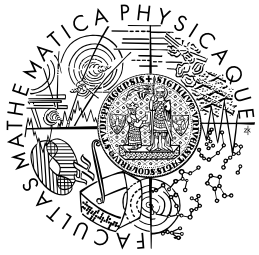

FACULTY
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Charles University

BACHELOR THESIS

Jan Kubica

Fully symmetric operations

Department of Algebra

Supervisor of the bachelor thesis: prof. Mgr. Libor Barto, Ph.D.
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Title: Fully symmetric operations

Author: Jan Kubica

Department: Department of Algebra

Supervisor: prof. Mgr. Libor Barto, Ph.D., Department of Algebra

Abstract: The Constraint Satisfaction Problem over an algebra A is solvable by the basic linear programming relaxation if, and only if, A has a symmetric term operation of every arity. It is, however, unclear how to decide that the latter condition is satisfied for a given algebra. In this thesis, we prove that a three-element algebra, which has bounded width and a symmetric term operation of arities 1,2, and 3, necessarily has a symmetric term operation of every arity.

Keywords: symmetric term operations, constraint satisfaction problem, linear programming

Název práce: Úplně symetrické operace

Autor: Jan Kubica

Katedra: Katedra algebry

Vedoucí bakalářské práce: prof. Mgr. Libor Barto, Ph.D., Katedra algebry

Abstrakt: Problém splnitelnosti omezujících podmínek nad algebrou A je řešitelný pomocí základní relaxace lineárního programování právě tehdy, když má A symetrické termové operace všech arit. Není však jasné, jak rozhodnout, zda daná algebra tuto podmínku splňuje. V této práci dokazujeme, že tříprvková algebra, která má omezenou šířku a která obsahuje symetrické termové operace arit 1,2,3, už má nutně symetrické termové operace všech arit.

Klíčová slova: symetrické termové operace, problém splnitelnosti omezujících podmínek, lineární programování

Contents

Introduction

In this thesis, we are primarily interested in finite algebras. In particular, we would like to know when an algebra $\mathbb{A}$ has a symmetric term operation of every arity, that is an operation whose output only depends on the multiset of arguments. Currently, it is unknown whether the existence of such operations is a decidable problem or not. Hence, the goal of this thesis is to move towards answering the question of decidability of this problem.

There are a couple of reasons why we would like to be able to decide this problem. Firstly, symmetric operations are interesting on their own and secondly, almost a practical motivation to study this question comes from the area of fixed template constraint satisfaction problems: It is known (see [1]) that a finite algebra $\mathbb{A}$ has a symmetric term operation of every arity if, and only if, the constraint satisfaction problem over $\mathbb{A}$ can be solved by the so-called basic linear programming relaxation.

The idea for deciding this problem is the following:

Conjecture 1. *Let $\mathbb{A}$ be a finite algebra. If $\mathbb{A}$ contains a symmetric term operation of arity $1, 2, ..., |A|$, then $\mathbb{A}$ contains a symmetric term operation of every arity.*

In this thesis, we prove this conjecture to be true for algebras of size 3.

Theorem 1. *Let $\mathbb{A}$ be a 3-element algebra that contains a symmetric term operation of arity 1,2, and 3, then $\mathbb{A}$ has a symmetric term operation of every arity.*

The strategy for proving the latter theorem was proposed by Zarathustra Brady through personal communication. Brady showed that in the case of 3-element algebras, to obtain certain weaker symmetric operations, it is sufficient enough to prove certain combinatorial lemma about linear orders and to prove the existence of term operations satisfying specific identities, which follows from [2]. The proof of the combinatorial lemma can be found in Chapter 2. The proof of this lemma constitutes the author's contribution. It is also important to note that the proof of the combinatorial lemma was independently discovered by Marcin Kozik. Nevertheless, it was not known to the author of this thesis at the time.

Later, it turned out that Brady proved Conjecture 1 for 3-element algebras (see [3]). A detailed version of his proof is given in the last chapter of this thesis and is another author's contribution.

It is also worth noting that Zarathustra Brady and Holden Mui proved an even stronger result (see [4]). If an algebra $\mathbb{A}$ of size at most 4 has a symmetric term operation of every arity up to $|A|$, then $\mathbb{A}$ has a symmetric term operation of every arity.

1. Preliminaries

In this chapter, we present important notations and a comprehensive list of definitions that will be used throughout this thesis. We recommend familiarizing yourself with all these definitions before proceeding to the subsequent chapters, as the concepts introduced here will be referenced frequently in the text.

1.1 Notation

We begin by introducing some notation. $\mathbb{N}$ denotes the set of positive integers. Non-negative integers are labeled $\mathbb{N}_0$. $[n]$ denotes the set $\{1, 2, 3, ..., n\}$, where $n \in \mathbb{N}$. The power set of a set X is denoted by 2^X. For any set X and any $k \in \mathbb{N}_0$, the set $\binom{X}{k} \subseteq 2^X$ denotes the family of all subsets of the set X that contain exactly k elements. Furthermore, the cardinality of a set X is denoted by $|X|$. S_n stands for the set of all permutations (bijective functions from $[n]$ to $[n]$) on the set $[n]$.

1.2 Orders

Many objects in mathematics are defined as a relation with some additional properties including functions, equivalences, congruences, orders, and so on. For this thesis, we include a definition of strict order and strict linear order. Formally, it goes as follows:

Definition 1 (**Strict orders and strict linear orders**)**.** A binary relation $\prec$ on a domain D is called a *strict order* if it is irreflexive and transitive. In addition, a strict order $\prec$ is called *linear* (or *total*) if for every $x, y \in D$ either $x \prec y$ or $x = y$ or $y \prec x$.

Furthermore, if X is a set and $<$ is a strict linear order on the set X, then the binary relation $<^{-1}$ on X, defined as $<^{-1} := \{(a, b) \in X^2 \mid b < a\}$, is also a linear order on X. In addition, we define a couple of needed notions for strict orders on a power set of some set.

Definition 2 (**Graded linear orders**)**.** Let D be a set and let $\prec$ be a strict linear order on 2^D, then $\prec$ is called *graded* if the following is true for any $U, V \subseteq D$: if $|U| < |D|$, then $U \prec V$.

Definition 3 (**Image of a strict linear order**)**.** Let $\pi : A \to B$ be a function and let $\prec$ be a strict linear order on 2^A, then we define the *image of* $\prec$ *under* π as a binary relation on $2^{\pi(A)}$ in the following way: $X \; \pi(\prec) \; Y \equiv \pi^{-1}(X) \; \prec \; \pi^{-1}(Y)$, where $\pi^{-1}(V) := \{x \in A \mid \pi(x) \in V \subseteq B\}$.

A key observation is that an image of a strict linear order on the power set of some set under a surjective function is also a strict linear order on the power set of the function's co-domain.

Definition 4 (**Strict orders extending inclusion**)**.** Let D be a set. We say that a strict order $\prec$ on $X \subseteq 2^D$ *extends inclusion* $\subset$ if for all $U, V \subseteq D$ the following implication holds: if $U \subset V$, then $U \prec V$.

An important example of a linear order is the lexicographic order on subsets of natural numbers. It is a routine exercise to show that a lexicographic order, as defined below, is a strict linear order on $2^{\mathbb{N}}$.

Definition 5. *Lexicographic order* is a binary relation $<_{\text{lex}}$ on the set $2^{\mathbb{N}}$ defined as

$$X <_{\text{lex}} Y \equiv \min(X \setminus (X \cap Y)) < \min(Y \setminus (X \cap Y)),$$

where $\min(\emptyset) := 0$ and $<$ is the usual linear order on $\mathbb{N}$.

The last notion that we define in this sub-section is a partition of a set, as it is frequently used in the thesis.

Definition 6 (**Partitions**)**.** Let X be any non-empty set. We say that a non-empty subset P of the set X is a *partition* of the set X if the following three conditions are all satisfied:

- every set $A \in P$ is non-empty
- the sets in P are pair-wise disjoint
- $\bigcup P = X$

1.3 Algebras and term operations

Let us continue with a definition of algebras and operations on a certain domain.

Definition 7 (**Algebras and operations**)**.** An *operation* on a non-empty set A is a mapping whose domain is A^n, where $n \in \mathbb{N}_0$, and whose co-domain is A. Consider a set of function symbols $\mathcal{F} = \{f_i \mid i \in I\}$, called a *signature of* $\mathbb{A}$, where each symbol has an assigned arity $\text{ar}(f_i) \in \mathbb{N}_0$. The pair $\mathbb{A} := (A, (f_i^{\mathbb{A}})_{i \in I})$ is called an *algebra over (in) the signature* $\mathcal{F}$ if A is a non-empty set (the *universe* of $\mathbb{A}$) and for every $i \in I$, the function $f_i^{\mathbb{A}} : A^{\text{ar}(f_i)} \to A$ is an operation called a *basic operation* of $\mathbb{A}$. Algebra $\mathbb{A} = (A, (f_i^{\mathbb{A}})_{i \in I})$ over a signature $\mathcal{F}$ is said to be finite if $|A| \in \mathbb{N}$.

Furthermore, we define term operations of an algebra $\mathbb{A}$ as operations that are obtained by composing the basic operations of the algebra. Formally, these operations are defined via terms, which are syntactically meaningful expressions that involve variables and symbols from the given signature. These terms are interpreted as operations by substituting elements from the universe of the given algebra.

Definition 8 (**Terms**)**.** Let $X = \{x_1, x_2, ...\}$ be a set of variables and let $\mathcal{F}$ be a fixed signature disjoint with X. We say that every variable in X is an $\mathcal{F}$-*term*, every function symbol $f \in \mathcal{F}$ of arity 0 is an $\mathcal{F}$-term, and if $u_1, ..., u_k$, where $k \in \mathbb{N}$, are $\mathcal{F}$-terms and $f \in \mathcal{F}$ is a k-ary function symbol, then the concatenation $fu_1, ..., u_k$ is an $\mathcal{F}$-term.

Definition 9 (**Term operations**)**.** Let $X = \{x_1, x_2, ...\}$ be a set of variables and let $\mathcal{F}$ be a signature disjoint with X. If $\mathbb{A} = (A, (f^{\mathbb{A}})_{f \in \mathcal{F}})$ is an algebra over the signature $\mathcal{F}$ and t is an $\mathcal{F}$-term, then we associate to the pair $(t, \mathbf{x}) = (t, (x_1, ..., x_n))$, where $\mathbf{x}$ is an n-tuple of variables from X and $n \in \mathbb{N}_0$, an n-ary *term operation* $t^{\mathbb{A}} : A^n \to A$ on A which is defined as follows:

- If $n = 0$ and $t = c$ for some 0-ary $c \in \mathcal{F}$, then $t^{\mathbb{A}} = c^{\mathbb{A}}$.
- If $n > 0$ and $t = x_i$ for some $i \in [n]$, then $t^{\mathbb{A}}(\mathbf{a}) = a_i$, where $\mathbf{a} = (a_1, ..., a_n) \in A^n$.
- If $n > 0$ and $t = fu_1, ..., u_k$ for some k-ary function symbol $f \in \mathcal{F}$ and some $\mathcal{F}$-terms $u_1, ..., u_k$, then $t^{\mathbb{A}}(\mathbf{a}) = f^{\mathbb{A}}(u_1^{\mathbb{A}}(\mathbf{a}), ..., u_k^{\mathbb{A}}(\mathbf{a}))$, where $\mathbf{a} \in A^n$.

Lastly, the set of all term operations of an algebra $\mathbb{A}$ is denoted by $\mathrm{Clo}(\mathbb{A})$.

Definition 10 (**Generalized composition**)**.** Let n, k be natural numbers and suppose f is an operation on a set A of arity n and $g_1, ..., g_n$ are operations on a set A of arity k, then we define a new k-ary operation $f \circ (g_1, ..., g_n)$ by

$$(x_1, ..., x_k) \mapsto f(g_1(x_1, ..., x_k), ..., g_n(x_1, ..., x_k)) : A^k \to A$$

called the *generalized composite* of f with $g_1, ..., g_n$.

For examples of algebras and a comprehensive explanation of the concepts mentioned above, refer to [5].

A special case of term operation of some algebra is a symmetric term operation, where the output remains unchanged even if the input tuple of values is permuted. Another important type of term operation is a k-symmetric term operation. These operations are symmetric for inputs that consist of at most k distinct values from the universe of $\mathbb{A}$.

Definition 11 (**Symmetric term operations**)**.** Let $t^{\mathbb{A}} : A^n \to A$ be a term operation of an algebra $\mathbb{A}$, where $n \in \mathbb{N}$, then $t^{\mathbb{A}}$ is said to be *symmetric* if for all $\pi \in S_n$ and for every $a_1, ..., a_n \in A : t^{\mathbb{A}}(a_{\pi(1)}, ..., a_{\pi(n)}) = t^{\mathbb{A}}(a_1, ..., a_n)$.

Definition 12 (**k-symmetric term operations**)**.** A n-ary term operation $t^{\mathbb{A}}$ of an algebra is called *k-symmetric* if for all $\pi \in S_n$ and for each $a_1, ..., a_n \in A$ the following implication holds: $|\{a_1, ..., a_n\}| \leq k \implies t^{\mathbb{A}}(a_{\pi(1)}, ..., a_{\pi(n)}) = t^{\mathbb{A}}(a_1, ..., a_n)$, where $k, n \in \mathbb{N}$.

1.4 LO-system of operations

Definition 13 (**Minors of an operation**)**.** Let $t : A^n \to A$ be an n-ary operation on a non-empty domain A and let $\pi : [n] \to [m]$ be a mapping, then we define the *minor of t given by* π, denoted as t^{π}, as an m-ary operation on A as follows:

$$\forall x_1, ..., x_m \in A : t^{\pi}(x_1, ..., x_m) := t(x_{\pi(1)}, ..., x_{\pi(n)}).$$

An important observation is that for any mapping π, the minor of a term operation of some algebra given by π is again a term operation.

Definition 14 (**LO-system of operations**)**.** Let $\mathbb{L}(X)$ denote the set of all strict linear orders on 2^X which extend inclusion $\subset$, then an LO-system of operations is a family of operations
$(t_{n,X,\prec})_{n \in \mathbb{N}, \emptyset \subset X \subseteq [n], \prec \in \mathbb{L}(X)}$ on a domain A such that

- for all $n \in \mathbb{N}, X \in 2^{[n]}, \prec \in \mathbb{L}(X)$, the operation $t_{n,X,\prec}$ has arity n,

- for all $m, n \in \mathbb{N}, X \in 2^{[n]}, \prec \in \mathbb{L}(X)$ and for every mapping $\pi : [n] \to [m]$:

$$t^{\pi}_{n,X,\prec} = t_{m,\pi(X),\pi(\prec)}.$$

It follows from known results that symmetric operations up to arity $|A|$ imply an important property of $\mathbb{A}$, called bounded width, which e.g. characterizes constraint satisfaction problems solvable by consistency methods, see [1]. Zarathustra Brady proved that bounded width is equivalent to the existence of an LO-system of term operations. In particular, we have the following consequence, which is essential in our proof of Conjecture 1 for 3-element algebras.

Theorem 2 ([3]). *Let $\mathbb{A} = (A, (f)_{f \in \mathcal{F}})$ be a finite algebra in a signature $\mathcal{F}$. If $\mathbb{A}$ has symmetric term operation of every arity up to $|A|$, then Clo($\mathbb{A}$) contains LO-system of operations $(t_{n,X,\prec})_{n \in \mathbb{N}, \emptyset \subset X \subseteq [n], \prec \in \mathbb{L}(X)}$.*

2. On the way to symmetric operations

2.1 Six orders lemma

In the introduction, we discussed a combinatorial lemma proposed by Brady and independently proved by Kozik. In this chapter, we present the author's proof of this lemma. Additionally, we will explore how to use this lemma to find almost symmetric term operations of large arities of some three-element algebra. This will demonstrate the key ideas that can be found in Brady's proof of Theorem 3.

Lemma 1 (**Six orders lemma**). *For every n natural and every subset X of the set $2^{[n]}$, there exist 3 linear orders $<_1^+, <_2^+, <_3^+$ on the set X such that if $\{A, B, C\} \subseteq X$ form any partition of the set $[n]$ and if $A <_{\text{lex}} B <_{\text{lex}} C$, then*

$$A <_1^+ B <_1^+ C,$$

$$C <_2^+ A <_2^+ B,$$

$$B <_3^+ C <_3^+ A.$$

Moreover, if we define $<_1^-, <_2^-$, and $<_3^-$ as the inverses of $<_1^+, <_2^+, <_3^+$, respectively, and if $\{A, B, C\} \subseteq X$ is an arbitrary partition of the set $[n]$, then the restrictions of $<_i^a$, where $i \in [3]$ and $a \in \{+, -\}$, onto the set $\{A, B, C\}$ are all the six linear orders on the 3-element set $\{A, B, C\}$.

The idea for the proof. We define the linear orders in such a way that if $\{A, B, C\} \subseteq X$ is an arbitrary partition of the set $[n]$ with $A <_{\text{lex}} B <_{\text{lex}} C$, then the following is satisfied:

- $A <_1^+ B <_1^+ C \ \wedge\ C <_1^- B <_1^- A$
- $C <_2^+ A <_2^+ B \ \wedge\ B <_2^- A <_2^- C$
- $B <_3^+ C <_3^+ A \ \wedge\ A <_3^- C <_3^- B$

Note that this condition must hold for all partitions ordered by the lexicographic order $<_{\text{lex}}$.

Proof. Let $n \in \mathbb{N}$ and $X \subseteq 2^{[n]}$ be arbitrary. For the first order, put $<_1^+ := <_{\text{lex}}$, as defined in Definition 5. It follows trivially that this relation is a linear order. Suppose that $\{A, B, C\} \subseteq X$ is any partition of the set $[n]$ such that $A <_{\text{lex}} B <_{\text{lex}} C$, then we have $A <_1^+ B <_1^+ C$. Consequently, it follows that $C <_1^- B <_1^- A$, since $<_1^-$ is the inverse of $<_1^+$.

Now, let us define the order $<_2^+$ on X in the following way:

$$\begin{aligned} A <_2^+ B \quad \equiv \quad & (1 \in A \ \wedge\ 1 \in B \ \wedge\ B^c <_{\text{lex}} A^c) \vee \\ & (1 \in A \ \wedge\ 1 \notin B \ \wedge\ \min(B) \leq \min(A^c)) \vee \\ & (1 \notin A \ \wedge\ 1 \in B \ \wedge\ \min(B^c) < \min(A)) \vee \\ & (1 \notin A \ \wedge\ 1 \notin B \ \wedge\ B <_{\text{lex}} A) \end{aligned}$$

Note that the complement of any subset A of the set $[n]$ is always taken with respect to the set $[n]$. More precisely, for any $A \in 2^{[n]} : A^c := [n] \setminus A$.

We start with the proof of irreflexivity of $<_2^+$. Let A be an arbitrary element of X. By way of contradiction, we assume that $A <_2^+ A$. If $1 \in A$, then by definition, $A^c <_{\text{lex}} A^c$. However, this cannot be true since $<_{\text{lex}}$ is irreflexive. Therefore, $1 \notin A$, which yields $A <_{\text{lex}} A$ – a contradiction.

Secondly, we show that $<_2^+$ is transitive. Let $A, B, C \in X$ be arbitrary. If $A <_2^+ B$ and $B <_2^+ C$, then we have to examine 4 cases:

- Assume that $1 \in A \cap B$. Since $A <_2^+ B$, then we have $B^c <_{\text{lex}} A^c$. Now, if $1 \in C$, then the condition $B <_2^+ C$ implies that $C^c <_{\text{lex}} B^c$. Using the transitivity of $<_{\text{lex}}$, we get $C^c <_{\text{lex}} A^c$. Therefore, $1 \in A \cap C$ and $C^c <_{\text{lex}} A^c$, which yields $A <_2^+ C$. If $1 \notin C$, then $\min(C) \leq \min(B^c)$. Since $A <_2^+ B$, we have $B^c <_{\text{lex}} A^c$. This implies that $\min(B^c) \leq \min(A^c)$. By the transitivity of $\leq$, we also have $\min(C) \leq \min(A^c)$. Therefore, it follows that $1 \in A \setminus C$ and $\min(C) \leq \min(A^c)$, which shows that $A <_2^+ C$.

- Suppose $1 \in A \setminus B$. Then $\min(B) \leq \min(A^c)$. If $1 \in C$, then $\min(C^c) < \min(B)$. Therefore, $\min(C^c) < \min(A^c)$, which implies that $C^c <_{\text{lex}} A^c$. This shows that $1 \in A \cap C$ and $C^c <_{\text{lex}} A^c$. Thus, $A <_2^+ C$. If $1 \notin C$, then we get $C <_{\text{lex}} B$, which means that $\min(C) \leq \min(B)$. Therefore, $\min(C) \leq \min(A^c)$. This implies that $1 \in A, \setminus C$ and $\min(C) \leq \min(A^c)$. Thus, $A <_2^+ C$.

- If $1 \in B \setminus A$, then $\min(B^c) < \min(A)$. If $1 \in C$, then $C^c <_{\text{lex}} B^c$, which implies that $\min(C^c) \leq \min(B^c)$. Thus, $1 \in C \setminus A$ and $\min(C^c) < \min(A)$. Therefore, $A <_2^+ C$. If $1 \notin C$, then $\min(C) \leq \min(B^c)$. Hence, using the transitivity of $<$, we have $\min(C) < \min(A)$. This means that $1 \notin A \cup C$ and $C <_{\text{lex}} A$, which yields $A <_2^+ C$.

- If $1 \notin A \cup B$, then $B <_{\text{lex}} A$, which shows that $\min(B) \leq \min(A)$. If $1 \in C$, then $\min(C^c) < \min(B)$ since $B <_2^+ C$. This yields $\min(C^c) < \min(A)$ and $1 \in C \setminus A$. Thus, $A <_2^+ C$. If $1 \notin C$, then $C <_{\text{lex}} B$. Therefore, $C <_{\text{lex}} A$. Hence, $1 \notin A \cup C$ and $\min(C) \leq \min(A)$, which implies that $A <_2^+ C$.

Furthermore, we prove that $<_2^+$ is a linear order. Let $A, B \in X$ be arbitrary. If $A \neq B$, then there are 4 cases:

- If $1 \in A \ \wedge \ 1 \in B$, then either $B^c <_{\text{lex}} A^c$ or $A^c <_{\text{lex}} B^c$. If it is the first case, then $A <_2^+ B$. If it is the second case, then $B <_2^+ A$.

- If $1 \in A$ and $1 \notin B$, then either $\min(B) \leq \min(A^c)$ or $\min(A^c) < \min(B)$. The first case yields $A <_2^+ B$ and the second one yields $B <_2^+ A$.

- If $1 \notin A$ and $1 \in B$, then either $\min(B^c) < \min(A)$ or $\min(A) \leq \min(B^c)$. The first case yields $A <_2^+ B$ and the second one yields $B <_2^+ A$.

- If $1 \notin A$ and $1 \notin B$, then either $B <_{\text{lex}} A$ or $A <_{\text{lex}} B$. This implies that either $A <_2^+ B$ if $B <_{\text{lex}} A$, or $B <_2^+ A$ if $A <_{\text{lex}} B$.

Thus, $<_2^+$ is a linear order. Furthermore, recall that $<_2^-$ is the inverse of $<_2^+$. We show that both $<_2^+$ and $<_2^-$ satisfy the condition outlined in the idea of the proof. Assume that $\{A, B, C\} \subseteq X$ is an arbitrary partition of the set $[n]$ and assume that $A <_{\text{lex}} B <_{\text{lex}} C$. Since the sets $A, B,$ and C are pair-wise disjoint, we have $1 \in A$ and $1 \notin B \cup C$. If this were not the case, then one of those two sets would be smaller than A with respect to the lexicographic order $<_{\text{lex}}$. It is evident that $\min(A^c) = \min(B)$. Indeed: By way of contradiction, suppose that $\min(A^c) \neq \min(B)$. If $\min(A^c) < \min(B)$, then $\min(A^c) \notin B$ and also $\min(A^c) \notin A$ because $\min(A^c) \in A^c$. Therefore, $\min(B \cup C) = \min(A^c) \in C \setminus B$, which yields $C <_{\text{lex}} B$. However, this contradicts the assumption that $B <_{\text{lex}} C$. On the other hand, if $\min(B) < \min(A^c)$, then $\min(B) \in A \cap B$, which contradicts the assumption that $\{A, B, C\}$ is a partition of $[n]$. Both scenarios result in a contradiction, meaning $\min(A^c) = \min(B)$. Thus, $\min(B) \leq \min(A^c)$ and $1 \in A \setminus B$. Therefore, we conclude that $A <_2^+ B$. Furthermore, $1 \notin B \cup C$ and $B <_{\text{lex}} C$, which shows that $C <_2^+ B$. Lastly, $\min(A^c) = \min(B) < \min(C)$ and $1 \in A \setminus C$. Hence, $C <_2^+ A$. This shows that $C <_2^+ A <_2^+ B$. Since $<_2^-$ is the inverse of $<_2^+$, we obtain $B <_2^- A <_2^- C$.

Lastly, we define binary relation $<_3^+$ on X as follows:

$$\begin{aligned} A <_3^+ B \quad \equiv \quad & (1 \notin A \ \wedge \ 1 \in B) \vee \\ & (1 \in A \ \wedge \ 1 \in B \ \wedge \ B <_{\text{lex}} A) \vee \\ & (1 \notin A \ \wedge \ 1 \notin B \ \wedge \ A <_{\text{lex}} B) \end{aligned}$$

Firstly, we show that $<_3^+$ is an irreflexive binary relation. This is trivial because for any $A \in X$, we have either $1 \in A$ or $1 \notin A$. In both cases, $A <_3^+ A \iff A <_{\text{lex}} A$. However, the relation $<_{\text{lex}}$ is irreflexive.

Next, we prove the relation $<_3^+$ to be transitive. Let $A, B, C \in X$ be arbitrary and assume that $A <_3^+ B$ and $B <_3^+ C$, then we have to examine the following cases:

- If $1 \notin A$ and $1 \in B$, then $1 \in C$ since $B <_3^+ C$. Hence, $1 \notin A$ and $1 \in C$. Therefore, we conclude that $A <_3^+ C$.
- If $1 \in A$ and $1 \in B$, then $B <_{\text{lex}} A$ and $1 \in C$ as $B <_3^+ C$. Thus, $C <_{\text{lex}} B$, which gives $C <_{\text{lex}} A$ because $<_{\text{lex}}$ is transitive. Hence, $1 \in A \cap C$ and $C <_{\text{lex}} A$, which implies that $A <_3^+ C$.
- Lastly, if $1 \notin A$ and $1 \notin B$, then $A <_{\text{lex}} B$ and we have either $1 \in C$ or $1 \notin C$. If $1 \in C$, then $A <_+^3 C$ since $1 \notin A$. On the other hand, if $1 \notin C$, then $B <_{\text{lex}} C$, which yields $A <_{\text{lex}} C$. This proves that $A <_+^3 C$.

Next, we show that $<_3^+$ is linear. The proof is again straightforward as one can always decide whether 1 is an element of the given set or not and can utilize the fact that $<_{\text{lex}}$ is a linear order. Let $A, B \in X$ be arbitrary. If $A \neq B$, then there are 4 possibilities:

- If $1 \in A$ and $1 \in B$, then either $B <_{\text{lex}} A$ or $A <_{\text{lex}} B$, which gives either $A <_3^+ B$ if $B <_{\text{lex}} A$, or $B <_3^+ A$ if $A <_{\text{lex}} B$.
- If $1 \in A$ and $1 \notin B$, then $B <_3^+ A$.

- If $1 \notin A$ and $1 \in B$, then $A <_3^+ B$
- If $1 \notin A$ and $1 \notin B$, then either $A <_{\text{lex}} B$ or $B <_{\text{lex}} A$, which implies that either $A <_3^+ B$ if $A <_{\text{lex}} B$, or $B <_3^+ A$ if $B <_{\text{lex}} A$.

Therefore, the relation is a linear order on X. Again, recall that $<_3^-$ is the inverse of $<_3^+$. We now verify the condition from the idea of the proof for $<_3^+$ and $<_3^-$. Let $\{A, B, C\} \subseteq X$ be an arbitrary partition of the set $[n]$ and assume that $A <_{\text{lex}} B <_{\text{lex}} C$. The sets A, B, C are pair-wise disjoint, and thus the set A must contain 1, otherwise some other set containing 1 would be smaller with respect to the lexicographic order $<_{\text{lex}}$. Then we have $1 \notin B$ and $1 \notin C$. Hence, $B <_3^+ A$ and $C <_3^+ A$. Since $B <_{\text{lex}} C$, we have $B <_3^+ C$ by definition. Therefore, $B <_3^+ C <_3^+ A$. Since $<_3^-$ is the inverse of $<_3^+$, we conclude that $A <_3^- C <_3^- B$. □

2.2 Almost symmetric operations

In this section, we prove a weaker result proposed by Brady that provides the reader with intuition and the idea behind finding symmetric term operations of large arities. Firstly, notice that the following lemma is a direct corollary of Lemma 1. We simply put $X := \binom{[3n]}{n}$.

Lemma 2. *For every n natural, there exist six linear orders $<_1, <_2, <_3, <_4, <_5, <_6$ on the set $\binom{[3n]}{n}$ such that for every partition $\{A, B, C\} \subseteq \binom{[3n]}{n}$ of the set $[3n]$, the restriction of $<_i$, where $i \in [6]$, onto $\{A, B, C\}$ are all the six linear orders on the 3-element set $\{A, B, C\}$.*

We begin by extending the linear orders from Lemma 2 onto $2^{[3n]}$ in such a way that the newly defined strict linear orders extend inclusion. Then we show that there are somewhat symmetric operations of arity $3n$ provided that the 3-element algebra $\mathbb{A}$ has a symmetric term operation of arity 6 and an LO-system of term operations.

Let n be a natural number. By Lemma 2, there are 6 linear orders

$$<_1, <_2, <_3, <_4, <_5, <_6$$

on the set $\binom{[3n]}{n}$ such that if $\{A, B, C\} \subseteq \binom{[3n]}{n}$ is an arbitrary partition of the set $[3n]$, then the restrictions of $<_i$, where $i \in [6]$, onto the set $\{A, B, C\}$ are all the six linear orders on the 3-element set $\{A, B, C\}$. Now, we fix $i \in [6]$ and define a binary relation on $2^{[3n]}$ as follows:

$$\begin{aligned} U \prec_i V \quad \equiv \quad & (|U| < |V|) \vee \\ & (|U| = |V| \neq n \ \wedge \ U^c <_i V^c) \vee \\ & (|U| = |V| = n \ \wedge \ U <_i V) \end{aligned}$$

Now, we prove the following lemma:

Lemma 3. *For any n natural and for any $i \in [6]$, the binary relation $\prec_i$ defined above is a strict linear order that extends inclusion.*

Proof. Fix $i \in [6]$. We have to prove that the relation is irreflexive, transitive, linear order and that it extends inclusion (see Definition 4).

Let U be an arbitrary subset of $[3n]$. Since $|U| = |U|$, the first case in the definition of $\prec_i$ is not an option. However, $<_i$ is irreflexive, which shows that $\neg(U \prec_i U)$.

For transitivity, let $U, V, W \subseteq [3n]$ be arbitrary and assume that $U \prec_i V$ and $V \prec_i W$, then we have to examine the two following cases:

- If $|U| < |V|$, then $|U| < |W|$ since $V \prec_i W$ implies that $|V| \leq |W|$. Thus, $U \prec_i W$.
- If $|U| = |V|$, then, as in the previous item, we have either $|V| < |W|$ or $|V| = |W|$. In the first case, since $|U| < |W|$, it follows that $U \prec_i W$. In the second case, where $|V| = |W|$, either all 3 sets U, V, and W have the same size n, or their sizes differ from n. In both scenarios, we can apply the transitivity of $<_i$ to conclude that $U \prec_i W$.

Furthermore, we prove that the relation is a strict linear order. Let $U, V \subseteq [3n]$ be arbitrary with $U \neq V$.

- If $|U| < |V|$, then $U \prec_i V$.
- If $|V| < |U|$, then $V \prec_i U$.
- If $|V| = |U| = n$, then either $U <_i V$ or $V <_i U$, which yields either $U \prec_i V$ if $U <_i V$, or $V \prec_i U$ if $V <_i U$.
- If $|U| = |V| \neq n$, then either $U^c <_i V^c$ or $V^c <_i U^c$, which yields either $U \prec_i V$ if $U^c <_i V^c$, or $V \prec_i U$ if $V^c <_i U^c$.

Lastly, we show that the strict linear order extends inclusion. One can see that $\prec_i$ is by definition graded (see Definition 2). Thus, if U, V are arbitrary subsets of $[3n]$ such that $U \subset V$, then $|U| < |V|$ (since they are finite). It follows that $U \prec_i V$. □

Now, we have everything we need to show that if $\mathbb{A}$ is a 3-element algebra that has an LO-system of term operations and a symmetric term operation of arity six, then $\mathbb{A}$ also has term operations of arity $3n$, where $n \in \mathbb{N}$, that is symmetric on balanced tuples. Those are tuples containing the same number of ones, twos, and threes, where $\{1, 2, 3\}$ is the universe of the algebra. This is the weaker version of symmetric operations that Brady proposed.

Lemma 4. *Let $\mathbb{A} = (\{1, 2, 3\}, (f)_{f \in \mathcal{F}})$ be an algebra in a signature $\mathcal{F}$ that has symmetric term operation s_6 of arity 6 and that has an LO-system of term operations*

$$(t_{n,X,\prec})_{n \in \mathbb{N}, \emptyset \subset X \subseteq [n], \prec \in \mathbb{L}(X)}.$$

Let n be a natural number and $\prec_1, \prec_2, \prec_3, \prec_4, \prec_5, \prec_6$ as above, then there exists $3n$-ary term operation such that for every permutation $\pi : [3n] \to [3n]$ and for every $3n$-tuple $(x_1, ..., x_{3n}) \in A^{3n}$ that contains n ones, n twos, and n threes, called a balanced tuple, we have $s_{3n}(x_{\pi(1)}, ..., x_{\pi(3n)}) = s_{3n}(x_1, ..., x_{3n})$.

Proof. Choose arbitrary $n \in \mathbb{N}$. Since $\prec_i$ is a strict linear order that extends inclusion, for every $i \in [6]$, then there is a term operation $t_{3n,[3n],\prec_i}$ in the LO-system of term operations. Thus, we can define (see Definition 10) $s_{3n} : A^{3n} \to A$ as a generalized composite of s_6 with these specifically chosen LO-operations:

$$s_{3n} := s_6 \circ (t_{3n,[3n],\prec_1}, t_{3n,[3n],\prec_2}, t_{3n,[3n],\prec_3}, t_{3n,[3n],\prec_4}, t_{3n,[3n],\prec_5}, t_{3n,[3n],\prec_6})$$

Choose an arbitrary balanced tuple $(x_1, ..., x_{3n})$. Firstly, we notice that

$$s_{3n}(x_1, ..., x_{3n}) = s_{3n}^{(x_1,...,x_{3n})}(1, 2, 3)$$

provided that $(x_1, ..., x_{3n})$ is viewed as a mapping from $[3n]$ into $[3]$ (recall Definition 13). Now, we use the definition of s_{3n} to rewrite the right-hand side of the last equality as

$$s_6(t^{\mathbf{x}}_{3n,[3n],\prec_1}(1, 2, 3), t^{\mathbf{x}}_{3n,[3n],\prec_2}(1, 2, 3), ..., t^{\mathbf{x}}_{3n,[3n],\prec_6}(1, 2, 3)),$$

where $\mathbf{x} = (x_1, ..., x_{3n})$. Furthermore, we use the definition of an LO-system of operations (see Definition 14), to rewrite the latter expression as

$$s_6(t_{3,[3],\mathbf{x}(\prec_1)}(1, 2, 3), t_{3,[3],\mathbf{x}(\prec_2)}(1, 2, 3), ..., t_{3,[3],\mathbf{x}(\prec_6)}(1, 2, 3))$$

Furthermore, put

$$A_1 := \mathbf{x}^{-1}(\{1\}),\ A_2 := \mathbf{x}^{-1}(\{2\}),\ A_3 := \mathbf{x}^{-1}(\{3\}).$$

If we can show that the last expression depends solely on the sizes of A_1, A_2, and A_3 and not on $\mathbf{x}$, then we are done. Since $\mathbf{x}$ is balanced, then $\{A_1, A_2, A_3\} \subseteq \binom{[3n]}{n}$ is a partition of the set $[3n]$. Consequently, we know that

$$A_1 \cup A_2 = \mathbf{x}^{-1}(\{1, 2\}),\ A_1 \cup A_3 = \mathbf{x}^{-1}(\{1, 3\}),\ A_2 \cup A_3 = \mathbf{x}^{-1}(\{2, 3\}),$$

$$\emptyset = \mathbf{x}^{-1}(\emptyset),\ [3n] = \mathbf{x}^{-1}([3]).$$

Additionally, notice that since $\mathbf{x}$ is balanced, then

$$A_1 = (A_2 \cup A_3)^c,\ A_2 = (A_1 \cup A_3)^c,\ \text{and } A_3 = (A_1 \cup A_2)^c$$

and that $|A_1| = |A_2| = |A_3| = n$ and $|A_1 \cup A_2| = |A_1 \cup A_3| = |A_2 \cup A_3| = 2n$. There are exactly 6 ways in which the set $\{A_1, A_2, A_3\}$ can be ordered by $\prec_i$, where $i \in [6]$. To avoid repetitiveness, for any permutation $\pi \in S_3$, we define the following strict linear order

$$\emptyset \prec^*_\pi \{\pi(1)\} \prec^*_\pi \{\pi(2)\} \prec^*_\pi \{\pi(3)\} \prec^*_\pi$$

$$\prec^*_\pi \{\pi(2), \pi(3)\} \prec^*_\pi \{\pi(1), \pi(3)\} \prec^*_\pi \{\pi(1), \pi(2)\} \prec^*_\pi [3n].$$

This newly defined relation is a strict linear order that extends inclusion. Furthermore, for every $i \in [6]$, we define $\ll_i := \mathbf{x}(\prec_i)$. Since $\prec_i \restriction_{\{A_1,A_2,A_3\}^2}$, where $i \in [6]$, are all the six linear orders on $\{A_1, A_2, A_3\}$, then, for all $i \in [6]$, there exists exactly one $\pi \in S_3$ such that

$$A_{\pi(1)} \prec_i A_{\pi(2)} \prec_i A_{\pi(3)}.$$

If we fix $i \in [6]$, then using the definition of $\prec_i$, we get

$$(A_{\pi(2)} \cup A_{\pi(3)}) \prec_i (A_{\pi(1)} \cup A_{\pi(3)}) \prec_i (A_{\pi(1)} \cup A_{\pi(2)}).$$

Since $\prec_i$ is graded, we have

$$\emptyset \prec_i A_{\pi(1)} \prec_i A_{\pi(2)} \prec_i A_{\pi(3)} \prec_i$$

$$\prec_i (A_{\pi(2)} \cup A_{\pi(3)}) \prec_i (A_{\pi(1)} \cup A_{\pi(3)}) \prec_i (A_{\pi(1)} \cup A_{\pi(2)}) \prec_i [3n].$$

Consequently, we obtain

$$\emptyset \ll_i \{\pi(1)\} \ll_i \{\pi(2)\} \ll_i \{\pi(3)\} \ll_i$$

$$\ll_i \{\pi(2), \pi(3)\} \ll_i \{\pi(1), \pi(3)\} \ll_i \{\pi(1), \pi(2)\} \ll_i [3].$$

Therefore, we know the following:

- For every $i \in [6]$, $\ll_i$ is one of the six strict graded linear orders $\prec^*_\pi$ on the 8-element set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, [3]\}$ shown above.
- Due to Lemma 2, the mapping $i \mapsto \pi : [6] \longrightarrow S_3$, where $A_{\pi(1)} \prec_i A_{\pi(2)} \prec_i A_{\pi(3)}$, is a bijection.

Thus, $t_{3,[3],\ll_i}$, where $i \in [6]$, are always the same 6 LO-operations provided that $\mathbf{x}$ is balanced. Since these LO-operations are always evaluated on the fixed tuple $(1,2,3)$, there exists $(u_1, u_2, u_3, u_4, u_5, u_6) \in A^6$ such that for every balanced $\mathbf{x} \in A^n$, we have the following:

$$(t_{3,[3],\ll_1}(1,2,3), t_{3,[3],\ll_2}(1,2,3), ..., t_{3,[3],\ll_6}(1,2,3)) = (u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, u_{i_5}, u_{i_6}),$$

where $i_j \in [6]$ are pair-wise different. Since s_6 is symmetric,

$$s_{3n}(x_1, ..., x_{3n}) = s_6(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, u_{i_5}, u_{i_6}) = s_6(u_1, u_2, u_3, u_4, u_5, u_6).$$

This example emphasizes the key ideas that are used repeatedly in the upcoming proof of Theorem 3. Given that Brady's proof involves an extensive case analysis, we have chosen to present this simplified version as well, allowing the reader to become familiar with the argument. □

3. Symmetric operations

3.1 Brady's proof

Let us begin by defining new binary relations using the linear orders from Lemma 1. Let n be an arbitrary natural number. Put $X := 2^{[n]}$ and let

$$<_1^+, <_1^-, <_2^+, <_2^-, <_3^+, <_3^-$$

be the linear orders from Lemma 1 on X. Let $i \in [3]$ and $a, b \in \{+, -\}$ be arbitrary, then we define a binary relation on $2^{[n]}$ in the following way:

$$\begin{aligned} U <_{iab} V \quad \equiv \quad & (|U| < |V|) \vee \\ & (|U| = |V| < \frac{n}{2} \wedge U <_i^a V) \vee \\ & (|U| = |V| = \frac{n}{2} \wedge U <_1^b V) \vee \\ & (|U| = |V| > \frac{n}{2} \wedge V^c <_i^a U^c) \end{aligned}$$

Lemma 5. *For any n natural, the binary relations $<_{iab}$ on $2^{[n]}$, where $i \in [3]$ and $a, b \in \{+, -\}$, are graded linear strict orders that extend the inclusion relation $\subset$.*

Proof. Fix $i \in [3]$ and fix $a, b \in \{+, -\}$. Firstly, we prove the relation $<_{iab}$ to be irreflexive. By way of contradiction, assume that there is a subset U such that $U <_{iab} U$. Since $|U| = |U|$, the first condition in the definition of $<_{iab}$ isn't fulfilled. Thus, one of the following must be true: $U <_i^a U$ or $U <_1^b U$ or $U^c <_i^a U^c$. However, these are all strict linear orders; thus, we get a contradiction because linear orders are irreflexive. Next, we prove that $<_{iab}$ is transitive. Let U, V, W be arbitrary subsets of the set $[n]$ and suppose that $U <_{iab} V$ and $V <_{iab} W$, then there are four cases:

- If $|U| < |V|$, then it follows that either $|V| < |W|$ or $|V| = |W|$. The case where $|V| > |W|$ cannot occur since the order $<_{iab}$ is clearly graded and we assume that $V <_{iab} W$. In both of these cases, $|U| < |W|$ because $<$ is transitive. This implies that $U <_{iab} W$.

- If $|U| = |V| < \frac{n}{2} \wedge U <_i^a V$, then, as previously stated, either $|V| < |W|$ or $|V| = |W|$. If $|V| < |W|$, then it follows that $|U| < |W|$, which yields $U <_{iab} W$. If $|V| = |W| < \frac{n}{2}$, then $V <_i^a W$. In this case, we have $|U| = |W| = \frac{n}{2}$ and $U <_i^a V <_i^a W$. Since $<_i^a$ is transitive, we conclude that $U <_{iab} W$.

- If $|U| = |V| = \frac{n}{2} \wedge U <_1^b V$, then either $|V| < |W|$ or $|V| = |W|$. If $|V| < |W|$, then it follows that $|U| < |W|$, which implies $U <_{iab} W$. If $|V| = |W|$, then we have $|W| = \frac{n}{2}$ and $V <_1^b W$. This implies that $|U| = |W| = \frac{n}{2}$ and $U <_1^b W$. Consequently, $U <_{iab} W$ holds since $<_1^b$ is transitive.

- Lastly, if $|U| = |V| > \frac{n}{2} \wedge V^c <_i^a U^c$, then either $|V| < |W|$ or $|V| = |W|$. If $|V| < |W|$, then $|U| < |W|$, which shows that $U <_{iab} W$. If $|V| = |W|$, then we have $|V| = |W| > \frac{n}{2}$ and $W^c <_i^a V^c$. Given that the relation $<_i^a$ is transitive, we can conclude that $|U| = |W| > \frac{n}{2}$ and $W^c <_i^a U^c$, which implies that $U <_{iab} W$.

Furthermore, we prove the order $<_{iab}$ to be linear. Choose arbitrary $U, V \subseteq 2^{[n]}$.

- If $|U| < |V|$, then $U <_{iab} V$.
- If $|V| < |U|$, then $V <_{iab} U$.
- If $|U| = |V| < \frac{n}{2}$, then either $U <_i^a V$ or $V <_i^a U$ since the relation $<_i^a$ is linear. This yields either $U <_{iab} V$, in the case that $U <_i^a V$, or $V <_{iab} U$ provided that $V <_i^a U$.
- If $|U| = |V| = \frac{n}{2}$, then we have either $U <_1^b V$ or $V <_1^b U$ because $<_1^b$ is linear. This results in either $U <_{iab} V$ if $U <_1^b V$, or $V <_{iab} U$ if $V <_1^b U$.
- If $|U| = |V| > \frac{n}{2}$, then either $V^c <_i^a U^c$ or $U^c <_i^a V^c$ as $<_i^a$ is linear. This implies the either $U <_{iab} V$ if $V^c <_i^a U^c$, or $V <_{iab} U$ if $U^c <_i^a V^c$.

Lastly, we show that $<_{iab}$ extends inclusion. The linear order $<_{iab}$ is graded by definition. So, let U, $V \in 2^{[n]}$ be arbitrary and assume that $U \subset V$. Since the inclusion is strict, it follows that $|U| < |V|$. Because $<_{iab}$ is graded, we conclude that $U <_{iab} V$. □

We will now explain how the new linear orders are used to prove the existence of 3-symmetric operations of every arity (see Definition 12). Since we assume that the algebras we are working with contain exactly three elements, these 3-symmetric operations are indeed symmetric.

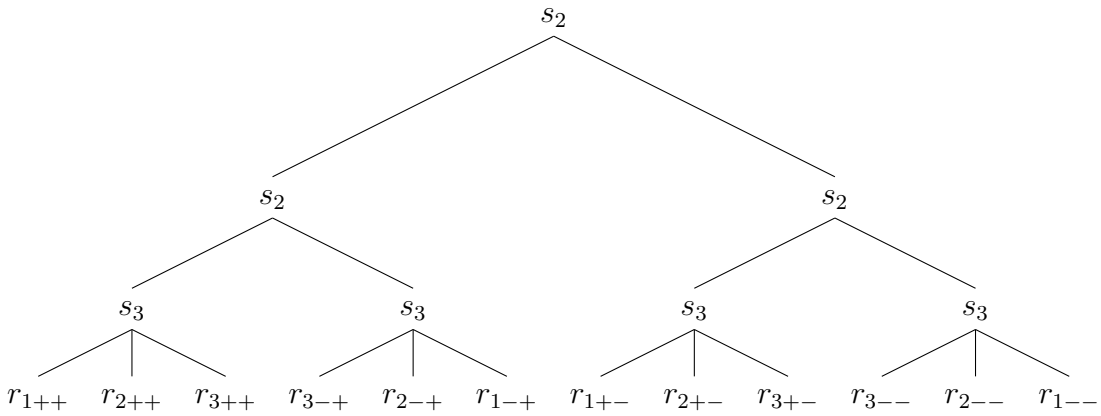
Theorem 3. *Let $\mathbb{A} = (\{1,2,3\}, (f)_{f\in\mathcal{F}})$ be a 3-element algebra in a signature $\mathcal{F}$ such that Clo($\mathbb{A}$) contains symmetric operation s_i of arity i, where $i \in [3]$, then Clo($\mathbb{A}$) contains a symmetric operation of every arity.*

Proof. Let n be a natural number, and let $<_{iab}$ be the twelve strict graded linear orders on the set $2^{\mathbb{N}}$ that extend inclusion $\subset$, where $i \in [3]$ and $a, b \in \{+,-\}$. According to Corollary 2, there exists an LO-system of term operations $(g_{n,X,\prec})_{n\in\mathbb{N},\emptyset\subset X\subseteq[n],\prec\in\mathbb{L}(X)}$. We can then define an n-ary term operation $r_{iab} := g_{n,[n],<_{iab}}$.

Using these auxiliary n-ary operations, we define an n-ary term operation

$$t := s_2 \circ \Big[s_2 \circ \Big(s_3 \circ (r_{1++}, r_{2++}, r_{3++}), s_3 \circ (r_{3-+}, r_{2-+}, r_{1-+}) \Big), \\ s_2 \circ \Big(s_3 \circ (r_{1+-}, r_{2+-}, r_{3+-}), s_3 \circ (r_{3--}, r_{2--}, r_{1--}) \Big) \Big]$$

(see Definition 10). The following figure depicts the operation t using a tree structure. Note that, for the sake of readability, we have removed the bottom layer of leaves displaying the variables; formally, this layer is supposed to be included.



We claim that t is 3-symmetric. Additionally, as in the previous section, for every $\pi \in S_6$, we define two strict orders, $\prec^1_\pi$ and $\prec^2_\pi$, on the 8-element set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, [3]\}$ as follows:

$$\begin{aligned}
&\emptyset \prec^1_\pi \{\pi(1)\} \prec^1_\pi \{\pi(2)\} \prec^1_\pi \{\pi(3)\} \prec^1_\pi \{\pi(1),\pi(2)\} \\
&\prec^1_\pi \{\pi(1),\pi(3)\} \prec^1_\pi \{\pi(2),\pi(3)\} \prec^1_\pi [3] \text{ and} \\
&\emptyset \prec^2_\pi \{\pi(1)\} \prec^2_\pi \{\pi(2)\} \prec^2_\pi \{\pi(1),\pi(2)\} \prec^2_\pi \\
&\{\pi(3)\} \prec^2_\pi \{\pi(1),\pi(3)\} \prec^2_\pi \{\pi(2),\pi(3)\} \prec^2_\pi [3].
\end{aligned}$$

These are clearly strict linear orders that extend inclusion. Therefore, we can define the following:

$$\begin{aligned}
u^1_\pi &:= g_{3,[3],\prec^1_\pi}(1,2,3) \\
u^2_\pi &:= g_{3,[3],\prec^2_\pi}(1,2,3).
\end{aligned}$$

Similarly, for any permutation $\sigma \in S_2$, we define a strict order $\prec^3_\sigma$ on the 4-element set $\{\emptyset, \{1\}, \{2\}, [2]\}$ by the following:

$$\emptyset \prec^3_\sigma \{\sigma(1)\} \prec^3_\sigma \{\sigma(2)\} \prec^3_\sigma [2]$$

This is a strict linear order that extends inclusion. Therefore, we put:

$$u^3_\sigma := g_{2,[2],\prec^3_\sigma}(1,2,3)$$

Note that permutations will always be written as products of pair-wise disjoint cycles omitting the 1-cycles. Let $\mathbf{x} := (x_1, ..., x_n)$ be a tuple of values from the universe of $\mathbb{A}$, meaning that $\emptyset \subset \{x_1, ..., x_n\} \subseteq A$. If $\mathbf{x}$ is viewed as a mapping from $[n]$ to $\{1,2,3\}$, we define the pre-images:

$$U_1 := \mathbf{x}^{-1}(\{1\}),\ U_2 := \mathbf{x}^{-1}(\{2\}),\ \text{and}\ U_3 := \mathbf{x}^{-1}(\{3\}).$$

Then we have $t(x_1, ..., x_n) = t^{\mathbf{x}}(1,2,3)$, which is equal to

$$\begin{aligned}
s_2\Big[s_2\Big(&s_3(r^{\mathbf{x}}_{1++}(1,2,3), r^{\mathbf{x}}_{2++}(1,2,3), r^{\mathbf{x}}_{3++}(1,2,3)), \\
&s_3(r^{\mathbf{x}}_{3-+}(1,2,3), r^{\mathbf{x}}_{2-+}(1,2,3), r^{\mathbf{x}}_{1-+}(1,2,3))\Big), \\
s_2\Big(&s_3(r^{\mathbf{x}}_{1+-}(1,2,3), r^{\mathbf{x}}_{2+-}(1,2,3), r^{\mathbf{x}}_{3+-}(1,2,3)), \\
&s_3(r^{\mathbf{x}}_{3--}(1,2,3), r^{\mathbf{x}}_{2--}(1,2,3), r^{\mathbf{x}}_{1--}(1,2,3))\Big)\Big].
\end{aligned}$$

Therefore, the value of $t^{\mathbf{x}}(1,2,3)$ is determined by $r^{\mathbf{x}}_{iab}(1,2,3) = g_{3,\mathbf{x}([n]),\mathbf{x}(<_{iab})}$. Thus, it is determined by $\mathbf{x}(<_{iab})$ and the sizes $|U_i|$, where $i \in [3]$. Throughout this section, $\mathbf{x}(<_{iab})$ will be referred to as $\ll_{iab}$.

Let $i \in [3]$, $a, b \in \{+,-\}$ be arbitrary. Without loss of generality, we can assume that $|U_3| \leq |U_2| \leq |U_1|$. If this is not the case, we can simply rename the elements of the algebra $\mathbb{A}$ accordingly. We will examine a total of 11 cases, showing that the operation t produces the same output regardless of how the sets U_1, U_2, U_3 are ordered with respect to the twelve newly defined strict linear orders $<_{iab}$. We begin with the degenerate cases where the input tuple contains only one or two of the elements of A.

Case 1. $|U_2| = |U_3| = 0$

This case is trivial as the input tuple contains only one element: element 1.

Case 2. $0 = |U_3| < |U_2| \ \wedge \ |U_2| < |U_1|$

Since $|U_3| = 0$, we have $\{\emptyset, U_1, U_2, U_3, U_1 \cup U_2, U_1 \cup U_3, U_2 \cup U_3, U_1 \cup U_2 \cup U_3\} = \{\emptyset, U_1, U_2, U_1 \cup U_2\}$. Given that $0 = |\emptyset| < |U_2| < |U_1| < |U_1 \cup U_2| = |U_1| + |U_2|$, and that $<_{iab}$ is graded, we conclude that

$$\emptyset <_{iab} U_2 <_{iab} U_1 <_{iab} U_1 \cup U_2.$$

Thus, using Definition 3, we get

$$\emptyset \ll_{iab} \{2\} \ll_{iab} \{1\} \ll_{iab} [2],$$

which is precisely the order $\prec^3_{(1\ 2)}$. Therefore,

$$r^{\mathbf{x}}_{iab}(1,2,3) = g^{\mathbf{x}}_{n,[n],<_{iab}}(1,2,3) = g_{3,[2],\ll_{iab}}(1,2,3) = g_{3,[2],\prec^3_{(1\ 2)}}(1,2,3) = u^3_{(1\ 2)}.$$

Thus, for every $\mathbf{x} \in A^n$ satisfying $0 = |U_3| < |U_2| < |U_1|$, we obtain the following:

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^3_{(1\ 2)}, u^3_{(1\ 2)}, u^3_{(1\ 2)}), s_3(u^3_{(1\ 2)}, u^3_{(1\ 2)}, u^3_{(1\ 2)})\Big), \\ &s_2\Big(s_3(u^3_{(1\ 2)}, u^3_{(1\ 2)}, u^3_{(1\ 2)}), s_3(u^3_{(1\ 2)}, u^3_{(1\ 2)}, u^3_{(1\ 2)})\Big)\Big]. \end{aligned}$$

Case 3. $0 = |U_3| < |U_2| = |U_1| = \frac{n}{2}$

The assumption of this case implies that the sizes of the sets satisfy $|U_1|, |U_2| < |U_1 \cup U_2| = |U_1| + |U_2| = n$. Therefore, we can conclude that $U_1, U_2 <_{iab} U_1 \cup U_2$. Since $|U_1| = |U_2| = \frac{n}{2}$, the value of $r^{\mathbf{x}}_{iab}(1,2,3)$ depends solely on b. Let π be the permutation in S_2 such that $U_{\pi(1)} <^+_1 U_{\pi(2)}$. Since $<^-_1$ is the inverse of $<^+_1$, it follows that $U_{\pi(2)} <^-_1 U_{\pi(1)}$. Therefore,

$$\emptyset \ll_{ia+} \{\pi(1)\} \ll_{ia+} \{\pi(2)\} \ll_{ia+} [2]$$

$$\emptyset \ll_{ia-} \{\pi(2)\} \ll_{ia-} \{\pi(1)\} \ll_{ia-} [2].$$

Thus, we get the following values for the auxiliary operations:

$$r^{\mathbf{x}}_{ia+}(1,2,3) = g^{\mathbf{x}}_{n,[n],<_{ia+}}(1,2,3) = g_{3,[2],\prec^3_\pi}(1,2,3) = u^3_\pi$$

$$r^{\mathbf{x}}_{ia-}(1,2,3) = g^{\mathbf{x}}_{n,[n],<_{ia-}}(1,2,3) = g_{3,[2],\prec^3_{(1\ 2)\circ\pi}}(1,2,3) = u^3_{(1\ 2)\circ\pi}.$$

Therefore, we obtain

$$\begin{aligned} t(\mathbf{x}) &= s_2\Big[s_2\Big(s_3(u^3_\pi, u^3_\pi, u^3_\pi), s_3(u^3_\pi, u^3_\pi, u^3_\pi)\Big), \\ &\qquad s_2\Big(s_3(u^3_{(1\,2)\circ\pi}, u^3_{(1\,2)\circ\pi}, u^3_{(1\,2)\circ\pi}), s_3(u^3_{(1\,2)\circ\pi}, u^3_{(1\,2)\circ\pi}, u^3_{(1\,2)\circ\pi})\Big)\Big] \\ &= s_2\Big[s_2\Big(s_3(u^3_{\mathrm{id}}, u^3_{\mathrm{id}}, u^3_{\mathrm{id}}), s_3(u^3_{\mathrm{id}}, u^3_{\mathrm{id}}, u^3_{\mathrm{id}})\Big), \\ &\qquad s_2\Big(s_3(u^3_{(1\,2)}, u^3_{(1\,2)}, u^3_{(1\,2)}), s_3(u^3_{(1\,2)}, u^3_{(1\,2)}, u^3_{(1\,2)})\Big)\Big]. \end{aligned}$$

The second equality follows from the fact that s_2 is symmetric. Thus, the operation t must output the same value for every n-tuple $\mathbf{x} \in A^n$ corresponding to this case.

Case 4. $0 < |U_3| < |U_2| < |U_2 \cup U_3| < |U_1|$

Since $<_{iab}$ is graded, we have $\emptyset <_{iab} U_3 <_{iab} U_2 <_{iab} U_2 \cup U_3 <_{iab} U_1$ and since $\{U_1, U_2, U_3\}$ is a partition of the set $[n]$, we conclude that $|U_2 \cup U_3| = |U_2| + |U_3| < |U_1| + |U_3| = |U_1 \cup U_3| < |U_1| + |U_2| = |U_1 \cup U_2| < n = |[n]|$. This shows that $U_2 \cup U_3 <_{iab} U_1 \cup U_3 <_{iab} U_1 \cup U_2 <_{iab} [n]$. Thus, we have the following:

$$\emptyset <_{iab} U_3 <_{iab} U_2 <_{iab} U_2 \cup U_3 <_{iab} U_1 <_{iab} U_1 \cup U_3 <_{iab} U_1 \cup U_2 <_{iab} [n]$$

because the order $<_{iab}$ is graded. This demonstrates that

$$\emptyset \ll_{iab} \{3\} \ll_{iab} \{2\} \ll_{iab} \{2,3\} \ll_{iab} \{1\} \ll_{iab} \{1,3\} \ll_{iab} \{1,2\} \ll_{iab} [3],$$

which is exactly the order $\prec^2_{(1\,3)}$. Therefore, we have

$$r^{\mathbf{x}}_{iab}(1,2,3) = g^{\mathbf{x}}_{n,[n],<_{iab}}(1,2,3) = g_{3,[3],\ll_{iab}}(1,2,3) = g_{3,[3],\prec^2_{(1\,3)}}(1,2,3) = u^2_{(1\,3)}.$$

Hence, for every $\mathbf{x} \in A^n$ corresponding to this case, we get

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)}), s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)})\Big), \\ &s_2\Big(s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)}), s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)})\Big)\Big]. \end{aligned}$$

Case 5. $0, |U_3|, |U_2|, |U_1|, |U_2 \cup U_3|$ are pair-wise different and $|U_1| < |U_2 \cup U_3|$.

This case is almost identical to the previous one, so we will proceed faster. It follows that

$$0 = |\emptyset| < |U_3| < |U_2| < |U_1| < |U_2 \cup U_3| < |U_1 \cup U_3| < |U_1 \cup U_2| < |[n]| = n.$$

From this, we get

$$\emptyset \ll_{iab} \{3\} \ll_{iab} \{2\} \ll_{iab} \{1\} \ll_{iab} \{2,3\} \ll_{iab} \{1,3\} \ll_{iab} \{1,2\} \ll_{iab} [3]$$

This corresponds exactly to the strict linear order $\prec^1_{(1\,3)}$. Similarly, as in the second case, it follows that

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)}), s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)})\Big), \\ &s_2\Big(s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)}), s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)})\Big)\Big] \end{aligned}$$

for every $\mathbf{x} \in A^n$ corresponding to this case.

Case 6. $0 < |U_3| < |U_2| < |U_1| = |U_2 \cup U_3|$

Then we have

$$0 < |U_3| < |U_2| < |U_1| = |U_2| + |U_3| < |U_1| + |U_3| < |U_1| + |U_2| < n.$$

Since $<_{iab}$ is graded, we have

$$\emptyset <_{iab} U_3 <_{iab} U_2 <_{iab} U_1, U_2 \cup U_3 <_{iab} U_1 \cup U_3 <_{iab} U_1 \cup U_2 <_{iab} [n].$$

Furthermore, $n = |[n]| = |U_1 \cup U_2 \cup U_3| = |U_1| + |U_2| + |U_3| = 2(|U_2| + |U_3|) = 2|U_2 \cup U_3|$, which implies that $|U_1| = |U_2 \cup U_3| = \frac{n}{2}$. Thus, the value of $r^{\mathbf{x}}_{iab}(1,2,3)$ depends solely on b; and hence there are two cases:

- If $U_1 <_1^+ U_2 \cup U_3$, then $U_2 \cup U_3 <_1^- U_1$. Thus,

$$\begin{aligned}
&\emptyset <_{ia+} U_3 <_{ia+} U_2 <_{ia+} U_1 <_{ia+} U_2 \cup U_3 \\
&<_{ia+} U_1 \cup U_3 <_{ia+} U_1 \cup U_2 <_{ia+} [n], \\
&\emptyset <_{ia-} U_3 <_{ia-} U_2 <_{ia-} U_2 \cup U_3 <_{ia-} U_1 \\
&<_{ia-} U_1 \cup U_3 <_{ia-} U_1 \cup U_2 <_{ia-} [n].
\end{aligned}$$

  The previous implies that

$$\begin{aligned}
&\emptyset \ll_{ia+} \{3\} \ll_{ia+} \{2\} \ll_{ia+} \{1\} \ll_{ia+} \\
&\{2,3\} \ll_{ia+} \{1,3\} \ll_{ia+} \{1,2\} \ll_{ia+} [3], \\
&\emptyset \ll_{ia-} \{3\} \ll_{ia-} \{2\} \ll_{ia-} \{2,3\} \ll_{ia-} \\
&\{1\} \ll_{ia-} \{1,3\} \ll_{ia-} \{1,2\} \ll_{ia-} [3].
\end{aligned}$$

  These are the strict linear orders $\prec^1_{(1\,3)}$ and $\prec^2_{(1\,3)}$, respectively. Thus,

$$\begin{aligned}
r^{\mathbf{x}}_{ia+}(1,2,3) &= g^{\mathbf{x}}_{n,[n],<_{ia+}}(1,2,3) = g_{3,[3],\prec^1_{(1\,3)}}(1,2,3) = u^1_{(1\,3)}, \\
r^{\mathbf{x}}_{ia-}(1,2,3) &= g^{\mathbf{x}}_{n,[n],<_{ia-}}(1,2,3) = g_{3,[3],\prec^2_{(1\,3)}}(1,2,3) = u^2_{(1\,3)}.
\end{aligned}$$

  Therefore,

$$\begin{aligned}
t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)}), s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)})\Big), \\
&s_2\Big(s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)}), s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)})\Big)\Big].
\end{aligned}$$

- If $U_2 \cup U_3 <_1^+ U_1$, then, similarly as in the previous sub-case, we obtain the following:

$$\begin{aligned}
t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)}), s_3(u^2_{(1\,3)}, u^2_{(1\,3)}, u^2_{(1\,3)})\Big), \\
&s_2\Big(s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)}), s_3(u^1_{(1\,3)}, u^1_{(1\,3)}, u^1_{(1\,3)})\Big)\Big]
\end{aligned}$$

Since s_2 is symmetric, we conclude that the operation t is symmetric on n-tuples corresponding to this case.

Case 7. $0 < |U_3| < |U_1| = |U_2|$

This implies that

$$0 < |U_3| < |U_2| = |U_1| < |U_2| + |U_3| = |U_2 \cup U_3| = |U_1| + |U_3| = |U_1 \cup U_3| < n.$$

It follows that

$$\emptyset <_{iab} U_3 <_{iab} U_1, U_2 <_{iab} U_2 \cup U_3, U_1 \cup U_3 <_{iab} U_1 \cup U_2 <_{iab} [n]$$

since the order $<_{iab}$ is graded. It is self-evident that $|U_1| = |U_2| < \frac{n}{2}$. Hence, there are two options: either $U_1 <_{i+b} U_2$ or $U_2 <_{i+b} U_1$. Let π be the element of $\{\mathrm{id},\ (1\ 2)\} \subset S_3$ such that $U_{\pi(1)} <_{i+b} U_{\pi(2)}$. This gives $U_{\pi(1)} = (U_{\pi(2)} \cup U_3)^c <_{i+b} (U_{\pi(1)} \cup U_3)^c = U_{\pi(2)}$ and $|U_{\pi(2)} \cup U_3| = |U_{\pi(1)} \cup U_3| > \frac{n}{2}$. Using the definition of $<_{i+b}$, we see that $U_{\pi(1)} \cup U_3 <_{i+b} U_{\pi(2)} \cup U_3$. Furthermore, because $<_{i-b}$ is the inverse of $<_{i+b}$, we have $U_{\pi(2)} <_{i-b} U_{\pi(1)}$ and $U_{\pi(2)} \cup U_3 <_{i-b} U_{\pi(1)} \cup U_3$. Therefore, we obtain the following:

$$\begin{aligned}
&\emptyset <_{i+b} U_3 <_{i+b} U_{\pi(1)} <_{i+b} U_{\pi(2)} <_{i+b} \\
&U_{\pi(1)} \cup U_3 <_{i+b} U_{\pi(2)} \cup U_3 <_{i+b} U_1 \cup U_2 <_{i+b} [n], \text{ and} \\
&\emptyset <_{i-b} U_3 <_{i-b} U_{\pi(2)} <_{i-b} U_{\pi(1)} <_{i-b} \\
&U_{\pi(2)} \cup U_3 <_{i-b} U_{\pi(1)} \cup U_3 <_{i-b} U_1 \cup U_2 <_{i-b} [n].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\emptyset \ll_{i+b} \{3\} \ll_{i+b} \{\pi(1)\} \ll_{i+b} \{\pi(2)\} \ll_{i+b} \\
&\{\pi(1), 3\} \ll_{i+b} \{\pi(2), 3\} \ll_{i+b} \{1, 2\} \ll_{i+b} [3], \text{ and} \\
&\emptyset \ll_{i-b} \{3\} \ll_{i-b} \{\pi(2)\} \ll_{i-b} \{\pi(1)\} \ll_{i-b} \\
&\{\pi(2), 3\} \ll_{i-b} \{\pi(1), 3\} \ll_{i-b} \{1, 2\} \ll_{i-b} [3].
\end{aligned}$$

From this, it follows that

$$\ll_{i+b} = \prec^1_{(\pi(1)\ 3\ \pi(2))\circ\pi} \text{ and that } \ll_{i-b} = \prec^1_{(\pi(1)\ 3)\circ\pi}.$$

Therefore, we get

$$\begin{aligned}
r^{\mathbf{x}}_{i+b}(1,2,3) &= g^{\mathbf{x}}_{n,[n],\ll_{i+b}}(1,2,3) = g_{3,[3],\prec^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}}(1,2,3) = u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}, \text{ and} \\
r^{\mathbf{x}}_{i-b}(1,2,3) &= g^{\mathbf{x}}_{n,[n],\ll_{i-b}}(1,2,3) = g_{3,[3],\prec^1_{(\pi(1)\ 3)\circ\pi}}(1,2,3) = u^1_{(\pi(1)\ 3)\circ\pi}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}), \\
&\quad s_3(u^1_{(\pi(1)\ 3)\circ\pi}, u^1_{(\pi(1)\ 3)\circ\pi}, u^1_{(\pi(1)\ 3)\circ\pi})\Big), \\
&\quad s_2\Big(s_3(u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ 3\ \pi(2))\circ\pi}), \\
&\quad s_3(u^1_{(\pi(1)\ 3)\circ\pi}, u^1_{(\pi(1)\ 3)\circ\pi}, u^1_{(\pi(1)\ 3)\circ\pi})\Big)\Big] \\
= s_2\Big[&s_2\Big(s_3(u^1_{(1\ 3\ 2)}, u^1_{(1\ 3\ 2)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u_{(1\ 3)})\Big), \\
&\quad s_2\Big(s_3(u^1_{(1\ 3\ 2)}, u^1_{(1\ 3\ 2)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big)\Big].
\end{aligned}$$

Where the last equality follows from the fact that s_2 is symmetric. Hence, the result holds for every $\mathbf{x}$ of this type.

Case 8. $0 < |U_2| = |U_3| < |U_1| < |U_2 \cup U_3|$

It follows that

$$0 < |U_2| = |U_3| < |U_1| < |U_2 \cup U_3| < |U_1 \cup U_3| = |U_1 \cup U_2| < n.$$

Since $<_{iab}$ is graded, we conclude that

$$\emptyset <_{iab} U_2, U_3 <_{iab} U_1 <_{iab} U_2 \cup U_3 <_{iab} U_1 \cup U_3, U_1 \cup U_2 <_{iab} [n].$$

Thus, $r^{\mathbf{x}}_{iab}(1,2,3)$ depends only on a because $|U_2| = |U_3| < \frac{n}{2}$. Hence, it leads to 2 cases: either $U_2 <_{i+b} U_3$ or $U_3 <_{i+b} U_2$. Consider the permutation $\pi \in \{\mathrm{id}, (2\ 3)\} \subset S_3$ such that $U_{\pi(2)} <_{i+b} U_{\pi(3)}$. This implies that $U_{\pi(2)} = (U_1 \cup U_{\pi(3)})^c <_{i+b} (U_1 \cup U_{\pi(2)})^c = U_{\pi(3)} \ \wedge\ |U_1 \cup U_{\pi(2)}| = |U_1 \cup U_{\pi(3)}| > \frac{n}{2}$, which yields $U_1 \cup U_{\pi(2)} <_{i+b} U_1 \cup U_{\pi(3)}$. Thus, we conclude that

$$\begin{aligned}&\emptyset <_{i+b} U_{\pi(2)} <_{i+b} U_{\pi(3)} <_{i+b} U_1 <_{i+b} U_2 \cup U_3 <_{i+b}\\ &U_1 \cup U_{\pi(2)} <_{i+b} U_1 \cup U_{\pi(3)} <_{i+b} [n].\end{aligned}$$

Therefore,

$$\begin{aligned}&\emptyset \ll_{i+b} \{\pi(2)\} \ll_{i+b} \{\pi(3)\} \ll_{i+b} \{1\} \ll_{i+b}\\ &\{2,3\} \ll_{i+b} \{1,\pi(2)\} \ll_{i+b} \{1,\pi(3)\} \ll_{i+b} [3],\end{aligned}$$

which means that $\ll_{i+b} = \prec^1_{(1\ \pi(2)\ \pi(3))\circ\pi}$. Furthermore, due to the fact that $<_{i-b}$ is the inverse of $<_{i+a}$, we have $U_{\pi(3)} <_{i-b} U_{\pi(2)}$. Therefore, similarly, to the above, we have $\ll_{i-b} = \prec^1_{(1\ \pi(3))\circ\pi}$. Hence,

$$\begin{aligned}r^{\mathbf{x}}_{i+b}(1,2,3) &= g^{\mathbf{x}}_{n,[n],<_{i+b}}(1,2,3) = g_{3,[3],\ll_{i+b}}(1,2,3) = u^1_{(1\ \pi(2)\ \pi(3))\circ\pi} \text{ and}\\ r^{\mathbf{x}}_{i-b}(1,2,3) &= g^{\mathbf{x}}_{n,[n],<_{i-b}}(1,2,3) = g_{3,[3],\ll_{i-b}}(1,2,3) = u^1_{(1\ \pi(3))\circ\pi}.\end{aligned}$$

Therefore,

$$\begin{aligned}t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi})\Big),\\ &s_2\Big(s_3(u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi})\Big)\Big]\\ = s_2\Big[&s_2\Big(s_3(u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big),\\ &s_2\Big(s_3(u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big)\Big].\end{aligned}$$

Where the last equality follows from the fact that s_2 is symmetric.

Case 9. $0 < |U_2| = |U_3| < |U_2 \cup U_3| < U_1$

This case is almost identical to the previous one, so we proceed at a much faster pace. Since $0 < |U_2| = |U_3| < |U_2 \cup U_3| < |U_1|$, it follows that

$$0 < |U_2| = |U_3| < |U_2 \cup U_3| < |U_1| < |U_3 \cup U_1| = |U_2 \cup U_1| < [n].$$

Since $<_{iab}$ is graded, we have the following:

$$\emptyset <_{iab} U_2, U_3 <_{iab} U_2 \cup U_3 <_{iab} U_1 <_{iab} U_1 \cup U_3, U_1 \cup U_2 <_{iab} [n].$$

This ordering again depends on a as $|U_2| = |U_3| < \frac{n}{2}$. Let $\pi \in \{\mathrm{id}, (2\ 3)\} \subset S_3$ be such that $U_{\pi(2)} <_{i+b} U_{\pi(3)}$, then

$$\ll_{i+b} = \prec^2_{(1\ \pi(2)\ \pi(3))\circ\pi} \text{ and } \ll_{i-b} = \prec^2_{(1\ \pi(3))\circ\pi}.$$

Hence,

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[s_2\Big(&s_3(u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi})\Big),\\ &s_2\Big(s_3(u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi})\Big)\Big]\\ = s_2\Big[s_2\Big(&s_3(u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}), s_3(u^2_{(1\ 3)}, u^2_{(1\ 3)}, u^2_{(1\ 3)})\Big),\\ &s_2\Big(s_3(u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}), s_3(u^2_{(1\ 3)}, u^2_{(1\ 3)}, u^2_{(1\ 3)})\Big)\Big]. \end{aligned}$$

Again, the last equality follows from the fact that s_2 is symmetric.

Case 10. $0 < |U_2| = |U_3| < |U_1| = |U_2| + |U_3|$

It follows from the assumption that

$$\begin{aligned} &0 < |U_2| = |U_3| < |U_1| = |U_2 \cup U_3| = |U_2| + |U_3| <\\ &|U_1| + |U_3| = |U_1 \cup U_3| = |U_1| + |U_2| = |U_1 \cup U_2| < n \end{aligned}$$

Since $<_{iab}$ is graded, we have

$$\emptyset <_{iab} U_2, U_3 <_{iab} U_1, U_2 \cup U_3 <_{iab} U_1 \cup U_3, U_1 \cup U_2 <_{iab} [n].$$

Because $\{U_1, U_2, U_3\}$ is a partition of the set $[n]$, we get

$$n = |[n]| = |\bigcup\{U_1, U_2, U_3\}| = |U_1| + |U_2| + |U_3|.$$

Furthermore, the condition $|U_1| = |U_2| + |U_3|$ implies that

$$n = |U_2| + |U_3| + |U_2| + |U_3|.$$

Since $|U_2| = |U_3|$, it follows that

$$n = |U_2| + |U_2| + |U_2| + |U_3| = 3|U_2| + |U_3|.$$

One can similarly derive the equality $n = |U_2| + 3|U_3|$. By solving this system of two linear equations with two unknowns, we get $|U_2| = |U_3| = \frac{n}{4} \ \wedge \ |U_1| = \frac{n}{2}$.

The order of U_2 and U_3 with respect to $<_{iab}$ depends solely on $a \in \{+, -\}$. Moreover, if π is the permutation in $\{\mathrm{id}, (2\ 3)\} \subset S_3$ such that $U_{\pi(2)} <_{i+b} U_{\pi(3)}$, then $U_{\pi(3)} <_{i-b} U_{\pi(2)}$ because $<^-_i$ is the inverse of $<^+_i$. It also follows that $(U_1 \cup U_{\pi(3)})^c = U_{\pi(2)} <_{i+b} U_{\pi(3)} = (U_1 \cup U_{\pi(2)})^c$. Thus, using again the fact that $<_{i-b}$ is the inverse of $<_{i+b}$, we get $U_1 \cup U_{\pi(2)} <_{i+b} U_1 \cup U_{\pi(3)}$ and $U_1 \cup U_{\pi(3)} <_{i-b} U_1 \cup U_{\pi(1)}$. Lastly, the order of U_1 and $U_2 \cup U_3$ with respect to $<_{iab}$ depends only on $b \in \{+, -\}$, for $|U_1| = |U_2 \cup U_3| = \frac{n}{2}$. Therefore, we have to distinguish two cases.

- If $U_1 <_{ia+} U_2 \cup U_3$, then

$$\begin{aligned}
&\emptyset <_{i++} U_{\pi(2)} <_{i++} U_{\pi(3)} <_{i++} U_1 <_{i++} U_2 \cup U_3 \\
&<_{i++} U_1 \cup U_{\pi(2)} <_{i++} U_1 \cup U_{\pi(3)} <_{i++} [n], \\
&\emptyset <_{i+-} U_{\pi(2)} <_{i+-} U_{\pi(3)} <_{i+-} U_2 \cup U_3 <_{i+-} U_1 \\
&<_{i+-} U_1 \cup U_{\pi(2)} <_{i+-} U_1 \cup U_{\pi(3)} <_{i+-} [n], \\
&\emptyset <_{i-+} U_{\pi(3)} <_{i-+} U_{\pi(2)} <_{i-+} U_1 <_{i-+} U_2 \cup U_3 \\
&<_{i-+} U_1 \cup U_{\pi(3)} <_{i-+} U_1 \cup U_{\pi(2)} <_{i-+} [n], \text{ and} \\
&\emptyset <_{i--} U_{\pi(3)} <_{i--} U_{\pi(2)} <_{i--} U_2 \cup U_3 <_{i--} U_1 \\
&<_{i--} U_1 \cup U_{\pi(3)} <_{i--} U_1 \cup U_{\pi(2)} <_{i--} [n].
\end{aligned}$$

  Thus,

$$\begin{aligned}
&\emptyset \ll_{i++} \{\pi(2)\} \ll_{i++} \{\pi(3)\} \ll_{i++} \{1\} \ll_{i++} \\
&\{2,3\} \ll_{i++} \{1,\pi(2)\} \ll_{i++} \{1,\pi(3)\} \ll_{i++} [3], \\
&\emptyset \ll_{i+-} \{\pi(2)\} \ll_{i+-} \{\pi(3)\} \ll_{i+-} \{2,3\} \ll_{i+-} \\
&\{1\} \ll_{i+-} \{1,\pi(2)\} \ll_{i+-} \{1,\pi(3)\} \ll_{i+-} [3], \\
&\emptyset \ll_{i-+} \{\pi(3)\} \ll_{i-+} \{\pi(2)\} \ll_{i-+} \{1\} \ll_{i-+} \\
&\{2,3\} \ll_{i-+} \{1,\pi(3)\} \ll_{i-+} \{1,\pi(2)\} \ll_{i-+} [3], \\
&\emptyset \ll_{i--} \{\pi(3)\} \ll_{i--} \{\pi(2)\} \ll_{i--} \{2,3\} \ll_{i--} \\
&\{1\} \ll_{i--} \{1,\pi(3)\} \ll_{i--} \{1,\pi(2)\} \ll_{i--} [3].
\end{aligned}$$

  Therefore, one can see that

$$\begin{aligned}
&\ll_{i++} = \prec^1_{(1\ \pi(2)\ \pi(3))\circ\pi},\ \ll_{i-+} = \prec^1_{(1\ \pi(3))\circ\pi}, \\
&\ll_{i+-} = \prec^2_{(1\ \pi(2)\ \pi(3))\circ\pi},\ \text{and}\ \ \ll_{i--} = \prec^2_{(1\ \pi(3))\circ\pi}\ .
\end{aligned}$$

  Thus,

$$\begin{aligned}
&r^{\mathbf{x}}_{i++}(1,2,3) = u^1_{(1\ \pi(2)\ \pi(3))\circ\pi},\ r^{\mathbf{x}}_{i-+}(1,2,3) = u^1_{(1\ \pi(3))\circ\pi}, \\
&r^{\mathbf{x}}_{i+-}(1,2,3) = u^2_{(1\ \pi(2)\ \pi(3))\circ\pi},\ r^{\mathbf{x}}_{i--}(1,2,3) = u^2_{(1\ \pi(3))\circ\pi}.
\end{aligned}$$

  Hence,

$$\begin{aligned}
t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi} u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}), \\
&\quad s_3(u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi})\Big), \\
&\quad s_2\Big(s_3(u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}), \\
&\quad s_3(u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi})\Big)\Big] \\
= s_2\Big[&s_2\Big(s_3(u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big), \\
&\quad s_2\Big(s_3(u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}), s_3(u^2_{(1\ 3)}, u^2_{(1\ 3)}, u^2_{(1\ 3)})\Big)\Big],
\end{aligned}$$

  where the second equality holds because s_2 is symmetric.

- If $U_2 \cup U_3 <_{ia+} U_1$, then it is easy to see that $\ll_{ia+}$ and $\ll_{ia-}$, written above in the previous item, swap places. More precisely, in this sub-case,

$$\begin{aligned}
&\ll_{i++} = \prec^2_{(1\ \pi(2)\ \pi(3))\circ\pi},\ \ll_{i-+} = \prec^2_{(1\ \pi(3))\circ\pi}, \\
&\ll_{i+-} = \prec^1_{(1\ \pi(2)\ \pi(3))\circ\pi},\ \text{and}\ \ \ll_{i--} = \prec^1_{(1\ \pi(3))\circ\pi}\ .
\end{aligned}$$

Therefore,

$$r^{\mathbf{x}}_{i++}(1,2,3) = u^2_{(1\ \pi(2)\ \pi(3))\circ\pi},\ r^{\mathbf{x}}_{i-+}(1,2,3) = u^2_{(1\ \pi(3))\circ\pi},$$
$$r^{\mathbf{x}}_{i+-}(1,2,3) = u^1_{(1\ \pi(2)\ \pi(3))\circ\pi},\ r^{\mathbf{x}}_{i--}(1,2,3) = u^1_{(1\ \pi(3))\circ\pi}.$$

Hence,

$$\begin{aligned}
t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^2_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\
&s_3(u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi}, u^2_{(1\ \pi(3))\circ\pi})\Big),\\
&s_2\Big(s_3(u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}, u^1_{(1\ \pi(2)\ \pi(3))\circ\pi}),\\
&s_3(u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi}, u^1_{(1\ \pi(3))\circ\pi})\Big)\Big]\\
= s_2\Big[&s_2\Big(s_3(u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}), s_3(u^2_{(1\ 3)}, u^2_{(1\ 3)}, u^2_{(1\ 3)})\Big),\\
&s_2\Big(s_3(u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big)\Big] =\\
= s_2\Big[&s_2\Big(s_3(u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(1\ 3)}, u^1_{(1\ 3)}, u^1_{(1\ 3)})\Big),\\
&s_2\Big(s_3(u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}, u^2_{(1\ 2\ 3)}), s_3(u^2_{(1\ 3)}, u^2_{(1\ 3)}, u^2_{(1\ 3)})\Big)\Big],
\end{aligned}$$

where the second and the third equality hold because s_2 is symmetric.

Case 11. $0 < |U_1| = |U_2| = |U_3| = \frac{n}{3} < \frac{n}{2}$

Since the order $<_{iab}$ is graded, we have

$$\emptyset <_{iab} U_1, U_2, U_3 <_{iab} U_1 \cup U_2, U_1 \cup U_3, U_2 \cup U_3 <_{iab} [n].$$

Furthermore, notice that $<_{iab}$ restricted onto the set $R := \{\emptyset, U_1, U_2, U_3, U_1 \cup U_2, U_1 \cup U_3, U_2 \cup U_3, [n]\}^2$ does not depend on b. This means that the restriction of $<_{ia+}$ onto R is equal to the restriction of $<_{ia-}$ onto R. In addition, we only have to examine $<_{iab}$ restricted on the set $\{U_1, U_2, U_3\}^2$. This is the case because the order of $U_1 \cup U_2, U_1 \cup U_3,$ and $U_2 \cup U_3$ is determined by the order of $U_1, U_2,$ and U_3. Indeed: $\{U_1, U_2, U_3\}$ is a partition of the set $[n]$ and $|U_1 \cup U_2| = |U_1 \cup U_3| = |U_2 \cup U_3| = \frac{2n}{3} > \frac{n}{2}$. Thus, if π is the permutation in S_3 such that $U_{\pi(1)} <_{iab} U_{\pi(2)} <_{iab} U_{\pi(3)}$, then $(U_{\pi(2)} \cup U_{\pi(3)})^c = U_{\pi(1)} <_{iab} U_{\pi(2)} = (U_{\pi(1)} \cup U_{\pi(3)})^c <_{iab} U_{\pi(3)} = (U_{\pi(1)} \cup U_{\pi(2)})^c$; and so $U_{\pi(1)} \cup U_{\pi(2)} <_{iab} U_{\pi(1)} \cup U_{\pi(3)} <_{iab} U_{\pi(2)} \cup U_{\pi(3)}$. Moreover, it follows from Lemma 1 that if $\{A, B, C\}$ is a partition of the set $[n]$ and if $A <^+_1 B <^+_1 C$, then $C <^+_2 A <^+_2 B$, $B <^+_3 C <^+_3 A$, $C <^-_1 B <^-_1 A$, $B <^-_2 A <^-_2 C$, and $A <^-_3 C <^-_3 B$. Thus, if π is the permutation in S_3 such that $U_{\pi(1)} <_{1+b} U_{\pi(2)} <_{1+b} U_{\pi(3)}$, then, by the above, we have:

$$\begin{aligned}
&U_{\pi(3)} <_{2+b} U_{\pi(1)} <_{2+b} U_{\pi(2)},\ U_{\pi(2)} <_{3+b} U_{\pi(3)} <_{3+b} U_{\pi(1)},\\
&U_{\pi(3)} <_{1-b} U_{\pi(2)} <_{1-b} U_{\pi(1)}, U_{\pi(2)} <_{2-b} U_{\pi(1)} <_{2-b} U_{\pi(3)},\ \text{and}\\
&U_{\pi(1)} <_{3-b} U_{\pi(3)} <_{3-b} U_{\pi(2)}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\ll_{1+b} = \prec^1_{\pi},\ \ll_{2+b} = \prec^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi},\ \ll_{3+b} = \prec^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}\ \text{and}\\
&\ll_{1-b} = \prec^1_{(\pi(1)\ \pi(3))\circ\pi},\ \ll_{2-b} = \prec^1_{(\pi(1)\ \pi(2))\circ\pi},\ \ll_{3-b} = \prec^1_{(\pi(2)\ \pi(3))\circ\pi}
\end{aligned}$$

and thus

$$r^{\mathbf{x}}_{1+b}(1,2,3) = u^1_{\pi},\ r^{\mathbf{x}}_{2+b}(1,2,3) = u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi},\ r^{\mathbf{x}}_{3+b}(1,2,3) = u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi},$$
$$r^{\mathbf{x}}_{1-b}(1,2,3) = u^1_{(\pi(1)\ \pi(3))\circ\pi},\ r^{\mathbf{x}}_{2-b}(1,2,3) = u^1_{(\pi(1)\ \pi(2))\circ\pi},\ \text{and}$$
$$r^{\mathbf{x}}_{3-b}(1,2,3) = u^1_{(\pi(2)\ \pi(3))\circ\pi}.$$

Therefore, we get

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi})\Big),\\ &s_2\Big(s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi})\Big)\Big]\\ = s_2\Big[&s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big),\\ &s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big)\Big] \end{aligned}$$

Let us now argue why the last equality holds. Suppose that $\pi \in S_3$ is arbitrary, then there are three cases:

- If π is a 3-cycle, then $\{\mathrm{id}\circ\pi, (\pi(1)\ \pi(2)\ \pi(3))\circ\pi, (\pi(1)\ \pi(3)\ \pi(2))\circ\pi\} = \{\mathrm{id}, (1\ 2\ 3), (1\ 3\ 2)\}$ and $\{(\pi(1)\ \pi(2))\circ\pi, (\pi(1)\ \pi(3))\circ\pi, (\pi(2)\ \pi(3))\circ\pi\} = \{(1\ 2), (1\ 3), (2\ 3)\}$ as can be easily verified. Therefore,

  $$\{u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}\} = \{u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}\}.$$

  Since s_3 is symmetric, we get

  $$s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}) = s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}).$$

  Also, $\{(\pi(1)\ \pi(2))\circ\pi, (\pi(1)\ \pi(3))\circ\pi, (\pi(2)\ \pi(3))\circ\pi\} = \{(1\ 2), (1\ 3), (2\ 3)\}$. Thus,

  $$\{u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi}\} = \{u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)}\}.$$

  Due to the fact that s_3 is symmetric, we have

  $$s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi}) = s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)}).$$

  Now, the equality

  $$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big),\\ &s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big)\Big] \end{aligned}$$

  follows from the fact that s_2 is symmetric.

- If π is a transposition, then $\{\mathrm{id}\circ\pi, (\pi(1)\ \pi(2)\ \pi(3))\circ\pi, (\pi(1)\ \pi(3)\ \pi(2))\circ\pi\} = \{(1\ 2), (1\ 3), (2\ 3)\}$ and $\{(\pi(1)\ \pi(2))\circ\pi, (\pi(1)\ \pi(3))\circ\pi, (\pi(2)\ \pi(3))\circ\pi\} = \{\mathrm{id}, (1\ 2\ 3), (1\ 3\ 2)\}$. Thus,

  $$\{u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi}\} = \{u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}\} \text{ and}$$
  $$\{u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}\} = \{u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)}\}.$$

Since s_3 is symmetric, we get

$$s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi}) = s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}) \text{ and}$$
$$s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}) = s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)}).$$

The equality

$$t(\mathbf{x}) = s_2\Big[s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big),$$
$$s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big)\Big]$$

again follows from the fact that s_2 is symmetric.

- If $\pi = \mathrm{id}$, then

$$\begin{aligned} t(\mathbf{x}) = s_2\Big[&s_2\Big(s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi})\Big),\\ &s_2\Big(s_3(u^1_{\pi}, u^1_{(\pi(1)\ \pi(3)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(2)\ \pi(3))\circ\pi}),\\ &s_3(u^1_{(\pi(2)\ \pi(3))\circ\pi}, u^1_{(\pi(1)\ \pi(2))\circ\pi}, u^1_{(\pi(1)\ \pi(3))\circ\pi})\Big)\Big]\\ = s_2\Big[&s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 3\ 2)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(2\ 3)}, u^1_{(1\ 2)}, u^1_{(1\ 3)})\Big),\\ &s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 3\ 2)}, u^1_{(1\ 2\ 3)}), s_3(u^1_{(2\ 3)}, u^1_{(1\ 2)}, u^1_{(1\ 3)})\Big)\Big]; \end{aligned}$$

and so

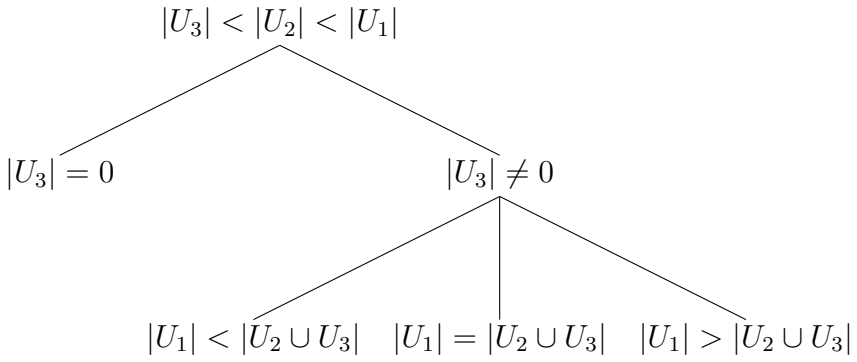
$$t(\mathbf{x}) = s_2\Big[s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big),$$
$$s_2\Big(s_3(u^1_{\mathrm{id}}, u^1_{(1\ 2\ 3)}, u^1_{(1\ 3\ 2)}), s_3(u^1_{(1\ 2)}, u^1_{(1\ 3)}, u^1_{(2\ 3)})\Big)\Big]$$

because the operations s_3 and s_2 are symmetric.

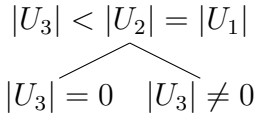
On the correctness of the case analysis

We assume that $|U_3| \leq |U_2| \leq |U_1|$. We have two options for each of these inequalities: strict or non-strict. Let us examine each scenario to understand how it leads to the 11 cases above.

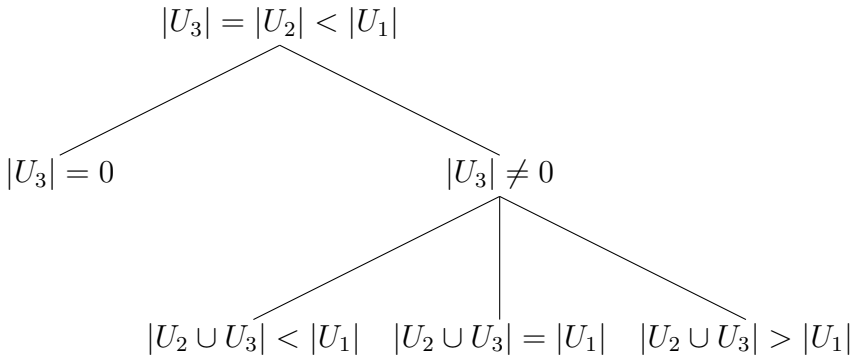
- If $|U_3| < |U_2|$ and $|U_2| < |U_1|$, then neither $|U_2|$ nor $|U_1|$ can be zero, as cardinality is always greater than or equal to zero. Therefore, we have two possibilities: either $|U_3| = 0$ or $|U_3| > 0$. In the first scenario, where $|U_3| = 0$, we end up with the case number 2. If, however, $0 < |U_3| < |U_2| < |U_1|$, then we know that $0 < |U_3| < |U_2| < |U_1|, |U_2| + |U_3| < |U_1| + |U_3| < |U_1| + |U_2|$. Based on this, we have three options: $|U_1| < |U_2 \cup U_3|$ (case 5) or $|U_1| = |U_2 \cup U_3|$ (case 6) or $|U_1| > |U_2 \cup U_3|$ (case 4).



- If $|U_3| < |U_2|$ and $|U_2| = |U_1|$, then neither $|U_2|$ nor $|U_1|$ is zero. However, $|U_3|$ can be zero, which corresponds to one of the degenerate cases. Therefore, we either have $|U_3| = 0$ (case 3) or $0 < |U_3| < |U_2| = |U_1|$ (case 7).



- If $|U_3| = |U_2|$ and $|U_2| < |U_1|$, then either $|U_3| = |U_2| = 0$ (case 1) or $|U_3| = |U_2| > 0$. If $|U_3| = |U_2| > 0$, then $0 < |U_3| = |U_2| < |U_1|, |U_2| + |U_3| < |U_1| + |U_3| = |U_1| + |U_2|$. Thus, we either have $|U_2 \cup U_3| = |U_2| + |U_3| < |U_1|$ (case 9) or $|U_2 \cup U_3| = |U_2| + |U_3| = |U_1|$ (case 10) or $|U_2 \cup U_3| = |U_2| + |U_3| > |U_1|$ (case 8).



- Finally, $|U_3| = |U_2|$ and $|U_2| = |U_1|$, which corresponds to case 11.

Note that this proof applies to any finite algebra; however, if $|A| > 3$, the conclusion is that t is only 3-symmetric and not symmetric. □

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