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**MASTER THESIS**

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# Countable dense homogeneous spaces

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Abstract: In this thesis we aim to study countable dense homogeneous spaces. First we present some known results regarding countable dense homogeneity and its relation to other topological properties. We then show examples of spaces that are countable dense homogeneous and spaces that are not. Lastly we investigate a generalization of countable dense homogeneity.
The main results of this thesis are: an example answering an open question regarding zero-dimensional metrizable countable dense homogeneous spaces and results in the last chapter generalizing some known results from the previous parts of the thesis.

Keywords: topological space, dense set, homeomorphism

Název práce: Spočetně hustě homogenní prostory

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Abstrakt: V této práci se zaměříme na studium spočetně hustě homogenních prostorů. Nejprve ukážeme známé výsledky týkající se spočetně husté homogenity a její vztah k jiným topologickým vlastnostem. Poté ukážeme jak příklady prostorů, které jsou, tak prostorů, které nejsou spočetně hustě homogenní. Nakonec budeme studovat zobecnění spočetně husté homogenity.
Hlavní přínosy této práce jsou: příklad odpovídající na otevřenou otázku týkající se nul-dimenzionálních metrizovatelných spočetně hustě homogenních prostorů a výsledky poslední kapitoly, které zobecňují některé známé výsledky prezentované v předchozích částech práce.

Klíčová slova: topologický prostor, hustá množina, homeomorfismus

# Contents

# Introduction

A space is countable dense homogeneous if for any two countable dense subsets there is an autohomeomorphism of the space sending one to the other.

Although the term countable dense homogeneity is often attributed to Bennet who used it in [5], first results were already obtained by Cantor who showed that $\mathbb{R}$ is countable dense homogeneous [8]. Later Brouwer [7] and Fréchet [16] independently showed that the same holds for $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Many contributions to this subject were made during 80's by B. Fitzpatrick and Z. Hao-xuan [11, 12, 13]. In more recent years notable contributions were made by J. van Mill [38, 30, 28]. In recent years many set theoretic notions have been introduced to the study of countable dense homogeneous spaces [18, 27].

In this thesis we aim to summarize known results about countable dense homogeneous spaces and show examples of such spaces. In Chapter 2 we establish these theoretical results as well as a general theoretical background. In Chapter 3 we provide many examples of spaces that are countable dense homogeneous and spaces that are not along with proofs of their respective properties. In Chapter 4 we study a generalization of countable dense homogeneity and we present some new results regarding this generalization.

Theorems, propositions etc. throughout this text can be divided into two categories, those that include citation after theorem number and those that do not. The theorems that include citation after the theorem number are, unless specifically stated otherwise, taken from the cited literature including their proofs. The nature of the theorems without citation are explained in the text on a case to case basis. These are sometimes small technical lemmas not mentioned in the literature however useful in the proofs, or well known and easy to prove facts that are usually stated by the authors as folklore and whose proofs are hard to track down to a particular author, sometimes these are also new results. For the cited theorems we sometimes fill in some small steps that are mentioned by the authors as easy or trivial, however we do not indicate this in the text as these really are easy but the nature of this text requires the proofs to be as detailed as possible. We also try to correct any misprints or typos that may appear in the original papers.

Any author's original contribution will be mentioned at a suitable place in the text.

# 1. Preliminaries

In this chapter we introduce some basic notions and definitions that will be used throughout this thesis. However, these are not the main focus of this text, therefore we do not include their proofs. We refer to [10] for basic topological notions, to [22] for basics of descriptive set theory and to [24] for basic notions of set theory.

## 1.1 Some basic notions and definitions

### 1.1.1 Polish spaces

**Definition 1.1** (Polish spaces). *We say that a topological space $X$ is Polish if it is separable and metrizable by a complete metric.*

Polish spaces are the main object of the study in classical descriptive set theory. Since these spaces have complete metric they behave somewhat nicely. Among other nice properties Polish spaces satisfy the Baire category theorem, which is widely used in their study. Some authors restrict their investigation of countable dense homogeneous spaces to Polish spaces as there is a well developed theory regarding Polish spaces. We do not restrict our attention only to Polish spaces but we will still need some facts about them.

One of the aimes of studying Polish spaces is to categorize them up to homeomorphism. We do not have such categorization in general however, if we restrict our attention to some subclass of Polish spaces we can obtain many results. One of such well understood classes of Polish spaces are zero-dimensional Polish spaces.

**Theorem 1.2** ([22]). *The Cantor space $2^\omega$ is the unique, up to a homeomorphism, nonempty compact metrizable zero-dimensional space without isolated points.*

This result has an interesting and very useful corollary.

*Corollary* 1.3 ([22]). Let $A$ be a non-empty clopen subset of $2^\omega$ then $A \approx 2^\omega$.

The Cantor space has one more property that will be used later. It is the universal space for zero-dimensional metrizable separable spaces.

**Proposition 1.4** ([10]). *Let $X$ be a separable metrizable zero-dimensional space. Then there exists an embedding $f : X \to 2^\omega$.*

Later on we will also need the characterization of the Cantor space without one point and the Baire space.

**Theorem 1.5** ([22]). *The space $2^\omega \setminus \{0\}$ is the unique, up to a homeomorphism, Polish nonempty noncompact locally compact zero-dimensional space without isolated points.*

**Theorem 1.6** ([22]). *The Baire space $\omega^\omega$ is the unique, up to a homeomorphism, Polish nonempty zero-dimensional space for which all compact subsets have empty interior.*

We will later show that in some sense there are no more zero-dimensional Polish spaces, which is well-known.

Even though $\mathbb{Q}$ is not a Polish space we have a very similar characterization of $\mathbb{Q}$.

**Theorem 1.7** ([36]). *The rationals $\mathbb{Q}$ is the unique, up to a homeomorphism, metrizbale countable space without isolated points.*

Another subject of study of classical descriptive set theory are Borel sets. These are in some sense nice well-behaved sets in a given Polish space.

**Definition 1.8** (Borel hierarchy). *Let $X$ be a Polish space, for every ordinal number $1 \leq \xi < \omega_1$ we define the family of sets $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Sigma}^0_\xi$ as follows.*

$$\mathbf{\Sigma}^0_1 = \{U \subset X; U \text{ is open}\}$$

$$\mathbf{\Pi}^0_\xi = \{X \setminus A; A \in \mathbf{\Sigma}^0_\xi\}$$

*and for $\xi > 1$*

$$\mathbf{\Sigma}^0_\xi = \left\{ \bigcup_{n\in\omega} A_n; A_n \in \mathbf{\Pi}^0_{\xi_n}, \xi_n < \xi \right\}.$$

*We also define $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$.*

We can also ask what happens with a Polish space if it is mapped by a continuous mapping. This gives a rise to another class of interesting subsets of a given Polish space.

**Definition 1.9** (Analytic and co-analytic sets). *Let $X$ be a Polish space, we say that $A \subset X$ is analytic if there is $Y$ Polish and $\varphi : Y \to X$ continuous such that $\varphi(Y) = A$. We say that $B \subset X$ is co-analytic if $X \setminus B$ is analytic. We denote the family of analytic sets by $\mathbf{\Sigma}^1_1$ and the family of co-analytic sets by $\mathbf{\Pi}^1_1$.*

These sets do not behave as nicely as Borel sets however there are still many results concerning analytic sets.

**Theorem 1.10** ([22]). *Let $X$ be a Polish space and $A \subset X$ uncountable analytic. Then $A$ contains a copy of the Cantor space.*

When constructing subspaces with given properties in a Polish space it is interesting to ask how nice can the subspace be in the sense of Definition 1.8 and Definition 1.9.

### 1.1.2 Topological groups and their actions

Later on we will also need some facts about topological groups and their actions on topological spaces.

**Definition 1.11** (Topological groups). *A topological group is a group $(G, \cdot)$ together with a topology such that $\cdot : G^2 \to G$ and $^{-1} : G \to G$ are continuous maps.*

*Remark.* Note that for a topological group the map $^{-1}$ is actually a homeomorphism.

**Definition 1.12** (Continuous group actions). *Let $(G, \cdot)$ be a topological group and $X$ be a topological space. A group action $\alpha$ of $G$ on $X$ is called continuous action if the mapping $\alpha : G \times X \to X$ is continuous.*

*We say that the action is transitive if $\alpha((G, x)) = X$ for every $x \in X$.*

*Remark.* If the action $\alpha$ is transitive then for any $x, y \in X$ there is $g \in G$ such that $\alpha((g, x)) = y$. The transitivity of the action is closely related to the homogeneity of the space $X$.

Note that the topology on a group turning it into a topological group is by no means unique. The discrete and the indescrete topologies make any group into a topological group. Hence, it is quite natural to impose some additional topological properties on topological groups to get a better behaviour. We will mostly work with groups that have a Polish topology.

**Definition 1.13** (Polish groups). *We say that a topological group $(G, \cdot)$ is a Polish group if the topology on $G$ is Polish.*

We will work with the group of homeomorphisms of a given topological space and its subgroups. For a topological space $X$ we will denote $\mathcal{H}(X)$ the group of all autohomeomorphisms. For a topological space $X$ the group $\mathcal{H}(X)$ induces a very natural action on $X$ that maps the point $(h, x) \in \mathcal{H}(X) \times X$ to $h(x)$. There is also a very natural topology on $\mathcal{H}(X)$. In some special cases $\mathcal{H}(X)$ with this topology is a topological group and it acts continuously on $X$.

**Definition 1.14** (Compact open topology). *Let $X$ be a topological space. A compact open topology on $\mathcal{H}(X)$ is a topology generated by the subbase $\{V(K, U); K \subset X$ compact, $U \subset X$ open $\}$, where $V(A, B) = \{h \in \mathcal{H}(X); h(A) \subset B\}$ for $A, B \subset X$.*

**Theorem 1.15** ([2]). *Let $X$ be a Hausdorff compact or Hausdorff locally compact and locally connected space. Then $\mathcal{H}(X)$ with the compact open topology is a topological group.*

**Theorem 1.16.** *Let $X$ be a Hausdorff compact or Hausdorff locally compact and locally connected space. Then the action of $\mathcal{H}(X)$ on $X$ is continuous.*

*Proof.* By Theorem 1.15 the group $\mathcal{H}(X)$ with the compact open topology is a topological group.

Denote $\alpha$ the action of $\mathcal{H}(X)$ on $X$. Now let $(h, x) \in \mathcal{H}(X) \times X$ and let $U$ be an open neighborhood of $h(x)$. In both cases the space is locally compact, thus we can find an open neighborhood $G$ of $x$ such that $\overline{G}$ is compact and $h(\overline{G}) \subset U$. We also have that $(h, x) \in V(\overline{G}, U) \times G$ and $\alpha(V(\overline{G}, U) \times G) \subset U$, hence the action is continuous. □

*Remark.* Note that for the continuity of the action we only need the space to be locally compact. The other assumptions are to ensure that $\mathcal{H}(X)$ is actually a group.

For a special class of spaces we have that the compact open topology on $\mathcal{H}(X)$ is Polish.

**Theorem 1.17** ([22])**.** *Let $X$ be a metrizable compact or metrizable locally compact and locally connected space. Then $\mathcal{H}(X)$ with the compact open topology is a Polish group.*

The actions of Polish groups on Polish spaces were studied by Effros [9], who showed the following.

**Theorem 1.18** (Effros theorem, [9])**.** *Let $G$ be a Polish group acting continuously on a Polish space $X$. Then the following are equivalent:*

1. *For each $x \in X$, the map $gG_x \mapsto gx$ from $G/G_x$ onto $Gx$ is a homeomorphism.*
2. *Each orbit is of second category in itself.*
3. *Each orbit is $G_\delta$ in $X$.*

Using the fact that the quotient map is open, Ungar [39] showed the following.

**Theorem 1.19** ([39])**.** *Let $G$ be a Polish group acting continuously on a Polish space $X$. If $G$ acts transitively on $X$ the map $g \mapsto gx$ from $G$ onto $X$ is open.*

Theorem 1.18 was used by Ungar [39, 38] when studying Polish countable dense homogeneous spaces. We will present his results later.

## 1.2 Topological homogeneity

Now we introduce the notion of (topological) homogeneity and other homogeneity notions. Later we will show that these are closely connected to the notion of countable dense homogeneity.

**Definition 1.20** (topological homogeneity)**.** *Let $X$ be a topological space, we say that $X$ is homogeneous, if for any two points $x, y \in X$ there is $h \in \mathcal{H}(X)$ such that $h(x) = y$.*

Informally, homogeneous spaces are spaces that are the same at every point.

Very natural generalization of homogeneity is the notion of $n$-homogeneity and strong $n$-homogeneity.

**Definition 1.21** (strong n-homogeneity)**.** *Let $X$ be a topological space, we say that $X$ is strongly $n$-homogeneous for $n \in \mathbb{N}$, if for any two $n$-tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ of distinct points of $X$, there is $h \in \mathcal{H}(X)$ such that $h(x_i) = y_i$ for $i \leq n$.*

**Definition 1.22** (n-homogeneity)**.** *Let $X$ be a topological space, we say that $X$ is $n$-homogeneous for $n \in \mathbb{N}$, if for any two $n$-point subsets $F, G \subset X$ there is $h \in \mathcal{H}(X)$, such that $h(F) = G$.*

The notion of countable dense homogeneity might seem like an even more general concept, and even though there are relations in some cases, as we will later show, in general the concept of countable dense homogeneity stands on its own.

**Definition 1.23** (Countable dense homogeneity). *Let $X$ be a separable topological space, we say that $X$ is countable dense homogeneous (CDH), if for any two dense countable sets $A, B$ there is $h \in \mathcal{H}(X)$, such that $h(A) = B$.*

*Remark.* Sometimes we will write CDH property as an abbreviation for countable dense homogeneity. We will also sometimes write $X$ has the CDH property instead of $X$ is CDH.

On a first glance countable dense homogeneity might seem very restrictive, we will show that in fact many classical spaces are actually CDH.

We introduce one more notion of homogeneity, which is very closely related to the notion of CDH-spaces

**Definition 1.24** (Strong local homogeneity). *Let $X$ be a topological space, we say that $X$ is strongly locally homogeneous (SLH), if it has a basis $\mathcal{B}$ such that for any $U \in \mathcal{B}$ and every $x, y \in U$, there is $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h$ is the identity on $X \setminus U$.*

*Remark.* Similarly as for CDH property, for strong local homogeneity we will sometimes use abbreviation SLH property and we will sometimes write $X$ has the SLH property instead of $X$ is SLH.

# 2. Theoretical results about CDH-spaces

In this chapter we present some well-known properties of CDH-spaces.

## 2.1 General results about CDH-spaces

In this section we show some topological properties of CDH-spaces without assuming any topological properties other than the CDH property. However there is not much we can say in general about CDH-spaces. It is also worth noting that in general the CDH property is closed under very few topological operations. However as we will show later, in some special cases, such as zero-dimensional Polish spaces, one has much better results regarding CDH-spaces.

We start by establishing the only separation axioms we have for CDH-spaces.

**Theorem 2.1** ([13])**.** *Let $X$ be CDH, then $X$ is $T_1$.*

*Proof.* Denote $\mathcal{C}$ the collection of all countable dense subsets of $X$. For $x, y \in X$ define $x \leq y$ if $x \in \overline{\{y\}}$, and $x < y$ if $x \leq y$ and $x \neq y$. Further define an equivalence relation on $X$ by $x \sim y$ if and only if $x \leq y$ and $y \leq x$, and we denote $Q$ the set of all nontrivial equivalence classes of $\sim$.

First we show that $X$ is $T_0$, so suppose $Q \neq \emptyset$. Let $K \in \mathcal{C}$. For every $A \in Q$ such that $A \cap K \neq \emptyset$, choose $p_A \in A \cap K$. The set of $p_A$ is at most countable, since K is countable. Denote $K' = (K \setminus \bigcup Q) \cup \{p_A, A \in Q\}$, then $K'$ is countable dense from the definition of $Q$. Choose $A \in Q$ and $p, q \in A$, $p \neq q$. Define $K'' = K' \cup \{p, q\}$, then $K'' \in \mathcal{C}$. However, there is no $h \in \mathcal{H}(X)$ such that $h(K') = K''$, since $K'$ intersects every set of $Q$ in at most one point but $K'' \cap A = \{p, q\}$. Therefore $Q = \emptyset$ and $X$ is $T_0$. This now implies that $\sim$ is antisymmetric.

Now assume that $X$ is not $T_1$ and denote $A = \{x \in X; \{x\} \neq \overline{\{x\}}\}$ and $B = \{y \in X; y < x \text{ for some } x \in A\}$. Let $x, y \in X$, we call $y$ the primitive of $x$ if $x < y$ and $y \notin B$, this implies $y \in A$ and $x \in B$. Assume that there are $x, y \in X$ such that $y$ is the primitive of $x$.

Let $K \in \mathcal{C}$ such that $\{x, y\} \subset K$. Let $K_1 = \{x \in K \cap B;\, x \text{ and a primitive of } x \text{ are in } K\}$ and let $K_2 = K \setminus K_1$. Then $K_2 \in \mathcal{C}$ but $K_2$ contains no point and its primitive. Therefore, there is no $h \in \mathcal{H}(X)$ such that $h(K) = K_2$, which can not be. This means that no point of $X$ has a primitive, which implies $A \subset B$.

Next assume $B \setminus A \neq \emptyset$. Then there is a $K \in \mathcal{C}$ such that $K \cap (B \setminus A) \neq \emptyset$. For each $x \in K \cap (B \setminus A)$ choose $p_x \in A$, $x < p_x$. Let $K' = (K \setminus (B \setminus A)) \cup \{p_x; x \in K \cap (B \setminus A)\}$. Then $K' \in \mathcal{C}$, but since $K' \cap (B \setminus A) = \emptyset$, there is no $h \in \mathcal{H}(X)$ such that $h(K') = K$, which is a contradiction. Therefore we have $B = A$.

It follows that for any $x \in A$ there are $y, z \in A$ such that $y < x < z$. Inductively we can construct double-ended sequence $y_i$, $i \in \mathbb{Z}$ such that $y_i < y_j$, whenever $i < j$, $i, j \in \mathbb{Z}$. It now follows that every $K \in \mathcal{C}$ contains such double-ended sequence. If there is $K \in \mathcal{C}$ that does not contain such sequence we can add such sequence to get $K' \in \mathcal{C}$ but then there is no $h \in \mathcal{H}(X)$ satisfying $h(K) = K'$.

Now let $K \in \mathcal{C}$ and let $K \cap A = \{k_i,\, i \in \mathbb{N}\}$. Let $L_1 = (K \cap A) \setminus \{k \in K;\, k < k_1\}$. Then $L_1 = \bigcup_{i=1}^{\infty} K_i^{(1)}$, where $K_i^{(1)} = L_1 \cap \{k_i\}$. Let $L_2 = K_1^{(1)} \cup K_2^{(1)} \cup \left(\bigcup_{i=3}^{\infty} K_i^{(1)} \setminus \{k \in K;\, k < k_2\}\right)$. Then $L_2 = \bigcup_{i=1}^{\infty} K_i^{(2)}$, where $K_i^{(2)} = K_i^{(1)} \cap L_2$. We can now inductively define $L_i$ for $i \in \mathbb{N}$ and $K_i^{(n)}$ for $i, n \in \mathbb{N}$. Then $K' = (K \setminus A) \cup \bigcup_{i=1}^{\infty} K_i^{(i)} \in \mathcal{C}$, however $K'$ does not contain any double-ended sequence, which is a contradiction and therefore $X$ is $T_1$.

□

Next we aim to show the result of Fitzpatrick and Lauer [12] that components of CDH-spaces are also CDH. For that we first need two technical lemmas. Even thought their main purpose is to be used in the proof of the theorem they can be used to show that some spaces cannot be CDH.

**Lemma 2.2** ([12]). *Let $X$ be CDH, then $X$ has at most countably many nondegenerate components.*

*Proof.* Let $S$ be a countable dense subset of $X$. There is at most countably many nondegenerate components intersecting $S$. We can choose two distinct points from every nondegenerate component intersecting $S$ and add them to $S$. We obtain countable dense set $S'$, such that if $K$ is a nondegenerate component intersecting $S'$ then $S'$ contains at least two points of $K$. If there are uncountably many components, choose one that does not intersect $S'$ and choose a point $x$ from it. Then $S' \cup \{x\}$ is a countable dense subset, but there is no $h \in \mathcal{H}(X)$ such that $h(S') = S' \cup \{x\}$. □

*Corollary* 2.3. Let $X$ be a nontrivial connected space then $X \times 2^\omega$ is not CDH.

*Proof.* For every $y \in 2^\omega$ we have that $X \times y$ is a nondegenerate component of $X \times 2^\omega$ because $X$ is nontrivial and connected. Since $2^\omega$ is uncountable, $X \times 2^\omega$ has uncountably many nondegenerate components, therefore by Lemma 2.2 it cannot be CDH. □

In the next chapter we will show that $2^\omega$ is CDH and that there exist CDH connected spaces. Therefore Corollary 2.3 actually shows that CDH-spaces are not closed under finite products.

**Lemma 2.4** ([12]). *If $X$ is CDH, then every nondegenarte component of $X$ is open.*

*Proof.* Let $K$ be a nondegenerate component of $X$. By lemma Lemma 2.2 we can denote $K_1, K_2, \ldots$ the nondegenerate components of $X \setminus K$, if there are any. Let $S$ be a countable dense subset of $X$. Assume $K$ is not open. Then the boundary $\partial K$ is nonempty. Let $S' = S \setminus \partial K$. Then $S'$ is a countable dense subset of $X$ that does not intersect $\partial K$. If for $n \in \mathbb{N}$ we have $\partial K_n \neq \emptyset$, we choose a point in $\partial K_n$ and add it to $S'$. We obtain a countable dense subset $S_1$ such that for one nondegenerate component $K$ of $X$ with nonempty boundary we have $S_1 \cap \partial K = \emptyset$ and such that if $K_n$ is any other nondegenerate component of $X$ with nonempty boundary, we have $S_1 \cap \partial K_n \neq \emptyset$. Now let $x \in \partial K$. Then $S_1 \cup \{x\}$ is a countable dense subset of $X$, however there is no $h \in \mathcal{H}(X)$ such that $h(S_1) = S_1 \cup \{x\}$.

□

*Corollary* 2.5. Let $X$ be a a nontrivial connected space. Then $X \times \mathbb{Q}$ is not CDH.

*Proof.* For every $y \in \mathbb{Q}$ we have that $X \times y$ is a nondegenerate component of $X \times \mathbb{Q}$. Since $y$ is not isolated in $\mathbb{Q}$, the component is not open. Thus by Lemma 2.4 $X \times \mathbb{Q}$ is not CDH. □

*Remark.* Note that in the proof of Corollary 2.5 we cannot use Lemma 2.2 as the space $X \times \mathbb{Q}$ only has countably many components.

Lemma 2.4 does not work for degenerate components, we will show later that $2^\omega$ is CDH.

**Theorem 2.6** ([12])**.** *Let $X$ be CDH, then every component of $X$ is also CDH.*

*Proof.* Let $K$ be a component of $X$. If $K$ is degenerate then $K$ is CDH, so assume $K$ is nondegenerate. Since $K$ is open subset of separable space, it is separable. Let $A, B$ be two countable subsets of $K$

Let $\mathcal{C} = \{h(K); h \in \mathcal{H}(X)\}$. The set $\mathcal{C}$ is at most countable since the number of components of $X$ is at most countable. Let $\{C_i\}_{i=1}^{\alpha}$, where $\alpha \in \mathbb{N} \cup \{\infty\}$, be an enumeration of the elements of $\mathcal{C}$. For every $i$ let $h_i \in \mathcal{H}(X)$ be such that $h_i(K) = C_i$ and also denote $A_i = h_i(A)$ and $B_i = h_i(B)$

Now let $S$ be a countable dense subset of $X$ and denote

$$S_A = (S \setminus \bigcup_{i=1}^{\alpha} C_i) \cup (\bigcup_{i=1}^{\alpha} A_i),$$

and

$$S_B = (S \setminus \bigcup_{i=1}^{\alpha} C_i) \cup (\bigcup_{i=1}^{\alpha} B_i).$$

Since $S_B$ and $S_A$ are two countable dense subsets, there exists $h \in \mathcal{H}(X)$, such that $h(S_A) = S_B$, there also exists $i$ such that $h(K) = C_i$. Let $f = h|_K$, $f : K \to C_i$ and $g = h_i|_K$, $g : K \to C_i$, then we have $g^{-1} \circ f(A) = B$, so $K$ is CDH.

□

As we will see throughout this text, the CDH property is not preserved by many topological operations. The following theorem shows that topological sum preserves the CDH property. The theorem is well-known and easy to proof, however we do not know the original author.

**Proposition 2.7.** *Let $X_1$ and $X_2$ be CDH-spaces. Then $X_1 \oplus X_2$ is also a CDH-space.*

*Proof.* Clearly, $X_1 \oplus X_2$ is separable. Let $A, B$ be two countable dense subsets of $X_1 \oplus X_2$. Then $A_1 = X_1 \cap A$ and $B_1 = X_1 \cap B$ are countable dense subsets of $X_1$ and $A_2 = X_2 \cap A$ and $B_2 = X_2 \cap B$ are countable dense subsets of $X_2$. Therefore there exists $h_1 \in \mathcal{H}(X_1)$ and $h_2 \in \mathcal{H}(X_2)$ such that $h_1(A_1) = B_1$ and $h_2(A_2) = B_2$. Now define $h : X \to X$ by

$$h(x) = \begin{cases} h_1(x), x \in X_1 \\ h_2(x), x \in X_2 \end{cases}$$

for $x \in X$. Then $h \in \mathcal{H}(X)$ and $h(A) = B$.

□

## 2.2 Cardinality of CDH-spaces

Using the structure of a CDH-space provided by the existence of many homeomorphisms Arhangel'skii and van Mill [3] showed that we can bound its cardinality assuming it is Hausdorff.

**Theorem 2.8** ([3])**.** *If $X$ is a CDH Hausdorff space then $|X| \leq \mathfrak{c}$.*

*Proof.* Let $M \subset X$ be a countable dense subset and put $Y = X \setminus M$. For each $y \in Y$ let $Z_y = M \cup \{y\}$.

Since $X$ is CDH, for every $y \in Y$ we can find $f_y \in \mathcal{H}(X)$ such that $f_y(M) = Z_y$. Let $x_y = f_y^{-1}(y)$, put $M_y = M \setminus \{x_y\}$ and denote $g_y = f_y|_{M_y}$. Since $M$ is dense in $Y$, we have that $y$ is not isolated in $Z_y$ and therefore $x_y$ is not isolated in $M_y$, hence $M_y$ is dense in $M$.

Now assume $|Y| > 2^\omega$. The set $M$ is countable, hence the cardinality of all possible mappings from any subset of $M$ to $M$ does not exceed $2^\omega$. Therefore by the pigeon hole principle there are $p, q \in Y$ such that $g_p = g_q$, $x_p = x_q$ and $M_p = M_q$.

However, this is not possible since it means that one mapping $h = g_q = g_p$ from dense subset $H = M_p = M_q$ of a Hausdorff space $X$ can be extended in two different ways $f_p$ and $f_q$, which is not possible. □

Without the assumption of Hausdorff property this does not hold. In fact we can have arbitrarily large non-Hausdorff CDH-spaces as follows from the following well-known example.

**Example 2.9.** *Let $X$ be an uncountable set, then $X$ equipped with the cofinite topology is CDH.*

*Proof.* The space $X$ is clearly separable, since any countable subset $D \subset X$ is dense in $X$. Now let $C, D \subset X$ be two countable sets. We can find a bijection $h : X \to X$ such that $h(D) = C$. Since bijections preserve cardinality, we also have that $h$ is a homeomorphism. Therefore $X$ is CDH. □

*Remark.* Note that all spaces defined this way are compact and not $T_2$.

The bound on the cardinality of Hausdorff CDH-spaces contrasts with the following result regarding countable CDH-spaces. It is not clear who is the original author of this result, however it is a classical result for CDH-spaces.

**Theorem 2.10.** *A countable space $X$ is CDH if and only if it has the discrete topology.*

*Proof.* It is clear that discrete countable space $X$ is CDH since the only countable dense subset of $X$ is $X$ itself.

Now suppose that $X$ is countable space that does not have the discrete topology. This means that there exists $x \in X$ such that $x$ is not an isolated point, therefore $x \in \overline{X \setminus \{x\}}$. Hence $X \setminus \{x\}$ is a countable dense subset of $X$. Since $X$ is countable, $X$ itself is also a countable dense subset of $X$. However there is no $h \in \mathcal{H}(X)$ such that $h(X) = X \setminus \{x\}$, therefore $X$ is not CDH. □

*Remark.* The same also holds for finite spaces.

This means that anything of interest can happen only for cardinalities between $\aleph_0$ and $\mathfrak{c}$. This sheds some light on why it is important to use also some set theoretic results and notions when studying CDH-spaces.

## 2.3 Connection between countable dense homogeneity and other homogeneity notions

Countable dense homogeneous spaces must have many homeomorphisms in order to send any countable dense subset to any other countable dense subset. Therefore it is natural to expect some level of homogeneity. In this, section we show some results demonstrating the connections between the CDH property and other notions of homogeneity. However, since the CDH property is closed under topological sum, it is not reasonable to expect all CDH-spaces to be homogeneous.

Fitzpatrick and Lauer [12] showed that if we assume connectedness, therefore excluding spaces obtained via topological sum, we get homogeneity.

**Theorem 2.11** ([12])**.** *Let $X$ be CDH and connected, then $X$ is homogeneous.*

*Proof.* Suppose $X$ is a connected CDH-space that is not homogeneous. Denote $\mathcal{C}$ the collection of all countable dense sets, $[x] = \{h(x);\, h \in \mathcal{H}(X)\}$ for $x \in X$, and $G = \{[x];\, x \in X\}$.

First we show that if for $x \in X$ the set $[x]$ contains an open subset, then $[x]$ is itself open. Let $G \subset [x]$ be open. If $y \in [x]$ then from the definition of $[x]$ there is some $h \in \mathcal{H}(X)$ such that $h(y) \in G$. Since $h$ is continuous, there is some open $U$ such that $h(U) \subset G$. This means $U \subset [x]$. Therefore $[x]$ is open.

Now we show that there is no $x \in X$ such that $[x]$ is open. For suppose there is such $x \in X$. Then $X \setminus [x]$ is closed and nonempty. Since $X$ is connected, we have $\overline{[x]} \cap X \setminus [x]$ is nonempty. Let $y \in \overline{[x]} \cap X \setminus [x]$. For $h \in \mathcal{H}(X)$ we have $h([x]) = [x]$ and also $h(\partial[x]) = \partial[x]$, this implies that $[y] \subset \partial[x]$. Let $S \in \mathcal{C}$, then $S \setminus [y] \in \mathcal{C}$ and also $K = (S \setminus [y]) \cup \{y\} \in C$. However, there is no $h \in \mathcal{H}(X)$ such that $h(K) = S \setminus [y]$, which is a contradiction. This shows that for every $x \in X$ we have $\operatorname{Int}[x] = \emptyset$ and thus $X \setminus [x]$ is dense in $X$.

We also have that if $S \in \mathcal{C}$ and $x \in X$ then $S \cap [x]$ is not dense and $S \cap (X \setminus [x])$ is also not dense.

Now let $S \in \mathcal{C}$ and $x \in X$. Since $S \cap (X \setminus [x])$ is not dense, let $D = X \setminus \overline{(S \cap (X \setminus [x]))} \neq \emptyset$. Then $D$ is open and $D \subset (X \setminus S) \cup [x]$ so $D \cap S \subset [x]$. Let $T = ((S \cap D) \cup (S \cap (X \setminus [x]))) \setminus \partial D$.

If $p \in X \setminus T$, $p \in \overline{D}$ and $G$ is an open neighborhood of $p$, then $(G \cap D) \cap S = G \cap (D \cap S)$ is nonempty.

If $p \in X \setminus T$, $p \notin \overline{D}$ and $G$ is an open neighborhood of $p$ disjoint from $D$. We have $p \in \overline{S \cap (X \setminus [x])}$ so $G \cap (S \cap (X \setminus [x]))$ is nonempty.

This implies that $T$ is dense subset of $X$ and since $T \subset S$, we have $T \in \mathcal{C}$. Since $D$ is open and $X \setminus [x]$ is dense, there is $q \in D \cap (X \setminus [x])$. Now, let $Q = T \cup \{q\} \in \mathcal{C}$, then there is $h \in \mathcal{H}(X)$ such that $h(Q) = T$. This also means $h(T) \subset T$. Since $h(q) \notin [x]$ and $h(q) \in T \subset S$, we have $h(q) \in S \cap (X \setminus [x])$ and also $h(q) \in (S \cap (X \setminus [x])) \setminus \partial D$.

Because $h(q) \in \overline{S \cap (X \setminus [x])}$ and $h(q) \notin \partial D$, we have $h(q) \in X \setminus \overline{D}$. Note that $(X \setminus \overline{D}) \cap T \subset X \setminus [x]$. From continuity of $h$, there is an open neighborhood $U$ of $x$ such that $h(U) \subset X \setminus \overline{D}$. Since $T$ is dense and $D$ is open, there is some $z \in U \cap D \cap T$. We have $z \in D$ and also $z \in T \subset S$, therefore $z \in [x]$ and so also $h(z) \in [x]$. However, we also have $h(z) \in X \setminus \overline{D} \subset X \setminus [x]$, which is a contradiction.

□

Bennet [5] showed that a connected and first countable CDH-space is homogeneous. His result appeared fifteen years earlier then Theorem 2.11, however it is not as general.

*Remark.* We can see that in order to get a contradiction in the proof of Theorem 2.11 it is enough to assume that there exists some $x \in X$ such that $[x]$ is not open. This means that $[x]$ is clopen for every $x \in X$, since we have $[x] = X \setminus \bigcup_{y \notin [x]} [y]$.

Using the remark we can proof the following well-known result.

*Corollary* 2.12. Every CDH-space $X$ can be written as a disjoint topological sum $X = \oplus_{i=0}^{\alpha} X_i$, where $\alpha \in \omega + 1$ and $X_i$'s are pairwise nonhomeomorphic homogeneous and CDH-subspaces of $X$ that are clopen in $X$.

*Proof.* For $x \in X$ let $[x] = \{h(x); h \in \mathcal{X}\}$ and let $G = \{[x]; x \in X\}$. Note that $[x]$ is clopen in $X$ thus the set $G$ is at most countable and the elements of $G$ are pairwise nonhomeomorphic. Thus we can write $G = \{X_i; i < \alpha\}$ for some $\alpha \in \omega + 1$. Since all $X_i$'s are clopen we have that $X = \oplus_{i=1}^{\alpha} X_i$. Also note that the spaces $X_i$ are clopen in $X$ and fixed by all homeomorphisms of $X$, thus separable and CDH. □

Similarly, we can show many decompositions for CDH-spaces in the sense of the following theorem. It does not appear in the literature in such a general form, however many special cases are known.

**Theorem 2.13.** *Let $X$ be a CDH-space and let $G$ be an open set preserved by all homeomorphisms of $X$ i.e. $h(G) = G$ for any $h \in \mathcal{H}(X)$. Then $G$ is clopen in $X$ and CDH.*

*Proof.* Since $G$ is an open subset of a separable space it is separable. The set $X \setminus \overline{G}$ is also open, thus separable. Let $A \subset G$ be countable dense in $G$ and $B \subset X \setminus \overline{G}$ be countable dense in $G$. Suppose $x \in \partial G$ then $C_1 = A \cup B$ and $C_2 = A \cup B \cup \{x\}$ are two countable dense subsets of $X$. Since $G$ is preserved by all homeomorphisms $X$, $\partial G$ is also preserved by all homeomorphisms of $X$. However, we have $C_1 \cap \partial G = \emptyset$ and $C_2 \cap \partial G = \{x\}$, which is a contradiction with the CDH property of $X$. Thus we have $\overline{G} = G$.

Since $G$ is clopen and preserved by all homeomorphism of $X$, it is clearly CDH. □

Although the following does not directly relate to homogeneity properties of $X$, we state the theorem here as we will need it later in this section. We first need a well-known proposition regarding locally connected spaces.

**Definition 2.14** (Connected im kleinen)**.** *Let $X$ be a topological space and $x \in X$. We say that $X$ is connected im kleinen at $x$ if each open neighborhood $U$ of $x$ contains an open neighborhood $V$ of $x$ such that any pair of points of $V$ lie in some connected subset of $U$.*

**Proposition 2.15** ([40])**.** *Let $X$ be a topological space. If $X$ is connected im kleinen at every point then $X$ is locally connected.*

*Proof.* Let $U$ be an open subset of $X$ and let $C$ be a component of $U$. If $x \in C$ then there exists open set $V_x$ containing $x$ such that each two points of $V_x$ lie in some connected subset of $U$. This implies $V_x \subset C$ and thus $C$ is open. □

**Theorem 2.16** ([11])**.** *Let $X$ be connected, locally compact, metrizable and CDH, then $X$ is locally connected.*

*Proof.* Since $X$ is locally compact and second countable, we have that $X$ is Polish, therefore we can assume that we work with a complete metric.

Assume that $X$ is not locally connected. Then by Proposition 2.15 there is a point in $X$ at which $X$ is not connected im kleinen. Since $X$ is connected and therefore by 2.11 is homogeneous, $X$ is not connected im kleinen at any point.

Suppose that every open set of $X$ contains an open subset $V$ such that some component of $\overline{V}$ has an interior. We will show that there is some point in $X$ at which $x$ is connected im kleinen. Let $U_1$ be an open set of diameter less than 1. Let $V_1$ be an open subset of $U_1$ such that some component $C_1$ of $\overline{V_1}$ has nonempty interior. Let $U_2$ be open subset such that $\operatorname{diam} U_2 < 1/2$ and $U_2 \subset \operatorname{Int} C_1$. Let $V_2$ be an open subset of $U_2$ such that some component $C_2$ of $\overline{V_2}$ has nonempty interior. We inductively build sequence $\{U_n\}_{n=1}^{\infty}$. Since $X$ is complete there is some $x \in \bigcap_{n=1}^{\infty} \overline{U_n}$ and from the construction we also have $x \in \bigcap_{n=1}^{\infty} U_n$. We have that $C_i$, $i \in \mathbb{N}$ is a connected neighborhood basis for $x$ which is a contradiction.

Therefore there exists an open set $U$ such that if $C$ is a component of a closure of some open subset of $U$ then $C$ has empty interior. By the homogeneity of $X$, every point of $X$ belong to such a set. Since $X$ is separable, there exists a countable base $\mathcal{G} = \{V_i, i \in \mathbb{N}\}$, such that if $C$ is a component of any $\overline{V_i}$, then $C$ has empty interior. From Baire Category Theorem it then follows that if a set $M$ is at most countable union of sets $C$, each of which is a component of $\overline{V_i}$ for some $i$, then $M$ has empty interior.

Let $x_1$ be a point of $V_1$. Let $x_2$ be a point of $V_2$ not in any component of any $V_i$ containing $x_1$, such point exists since the union of all such components has empty interior. Let $x_3$ be a point in $V_3$ not in any component of any $V_i$ containing $x_1$ or $x_2$. Continuing this process we get countable dense set $A$ in $X$, such that for any $V_n$ and any component $C$ of $V_n$ the intersection $C \bigcap A$ has at most one point.

Now let $x \in X$. By the local compactness of $X$ and using "Boundary bumping theorem", there exists sequence $M_i$, $i \in \mathbb{N}$ of nondegenerate connected sets containing $x$, such that $\operatorname{diam} M_i < 1/i$ for $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $B_i$ be a countable dense subset of $M_i$ and let $B = A \cup \bigcup_{i=1}^{\infty} M_i$. Then $B$ is countable dense subset of $X$ such that any open set containing $x$ has a component which contains infinitely many points of $B$. This implies that there is no $h \in \mathcal{H}(X)$ such that $h(A) = h(B)$. □

Next we will show Ungar's Theorem connecting $n$-homogeneity, strong $n$-homogeneity and CDH-spaces. We will proceed in several steps, proving first few technical lemmas and three theorems, which we will then use to prove Ungar's theorem.

*Notation.* Let $X$ be a set and $n \in \mathbb{N}$, we denote

$$F^n(X) = X \setminus \bigcup_{\substack{0 \leq i,j \leq n \\ i \neq j}} \{(x_1, \ldots, x_n); x_i = x_j\}.$$

This set is sometimes called the $n$-th configuration space of $X$.

Note that if $X$ is Polish and $n \in \mathbb{N}$ then the set $F^n(X)$ is open in $X^n$ thus also Polish. The $n$-th configuration space is very useful tool for studying homogeneity of the space $X$.

*Remark.* Let $X$ be a topological space and $n \in \mathbb{N}$. When we say that the group $\mathcal{H}(X)$ acts on $F^n(X)$, we always mean the action that for $h \in \mathcal{H}(X)$ and $(x_1, \ldots, x_n) \in F^n$ maps $(h, (x_1, \ldots, x_n))$ to $(h(x_1), \ldots, h(x_n))$.

Note that this action is transitive if and only if $X$ is strongly $n$-homogeneous.

The following lemma is a well-known result for connected spaces.

**Lemma 2.17** ([10])**.** *Let $X, Y$ be topological spaces such that $Y$ is connected. Let $f : X \to Y$ be continuous, open and onto such that $f^{-1}(y)$ is connected for any $y \in Y$. Then $X$ is also connected.*

*Proof.* For a contradiction assume that $X$ is not connected. Let $A, B \subset X$ be two disjoint nonempty open sets such that $A \cup B = X$. Then we have that $f(A)$ and $f(B)$ are two nonempty open subsets of $Y$. By connectedness of $Y$ and the fact that $f$ is onto, there exists $y \in f(A) \cap f(B)$. Then we have that $f^{-1}(y) \cap A \neq \emptyset$ and $f^{-1}(y) \cap B \neq \emptyset$, but this contradicts the fact that $f^{-1}(y)$ is connected. □

Using the following Lemma one gets a relation between the connectedness of the space $X$ and the connectedness of the space $F^n(X)$.

**Lemma 2.18** ([39])**.** *Let $X$ be a topological space and $n \in \mathbb{N}$. If no set of $n-1$ points separates $X$ then $F^n(X)$ is connected.*

*Proof.* We proceed by induction. For $n = 1$ it follows from the definition of a connected space.

Suppose the theorem holds for some $k \in \mathbb{N}$ and consider the case $n = k + 1$. Let $\pi : F^{k+1}(X) \to F^k$ be the projection to the first $k$ coordinates. If no $k$ points separate $X$ we have that the set $\pi^{-1}(x_1, \ldots, x_k) = \{(x_1, \ldots, x_k, x); x \in X \setminus \bigcup_{i=1}^{k} \{x_i\}\}$ is connected for any $(x_1, \ldots, x_k) \in F^k$. The map $\pi$ is open and onto the set $F^k(x)$ such that the inverse of any point in $F^k(X)$ is connected. Thus by Lemma 2.17 $F^{k+1}(X)$ is connected. □

We now present a lemma that can be used to find special countable dense subsets of $X$. As the proof is quite technical we proceed in two steps, proving first what can be seen as a version of Kuratowski-Ulam Theorem. We will need the following notation.

*Notation.* Let $X$ be a set and $A \subset F^n(X)$ for some $n > 1$.

For $b \in X$ and $i \in \{1, \ldots, n\}$ we denote

$$D(i, b, A) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n); (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots x_n) \in A\}$$

and

$$D(b, A) = \bigcup_{i=1}^{n} D(i, b, A).$$

For $k \in \{1, \ldots, n-1\}$ and $\{b_1, \ldots, b_k\} \subset X$ we also denote

$$D(b_1, \ldots, b_k, A) = D\left(b_1, D\left(\ldots\left(D(b_k, A)\right)\right)\right).$$

*Remark.* Note that for a set $X$, $A \subset F^n(X)$ for some $n > 1$ and $\{b_1, \ldots, b_k\} \subset X$ for some $1 \leq k \leq n-1$ we have that $D(b_1, \ldots, b_k, A) \subset F^{n-k}(X)$. The subsets $D(b_1, \ldots, b_k, A)$ can be viewed as sections of the set $A$.

**Lemma 2.19** ([38])**.** *Let $X$ be a Polish space and let $A$ be a first category subset of $F^n(X)$ for some $n > 1$. Then $B = \{x; D(x, A)$ is not first category$\}$ is first category. Furthermore if $A_i$ is a first category subset of $F^i(X)$ for $i = 2, \ldots, k$ for some $k \in \mathbb{N}$, and $C$ is a second category subset of $X$, then there exists $x \in C$ such that $D(x, A_i)$ is a first category subset of $F^{i-1}(X)$ for each $i = 2, \ldots, k$.*

*Proof.* Assume $B$ is of second category. Since $A$ is of first category we can write $A = \bigcup_{i \in \mathbb{N}} E_i$ such that $\mathrm{Int}(\overline{E_i}) = \emptyset$. If $b \in B$ we have

$$D(b, A) = \bigcup_{i=1}^{n} D(i, b, A) = \bigcup_{i=1}^{n} \bigcup_{j \in \mathbb{N}} D(i, b, E_j)$$

there is $i_b, j_b$ such that $\mathrm{Int}(\overline{D(i_b, b, E_{j_b})}) \neq \emptyset$.

Let $\mathcal{U} = \{U_i; i \in \mathbb{N}\}$ be a countable base for $F^{n-1}(X)$ and let $U_b$ the first element of $\mathcal{U}$ contained in $\mathrm{Int}(\overline{D(i_b, b, E_{j_b})})$. Let $B_i = \{b \in B; U_b = U_i\}$. Then $B = \bigcup_{i \in \mathbb{N}} B_i$ and since $B$ is of second category there is $N \in \mathbb{N}$ such that $B_N$ is of second category. Let $B_N(i, j) = \{b \in B_N; i_b = i, j_b = j\}$. Again there is $I, J$ such that $B_N(I, J)$ is of second category. This means that if $b \in B_N(I, J)$ then $\mathrm{Int}(\overline{D(I, b, E_J)}) \supset U_N$. Since $U_N$ is open in $F^{n-1}(X)$ and $F^{n-1}(X)$ is open in $X^{n-1}$ there are open sets $V_1, \ldots, V_{I-1}, V_{I+1}, \ldots, V_n \subset X$ such that $V_1 \times \cdots \times V_{I-1} \times V_{I+1} \times \cdots \times V_n \subset U_N$. This means that

$$V_1 \times \cdots \times V_I \times \overline{B(I, J)} \times V_{I+1} \times \ldots V_n \subset \overline{E_J}.$$

This is a contradiction since the set on the right side has an empty interior, however the set on the left side has a non-empty interior.

Now let $A_i$ be a first category subset of $F^i(X)$ for $i = 2, \ldots, k$ for some $k \in \mathbb{N}$. Then the set

$$D = \bigcap_{i=2}^{k} \{x \in X; D(x, A_i) \text{ is of first category in } F^{i-1}\}$$

is of second category, since by the previous it is a finite intersection of sets of second category. Therefore we have $D \cap C \neq \emptyset$. □

**Lemma 2.20** ([38])**.** *Let $X$ be a Polish space and let $A$ be a first category subset of $F^n(X)$. Then there exists a countable dense subset of $B \subset X$ such that $F^n(B) \cap A = \emptyset$*

*Proof.* Denote $S_n$ the group of permutations on the set $\{1, \ldots, n\}$. Note that for any $\varphi \in S_n$ the mapping $\hat{\varphi} : F^n(X) \to F^n(X)$ defined as $\hat{\varphi}(x_1, \ldots, x_n) = (x_{\varphi(1)}, \ldots, x_{\varphi(n)})$ for any $(x_1, \ldots, x_n) \in F^n(X)$ is a homeomorphism. Therefore the set

$$\tilde{A} = \bigcup_{\varphi \in S_n} \hat{\varphi}(A)$$

is of first category in $F^n(X)$.

Let $\mathcal{U} = \{U_i; i \in \mathbb{N}\}$ be a countable base for $X$. We will define the set $B$ inductively. We will choose the first $n-1$ as follows. By Lemma 2.19 choose $b_1 \in U_1$ such that $D(b_1, \tilde{A})$ is a first category subset of $F^{n-1}$. Now suppose we have $(b_1, \dots b_s) \in U_1 \times \cdots \times U_s$ for some $s < n-2$ such that for any decreasing sequence of natural numbers $s \geq k_1, \dots k_l \geq 1$ for some $l \leq s$ we have that the set $D(b_{k_1}, \dots, b_{k_l}, \tilde{A})$ is a first category subset of $F^{n-l-1}$. For $k = 1, \dots, s$ let

$$E_k = \bigcup \{D(b_{j_1}, \dots b_{j_k}, \tilde{A}); s \geq j_1 > \cdots > j_k \geq 1\} \subset F^{n-k}.$$

Note that since the unions are finite, we have that the sets $E_k$ are of first category for any $k = 1, \dots, s$. Hence by Lemma 2.19 we can find $b_{s+1} \in U_{s+1}$ such that $D(b_{s+1}, \tilde{A})$ and $D(b_{s+1}, E_k)$ are first category for each $k = 1, \dots s$. This point clearly satisfies the inductive hypothesis.

In order to define $b_n$ let $E_k$'s be as above. Using Lemma 2.19 we can find $b_n \in U_n \setminus E_{n-1}$ such that $D(b_n, \tilde{A})$, respectively $D(b_n, E_k)$ is first category in $F^{n-1}(X)$, respectively $F^{n-k-1}(X)$ for each $k = 1, \dots n-2$. Now we proceed as before. Assume $b_1, \dots, b_s$ has been defined for some $s > n$. We let $E_k$ be as above for $k = 1, \dots, n-1$. By Lemma 2.19 we can find $b_{s+1} \in U_{s+1} \setminus E_{n-1}$ such that $D(b_{s+1}, \tilde{A})$ and $D(b_{s+1}, E_k)$ is first category for each $k = 1, \dots, n-2$.

Let $B = \{b_i; i \in \mathbb{N}\}$. Clearly, $B$ is dense. Let $(b_{k_1}, \dots, b_{k_n}) \in F^n(B)$. We can assume $k_1 < \cdots < k_n$ otherwise we can use suitable permutation in $S_n$ and the definition of $\tilde{A}$. Thus we have

$$D(b_{k_1}, \dots b_{k_{n-1}}, \tilde{A}) \subset E_{n-1}.$$

However, we have that $b_{k_n} \in U_{k_n} \setminus E_{n-1}$ and thus $(b_{k_1}, \dots, b_{k_n}) \notin \tilde{A}$. $\square$

Now we are ready to proof the theorems connecting many notions of homogeneity.

The following Theorem was originally formulated for compact metric spaces or locally connected and locally compact metric spaces. However some authors realised [4] that one only needs a Polish topology on $\mathcal{H}(X)$, which is clear from the proof.

**Theorem 2.21** ([39])**.** *Let $X$ be Polish and $n$-homogeneous space such that $F^n(X)$ is connected. Suppose there is a topology on $\mathcal{H}(X)$ such that $\mathcal{H}(X)$ is a Polish group and the action of $\mathcal{H}(X)$ on $X$ is continuous. Then $X$ is strongly $n$-homogeneous.*

*Proof.* Let $G$ be the symmetric group on $\{1, \dots, n\}$ with the discrete topology. Then we have that $H \times G$ is a Polish group with multiplication defined coordinatewise. Define an action $H \times G$ on $F^n(X)$ as

$$(h, \pi)(x_1, \dots, x_n) = (h(x_{\pi(1)}), \dots, h(x_{\pi(n)})).$$

This is a continuous action and since $X$ is $n$-homogeneous, it is also transitive. Therefore by Theorem 1.19 the map $T_{(x_1, \dots, x_n)} : H \times G \to F^n(X)$ is open for any $(x_1, \dots, x_n) \in F^n(X)$. Now fix $(x_1, \dots, x_n) \in F^n(X)$ then for any $\pi \in G$ we have that $A_\pi = T_{(x_1, \dots, x_n)}(H \times \{\pi\})$ is open in $F^n(X)$. Since the action of $H \times G$ is transitive we have that $F^n(X) = \bigcup_{\pi \in G} A_\pi$. We also have that $A_\pi$ is the orbit of the point $(x_{\pi(1)}, \dots, x_{\pi(n)})$ by the action of $H$. Since orbits are either disjoint or equal and $F^n(X)$ is connected we have that all the orbits are equal to $F^n(X)$. By taking $\pi = \mathrm{Id}$ we get that $X$ is strongly $n$-homogeneous. $\square$

The following Theorem was originally also formulated only for locally compact spaces. However it has been pointed out by some authors [4] that again we only need $X$ and $\mathcal{H}(X)$ to be Polish.

We made some minor simplifications in the proof.

**Theorem 2.22** ([38])**.** *Let $X$ be a Polish and countable dense homogeneous space, such that no finite set separates $X$. Suppose there is a topology on $\mathcal{H}(X)$ such that $\mathcal{H}(X)$ is a Polish group and the action of $\mathcal{H}(X)$ on $X$ is continuous. Then $X$ is strongly $n$-homogeneous for any $n \in \mathbb{N}$.*

*Proof.* We will show that the action of $H = \mathcal{H}(X)$ on $F^n(x)$ is transitive. Let $\hat{x} = (x_1, \dots, x_n) \in F^n(X)$ be arbitrary. Denote by $H(\hat{x}) = \{(h(x_1), \dots, h(x_n)); h \in H\}$ the orbit of $\hat{x}$.

Let $A \subset X$ be a countable dense set. Then $B \cup \{x_1, \dots, x_n\}$ is also a countable dense set. From the countable dense homogeneity there exists $h \in H$ such that $h(B) = A$, hence $(h(x_1), \dots, h(x_n)) \in F^n(A)$ which is a countable set. Since $\hat{x}$ was arbitrary, this implies that every orbit of the action of $H$ on $F^n(X)$ intersects the set $F^n(A)$ for any $A \subset X$ countable dense.

Now suppose $H(\hat{x})$ is meager in $F^n(X)$. Then by 2.20 there is $A \subset X$ countable dense such that $H(\hat{x}) \cap F^n(A) = \emptyset$. However, this is a contradiction to what we showed above. This means that any orbit of the action of $H$ on $F^n(X)$ is a second category subset of $F^n(X)$.

Since $H(\hat{x})$ is of second category, we have that $\mathrm{Int}(\overline{H(\hat{x})}) \neq \emptyset$. Therefore, also $H(\hat{x}) \cap \mathrm{Int}(\overline{H(\hat{x})}) \neq \emptyset$. Suppose there is $(y_1, \dots, y_n) = \hat{y} \in H(\hat{x}) \setminus \mathrm{Int}(\overline{H(\hat{x})})$. Then we have $(h(y_1), \dots, h(y_n)) \notin \{(h(z_1), \dots, h(z_n)); (z_1, \dots, z_n) \in \mathrm{Int}(\overline{H(\hat{x})})\} = \mathrm{Int}(\overline{H(\hat{x})})$ for any $h \in H$, where the equality of the sets follows from the fact that $h$ acts as homeomorphism on $F^n(X)$. This means that $H(\hat{x}) = \{(h(y_1), \dots, h(y_n)); h \in H\} \cap \mathrm{Int}(\overline{H(\hat{x})}) = \emptyset$, which is a contradiction. Hence $H(\hat{x}) \subset \mathrm{Int}(\overline{H(\hat{x})})$.

Suppose $(z_1, \dots z_n) = \hat{z} \in \overline{H(\hat{x})}$ and let $(u_1, \dots, u_n) \in H(\hat{z})$. Then there exist $g \in H$ and a sequence $\{h_i\}_{i=1}^{\infty}$ of points of $H$ such that $\hat{u} = (g(z_1), \dots g(z_n))$ and $(h_i(x_1), \dots h_i(x_n)) \to z$. This means that $(g(h_i(x_1)), \dots, g(h_i(x_1))) \to \hat{u}$ so $H(\hat{z}) \subset \overline{H(\hat{x})}$ and also $\mathrm{Int}(\overline{H(\hat{z})}) \subset \overline{H(\hat{x})}$. Hence we have

$$\overline{H(\hat{x})} \subset \bigcup \{\mathrm{Int}(\overline{H(\hat{z})}); \hat{z} \in \overline{H(\hat{x})}\} \subset \overline{H(\hat{x})}.$$

Hence $\overline{H(\hat{x})}$ is clopen and since $F^n(X)$ is connected for any $\hat{x} \in F^n(X)$ by 2.17, the set $H(\hat{x})$ is dense. By 1.18 we also have that $H(\hat{x})$ is $G_\delta$. Since $X$ is Polish and orbits are either disjoint or equal we have that the action of $H$ on $F^n(X)$ is transitive. □

We not only have that Polish CDH-spaces with a Polish group of homeomorphisms are strongly $n$-homogeneous for any $n \in \mathbb{N}$, but Ungar also showed the inverse [38]. The following has been formulated in [38] again only for locally compact metric spaces, however it again holds for any Polish space such that $\mathcal{H}(X)$ admits a Polish topology. The proof resembles the very standard back and forth argument.

**Theorem 2.23** ([38])**.** *Let $X$ be Polish and strongly $n$-homogeneous for any $n \in \mathbb{N}$. Suppose there is a topology on $\mathcal{H}(X)$ such that $\mathcal{H}(X)$ is a Polish group and the action of $\mathcal{H}(X)$ on $X$ is continuous. Then $X$ is CDH.*

*Proof.* Let $A = \{a_i; i \in \mathbb{N}\}$ and $B = \{b_i; i \in \mathbb{N}\}$ be two countable dense sets in $X$. Let $\rho$ be a complete metric on $\mathcal{H}(X)$. Denote $N(h, \varepsilon) = \{g \in \mathcal{H}(X); \rho(h, g) < \varepsilon\}$ for $h \in \mathcal{H}(X)$ and $\varepsilon > 0$. Let $c_1 = a_1$ and $d_1 = b_1$. Since $X$ is homogeneous, there exists $h_1 \in \mathcal{H}(X)$ such that $h_1(c_1) = d_1$. Let $c_2 = a_2$ then by Theorem 1.19 we have that $\{(g(c_1), g(c_2)); g \in N(h_1, 1)\}$ is open in $F^2(X)$ and it contains $(d_1, h_1(c_2))$. Set $d_2$ as the first $b_i$ such that $b_i \neq d_1$ and $(d_1, b_i) \in \{(g(c_1), g(c_2)); g \in N(h_1, 1)\}$. Then there exists $h_2 \in N(h_1, 1)$ such that $(h_2(c_1), h_2(c_2)) = (d_1, d_2)$. Let $0 < \varepsilon_2 < 1/2$ be such that $\overline{N(h_2, \varepsilon_2)} \subset N(h_1, 1)$. Set $d_3$ as the first $b_i$ such that $b_i \neq d_1$ and $b_i \neq d_2$. Since $\mathcal{H}(X)$ is a topological group there exists a $\delta_2 > 0$ such that $\{g^{-1}; g \in N(h_2^{-1}, \delta_2)\} \subset N(h_2, \varepsilon_2)$. Again using Theorem 1.19 we have that $\{(g(d_1), g(d_2), g(d_3); g \in N(h_2^{-1}, \delta_2)\}$ is open subset of $F^3(X)$. Set $c_3$ as the first $a_i$ such that $a_i \neq c_1$, $a_i \neq c_2$ and $(c_1, c_2, a_i) \in \{(g(d_1), g(d_2), g(d_3); g \in N(h_2^{-1}, \delta_2)\}$. Then there exists $h_3$ such that $h_3^{-1} \in \underline{N(h_2^{-1}, \delta_2)}$ and $(h_3(c_1), h_3(c_2), h_3(c_3)) = (d_1, d_2, d_3)$. Let $\varepsilon_3 < 1/3$ be such that $\overline{N(h_3, \varepsilon_3)} \subset N(h_2, \varepsilon_2)$. We can the proceed by a straightforward induction to construct a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ and numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ such that $0 < \varepsilon_i < 1/i$ and $\overline{N(h_{i+1}, \varepsilon_{i+1})} \subset N(h_i, \varepsilon_i)$. Since $d$ is a complete metric there exists $h \in \bigcap_{i \in \mathbb{N}} N(h_i, \varepsilon_i)$ and by construction we get $h(A) = B$. □

Connecting Theorem 2.21, Theorem 2.22 and Theorem 2.23 we have the following.

**Theorem 2.24** ([38], [39]). *Let $X$ be a locally compact separable metrizable space such that no finite set separates $X$. Then the following statements are equivalent.*

1. *$X$ is CDH.*
2. *$X$ is $n$-homogeneous for every $n$.*
3. *$X$ is strongly $n$-homogeneous for every $n$.*

*Proof.* Note that $X$ is Polish and by Theorem 2.16, Theorem 1.16 and Theorem 1.17 we have that $\mathcal{H}(X)$ is Polish and acts continuously on $X$.

- "1. $\Longrightarrow$ 3.": Follows by Theorem 2.22
- "3. $\Longrightarrow$ 1.": Follows by Theorem 2.23.
- "2. $\Longrightarrow$ 3.": Follows by Theorem 2.21 and Lemma 2.18.
- "3. $\Longrightarrow$ 2.": Holds true in general.

□

Van Mill [29] showed that the implication "1 $\Longrightarrow$ 3" in Theorem 2.24 holds level by level and for any space. Results in [29] are formulated for a more general case of actions of groups on spaces. For keeping the text uniform in its approach we present slightly less general results that are more in line with the rest of the text, however the proofs remain the same. We again proceed in several steps proving first some technical results. However, the first one is interesting in itself as it reveals that CDH-spaces have richer structure of homeomorphisms then one might expect.

**Theorem 2.25** ([29]). *Let $X$ be CDH. If $F \subset X$ is finite and $D, E \subset X \setminus F$ are countable dense in $X$ then there is $h \in \mathcal{H}(X)$ such that $h(D) \subset E$ and $h(x) = x$ for every $x \in F$.*

*Proof.* Let $h_0 \in \mathcal{H}(X)$ be arbitrary. Suppose we constructed $\{h_\beta, \beta < \alpha\}$ for some $\alpha < \omega_1$. Since $X$ is CDH, we can find $h_\alpha \in \mathcal{H}(X)$ such that

$$h_\alpha(F \cup E) = \bigcup_{\beta<\alpha} h_\beta(D).$$

For $1 \leq \alpha < \omega_1$, let $T_\alpha$ be a nonempty finite subset of $[1, \alpha)$ such that $h_\alpha(F) \subset \bigcup_{\beta \in T_\alpha} h_\beta(D)$. For mapping $T : [1, \omega_1) \to [\omega_1]^{<\omega}$ defined by $T(\alpha) = T_\alpha$ there exists $A \in [\omega_1]^{<\omega}$ such that the fibre $T^{-1}(A)$ is uncountable.

Suppose that for every $A \in [\omega_1]^{<\omega}$ we have that $T^{-1}(A)$ is countable. We can find increasing sequence of countable ordinal numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that $T^{-1}(A) \subset \alpha_{n+1}$ for any finite $A \subset [1, \alpha_n]$ . Define $\alpha = \sup_n \alpha_n$, we have that $A = T(\alpha) \subset [1, \alpha)$ and we can find $n$ such that $A \subset \alpha_n$, hence $\alpha \in T^{-1}(A) \subset [1, \alpha_{n+1})$, which is a contradiction.

Let $A \in [\omega_1]^{<\omega}$ be such that $B = T^{-1}(A)$ is uncountable. Then $h_\alpha(F) \subset \bigcup_{\beta \in A} h_\beta(D)$ for every $\alpha \in B$. Since $\bigcup_{\beta \in A} h_\beta(D)$ is countable and $B$ is uncountable, by the pigeonhole principle we can find $C \subset B$ uncountable such that for any $\alpha, \beta \in C$ we have $h_\alpha|_F = h_\beta|_F$. Hence if we take $\alpha, \beta \in C$ such that $\alpha < \beta$ we have $(h_\beta^{-1} \circ h_\alpha)\Big|_F = \mathrm{Id}_F$ and $(h_\beta^{-1} \circ h_\alpha)(D) \subset E$. □

The following connects the strong $n$-homogeneity of a space $X$ with an action of a special subgroup of $\mathcal{H}(X)$ on the space $X \setminus F$ for some $F \in [X]^{n-1}$.

**Lemma 2.26** ([29]). *Let $X$ be an infinite topological space and $n \geq 1$. Then the following are equivalent:*

1. *$X$ is strongly $n$-homogeneous.*
2. *For every $F \in [X]^{n-1}$, the group $\{h \in \mathcal{H}(X); h(x) = x$ for any $x \in F\}$ acts transitively on $X \setminus F$.*

*Proof.* The first implication "1 $\implies$ 2" is clear from the definition of strong $n$-homogeneity.

We will prove the other implication by induction. For $n = 1$ there is nothing to prove. So suppose the implication holds for $n - 1$ for some $n \geq 2$ and suppose 2 holds for $n$. Since $X$ is infinite 2 also holds for $n - 1$ and so by the induction hypothesis $X$ is strongly $n - 1$-homogeneous. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be arbitrary $n$-tuples of distinct points of $X$. We can find $g_0$ such that $g_0(x_i) = y_i$ for all $i < n$. Let $F = \{y_1, \ldots y_{n-1}\}$. Note that $g_0(x_n) \notin F$. Therefore, by the assumption we can find $g_1 \in \{h \in \mathcal{H}(X); h(x) = x$ for any $x \in F\}$ such that $g_1(g_0(x_n)) = y_n$. For $g = g_1 \circ g_0$ we then have that $g(x_i) = y_i$ for any $i \leq n$. □

**Lemma 2.27** ([29]). *Let $X$ be a CDH-space without isolated points. Then for every finite subset $F \subset X$ and any $x \in X \setminus F$ the set $\{h(x); h \in \mathcal{H}$ $h(y) = y$ for any $y \in F\}$ is open.*

*Proof.* Let $x \in X \setminus F$ and let $Y = \{h(x); h \in \mathcal{H}\ h(y) = y \text{ for any } y \in F\}$. Assume that $Y$ has an empty interior. Note that $F$ has an empty interior since $X$ has no isolated points. Then there is a countable dense subset $D$ of $X$ such that $D \subset X \setminus (Y \cup F)$. By Theorem 2.25 there exists $h \in \mathcal{H}(X)$ such that $h|_F = \mathrm{Id}|_F$ and $h(D \cup \{x\}) \subset D$ which is a contradiction since $h(x) \in Y$ and $Y \cap D = \emptyset$. Therefore $Y$ has nonempty interior. Note that $Y$ is an orbit induced by the action of the group $\{h \in \mathcal{H}; h(y) = y \text{ for any } y \in F\}$. This implies that it is open. □

**Theorem 2.28** ([29])**.** *Let $X$ be a CDH-space and $n \in \mathbb{N}$. If no set of size $n-1$ separates $X$ then $X$ is strongly $n$-homogeneous.*

*Proof.* By Lemma 2.26 it is enough to show that the group $\{h \in \mathcal{H}(X); h(x) = x$ for any $x \in F\}$ acts transitively on $X \setminus F$ for any $F \in [X]^{n-1}$. By Lemma 2.27 the orbits of this action are open and hence clopen. Since $X \setminus F$ is connected we get that the action is transitive. □

We have seen that a CDH-space admits some level of homogeneity. On the other hand it is reasonable to expect that spaces having "many" homeomorphisms would be CDH. One such class of spaces with "many" homeomorphisms are SLH-spaces. SLH-spaces are in general not CDH, however if we restrict our attention to SLH Polish spaces then the situations improves.

First we present a theorem regarding convergence of sequences of homeomorphisms of a Polish space. We will use this theorem later on to show that SLH Polish spaces are CDH. We include the proof for the sake of completeness.

**Theorem 2.29** (Inductive convergence theorem [1])**.** *Let $(X, d)$ be a complete metric space , let $\{h_i\}_{i=1}^{\infty} \in \mathcal{H}(X)^{\mathbb{N}}$, denote $g_i = (h_i \circ \cdots \circ h_1)$ for $i \in \mathbb{N}$, and let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a sequence of open coverings of $X$. Assume that the following conditions are satisfied:*

1. $\sup_{G \in \mathcal{G}_i} \mathrm{diam}(G) < 1/2^i$ *for* $i \in \mathbb{N}$,
2. $\sup_{G \in \mathcal{G}_i} \mathrm{diam}(g_i^{-1}(G)) < 1/2^i$ *for* $i \in \mathbb{N}$,
3. *for each $x \in X$ and each $i \in \mathbb{N}$ there exist $G_x \in \mathcal{G}_i$ such that $x \in G_x$ and $h_{i+1}(x) \in G_x$.*

*Then the sequence $\{g_i\}_{i=1}^{\infty}$ converges to some $g \in \mathcal{H}(X)$.*

*Proof.* Let $p \in X$ and $i \in \mathbb{N}$. We can find $G \in \mathcal{G}_i$ such that $g_i(p) \in G$ and also $g_{i+1}(p) = h_{i+1}(g_i(p)) \in G$. Then we have $d(g_i(p), g_{i+1}(p)) < 1/2^i$. Therefore, $\{g_i\}_{i=1}^{\infty}$ is pointwise Cauchy and there exists a pointwise limit $g$. Since $d(g_i(p), g_{i+1}(p)) < 1/2^i$ for all $p \in X$ the convergence is uniform and we have that $g$ is continuous.

Let $q \in X$ and $i \in \mathbb{N}$. Then there is $G \in \mathcal{G}_i$ such that $h_{i+1}^{-1}(q) \in G$ and $q = h_{i+1}(h_{i+1}^{-1}(q)) \in G$. Since $\mathrm{diam}(g_i^{-1}(G)) < 1/2^i$ we have that $d(g_i^{-1}(q), g_{i+1}^{-1}(q)) = d(g_i^{-1}(q), g_i^{-1} \circ h_{i+1}^{-1}(q)) < 1/2^i$. By a similar argument as above we get that $g_i^{-1}$ converge to a continuous function of $X$ into itself.

This implies that $g$ is a homeomorphism.

□

Now we prove the theorem that is one of the main tools for showing that a space is CDH. The limitation of this theorem is obvious as it can be used only for Polish spaces. It can be found in [5], we present a proof mentioned in [4].

**Theorem 2.30** ([5]). *If $X$ is SLH and Polish, then $X$ is CDH.*

*Proof.* Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be two dense subsets of $X$. By the strong local homogeneity, we have that for every $x \in X$, any neighborhood $U$ of $x$, and any dense $G \subset X$ there is $g \in \mathcal{H}(X)$ such that $g(x) \in G$ and $g|_{X\setminus U} = \mathrm{Id}_{X\setminus U}$. We construct a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ having the following properties:

1. $h_n \circ \dots \circ h_1(a_i) = h_{2i} \circ \dots \circ h_1(a_i) \in B$ for each $i \in \mathbb{N}$ and $n \geq 2i$,

2. $(h_n \circ \dots \circ h_1)^{-1}(b_i) = (h_{2i+1} \circ \dots \circ h_1)^{-1}(b_i) \in A$ for each $i \in \mathbb{N}$ and $n \geq 2i+1$.

We proceed inductively. So assume we have $h_1, \dots, h_{2i-1}$ for some $i \in \mathbb{N}$.

If $h_{2i-1} \circ \dots \circ h_1(a_i) \in B$, take $h_{2i}$ to be the identity on $X$. Otherwise we can choose a small neighborhood $U_{2i}$ of $h_{2i-1} \circ \dots \circ h_1(a_i)$ such that it is disjoint from the set

$$\{b_1, \dots, b_{i-1}\} \cup h_{2i-1} \circ \dots \circ h_1(\{a_1, \dots, a_{i-1}\}).$$

Take $h_{2i} \in \mathcal{H}(X)$ such that $h_{2i}|_{X\setminus U_{2i}} = \mathrm{Id}_{X\setminus U_{2i}}$ and

$$h_{2i} \circ h_{2i-1} \circ \dots h_1(a_i) \in B.$$

If $(h_{2i} \circ \dots \circ h_1)^{-1}(b_i) \in A$, take $h_{2i+1}$ to be the identity on $X$. Otherwise we can choose a small neighborhood $U_{2i+1}$ of $b_i$ such that it is disjoint from the set

$$\{b_1, \dots, b_{i-1}\} \cup h_{2i} \circ \dots \circ h_i(\{a_1, \dots, a_i\}).$$

Take $h_{2i+1} \in \mathcal{H}(X)$ such that $h_{2i+1}|_{X\setminus U_{2i+1}} = \mathrm{Id}_{X\setminus U_{2i+1}}$ and

$$h_{2i+1}^{-1}(b_i) \in (h_{2i} \circ \dots \circ h_1)(A).$$

Since we can choose $U_{2i}$ and $U_{2i+1}$ arbitrarily small, we can choose them small enough so the conditions of 2.29 are satisfied, then the left product of the sequence $\{h_i\}_{i=1}^{\infty}$ converge to a homeomorphism $h$ such that $h(A) = B$.

□

For non-Polish spaces it can be quite tricky to show the CDH property.

It is well-known that using the same techniques we can show even more. Since the proof is very similar to the proof of Theorem 2.30 we don't present all the details.

**Theorem 2.31.** *Let $X$ be SLH and Polish. If $A_1, A_2, B_1, B_2 \subset X$ are countable dense subsets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$ then there is $h \in \mathcal{H}(X)$ such that $h(A_1) = B_1$ and $h(A_2) = B_2$.*

*Proof.* Let $A_1 = \{a_1^i; i \in \mathbb{N}\}$, $A_2 = \{a_2^i; i \in \mathbb{N}\}$, $B_1 = \{b_1^i; i \in \mathbb{N}\}$ and $B_2 = \{b_2^i; i \in \mathbb{N}\}$. We construct a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ having the following properties:

1. $h_n \circ \dots \circ h_1(a_1^i) = h_{2i} \circ \dots \circ h_1(a_1^i) \in B_1$ for each $i \in \mathbb{N}$ and $n \geq 2i$,

2. $h_n \circ \dots \circ h_1(a_2^i) = h_{2i} \circ \dots \circ h_1(a_2^i) \in B_2$ for each $i \in \mathbb{N}$ and $n \geq 2i$,

3. $(h_n \circ \cdots \circ h_1)^{-1}(b_1^i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_1^i) \in A_1$ for each $i \in \mathbb{N}$ and $n \geq 2i+1$,

4. $(h_n \circ \cdots \circ h_1)^{-1}(b_2^i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_2^i) \in A_2$ for each $i \in \mathbb{N}$ and $n \geq 2i+1$.

Assume we have $h_1, \ldots, h_{2i-1}$ for some $i \in \mathbb{N}$. For simplicity assume $h_{2i-1} \circ \cdots \circ h_1(a_1^i) \notin B_1$ and $h_{2i-1} \circ \cdots \circ h_1 a_2^i \notin B_2$. Otherwise we can proceed similarly in all the cases. Choose small disjoint neighborhoods $U_{2i}^1$ of $h_{2i-1} \circ \cdots \circ h_1(a_1^i)$ and $U_{2i}^2$ of $h_{2i-1} \circ \cdots \circ h_1(a_2^i)$ both disjoint from the sets

$$\{b_1^1, \ldots, b_1^{i-1}\} \cup h_{2i-1} \circ \cdots \circ (\{a_1^1, \ldots, a_1^{i-1}\}),$$

and

$$\{b_2^1, \ldots, b_2^{i-1}\} \cup h_{2i-1} \circ \cdots \circ (\{a_2^1, \ldots, a_2^{i-1}\}).$$

We can find $h_{2i} \in \mathcal{H}(X)$ such that

$$h_{2i}|_{X \setminus (U_{2i}^1 \cup U_{21}^2)} = \mathrm{Id}|_{X \setminus (U_{2i}^1 \cup U_{21}^2)},$$

$$h_{2i} \circ h_{2i-1} \circ \cdots \circ h_1(a_1^i) \in B_1$$

and

$$h_{2i} \circ h_{2i-1} \circ \cdots \circ h_1(a_2^i) \in B_2.$$

Similarly as in the proof of Theorem 2.30 we can construct the inverse mappings as well and use Theorem 2.29 to ensure convergence of the sequence. □

*Remark* 2.32. Note that for a Polish space $X$ in Theorem 2.30 and Theorem 2.31 we do not need to use the whole group $\mathcal{H}(X)$. It is enough to use a subgroup such that the SLH property holds using only homeomorphisms from the given subgroup and it is closed under convergence in Theorem 2.29.

Theorem 2.31 has a very interesting well-known corollary.

*Corollary* 2.33. Let $X$ be SLH and Polish and let $C \subset X$ be a countable dense subset. Then $X \setminus C$ is CDH.

*Proof.* Without loss of generality we may assume that $X$ has no isolated points, since by Corollary 2.12 the set of isolated points is clopen in $X$.

Let $A, B \subset X \setminus C$ be countable dense sets in $X \setminus C$. Then $A$ and $B$ are also dense in $X$ since $X \setminus C$ is dense in $X$ from the Baire Category Theorem and because $X$ has no isolated points. By Theorem 2.31 there is $h \in \mathcal{H}(X)$ such that $h(A) = B$ and $h(C) = C$. But this means that $h(X \setminus C) = X \setminus C$ and we are done. □

For a CDH-space $X$ and any two countable dense sets $A, B \subset X$ we have that $X \setminus A \approx X \setminus B$. By Corollary 2.33 this space is CDH if $X$ is SLH and Polish. It is very natural to ask if this happens for any CDH-space $X$. In the next chapter we will show that in general this is not the case.

## 2.4 Zero-dimensional Polish CDH-spaces

As we indicated above, the situation for zero-dimensional Polish CDH-spaces is much better. In fact one could say that the issue of countable dense homogeneity is solved for zero-dimensional Polish spaces.

The following theorem is a consequence of the topological characterizations of the classical zero-dimensional Polish spaces.

**Theorem 2.34.** *Let $X$ be an infinite, zero-dimensional, Polish and homogeneous space. Then $X$ is homeomorphic to either $2^\omega$, $\omega$, $\omega^\omega$ or $2^\omega \setminus \{0\}$.*

*Proof.* Suppose $X$ has an isolated point. Then from the homogeneity, every point of $X$ is isolated. Since $X$ is Polish and infinite we have that $X \approx \omega$.

Now suppose $X$ has no isolated points. We distinguish two cases.

First suppose that there exists some compact set in $X$ that has nonempty interior. From the homogeneity this means that every point in $X$ is in a compact set with nonempty interior and since $X$ is metrizable it follows that $X$ is locally compact. If $X$ is also compact by Theorem 1.2 we have $X \approx 2^\omega$. If $X$ is not compact by Theorem 1.5 we have $X \approx 2^\omega \setminus \{0\}$.

Now suppose that all compact sets have empty interior by Theorem 1.6 this means that $X \approx \omega^\omega$. □

**Proposition 2.35.** *Spaces $2^\omega$, $\omega$, $2^\omega \setminus \{0\}$ and $\omega^\omega$ are all CDH.*

We will show this later however this can be used to characterize all Polish, zero-dimensional CDH-spaces. This result is due to [21], however we can find some steps of the proof also in [14]. We present a different proof.

Let $\mathcal{C} = \{X;\ X \approx \kappa\ \oplus (\lambda \times 2^\omega) \oplus (\mu \times \omega^\omega),\ \text{where } 0 \leq \kappa, \lambda, \mu \leq \omega\}$.

**Theorem 2.36** ([21])**.** *Let $X$ be zero-dimensional, CDH and Polish then $X \in \mathcal{C}$.*

*Proof.* By 2.12 we can write $X = \oplus_{i=0}^{\alpha} X_i$, where $\alpha \in \omega+1$. Since all $X_i$'s are homogeneous zero-dimensional CDH-spaces we can use Theorem 2.34, Proposition 2.35 and the fact that $2^\omega \setminus \{0\} \approx \omega \times 2^\omega$. □

This theorem has several very interesting corollaries showing that the behaviour of zero-dimensional Polish spaces in regards to the CDH property is not as bad as for general spaces.

*Corollary* 2.37. Let $X$ be zero-dimensional Polish space then $X$ is SLH if and only if $X$ is CDH

*Remark.* We will later show that all the classical zero-dimensional spaces are SLH.

Also since the class $\mathcal{C}$ is closed under countable and finite products we have the following.

*Corollary* 2.38. Let $X_i$, $0 \leq i \leq \alpha$, where $\alpha \in \omega + 1$, be zero-dimensional, Polish CDH-spaces. Then also $X = \prod_{i=0}^{\alpha} X_i$ is CDH.

However it is important to note that the requirement for the spaces to be Polish is crucial. We will show later on that without this assumption the situation is more complicated.

# 3. Examples of CDH- and non-CDH-spaces

In this chapter we introduce spaces that are CDH and some spaces that are not, putting to a good use the theory we presented. We start this chapter by showing that $\mathbb{R}$ is CDH. Although this will follow from some of the later examples, we present the original argument by Cantor [8] using the "back and forth" argument, it can also be found in Brouwer's paper [7].

**Example 3.1** ([7]). *$\mathbb{R}$ is CDH.*

*Proof.* Let $A = \{a_i, i \in \mathbb{N}\}$ and $B = \{b_i, i \in \mathbb{N}\}$ be two countable dense subsets of $\mathbb{R}$. We define function $f : A \to B$ inductively as follows. Define $f(a_1) = b_1$. Now suppose we defined $f(a_i)$ for $i \in \{1, \dots n\}$ for some $n \in \mathbb{N}$. We find the smallest index $j \in \mathbb{N}$ such that $b_j \in B \setminus \{f(a_i), i \leq n\}$ and $b_j$ has the same order relation to all the points $f(a_i)$, $i \leq n$ as $a_{n+1}$ has to all the points $a_i$, $i \leq n$. Then we define $f(a_{n+1}) = b_j$.

Clearly $f$ is one-to-one. Suppose $j, n \in \mathbb{N}$ are such that $b_j \notin \{f(a_1), \dots, f(a_n)\}$ and $\{b_1, \dots, b_{j-1}\} \subset \{f(a_1), \dots, f(a_n)\}$. Let $k \in \mathbb{N}$ be the smallest index larger than $n$ such that $a_k$ has the same order relation to all the points $a_1, \dots a_n$ as $b_j$ has to all the points $f(a_1), \dots, f(a_n)$. For any $l \in \{n+1, \dots, k-1\}$ if $a_l < a_k$, then since $l < k$ there is $i \in \{1, \dots, j-1\}$ such that $a_l < a_i < a_k$, so we have $f(a_l) < f(a_i) < b_j$, similarly if $a_l > a_k$, then there is $i \in \{1, \dots, j-1\}$ such that $f(a_l) > f(a_i) > b_j$. This means that $b_j$ has the same order relation to all $f(a_i)$, $i < k$, such as $a_k$ has to all $a_i$, $i < k$, which means $f(a_k) = b_j$. This means that $f$ is order preserving bijection. Since the topology on $\mathbb{R}$ is the order topology we get that $f$ is a homeomorphism between $A$ and $B$. Now for any $r \notin A$ we can find $a_i^1, a_i^2 \in A$, $i \in \mathbb{N}$ such that for any $i \in \mathbb{N}$ we have $[a_i^1, a_i^2] \supset [a_{i+1}^1, a_{i+1}^2]$ and $\{r\} = \bigcap_{i=1}^{\infty} [a_i^1, a_i^2]$. We can than define $\{F(r)\} = \bigcap_{i=1}^{\infty} [f(a_i^1), f(a_i^2)]$ and for any $a \in A$ $F(a) = f(a)$. This definition is correct since $f$ is an order preserving homeomorphism and $\mathbb{R}$ is complete. We get that $F$ is also order preserving bijection so it is a homeomorphism. □

Ford [15] showed that normed linear spaces are SLH.

**Proposition 3.2** ([15]). *Every normed linear space $(X, \|\cdot\|)$ is SLH.*

*Proof.* From the properties of normed linear spaces, it is enough to show the SLH property only for the unit ball as a neighborhood of zero. Let $x_0 \in B(0,1) \setminus \{0\}$. For $x \in X$ we can define

$$f(x) = \begin{cases} x + x_0(1 - \|x\|) & \text{if } \|x\| < 1, \\ x & \text{if } \|x\| \geq 1. \end{cases}$$

By distinguishing possible cases, we can show that for all $x, y \in X$ we have

$$(1 + \|x_0\|)\|x - y\| \geq \|f(x) - f(y)\| \geq (1 - \|x_0\|)\|x - y\|.$$

These inequalities show that $f$ is injective and continuous and that the inverse is continuous as well.

To show surjectivity, note that for any $x \in B(0,1) \setminus \{x_0\}$ and the map $g(t) : [0,1] \to \mathbb{R}$, defined as $g(t) = 1 - \|x - tx_0\| - t$, there exists $t_0 \in (0,1)$ such that $g(t_0) = 0$, since $g$ is continuous, $g(1) < 0$ and $g(0) > 0$. This implies $f(x - tx_0) = x - tx_0 + x_0(1 - \|x - tx_0\|) = x$. Therefore $f$ is surjective and so it is a homeomorphism. □

**Example 3.3.** *By Proposition 3.2 we have that every separable Banach space is CDH.*

Note that by Example 3.3 we have that $\mathbb{R}$ is CDH and also that $\mathbb{R}^n$ is CDH.

Now we would like to show that the classical zero-dimensional spaces are all CDH. We will do this by using Theorem 2.30. By the following, homogeneity implies strong local homogeneity for zero-dimensional spaces.

**Proposition 3.4** ([31]). *Let $X$ be zero-dimensional, homogeneous and Hausdorff. Then $X$ is SLH.*

*Proof.* Let $A \subset X$ be clopen and let $x, y \in A$. Then there is $h \in \mathcal{H}(X)$ such that $h(x) = h(y)$. Since $X$ is Hausdroff and 0-dimensional we can find a clopen neighborhood $B$ of $x$ such that $B \subset A$, $h(B) \subset A$, and $B \cap h(B) = \emptyset$. Then we can define $g \in \mathcal{H}(X)$ as follows

$$g(z) = \begin{cases} h(z), & z \in B, \\ h^{-1}(z), & z \in h(B), \\ z, & z \in X \setminus (B \cup h(B)) \end{cases}$$

for any $z \in X$. Since $B$ and $h(B)$ are clopen and disjoint, we have that $g$ is a homeomorphism. Since clopen sets form a base for the topology of $X$, we have that $X$ is SLH. □

Note that even though $\mathbb{Q}$ is SLH by Proposition 3.4, it is not CDH by Theorem 2.10. This does not contradict Theorem 2.30 since $\mathbb{Q}$ is not polish.

Using Proposition 3.4 we can finally show that the classical zero-dimensional spaces are CDH.

**Example 3.5.** *By Proposition 3.4 and and Theorem 2.30 we have that $\omega$, $\omega^\omega$, $2^\omega$ and $2^\omega \setminus \{0\}$ are CDH, since these spaces are Polish zero-dimensional and homogeneous.*

We can use the CDH property of $\mathbb{R}$ to show the CDH property even for some other topologies on $\mathbb{R}$, namely the Sorgenfrey line. This example is well-known and easy to show.

**Example 3.6.** *The Sorgenfrey line $S$ is CDH.*

*Proof.* Let $A, B \subset S$ be two countable dense sets. Then both sets are also dense in $\mathbb{R}$. In 3.1 we constructed $h \in \mathcal{H}(\mathbb{R})$ that preserves orientation and satisfies $h(A) = B$. Since $h$ preserves orientation, we also have $h \in \mathcal{H}(C)$. □

Again using the properties of $\mathbb{R}$ we can show even more.

**Example 3.7.** *Let $S$ be the Sorgenfrey line and let $C \subset S$ then the space $S \setminus C$ is CDH.*

*Proof.* Let $A, B \subset S \setminus C$ be countable dense sets. Then $A, B$ and $C$ are also dense countable subsets of $\mathbb{R}$. By Remark 2.32 we can use Theorem 2.31 with the subgroup $\{h \in \mathcal{H}(\mathbb{R}); h \text{ preserves orientation}\}$ to find an orientation preserving homeomorphism $h$ such that $h(A) = B$ and $h(C) = C$. Since $h$ is orientation preserving, we have that $h \in \mathcal{H}(S)$ and we are done. □

So far we have made an effort to present a theory regarding spaces that are CDH. However it is also useful to have tools to show that some spaces cannot be CDH. One such can be found in [17].

**Proposition 3.8** ([17])**.** *Let $X, Y$ be two crowded, Hausdorff spaces with countable $\pi$-weight. If $X \times Y$ is CDH, then $X$ contains subspace homeomorphic to $2^\omega$ if and only if $Y$ contains subspace homeomorphic to $2^\omega$.*

*Proof.* Assume $X$ contains a subset homeomorphic to $2^\omega$ but $Y$ does not. Then $X \times Y$ contains a subset homeomorphic to $2^\omega$ and we can find a countable dense subset $D \subset X \times Y$ and a set $Q \subset D$ such that $\mathrm{Cl}_{X\times Y} Q \approx 2^\omega$.

Let $\mathcal{B} = \{U_n \times V_n,\ n \in \omega\}$ be a countable $\pi$-base of $X \times Y$, where for each $n \in \omega$ $U_n$ is open in $X$ and $V_n$ is open in $Y$. Let $\pi_X$, respectively $\pi_Y$, denote the projections from $X \times Y$ to $X$, respectively $Y$. Inductively, choose a set $E = \{a_n,\ n \in \omega\} \subset X \times Y$ such that $\pi_X(a_n) \in U_n$ and $\pi_Y(a_n) \in V_n \setminus \{\pi_Y(a_1), \ldots, \pi_Y(a_{n-1})\}$, for all $n \in \omega$. Then $\pi_Y|_E$ is one-to-one and $E$ is countable dense in $X \times Y$.

Assume there is $h \in \mathcal{H}(X \times Y)$ such that $h(D) = E$. Then there is $R \subset D$ such that $K = \mathrm{Cl}_{X\times Y} R \approx 2^\omega$. Then $T = \pi_Y(K)$ is compact subset of $Y$ of countable weight. Since $T$ is not homeomorphic to $2^\omega$, it contains an isolated point $p$. Then we have $X \times \{p\} \cap K$ is a clopen subset of $K$. Since $\pi_Y|_R : R \to Y$ is one-to-one and $R$ is dense in $K$, we get that $X \times \{p\} \cap K$ is a single point, which is a contradiction. □

*Remark* 3.9. Note that in the proof of Proposition 3.8 we constructed a countable dense set $E$ such that for no $D \subset E$ we have $\overline{D} \approx 2^\omega$.

For quite some time it was not known whether the space $\mathbb{Q}^\omega$ is CDH or not. Note that this space is not Polish. It was finally solved in [14] in the negative. Proposition 3.8 can be used to show this too in a more convenient way. This has been done in [17].

*Corollary* 3.10. If $X$ is crowded Hausdorff space of countable $\pi$-weight and contains a copy of $2^\omega$, then $X \times \mathbb{Q}$ is not CDH.

**Example 3.11** ([17])**.** *$\mathbb{Q}^\omega$ is not CDH.*

*Proof.* This follows by Corollary 3.10 and the fact that $\mathbb{Q} \times \mathbb{Q}^\omega \approx \mathbb{Q}^\omega$. □

We have seen that for a Polish SLH space $X$ and a countable dense subset $C \subset X$ we have that the remainder $X \setminus C$ is CDH. This is however not true in general for a CDH-space X. This example does not appear in the literature.

**Example 3.12.** *There exists a CDH-space $X$ such that for $C \subset X$ countable dense $X \setminus C$ is not separable.*

*Proof.* Let $\hat{\mathcal{U}}$ be a non-principal ultrafilter on $\omega$. Define $\mathcal{U} = \{A \subset \omega_1; A \cap \omega \in \hat{\mathcal{U}}\}$. Then $\mathcal{U}$ is clearly a non-principal ultrafilter on $\omega_1$. Let $X$ be the topological space $(\omega_1, \mathcal{U})$. This is clearly separable since $\omega$ is a countable dense subset of $X$. On the other hand the set $X \setminus \omega$ is an uncountable and discrete space since $\omega \cup \{x\}$ is open in $X$ for any $x \in X \setminus \omega$. It remains to show that $X$ is CDH.

To this end, let $A, B$ be two countable dense subsets of $X$. Since $\mathcal{U}$ is a non-principal we have that $A \cap B$ is countable and $A \cap B \in \mathcal{U}$. We can find $D_1, D_2$ countable disjoint sets such that $(D_1 \cup D_2) \cap (A \cup B) = \emptyset$. Since $\mathcal{U}$ is an ultrafilter, we can find $C \subset (A \cap B)$ countable such that $C \in \mathcal{U}$ and $(A \cap B) \setminus C$ is countable. Denote $\hat{A} = A \setminus C$ and $\hat{B} = B \setminus C$. We have that $\hat{A}$ and $\hat{B}$ are countable. We can find a bijection $f : \hat{A} \cup \hat{B} \cup D_1 \cup D_2 \to \hat{A} \cup \hat{B} \cup D_1 \cup D_2$ such that

$$f(\hat{A}) = \hat{B},$$

$$f\left(\left(\hat{B} \setminus (\hat{A} \cap \hat{B})\right) \cup D_2\right) = D_2$$

and

$$f(D_1) = D_1 \cup \left(\hat{A} \setminus (\hat{A} \cap \hat{B})\right).$$

We can also define $f(x) = x$ for $x \in X \setminus \left(\hat{A} \cup \hat{B} \cup D_1 \cup D_2\right)$. Then we have $C \subset \{x \in X; f(x) = x\}$, therefore $\{x \in X; f(x) = x\} \in \mathcal{U}$. And for any $A \in \mathcal{U}$ we have that

$$\begin{aligned} f(A) &= f(A \cap \{x \in X; f(x) = x\}) \cup f(A \setminus \{x \in X; f(x) = x\}) \\ &= A \cap \{x \in X; f(x) = x\} \cup f(A \setminus \{x \in X; f(x) = x\}) \in \mathcal{U}. \end{aligned}$$

Therefore $f$ preserves $\mathcal{U}$. Similarly we can show that $f^{-1}$ preserves $\mathcal{U}$ and thus $f$ is a homeomoprhism of $X$ onto $X$ and by definition $f(A) = B$.

□

## 3.1 λ-sets

In this section, we introduce the concept of λ-sets an show its connection to CDH-spaces. Majority of this section is based on [18]. At the end of this section we use these concepts to answer one of the open questions posed in [20].

**Definition 3.13** (λ-sets)**.** *A set $X \subset 2^\omega$ is called a λ-set if every countable set of $X$ is $G_\delta$ in $X$ i.e. for any $D \subset X$ countable $D = \bigcap_{n=1}^{\infty} (G_n \bigcap X)$, where $G_n$ are open in $2^\omega$.*

*Remark.* The term λ-set is sometimes used for any zero-dimensional space such that all countable sets are $G_\delta$. We will use both concepts, however, it will always be clear from the context which one we mean.

*Remark.* Being a λ-set is clearly a hereditary property. This also shows that no λ-set $X$ contains a homeomorphic copy of the Cantor space since dense countable subsets of the Cantor space are not $G_\delta$.

**Definition 3.14** (bounding number)**.** *Let $f, g \in \omega^\omega$ we say $f \leq^* g$ if the set $\{n \in \omega; f(n) \leq g(n)\}$ is cofinite. We define bounding number $\mathfrak{b}$ to be the minimal cardinality of a set $F \subset \omega^\omega$ that is unbounded with respect to $\leq^*$.*

*Remark.* There exists an unbounded family $F = \{f_\alpha; \alpha < \mathfrak{b}\}$ such that if $\alpha \leq \beta < \mathfrak{b}$ then $f_\alpha \leq^* f_\beta$.

**Lemma 3.15** ([24]). $\aleph_1 \leq \mathfrak{b}$.

*Proof.* Let $F = \{f_i; i \in \omega\} \subset \omega^\omega$ be any countable set. Define $g(k) = \max\{f_n(k); n \leq k\}$. Then for each $n \in \omega$ we have $f_n \leq^* g$, since $f_n(k) \leq g(k)$ for each $k \geq n$. Thus we have $\mathfrak{b} > \aleph_0$. □

We have seen that Polish zero-dimensional spaces behave very well with respect to the CDH property. If we want any "non-standard" behaviour we need to focus on non Polish zero-dimensional spaces. One such class of spaces are meager spaces and if we restrict our attention to those the notion of $\lambda$-set arises very naturally in the study of CDH-spaces.

**Theorem 3.16** ([14]). *If $X$ is metric, meager in itself CDH-space then $X$ is a $\lambda$-set.*

*Proof.* First we show that there exists a countable dense subset of $X$ that is $G_\delta$. Suppose $X = \bigcup_{i=0}^{\infty} F_i$, where each $F_i$ is closed nowhere dense. Let $\mathcal{U} = \{U_i, i \in \omega\}$ be a countable basis for $X$. Choose $x_0 \in U_0$ and for each $n > 0$ choose $x_n \in U_n \setminus \bigcup_{i=0}^{n-1} F_i$ and let $C = \{x_i, i \in \omega\}$. Then $C$ is countable dense from the construction. And we have

$$C = \bigcap_{i=0}^{\infty} \bigcap_{n=1}^{\infty} \Big( (X \setminus F_i) \cup B_{\frac{1}{n}} (F_i \cap C) \Big).$$

Now let $A \subset X$ be countable. Let $B$ be countable dense subset of $X$. Since $X$ is CDH, we have that $A \cup B$ is $G_\delta$ in $X$. We also have that $A$ is $G_\delta$ subset of $A \cup B$, therefore $A$ is $G_\delta$ subset of $X$. □

On the other hand all "reasonable" $\lambda$-sets are meager. therefore not Polish. It is therefore reasonable to study those in regard to the CDH property.

**Proposition 3.17.** *Every $T_1$ and separable $\lambda$-set $X$ without isolated points is meager in itself.*

*Proof.* Since $X$ is separable, there exists $D = \{d_i, i \in \omega\} \subset X$ dense subset. Then there are sets $G_i$, $i \in \omega$ open such that $D = \bigcap_{i=0}^{\infty} G_i$. This implies that all the sets $G_i$ are dense in $X$, therefore the sets $F_i = X \setminus (G_i \bigcap X) = X \setminus G_i$ are closed and nowhere dense in $X$. Since $X$ does not have isolated points and is $T_1$, all the sets $\{d_i\}$ are closed and nowhere dense in $X$ and we can write $X = (\bigcup_{i=1}^{\infty} F_i) \cup (\bigcup_{i=1}^{\infty} \{d_i\})$. □

By Theorem 2.10 any interesting behavior for CDH-spaces can only happen for cardinalities above $\aleph_0$. From the definition of a $\lambda$-set it is not clear that an uncountable $\lambda$-set even exists. The notion of a $\lambda$-set is due to Kuratowski [25]. Lusin showed the existence of an uncountable $\lambda$-set [26] and Rothberger [35] showed that for any $\kappa \leq \mathfrak{b}$ there is a $\lambda$-set of cardinality $\kappa$. Here we present a construction due to Miller [33] of $\lambda$-sets in $\omega^\omega$ of cardinalities $\leq \mathfrak{b}$. From that we can also construct $\lambda$-sets in $2^\omega$.

**Lemma 3.18.** *Let $X \subset \omega^\omega$ be a $\lambda$-set then there is a $\lambda$-set in $2^\omega$ with the same cardinality.*

*Proof.* Let $C$ be a countable dense subset of $2^\omega$. Then we have $2^\omega \setminus C \approx \omega^\omega$, since $2^\omega \setminus C$ is a zero-dimensional Polish space such that all compact sets in $2^\omega \setminus C$ have empty interior. We will show if $X \subset 2^\omega \setminus C$ is a $\lambda$-set in $2^\omega \setminus C$ then it is a $\lambda$-set in $2^\omega$.

Let $A \subset X$ be countable, then there are sets $H_i \subset 2^\omega \setminus C$, $i \in \omega$ such that $H_i$ is open in $2^\omega \setminus C$ for all $i \in \omega$ and $A = \bigcap_{i\in\omega}(H_i \cap X)$. For each $i \in \omega$ we can find $G_i$ open in $2^\omega$ such that $G_i \cap (2^\omega \setminus C) = H_i$. We can also find sets $O_i \subset 2^\omega$, $i \in \omega$ open in $2^\omega$ such that $2^\omega \setminus C = \bigcap_{i\in\omega} O_i$.

Then we have $A = \bigcap_{i\in\omega} (G_i \cap O_i \cap X)$. □

**Lemma 3.19** ([33])**.** *Let $X \subset \omega^\omega$ such that the cardinality of $X$ is strictly less than $\mathfrak{b}$. Then $X$ is a $\lambda$-set.*

*Proof.* Let $C \subset X$ be any countable subset and enumerate $X \setminus C = \{x_\alpha; \alpha < \kappa\}$ for some $\kappa < \mathfrak{b}$. Let $\{y_i\}_{i=1}^\infty$ be a sequence of points of $C$ such that every point occurs infinitely many times. For every $i \in \omega$ let $\{U_n(y_i); n < \omega\}$ be a decreasing neighborhood basis of $y_i$ in $\omega^\omega$. Now for every $\alpha < \kappa$ define $g_\alpha \in \omega^\omega$ such that for every $i \in \omega$ we have $x_\alpha \notin U_{g_\alpha(i)}(y_i)$. Since $\kappa < \mathfrak{b}$, there is $g \in \omega^\omega$ such that for any $\alpha < \kappa$ we have $g_\alpha \leq^* g$. Now define

$$U = \bigcap_{n\in\omega} \left( \bigcup_{i>n} U_{g(i)}(y_i) \right).$$

Then $U$ is $G_\delta$ in $\omega^\omega$ and we have $C \subset U$ and $(X \setminus C) \cap U = \emptyset$, therefore $U \cap X = C$. □

**Theorem 3.20** ([33])**.** *There exists $X \subset 2^\omega$ a $\lambda$-set with cardinality $\mathfrak{b}$.*

*Proof.* We will find such $\lambda$-set in $\omega^\omega$. By Lemma 3.18 we will then have a $\lambda$-set in $2^\omega$ with the same cardinality.

Let $F = \{f_\alpha; \alpha < \mathfrak{b}\}$ be an unbounded family such that if $\alpha \leq \beta < \mathfrak{b}$ then $f_\alpha \leq^* f_\beta$. Let $C \subset F$ be countable. Since $F$ is unbounded and $C$ is countable and therefore bounded, there exists $\alpha < \mathfrak{b}$ such that for every $h \in C$ we have $f_\alpha \not\leq^* h$. This also holds for $\beta > \alpha$, since $f_\beta \geq^* f_\alpha$. Now let

$$K = \{g \in \omega^\omega; f_\alpha \not\leq^* g\}.$$

We have that

$$K = \bigcap_{\substack{A\subset\omega \\ \text{cofinite}}} \bigcup_{n\in A} \{g \in \omega^\omega; f_\alpha(n) > g(n)\},$$

so $K$ is a $G_\delta$ subset of $\omega^\omega$. We also have that $C \subset K \cap F \subset \{f_\beta; \beta < \alpha\}$ therefore the cardinality of $K \cap F$ is strictly less then $\mathfrak{b}$. Therefore by Lemma 3.19, $C$ is relatively $G_\delta$ in $K \cap F$. Since $K$ is $G_\delta$ in $\omega^\omega$ we also have that $C$ is relatively $G_\delta$ in $F$. □

*Remark.* Note that since being a $\lambda$-set is a hereditary property, Theorem 3.20 actually shows that for any $\kappa \leq \mathfrak{b}$ there exists a $\lambda$-set in $2^\omega$ of cardinality $\kappa$.

Before we proceed to the study of $\lambda$-sets in relation to the CDH property, we present two useful results regarding $\lambda$-sets. We will use those later.

**Lemma 3.21** ([18])**.** *Let $X, Y$ be $\lambda$-sets then also $X \times Y$ is a $\lambda$-set.*

*Proof.* Let $A \subset X \times Y$ be countable. Let $\pi_X$ be the projection to the first coordinate. Then $X \setminus \pi_X(A)$ is $F_\sigma$ in $X$ by assumptions. Therefore also $X \setminus \pi_X(A) \times Y$ is $F_\sigma$ in $X \times Y$. We also have that for all $p \in \pi_X(A)$ the set $\{p\} \times Y \setminus A$ is $F_\sigma$ in $p \times Y$ and hence also in $X \times Y$. We can also write

$$(X \times Y) \setminus A = ((X \setminus \pi_X(A)) \times Y) \cup \left( \bigcup_{p \in \pi_X(A)} ((\{p\} \times Y) \setminus A) \right).$$

Note that the last union is countable, therefore the set $(X \times Y) \setminus A$ is $F_\sigma$. ☐

**Lemma 3.22** ([18])**.** *Let $X$ be a separable metric space such that $X = \bigcup_{i \in \omega} F_i$, where $F_i$ is a closed $\lambda$-set for each $i \in \omega$. Then $X$ is a $\lambda$-set.*

*Proof.* The space $X$ is zero-dimensional by the sum theorem for zero-dimensional spaces [10]. Thus for every $i \in \omega$ there are clopen sets $G_1^i \supset G_2^i \supset \cdots \supset F_i$ such that $F_i = \bigcap_{n \in \omega} G_n^i$. Let $L_1 = F_1$ and for $i \in \omega$, $i > 1$ let

$$L_i = \left(F_i \setminus \left(G_1^1 \cup \cdots \cup G_1^{i-1}\right)\right) \cup \left(\left(F_{i-1} \cap \left(G_1^1 \cup \ldots G_1^{i-2}\right)\right) \setminus \left(G_2^1 \cup \cdots \cup G_2^{i-2}\right)\right)$$
$$\cup \cdots \cup \left(\left(F_2 \cap G_{i-2}^1\right) \setminus G_{i-1}^1\right).$$

Then $\mathcal{L} = \{L_i; i \in \omega\}$ is a disjoint cover of $X$ by closed $\lambda$-sets. Let $A \subset X$ be countable. Then $L_i \setminus A$ is $F_\sigma$ in $L_i$ for every $i \in \omega$ since $L_i$ is a $\lambda$-set. But then also $X \setminus A = \bigcup_{i \in \omega} L_i \setminus A$ is $F_\sigma$ since $L_i$ is closed for every $i \in \omega$. ☐

Since by Proposition 2.7 the CDH property is preserved under topological sum, we cannot expect all uncountable $\lambda$-sets to be CDH. We can start with an uncountable $\lambda$-set $X \subset 2^\omega$ and take $X \oplus \mathbb{Q} \subset 2^\omega \oplus 2^\omega \approx 2^\omega$, which will still be a $\lambda$-set however not a CDH-space, since all homeomorphisms have to preserve $\mathbb{Q}$ and $X$. Therefore we need only to consider $\lambda$-sets that are "uniformly" spread throughout $2^\omega$.

**Definition 3.23** (tail equivalence)**.** *For $x, y \in 2^\omega$ we define*

$$x \overset{\text{tail}}{\sim} y \text{ if } \exists m, n \in \omega \, \forall k \in \omega \, x(m+k) = y(n+k).$$

*For set $X \subset 2^\omega$ we define its saturation $X^* = \{y \in 2^\omega; \exists x \in X \, x \overset{\text{tail}}{\sim} y\}$. We say that a set $X \subset 2^\omega$ is saturated if $X = X^*$.*

*Remark.* For every $x \in 2^\omega$ the equivalence class $[x]$ of $x$ in the tail equivalence relation is countable and dense subset of $2^\omega$.

Saturated sets are "uniformly" spread throughout $2^\omega$. However now we again run into the issue of existence of such uncountable $\lambda$-sets. Hernández-Gutiérrez, Van Mill and Hrušák [18] showed that uncountable saturated $\lambda$-sets exist. Using the fact that tail equivalence is a Borel equivalence. This follows easily from the definition of tail equivalence.

**Lemma 3.24.** *The set* $E = \{(x, y);\ x \overset{\text{tail}}{\sim} y\} \subset 2^\omega \times 2^\omega$ *is a Borel set.*

*Proof.* Let $m, n, k \in \omega$ and define $A_{(m,n,k)} = \{(x, y);\ x(m+k) = y(n+k)\}$. These sets are clearly closed and we have

$$E = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_{(m,n,k)}.$$

□

We do not present a proof of the following since it is out of the scope of this text. However, it can be used to construct an uncountable saturated λ-set.

**Proposition 3.25** ([37]). *Suppose* $R$ *is an equivalence relation on* $2^\omega$ *such that the set* $R$ *is a Borel subset of* $2^\omega \times 2^\omega$. *Then there exists an embedding* $\varphi : 2^\omega \to 2^\omega$ *such that for* $x, y \in 2^\omega$ *and* $x \neq y$ *we have* $(\varphi(x), \varphi(y)) \notin R$.

**Lemma 3.26** ([18]). *For every* $\lambda$*-set* $X \subset 2^\omega$ *there is a saturated* $\lambda$*-set* $Y \subset 2^\omega$ *of the same cardinality.*

*Proof.* By Proposition 3.25 there is $\varphi : 2^\omega \to 2^\omega$ such that if $x, y \in 2^\omega$, $x \neq y$ then $\varphi(x) \overset{\text{tail}}{\nsim} \varphi(y)$. Now let $Y = \varphi(X)^*$. We have that $\varphi(2^\omega) \bigcap Y = \varphi(X)$, thus $\varphi(X)$ is closed in $Y$. For $\sigma \in \omega^{<\omega}$ and $k \in \omega$ define an embedding $h_{\sigma,k} : 2^\omega \to 2^\omega$ by $h_{\sigma,k}(x) = (\sigma, x(k), x(k+1), \dots)$. Then we can write

$$Y = \varphi(X)^* = \bigcup_{k \in \omega} \bigcup_{\sigma \in \omega^{<\omega}} h_{\sigma,k}(\varphi(X)),$$

so by Lemma 3.22, $Y$ is a λ-set and since the unions are countable it has the same cardinality as $X$. □

*Remark.* Since being a λ-set is a hereditary property, Lemma 3.26 actually shows that for any $\kappa \leq \mathfrak{b}$ there is a saturated λ-set of cardinality $\kappa$

In [18] it was shown that uncountable saturated λ-sets are CDH and that the CDH property is closed under finite powers on these sets, however not under countable infinite powers.

In [18] the following is proved. We do not include the proof, since it is quite technical and complicated .

**Theorem 3.27.** *[18] Let* $X \subset 2^\omega$ *be an uncountable saturated* $\lambda$*-set. Then* $X$ *is CDH.*

We can use Theorem 3.27 to show the following. The first two parts were proved in [18] the last one is an easy consequence of properties of λ-sets and Proposition 3.8.

**Theorem 3.28.** *[18] Let* $X \subset 2^\omega$ *be an uncountable saturated* $\lambda$*-set. Then we have the following:*

1. $X^n$ *is CDH for every* $n \in \omega$,
2. $X^\omega$ *is not CDH,*

3. *$X^n \times 2^\omega$ is not CDH for any $n \in \omega$, $n \geq 1$.*

*Proof.* 1. We will show that $X^n$ is homeomorphic to a saturated $\lambda$-set in $2^\omega$. By Lemma 3.21, $X^n$ is a $\lambda$-set. Define $\varphi : (2^\omega)^n \to 2^\omega$ by

$$\varphi((x_0, \dots, x_{n-1}))(nk + j) = x_j(k)$$

for any $(x_0, \dots, x_{n-1}) \in (2^\omega)^n$, $j \in \{0, \dots, n-1\}$ and $k \in \omega$. This is clearly a homeomorphism between $(2^\omega)^n$ and $2^\omega$.

Now suppose $y = \varphi((x_0, \dots, x_{n-1}))$ for some $(x_0, \dots, x_{n-1}) \in X^n$ and $z \overset{\text{tail}}{\sim} y$. Then there are $m_1, m_2 \in \omega$ such that for any $l \in \omega$ we have $y(m_1 + l) = z(m_2 + l)$. Suppose that $m_1 = k_1 n + j_1$ and $m_2 = k_2 n + j_2$ for some $k_1, k_2 \in \omega$ and $j_1, j_2 \in \{0, \dots, n-1\}$. Denote

$$\alpha(q) = q \mod n$$

for $q \in \mathbb{N}$. Then for $j \in \omega$ and $i \in \{0, \dots, n-1\}$ we have that

$$z((k_2 + 1 + j)n + i) = x_{\alpha(j_1+n-j_2+i)}(k_1 + 1 + j).$$

This means that for $i \in \{0, \dots, n-1\}$ we have

$$(z(i), z(n+i), \dots) \overset{\text{tail}}{\sim} (x_{\alpha(j_1+n-j_2+i)}(k_1 + 1), x_{\alpha(j_1+n-j_2+i)}(k_1 + 2), \dots).$$

Since $X$ is saturated this implies that $z \in \varphi(X^n)$. Thus $X^n$ is homeomorphic to a saturated $\lambda$-set in $2^\omega$ and therefore CDH by Theorem 3.27

2. Note that $X^\omega \approx X \times X^\omega$ and $X^\omega$ contains a copy of the Cantor space. However, $X$ being a $\lambda$-set does not contain a copy of the Cantor space. Both $X$ and $X^\omega$ are clearly crowded spaces with countable $\pi$-weight. Thus, by Proposition 3.8 $X^\omega$ is not CDH.

3. By Lemma 3.21, $X^n$ is a $\lambda$-set therefore it does not contain a copy of the Cantor space. Using again Proposition 3.8 we get that $X^n \times 2^\omega$ is not CDH. □

We can also show that a reminder of countable dense subset in uncountable saturated $\lambda$-set is CDH as well. This is not mentioned in [18], however the proof is very easy.

**Proposition 3.29.** *Let $X \subset 2^\omega$ be an uncountable saturated $\lambda$-set and let $C \subset X$ be countable dense then $X \setminus C$ is CDH.*

*Proof.* Let $x \in X$, then $[x] = \{x\}^*$ is a countable dense subset of $X$ and $X \setminus [x]$ is uncountable saturated $\lambda$-set, therefore CDH by Theorem 3.27. Since $X$ is CDH, we have that $X \setminus C \approx X \setminus [x]$. Thus $X \setminus C$ is CDH for any $C \subset X$ countable dense. □

Uncountable $\lambda$-sets can be used to answer one of the open questions posed in [20, Problem 14]. This question has originally been asked by Medini in [27].

**Example 3.30.** *There is a zero-dimensional CDH metric space $X$ such that $X^2$ is not CDH.*

*Proof.* Let $Y$ be an uncountable saturated $\lambda$-set and consider $X = Y \oplus 2^\omega$. By Example 3.5. $2^\omega$ is CDH. By Theorem 3.27, we also have that $Y$ is CDH, therefore by Proposition 2.7, $X$ is CDH. Clearly $X$ is zero-dimensional and metrizable. We have that $X^2 \approx (Y \times 2^\omega) \oplus Y^2 \oplus 2^\omega$. By Theorem 3.28, item 3, $Y \times 2^\omega$ is not CDH. Thus, it is enough to show that for any $h \in \mathcal{H}(X^2)$ we have $h(Y \times 2^\omega) = Y \times 2^\omega$.

Let $h \in \mathcal{H}(X^2)$. Since any clopen subset of $Y \times 2^\omega$ contains the Cantor set and $Y^2$ does not contain the Cantor set, we have that $h(Y \times 2^\omega) \bigcap Y^2 = \emptyset$. By the same argument, we have $h(2^\omega) \bigcap Y^2 = \emptyset$. Now suppose $h(2^\omega) \bigcap (Y \times 2^\omega) \neq \emptyset$. Denote by $\pi_1 : Y \times 2^\omega \to Y$ the projection to the first coordinate, then we have that $\pi_1(h(2^\omega) \bigcap (Y \times 2^\omega))$ is compact, zero-dimensional metric space. It is also a clopen subset of $Y$ therefore it is a crowded space. But this means that $\pi_1(h(2^\omega) \bigcap Y \times 2^\omega) \approx 2^\omega$ which cannot be since $Y$ does not contain the Cantor set. This and the fact that $h(2^\omega) \bigcap Y^2 = \emptyset$ implies that $h(2^\omega) = 2^\omega$ and thus $h(Y \times 2^\omega) = Y \times 2^\omega$, therefore $X^2$ is not CDH. □

Medini in his paper [27] also asks if we can consistently say anything about the descriptive quality of such space. We will show that our example is actually consistently co-analytic. Note that since $\lambda$-sets do not contain copy of the Cantor space the space in Example 3.30 is not analytic, as uncountable analytic sets contain a copy of the Cantor space. Also note that assuming Projective Determinacy all projective uncountable sets contain a copy of the Cantor space, thus our example cannot be co-analytic in ZFC. We will not go into more details regarding the set theory as we are more focused on topology.

We will use the following well-known theorem.

**Theorem 3.31** ([32])**.** *Suppose MA + ¬CH +* $\omega_1 = (\omega_1)^L$*. Then every* $A \subset 2^\omega$ *of cardinality* $\omega_1$ *is* $\mathbf{\Pi}^1_1$*.*

**Example 3.32.** *Suppose MA + ¬CH +* $\omega_1 = (\omega_1)^L$*. Then there exists CDH-space* $X \subset 2^\omega$ *such that* $X^2$ *is not CDH and* $X$ *is* $\mathbf{\Pi}^1_1$*.*

*Proof.* Let $Y$ be an uncountable saturated $\lambda$-set of cardinality $\omega_1$. Then by Theorem 3.31, $Y$ is co-analytic, which implies that also $Y \oplus 2^\omega$ is co-analytic. □

# 4. Spaces with few types of countable dense subsets

As we have seen, many space are CDH. However, there are also very natural spaces that fail to be CDH such as $[0,1]$. The notion of CDH-spaces can be generalized to deal with many of those spaces.

**Definition 4.1** (Nearly CDH-space). *Let $X$ be a separable space. We say that $X$ is $\frac{1}{n}$-CDH, for $n \geq 2$ if $X$ has exactly $n$ many types of countable dense subsets. We say that $X$ is $\frac{1}{\omega}$-CDH if $X$ has countably many types of countable dense subsets.*

*Remark.* Note that in the definition we require a $\frac{1}{n}$-CDH space to have exactly $n$ many types of countable dense subsets. To simplify notation we will say that a space $X$ is at most $\frac{1}{n}$-CDH, if there are at most $n$ many types of countable dense subsets. We will also say that a space $X$ is at most $\frac{1}{\omega}$-CDH if there are at most countably many types of countable dense subsets.

*Remark.* If a space $X$ is $\frac{1}{n}$-CDH for some $n \geq 2$ and $C_0, \ldots, C_n$ are countable dense subsets of $X$ then, by the pigeonhole principle, there exist $0 \leq i < j \leq n$ and $h \in \mathcal{H}(X)$ such that $h(C_i) = C_j$. On the other hand, if for any countable dense subsets $C_0, \ldots, C_n$ there is $0 \leq i < j \leq n$ and $h \in \mathcal{H}(X)$ such that $h(C_i) = C_j$ then the space $X$ is at most $\frac{1}{n}$-CDH.

These notions were studied by Kennedy in 80's in [34], [23]. She mainly focused on the connection of these notions and homogeneity properties of the given space. More recent results can be found in [19]. In [19] the following turned out to be very useful tool in the study of spaces with at most countable number of types of countable dense sets.

**Lemma 4.2** ([6]). *The number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is $\mathfrak{c}$.*

*Proof.* Since $\mathbb{R}$ is CDH, every countable subset of $\mathbb{R}$ can be embedded into $\mathbb{Q}$, so the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at most $|\mathcal{P}(\mathbb{Q})| = \mathfrak{c}$.

Let $X \subset \mathbb{R}$. Let $P$ be the largest crowded subset of $X$ and let $S = X \setminus P$ be the scattered part of $X$. We define scattered signature of $H(X)$ of $X$ as follows: $H(X)$ is a set of ordinal numbers, and $\alpha \in H(X)$ if and only if there is some $p \in P$ such that $p$ has Cantor-Bendixon rank $\alpha$ in $S \cup \{p\}$.

Let $A = \{\alpha_n; n \in \mathbb{N}\}$ be a countable subset of $\omega_1$. We show that there is a countable subset of $\mathbb{R}$ with scattered signature $A$. On the interval $[n+1/4, n+1/2]$, embed $\omega^{\alpha_n} + 1$, making sure that the point $\omega^n$ maps to $n + 1/2$. This can be done since by Proposition 1.4 the space $\omega^{\alpha_n} + 1$ can be embedded into The cantor set, which is homogeneous and can be embedded into $[n + 1/4, n + 1/2]$ in such a way that it contains the point $n + 1/2$. Include all the points $\mathbb{Q} \cap [n+1/2, n+3/4]$ and call the resulting set $X$. It follows that the set $X$ has scattered rank $A$. There are $\mathfrak{c}$ countable subsets of $\omega_1$. This shows that the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at least $\mathfrak{c}$. □

*Corollary* 4.3. There exist $\mathfrak{c}$ many nonhomeomorphic nowhere dense subsets of $\mathbb{Q}$.

*Proof.* By Lemma 4.2, there are $\mathfrak{c}$ many nonhomeomorphic countable subsets of $\mathbb{R}$. Every countable subset of $\mathbb{R}$ can be embedded into $\mathbb{Q}$, since $\mathbb{R}$ is CDH. Also we have that $\mathbb{Q}$ contains a nowhere dense homeomorphic copy of $\mathbb{Q}$, since there exists countable dense subset of $\mathbb{R}$ containing a nowhere dense copy of $\mathbb{Q}$. Thus we can embed every countable subset of $\mathbb{R}$ into $\mathbb{Q}$ such that it is nowhere dense in $\mathbb{Q}$. ☐

In this chapter we generalize some results mentioned in Chapter 2. These doesn't seem to be published, however many of the proofs are only a slight modification of known proofs.

One of the drawbacks of studying $\frac{1}{n}$-CDH spaces in general is that we do not have any separation axioms as the following example shows.

**Example 4.4.** *The space $X = \{x, y\}$ with indiscrete topology is $\frac{1}{2}$-CDH.*

*Proof.* Countable dense subsets of $X$ are $\{x\}$, $\{y\}$ and $\{x, y\}$. Clearly the sets $\{x\}$ and $\{y\}$ are of the same type. ☐

We first show a parallel with Theorem 2.6 for $\frac{1}{n}$-CDH spaces. We will proceed in similar steps first showing that the number of components for $\frac{1}{n}$-CDH space is bounded

**Lemma 4.5.** *Let $X$ be $\frac{1}{n}$-CDH for some $n \geq 2$. Then $X$ has at most countable many nondegenerate components. Of those nondegenerate components at most $n-1$ is not open.*

*Proof.* As in proof of Lemma 2.2, we can find a countable dense subset $C_0$ of $X$ such that if $K$ is a nondegenerate component of $X$ intersecting $C_0$ then $C_0 \cap K$ has at least two points. Suppose $X$ has y many nondegenerate components. Then we can find $K_1, \ldots, K_n$ nondegenerate components of $X$ such that for any $1 \leq i \leq n$ we have $C_0 \cap K_i = \emptyset$. For $1 \leq i \leq n$ let $x_i \in K_i$, and define $C_i = C_0 \cup \{x_1, \ldots, x_i\}$. Then there are $0 \leq j_0 < j_1 \leq n$ and a homeomorphism $h \in \mathcal{H}(X)$ such that $h(C_{j_0}) = C_{j_1}$. However this is a contradiction as $C_{j_0}$ intersects exactly $j_0$ nondegenerate components at one point and $C_{j_1}$ intersects exactly $j_1$ nondegenerate components at one point and $j_0 < j_1$.

Suppose $K_1, \ldots, K_n$ are nondegenerate non-open components of $X$, and let $x_i \in \partial K_i$ for $i = 1, \ldots, n$. By the previous we can denote $B_1, B_2, \ldots$ nondegenerate components of $X \setminus \bigcup_{i=1}^{n} K_i$ if there are any. Let $C$ be a countable dense subset of $X$ and let $C_0 = C \setminus \bigcup_{i=1}^{n} \partial K_i$. Then $C_0$ is a countable dense subset of $X$ if for any $i \in \mathbb{N}$ we have $\partial B_i \neq \emptyset$ we can add a point from $\partial B_i$ to $C_0$. Then $C_0$ is a countable dense subset such that $C_0$ does not intersect exactly $n$ nondegenerate non-open components at their boundary. Now define $C_i = C_0 \cup \{x_1, \ldots, x_i\}$. Clearly, all of the sets $C_i$ for $i = 0, \ldots, n$ are countable dense subsets of a different type, which is a contradiction. ☐

By the following example, we cannot improve this result to $\frac{1}{\omega}$-CDH spaces.

**Example 4.6.** *There exists $\frac{1}{\omega}$-CDH space with uncountably many nondegenerate components.*

*Proof.* Let $\{x_i; i \in I\}$ be an uncountable set. Let

$$X = (\{x_i; i \in I\} \times \{0,1\}) \cup ((0,1) \times \mathbb{N}).$$

On $(0,1) \times \mathbb{N}$ we define the topology as the product of the standard topologies on $(0,1)$ and $\omega$. For the points $y = (x_i, j)$ for $i \in I$ and $j \in \{0,1\}$ we define their neighborhood basis as

$$(\{x_i\} \times \{0,1\}) \cup ((0,1) \times \{k \in \mathbb{N}; k \geq n\})$$

for each $n \in \mathbb{N}$. Informally we can describe this space as taking copies of segments with shrinking lengths converging to a point and then splitting this point into uncountably many copies of the space $\{x, y\}$ with the indiscrete topology.

Components of $X$ are of the form $(0,1) \times \{k\}$ for some $k \in \mathbb{N}$ and $\{x_i\} \times \{0,1\}$ for some $i \in I$. Thus $X$ has uncountably many nondegenerate components. Since the spaces $(0,1) \times \{k\}$ for $k \in \mathbb{N}$ are CDH, every countable dense set $C \subset X$ is uniquely determined by its intersection with the set $\{x_i; i \in I\} \times \{0,1\}$. More precisely it is determined by the cardinality of this intersection and by the cardinality of the set $\{i \in I; |C \cap \{(x_i, 0), (x_i, 1)\}| = 2\}$. This implies that there is exactly $\omega$ types of countable dense sets. □

*Remark.* In Example 4.6, instead of uncountably many copies of $\{x, y\}$ we could glue an uncountable set with the cocountable topology. Thus obtaining a space with nonseparable nondegenerate component. This shows that we cannot prove Theorem 2.6 for $\frac{1}{\omega}$-CDH spaces.

Although we do not have Lemma 4.5 for $\frac{1}{\omega}$-CDH spaces in general, we can show it for metrizable spaces.

**Lemma 4.7.** *Let $X$ be a metrizable $\frac{1}{\omega}$-CDH space. Then $X$ has at most countably many nondegenerate components.*

*Proof.* Let $K$ be a nondegenerate component of $X$ and let $C \subset K$ be a countable dense subset in $K$. Since $K$ is connected, thus has no isolated points, $C$ is crowded, therefore by Theorem 1.7 we have $C \approx \mathbb{Q}$.

Let $C$ be a countable dense subset of $X$. For any nondegenerate component $K$ of $X$ intersecting $C$ we can add to $C$ a countable dense subset of $K$. Thus obtaining a countable dense set $C$ of $X$ such that if $K$ is a nondegenerate component of $X$ then $K \cap C$ is either empty or dense in $K$, thus homeomorphic to $\mathbb{Q}$. Now suppose $X$ has uncountably many nondegenerate components. This means that there exists nondegenerate component $K$ of $X$ such that $K \cap C = \emptyset$. Let $B \subset K$ be a countable dense in $K$. By Corollary 4.3, there exists $\{B_i \subset B; i \in (0,1)\}$ such that $B_i$ are nowhere dense in $B$, thus nowhere dense in $K$, and pairwise nonhomeomorphic. But then the sets $C_i = C \cup C_i$ for $i \in (0,1)$ are pairwise nonhomeomorphic, which is a contradiction. □

*Remark.* Note that the proof of Lemma 4.7 actually shows that for $X$ metrizable $\frac{1}{\omega}$-CDH space any countable dense subset $C \subset X$ intersects every nondegenerate component of $X$.

In proving Theorem 2.6, we used that components of CDH spaces are open and therefore separable. We need separability of components even for $\frac{1}{n}$-CDH if we want to show that components of $\frac{1}{n}$-CDH spaces are $\frac{1}{n}$-CDH. However, as the components of $\frac{1}{n}$-CDH spaces do not need to be open we do not get separability as easily. Nonetheless we can still show separability for the components.

**Lemma 4.8.** *Let $X$ be $\frac{1}{n}$-CDH for $n \geq 2$ and denote*

$$\mathcal{K} = \{K; K \textit{ is nondegenerate, non-open component of } X\}.$$

*Then the set $\bigcup_{K \in \mathcal{C}} \partial K$ has at most $n-1$ elements. Moreover if the set $\bigcup_{K \in \mathcal{C}} \partial K$ has exactly $n-1$ elements then the space $X \setminus \bigcup_{K \in \mathcal{C}} \partial K$ is CDH.*

*Proof.* To strive for a contradiction, suppose there is $\{x_1, \ldots x_n\} \subset \bigcup_{K \in \mathcal{C}} \partial K$. Let $C$ be a countable dense subset of $X$. By Lemma 4.5, the set $\mathcal{K}$ is finite, therefore the set $C_0 = C \setminus \bigcup_{K \in \mathcal{C}} \partial K$ is countable dense in $X$. For $i = 1, \ldots, n$ define $C_i = C_0 \cup \{x_1, \ldots, x_i\}$. Then clearly all the sets $C_i$ are countable dense sets of different types, which is a contradiction.

Now suppose that $\bigcup_{K \in \mathcal{C}} \partial K = \{x_1, \ldots, x_{n-1}\}$ and let $A, B \subset X \setminus \bigcup_{K \in \mathcal{C}} \partial K$ be countable dense subsets. Then clearly $A$ and $B$ are countable dense subsets of $X$. Now let $B_i = B \cup \{x_1, \ldots, x_i\}$ for $i = 1, \ldots, n-1$. Then there are two distinct sets $C_1, C_2 \in \{A, B, B_1, \ldots, B_{n-1}\}$ and $h \in \mathcal{H}(X)$ such that $h(C_1) = C_2$. However, since the only sets that do not intersect $\bigcup_{K \in \mathcal{C}} \partial K$ are $A, B$ and all the sets $B_i$ intersect $\bigcup_{K \in \mathcal{C}} \partial K$ in a different number of points, the only possibility is $\{C_1, C_2\} = \{A, B\}$. Without loss of generality we may assume $C_1 = A$ and $C_2 = B$. Note that the set $\bigcup_{K \in \mathcal{C}} \partial K$ is preserved by any element of $\mathcal{H}(X)$. Thus we have

$$h|_{X \setminus \bigcup_{K \in \mathcal{C}} \partial K} \in \mathcal{H}(X \setminus \bigcup_{K \in \mathcal{C}} \partial K)$$

and

$$h|_{X \setminus \bigcup_{K \in \mathcal{C}} \partial K}(A) = B.$$

□

**Lemma 4.9.** *Let $X$ be $\frac{1}{n}$-CDH for some $n \geq 2$. Then every component of $X$ is separable.*

*Proof.* Let $K$ be a component of $X$. If $K$ is degenerate then $K$ is clearly separable. So suppose $K$ is nondegenerate. Let $C$ be a countable dense subset of $X$ and define $C_K = (C \cap K) \cup \partial K$. By Lemma 4.8, the set $C_K$ is countable and we have $\overline{C_K} \supset \operatorname{Int}(K) \cup \partial K = \overline{K} = K$. □

We now have everything that we need to prove the following.

**Theorem 4.10.** *Let $X$ be $\frac{1}{n}$-CDH for some $n \geq 2$. Then every component of $X$ is at most $\frac{1}{n}$-CDH.*

*Proof.* Let $K$ be a component of $X$. If $K$ is degenerate then it is clearly at most $\frac{1}{n}$-CDH. So suppose $K$ is nondegenerate. By Lemma 4.9 the space $K$ is separable. Let $A_1, \ldots A_n$ be countable dense subsets of $K$. And let $\mathcal{C} = \{h(K); h \in \mathcal{H}(X)\}$.

By Lemma 4.5, the set $\mathcal{C}$ is at most countable. Let $\{C_i\}_{i=1}^{\alpha}$, where $\alpha \in \mathbb{N} \cup \{\infty\}$, be an enumeration of the elements of $\mathcal{C}$. For every $i$ let $h_i \in \mathcal{H}(X)$ such that $h_i(K) = C_i$, also denote $A_j^i = h_i(A_j)$ for $j = 1, \ldots, n$.

Let $S$ be a countable dense subset of $X$ and for $j = 1, \dots, n$ define

$$S_j = (S \setminus \bigcup_{i=1}^{\alpha} C_i) \cup (\bigcup_{i=1}^{\alpha} A_i^j).$$

Then $S_j$ is a countable dense subset of $X$ for every $j = 1, \dots, n$. Thus there exist $1 \le j_1 < j_2 \le n$ and $h \in \mathcal{H}(X)$ such that $h(S_{j_1}) = S_{j_2}$. There also exists $i$ such that $h(K) = C_i$. Let $f = h|_K : K \to C_i$ and $g = h_i|_K : K \to C_i$ then we have $g^{-1} \circ f(C_{j_1}) = C_{j_2}$. ☐

*Remark.* We cannot improve Theorem 4.10 to specify the exact number of types of countable dense subsets of the components. The space $[0, 1) \oplus (0, 1)$ is clearly $\frac{1}{2} - CDH$, however one of the components is CDH and the other $\frac{1}{2}$-CDH.

Spaces with a bounded number of types of countable dense subset still have to have many homeomorphisms. Using the structure provided by these homeomorphisms, we can again bound the cardinality of Hausdorff spaces that are at most $\frac{1}{\omega}$-CDH.

**Theorem 4.11.** *Let $X$ be at most $\frac{1}{\omega}$-CDH. If $X$ is Hausdorff then $|X| \le \mathfrak{c}$*

*Proof.* Suppose $X > \mathfrak{c}$. Let $M \subset X$ be a countable dense subset and put $Y = X \setminus M$. For each $y \in M$ let $Z_y = M \cup \{y\}$. Since $X$ is either $\frac{1}{n}$-CDH or $\frac{1}{\omega}$-CDH and $|X| > \mathfrak{c}$, there exist set $B \subset Y$ such that $|B| > \mathfrak{c}$ and for any $y_1, y_2 \in B$ there is $h \in \mathcal{X}$ such that $h(Z_{y_1}) = Z_{y_2}$. Fix some point $b \in B$ and denote $A = Z_b$. For any $y \in B \setminus \{b\}$ we can find $h_y \in \mathcal{H}(X)$ such that $h_y(A) = Z_y$. Let $x_y = h_y^{-1}(y)$, put $A_y = A \setminus x_y$ and denote $g_y = h_y|_{A_y} : A_y \to M$. We have that $A_y$ is dense in $A$ thus also in $X$.

Since $Y > \mathfrak{c}$ and both the sets $M$ and $A$ are countable and the number of subsets of a countable set and number of mappings between two countable sets does not exceed $\mathfrak{c}$, there are $p, q \in B \setminus b$ such that $p = q$, $g_p = g_q$, and $A_p = A_q$. However, then the homeomorphism $f = g_p = g_q$ defined on a dense subset of a Hausdorff space $X$ can be extended in two different ways on the whole space, which is a contradiction. ☐

We have shown that if for a CDH-space we have a local topological property then we can find a clopen subset such that all the points of that subset have the desired local property. We cannot do this for $\frac{1}{n}$-CDH or $\frac{1}{\omega}$-CDH spaces. the following example is $\frac{1}{2}$-CDH, however the set of isolated points is not clopen.

**Example 4.12.** *The space $X = \{1/n; n \in \mathbb{N}\} \cup \{0\}$ with the standard topology is $\frac{1}{2}$-CDH*

*Proof.* A countable dense subset of $X$ is either $\{1/n; n \in \mathbb{N}\}$ or the whole space $X$. ☐

However, we still have some control over the set of isolated points of $\frac{1}{n}$-CDH space.

**Theorem 4.13.** *Let $X$ be $\frac{1}{n}$-CDH and let $I$ be the set of isolated points of $X$. Then the set $\partial I$ has at most $n - 1$ points and the space $X \setminus \bar{I}$ is at most $\frac{1}{n}$-CDH.*

*Proof.* The space $X \setminus \bar{I}$ is clearly separable. Let $C_1, \ldots, C_n$ be countable dense subsets of $X \setminus \bar{I}$. Then $B_i = C_i \cup I$ for $i = 1, \ldots, n$ is countable dense subset of $X$ thus there exist $1 \leq i_1 < i_2 \leq n$ and $h \in \mathcal{H}(X)$ such that $h(B_{i_1}) = B_{i_2}$. Note that the set $\bar{I}$ is preserved by any homeomorphism thus we have $g = h|_{X\setminus\bar{I}} \in \mathcal{H}(X \setminus \bar{I})$ and $g(C_{i_1}) = C_{i_2}$.

Suppose that $\{x_1, \ldots, x_n\} \subset \partial I$. Let $C$ be a countable dense subset of $X$ and let $C_0 = C \setminus \partial I$. Then $C_0$ is also a countable dense in $X$. Now for $i \in \{1, \ldots, n\}$ define $C_i = C_0 \cup \{x_1, \ldots, x_i\}$. Then for all $j \in \{0, \ldots, n\}$ the sets $C_j$ are countable dense and pairwise nonhomeomorphic, since for every $j \in \{0, \ldots, n\}$ the set $C_j$ intersect $\partial I$ in exactly $j$ many points. $\square$

*Remark.* In the same way we could get a parallel with Theorem 2.13.

In Example 3.30 we constructed a space whose square is not CDH and it is very natural to ask how many types of countable dense sets does the square of the space have.

**Example 4.14.** *Let $X$ be an uncountable $\lambda$-set. Then the space $X \times 2^\omega$ has $\mathfrak{c}$ many types of countable dense set.*

*Proof.* Since $|X \times 2^\omega| = \mathfrak{c}$, it is enough to find $\mathfrak{c}$ many countable dense subset of $X \times 2^\omega$ of a different type.

Let $x \in 2^\omega$ and let $A, B \subset 2^\omega$ be open disjoint such that $A \cup B \cup \{x\} = 2^\omega$ and $\partial A = \partial B = x$. Note that by Theorem 1.5 we have $A \approx B \approx 2^\omega \setminus \{0\}$.

By Remark 3.9, we can find $Q_0 \subset X \times A$ countable dense such that for no $E \subset Q_0$ we have $\overline{E} \approx 2^\omega$. Let $\{U_n\}_{n\in\omega}$ be a countable base of the space $X \times B$. For $n \in \omega$ let $F_n \subset U_n$ be countable such that $\overline{F_n} \approx 2^\omega$. Let $Q_1 = \bigcup_{n\in\omega} F_n$ and put $D = Q_0 \cup Q_1$. Then the set $D$ is by construction a countable dense subset of $X \times 2^\omega$. By Corollary 4.3, there is a collection $\{C_r; r \in (0, 1)\}$ of countable pairwise nonhomeomorphic nowhere dense subsets of $X \times \{x\}$. For $r \in (0, 1)$ let $D_r = D \cup C_r$.

Now let $0 < p < r < 1$ and suppose there is $h \in \mathcal{H}(X \times 2^\omega)$ such that $h(D_p) = D_r$. Let $a \in X \times \{x\}$. First suppose $h(a) \in X \times A$, then we can find an open neighborhood $V_0$ of $a$ such that $h(V_0) \subset X \times A$. We have that $V_0 \cap X \times B \neq \emptyset$ thus there exists $n \in \omega$ such that $U_n \subset V_0$ thus also $F_n \subset V_0$ which means $h(F_n) \subset Q_0$. But this contradicts the fact that for no countable subset $E \subset Q_0$ we have $\overline{E} \approx 2^\omega$. Now suppose $h(a) \in X \times B$. Then there is an open neighborhood $V_1$ of $a$ such that $h(V_1) \subset X \times B$, thus also $h(V_1 \cap X \times A) \subset X \times B$. Again, we can find $n \in \omega$ and $F_n \subset h(V_1 \cap X \times A)$. However, this means $h^{-1}(F_n) \subset Q_0$, which cannot be. This means that $h(X \times \{x\}) = X \times \{x\}$, thus $h(C_p) = C_r$, which is a contradiction. Thus all the sets $D_r$ for $r \in (0, 1)$ are of a different type. $\square$

Example 4.14 shows that for an uncountable $\lambda$-set $X$ the space $(X \oplus 2^\omega)^2$ has $\mathfrak{c}$ many types of countable dense sets. Thus this example also partially solves [20, Problem 28] *For which cardinals $\kappa$ is there a zero-dimensional CDH space $X$ such that $X^2$ has exactly $\kappa$ many types of countable dense subsets*, which also has been originally asked by Medini [27].

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