^A$ are added to the cycle $\mathcal{C}_1$. The edges C^2 added to $\mathcal{C}_2$ are either the edges of $C(\mathcal{R}_0 \cup \mathcal{B}_0)^A$ or $C(\mathcal{R}_0 \cup \mathcal{B}_0)^B$—we choose the set with fewer edges with weight one in common with $C^1 = C(\mathcal{R}_0 \cup \mathcal{G}_0)^A$. Finally, the edges added to $\mathcal{C}_3$ are chosen so that every edge of C is covered an odd number of times; explicitly, the edges $C^3 = C^1 \triangle C^2 \triangle C$ are added to $\mathcal{C}_3$. Note that C^3 is either $C(\mathcal{G}_0 \cup \mathcal{B}_0)^A$ or $C(\mathcal{G}_0 \cup \mathcal{B}_0)^B$. In particular, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form cycles.

We now estimate the number of the edges of C of weight one contained in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Let C_* be the edges of weight one contained in the circuit C, $\ell = |C_*|$ and $C^i_* = C^i \cap C_*$ for $i = 1, 2, 3$. By the choice of C^2, the number of edges of weight one in $C^1 \cap C^2$ is $|C^1_* \cap C^2_*| \leq |C^1_*|/2$. Consequently, the number of edges of C of weight one contained in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is:

$$\begin{aligned} |C^1_*| + |C^2_*| + |C^1_* \triangle C^2_* \triangle C_*| &= \\ |C^1_* \cup C^2_*| + |C^1_* \cap C^2_*| + |C_* \setminus (C^1_* \cup C^2_*)| + |C^1_* \cap C^2_*| &= \\ |C_*| + 2|C^1_* \cap C^2_*| \,. \end{aligned}$$

Since $|C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*| \le |C(\mathcal{R}_0 \cup \mathcal{G}_0)^B_*|$, the number of edges contained in the set $C^1_* = C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*$ is at most $\ell/2$. By the choice of C^2, $|C^1_* \cap C^2_*| \le |C^1_*|/2$. Consequently, it holds that

$$|C^1_* \cap C^2_*| \le |C^1_*|/2 \le \ell/4 \tag{11.1}$$

and eventually conclude that the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most $\ell + 2\lfloor \ell/4 \rfloor$ edges of the circuit C with weight one.

If $\ell = 4$, the estimate given in (11.1) can be further refined. Let C' be the circuit of G' corresponding to C. Clearly, C' is a circuit of length four. Color a vertex v of the circuit C'

red if v has degree three and is incident with a red edge, or v has degree four and is incident with green and blue edges,

green if v has degree three and is incident with a green edge, or v has degree four and is incident with red and blue edges,

blue if v has degree three and is incident with a blue edge, or v has degree four and is incident with red and green edges, and

white otherwise.

Observe that either C' contains a white vertex or it contains an even number of red vertices, an even number of green vertices and an even number of blue vertices. If C' contains a white vertex, it is easy to verify that

$$|C^1_*| = |C(\mathcal{R}_0 \cup \mathcal{G}_0)^A_*| \le 1 \tag{11.2}$$

for a suitable permutation of red, green and blue colors. The same holds if C' contains two adjacent vertices of the same color (see Figure 11.5).

If the circuit of F'_0 corresponding to C contains an edge of weight two or more, then C contains a white vertex and the estimate (11.2) holds. Otherwise, all vertices of C have degree three in G_0 and thus the circuit C is also contained in F'_0. Since the edges of C have weight zero and one only, the pattern of C is one of the patterns listed in Lemma 11.5. A close inspection of possible patterns of C' yields that the cycle C' contains a white vertex or it contains two adjacent vertices with the same color. We conclude that the estimate (11.2) applies. Hence, if $\ell = 4$, the estimate (11.1) can be improved to 0.

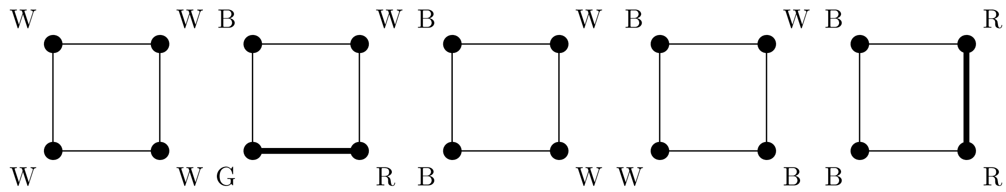

Figure 11.5: An improvement for circuits of length four considered in the proof of Theorem 11.10. The letters R, G, B and W stand for red, green, blue and white colors. Note that it is possible to freely permute the red, green and blue colors. The edges included to $C(\mathcal{R}_0 \cup \mathcal{G}_0)^A$ are bold. Symmetric cases are omitted.

We now estimate the length of the cycle cover of G_0 formed by the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Since each red, green and blue edge is covered by exactly two of the cycles, we conclude that:

$$\begin{aligned} 2(r_0+g_0+b_0)+2d_2+3d_3+4d_4+7d_5+8d_6+9d_7+\sum_{\ell=8}^{\infty}\frac{3\ell}{2}d_\ell &= \\ 2(r_0+g_0+b_0)+\frac{3}{2}\sum_{\ell=2}^{\infty}\ell d_\ell-d_2-3d_3/2-2d_4-d_5/2-d_6-3d_7/2 &= \\ \frac{3m}{2}+\frac{r_0+g_0+b_0}{2}-d_2-3d_3/2-2d_4-d_5/2-d_6-3d_7/2\ . & \end{aligned} \tag{11.3}$$

Note that we have used the fact that the sum $r_0+g_0+b_0+\sum_{\ell=2}^{\infty}\ell d_\ell$ is equal to the number of the edges of G.

The second cycle cover. The second cycle cover is constructed in an auxiliary graph G'' which we now describe. Every vertex v of G is eventually split to a vertex of degree three or four in G'. The vertex of degree four is then expanded. Let $r(v)$ be the vertex of degree three obtained from v or one of the two vertices obtained by the expansion of the vertex of degree four obtained from v. By the construction of F_0, each $r(v)$ is contained in a circuit of F_0. The graph G'' is constructed from the graph G_0 as follows: every vertex of G_0 of degree two not contained in F_0 that is obtained by splitting from a vertex v is identified with the vertex $r(v)$. The edges of weight zero contained in the cycle F_0 are then contracted. Let F be the cycle of G'' corresponding to the cycle F_0 of G_0. Note that F is formed by disjoint circuits and it contains d_ℓ circuits of weight/length ℓ.

Observe that G'' can be obtained from G by splitting some of its vertices (perform exactly those splittings yielding vertices of degree two contained in the circuits of F_0) and then expanding some vertices. In particular, every cycle of G'' is also a cycle of G. Edges of weight one of G'' one-to-one correspond to edges of weight one of G_0, and edges of weight zero of G'' correspond to edges of weight zero of G_0 not contained in F_0. Hence, the weight of a cycle in G'' is the length of the corresponding cycle in G.

The edges not contained in F are red, green and blue (as in G_0). Each circuit of F is incident either with an odd number of red edges, an odd number of green edges and an odd number of blue edges, or with an even number of red edges, an even number of green edges and an even number of blue edges (chords are counted twice). Let $H = G''/F$. If H contains a red circuit (which can be a loop), recolor such a circuit to blue. Similarly, recolor green circuits to blue. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the resulting sets of red, green and blue edges and r, g and b their weights. Clearly, $r+g+b = r_0+g_0+b_0$. Also note that each circuit of F is still incident either with an odd number of red edges, an odd number of green edges and an odd number of blue edges, or with an even number of red edges, an even number of green edges and an even number of blue edges. Since the red edges form an acyclic subgraph of $H = G''/F$, there are at most $\sum_{\ell=2}^{\infty} d_\ell - 1$ red edges and thus the total weight r of red edges is at most $\sum_{\ell=2}^{\infty} d_\ell$ (we forget "-1" since it is not important for our further estimates). A symmetric argument yields that $g \le \sum_{\ell=2}^{\infty} d_\ell$.

Let us have a closer look at circuits of F with weight two. Such circuits correspond to pairs of parallel edges of G'' (and thus of G). By our assumption, the only parallel edges contained in G are pairs of edges between two vertices v_1 and v_2 of degree three and triples of edges between vertices v_1 and v_2 of degree three and five.

In the former case, both v_1 and v_2 have degree three in G''. Consequently, each of them is incident with a single colored edge. By the assumption on the edge-coloring, the two edges have the same color.

In the latter case, the third edge v_1v_2 which corresponds to a loop in G''/F is blue. Hence, the other two edges incident with v_2 must have the same color, which is red, green or blue.

In both cases, the vertex of H corresponding to the circuit v_1v_2 is an isolated vertex in the subgraph of H formed by red edges or in the subgraph formed by green edges (or both). It follows we can improve the estimate on r and g:

$$r + g \le 2\sum_{\ell=2}^{\infty} d_\ell - d_2 = d_2 + 2\sum_{\ell=3}^{\infty} d_\ell \tag{11.4}$$

We are now ready to construct the cycle cover of the graph G''. Its construction closely follows the one presented in the proof of Theorem 11.3. The cycle cover is formed by three cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ contain all red and green edges and the cycle $\mathcal{C}_3$ contains all red and blue edges. We now explain how to alter the definition of the sets $C(E)^A$ and $C(E)^B$ to the setting needed in the construction of these three cycles. Let C be a circuit of F. Consider a set E of edges disjoint from C with an even number of end-vertices on the circuit C. The set $C(E)$ is defined to be the set of the vertices of C incident with an *odd* number of edges of E. Clearly, $|C(E)|$ is even. As before, it is possible to partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that

- each vertex of $C(E)$ is incident with one edge of $C(E)^A$ and one edge of $C(E)^B$, and
- each vertex of C not contained in $C(E)$ is incident with either two edges of $C(E)^A$ or two edges of $C(E)^B$.

As before, we always assume that $|C(E)^A| \leq |C(E)^B|$. Note that if all the vertices of C have degree three, the new definition coincides with the earlier one.

For every circuit C of F, the edges of $C(\mathcal{R}\cup\mathcal{G})^A$ are added to the cycle $\mathcal{C}_1$, the edges of $C(\mathcal{R}\cup\mathcal{G})^B$ to the cycle $\mathcal{C}_2$, and the edges of $C(\mathcal{R}\cup\mathcal{B})^A$ to the cycle $\mathcal{C}_3$. Clearly, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are cycles of G'' and correspond to cycles of G whose length is equal to the the weight of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ in G''.

We now estimate the total weight of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each red edge is covered three times, each green edge twice and each blue edge once. Each edge of F is contained in either $\mathcal{C}_1$ or $\mathcal{C}_2$ and for every circuit C of F at most half of its edges are also contained in $\mathcal{C}_3$. We conclude that the total length of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ can be bounded as follows (note that we apply (11.4) to estimate the sum $r+g$ and we also use the fact that the number of the edges of F is at most $2m/3$ by Lemma 11.5):

$$\begin{aligned}
3r+2g+b+\sum_{\ell=2}^{\infty}\left\lfloor\frac{3\ell}{2}\right\rfloor d_\ell &= \\
m+2r+g+\sum_{\ell=2}^{\infty}\left\lfloor\frac{\ell}{2}\right\rfloor d_\ell &\leq \\
m-d_2+\sum_{\ell=2}^{\infty}\left(\left\lfloor\frac{\ell}{2}\right\rfloor+3\right) d_\ell &= \\
\frac{13m}{8}-\frac{5(r_0+g_0+b_0)}{8}-d_2+\sum_{\ell=2}^{\infty}\left(\left\lfloor\frac{\ell}{2}\right\rfloor+3-\frac{5\ell}{8}\right) d_\ell &\leq \\
\frac{43m}{24}-\frac{5(r_0+g_0+b_0)}{8}-d_2+\sum_{\ell=2}^{\infty}\left(\left\lfloor\frac{\ell}{2}\right\rfloor+3-\frac{7\ell}{8}\right) d_\ell &\leq \\
\frac{43m}{24}-\frac{5(r_0+g_0+b_0)}{8}-d_2+\sum_{\ell=2}^{6}\left(\left\lfloor\frac{\ell}{2}\right\rfloor+3-\frac{7\ell}{8}\right) d_\ell &= \\
\frac{43m}{24}-\frac{5(r_0+g_0+b_0)}{8}+5d_2/4+11d_3/8+3d_4/2+5d_5/8+3d_6/4\,. & \qquad (11.5)
\end{aligned}$$

The last inequality follows from the fact that $\left\lfloor\frac{\ell}{2}\right\rfloor+3-\frac{7\ell}{8}\leq 0$ for $\ell\geq 7$.

The length of the shortest cycle cover of G with three cycles exceeds neither the bound given in (11.3) nor the bound given in (11.5). Hence, the length of such a cycle cover of G is bounded by any convex combination of the two bounds, in particular, by the following:

$$\frac{5}{9} \cdot \left(\frac{3m}{2} + \frac{r_0 + g_0 + b_0}{2} - d_2 - 3d_3/2 - 2d_4 - d_5/2 - d_6 - 3d_7/2 \right) +$$

$$\frac{4}{9} \cdot \left(\frac{43m}{24} - \frac{5(r_0 + g_0 + b_0)}{8} + 5d_2/4 + 11d_3/8 + 3d_4/2 + 5d_5/8 + 3d_6/4 \right) =$$

$$\frac{44m}{27} - 2d_3/9 - 4d_4/9 - 2d_6/9 - 5d_7/6 \le \frac{44m}{27} \; .$$

The proof of Theorem 11.10 is now completed.

■

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