"QUADRATICITY OF THE OPERADIC CATEGORIES OF GRAPHS": A REPORT FOR THE THESIS COMMITTEE AND THE AUTHOR

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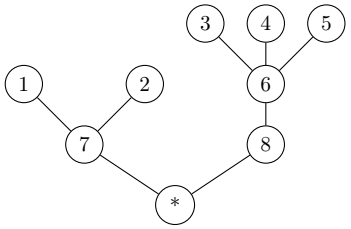
The paper under review is a qualification work submitted at Charles University in 2024 in partial fulfillment of the requirements for the master's degree program in mathematics. In general terms, the work belongs to the field of operad theory. More specifically, the thesis concerns the structure of a certain operad defined in the framework of Batanin-Markl's operadic category of ordered graphs. The work comprises a survey of the basic facts about operadic categories, based primarely on the original papers [1, 2], and features author's own contribution in the form of a theorem stating that the operad in question does not admit a quadratic presentation.

1. Synopsis

Let T be a finite set with a marked element $* \in T$ and $f : T \to T$ be a nilpotent endomorphism of T, i.e. there exists $n \geq 0$ such that $\underbrace{(f \circ \cdots \circ f)}_{n}(v) = *$ for all $v \in T$. The data of such an f is equivalent to that of a non-planar rooted tree with T as the set of vertices. Specifically, $f : T \to T$ is to be interpreted as the successor function that, for any $v \neq *$, returns the vertex $f(v)$ immediately following v in the unique path from v to the root $*$, as illustrated in the example below.



$f(3) = f(4) = f(5) = 6$

$f(1) = f(2) = 7, \quad f(6) = 8$

$f(7) = f(8) = *$

$f(*) = *$

For $k \geq 1$, let $T_k := \{v \in T | f^{(k)}(v) = *,\ f^{(k-1)}(v) \neq *\}$, and let $T_0 := \{*\}$. Then $T = \bigsqcup_{k \geq 0} T_k$, and $f : T \to T$ can be presented by the chain of maps

$$T_n \overset{f|_{T_n}}{\to} T_{n-1} \overset{f|_{T_{n-1}}}{\to} \cdots \to T_1 \overset{f|_{T_1}}{\to} \{*\} \tag{1}$$

that fully encodes the corresponding tree. Planar rooted trees can be encoded similarly by imposing, in addition, a linear order on each T_k, the set of vertices at the height k in the tree, and requiring each $f|_{T_k}$ to be order-preserving. More abstractly, passing to the skeletal category of finite sets $sFin$, one observes that the data of a planar rooted tree is equivalent, by means of the above correspondence, to a finite chain of morphisms

$$\bar{m}_n \to \bar{m}_{n-1} \to \cdots \to \bar{m}_1 \to \bar{1} \tag{2}$$

in $sFin$, where each $\bar{m} \in Ob(sFin)$ is the standard linearly ordered set $\{1, 2 \ldots m\}$ and morphisms are maps of sets (not necessarily order-preserving). In particular, a corolla with m linearly ordered leaves labeled

by $1, 2 \dots m$ is encoded by the terminal morphism $\bar{m} \to \bar{1}$. Note that any isomorphism in (2), if present, encodes now a choice of a linear order on the vertices of the corresponding tree at a certain height.

As an upshot, any construction involving planar rooted trees, such as the operation of tree grafting and the definition of a symmetric operad as an algebra over the monad of planar rooted trees, can be recast in terms of maps in $sFin$. One may raise the level of abstraction a bit further and consider chains of morphisms (2) in categories other than $sFin$, treating such chains as data encoding compositional patterns for other tree-like, or flowchart-like, structures. The notion of an *operadic category* [1, 2] is meant to provide a reasonable level of generality for carrying out such a task. Descriptively, an operadic category is a category of "generalized arities", or placeholders for the arguments of a would-be operation, inasmuch the category $sFin$ is the category where each object $\bar{m}$ is identified, in our context, with a cluster of m input sockets for a generic m-ary operation with a single output, or an (ordered) corolla with m leaves labeled by $1, 2, \dots, m$.

More precisely, an operadic category C is a category equipped with a *cardinality functor* $|-| : C \to sFin$, a choice of an object $f^{-1}(i) \in Ob(C)$, called the *i-th fiber* of f, made for all morphisms $f : T \to S$ in C and all $i \in |S|$, and a choice of a *local terminal object* U_c made for each connected component c of C. The data is subject to a number of conditions modeled upon the behavior of maps and fibers of maps in $sFin$. In particular, the prototypical example of an operadic category is the category $sFin$ itself with the cardinality functor being the identity.

Given an operadic category C and a closed symmetric monoidal category V, an *C-operad $\mathcal{P}$ in V* is a collection of objects $\{\mathcal{P}(T)\}_{T \in Ob(C)}$ and morphisms $\gamma_f : \bigotimes_{i \in |S|} \mathcal{P}(f^{-1}(i)) \otimes \mathcal{P}(S) \to \mathcal{P}(T)$ in V defined for all $f : T \to S$ in C, subject to associativity and unitality conditions akin to those appearing in the definition of an ordinary symmetric operad. The latter can be recovered upon taking $C = sFin$. Some other associative structures, such as non-Σ operads, braided operads, monoids in V, and V-valued presheaves on C, can also be realized as C-operads in V, each for an appropriate choice of C. Under some technical conditions on C and V, one can define the notion of a free C-operad, a C-operadic ideal and a presentation of a C-operad in terms of generators and relations.

What is of interest are C-operads for operadic categories of "generalized arities" endowed some underlying combinatorial structure, such as the operadic category of *ordered graphs* discussed in the thesis. An ordered graph is a finite graph Γ, possibly with open-ended half-edges (*legs*), such that the set of vertices V, each of the sets F_v of the half-edges incident to a vertex $v \in V$, and the set of legs L of Γ, are all linearly ordered (independently of each other). A morphism of ordered graphs $f : \Gamma \to \Gamma'$ is a map of graphs that can be realized via a sequence of elementary moves applied to Γ. The possible moves are gluing two vertices by contracting an edge, gluing two vertices by forming a loop out of an edge, gluing two disconnected vertices, removing a loop, removing an edge e if there exists another one e' with the same endpoints, changing the linear order on F_v for any $v \in V$, and re-indexing the vertices. Note that such a morphism is surjective on vertices, preserves the adjacency relation and leaves the set of legs L with its linear order intact. The category of ordered graphs Gr defined in this way can be equipped with a cardinality functor returning the set of vertices $V \in Ob(sFin)$ for an ordered graph Γ, a naturally defined fiber $f^{-1}(v)$ for all $v \in |\Gamma| = V$, and ordered corollas as local terminal objects, making Gr an operadic category. Connected simple ordered graphs, that is, connected ordered graphs with no loops and no more than one edge between any two vertices, form an operadic subcategory $SGrc$ of Gr upon restricting to those and only those morphisms in Gr that have connected simple graphs as fibers.

Given a symmetric monoidal category V with the unit object I and an initial object 0, consider the *constant* $SGrc$-operad 1_{SGrc} with $1_{SGrc}(\Gamma) := I$ for any $\Gamma \in Ob(SGrc)$ with at least one edge and $1_{SGrc}(\Gamma) := 0$ otherwise. The author shows (Theorem 73) that this $SGrc$-operad does not admit a quadratic presentation. This differs from the common cases of symmetric and non-Σ operads, where the constant operad 1_C, being the operad of commutative and associative algebras in V respectively, is known to be quadratic. The non-quadraticity of 1_{SGrc} is due to the following. On the one hand (Proposition 72), defining 1_{SGrc} in terms of generators and relations requires introducing a separate generator for each isomorphism class of an *elementary graph* (Definition 30). On the other hand, in $SGrc$ there exist (infinitely many) elementary graphs with different cardinalities. In particular, one can find a connected simple graph (such as the "Czech flag" graph used by the author) that can be presented as both a composite of 2 and a composite of 3 graphs with elementary fibers, rendering a homogeneous quadratic relation for 1_{SGrc} impossible. The reason why it does

not happen in $sFin$ for $1_{sFin} = Com$ is that there is only one elementary object $\bar{2}$ in $sFin$, and there is indeed a well-known presentation of Com that requires only one binary generator.

2. Corrections and suggestions

(1) `page 2, paragraph 1:` *"with a rule for their composition"* $\mapsto$ with a rule for their composition and a characterization of the symmetries of operations under permutations of their arguments [or something along this lines]
(2) `page 2, paragraph 3:` *"loop space"* $\mapsto$ based loop space
(3) `page 2, paragraph 4:` *"The renaissance was arguably started by the discovery of an operad structure on the moduli space of stable complex algebraic curves."* $\dashrightarrow$ Most experts would likely add here, at least, Tamarkin's proof of Kontsevich's formality theorem and Ginzburg-Kapranov's theory of Koszul duality for operads.
(4) `page 10, Definition 20:` *"*$Objects(O) \to \mathbb{N}$*"* $\mapsto$ $Objects(O) \to \mathbb{N} \cup \{0\}$
(5) `page 17, Definition 41:` This is not a valid definition of Grc and it needs to be fixed. Unwinding the original construction [2, Example 4.6], Grc can be defined as the subcategory of Gr, where objects are all connected graphs and morphisms are all those, and only those, morphisms in Gr that have connected fibers. This should be accompanied by a verification that Grc defined in this way is indeed a subcategory of Gr. That is, the composite of two morphisms of connected ordered graphs with connected fibers has connected fibers.
(6) `page 18, Theorem 48:` Using Theorem 48, one can show that an elementary graph is 2-connected and is a triangulation (any edge belongs to a 3-cycle), but these conditions alone are not sufficient (an example is not too hard to find). It would be interesting to see if there is a reasonably simple topological characterization of elementary graphs.
(7) `page 22, Definition 62:` *"leaves"* $\mapsto$ legs
(8) `page 23, Theorem 68:` The statement accounts only for the objects $\mathbb{F}(E)(X)$. For the sake of completeness, what needs to be spelled out is how $\mathbb{F}(E)$ is defined for the maps $F \triangleright X \to Y$ in $SGrc$, as well as how $\mathbb{F}(-)$ is defined on morphisms of collections $E \to E'$.
(9) `page 23, paragraph -3:` *" the intersection of all congruences containing R"* $\dashrightarrow$ To make sense of the "intersection of all congruences", the category V should be assumed to be complete.
(10) `page 24, paragraph -2:` *"a sub-*$SGrc_{iso}$*-presheaf R of* $\mathbb{F}^2(E)$*"* $\mapsto$ a sub-$SGrc_{iso}$-presheaf R of $\mathbb{F}^2(E) \times \mathbb{F}^2(E)$

3. Conclusion

In the reviewer's opinion, the thesis, while being fairly concise, does properly reflect author's competence within the subject area in general. The author's own contribution is of certain interest. If accompanied by additional results in the same vein, it could become a basis for an original research publication. The reviewer ranks the work as "excellent" (grade 1 on the 1-4 scale).

Reference

[1] Michael Batanin and Martin Markl, *Koszul duality for operadic categories*, Compositionality **5** (2023).
[2] ———, *Operadic categories as a natural environment for Koszul duality*, Compositionality **5** (2023).