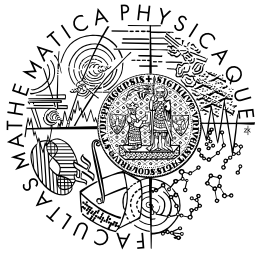

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MASTER THESIS

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Quadraticity of the operadic categories of graphs

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Study programme: Mathematics
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Title: Quadraticity of the operadic categories of graphs

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Abstract: In this thesis we define the operadic category of simple graphs and study its constant operad. We then construct a discrete operadic fibration from this category to the category of graphs and use it to establish additional properties necessary for the study of Koszulity. We then define the notion of an elementary object of a graded operadic category and investigate these objects in the category of simple graphs. At the end, we prove that the constant operad over simple graphs is not quadratic, therefore it is not Koszul. This is surprising, because the constant operad over any previously known operadic category of graphs is Koszul.

Keywords: Operadic category, Operad, Algebra, Graph, Quadraticity

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Introduction

An operad is a family of abstract operations together with a rule for their composition. Classical operads are a formalization of a way to compose multivariable functions of the form $X^n \to X$ and they come in two different flavors depending on how they choose to handle compositions. In the language of functions, either with the simultaneous composition, which for a function with n inputs $f : X^n \to X$ and n functions $g_i : X^{m_i} \to X$, $i = 1, \ldots, n$ gives the composition rule

$$\alpha(f, g_1, \ldots, g_n) = f(g_1, \ldots, g_n) : X^{m_1+\cdots+m_n} \to X,$$

or with the binary composition, which for functions $f : X^n \to X$, $g : X^k \to X$ and $1 \leq i \leq n$ gives the composition rule

$$f \circ_i g = f(id, \ldots, id, g, id, \ldots, id) : X^{n+k-1} \to X,$$

where g goes into the ith input of f. The former are usually called *May operads*, while the latter are called *Markl operads*. Under some assumptions these two approaches are equivalent, an interested reader should see [1, Sections 1,2] for the classical operads and [2, Section 6] for general operads. In this thesis we will work exclusively with Markl operads.

Every operad has an associated notion of an *operad algebra* or simply an *algebra*. An algebra is to an operad what a representation is to a group and these algebras were the primary motivation for the study of operads when they were initially conceived in the early 1970's. The titular example is the recognition of a loop space in topology.

Given a topological space X, there is a natural operation on its loop space $\Omega X \times \Omega X \to \Omega X$, gluing the two loops. However, this operation is not associative, it is only associative up to reparametrization. It turns out that the best and natural way to describe the algebraic structure of ΩX is via an action of the *little intervals operad* $\mathcal{D}_1$, more can be found in [1, Examples 4, 27]. This description can further be improved by replacing $\mathcal{D}_1$ by the operad $\mathcal{K}$ of the associahedra. The magic property of $\mathcal{K}$ is that it characterizes loop spaces: a connected topological space has a weak homotopy type of a loop space if and only if it is a $\mathcal{K}$-algebra. This reveals the second major early motivation for studying operads, as a tool for characterizing homotopy invariant structures on topological spaces.

In the early 1990's the so called *renaissance* of operads began. We will mention some of its relevant results, visit [1] for more. The renaissance was arguably started by the discovery of an operad structure on the moduli space of stable complex algebraic curves. This sparked interest in the study of operads as of themselves, not just as a necessary tool to describe their algebras.

If we restrict ourselves to stable complex algebraic curves of genus zero, the operad structure has an additional symmetry. Our previous notion of an operad formalized the composition of multivariable functions of the form $X^n \to X$, which have n inputs and one output. However, for composition of algebraic curves it is not important what is an input and what is an output. Operads with this additional symmetry are called *cyclic operads*. The composition of stable complex algebraic curves of an arbitrary genus is then described by *modular operads*.

The most common types of algebras, such as associative algebras, commutative associative algebras and Lie algebras, are also algebras over operads. Therefore the development of *operadic cohomology* gives us a unified view of some classical cohomology constructions, such as the Hochschild cohomology of associative algebras, Harrison cohomology of commutative associative algebras and Chevalley-Eilenberg cohomology of Lie algebras.

Another result, this time concerning model categories, states that operads in a reasonable monoidal model category form a model category themselves. Moreover, algebras over cofibrant operads are homotopy invariant. The importance of this result is twofold.

Firstly, each object in a model category has in some sense unique cofibrant replacement, which is a cofibrant object weakly equivalent to the original. Thus for each operad there exists a weakly equivalent operad whose algebras are homotopy invariant. We have seen this before, the associahedra operad is the cofibrant replacement of the little interval operad, which is not cofibrant itself.

Secondly, this allowed the development of homotopy invariant structures in algebraic model categories, such as in the category of complexes. For an algebraic operad, its cofibrant replacement, that is in some way the smallest, is called its *minimal model*. For operads that are *Koszul*, it is possible to explicitly describe their minimal models via the *operadic bar construction*. For example, the operad whose algebras are the usual associative algebras is Koszul and algebras over its minimal model are A_∞-algebras, an example of which is the singular complex of a loop space.

Operadic categories were invented as a tool for systematic study of various types of operads and PROPs. The latter formalize compositions of functions with several inputs and *several* outputs. An operadic category is a combinatorial structure which contains information describing operads of a given type together with their algebras. The concept of operadic categories allows us to construct new types of operads.

We will focus mainly on operadic categories, whose objects are certain graphs. This is an important class of operadic categories, since classical operads are governed by the operadic category of rooted trees, cyclic operads are governed by the operadic category of ordinary trees and modular operads are governed by the operadic category of genus graded graphs.

Plan of the thesis

In Section 1 we recall the notion of an operadic category introduced in [3]. In Section 2 we recall additional axioms from [2] we require our operadic categories to fulfill, so their free operads are well behaved. In Section 3 we recall elementary morphisms defining the binary composition of Markl operads which we introduce in Section 4. We then say what is an elementary object of a graded operadic category. In Section 5 we recall a version of graphs relevant for the operadic context from [2] and define its full operadic subcategory of *simple graphs*. We then characterize elementary objects in this category.

In Section 6 we recall discrete operadic fibrations between two operadic categories and how this notion allows to establish some axioms recalled in Section 2. Then we construct a discrete operadic fibration from the category of simple graphs

to the category of graphs and use it to prove that the category of simple graphs fulfills all the relevant axioms from Section 2. We also construct a discrete operadic fibration from the category of trees into simple graphs. And finally, in Section 7, we recall some results on the construction of free Markl operads alongside with the notion of *quadraticity*. We then spend the rest of the section proving, that the constant operad over the category of simple graphs is *not* quadratic.

Material in Sections 1, 2, 3, 4 and the first half of Section 5 is taken from [2], except for Lemmata 6 and 23 and Definition 30 which are our own contributions. Definitions and closely related lemmata in these sections are recalled almost verbatim. Material in the second half of Section 5 starting with Definition 44, Sections 6 and 7 is our own contributions, except for Definition 52 and Proposition 53 which are recalled from [2] and Propositions 64, 68, 66 and Definitions 65, 71 which are recalled from [4]. Definition 52 and Propositions 64, 68 are taken almost verbatim.

1. Operadic categories

In this section we will provide the definition and some basic examples of *operadic categories*. A more detailed look can be found in [3] or [2].

Let $\mathtt{Fin}$ be the skeletal category of finite sets. The objects of this category are the linearly ordered sets $\bar{n} = \{1, \dots, n\}, n \in \mathbb{N}$. Morphisms are arbitrary maps between these sets. We define the ith fiber $f^{-1}(i)$ of a morphism $f : T \to S$, $i \in S$, as the pullback of f along the map $j : \bar{1} \to S$ which picks up the element i,

$$\begin{array}{ccc} f^{-1}(i) & \xrightarrow{!_f} & \bar{1} \\ {\scriptstyle inc_f}\downarrow & \lrcorner & \downarrow{\scriptstyle j} \\ T & \xrightarrow[f]{} & S \end{array}$$

so it is the object $f^{-1}(i) = \bar{n}_i \in \mathtt{Fin}$ which is isomorphic as a linearly ordered set to the preimage $\left\{j \in T \mid f(j) = i\right\}$. Any commutative triangle in $\mathtt{Fin}$

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array}$$

then induces a map $f_i : h^{-1}(i) \to g^{-1}(i)$ for any $i \in R$. This assignment defines a functor $Fib_i : \mathtt{Fin}/R \to \mathtt{Fin}$. The above structure on the category $\mathtt{Fin}$ motivates the following abstract definition.

Definition 1. An *operadic category* is a category $\mathtt{O}$ equipped with a *cardinality functor* $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$ having the following properties. We require that each connected component of $\mathtt{O}$ has a chosen local terminal object U_c, $c \in \pi_0(\mathtt{O})$. We will sometimes call these chosen local terminal objects the *trivial* ones. We also assume that for every $f : T \to S$ in $\mathtt{O}$ and every element $i \in |S|$, there is given an object $f^{-1}(i) \in \mathtt{O}$, which we will call *the i-th fiber* of f, such that $|f^{-1}(i)| = |f|^{-1}(i)$. We also require that

(i) For any $c \in \pi_0(\mathtt{O})$, $|U_c| = 1$.

(ii) For any $T \in \mathtt{O}$ and each $i \in |T|$, the fiber $\mathbb{1}_T^{-1}(i)$ of the identity $\mathbb{1}_T : T \to T$ is a chosen local terminal object.

(iii) For any commutative diagram in $\mathtt{O}$

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array} \tag{1.1}$$

and every $i \in |R|$, one is given a map

$$f_i : h^{-1}(i) \to g^{-1}(i)$$

such that $|f_i| : |h^{-1}(i)| \to |g^{-1}(i)|$ is the map $|h|^{-1}(i) \to |g|^{-1}(i)$ of sets induced by

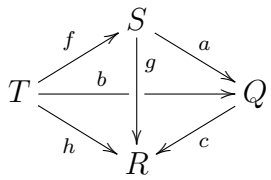
$$\begin{array}{ccc} |T| & \xrightarrow{|f|} & |S| \\ & {\scriptstyle |h|}\searrow \quad \swarrow{\scriptstyle |g|} & \\ & |R| \,. & \end{array}$$

We moreover require that this assignment forms a functor $Fib_i : \mathtt{O}/R \to \mathtt{O}$ called the *fiber functor*. Moreover, if $R = U_c$, the functor Fib_1 must be the domain functor $\mathtt{O}/R \to \mathtt{O}$. The last condition says that the unique fiber of the canonical morphism $!_T : T \to U_c$ is T.

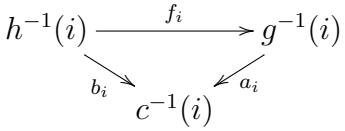
(iv) In the situation of Axiom (iii), for any $j \in |S|$ and i such that $j = inc(i)$ for $inc : |g|^{-1}(|g|(j)) \hookrightarrow |S|$, one has the equality

$$f^{-1}(j) = f^{-1}_{|g|(j)}(i).$$

(v) Let



be a commutative diagram in $\mathtt{O}$ and let $j \in |Q|, i = |c|(j)$ and $j' \in |c^{-1}(i)|$ such that $inc(j') = j$ for $inc : |c^{-1}(i)| \hookrightarrow |Q|$. Then by Axiom (iii) the diagram



commutes, so it induces a morphism $(f_i)_{j'} : b_i^{-1}(j') \to a_i^{-1}(j')$. By Axiom (iv) we have

$$a^{-1}(j) = a_i^{-1}(j') \text{ and } b^{-1}(j) = b_i^{-1}(j').$$

We then require the equality $f_j = (f_i)_{j'}$.

An *operadic functor* between operadic categories is a functor $F : \mathtt{O} \to \mathtt{P}$ over $\mathtt{Fin}$ which preserves fibers in the sense that $F\Big(f^{-1}(i)\Big) = F(f)^{-1}(i)$, for any $f : T \to S \in \mathtt{O}$ and $i \in |S| = |F(S)|$. We also require that F preserves the chosen local terminal objects, and that $F(f_i) = F(f)_i$ for f as in (1.1). This gives the category $\mathtt{OpCat}$ of operadic categories and their operadic functors.

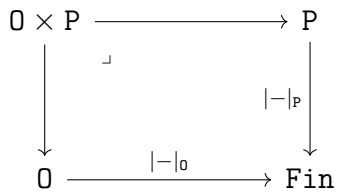
Let us now list some basic examples of operadic categories.

Example 2. The category $\mathtt{Fin}$ itself is an operadic category and the cardinality functor $|-| : \mathtt{O} \to \mathtt{Fin}$ is an operadic functor. Therefore $\mathtt{Fin}$ is the terminal object in the category of operadic categories.

Example 3. The subcategory $\mathtt{Fin}_{\text{semi}} \subset \mathtt{Fin}$ of nonempty finite sets and surjections is an operadic category.

Example 4. The category Δ_{alg} of finite ordinals and order preserving maps also forms an operadic category.

Example 5. There exists a Cartesian product in the category of operadic categories and is given by the following pullback.



The product $\Delta_{alg} \times \mathtt{O}$ is isomorphic to the subcategory $\mathtt{O}_{\mathtt{ord}} \subset \mathtt{O}$ with the same objects as $\mathtt{O}$ and with morphisms for which $|f|$ is order preserving, therefore $\mathtt{O}_{\mathtt{ord}}$ has a structure of an operadic category.

We need to formulate a criterion when a subcategory of an operadic category carries an induced operadic category structure.

Lemma 6. *Let* $\mathtt{C}$ *be a full subcategory of an operadic category* $\mathtt{O}$ *such that*

(i) The chosen local terminal objects of $\mathtt{O}$ *coincide with the chosen local terminal objects of* $\mathtt{C}$*,*

(ii) for any morphism f *in* $\mathtt{O}$ *whose both domain and codomain are in* $\mathtt{C}$*, all fibers of* f *are also in* $\mathtt{C}$*.*

Then $\mathtt{C}$ *has a structure of an operadic category, i.e.* $\mathtt{C}$ *is a full operadic subcategory of* $\mathtt{O}$*.*

Proof. The cardinality functor restricts to $\mathtt{C}$. The chosen local terminal objects come from (i), the fibers of each morphism exist and are defined by (ii) and axioms $(i) - (v)$ of operadic categories hold, since they hold in $\mathtt{O}$ and $\mathtt{C}$ is a *full* subcategory of $\mathtt{O}$, hence the properties of induced maps also hold in $\mathtt{C}$. □

Definition 7. A *quasibijection* is a morphism $f : T \to S$ in $\mathtt{O}$ such that, for each $i \in |S|$, we have $f^{-1}(i) = U_{c_i}$ for some $c_i \in \pi_0(\mathtt{O})$. Note that an isomorphism is not necessary a quasibijection and that a quasibijection is not necessary an isomorphism. We will denote by $\mathtt{O}_{\mathtt{qb}} \subset \mathtt{O}$ the subcategory with the same objects as $\mathtt{O}$ and with quasibijections as morphisms. Unlike $\mathtt{O}_{\mathtt{ord}}$ from Example 5, $\mathtt{O}_{\mathtt{qb}}$ does not have a natural structure of an operadic category.

2. Relevant axioms of operadic categories

The aim of this section is to recall some relevant axioms of operadic categories necessary for defining Markl operads in Section 4 and constructing free Markl operads in Section 7. For more in depth analysis containing all proofs cf. [2, Section 2].

Lemma 8. *Consider the commutative diagram in an operadic category*

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S'' \\ {\scriptstyle f'}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f''} \\ T' & \xrightarrow{\sigma} & T''. \end{array} \tag{2.1}$$

Let $j \in |T''|$ and $|\sigma|^{-1}(j) = \{i\}$ for some $i \in |T'|$. Diagram (2.1) determines:

(i) the map $f'_j : f^{-1}(j) \to \sigma^{-1}(j)$ whose unique fiber equals ${f'}^{-1}(i)$, and

(ii) the induced map $\pi_j : f^{-1}(j) \to {f''}^{-1}(j)$.

If $\sigma^{-1}(j)$ is trivial, in particular if σ is a quasibijection, then π induces a map

$$\pi_{(i,j)} : {f'}^{-1}(i) \to {f''}^{-1}(j) \tag{2.2}$$

which is a quasibijection if π is.

In the situation of Lemma 8 with σ a quasibijection, one has the *derived sequence*

$$\left\{\pi_{(i,j)} : {f'}^{-1}(i) \to {f''}^{-1}(j), j = |\sigma|(i)\right\}_{i \in |T'|} \tag{2.3}$$

consisting of quasibijections if π is a quasibijection.

Central constructions of this work will require the following:

Definition 9 (Blow-up axiom)**.** Let $\mathtt{O}$ be an operadic category. Consider the corner

$$\begin{array}{ccc} S' & & \\ {\scriptstyle f'}\downarrow & & \\ T' & \xrightarrow[\sim]{\sigma} & T'' \end{array} \tag{2.4}$$

in which σ is a quasibijection and $f' \in \mathtt{O}_{\mathtt{ord}}$. Assume we are given objects F''_j, $j \in |T''|$ together with a collection of maps

$$\left\{\pi_{(i,j)} : {f'}^{-1}(i) \to F''_j,\ j = |\sigma|(i)\right\}_{i \in |T'|}. \tag{2.5}$$

Then the corner (2.4) can be completed uniquely into the commutative square

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f''} \\ T' & \xrightarrow[\sim]{\sigma} & T'' \end{array}$$

in which $f'' \in \mathtt{O}_{\mathtt{ord}}$, ${f''}^{-1}(j) = F''_j$ for $j \in |T''|$, and such that the derived sequence (2.3) induced by π coincides with (2.5).

The requirement that $f', f'' \in \mathtt{O}_{\mathtt{ord}}$ is crucial, otherwise the factorization would not be unique even in "simple" operadic categories such as $\mathtt{Fin}$. It will sometimes suffice to assume the blow-up for $\sigma = \mathbb{1}$ only, i.e. to assume

Definition 10 (Weak blow-up axiom)**.** For any $f' : S' \to T$ in $\mathtt{O}_{\mathtt{ord}}$ and morphisms $\pi_i : {f'}^{-1}(i) \to F''_i$ in $\mathtt{O}$, $i \in |T|$, there exists a unique factorization of f'

$$\begin{array}{ccc} S' & \xrightarrow{\omega} & S'' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle f''} & \\ & T & \end{array}$$

such that $f'' \in \mathtt{O}_{\mathtt{ord}}$ and $\omega_i = \pi_i$ for all $i \in |T|$.

Corollary 11. *Assume the weak blow-up axiom holds in* $\mathtt{O}$*. Then*

$$\mathtt{O}_{\mathtt{qb}} \cap \mathtt{O}_{\mathtt{ord}} = \mathtt{O}_{\mathtt{disc}},$$

where $\mathtt{O}_{\mathtt{disc}}$ *is the discrete category with the same objects as* $\mathtt{O}$*.*

Example 12. Corollary 11 shows how the blow-up axiom determines the nature of an operadic category. For an example in which the blow-up axiom does not hold, see Batanin's category of n-trees [2, Example 2.7.].

Definition 13 (Factorizability)**.** An operadic category $\mathtt{O}$ is *factorizable* if each morphism $f \in \mathtt{O}$ decomposes, not necessarily uniquely, as $\phi\sigma$ for some $\phi \in \mathtt{O}_{\mathtt{ord}}$ and $\sigma \in \mathtt{O}_{\mathtt{qb}}$ or, symbolically, $\mathtt{O} = \mathtt{O}_{\mathtt{ord}}\mathtt{O}_{\mathtt{qb}}$.

An operadic category $\mathtt{O}$ is *strongly factorizable* if each morphism $f : T \to S$ decomposes *uniquely* as $\phi\sigma$ for some $\phi \in \mathtt{O}_{\mathtt{ord}}$ and $\sigma \in \mathtt{O}_{\mathtt{qb}}$ such that the induced map between the fibers

$$\sigma_i : f^{-1}(i) \to \phi^{-1}(i)$$

is an order-preserving quasibijection for each $i \in |S|$.

Similar result as in Corollary 11 can be obtained from strong factorizability.

Corollary 14. *Assume the operadic category* $\mathtt{O}$ *is strongly factorizable. Then*

$$\mathtt{O}_{\mathtt{qb}} \cap \mathtt{O}_{\mathtt{ord}} = \mathtt{O}_{\mathtt{disc}},$$

where $\mathtt{O}_{\mathtt{disc}}$ *is the discrete category with the same objects as* $\mathtt{O}$*. Moreover, the morphisms between fibers induced by the quasibijection* σ *in the unique factorization* $f = \phi\sigma$ *are always the identities.*

Definition 15 (Unique fiber condition)**.** The *unique fiber condition* for an operadic category $\mathtt{O}$ requires that, if the fiber of the unique morphism $\phi : T \to t$ to a local terminal object t is T, then t is a chosen local terminal object U_c for some $c \in \pi_0(\mathtt{O})$.

Definition 16 (Rigidity)**.** An operadic category $\mathtt{O}$ is *rigid* if, given $\phi \in \mathtt{O}_{\mathtt{ord}}$, the only isomorphism σ that makes

$$\begin{array}{ccc} S & = & S \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ T & \xrightarrow[\cong]{\sigma} & T \end{array}$$

commutative is the identity $\mathbb{1} : T = T$.

Example 17. The category $\mathtt{Fin}$ is itself *not* rigid, but its subcategory $\mathtt{Fin}_{\mathrm{semi}}$ from Example 3 is.

Definition 18 (Constant freeness)**.** An operadic category $\mathtt{O}$ is *constant-free* if $|f|$ is surjective for each $f \in \mathtt{O}$. Equivalently, $\mathtt{O}$ is constant-free if the cardinality functor $\mathtt{O} \to \mathtt{Fin}$ factorizes through the operadic category $\mathtt{Fin}_{\mathrm{semi}}$.

Lemma 19. *If a constant-free operadic category fulfills the the unique fiber condition and the weak blow-up axiom, then it is rigid. Schematically*

$$\mathsf{WBU}\ \&\ \mathsf{CFree}\ \&\ \mathsf{UFib} \Longrightarrow \mathsf{Rig}.$$

Definition 20. A *grading* on an operadic category $\mathtt{O}$ is a map $e : \mathrm{Objects}(\mathtt{O}) \to \mathbb{N}$ of sets with the property that

$$e(T) + e(F_1) + \cdots + e(F_k) = e(S)$$

for each $f : S \to T$ with fibers $F_1, \ldots, F_k$. In this situation we define the *grade* $e(f)$ of f by $e(f) := e(S) - e(T)$.

Example 21. Each constant-free operadic category $\mathtt{O}$ can be given the *canonical grading* $e(T) := |T| - 1$.

Definition 22 (Strict grading)**.** A graded operadic category $\mathtt{O}$ is *strictly graded* if a morphism $f \in \mathtt{O}$ is an isomorphism if and only if $e(f) = 0$.

Lemma 23. *Let* $\mathtt{C}$ *be a full operadic subcategory of* $\mathtt{O}$*. If* $\mathtt{O}$ *is constant-free, strictly graded and satisfies the unique fiber condition, then* $\mathtt{C}$ *is also constant-free, strictly graded and satisfies the unique fiber condition.*

Proof. Let $\phi : T \to t$ be the unique morphism to a local terminal object in $\mathtt{C}$ with fiber T. Then ϕ is also the unique morphism to a local terminal object in $\mathtt{O}$, since $t \cong U_c$ in $\mathtt{C}$ for some $c \in \pi_0(\mathtt{C})$, thus also $t \cong U_c$ in $\mathtt{O}$, so t is also a local terminal object in $\mathtt{O}$. Then since $\mathtt{O}$ satisfies the unique fiber condition $t = U_c$, hence $\mathtt{C}$ also satisfies the unique fiber condition.

A morphism ϕ is an isomorphism in $\mathtt{C}$ if and only if ϕ is an isomorphism in $\mathtt{O}$, since $\mathtt{C}$ is a full subcategory of $\mathtt{O}$ and that happens if and only if $e(\phi) = 0$, since $\mathtt{O}$ is strictly graded. Hence $\mathtt{C}$ is also strictly graded. Since $\mathtt{C}$ is an operadic subcategory of $\mathtt{O}$ it is automatically constant-free. □

The above axioms will be required in many constructions in this work, therefore we shall put them together for future references in the following assumptions.

Assumptions 1. *We assume that* $\mathtt{O}$ *is a constant-free strictly graded rigid factorizable operadic category in which all quasibijections are invertible, the blow-up axiom and unique fiber condition are fulfilled. In short*

$$\mathsf{SGrad}\ \&\ \mathsf{Rig}\ \&\ \mathsf{BU}\ \&\ \mathsf{QBI}\ \&\ \mathsf{CFree}\ \&\ \mathsf{UFib}\ \&\ \mathsf{Fac}.$$

3. Elementary morphisms

The aim of this section is to introduce *elementary morphisms*, which play an important role in the definition of Markl operads in Section 4. For more information see [2, Section 5]. All operadic categories are assumed to be *graded.*

Definition 24. A morphism $\phi : T \to S \in \mathtt{O}_{\mathtt{ord}}$ in an operadic category $\mathtt{O}$ is *elementary* if all its fibers are the chosen local terminal objects except precisely one whose grade is ≥ 1. If $\phi^{-1}(i)$ is, for $i \in |S|$, the unique nontrivial fiber, we will sometimes write ϕ as the pair (ϕ, i). If we want to name the unique nontrivial fiber $F := \phi^{-1}(i)$ explicitly, we will write $F \triangleright_i T \xrightarrow{\phi} S$, or $F \triangleright T \xrightarrow{\phi} S$ when the concrete $i \in |S|$ is not important.

Definition 25. Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms. If $|\psi|(j) = i$ we say that the fibers of ϕ and ψ are *joint*. If $|\psi|(j) \neq i$ we say that ϕ and ψ have *disjoint fibers* or, more specifically, that the fibers of ϕ and ψ are (i, j)-*disjoint*.

Lemma 26. *If the fibers of elementary morphisms ϕ and ψ in Definition 25 are joint, then the composite $\xi = \psi(\phi)$ is elementary as well, with the nontrivial fiber over i, and the induced morphism $\phi_i : \xi^{-1}(i) \to \psi^{-1}(i)$ is elementary with the nontrivial fiber over j that equals $\phi^{-1}(j)$. For $l \neq i$ the morphism ϕ_l equals the identity $U_c \to U_c$ of trivial objects.*

If the fibers of ϕ and ψ are (i, j)-disjoint then the morphism $\xi = \psi(\phi)$ has exactly two nontrivial fibers and these are fibers over i and $k := |\psi|(j)$. Moreover, there is a canonical induced quasibijection

$$\phi_i : \xi^{-1}(i) \to \psi^{-1}(i) \in \mathtt{O}_{\mathtt{ord}} \tag{3.1}$$

and we have the equality

$$\xi^{-1}(k) = \phi^{-1}(j).$$

Definition 27. We will call the pair $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ of morphisms in Definition 25 with disjoint fibers *harmonic* if $\xi^{-1}(i) = \psi^{-1}(i)$ and the map ϕ_i in (3.1) is the identity.

Corollary 28. *Assume the blow-up axiom holds, then all pairs of elementary morphisms with disjoint fibers are harmonic.*

Corollary 29. *Consider the following commutative diagram of elementary morphisms.*

$$\begin{array}{ccccc} & & P' & & \\ & \overset{(\phi',j)}{\nearrow} & & \overset{(\psi',i)}{\searrow} & \\ T & & & & S \\ & \underset{(\phi'',l)}{\searrow} & & \underset{(\psi'',k)}{\nearrow} & \\ & & P'' & & \end{array} \tag{3.2}$$

Assume that $|\psi''|(l) = i$, $|\psi'|(j) = k$ *and* $i \neq k$. *Let* F', F'', G' *and* G'' *be the only nontrivial fibers of* ϕ', ϕ'', ψ' *and* ψ'', *respectively. Then there exist canonical quasibijections*

$$\sigma' : F' \longrightarrow G'' \quad \textit{and} \quad \sigma'' : F'' \longrightarrow G'. \tag{3.3}$$

If both pairs of elementary morphisms in (3.2) *are harmonic, then* $F' = G''$, $F'' = G'$ *and* σ', σ'' *are the identities.*

Definition 30. An object T is called *elementary* if for any elementary morphism $\phi : T \to S$ the codomain S is a local terminal object and $\phi = !_T$ its unique morphism.

4. Markl operads

In this section we introduce Markl operads in the language of operadic categories. For more in depth study of this topic see [2, Section 6]. Recall that all operadic categories satisfy Assumptions 1, i.e.

$$\mathsf{SGrad}\ \&\ \mathsf{Rig}\ \&\ \mathsf{BU}\ \&\ \mathsf{QBI}\ \&\ \mathsf{CFree}\ \&\ \mathsf{UFib}\ \&\ \mathsf{Fac}.$$

Let $(\mathtt{V}, \otimes, \mathbb{K})$ be a cocomplete strict symmetric monoidal category. Archetypal examples of such category are the category $\mathtt{Vect}$ of vector spaces over a field $\mathbb{K}$ of characteristic 0 with the standard tensor product and $\mathbb{K}$ as the monoidal unit, or the category $\mathtt{Set}$ of sets with the Cartesian product and a one element set as the monoidal unit.

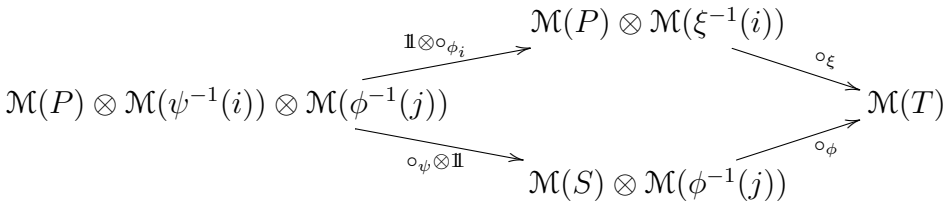
We will denote by $\mathtt{O}_{\mathtt{iso}} \subset \mathtt{O}$ the subcategory with the same objects as $\mathtt{O}$ and with all morphisms invertible. Thus by QBI we have that $\mathtt{O}_{\mathtt{qb}} \subset \mathtt{O}_{\mathtt{iso}}$ and SGrad gives us $\mathtt{O}_{\mathtt{iso}} = \{f : T \to S,\ e(f) = 0\}$.

Definition 31. A *Markl* $\mathtt{O}$*-operad* in $(\mathtt{V}, \otimes, \mathbb{K})$ is a presheaf $\mathcal{M} : \mathtt{O}_{\mathtt{iso}}^{\mathrm{op}} \to \mathtt{V}$ equipped, for each elementary morphism $F \triangleright T \xrightarrow{\phi} S$ as in Definition 24, with a "circle product"

$$\circ_\phi : \mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T). \tag{4.1}$$

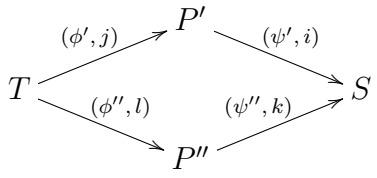
These operations must fulfill the following axioms.

(i) Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms such that $|\psi|(j) = i$ and let $\xi : T \to P$ be the composite $\psi\phi$. Then the diagram

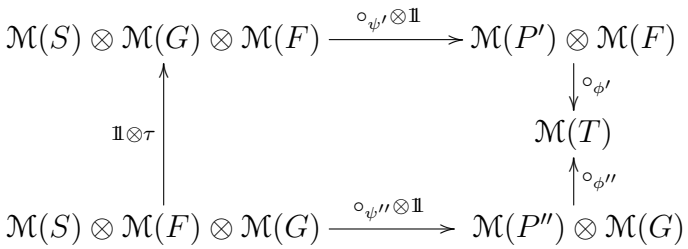


commutes.

(ii) Let us consider the diagram



of elementary morphisms with disjoint fibers as in Corollary 29. By the harmonicity implied by BU, the morphisms σ', σ'' in (3.3) are the identities; denote $F := F' = G''$ and $G := G' = F''$. With this notation, the diagram



commutes.

(iii) For every commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow[\cong]{\omega} & T'' \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} \\ S' & \xrightarrow[\sim]{\sigma} & S'' \end{array}$$

where ω is an isomorphism, σ a quasibijection, and $F' \triangleright_i T' \xrightarrow{\phi'} S'$, $F'' \triangleright_j T'' \xrightarrow{\phi''} S''$, the diagram

$$\begin{array}{ccc} \mathcal{M}(F'') \otimes \mathcal{M}(S'') & \xrightarrow{\circ_{\phi''}} & \mathcal{M}(T'') \\ {\scriptstyle \bar{\omega}^* \otimes \sigma^*} \downarrow \cong & & \cong \downarrow {\scriptstyle \omega^*} \\ \mathcal{M}(F') \otimes \mathcal{M}(S') & \xrightarrow[\circ_{\phi'}]{} & \mathcal{M}(T') \,, \end{array}$$

in which $\bar{\omega} : F' \to F''$ is the induced map (2.2) of fibers, commutes.

A Markl operad $\mathcal{M}$ is called *unital* if one is given, for each trivial U, a map $\eta_U : \mathbb{K} \to \mathcal{M}(U)$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}(U) \otimes \mathcal{M}(T) & \xrightarrow{\circ_!} & \mathcal{M}(T) \\ {\scriptstyle \eta_U \otimes \mathbb{1}} \uparrow & & \| \\ \mathbb{K} \otimes \mathcal{M}(T) & \xrightarrow{\cong} & \mathcal{M}(T) \end{array}$$

where $T \triangleright T \xrightarrow{!} U$ is the unique map, commutes whenever T is such that $e(T) \geq 1$.

Let $\mathtt{O}_{\mathtt{LTrm}} \subset \mathtt{O}$ be the full operadic subcategory consisting of local terminal objects of $\mathtt{O}$. Denote by $\mathbf{1}_{\mathtt{Term}} : \mathtt{O}_{\mathtt{LTrm}} \to \mathtt{V}$ the constant functor, i.e. the functor such that $\mathbf{1}_{\mathtt{Term}}(u) = \mathbb{K}$ for each $u \in \mathtt{O}_{\mathtt{LTrm}}$. Since $\mathtt{O}_{\mathtt{LTrm}}$ is equivalent, as a category, to the discrete groupoid of trivial objects, for a unital Markl operad $\mathcal{M}$ the collection $\{\eta_U : \mathbb{K} \to \mathcal{M}(U)\}$ of unit maps extends uniquely into a transformation

$$\eta : \mathbf{1}_{\mathtt{Term}} \to \iota^* \mathcal{M} \tag{4.2}$$

from the functor $\mathbf{1}_{\mathtt{Term}}$ to the restriction of $\mathcal{M}$ along the inclusion $\iota : \mathtt{O}_{\mathtt{LTrm}} \hookrightarrow \mathtt{O}$. The component $\eta_u : \mathbb{K} \to \mathcal{M}(u)$ of that extension is, for u local terminal, given by $\eta_u := !^* \eta_U$, where $! : u \to U$ is the unique map to a trivial U. Transformation (4.2) of course amounts to a family of maps $\eta_u : \mathbb{K} \to \mathcal{M}(u)$ given for each local terminal $u \in \mathtt{O}_{\mathtt{LTrm}}$, such that the diagram

$$\begin{array}{ccc} \mathcal{M}(u) & \xrightarrow{!^*} & \mathcal{M}(v) \\ {\scriptstyle \eta_u} \uparrow & & \uparrow {\scriptstyle \eta_v} \\ \mathbb{K} & = & \mathbb{K} \end{array}$$

commutes for each (unique) map $! : v \to u$ of local terminal objects. We will call the components $\eta_u : \mathbb{K} \to \mathcal{M}(u)$ of the transformation (4.2) the *extended units*.

For each T with $e(T) \geq 1$ and $F \triangleright T \xrightarrow{!} u$ with u a local terminal object, one has a map $\vartheta(T,u) : \mathcal{M}(F) \to \mathcal{M}(T)$ defined by the diagram

$$\begin{array}{ccc} \mathcal{M}(u) \otimes \mathcal{M}(F) & \xrightarrow{\circ_!} & \mathcal{M}(T) \\ {\scriptstyle !^* \otimes \mathbb{1}}\uparrow & & \\ \mathcal{M}(U) \otimes \mathcal{M}(F) & & \uparrow{\scriptstyle \vartheta(T,u)} \\ {\scriptstyle \eta_U \otimes \mathbb{1}}\uparrow & & \\ \mathbb{K} \otimes \mathcal{M}(F) & \xrightarrow{\cong} & \mathcal{M}(F)\,. \end{array} \tag{4.3}$$

Notice that the composite of the maps in the left column equals $\eta_u \otimes \mathbb{1}$, where η_u is a component of the extension (4.2).

Definition 32. A Markl operad $\mathcal{M}$ is *strictly unital* if all the maps $\vartheta(T,u)$ in (4.3) are identities. It is 1-*connected* if the unit maps $\eta_U : \mathbb{K} \to \mathcal{M}(U)$ are isomorphisms for each trivial U.

Example 33. A Markl operad $\mathcal{M}$ in $(\mathtt{V}, \otimes, \mathbb{K})$ such that $\mathcal{M}(T) = \mathbb{K}$ for each $T \in \mathtt{O}$ with all composition laws the identities is called the *constant* $\mathtt{O}$-operad and we will denote it as $\mathtt{1}_{\mathtt{O}}$.

5. The category of Graphs

The aim of this section is to recall the operadic category of graphs. This category and its subcategories, namely the category of connected simple graphs, are precisely the objects we intent to study. We will show that it satisfies Assumptions 1 and will characterize its elementary objects. For a more detailed look and full proofs see [2, Section 3].

Definition 34. A *preordered graph* Γ is a pair (g, σ) consisting of an order-preserving map

$$g : F \to V,\ V \neq \emptyset$$

in the category `Fin` together with an involution σ on F.

Elements of $\mathrm{Flg}(\Gamma) := F$ are the *flags* (also called *half-edges*) of Γ and elements of $\mathrm{Ver}(\Gamma) := V$ are its *vertices*. The elements of the set $\mathrm{Leg}(\Gamma)$ of fixed points of σ are called the *legs* of Γ while nontrivial orbits of σ are its *edges*. The *endpoints* of an edge $e = \{h_1, h_2\}$ are $g(h_1)$ and $g(h_2)$. For any $v \in V$, the set $g^{-1}(v)$ of flags adjacent to v inherits a linear order from F which we call the *local order* at v.

We will often use a short notation (F, V) or (F, g, V) for a preordered graph (g, σ) if we want to specify its set of vertices and flags only.

Definition 35. A *morphism* of preordered graphs $\Phi : \Gamma \to \Gamma'$ is a pair (ψ, ϕ) of morphisms of finite sets such that the diagram

$$\begin{array}{ccc} F & \xleftarrow{\ \psi\ } & F' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ V & \xrightarrow{\ \phi\ } & V' \end{array} \tag{5.1}$$

commutes. We moreover require ϕ to be a surjection and require ψ to be equivariant with respect to the involutions and to induce a bijection on fixed points. Thus ψ injectively maps flags to flags and bijectively legs to legs.

The pair (ψ, ϕ) must satisfy the following condition: If $\phi(i) \neq \phi(j)$ and e is an edge with endpoints i and j then there exists an edge e' in Γ' with endpoints $\phi(i)$ and $\phi(j)$ such that $e = \psi(e')$. Notice that we denote by the same symbol both the map of flags and the obvious induced map of edges. Preordered graphs and their morphisms form a category *of preordered graphs* `prGr`.

Definition 36. The *fiber* $\Phi^{-1}(i)$ of a morphism $\Phi = (\psi, \phi) : \Gamma \to \Gamma'$ in (5.1) over $i \in V'$ is a preordered graph whose set of vertices is $\phi^{-1}(i)$ and whose set of flags is $(\phi g)^{-1}(i)$. The involution τ of $\Phi^{-1}(i)$ is defined as

$$\tau(h) := \begin{cases} h & \text{if } h \in Im(\psi) \\ \sigma(h) & \text{if } h \notin Im(\psi), \end{cases}$$

where σ is the involution of Γ. Observe that $h \notin Im(\psi)$ if and only if $\sigma(h) \notin Im(\psi)$.

Definition 37. For each natural number $n \geq 0$, let 1_n (the *corolla*) be the graph $\bar{n} \to \bar{1}$ with the trivial involution.

The corollas are not local terminal objects in `prGr` since there are exactly $n!$ morphisms from any graph Γ with n legs to 1_n, therefore preordered graphs do not form an operadic category, however any such morphism is completely determined by a linear order of the legs of Γ. This leads us to the following definition.

Definition 38. The category of *ordered graphs* `Gr` is the coproduct of the categories $\texttt{prGr}/1_n$ for $n \geq 0$.

Proposition 39. *The category of ordered graphs* `Gr` *is an operadic category.*

We want to show that the category of graphs fulfills axioms of Assumptions 1 or at least most of them.

Lemma 40. *The operadic category* `Gr` *is graded by the number of internal edges of a graph.*

The grading from Lemma 40 is not strict. Consider a unique morphism $!_{p+q}$ of the graph obtained as the disjoint union of corollas $1_p \cup 1_q$ to 1_{p+q} and recall that $1_n, n \geq 0$, are the chosen local terminal objects of `Gr`. This is not an isomorphism, but $e(!_{p+q}) = 0$.

For this reason let us consider the following subcategory.

Definition 41. The category of *connected graphs* `Grc` is a subcategory of `Gr`. A graph is *connected* if there exists a path between any two different vertices.

Lemma 42. *The category of connected graphs* `Grc` *forms an operadic category with structure induced from* `Gr`.

Proposition 43. *The operadic category* `Gr` *satisfies all Assumptions 1 except for the strict grading. The operadic category* `Grc` *satisfies all Assumptions 1.*

We can finally proceed to the definition of the central object of our thesis, the category of *simple graphs*.

Definition 44. The category of *simple graphs* `SGr` is a full subcategory of `Gr`. A graph is *simple* if there is no more than one edge between any two different vertices and no loops, i.e. edges from a vertex to itself. Similarly the category of *connected simple graphs* `SGrc` is a full subcategory of `Grc`.

Lemma 45. *The category* `SGr` *is a full operadic subcategory of* `Gr` *and* `SGrc` *is a full operadic subcategory of* `Grc`.

Proof. we will use Lemma 6. Part (i) is fulfilled trivially since the corollas are simple graphs. For (ii) consider a morphism $\phi : T \to S$ of simple graphs. If one of its fibers contains a multiple edge or a loop then by Definition 36 of fibers of graphs there must be a multiple edge or a loop in T. Hence (ii) is fulfilled. The proof for connected simple graphs is analogous. □

Definition 46. The category of *trees* `Tr` is a full subcategory of `SGrc` consisting of connected graphs without cycles.

Lemma 47. *The category* `Tr` *is a full operadic subcategory of* `SGrc`.

Proof. Completely analogous to the proof of Lemma 45 except the role of multiple edges and loops is played by cycles. □

Let us describe the elementary objects (Definition 30) in the category `SGrc`. We will use this description in Section 7 to investigate the structure of free Markl operads. Since we are working with categories of graphs, we will call these elementary objects *elementary graphs*.

Theorem 48. *Let* $\Gamma \in$ `SGrc`, $e(\Gamma) \geq 1$. *Then* Γ *is an elementary graph if and only if for each full subgraph* $\Gamma' \subsetneq \Gamma$ *such that* $e(\Gamma') \geq 1$ *there exists a vertex* $v \in \mathrm{Ver}(\Gamma) \setminus \mathrm{Ver}(\Gamma')$ *and vertices* $u_1, u_2 \in \mathrm{Ver}(\Gamma')$ *satisfying* $u_1 \neq u_2$ *such that there exist edges in* Γ *between* u_1 *and* v *and also between* u_2 *and* v.

Proof. ($\leftarrow$) Assume that Γ is not elementary, i.e. there exists an elementary morphism $\phi : \Gamma \to S$ with fiber F such that S is not local terminal. This means that $e(S) \geq 1$ and $e(F) \geq 1$. Now take as Γ' the fiber F but without its legs. Since $e(S) \geq 1$ we have $\Gamma' \subsetneq \Gamma$ and since $e(F) \geq 1$ we have $e(\Gamma') \geq 1$. Assume that there exist vertices u_1, u_2, v as in the theorem. Then there is a double-edge in S, concretely, since $\phi(u_1) \neq \phi(v)$ and there exist an edge between u_1 and v, there has to be an edge between $\phi(u_1)$ and $\phi(v)$. For the same reason there has to be a different edge between $\phi(u_2)$ and $\phi(v)$, because $u_1 \neq u_2$. But since $\phi(u_1) = \phi(u_2)$, this forms a double-edge between $\phi(u_1) = \phi(u_2)$ and $\phi(v)$, so $S \notin$ `SGrc`, a contradiction.

($\rightarrow$) Assume that there exists $\Gamma' \subsetneq \Gamma$ with no vertices u_1, u_2, v as in the theorem. Define then a morphism $\phi : \Gamma \to S$ that sends all vertices of Γ' to a single vertex and acts as an identity on all other vertices. Similarly all half-edges not in Γ' are preserved and all half-edges in Γ' are not. Then ϕ is an elementary morphism. Since $\Gamma' \subsetneq \Gamma$ it holds that $e(S) \geq 1$ so it is not a local terminal, $e(\Gamma') \geq 1$ so the fiber of ϕ has a grade ≥ 1. And crucially $S \in$ `SGrc`, because S contains no loop, since Γ' was a full subgraph, and no multiple edge, since that would give us vertices u_1, u_2, v as in the theorem and we assumed no such exist. Hence Γ is not elementary. □

Recall that a graph is called *complete* if there exists an edge between every two different vertices.

Corollary 49. *Each complete graph in* `SGrc` *is elementary.*

Corollary 50. *There is an infinite number of elementary graphs in* `SGrc`.

Remark 51. In Corollary 50 we can see a big difference between the category `SGrc` and the categories `Grc` and `Tr` where the only elementary graphs are those with precisely one internal edge. Modulo the number of legs, there are only two elementary graphs in `Grc`, the loop and the interval, and only one elementary graph in `Tr`, the interval.

To be able to work with the category `SGr` or `SGrc` further, we need to verify that is satisfies Assumptions 1. For that we need to introduce a concept of *discrete operadic fibration*.

6. Discrete operadic fibrations

The aim of this section is to recall the notion of discrete operadic fibrations $p : \mathtt{O} \to \mathtt{P}$, and show that the operadic category $\mathtt{O}$ fulfills at least some of the axioms of Assumptions 1 if $\mathtt{P}$ fulfills them. Then we will provide some examples of discrete operadic fibrations and we use them to prove that $\mathtt{SGrc}$ satisfies Assumptions 1. For more information and proofs see [2, Section 4].

Definition 52. An operadic functor $p : \mathtt{O} \to \mathtt{P}$ is a *discrete operadic fibration* if

(a) p induces a surjection $\pi_0(\mathtt{O}) \twoheadrightarrow \pi_0(\mathtt{P})$ and

(b) for any morphism $f : T \to S$ in $\mathtt{P}$ and any list of objects $t_1, \dots, t_k, s \in \mathtt{O}$, where $k = |S|$, such that

$$p(s) = S \text{ and } p(t_i) = f^{-1}(i) \text{ for all } i \in |S|,$$

there exists a unique $\sigma : t \to s$ in $\mathtt{O}$ such that

$$p(\sigma) = f \text{ and } t_i = \sigma^{-1}(i) \text{ for all } i \in |S|.$$

Proposition 53. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration. If $\mathtt{P}$ is factorizable operadic category satisfying the blow-up axiom and in which all quasibijections are invertible, then $\mathtt{O}$ is also a factorizable operadic category satisfying the blow-up axiom in which all quasibijections are invertible.*

Remark 54. Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration, it is *not* true that the unique fiber condition holds in $\mathtt{O}$ if it holds in $\mathtt{P}$, cf. [2, Example 4.12.].

Remark 55. Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration and assume $\mathtt{P}$ satisfies Assumptions 1. If we "manually" verify that $\mathtt{O}$ is constant-free, strictly graded and satisfies the unique fiber condition, then from Proposition 53 and Lemma 19 we conclude that $\mathtt{O}$ satisfies all Assumptions 1.

The following lemma provides a recipe for obtaining examples of discrete operadic fibrations and will also play an important role in proving that the category of simple graphs fulfills all the relevant axioms.

Lemma 56. *Let $i : \mathtt{O} \subset \mathtt{P}$ be a full operadic subcategory such that*

(i) The chosen local terminal objects of $\mathtt{P}$ coincide with the chosen local terminal objects of $\mathtt{O}$,

(ii) for any morphism $f : T \to S$ in $\mathtt{P}$ with fibers $t_1, \dots, t_k$ such that $S \in \mathtt{O}$ and $t_1, \dots, t_k \in \mathtt{O}$, it also holds that $T \in \mathtt{O}$.

Then i is a discrete operadic fibration.

Proof. The surjection property $\pi_0(\mathtt{O}) \twoheadrightarrow \pi_0(\mathtt{P})$ follows immediately from (i). To prove the second part, given $f : T \to S$ in $\mathtt{P}$ and a list of objects $t_1, \dots, t_k, S \in \mathtt{O}$ as in (b) in Definition 52, we have by (ii) that $T \in \mathtt{O}$, thus the unique σ we are looking for is f itself. □

Example 57. The inclusion $\mathtt{Grc} \to \mathtt{Gr}$ is a discrete operadic fibration. Indeed, consider a morphism whose codomain is connected, but whose domain is not. Then there exist two vertices in the domain, not connected by any path, that are mapped to a single vertex in the codomain. But then the fiber over this vertex contains two vertices not connected by a path, i.e. such fiber is not connected. Thus we checked (ii) in Lemma 56 while (i) is obvious.

Example 58. As in Example 57, the inclusion $\mathtt{Tr} \to \mathtt{Grc}$ is a discrete operadic fibration. This is because, given a morphism whose domain contains a cycle, but whose codomain does not, since our definition of morphisms of graphs does not allow us to "forget" an edge between two vertices *not* mapped to a single one, the entire cycle needs to be mapped onto a single vertex in the codomain. But then the fiber over this vertex contains a cycle, i.e. this fiber is not a tree. Thus again we checked (ii) in Lemma 56 while (i) is obvious.

Since trees are *simple graphs*, the same reasoning as in Example 58 gives the following statement.

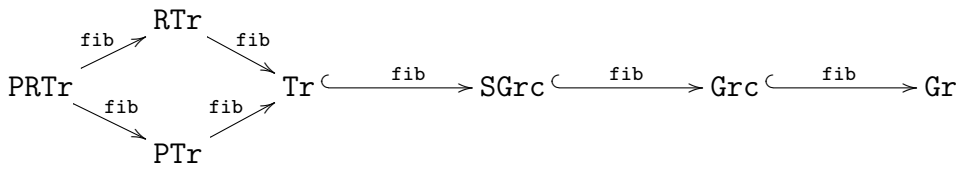
Corollary 59. *The inclusion* $\mathtt{Tr} \to \mathtt{SGrc}$ *is a discrete operadic fibration.*

A more relevant question is however whether SGr is a *domain* of some discrete operadic fibration.

Theorem 60. *The inclusion* $\mathtt{SGr} \to \mathtt{Gr}$ *is a discrete operadic fibration. The same is true for* $\mathtt{SGrc} \to \mathtt{Grc}$*.*

Proof. To prove the theorem, we show that the assumptions of Lemma 56 are fulfilled. To prove (ii) assume that for a morphism in Gr its domain contains either a multiple edge or a loop. Then either this multiple edge or a loop is present also in the codomain, or by Definition 36 of a fiber, the multiple edge or a loop will appear in one of the fibers. Hence we have (ii) and (i) follows immediately. A similar argument applies for connected simple graphs. □

As in Example 57 and Example 58 we construct discrete operadic fibrations from the categories of *rooted trees*, *planar trees* and *planar rooted trees*, which are full operadic subcategories of Tr. The detailed construction can be found in [2, Section 4] and, together with our constructions, we obtain the diagram



(6.1)

in which fib denotes discrete operadic fibrations.

Now we can finally prove that SGrc fulfills the relevant axioms.

Proposition 61. *The operadic category* SGrc *fulfills all axioms of Assumptions 1. Moreover all categories featured in diagram* (6.1) *do so.*

Proof. By Proposition 43 the category `Grc` satisfies all Assumptions 1. Since `SGrc` is a full operadic subcategory of `Grc` we get by Lemma 23 that `SGrc` is constant-free, strictly graded and satisfies the unique fiber condition. Then from Theorem 60 we get that $\texttt{SGrc} \to \texttt{Grc}$ is a discrete operadic fibration and finally using Remark 55 we conclude that that `SGrc` satisfies all Assumptions 1. Other categories mentioned in diagram (6.1) could be attended to similarly. □

7. Free Markl operads

This section provides some definitions and results on construction of free strictly unital 1-connected Markl operads. The most general construction can be found in [4, Section 3], we will restrict ourselves to the category $\mathtt{SGrc}$. Let us recall that $(\mathtt{V}, \otimes, \mathbb{K})$ is a cocomplete strict symmetric monoidal category and 0 its initial object.

To construct a free object, we need to understand the underlining structure. In Definition 31, Markl operads were defined as $\mathtt{O}_{\mathtt{iso}}^{\mathrm{op}}$-presheaves with values in $\mathtt{V}$, however we will need a more delicate additional structure.

Definition 62. A category $\mathtt{QSGrc}$ has, as the category $\mathtt{SGrc}$, as objects connected simple graphs, but without global orders of their leaves and grade at least 1. Its morphisms are isomorphisms of graphs which do not need to preserve the global orders.

Definition 63. A 1-*connected* $\mathtt{SGrc}$-*collection* in $(\mathtt{V}, \otimes, \mathbb{K})$ is a $\mathtt{QSGrc}$-presheaf in $\mathtt{V}$. We will denote by $\mathtt{Coll}_1^{\mathtt{V}}$ the corresponding category.

Proposition 64. *One has a forgetful functor* $\square : \mathtt{SUMOp}_1^{\mathtt{V}}(\mathtt{SGrc}) \to \mathtt{Coll}_1^{\mathtt{V}}(\mathtt{SGrc})$ *from the category of* 1*-connected strictly unital Markl* $\mathtt{SGrc}$*-operads to the category of* 1*-connected* $\mathtt{SGrc}$*-collections in* $\mathtt{V}$ *defined on objects by*

$$\square\mathcal{M}(T) := \begin{cases} \mathcal{M}(T) & \text{if } e(T) \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Our next goal is to construct a left adjoint $E \mapsto \mathbb{F}(E)$ to the forgetful functor of Proposition 64. The main building blocks of such a functor are the subjects of the following definition.

Definition 65. A *tower* is a sequence

$$\mathfrak{T} := T \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} T_{k-1} \xrightarrow{f_k} U_c \tag{7.1}$$

of elementary morphisms as in Definition 24, with f_k the unique morphism to a chosen local terminal object U_c.

Let $F_1, \ldots, F_k$ be the unique nontrivial fibers of $f_1, \ldots, f_k$. We will call $F_1, \ldots, F_k$ the *fiber sequence* of the tower $\mathfrak{T}$. The number k is the *height* of the tower $\mathfrak{T}$.

A *labeled tower* is a pair $(\ell, \mathfrak{T})$ where $\mathfrak{T}$ is a tower as in (7.1) and ℓ is an isomorphism (the *labeling*) $\ell : X \to T$. We denote by $\mathtt{lTw}(X)$ the category whose objects are labeled towers of this form.

The full description of morphisms in the category $\mathtt{lTw}(X)$ can be found in [4], we will only require the following property.

Proposition 66. *All morphisms in* $\mathtt{lTw}(X)$ *are invertible and preserve the height of towers, thus the category* $\mathtt{lTw}(X)$ *is a graded grupoid,*

$$\mathtt{lTw}(X) = \coprod_{k \geq 1} \mathtt{lTw}^k(X),$$

where $\mathtt{lTw}^h(X)$ *is the subcategory of labeled towers of height* k*.*

Definition 67. Let $E \in \mathtt{Coll}_1^{\mathtt{V}}$ be a 1-connected collection. For a labeled tower $(\ell, \mathfrak{T})$ with $\mathfrak{T}$ as in (7.1) we define

$$E(\ell, \mathfrak{T}) := E(t_1) \otimes \cdots \otimes E(t_k) \in \mathtt{V}. \tag{7.2}$$

This assignment extends to a functor $E : \mathtt{lTw}(X) \to \mathtt{V}$.

Theorem 68. *Let $E \in \mathtt{Coll}_1^{\mathtt{V}}$ be a 1-connected collection in $\mathtt{V}$. Then the formula*

$$\mathbb{F}(E)(X) := \begin{cases} \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}(X)} E(\ell, \mathfrak{T}) & \text{if } e(X) \geq 1 \\ \mathbb{K} & \text{if } e(X) = 0 \end{cases}$$

defines a left adjoint $E \mapsto \mathbb{F}(E)$ to the forgetful functor $\square$ of Proposition 64. Therefore $\mathbb{F}(E)$ is the free 1-connected strictly unital Markl operad generated by E.

Remark 69. Assume that $X \in \mathtt{SGrc}$ is such that $e(X) \geq 1$. Then it follows from Proposition 66 that $\mathbb{F}(E)(X)$ is graded by the height k of the underlying towers so that it decomposes as

$$\mathbb{F}(E)(X) \cong \coprod_{k\geq 1} \mathbb{F}^k(E)(X).$$

Next we will need the notion of *quadraticity* of Markl operads. From now on the symmetric monoidal category $\mathtt{V}$ will be the category $\mathtt{Set}$ of sets with the Cartesian product and a chosen one-element set $\mathbb{1}$ as the monoidal unit.

Definition 70. A *congruence* $\mathcal{C}$ in a Markl operad $\mathcal{M}$ is a sub-$\mathtt{SGrc}_{\mathrm{iso}}$-presheaf of $\mathcal{M}\times\mathcal{M}$ which is an equivalence relation and respects the circle product (4.1), i.e.

$$\circ_\phi((a, b) \times (c, d)) \in \mathcal{C}(T) \text{ if } (a, b) \in \mathcal{C}(S) \text{ or } (c, d) \in \mathcal{C}(F) \text{ for } F \triangleright T \xrightarrow{\phi} S.$$

For a sub-$\mathtt{SGrc}_{\mathrm{iso}}$-presheaf R we denote by (R) the intersection of all congruences containing R. We call (R) the congruence *generated* by R.

Definition 71. A 1-connected Markl operad $\mathcal{M}$ is *quadratic* if it is of the form

$$\mathcal{M} = \mathbb{F}(E)/(R),$$

where $E \in \mathtt{Coll}_1^{\mathtt{V}}$ is a 1-connected collection and R is a sub-$\mathtt{SGrc}_{\mathrm{iso}}$-presheaf of $\mathbb{F}^2(E) \times \mathbb{F}^2(E)$.

Proposition 72. *Let $E \in \mathtt{Coll}_1^{\mathtt{V}}$ be a 1-connected collection in $\mathtt{V}$, $\mathbb{F}(E)$ a free operad generated by E as in Theorem 68 and X an elementary object such that $e(X) \geq 1$. Then*

$$\mathbb{F}(E)(X) \cong E(X).$$

Proof. By Definition 30 of elementary objects the only elementary morphism going from X, or any isomorphic object, goes to a local terminal, thus the only towers in $\mathtt{lTw}(X)$ are those of height one. Take any such tower $(\ell, \mathfrak{T})$,

$$X \xrightarrow{\cong} T \xrightarrow{\tau} U_c.$$

Its fiber sequence is just T so $E(\ell, \mathfrak{T}) = E(T)$ and since E is a functor we have $E(T) \cong E(X)$. Hence by Theorem 68 we have

$$\mathbb{F}(E)(X) = \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}(X)} E(\ell, \mathfrak{T}) \cong \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}(X)} E(X) \cong E(X).$$

□

Recall the notion of a constant $\mathtt{SGrc}$-operad introduced in Example 33 having $\mathtt{1}_{\mathtt{SGrc}}(\Gamma) := \mathbb{1}$ for each $\Gamma \in \mathtt{SGrc}$ and the constant composition laws.

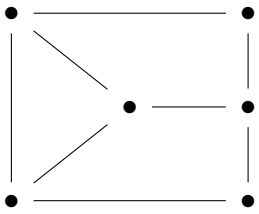
Theorem 73. *The constant* $\mathtt{SGrc}$*-operad* $\mathtt{1}_{\mathtt{SGrc}}$ *is* not *quadratic.*

Proof. Assume that $\mathtt{1}_{\mathtt{SGrc}}$ is quadratic, i.e. there exists a 1-connected collection $E \in \mathtt{Coll}_1^{\mathtt{V}}$ and a sub-$\mathtt{SGrc}_{\mathrm{iso}}$-presheaf R of $\mathbb{F}^2(E)$ such that

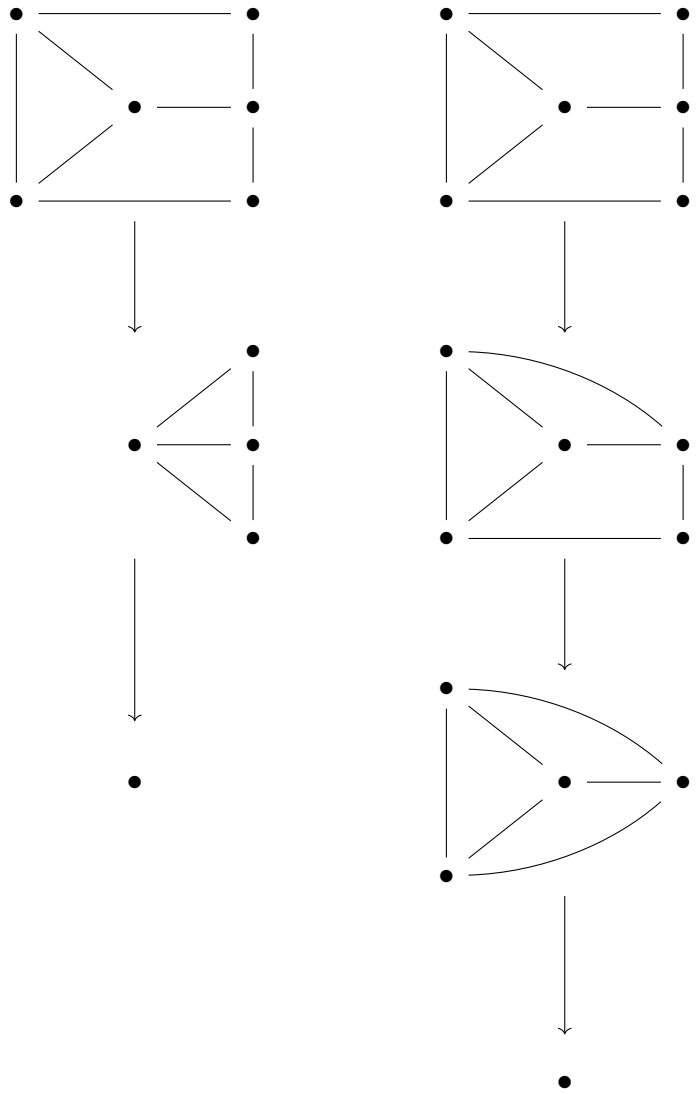
$$\mathtt{1}_{\mathtt{SGrc}} = \mathbb{F}(E)/(R).$$

We want to show that E is nonzero for elementary graphs. By Proposition 72, for an elementary graph X we have that $\mathbb{F}(E)(X) \cong E(X)$. Therefore if $E(X) = 0$ then $\mathbb{F}(E)(X) = 0$ and no quotient of it can be equal to $\mathtt{1}_{\mathtt{SGrc}}(X) = \mathbb{1}$. Thus $E(X) \neq 0$ for any elementary graph X.

For the following considerations neither the orders nor legs are relevant so we omit them from the pictures. Consider the following graph $\Gamma \in \mathtt{SGrc}$



and the two towers constructed in the diagram below.



Denote the left tower by $(id, \mathfrak{T}_1)$ and the right tower by $(id, \mathfrak{T}_2)$. We shall inspect their fiber sequences. The fiber sequence of $(id, \mathfrak{T}_1)$ is

(7.3)

and the fiber sequence of $(id, \mathfrak{T}_2)$ is

(7.4)

We conclude by Theorem 48 that all graphs in both fiber sequences (7.3) and (7.4) are elementary. So by formula (7.2) and by the fact that E must be nonzero for elementary graphs, we get $E(id, \mathfrak{T}_1) \neq 0$ and $E(id, \mathfrak{T}_2) \neq 0$. Using Theorem 68 and Remark 69 we compute

$$\mathbb{F}^2(E)(\Gamma) = \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}^2(\Gamma)} E(\ell, \mathfrak{T}) \neq 0.$$

The non-vanishing holds, because $(id, \mathfrak{T}_1) \in \mathtt{lTw}^2(\Gamma)$. For the same reason, since $(id, \mathfrak{T}_2) \in \mathtt{lTw}^3(\Gamma)$ it holds that

$$\mathbb{F}^3(E)(\Gamma) = \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}^3(\Gamma)} E(\ell, \mathfrak{T}) \neq 0.$$

Since R is a sub-$\mathtt{SGrc}_{\mathrm{iso}}$-presheaf of $\mathbb{F}^2(E) \times \mathbb{F}^2(E)$, the congruence (R) only identifies elements of the same degree, never elements of different degrees. Therefore $(\mathbb{F}(E)/(R))(\Gamma)$ contains at least two elements, since a quotient of a non-empty set always contains at least one element, $\mathbb{F}^2(E)(\Gamma)$ and $\mathbb{F}^3(E)(\Gamma)$ are non-empty and their respective elements can not be identified since they have different degrees. However

$$\mathbb{F}(E)/(R)(\Gamma) = \mathtt{1}_{\mathtt{SGrc}}(\Gamma) = \mathbb{1}$$

is a one element set, a contradiction. □

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