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DIPLOMOVÁ PRÁCE

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Delta-monotonní funkce více proměnných

Katedra matematické analýzy

Vedoucí diplomové práce: **Prof. RNDr. Jan Malý, DrSc.**

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Abstrakt: V této práci studujeme monotonní zobrazení mezi Banachovými prostory, konečné a nekonečné dimenze, a zobrazení která jsou rozdíly monotonních (DM). Dokazujeme odhad Radó-Reichelderferova typu pro monotonní zobrazení v konečné dimenzi, jenž se přenáší i na rozdíly monotonních zobrazení. Tím podáváme alternativní důkaz o Fréchetovské diferencovatelnosti s.v. DM zobrazení. Dokazujeme odhad Morreyova typu pro distributivní derivaci monotonních zobrazení, který lze přenést i na DM zobrazení. Je ukázáno, že zobrazení mezi konečně dimenzionálními prostory, které je lokálně DM, je DM i globálně. Zavádíme a studujeme novou třídu tzv. UDM zobrazení mezi Banachovými prostory, která je zobecněním křivek s konečnou variací.
Klíčová slova: zobrazení s konečnou variací; monotonní zobrazení; DM zobrazení; UDM zobrazení; Radó-Reichelderferova podmínka.

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Abstract: We study the classes of mappings between finite and infinite dimensional Banach spaces which are monotone and mappings which are differences of monotone mappings (DM). We prove the Radó-Reichelderfer estimate for the monotone mappings in finite dimensional spaces which remains valid for the DM mappings. This provides an alternative proof of the Fréchet differentiability a.e. of DM mappings. We establish the Morrey-type estimate for the distributional derivative of monotone mappings. We prove that a locally DM mapping between finite dimensional spaces is also globally DM. We introduce and study the new class of so called UDM mappings between Banach spaces, which generalizes the concept of curves of finite variation.
Keywords: mappings of bounded variation; monotone mapping; DM mapping; UDM mapping; Radó-Reichelderfer condition.

Preface

This thesis is devoted to the study of monotone mappings and their differences on finite and infinite dimensional Banach spaces. The class of the monotone mappings from a reflexive Banach space to its dual space was at first introduced in the works by F. Browder and G. Minty as a powerful method for solving nonlinear partial differential equations. The most simple examples of monotone mappings are the Gateaux derivatives of the convex functionals, such mappings are called the potential monotone mappings and they have many specific properties. The proof of the surjectivity of such a potential monotone operator is based on a simple variational argument. The main contribution of the methods of Browder and Minty was in the removing of the assumption of the potentiality of the considered operators. It is well known fact that the smooth vector field in finite dimension is potential if and only if its Jacobian matrix is symmetric. Thus it seems that the assumption of potentiality is very restictive. Recall the famous Browder-Minty theorem about surjectivity of a monotone coercive continuous operator, which is sufficient for the proof of the existence of the weak solution for a wide class of nonlinear problems. The methods were developed and generalized by H. Brézis, R. Kačurovski, L. Leray, J. L. Lions, J. P. Aubin and by many others. In some cases the concept of accretive mappings, which represent a generalization of the Hilbert space monotonicity in a bit different way than the Banach space monotonicity, appeared to be more suitable for some types of problems. The accretive mappings and their differences are also briefly discussed here.

Later the monotone operators were studied from the point of view of applications in variational inequalities, differential inclusions, optimization theory and connections to the convex and the nonlinear analysis. This naturally led to the introducing of set-valued monotone operators which were investigated in the works by T. Rockafellar, S. Fitzpatrick, J. Borwein and others and they are still an active area of the contemporary research. Also the theoretic aspects of the monotone operators were treated and a lot of interesting and deep results about the description of the points of the multiplicity, the continuity and the differentiability and the maximality of the monotone operators were proved by T. Rockafellar, R. Phelps, J. Borwein, D. Preiss, L. Veselý and L. Zajíček. The real analytic approach to the monotone operators on Euclidean spaces is presented in the paper [1] by L. Alberti and G. Ambrosio. In this paper there are discussed relations to the geometric measure theory and to mappings of bounded variation. In the papers of L. Kovalev there is studied a special class of monotone mappings, so called δ-monotone mappings which poses many nice properties. There are very interesting connections between δ-monotone mappings and quasiconformal mappings, quasisymmetric mappings and mappings of finite distortion.

The functions which can be represented as a difference of two convex functions (now standardly called d.c. functions) were probably at first introduced by Russian geometer A. D. Aleksandrov. He studied mainly the geometric properties of surfaces in three dimensional space which can be viewed as a graph of d.c. functions. The theory of d.c. functions was further developed by P. Hartman who proved the composition theorem for the d.c. functions. There is a simple characterization of the real d.c. functions of one real variable. The class of d.c. functions coincides with the indefinite integrals of function of locally finite variation as follows from the classical C. Jordan's result. The similar characterization for vector valued d.c. functions of one real variable (d.c. curves) is shown in [25], but no effective characterization of the d.c. functions defined on higher dimensional spaces is known. The d.c. functions were applied in the nonsmooth optimization by V. F. Demyanov and A. M. Rubinov. The contemporary research in the theory of d.c. functions with the connection to the optimization theory focuses rather on different algorithms for minimization problems of the mathematical programming evolving d.c. functions. The generalization of the d.c. functions to arbitrary Banach spaces is due to L. Veselý and L. Zajíček and is in detail studied in their paper [25]. The further development is contained in the papers [8], [26].

The class of the DM mappings seems to be studied for the first time in this thesis. Roughly speaking the basic examples of the DM mappings arise by the differentiating of the d.c. function. Thus the DM mappings is a system of mappings from a Banach space to its dual space which can be written as a difference of two monotone operators but the assumption of the potentiality is removed. In contrast to the one-dimensional case the requirement that the mapping is DM is more restrictive than that it is in BV_{loc}. The example is provided. It is well known that there exist a functions of bounded variation of two or more variables which are not bounded. Such mappings provide the simplest examples of BV mappings which are not DM. The DM mappings of course inherit many important properties of the monotone mappings such as the continuity, the differentiability and belonging to the space of the functions of bounded variation. In this thesis we also discuss the so called Radó-Reichelderfer and the Morrey type estimates for the monotone mappings which remain valid for the DM mappings. Unfortunately the class of the DM mappings lacks some stability-type properties of the d.c. mappings, for instance an analogue of the composition theorem for the d.c. mappings does not hold. The example of the DM mapping whose composition with the linear mapping is not DM is presented.

This was the main motivation for the introducing the class of the UDM mappings which generalizes the concept of curves with the locally finite variation. The advantage of such mappings is that they can be defined between arbitrary Banach spaces and enjoy some nice properties in comparison with DM mappings. As a difference to the d.c. mappings not all monotone mappings are UDM mappings. The counterexample is provided. In the case of the d.c. mapping it is obvious that every convex function is a d.c. function. The source of such difficulties is probably in the fact that the basic concept for the DM and UDM mappings is an monotone operator between a Banach space and its dual space. This operator seems to be a more complicated object than a convex function which is the central concept for the theory of the d.c. mappings.

As declared the DM and the UDM mappings are probably for the first time studied in this thesis, thus many natural questions remained unanswered and some of them are written in the last section as open problems.

List of notation

$\mathcal{C}(U)$	space of continuous functions on a topological space U,
$\mathcal{C}_c(U)$	space of compactly supported functions on a topological space U,
$\mathcal{C}_0(U)$	space of continuous functions lying in the closure of the space $\mathcal{C}_c(U)$ with respect to the metric of the uniform convergence,
$L^p(S, \mathcal{S}, \mu)$	space of measurable functions on the measurable space $(S, \mathcal{S}, \mu)$, whose absolute value is integrable with the p-th power,
$W^{1,p}(\Omega)$	Sobolev space of functions on the open set Ω,
$\langle x^*; x\rangle$	duality pairing between a Banach space X and its dual space X^*, scalar product for X being a Hilbert space
$\delta f(x; v)$	directional derivative of a mapping f at a point x in a direction v
$\delta f(x)$	Gateuax derivative of a mapping f at a point x,
$f'(x)$	Fréchet derivative of a mapping f at a point x,
$\lvert\lvert x\rvert\rvert_X, \lvert\lvert x\rvert\rvert, \lvert x\rvert_X, \lvert x\rvert$	norm of a point x of a Banach space X,
$B(a, r)$	open ball with the center a and the radius r,
B_X	closed unit ball in the Banach space $(X, \lvert\cdot\rvert_X)$,
$\mathcal{L}^n(E)$, $\lvert E\rvert$	Lebesgue measure of a measurable set E,
$\mathcal{L}(X, Y)$	space of bounded linear operators between Banach spaces X and Y,
$f_{\vert C}$	restriction of a mapping $f : A \to B$ on the set $C \subset A$,
$\bigvee_a^b f$	variation of a mapping $f : (a; b) \to Y$,

$\bigvee(f, L)$	variation of the restriction $f_{\mid L}$, where $L \subset X$ is a line segment,
$A : T \to 2^S$	multi-mapping between sets T and S i.e. $At \subset S$ not necessarily singleton,
$\mathrm{Gr}(A)$	graph of the multi-mapping $A : T \to 2^S$ i.e. the set $\{(t, s); s \in At\}$,
$⨍_M u$, u_M	the average of L^1 function over a measurable set M with $0 < \lvert M\rvert < \infty$ i.e. $u_M := \frac{1}{\lvert M\rvert} \int_M u$,
$\mathrm{lip}(f, E)$	the lipschitz constant of a mapping f on the set E,
$U \subset\subset V$	$U \subset \overline{U} \subset V$.

1 Preliminaries

All Banach spaces considered in this text are real.

Definition 1 Let U be an open subset of $\mathbb{R}^n$. We say that $u \in L^1(U;\mathbb{R}^d)$ is a *function of bounded variation* if the distributional gradient of u is (representable by) a Radon $\mathbb{R}^{d\times n}$-valued measure in U. We denote this function space by $BV(U;\mathbb{R}^d)$ (if $d=1$ we write briefly $BV(U)$) and equip it with the norm

$$||u||_{BV(U;\mathbb{R}^d)} := ||u||_{L^1(U;\mathbb{R}^d)} + ||Du||_{\mathcal{M}(U;\mathbb{R}^{d\times n})}.$$

We say that u is a function of *locally bounded variation* and write $u \in BV_{loc}(U;\mathbb{R}^d)$ if $u \in BV(\tilde{U};\mathbb{R}^d)$ for each $\tilde{U} \subset\subset U$.

Definition 2 Let $u \in L^1_{loc}(U;\mathbb{R}^m)$ be a mapping. We say that $a \in \mathbb{R}^m$ is the L^1*-approximate limit* of u at x_0 if

$$\lim_{r\searrow 0} \fint_{B(x_0,r)} |u(y) - a| \, dy = 0. \tag{1}$$

The set of points $x_0 \in U$ where such a does not exist is called the L^1*-approximate discontinuity set* and denoted by S_u.

Remark 3 Notice that the points of the set $U \setminus S_u$ for which

$$\lim_{r\searrow 0} \fint_{B(x_0,r)} |u(y) - u(x)| \, dy = 0.$$

are standardly called the *Lebesgue points of the function* u and it is well known that the complement of the set of all Lebesgue points in Ω is a Lebesgue null set.

Definition 4 For given $u \in L^1_{loc}(U;\mathbb{R}^d)$ and $x \in U \setminus S_u$, denote by $\tilde{u}(x)$ the L^1-approximate limit of u at x. We say that u is L^1*-approximate differentiable* at x if there is a $d \times n$ matrix L such that

$$\lim_{r\searrow 0} \fint_{B(x,r)} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{r} \, dy = 0. \tag{2}$$

The set of such points x, where the mapping u is L^1-approximate differentiable is denoted by D_u and the matrix $L \in \mathbb{R}^{d\times n}$ is denoted by $\mathrm{D_{ap}}u(x)$ and called the L^1*-approximate differential*.

Remark 5 There are available other more general definitions of the differentiability. The definition which uses the densities of "bad" sets

$$\left\{ y \in U \setminus \{x\}; \frac{|u)y) - \tilde{u}(x) - L(y-x)}{|y-x|} > \varepsilon \right\},$$

leads to the *approximate differentiability* (see [9]). Since we work only with functions of bounded variation, that are differentiable in the stronger sense, we will use the definition based on (2).

THEOREM 6 (Calderón-Zygmund; see [3]) *Let u be a function of the class $BV(U;\mathbb{R}^d)$. Then u is L^1-approximate differentiable at $\mathcal{L}^n$- a.e. points of U and the L^1-approximate differential $\mathrm{D}_{\mathrm{ap}}u$ is the density of the absolutely continuous part of Du with respect to the Lebesgue measure.*

REMARK 7 It can be easily seen (for details consult [3]) that the function u is L^1-approximately differentiable at a point x with L being its L^1-approximate differential if and only if the rescaled functions

$$u_r(y) := \frac{u(x+ry) - \tilde{u}(x)}{r}$$

converge in the L^1_{loc}-topology to the linear mapping $y \mapsto Ly$.

THEOREM 8 (Kirzbraun; see [9]) *Let $S \subset \mathbb{R}^n$ be an arbitrary set and let $f : S \to \mathbb{R}^m$ be a Lipschitz continuous mapping. Then there is a mapping $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$, such that $\overline{f}_{|S} = f$ and $\operatorname{lip}(f,S) = \operatorname{lip}(\overline{f},\mathbb{R}^n)$.*

The following theorem is a known tool for handling with BV functions, since we were not able to find a suitable reference we sketch the proof.

THEOREM 9 *Let $\Omega \subset \mathbb{R}^n$ be an open set and u be a function of the class $BV_{loc}(\Omega;\mathbb{R}^m)$. Let $B := B(z,r) \subset\subset \Omega$ be a ball. For $\mathcal{L}^n$-a.e $\zeta \in B$ it is*

$$\left| \fint_B u(y)\,dy - u(\zeta) \right| \le 2r \int_0^1 \frac{|Du|(B_t)}{|B_t|}\,dt, \tag{3}$$

where $B_t := B(\zeta + t(z-\zeta), tr)$.

Proof. We have $B_1 = B$. Using the terminology from [23] we have that B_t *shrinks nicely* to the point ζ as $t \to 0$. It is not difficult to see that for $0 < s < t \le 1$ we have

$$\overline{B_s} \subset B_t. \tag{4}$$

Indeed, we can for a moment assume $z = 0, \zeta = \zeta_1 \mathbf{e}_1$, $0 \le \zeta_1 < r$, since for $1 \ge t > s > 0$ we have

$$\zeta_1(1-t) + tr > \zeta_1(1-s) + sr,$$

we conclude $B_s \subset\subset B_t$.

Let ζ be a Lebesgue point of u. By [23]

$$\lim_{\epsilon \to 0+} \fint_{B_\epsilon} u(y)\,dy = u(\zeta). \tag{5}$$

Assume for a moment that the function u is smooth (i.e. of the class $\mathcal{C}^\infty(\Omega)$). Denote

$$\phi(t) := \fint_{B_t} u(s)\,ds.$$

We have

$$\phi(t) = \fint_{B_1} u(\zeta + t(y-\zeta))\,dy$$

thus

$$\frac{d\phi}{dt} = \fint_{B_1} \nabla u(\zeta + t(y-\zeta))(y-\zeta)\,dy.$$

This implies for $0 < \epsilon < 1$

$$\phi(1) - \phi(\epsilon) = \int_\epsilon^1 \fint_{B_1} \nabla u(\zeta + t(y-\zeta))(y-\zeta)\,dy\,dt. \tag{6}$$

Since $\zeta, y \in B_1$ and $B_t \subset B_1$, $t \in (0;1]$ we infer from (6)

$$|\phi(1) - \phi(\epsilon)| \leq 2r \int_\epsilon^1 \fint_{B_1} |\nabla u(\zeta + t(y-\zeta))|\,dy\,dt.$$

For general u we can find a sequence of smooth functions $(u_j)_{j\in\mathbb{N}}$ such that

$$u_j \to u, \quad \text{in } L^1_{loc}(\Omega) \tag{7}$$

and

$$|Du_j|(B_1) \to |Du|(B_1). \tag{8}$$

(This is so called *strict convergence* see [3].) Let denote $\mu := |Du|$, $\mu_j := |Du_j|$. By the calculations for smooth functions we have

$$\begin{aligned}
|\phi_j(1) - \phi_j(\epsilon)| &\leq 2r \int_\epsilon^1 \fint_{B_1} |\nabla u_j(\zeta + t(y-\zeta))|\,dy\,dt, \\
&= 2r \int_\epsilon^1 \fint_{B_t} |\nabla u_j(\xi))|\,d\xi\,dt \\
&= 2r \int_\epsilon^1 \frac{\mu_j(B_t)}{|B_t|}\,dt,
\end{aligned} \tag{9}$$

where we have denoted

$$\phi_j(t) := \fint_{B_t} u_j(s)\,ds.$$

Let ϵ be fixed. From (7) we infer that the left hand side of (9) converges for $j \to \infty$ to

$$\left| \fint_{B_1} u(y)\,dy - \fint_{B_\epsilon} u(s)\,ds \right|. \tag{10}$$

Now we realize that $\mu_j \overset{*}{\rightharpoonup} \mu$. (This is a property of the strict convergence, see [3].) Thus we have

$$\mu(\overline{B_t}) \geq \limsup \mu_j(\overline{B_t}) \geq \limsup \mu_j(B_t) \tag{11}$$

and

$$\mu(B_t) \leq \liminf \mu(B_t). \tag{12}$$

We infer that $\mu(\partial B_t) > 0$ at most for countably many $t \in (0;1]$. Indeed, if it would not be true, since the boundaries if B_t and B_s are disjoint by (4), we would obtain a contradiction with $\mu B_1 < \infty$. Since for almost all $t \in (0;1]$ it is $\mu(\partial B_t) = 0$ we have by (11) and (12) for almost all $t \in (0;1]$ that

$$\mu_j(B_t) \to \mu(B_t). \tag{13}$$

We infer from (8) that

$$\frac{\mu_j(B_t)}{|B_t|} \leq \frac{1}{\epsilon^n r^n |B(0,1)|} \int_{B_t} d|Du_j| \leq \frac{1}{\epsilon^n r^n |B(0,1)|} \int_{B_1} d|Du_j| \leq \frac{K}{\epsilon^n r^n |B(0,1)|}.$$

Thus we have an integrable majorant and (13) combined with the dominated convergence theorem implies that the right hand side of (9) converge to

$$2r \int_\epsilon^1 \frac{|Du|(B_t)}{|B_t|}\, dt. \tag{14}$$

Thus by (10) and (14) we have

$$\begin{aligned} \left| \fint_{B_1} u(y)\, dy - \fint_{B_\epsilon} u(s)\, ds \right| &\leq 2r \int_\epsilon^1 \frac{|Du|(B_t)}{|B_t|}\, dt \\ &\leq 2r \int_0^1 \frac{|Du|(B_t)}{|B_t|}\, dt. \end{aligned}$$

From (5) we obtain by the limit passage for $\epsilon \to 0$

$$\left| \fint_B u(y)\, dy - u(\zeta) \right| \leq \int_0^1 2r \frac{|Du|(B_t)}{|B_t|}\, dt.$$

This concludes the proof. $\square$

Theorem 10 (Poincaré inequality; see [3]) *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then there is a constant γ such that for any $u \in BV(\Omega;\mathbb{R}^m)$ and for any ball $B(a,\rho) \subset \Omega$*

$$\int_{B(a,\rho)} |u - u_{B(a,\rho)}|\, dx \leq \gamma\rho \int_{B(a,\rho)} d|Du|.$$

The following theorem can be viewed as a generalization of the well known theorem about the Lebesgue points of a locally integrable function.

THEOREM 11 ([23]) *Let X be a normed linear space and $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n; X)$ a vector measure which is singular with respect to the Lebesgue measure. Then*

$$\lim_{r\to 0+} \frac{|\mu|(B(x,r))}{|B(x,r)|} = 0 \tag{15}$$

for $\mathcal{L}^n$-almost all $x \in \mathbb{R}^n$.

The following lemma presents a well known fact from the theory of Sobolev functions. Since we were not able to find a reference, we sketch the proof.

LEMMA 12 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and $B(0,R)$ be a given ball. Then there is a function u_R and $c > 0$ such that $u = u_R$ on $B(0,R)$, $u_R = 0$ in the exterior of the larger ball $B(0,cR)$ and $\operatorname{lip}(u_R, \mathbb{R}^n) \leq \operatorname{lip}(u, \mathbb{R}^n)$. (The constant c depends on the function u.)*

Proof. We define the function

$$v(x) := \begin{cases} u(x), & x \in B(0,R) \\ 0, & x \in \mathbb{R}^n \setminus B(0,cR), \end{cases}$$

where the constant c is chosen as large as

$$\operatorname{lip}(v, B(0,R) \cup \mathbb{R}^n \setminus B(0,cR)) \leq \operatorname{lip}(u, \mathbb{R}^n).$$

Theorem 8 gives a function u_R such that u_R is an extension of v and the Lipschitz constant of u_R does not exceed the Lipschitz constant of v. This concludes the proof. □

DEFINITION 13 Let X, Y be Banach spaces and let $D \subset X$ be an open convex set. A mapping $F : D \to Y$ is called a *d.c. mapping* if there exists a continuous convex function $f : D \to \mathbb{R}$ such that for every $y^* \in B_{Y^*}$ the function

$$f + y^* \circ F : D \to \mathbb{R}$$

is a continuous convex function.

We need some basic tools of the descriptive set theory.

DEFINITION 14 Let M be a complete separable metric spaces. The set $A \subset M$ is called *analytic set* if there is a complete separable metric space N and a Borel subset B of $M \times N$ such that

$$A = \pi_M B,$$

where $\pi_M \colon (m,n) \in M \times N \mapsto m$ is the projection of $M \times N$ onto M.

Proposition 15 *Let M, N be complete separable metric spaces, let*

$$g : M \to N$$

be a continuous mapping and let $A \subset M$ be an analytic set. Then $g(A) \subset N$ is an analytic set. Analytic subsets of $\mathbb{R}^n$ are Lebesgue-measurable.

2 Monotone multi-mappings

2.1 Basic properties

In the sequel the symbol $T : M \to 2^V$ will denote that Tm is a nonempty subset of V for every $m \in M$. We can find in a literature concerning monotone operators the convention to write $T : X \to 2^{X^*}$ and keep in the mind that Tx may be empty but we will not use this convention.

Definition 16 Let X be an arbitrary Banach space with the dual space X^*. We say that a multi-mapping $T : \mathrm{Dom}(T) \to 2^{X^*}$ is the *monotone multi-mapping* if for every $x_1, x_2 \in \mathrm{Dom}(T)$ and every $x_1^* \in Tx_1, x_2^* \in Tx_2$ the following inequality holds:

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0. \tag{16}$$

The set $\mathrm{Dom}(T)$ is the set of all point $x \in X$ such that $Tx \neq \emptyset$ and we call it the *effective domain of* T. We call T the *maximal monotone multi-mapping* if there is no proper enlargement S of T which preserves the monotonicity (the proper enlargement means that there is an element of the graph of S which is not contained in the graph of T). We denote the class of all maximal monotone multi-mappings by $\mathfrak{Mon}(X)$. We say that T is *strictly monotone* if for every $x_1 \neq x_2$ the inequality in (16) is strict.

An easy consequence of the definition is the following proposition.

Proposition 17 *The set of all monotone mappings forms a convex cone in the space of all mappings from the space X to the dual space X^*.*

Remark 18 It is common in the theory of monotone multi-mappings to identify the multi-mapping T with its graph

$$\mathrm{Gr}(T) := \{(x, x^*) \in X \times X^*, x^* \in Tx\}.$$

Having in mind this convention we can speak about monotone subsets of $X \times X^*$ as well as about monotone multi-mappings.

Remark 19 We can show by Zorn's lemma that each monotone multi-mapping has a maximal monotone extension (in general not unique). To see this take a fixed monotone multi-mapping T and let $\mathcal{T}$ be a system of all monotone multi-mappings whose graph contains the graph of T. We consider $\mathcal{T}$ partially ordered by the graph-inclusion. If $\mathcal{T}'$ is an arbitrary linearly ordered subsystem of $\mathcal{T}$ then the set $\bigcup \mathcal{T}'$ is the graph of a monotone multi-mapping which is an upper bound for the chain $\mathcal{T}'$. Thus by Zorn's lemma there is a maximal element $\overline{T}$ which is the desired maximal monotone extension of T.

Remark 20 It is easily seen from the definition of the monotone multi-mapping that the inversion of a monotone mapping T, defined as the mapping

$$T^{-1} : X^* \to 2^X,\ T^{-1}x^* = \{x \in X; x^* \in Tx\},$$

is monotone.

Definition 21 (Minty; see [17]) Let X be a Hilbert space and let $M \subset X$ be its arbitrary subset. Let $A : M \subset X \to 2^X$ be a multi-mapping. The *Cayley transformation*

$$\Gamma : X \times X \to X \times X,$$

is defined by the formula

$$\Gamma(x_1, x_2) := (x_1 + x_2, x_1 - x_2).$$

We define the mapping $\Gamma_\sharp A$ via the equality

$$\mathrm{Gr}(\Gamma_\sharp A) := \Gamma(\mathrm{Gr}(A)).$$

Further the mapping $\Gamma_\sharp^{-1} A$ is a mapping whose graph is $\Gamma^{-1}\mathrm{Gr}(A)$.

Proposition 22 (see [17]) *Let M be an arbitrary subset of a Hilbert space X and let $A : M \to 2^X$ be a monotone multi-mapping. Then $\Gamma_\sharp A$ is 1-Lipschitz. On the other hand, for a given Lipschitz continuous mapping $\phi : N \subset X \to X$ a monotone multi-mapping $\Gamma_\sharp^{-1}\phi$ is a monotone multi-mapping.*

Proof. Denote $B := \Gamma_\sharp A$. The definition of the operator $\Gamma_\sharp$ gives that

$$(\alpha, \xi) \in \mathrm{Gr}(B) \Leftrightarrow (\exists (a, x) \in \mathrm{Gr}(A),\ \alpha = a + x,\ \xi = -a + x). \qquad (17)$$

Take arbitrary two pairs $(\alpha_1, \xi_1)\,(\alpha_2, \xi_2) \in \mathrm{Gr}(B)$. A simple manipulation with (17) gives immediately

$$\frac{\alpha_i + \xi_i}{2} \in A\left(\frac{\alpha_i - \xi_i}{2}\right),\ i = 1, 2.$$

We write the monotonicity condition for A and obtain

$$0 \leq \left\langle \frac{\alpha_1 + \xi_1}{2} - \frac{\alpha_2 + \xi_2}{2}; \frac{\alpha_1 - \xi_1}{2} - \frac{\alpha_2 - \xi_2}{2} \right\rangle,$$

this easily gives

$$|\xi_1 - \xi_2| \leq |\alpha_1 - \alpha_2|.$$

Thus $B = \Gamma_\sharp A$ is 1-Lipschitz. The reverse correspondence is proved by the same argument. $\square$

LEMMA 23 (Minty; see [17], [21]) *Let X be a Hilbert space and let $T : X \to 2^X$ be a maximal monotone multi-mapping. Then the mappings*

$$M_1^T := (I + T)^{-1},\ M_2^T := (I + T^{-1})^{-1}$$

are non-expansive maximal monotone and the mapping

$$x \mapsto (M_1^T x, M_2^T x)$$

is bi-Lipschitz bijection from X onto $\mathrm{Gr}(T)$. *In terms of the Caley transform*

$$\Gamma(M_1^T(x), M_2^T(x)) = (x, M_1^T(x) - M_2^T(x)).$$

REMARK 24 The previous lemma asserts a nice property of the monotone multi-mappings. The graph of a maximal monotone multi-mapping can be viewed as a Lipschitz manifold in $X \times X$. If $X = \mathbb{R}^n$ this observation combined with Rademacher's theorem gives that there exists a tangent space to the graph of the maximal monotone multi-mapping at the point $(M_1^T x, M_2^T x)$ for almost every $x \in \mathbb{R}^n$. This information is crucial in the proof of Mignot theorem 59.

The following propositions are easy facts about monotone mappings and can be found in a more general form in the paper [1].

PROPOSITION 25 (see [1]) *Let $u : \mathrm{Dom}(u) \to 2^{\mathbb{R}^n}$ be a monotone mapping. Then the set of $x \in \mathrm{Dom}(u)$, where $u(x)$ is not a singleton is a Lebesgue null set.*

PROPOSITION 26 (see [1]) *Let $u : \mathrm{Dom}(u) \to 2^{\mathbb{R}^n}$ be a maximal monotone mapping. Assume that $x_j \to x$, $y_j \to y$ and $y_j \in u(x_j)$ (i.e. $x_j \in \mathrm{Dom}(u)$). Then $y \in u(x)$.*

In the following proposition all interiors and closures are meant with respect to the norm topology.

Theorem 27 (Rockafellar; see [20]) *Let $T : X \to 2^{X^*}$ be a maximal monotone multi-operator and suppose that* $\mathrm{int}(\mathrm{co}(\mathrm{Dom}(T))) \neq \emptyset$. *Then T is locally bounded at each point $x \in \mathrm{int}(\mathrm{Dom}(T))$ and satisfies the relations*

$$\mathrm{int}(\mathrm{co}(\mathrm{Dom}(T))) = \mathrm{int}(\mathrm{Dom}(T))$$

and

$$\overline{\mathrm{Dom}(T)} = \overline{\mathrm{int}(\mathrm{Dom}(T)},$$

hence $\mathrm{int}(\mathrm{Dom}(T))$ *and* $\overline{\mathrm{Dom}(T)}$ *are convex. Further for all* $x \in \mathrm{int}(\mathrm{Dom}(T))$ *the set Tx is weak* compact and convex.*

The following lemma, which is a form of an extension theorem for monotone mappings, will be useful.

Lemma 28 *Let $\Omega \subset \mathbb{R}^d$ be a bounded set and let $A : \Omega \to 2^{\mathbb{R}^d}$ be a bounded monotone multi-mapping. Then there is a monotone multi-mapping $\overline{A} : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ which is an extension of A i.e. $\overline{A}_{|\Omega} = A$. In particular for $A : \Omega \to \mathbb{R}^d$ single-valued there is a single-valued extension.*

Proof. We will work in the space $\mathbb{R}^d \times \mathbb{R}^d$. We denote the first copy of $\mathbb{R}^d$ by R_1 and the second one by R_2.

The set $N := \pi_1 \Gamma(\mathrm{Gr}(A))$, where we have denoted by π_1 the projection on R_1, is a bounded subset of R_1 since Ω is bounded, A is bounded on Ω and π_1 and Γ are the linear mappings.

By Proposition 22 we conlude that the mapping $\Gamma_\sharp A : N \to \mathbb{R}^d$ is a Lipschitz mapping. If we use Theorem 8 we get the mapping $\overline{\Gamma_\sharp A} : \mathbb{R}^d \to \mathbb{R}^d$ which is a Lipschitz extension of $\Gamma_\sharp A$ onto the whole $\mathbb{R}^d$.

Now we have to guarantee that if we return back to the monotone multi-mapping it will remain to be defined on the whole space. We can apply Lemma 12 to the function $\overline{\Gamma_\sharp A}$ and obtain a Lipschitz continuous mapping $(\overline{\Gamma_\sharp A})_R$ which has the Lipschitz constant bounded by the Lipschitz constant of $\overline{\Gamma_\sharp A}$ and which coincides with $\overline{\Gamma_\sharp A}$ in a ball $B(0, R)$ which contains Ω. Now we consider the mapping

$$\overline{A} := \Gamma_\sharp^{-1}[(\overline{\Gamma_\sharp A})_R],$$

because we have used the cut off procedure, we infer that the graph of $\overline{A}$ in the exterior of a sufficiently large ball $B(0, cR)$ coincides with the linear subspace M of $\mathbb{R}^d \times \mathbb{R}^d$, where $M = \Gamma(R_1 \times \{0\})$. The fact that for every $x \in \mathbb{R}^d$ there is a point y such that $(x, y) \in \mathrm{Gr}\overline{A}$ follows from Brouwer theorem, indeed we are looking for such y such that

$$x - y = f(x + y),$$

where we have denoted $f := (\overline{\Gamma_\sharp A})_R$. Let x be fixed. We are solving the equation

$$y = x - f(x + y) =: h(y).$$

Since f is continuous with compact support $|f(\cdot)|$ is bounded by some C. We have $|h(y)| \leq |x| + C$. Thus for $a \geq |x| + C$ we have $h : B(0,a) \to B(0,a)$. Brouwer theorem gives the desired point y . This gives that the multi-mapping $\overline{A}$ is defined on the whole R_1 (an argument using Theorem 27 is also possible).

Finally, for A being single-valued, an arbitrary selection of $\overline{A}$, which coincides with A on Ω, is surely a single-valued monotone extension of A. $\square$

The classical result says that in the one dimensional case a function defined on an open interval can be written as a difference of two nondecreasing functions if and only if it has the locally finite variation. Thus the following example is a bit surprising.

Example 29 Consider the plane with the axis x, y. Consider an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ and define the function $u : \mathrm{Dom}(u) \to \mathbb{R}^2$, where $\mathrm{Dom}(u)$ is the x-axis by the formula $u(x,y) := (x, f(x)$. A simple geometric argument implies the monotonicity of the mapping u.

Let us be more concrete. Let $\mathbb{R}^2 \supset D := (-1;1) \times \{0\}$, let $(r_n)_{n\in\mathbb{N}}$ be a countable dense subset of $(-1;1)$. Consider for $x \in (-1;1)$

$$v_n(x) := \frac{1}{2^n}(x - r_n) \sin \frac{1}{x - r_n}$$

and define

$$v := \sum_{n=1}^{\infty} v_n.$$

Finally let $u : D \to \mathbb{R}^2$ be given by

$$u(x,y) := (x, v(x)).$$

This function is continuous on D by the Weierstrass criterion, it is monotone but it does not have the finite variation over any line segment contained in D. Later we will see that this example is in fact pathological, since the domain of the function is very small. ♣

Definition 30 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function on a Banach space X which is not identically equal to $+\infty$ (such functions are called *proper convex functions*) and let $x \in X$ be a point where f is finite. We define the *subdifferential of f at x* as the set

$$\partial f(x) := \{x^* \in X^*; f(z) - f(x) \geq \langle x^*; z - x\rangle, z \in X\}. \tag{18}$$

The elements of $\partial f(x)$ are called the *subgradients* of f at x.

Remark 31 By Hahn-Banach theorem we can see that f is subdifferentiable in the interior of its effective domain, for details see [27].
It can be proved that a continuous convex function f is Gateaux differentiable at point x, if and only if $\partial f(x)$ consists of exactly one point $\delta f(x)$, for details see [27].

Example 32 The well known examples of monotone multi-mappings are the differentials of smooth convex functions, more generally subdifferentials of proper convex functions. Indeed, consider the multi-mapping ∂f, choose points $x, y \in \mathrm{Dom}(\partial f)$ and $x^* \in \partial f(x)$, $y^* \in \partial f(y)$. Write the inequality (18) for $z := y$ and for $z := x$, subtract and rewrite. We obtain exactly the monotonicity-inequality (16).

If we consider as the special case the indicator function of a convex set $C \subset X$ defined by

$$\delta_C(x) = \begin{cases} 0, \, x \in C \\ \infty, \, x \notin C, \end{cases}$$

then its subdifferential is equal to the normal cone of C, defined by

$$N(x, C) := \{\xi^* \in X^*; \langle \xi^*; c - x \rangle \leq 0, \, c \in C\}$$

as can be easily verified. ♣

We introduce now a definition which will be used later and for the sake of completeness we recall one deep result.

Definition 33 A multi-mapping $T : \mathrm{Dom}(T) \to 2^{X^*}$ is called *cyclically monotone* if for every $n \in \mathbb{N}$, for every choice $\{x_0, x_1, \ldots, x_n = x_0\} \subset \mathrm{Dom}(T)$ and each $x_j^* \in Tx_j$, $j = 0, \ldots, n$

$$\sum_{k=1}^{n} \langle x_k^*; x_k - x_{k-1} \rangle \geq 0.$$

The operator T is called *maximal cyclically monotone* if there is no proper enlargement of T which preserves the cyclical monotonicity.

Theorem 34 (Rockafellar; see [19]) *Let $T : \mathrm{Dom}(T) \subset X \to 2^{X^*}$ be a multi-mapping. Then the following conditions are equivalent:*
(i) T is maximal cyclically monotone,
(ii) there exists a proper lower semi-continuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $T = \partial f$.

Example 35 Another example of a monotone mapping can be gained as the projector mapping in a Hilbert space. Let C be a closed convex subset of the Hilbert space X. It is a well known fact from the elementary theory of Hilbert spaces that for every $u \in X$ there is a unique element $c \in C$ such that $\|u - c\| = \min_{y \in C} \|x - y\|$. We prove that the projector mapping $P : X \to C$ is monotone. It is geometrically obvious and easy to prove that

$$Pu = c \Leftrightarrow \{\langle u - c; x - c\rangle \leq 0, x \in C\}. \tag{19}$$

Take $u, v \in X$ and denote by Pu, Pv their projections. Let us write the inequality (19) for u with $x = Pv$ and for v with $x = Pu$. We have

$$\langle u - Pu; Pv - Pu\rangle \leq 0$$

and

$$\langle v - Pv; Pu - Pv\rangle \leq 0.$$

We sum these two inequalities to obtain

$$\langle u - v, Pv - Pu\rangle \leq \langle Pu - Pv; Pv - Pu\rangle \leq 0.$$

This gives the desired. ♣

Definition 36 Let $(X, \|\cdot\|)$ be a Banach space and let $\|\cdot\|_*$ be the canonical dual norm. The multi-mapping $j : X \to 2^{X^*}$ for which

$$(x, f) \in \mathrm{Gr}(j) \Longleftrightarrow \langle f; x\rangle = \|x\|^2 = \|f\|_*^2$$

is called the *duality mapping.*

Proposition 37 (see [27]) *Let $(X, \|\cdot\|)$ be a Banach space and let j be the duality mapping. Then j is maximal cyclically monotone, homogeneous and*

$$\partial \frac{1}{2}\|x\|^2 = j(x).$$

Definition 38 A Banach space X is called *smooth* if for every $x \in X$ there is only one $x^* \in X^*$ such that $\langle x^*; x\rangle = \|x\|$.

Definition 39 A multi-operator $A : \mathrm{Dom}(A) \to 2^X$ is called *accretive* if it satisfies the condition:

$$\forall x, y \in \mathrm{Dom}(A), \forall(\xi, \eta) \in Ax \times Ay \;\; \exists f \in j(x - y) : \;\; \langle f; \xi - \eta\rangle \geq 0. \tag{20}$$

If the inequality (20) is satisfied for all $f \in j(x - y)$ we call this multi-operator *fully accretive.*

Remark 40 We have introduced the definition of accretive operators which is used in the modern theory of the partial differential equations. The notion of the full accretivity was introduced in the pioneering Browder's work [4].

The following proposition is obvious.

Proposition 41 *Let X be a smooth Banach space. Then the duality mapping is single-valued.*

Remark 42 The previous proposition shows that the concepts of the full accretivity and the accretivity coincide in the class of the smooth Banach spaces.

Definition 43 We say that a mapping $a : M \to N$, where M, N are abstract sets, is a *selection of a multivalued mapping* $A : M \to 2^N$ if for every $m \in M$ $a(m)$ is a singleton and $a(m) \in A(m)$. We write briefly $a \in A$ for a being a selection of A.

Lemma 44 *Let $A : \Omega \subset X \to 2^{X^*}$, where Ω is a convex set, be a multivalued mapping. Let $x, h \in X$ be such points such that for every selection $a \in A$ the function*

$$t \mapsto \varphi(x, h, a; t) := \langle a(x + th) - a(x), h\rangle,$$

is defined on an interval $I_{x,h}$. Then A is monotone if and only if for each $x, h \in X$, $a \in A$ the function $\varphi(x, h, a; \cdot)$ is nondecreasing on $I_{x,h}$.

Proof It is clear that A is a monotone multivalued mapping if and only if each selection $a \in A$ is a monotone mapping. Let A be a monotone multi-mapping, $a \in A$ its arbitrary selection and choose $x, h \in X$, $0 \le s < t$ such that $\varphi(x, h, a; t)$ and $\varphi(x, h, a; s)$ are defined. We have

$$\varphi(x, h, a; t) - \varphi(x, h, a; s) = \langle a(x + th) - a(x), h\rangle - \langle a(x + sh) - a(x), h\rangle =$$

$$\frac{1}{t-s}\langle a(x + th) - a(x + sh), (t - s)h\rangle \ge 0,$$

since a is monotone.

Conversely let φ be nondecreasing, choose admissible $x, h \in X$ and a selection $a \in A$. We obtain

$$\langle a(x + h) - a(x), h\rangle = \varphi(x, h, a; 1) - \varphi(x, h, a; 0) \ge 0.$$

This gives the monotonicity of the multi-mapping A. □

Definition 45 We say that a multi-mapping $A : \Omega \subset X \to 2^{X^*}$ is *locally monotone* in Ω if for every point $x_0 \in \Omega$ there is a (relative) neighborhood $U(x_0)$ of x_0 in Ω such that $A_{|U(x_0)} : U(x_0) \to 2^{X^*}$ is a monotone multi-mapping.

Lemma 46 *Let $\Omega \subset X$ be a convex set. A multi-operator $A : \Omega \to 2^{X^*}$ is monotone if and only if it is locally monotone in Ω.*

Proof The necessity is obvious.

For the sufficiency recall that due to Lemma 44 it is sufficient to show that for every selection $a \in A$ and every $x, h \in X$ such that the function $\varphi(x, h, a; \cdot)$ is defined, $\varphi(x, h, a; \cdot)$ is nondecreasing on its domain. The assumption of the local monotonicity of A implies the local monotonicity of each selection $a \in A$. This gives that the function $\varphi(x, h, a; \cdot)$ is nondecreasing at every point of its domain. Thus φ is nondecreasing as follows from the standard calculus result. □

DEFINITION 47 Let (E, ϱ) be an arbitrary metric space and let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of E. We say that the sequence $(C_n)_{n\in\mathbb{N}}$ converges in the *Hausdorff sense* to a closed set $C \subset E$ if the following conditions are satisfied:
i) every cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in C_n$ belongs to C,
ii) for every point $x \in C$ there is a sequence of points $x_n \in C_n$ which converges to x.
We denote this type of the convergence by the symbol ${}_{\mathcal{H}}\lim$.

REMARK 48 Let (E, ϱ) be a metric space. The *Hausdorff distance* of closed sets F, F' is defined by

$$\delta_{\mathcal{H}}(F, F') := \max\{\sup_{x\in F} \varrho(x, F'); \sup_{x'\in F'} \varrho(x', F)\}.$$

If E is compact then $\delta_{\mathcal{H}}$ induces the Hausdorff convergence (see [1] and the references there in).

We can consider the the space E as the one-point compactification of the space $\mathbb{R}^n \times \mathbb{R}^n$ i.e. $E = (\mathbb{R}^n \times \mathbb{R}^n) \cup \{\infty\}$. Let i be the map which associates to any closed subset F of $\mathbb{R}^n \times \mathbb{R}^n$ the closed subset $F \cup \{\infty\}$ of E. We put for closed subsets of $\mathbb{R}^n \times \mathbb{R}^n$

$$d_{\mathcal{H}}(F, F') := \delta_{\mathcal{H}}(iF, iF').$$

Let $\mathcal{F}_0 := \mathcal{F}_0(\mathbb{R}^n \times \mathbb{R}^n)$ be the class of all closed subsets of $\mathbb{R}^n \times \mathbb{R}^n$. It can be verified that $(\mathcal{F}_0, d_{\mathcal{H}})$ is a compact metric space where the convergence is the convergence in the Hausdorff sense.

DEFINITION 49 We say that mappings $(u_n)_{n\in\mathbb{N}} \subset (E \to 2^F)$ (E, F are metric spaces) converge to a mapping $u : E \to 2^F$ *graphically* if

$${}_{\mathcal{H}}\lim_{n\to\infty} \mathrm{Gr}(u_n) = \mathrm{Gr}(u).$$

We use the notation

$${}_{g}\lim u_n = u.$$

PROPOSITION 50 (see [1]) *Let* $A : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ *be a maximal monotone multimapping. Then* $\mathrm{Gr}(A)$ *is a closed subset of* $\mathbb{R}^d \times \mathbb{R}^d$.

PROPOSITION 51 (see [1]) *Let* $A_n : \mathbb{R}^d \to 2^{\mathbb{R}^d}$, $n \in \mathbb{N}$ *be a sequence of maximal monotone mappings. Let* $A =_g \lim A_n$ *be finite. Then* $A \in \mathfrak{Mon}(\mathbb{R}^d)$.

PROPOSITION 52 (see [1]) *Let* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be a maximal monotone mapping. We define the so called* Yosida regularizations of A *as the mappings*

$$Y_\lambda A := (\lambda I + A^{-1})^{-1},\ \lambda > 0.$$

Then $Y_\lambda A$ *is* $1/\lambda$*-Lipschitz maximal monotone mapping and*

$$_g \lim_{\lambda \to 0+} Y_\lambda A = A.$$

The notion of Borel measurable mappings can be generalized for multi-mappings.

DEFINITION 53 The multi-mapping $T : \mathrm{Dom}(T) \subset \mathbb{R}^m \to 2^{\mathbb{R}^n}$ is called *Borel measurable multi-mapping* (briefly Borel multi-mapping) if for every open subset $O \subset \mathbb{R}^m$ the set $T^{-1}(O) \subset \mathbb{R}^n$ is Borel.

It can be proved by basic methods of the descriptive set theory that if the graph of a multi-mapping $T : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ is an analytic subset of $\mathbb{R}^m \times \mathbb{R}^n$ then T is a Borel multi-mapping. Indeed, let O be an open subset of $\mathbb{R}^n$ we have

$$T^{-1}(O) = \pi_m(\mathrm{Gr}(T) \cap (\mathbb{R}^m \times O)),$$

where π_m is the projection of $\mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$. This together with Proposition 15 implies that the set $T^{-1}(O)$ is an analytic subset of $\mathbb{R}^m$. Similarly we obtain that $T^{-1}(\mathbb{R}^n \setminus O)$ is an analytic subset of $\mathbb{R}^m$. It follows by a descriptive set theoretic argument(see [24]) that the set $T^{-1}(O)$ is a Borel subset of $\mathbb{R}^m$.

This result combined with the closeness of the graph of a maximal monotone multi-mapping $u : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ (Proposition 50) implies:

PROPOSITION 54 *Every maximal monotone multi-mapping* $u : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *is a Borel measurable multi-mapping.*

EXAMPLE 55 On the other hand, there is a monotone mapping $u : \mathbb{R}^n \to \mathbb{R}^n$ which is not Borel measurable. For simplicity we construct such an example for $n = 2$ but the similar construction can be done for arbitrary $n \in \mathbb{N}$. At first we realize that if $f : \mathbb{R} \to [0; 1]$ is an arbitrary function and the function $u_1 : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$u_1(x_1, x_2) = \begin{cases} 0,\ x_1 < 0 \\ f(x_2),\ x_1 = 0 \\ 1,\ x_1 > 0, \end{cases}$$

then the function $u := (u_1, 0) : \mathbb{R}^2 \to \mathbb{R}^2$ is a monotone mapping. This can be easily shown by distinguishing the cases

$$((x_1 < 0) \,\&\, (y_1 < 0)),$$

$$((x_1 < 0)\,\&\,(y_1 > 0)),$$
$$((x_1 < 0\,\&\,(y_1 = 0)),$$

and

$$((x_1 > 0\,\&\,(y_1 = 0)).$$

Let $N \subset \mathbb{R}$ be a set which is not Lebesgue measurable. Then the set $\{0\} \times N$ is a subset of $\mathbb{R}^2$ which is not Borel (see Proposition 15). Denote by $\mathbf{1}_N$ the characteristic function of the set N. We define

$$f(t) := \frac{1}{2}\mathbf{1}_N(t)$$

and take the monotone mapping u defined for this function f as before. Consider the open subset of $\mathbb{R}^2$ $O := (0;1) \times \mathbb{R}$. Then we have

$$u^{-1}(O) = \{0\} \times N.$$

Hence u is a monotone mapping which is not Borel measurable. ♣

2.2 Differential theory for monotone mappings

In this section Ω will denote an open convex set of a Banach space X.

2.2.1 Classical case

THEOREM 56 (see [21]) *Let $u : \Omega \to X^*$ be a Gateaux differentiable mapping. Then u is monotone if and only if its Gateaux derivative $\delta u(x) \in \mathcal{L}(X, X^*)$ is positive-semidefinite for each $x \in \Omega$. Further it is strictly monotone if the operator $\delta u(x)$ is positive definite for each $x \in \Omega$.*

Proof. Let u be monotone, we have

$$\langle u(x + th) - u(x); th\rangle \geq 0.$$

We divide by t^2 and take the limit for $t \to 0$. We obtain

$$\langle \delta u(x)h; h\rangle \geq 0$$

for arbitrary choice of x, h. Which gives the desired.

Conversely, let $\delta u(x)$ be positive-semidefinite for all x. Choose $x_0, x_1 \in X$ and define $f(t) := \langle u(x_t) - u(x_0); x_1 - x_0\rangle$, where $x_t := (1-t)x_0 + tx_1$. We have $f(0) = 0$ and we claim $f(1) \geq 0$. Since f is differentiable, we infer

$$f'(t) = \langle \delta u(x_t)(x_1 - x_0); x_1 - x_0\rangle \geq 0,$$

thus f is nondecreasing and consequently $f(1) \geq 0$. This gives monotonicity of u. The second part of the proof can be directly rephrased for the strict case. □

Remark 57 The strict monotonicity does not imply the positive-defineteness of the derivative matrix as a simple one-dimensional example $t \mapsto t^3$ shows.

A straightforward modification gives:

Theorem 58 *Let $T : X \to X$ be a Gateaux differentiable mapping from a smooth Banach space to itself. Then T is an accretive mapping if and only if its Gateaux differential $\delta T(x)$ is accretive, i.e. positive semidefinite in the sense that for every $h \in X$*

$$\langle j(h); \delta T(x)h \rangle \geq 0$$

at every point $x \in X$.

Proof. The proof is only paraphrasing of the the previous one and we use that the duality mapping is single-valued and homogeneous (see Proposition 41). □

Theorem 59 (Mignot; see [1], [16]) *Let $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ be a maximal monotone function and let D be the set of all $x \in \mathrm{Dom}(u)$ such that $u(x)$ is a singleton. Then u is differentiable at almost every $x_0 \in D$, i.e. there is a matrix $u'(x_0) \in \mathbb{R}^{n \times n}$ such that*

$$\lim_{\substack{x \to x_0 \\ y \in u(x)}} \frac{y - u(x_0) - u'(x_0)(x - x_0)}{|x - x_0|} = 0. \tag{21}$$

Remark 60 It is easily seen that for a single valued function u (21) reduces exactly to the Fréchet differentiability of u at x_0. The fact that for almost every point $x \in \mathrm{Dom}(u)$ there is a matrix $u'(x)$ satisfying (21) follows from a combination of Theorem 59 and Proposition 25. The standard proof of Theorem 59 uses the Cayley transformation and Rademacher theorem. We will provide later an alternative proof for the single valued case which uses the Radó-Reichelderfer property of monotone mappings.

2.2.2 Nonsmooth case

In the following section we introduce a characterization of monotone mappings on reflexive Banach spaces using the generalized differentiation. The methods were developed in the paper [10] for a finite dimensional case and here adapted for reflexive Banach spaces.

Let Ω be an open convex subset of a Banach space X we denote for arbitrary $F : \Omega \to X^*,\ x, h, \xi \in X$

$$\underline{d}\langle F; \xi \rangle(x, h) := \liminf_{t \to 0+} \left\langle \frac{F(x + th) - Fx}{t}; \xi \right\rangle$$

$$\overline{d}\langle F; \xi \rangle(x, h) := \limsup_{t \to 0+} \left\langle \frac{F(x + th) - Fx}{t}; \xi \right\rangle.$$

Analogously for a smooth Banach space X, an open convex set $\Omega \subset X$ and $F : \Omega \to X$ we define

$$\underline{d}\langle j(\xi); F\rangle(x,h) := \liminf_{t\to 0+} \left\langle j(\xi); \frac{F(x+th)-Fx}{t} \right\rangle$$

$$\overline{d}\langle j(\xi); F\rangle(x,h) := \limsup_{t\to 0+} \left\langle j(\xi); \frac{F(x+th)-Fx}{t} \right\rangle,$$

where $j : X \to X^*$ is the duality mapping.

Definition 61 We say that a closed subset $\partial_* F(x)$ of $\mathcal{L}(X, X^*)$ is the *generalized differential* of the mapping $F : \Omega \to X^*$ at the point $x \in \Omega$ if for each $\xi \in X$, $h \in X$

$$\underline{d}\langle F; \xi\rangle(x,h) \le \sup_{M\in\partial_* F(x)} \langle Mh; \xi\rangle. \tag{22}$$

Remark 62 Similarly for a smooth Banach space X we introduce the generalized differential as a closed subset $\partial^* F(x)$ of $\mathcal{L}(X)$ satisfying

$$\underline{d}\langle j(\xi); F\rangle(x,h) \le \sup_{M\in\partial^* F(x)} \langle j(\xi); Mh\rangle. \tag{23}$$

Remark 63 It is evident that the inequality (22) is equivalent to the inequality

$$\overline{d}\langle F; \xi\rangle(x,h) \ge \inf_{M\in\partial_* F(x)} \langle Mh; \xi\rangle. \tag{24}$$

It is suitable to remark that for $F : \Omega \to X^*$ resp. $F : \Omega \to X$ the whole space $\mathcal{L}(X, X^*)$ resp. $\mathcal{L}(X)$ is a generalized differential $\partial_* F(x)$ resp. $\partial^* F(x)$, but it can not provide any other information about properties of F. It is also obvious that if a set $\mathbf{A}$ is a generalized differential and $\mathbf{B} \supset \mathbf{A}$ then $\mathbf{B}$ is a generalized differential as well.

Definition 64 We say that the generalized differential $\partial_* F(x)$ is *regular* if for each $\xi, h \in X$

$$\overline{d}\langle F; \xi\rangle(x,h) = \sup_{M\in\partial_* F(x)} \langle Mh; \xi\rangle \tag{25}$$

or equivalently

$$\underline{d}\langle F; \xi\rangle(x,h) = \inf_{M\in\partial_* F(x)} \langle Mh; \xi\rangle. \tag{26}$$

Remark 65 The reformulation of the definition 64 for $\partial^* F(x)$ is straightforward.

Theorem 66 (Version of nonsmooth mean value theorem) *Let X be a reflexive Banach space and let $F : \Omega \subset X \to X^*$ be a mapping which is continuous on lines and admits a generalized differential. Then for every $a, b \in \Omega$*

$$F(b) - F(a) \in \overline{\text{co}}\{\partial_* F[a;b](b-a)\}, \tag{27}$$

where $[a;b]$ denotes the line segment between a, b.

Proof. Let $\xi \in X$ be fixed, consider the function $g : [0,1] \to \mathbb{R}$ given by the formula

$$g(t) := \langle F(a + t(b-a)) - F(a) + t(F(a) - F(b)); \xi\rangle.$$

Since $g(0) = g(1) = 0$ and g is continuous, there is a point of an extreme $t_0 \in (0,1)$. Suppose that t_0 is the point of a minimum, since the case of a maximum can be investigated analogously. We denote

$$\underline{d}g(t_0, \alpha) := \liminf_{t \to t_0} \frac{g(t_0 + \alpha t) - g(t_0)}{t}.$$

It is easy to compute

$$\underline{d}g(t_0, \alpha) = \underline{d}\langle F(a + t_0(b-a)); \xi\rangle(t_0, \alpha(b-a)) + \alpha\langle F(b) - F(a); \xi\rangle.$$

Hence for every $\alpha \in \mathbb{R}$ we have

$$\underline{d}\langle F(a + t_0(b-a)); \xi\rangle(t_0, \alpha(b-a)) \geq \alpha\langle F(a) - F(b); \xi\rangle.$$

If we take $\alpha := \pm 1$ we get

$$-\underline{d}\langle F(a+t_0(b-a)); \xi\rangle(t_0, a-b) \leq \langle F(b)-F(a); \xi\rangle \leq \underline{d}\langle F(a+t_0(b-a)); \xi\rangle(t_0, b-a).$$

Thus (22) gives

$$\inf_{M \in \partial_* F(a+t_0(b-a))} \langle M(b-a); \xi\rangle \leq \langle F(b) - F(a); \xi\rangle \leq \sup_{M \in \partial_* F(a+t_0(b-a))} \langle M(b-a); \xi\rangle,$$

which gives

$$\langle F(b) - F(a); \xi\rangle \in \langle \overline{\text{co}}\{\partial_* F(a + t_0(b-a))(b-a)\}; \xi\rangle,$$

consequently

$$\langle F(b) - F(a); \xi\rangle \in \langle \overline{\text{co}}\{\partial_* F[a;b](b-a)\}; \xi\rangle, \tag{28}$$

for all $\xi \in X$.

Suppose that (27) is not true. Since X is reflexive, the set $\overline{\text{co}}\{\partial_* F[a;b](b-a)\}$ is convex and closed and the singleton $F(b) - F(a)$ is compact, geometric form of Hahn-Banach theorem enables to find $z \in X$ and $\varepsilon > 0$ such that

$$\langle F(b) - F(a); z\rangle - \varepsilon > \sup_{\xi^* \in \overline{\text{co}}\{\partial_* F[a;b](b-a)\}} \langle \xi^*; z\rangle,$$

thus

$$\langle F(b) - F(a); z\rangle > \sup\{\langle \overline{\text{co}}\{\partial_* F[a;b]\}(b-a); z\rangle\}.$$

This contradicts (28). ∎

REMARK 67 Theorem 66 can be modified as: let $F : \Omega \to X$ be continuous on lines where X is a smooth and reflexive Banach space and suppose that there is a generalized differential $\partial^* F(x)$ for all $x \in \Omega$. Then for all $[a; b] \subset \Omega$

$$F(b) - F(a) \in \overline{\text{co}}\{\partial^* F[a; b](b - a)\}.$$

The proof is only imitating of the scenario of the proof of Theorem 66. Let us note only the main changes. Instead of the function g from the proof of Theorem 66 we work with the function

$$h(t) := \langle j(\xi); F(a + t(b - a)) - F(a) + t(F(a) - F(b))\rangle.$$

Then the proof goes in the same way. Hahn-Banach theorem is used for the existence of a separating functional $\xi^* \in X^*$ and then we realize that since X is reflexive there is a point $\xi \in X$ such that $j(\xi) = \xi^*$.

THEOREM 68 *Let X be a reflexive Banach space and let $F : \Omega \to X^*$ be a mapping continuous on lines which admits a generalized differential $\partial_* F(x)$ for every $x \in \Omega$ and assume that for each $x \in \Omega$ all the (linear) operators $M \in \partial_* F(x)$ are monotone. Then F is monotone.*

Proof. Choose arbitrary $x, h \in X$, by Theorem 66 we have

$$F(x + h) - F(x) \in \overline{\text{co}}\{\partial_* F[x; x + h]h\},$$

thus

$$\langle F(x + h) - F(x)); h\rangle \in \langle\overline{\text{co}}\{\partial_* F[x; x + h]h\}; h\rangle.$$

Thus there exists $z \in [x, x + h]$ and $N \in \overline{\text{co}}\partial_* F(z)$, such that

$$\langle F(x + h) - F(x); h\rangle = \langle Nh; h\rangle \geq \inf_{M \in \overline{\text{co}}\partial_* F(z)} \langle Mh; h\rangle = \inf_{M \in \partial_* F(z)} \langle Mh; h\rangle \geq 0.$$

Let us only comment the last equality. Since the function $M \mapsto \langle Mh; h\rangle$ is a continuous linear functional it is sufficient to realize that for a continuous concave function g the relation

$$\inf_A g = \inf_{\text{coA}} g = \inf_{\overline{\text{coA}}} = \inf_{\overline{\text{co}}A} g$$

holds, but the second equality follows from the continuity, the third one is a consequence of a well known property of the convex hull $\overline{\text{coA}} = \overline{\text{co}}A$ and the first one is easily obtained by the concavity of g. Thus the monotonicity of F is proved. □

REMARK 69 We again only note that it is possible to modify the previous result: Let X be a smooth and reflexive Banach space and let $F : \Omega \to X$ be a mapping continuous on lines which admits a generalized differential $\partial^* F(x)$ for every $x \in \Omega$ and suppose that for each $x \in \Omega$ (linear) operators $M \in \partial^* F(x)$ are accretive. Then F is accretive. The proof is a straightforward modification of the proof of Theorem 68 where the version of mean value theorem from Remark 67 is used.

THEOREM 70 *Let Ω be an open convex subset of a Banach space X and let $F : \Omega \to X^*$ be a monotone mapping which admits a regular generalized differential $\partial_* F(x)$ for every $x \in \Omega$. Then each $M \in \partial_* F(x)$ is a monotone linear mapping for every $x \in \Omega$.*

Proof. Suppose that there are points $x \in \Omega$, $\xi \in X$ and an operator $M \in \partial_* F(x)$ such that

$$\langle M\xi; \xi \rangle < 0.$$

The regularity of $\partial_* F(x)$ gives

$$\underline{d}\langle F; \xi \rangle(x, \xi) = \inf_{M \in \partial_* F(x)} \langle M\xi; \xi \rangle < 0.$$

Thus for a suitable positive t we have

$$\langle F(x + t\xi) - F(x); \xi \rangle < 0.$$

We have a contradiction with the assumption that F is monotone. □

REMARK 71 Let us only note that the modification for accretive operators is easy and can be omitted.

2.2.3 Relation to functions of bounded variation

THEOREM 72 (see [1]) *Let $u : \mathrm{Dom}(u) \to 2^{\mathbb{R}^n}$ be a monotone multi-mapping and let $\Omega \subset \mathrm{intDom}(u)$ be a measurable set. Then u, understood as an element of $L^\infty(\Omega; \mathbb{R}^n)$, is a mapping of the class $BV(\Omega; \mathbb{R}^n)$. Moreover*

$$\int_\Omega d|Du| \le C[\mathrm{diam}\Omega + \mathrm{osc}(u, \Omega)]^n, \tag{29}$$

where the constant $C = C(n)$ depends only on the dimension n.

THEOREM 73 *Let $u : \Omega \to \mathbb{R}^n$ be a continuous monotone function. Then the L^1-approximate differential maps Ω into positive-semidefinite $n \times n$ matrices.*

Proof. Since a monotone function has the bounded variation by Theorem 29, we have by Theorem 6 that u is a.e. L^1-approximate differentiable and thus the mapping $\mathrm{D_{ap}}u$ is defined almost everywhere in Ω.
Pick a point $x \in D_u$ and denote $L := \mathrm{D_{ap}}u(x)$. Consider the set

$$Z := \{z \in B(0, 1); \langle Lz; z \rangle < 0\}.$$

By the monotonicity of u and by the continuity, which guarantees $u = \tilde{u}$ in Ω, we obtain

$$0 \le \langle u(x + rh) - u(x); rh \rangle,$$

we divide by r^2, integrate, use Remark 7 and pass to the limit as r tends to 0.

$$0 \leq \int_Z \left\langle \frac{u(x+rh) - u(x)}{r}; h \right\rangle dh \to \int_Z \langle Lh; h \rangle \, dh < 0.$$

This is a contradiction. □

REMARK 74 The function $f\colon t \mapsto t - c(t)$, where c is the singular Cantor function satisfies $f(0) = f(1) = 0$, f is increasing on the interval $(1/3; 2/3)$, the L^1-approximate differential $\mathrm{D}_{\mathrm{ap}} f$ is positive almost everywhere but the function f is not increasing. This demonstrates that the reverse implication in Theorem 73 does not hold.

THEOREM 75 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a monotone function and let* $B_0 \subset \mathrm{intDom}(u)$ *be a ball. Then there are constants* $C = C(u, n, B_0)$ *and* $\tilde{C} = \tilde{C}(u, n, B_0)$ *such that for any ball* $B \subset B_0$

$$⨍_B d|Du| \leq \frac{C \mathrm{osc}(u, B)}{\mathrm{diam} B} \leq \frac{\tilde{C}}{\mathrm{diam} B}. \tag{30}$$

Proof. Let u be a given monotone function and $B \subset B_0$ a ball, let denote

$$\delta := \mathrm{diam} B, \;\; \lambda := \mathrm{osc}(u, B).$$

Choose a point $x_0 \in B$, consider the change of coordinates $\delta x' - x_0 = x$ and denote as B' the set of all points x' corresponding to all points $x \in B$. We define the monotone function $v(x') := \frac{u(x)}{\lambda}$. Now we have

$$\int_B |Du|(dx) = \frac{\lambda}{\delta} \int_B |Dv|(dx) = \frac{\lambda}{\delta} \int_{B'} \delta^n |Dv|(dx') \leq$$

$$C' \lambda \delta^{n-1} (\mathrm{osc}(v, B') + \mathrm{diam} B') = C'' \mathrm{osc}(u, B) (\mathrm{diam} B)^{n-1},$$

where we have used the estimate (29). The second inequality in (30) easily follows, since $\mathrm{osc}(u, B) \leq \mathrm{osc}(u, B_0)$. This concludes the proof. □

The previous theorem asserts a type of *Morrey estimate* for the derivative of a monotone function. Suppose that Du is representable by a locally integrable function f, then the inequality (30) can be read as $f \in M^{1,n-1}(B_0; \mathbb{R}^{n \times n})$. The Morrey spaces of functions have broad applications in the theory of the regularity of weak solutions of partial differential equations, see [13] for the more complete definition and basic facts about Morrey spaces. In our case we have in fact proved the following corollary:

COROLLARY 76 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a monotone mapping and let* $B_0 \subset\subset \mathrm{intDom}(u)$ *be a ball. Then the derivative* Du *of the mapping* u *belongs to the space of measures*

$$M_*^{1,n-1}(B_0;\mathbb{R}^{n\times n}) := \left\{ \tau \in \mathcal{M}(B_0;\mathbb{R}^{n\times n});\ \sup_{B(x,\rho)\subset B_0} \frac{1}{\rho^{n-1}} \int_{B(x,\rho)} d|\tau| < \infty \right\}.$$

Proof. The corollary follows immediately from Theorem 75. □

THEOREM 77 *Let* $\Omega \subset \mathbb{R}^n$ *be an open convex set. There exists a constant* C, *depending only on* Ω *such that*

$$\frac{\mathrm{osc}(u, B(x_0,r))}{r} \le C ⨍_{B(x_0,2r)} d|Du| \tag{31}$$

for every monotone function $u : \mathrm{Dom}(u) \to \mathbb{R}^n$, $\Omega \subset \mathrm{intDom}(u)$ *and every ball* $B(x_0,r) \subset B(x_0,2r) \subset \Omega$.

Proof. We abbreviate $B_1 := B(x_0,r)$ and $B_2 := B(x_0,2r)$ and denote $d := \mathrm{osc}(u,B_1)$.

The set $u(B_1)$ can be covered by a finite family $\mathcal{B}$ consisting of $N = N(n)$ balls of the diameter $2\rho := \frac{2}{5}d$. We can find a ball $B := B(z,\rho) \in \mathcal{B}$ such that

$$|B_1 \cap \{u \in B\}| \ge \frac{|B_1|}{N}. \tag{32}$$

We denote $E := B_1 \cap \{u \in B\}$. There are two points $y, \tilde{y} \in u(B_1)$ such that $|y - \tilde{y}| > 4\rho$. We can suppose that $|y - z| \ge 2\rho$ (at least one of the points $y, \tilde{y}$ satisfies this). Let $x \in B_1$ be such a point such that $u(x) = y$. Consider the cone

$$U := \{y'; \langle y' - y; z - y\rangle \le \frac{\sqrt{2}}{2}|y - y'||y - z|\}.$$

We define the set

$$E' := B_2 \cap \left\{ x'; \langle y - z; x' - x\rangle \ge \frac{\sqrt{2}}{2}|x' - x||y - z| \right\}. \tag{33}$$

Take a point $x' \in E'$ and set $y' = u(x')$. We have

$$\langle y - z; x' - x\rangle \ge \frac{\sqrt{2}}{2}|x' - x||y - z|. \tag{34}$$

Since u is monotone we have

$$\langle y' - y; x' - x\rangle = \langle u(x') - u(x); x' - x\rangle \ge 0, \tag{35}$$

Define
$$a := \frac{x - x'}{|x - x'|},$$
$$b := \frac{y - z}{|y - z|}$$
and
$$c := \frac{y - y'}{|y - y'|}.$$
Suppose that
$$\langle c; -b\rangle > \frac{\sqrt{2}}{2}.$$
Since by (34)
$$\langle b; a\rangle \geq \frac{\sqrt{2}}{2}$$
we have
$$\sqrt{2} < \langle b; a - c\rangle \leq |a - c|.$$
Taking the squares and using $|a| = |c| = 1$ we have
$$\langle a; c\rangle < 0.$$
This contradicts (35). Thus we have
$$\langle z - y; y' - y\rangle \leq \frac{\sqrt{2}}{2}|z - y||y' - y|. \tag{36}$$
The inequality (36) means that $y' = u(x') \in U$ consequently $u(E') \subset U$.

We have that B is the ball of radius ρ which is contained in the cone $\mathbb{R}^n \setminus U$. The center z lies on the axis of the the cone $\mathbb{R}^n \setminus U$. For each $y'' \in \partial U$ which is closest to the point z we have
$$|z - y''|^2 = |z - y|^2 - |y'' - y|^2$$
and
$$|z-y''|^2 = |z-y|^2 - 2\langle z-y; y''-y\rangle + |y''-y|^2 = |z-y|^2 + |y''-y|^2 - \sqrt{2}|z-y||y''-y|.$$
Hence we conclude
$$|y'' - y|^2 = \frac{1}{2}|z - y|^2.$$
Thus
$$|z - y''|^2 \geq 2\rho^2.$$
By triangle inequality
$$\mathrm{dist}(B, U) \geq \rho(\sqrt{2} - 1)$$

thus for $x \in E$ and $x' \in E'$ we have

$$|u(x) - u(x')| \geq \rho(\sqrt{2} - 1). \tag{37}$$

The relations (32) and (33) imply the existence of a constant $\alpha = \alpha(n)$ depending only on the dimension n such that

$$\alpha|E| \geq |B_2| \tag{38}$$

and

$$\alpha|E'| \geq |B_2|. \tag{39}$$

Using the estimates (38), (39) and (37) we have

$$\begin{aligned} d = 5\rho &\leq \frac{5}{\sqrt{2}-1} \frac{1}{|E||E'|} \int_E \int_{E'} \operatorname{dist}(U, B)\, dx dx' \\ &\leq \frac{5}{\sqrt{2}-1} \frac{1}{|E||E'|} \int_E \int_{E'} |u(x) - u(x')|\, dx dx' \\ &\leq \frac{k}{|B_2|^2} \int_{B_2} \int_{B_2} |u(x) - u(x')|\, dx\, dx'. \end{aligned} \tag{40}$$

The comparability conditions (38) and (39) give that the constant k depends only on the dimension.

The right hand side of (40) can be estimated using the triangle inequality and the Poincaré inequality (Theorem 10) as

$$\begin{aligned} &\frac{k}{|B_2|^2} \int_{B_2} \int_{B_2} |u(x) - u(x')|\, dx\, dx' \\ &\leq \frac{k}{|B_2|^2} \int_{B_2} \left(\int_{B_2} |u(x) - u_{B_2}|\, dx + \int_{B_2} |u(x') - u_{B_2}|\, dx \right) dx' \\ &\leq \frac{k}{|B_2|} \int_{B_2} \left(\gamma r \fint_{B_2} d|Du| + |u(x') - u_{B_2}| \right) dx' \\ &\leq 2k\gamma r \fint_{B_2} d|Du|. \end{aligned}$$

The proof is finished. □

Definition 78 Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u : \Omega \to \mathbb{R}^d$ be a mapping. We say that u satisfies the *weak Radó-Reichelderfer condition*, if there is a non-negative Radon measure $\mu \in \mathcal{M}^+(\Omega)$ depending only on the mapping u and the set Ω such that for every balls $B(a, r) \subset B(a, 2r) \subset \Omega$

$$\frac{\operatorname{osc}(u, B(a, r))}{r} \leq \fint_{B(a,2r)} d\mu. \tag{41}$$

We will say that u satisfies the weak Radó-Reichelderfer property with the *weight* $\mu \in \mathcal{M}^+(\Omega)$. We use the notation $u \in RR^1_*(\Omega; \mathbb{R}^d)$.

Theorem 77 implies easily the following corollary.

Corollary 79 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a given monotone function and let* $\Omega \subset \mathrm{intDom}(u)$*. Then* $u \in RR^1_*(\Omega; \mathbb{R}^n)$.

Proof. Corollary is an easy reformulation of Theorem 77. □

The following proposition follows easily from results from the paper [18].

Proposition 80 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. Let* $u : \Omega \to \mathbb{R}^d$ *be a mapping of class* $RR^1_*(\Omega; \mathbb{R}^d)$*. Then* u *is a mapping of the class* $BV(\Omega; \mathbb{R}^d)$.

Remark 81 Further details about the class RR^1_* can be found in the paper [6]. Let notice that Proposition 80 does not provide an alternative proof of Theorem 72 since in the proof of Theorem 77 there was used Poincaré inequality holding for mappings of the class BV.

The following propositions yield easy facts about the mappings resp. the measures of the class RR^1_* resp $M^{1,n-1}_*$. We formulate them in the not too general form.

Proposition 82 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a mapping, let* $B_0 \subset \mathrm{intDom}(u)$ *be a ball and let* $q : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuously differentiable and Lipschitz continuous mapping. If the mapping* u *is of the class* $RR^1_*(B_0; \mathbb{R}^n)$ *then the mapping* $q \circ u$ *is of the class* $RR^1_*(B_0; \mathbb{R}^n)$ *too.*

Proof. Let $B(x, \rho) \subset B_0$ be an arbitrary ball. Let u satisfy the weak Radó-Reichelderfer condition (41) with a weight μ and let $q : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous mapping with a Lipschitz constant ℓ_q. It is

$$\frac{\mathrm{osc}(q \circ u, B(x, \rho)}{\rho} \le \ell_q \frac{\mathrm{osc}(u, B(x, \rho)}{\rho} \le \fint_{B(x,2\rho)} d\mu_q,$$

where $\mu_q := \ell_q \mu$. □

Proposition 83 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a mapping, let* $B_0 \subset \mathrm{intDom}(u)$ *be a ball and let* $q : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear mapping. If* Du *is of the class* $M^{1,n-1}_*(B_0; \mathbb{R}^{n\times n}$ *then* $D(q \circ u)$ *is of the class* $M^{1,n-1}_*(B_0; \mathbb{R}^{n\times n})$ *as well.*

Proof. Denote again the Lipschitz constant of q by ℓ_q. Let Du belong to the space $M^{1,n-1}_*(B_0, \mathbb{R}^{n\times n})$. The general chain rule for BV functions (see [3]) implies $|D(q \circ u)| \le \ell_q |Du|$. Hence

$$\fint_{B(x,\rho)} d|D(q \circ u)| \le \ell_q \fint_{B(x,\rho)} d|Du| \le \frac{\ell_q C}{\rho},$$

which gives the result. □

We realize in the next proposition that the mappings of the class RR^1_* have very good differentiability properties. The following observation slightly generalizes the result which can be found in [15]. We join the proof for the sake of completeness.

PROPOSITION 84 *Let* $f \in RR^1_*(\Omega;\mathbb{R}^d)$. *Then* f *is Fréchet differentiable almost everywhere.*

Proof. Proposition 80 implies that f is of the class BV thus its distributional derivative is a Radon matrix valued measure. We can apply the Lebesgue-Radon-Nikodým theorem and write

$$Df = D^a f + D^s f = g\mathcal{L}^n + D^s f,$$

where $D^s f$ is the singular part of the measure Df with respect to the Lebesgue measure, $D^a f$ is the absolutely continuous part of Df with respect to the Lebesgue measure and $g \in L^1(\Omega;\mathbb{R}^{d\times n})$ is the Radon-Nikodým derivative of the measure $D^a f$ with respect to the Lebesgue measure. It is known (see [23]) that for $\mathcal{L}^n$- almost all points $x \in \Omega$

$$g(x) = \lim_{r\to 0+} \frac{D^a f(B(x,r))}{|B(x,r)|}. \tag{42}$$

Let $\mu \in \mathcal{M}^+(\Omega)$ be a weight from the definition of the Radó-Reichelderfer condition (41). We write again

$$\mu = \mu^a + \mu^s = \theta\mathcal{L}^n + \mu^s,$$

where $\theta \in L^1(\Omega)$ and μ^s resp. μ^a is the singular part resp. the absolutely continuous part with respect to $\mathcal{L}^n$.

Let choose one fixed representative for g. We will prove that $f'(z) = g(z)$ at such points $z \in \Omega$ which satisfy:
i) z is a Lebesgue point of f,
ii) z is a Lebesgue point of g,
iii) z is a Lebesgue point of θ,
iv) $\theta(z) < \infty$,
v) $D^s f$ satisfies the condition (15) from Theorem 11 in the point z,
vi) μ^s satisfies the condition (15) from Theorem 11 in the point z
vii) the measure $D^a f$ satisfies (42) in the point z.

Theorem 11 and Remark 3 imply that the set of such points z which fulfill the conditions *i*)-*vii*) has the full measure in Ω.

We can suppose that $f(z) = 0,\ g(z) = 0$ otherwise we pass to the function

$$x \mapsto f(x) - f(z) - g(z)(x - z).$$

Choose $\varepsilon \in (0; 1/4)$ and find $\delta > 0$ such that

$$0 < \rho \le \delta \Rightarrow \begin{cases} \displaystyle\fint_{B(z,\rho)} |\theta(z) - \theta(x)|\, dx \le \varepsilon^n \\ \displaystyle\fint_{B(z,\rho)} d|D^s f| \le \varepsilon^{n+1} \\ \displaystyle\left| \frac{|D^a f|(B(z,\rho))}{|B(z,\rho)|} \right| \le \varepsilon^{n+1} \end{cases} \tag{43}$$

Let $y \in B(z, \delta/2)$ and denote $r := 2|y - z|$. Thus we have $B(y, 2r\varepsilon) \subset B(z, r)$. We estimate

$$|f(y)| \le \left| f(y) - \fint_{B(y,\varepsilon r)} f(x)\, dx \right| + \fint_{B(y,\varepsilon r)} |f(x)|\, dx =: T_1 + T_2.$$

The first term can be estimated using the Radó-Reichelderfer condition and the assumption iii) and vi) as

$$\begin{aligned} T_1 &\le \operatorname{osc}(f, B(y, \varepsilon r)) \le \varepsilon r \fint_{B(y,2\varepsilon r)} d\mu \\ &= \varepsilon r \left(\fint_{B(y,2\varepsilon r)} d\mu^a + \fint_{B(y,2\varepsilon r)} d\mu^s \right) \\ &= \varepsilon r \left(\fint_{B(y,2\varepsilon r)} \theta\, dx + \fint_{B(y,2\varepsilon r)} d\mu^s \right) \\ &\le \varepsilon r \left(\fint_{B(y,2\varepsilon r)} |\theta(z) - \theta|\, dx + \theta(z) + \fint_{B(y,2\varepsilon r)} d\mu^s \right) \\ &\le \varepsilon r \left(\theta(z) + 2^{-n}\varepsilon^{-n} r^{-n} \int_{B(z,r)} |\theta(z) - \theta|\, dx + 2^{-n}\varepsilon^{-n} r^{-n} \int_{B(z,r)} d\mu^s \right) \\ &\le \varepsilon r (\theta(z) + 2^{-n}\varepsilon^{-n}\varepsilon^n + 2^{-n}\varepsilon^{-n}\varepsilon^n) \\ &\le \varepsilon r (\theta(z) + 1). \end{aligned}$$

The term T_2 is treated with the help of Theorem 9 applied on the point $\zeta := z$

and $u := |f|$. Thus $B_t = B(z, tr)$ and $|Du| \leq |Df|$ (see [3]). We have

$$\begin{aligned} T_2 &= \left| \fint_{B(y,\varepsilon r)} f(x)\,dx \right| \leq \fint_{B(y,\varepsilon r)} |f(x)|\,dx \leq \varepsilon^{-n} \fint_{B(z,r)} |f(x)|\,dx \\ &= \varepsilon^{-n} \left| \fint_{B(z,r)} |f(x)|\,dx - |f(z)| \right| \leq \frac{2r}{\varepsilon^n} \int_0^1 \frac{|Df|(B_t)}{|B_t|}\,dt \\ &\leq \frac{2r}{\varepsilon^n} \int_0^1 \left(\frac{|D^a f|(B_t) + |D^s f|(B_t)}{|B_t|} \right) dt \leq 4r\varepsilon, \end{aligned}$$

where we have used the last two inequalities from (43). Thus we have proved that

$$|f(y)| \leq M\varepsilon r = 2M\varepsilon|y - z|,$$

where the constant M does not depend on ε and y. The Fréchet differentiability of the mapping f at the point z is proved. □

Corollary 85 *Let* $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ *be a monotone mapping. Let* $\Omega \subset \mathrm{intDom}(u)$ *be an open set. Then* u *is Fréchet differentiable at almost every point of* Ω.

Proof. The corollary follows immediately from Corollary 79 and Proposition 84. □

3 Differences of monotone mappings

3.1 Properties of DM mappings

Definition 86 A multi-mapping $A : \Omega \subset X \to 2^{X^*}$ is called *DM multi- mapping* if there exist monotone multi-mappings $A^\uparrow$, $A^\downarrow : \Omega \to 2^{X^*}$ such that for all $x \in \Omega$ it is

$$Ax \subset A^\uparrow x - A^\downarrow x.$$

Single-valued DM multi-mappings are called DM mappings. We will also say that the mapping A has the *DM property.*

Remark 87 We will work mainly with single-valued DM mappings. It is an easy observation that the class of DM mappings is the smallest space generated by the cone of monotone mappings.

Proposition 88 *Every Lipschitz mapping from a Hilbert space to itself is a DM mapping.*

Proof. Let α be the Lipschitz constant of A. We use the Schwartz inequality to obtain

$$\langle(\alpha I - A)x - (\alpha I - A)y; x - y\rangle = \alpha\|x - y\|^2 - \|Ax - Ay\|\|x - y\|$$

$$\geq \alpha\|x - y\|^2 - \alpha\|x - y\|^2 = 0.$$

Finally we put $A = \alpha I - (\alpha I - A)$. □

The situation in general Banach spaces is more complicated and it has some interesting connections to its geometry. The analogous conclusion for Lipschitz maps holds only in the accretive setting:

PROPOSITION 89 *Let $A : X \to X$ be a Lipschitz mapping from a Banach space X to itself. Then A can be represented as a difference of two fully-accretive mappings.*

Proof. Let α be the Lipschitz constant of A. We can write $A = \alpha I - (\alpha I - A)$. We need to show the full accretivity of $\alpha I - A$. Let $x, y \in X$, $f \in j(x - y)$ be chosen, we have

$$\langle f; (\alpha I - A)x - (\alpha I - A)y\rangle = \alpha\langle f; x - y\rangle - \langle f; Ax - Ay\rangle \geq$$

$$\alpha\|x - y\|^2 - \|Ax - Ay\|\|f\|_* \geq \alpha\|x - y\|^2 - \alpha\|x - y\|\|f\|_* \geq 0.$$

This concludes the the proof. □

Corollaries 76 and 79 imply two necessary conditions for the DM property of a mapping u.

PROPOSITION 90 *Let $u : \mathrm{Dom}(u) \to \mathbb{R}^n$ be a DM mapping and let $B_0 \subset \mathrm{intDom}(u)$ be a ball. Then the derivative Du belongs to the space of measures*

$$M_*^{1,n-1}(B_0; \mathbb{R}^{n\times n})$$

and the function u itself belongs to the space $RR_*^1(B_0; \mathbb{R}^n)$.

Proof. Let $u = u^\uparrow - u^\downarrow$, where $u^\downarrow$ and $u^\uparrow$ are monotone mappings. The ball B_0 lies in the intersection $\mathrm{Dom}(u^\uparrow) \cap \mathrm{Dom}(u^\downarrow)$. Let $B(x, \rho) \subset B_0$ be an arbitrary ball.

At first we prove the Morrey estimate. We can write for the ball $B(x, \rho)$

$$⨍_{B(x,\rho)} d|Du| = \frac{1}{|B(x,\rho)|}|Du|(B(x,\rho) = \frac{1}{|B(x,\rho)|}|Du^\uparrow - Du^\downarrow|(B(x,\rho))$$

$$\leq \frac{1}{|B(x,\rho)|}|Du^{\uparrow}|(B(x,\rho) + \frac{1}{|B(x,\rho)|}|Du^{\downarrow}|(B(x,\rho) \leq \frac{C}{\rho},$$

where we have used the elementary property of variation of vector measures $|g_1 - g_2| \leq |g_1| + |g_2|$ and Theorem 75.

The proof of the Radó-Reichelderfer estimate is easy too. Let denote the non-negative Radon measures from Corollary 79 corresponding to the mappings $u^{\uparrow}$ resp. $u^{\downarrow}$ by $\mu^{\uparrow}$ resp. $\mu^{\downarrow}$. We have by using Corollary 79

$$\frac{\operatorname{osc}(u, B(x,\rho))}{\rho} = \frac{\operatorname{osc}(u^{\uparrow} - u^{\downarrow}, B(x,\rho)}{\rho} \leq \frac{\operatorname{osc}(u^{\uparrow}, B(x,\rho))}{\rho} + \frac{\operatorname{osc}(u^{\downarrow}, B(x,\rho))}{\rho}$$

$$\leq \fint_{B(x,2\rho)} d\mu^{\uparrow} + \fint_{B(x,2\rho)} d\mu^{\downarrow} = \fint_{B(x,2\rho)} d\mu,$$

where the non-negative Radon measure μ is defined by $\mu := \mu^{\uparrow} + \mu^{\downarrow}$. This concludes the proof. □

Later we will demonstrate that the conditions from Proposition 90 are not sufficient for the DM property.

COROLLARY 91 *Let $u : \operatorname{Dom}(u) \to \mathbb{R}^n$ be a DM mapping. Let $\Omega \subset \operatorname{intDom}(u)$ be an open set. Then u is Fréchet differentiable at almost every point of Ω.*

Proof. The corollary follows immediately from Corollary 85. □

PROPOSITION 92 *A mapping $A : X \to X^*$ is DM if and only if there exists a monotone mapping $k_A : X \to X^*$ such that for each $x, y \in X$*

$$|\langle Ax - Ay, x - y\rangle| \leq \langle k_A x - k_A y, x - y\rangle.$$

We will say that k_A is a (monotone) control mapping for DM mapping A.

Proof. Let A be DM, we define $k_A := A^{\uparrow} + A^{\downarrow}$. For this k_A we have by the monotonicity of $A^{\downarrow}$

$$\langle Ax - Ay; x - y\rangle = \langle A^{\uparrow}x - A^{\uparrow}y; x - y\rangle - \langle A^{\downarrow}x - A^{\downarrow}y; x - y\rangle \leq \langle k_A x - k_A y; x - y\rangle.$$

We apply the same for DM mapping $-A$ use the monotonicity of $A^{\uparrow}$ and obtain the control inequality

$$|\langle Ax - Ay; x - y\rangle| \leq \langle k_A x - k_A y; x - y\rangle.$$

Conversely, suppose the existence of a control mapping k_A and define $A^{\uparrow} := k_A$, $A^{\downarrow} := k_A - A$. We need only to show the monotonicity of $A^{\downarrow}$. For arbitrary $x, y \in X$ we have

$$\langle A^{\downarrow}x - A^{\downarrow}y; x - y\rangle = \langle k_A x - k_A y; x - y\rangle - \langle Ax - Ay; x - y\rangle \geq$$

$$\langle k_A x - k_A y; x - y\rangle - |\langle Ax - Ay; x - y\rangle| \geq 0.$$

This concludes the proof. □

Corollary 93 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of single-valued DM mappings from an open subset of a Banach space to its dual space and let k_n be a monotone control function for A_n. Assume that $A_n \to A$ and $k_n \to k$ (pointwise convergence). Then A is a DM mapping with the control function k.*

Proof. We pass to the limit in the inequality

$$\langle A_n x - A_n y; x - y\rangle \leq \langle k_n x - k_n y; x - y\rangle.$$

□

Remark 94 Suppose that A is a DM mapping and k is a monotone mapping such that $k - A$ is monotone. Then $\tilde{k} := 2k - A$ has the property that both of the mapping $\tilde{k} \pm A$ are monotone. Thus if we have a monotone mapping k such that at least one of the mappingf $k \pm A$ is monotone, then we can find a monotone control mapping for A.

Definition 95 We say that the mapping $A : \Omega \to X^*$ is *locally DM* if for every $x_0 \in \Omega$ there is its neighborhood $U(x_0)$ such that the restriction $A_{|U(x_0)} : U(x_0) \to X^*$ is DM.

Theorem 96 *Let $\Omega \subset \mathbb{R}^d$ be an open convex set and $A : \Omega \to \mathbb{R}^d$ be a locally DM mapping. Then A is a DM mapping.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ and $(K^n)_{n\in\mathbb{N}}$ be a nondecreasing sequences of compact convex subsets of Ω such that

$$\Omega = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} K^n,$$

$$K^1 \subset K_1 \subset K^2 \subset K_2 \subset \dots$$

and the distances $\operatorname{dist}(\partial K^j, \partial K_j)$ and $\operatorname{dist}(\partial K_j, \partial K^{j+1})$ are strictly positive.

Consider the set K_1, we find points $x_1^1, \dots, x_{j(1)}^1 \in K_1$ such that

$$K_1 \subset \bigcup_{i=1}^{j(1)} B\left(x_i^1, \frac{1}{4} r_i^1\right)$$

and A is DM as the mapping $A : B(x_i^1, r_i^1) \to \mathbb{R}^d$. Thus there is a monotone mapping $k_i^1 : B(x_i^1, r_i^1) \to \mathbb{R}^d$, for $i = 1, \dots, j(1)$ such that $k_i^1 - A : B(x_i^1, r_i^1) \to \mathbb{R}^d$ is a monotone mapping.

We define the sets $B_i^1 := B(x_i^1, \frac{3}{4} r_i^1) \cap K_1$, where $i = 1, \dots, j(1)$. Consider the restrictions $(k_i^1)_{|B_i^1}$ denoted again as k_i^1. These mappings are bounded (see Theorem 27) monotone mappings and they can be extended to monotone mappings $k_{i,1} : \mathbb{R}^d \to \mathbb{R}^d$ by Lemma 28. We put

$$k_1 := \sum_{i=1}^{j(1)} k_{i,1}.$$

This mapping is a monotone mapping such that $k_1 - A$ is monotone on the set K_1. It is sufficient to realize that $k_1 - A$ satisfies the monotonicity inequality (16) for arbitrary two points which are closer than $\varepsilon := \frac{1}{4}\min\{r_i^1; i = 1, \dots, j(1)\}$. But this is easy: take x, y, $|x-y| \le \varepsilon$, there is a ball $B(x_i^1, \frac{1}{4}r_i^1)$ which contains x and thus y is contained in $B(x_i^1, \frac{1}{2}r_i^1)$. We have the monotone mapping $k_{i,1}$ with the property that $k_{i,1} - A$ is a monotone mapping on the set $B(x_i, \frac{1}{2}r_i^1)$. This gives that k_1 has the desired property too.

Now we construct the mapping k^1 with the properties:
1) $k^1 : \mathbb{R}^d \to \mathbb{R}^d$ is a monotone mapping
2) $k^1 - A$ is a monotone mapping on the set K_2,
3) $k^1 = k_1$ on the set K^1.

Assume for a moment that we have constructed such function k^1. Then we can by the induction construct a sequence $(k^n)_{n\in\mathbb{N}}$ such that:
$1n$) $k^n : \mathbb{R}^d \to \mathbb{R}^d$ is a monotone mapping
$2n$) $k^n - A$ is a monotone mapping on the set K_{n+1},
$3n$) $k^n = k^{n+1}$ on the set K^{n+1}.
Hence the limit $\lim_{n\to\infty} k^n =: k$ exists uniformly on compact subsets of Ω and is a monotone function. Since $k = k^j$ on K^i for $j \ge i$, we have that $k - A$ is a monotone mapping on the whole set Ω.

It remains to construct the function k^1 :
we have the sets $K^1 \subset K_1 \subset K^2 \subset K_2$ and the mapping $k_1 : \mathbb{R}^d \to \mathbb{R}^d$ which is monotone and $k_1 - A$ is monotone on the set K_1, we seek for a mapping k^1 with the properties 1-3. Consider the set $\overline{K_2 \setminus K_1}$ and the mapping $k_1 - A$. This mapping is locally DM on this set and thus this set can be covered by a finite number of open balls $G_1^1, \dots, G_{m(1)}^1$ such that there are monotone mappings $\varrho_1^1, \dots, \varrho_{m(1)}^1$ with the property that $\varrho_i^1 + k_1 - A$ is monotone on the set G_i^1. We can suppose that $\operatorname{dist}(G_i^1, K^1) > \varepsilon_0 > 0$, $i = 1, \dots m(1)$ since $\operatorname{dist}(\partial K^1, \partial K_1) > 0$ by the assumption.

We define the sets $G_{1,i} := G_i^1 \cap (K_2 \setminus K_1)$. Let $i \in \{1, \dots, m(1)\}$ be fixed. Consider the function

$$\varrho_{1,i}(x) = \begin{cases} \varrho_i^1(x) - c_i, \ x \in G_{1,i} \\ 0, \ x \in K^1, \end{cases}$$

where the vector $c_i \in \mathbb{R}^d$ is chosen such that the mapping

$$\varrho_{1,i} : K^1 \cup G_{1,i} \to \mathbb{R}^d$$

is monotone.

Let justify the existence of such vector c_i. This vector is chosen suitably if and only if it satisfies

$$\langle \varrho_i^1(x) - c_i; x - z\rangle \ge 0, \ (x, z) \in G_{1,i} \times K^1.$$

Thus it suffices to take c_i which solves the inequality

$$\langle c_i; x - z\rangle \le -|\varrho_i^1(x)||x - z|,\ (x, z) \in G_{1,i} \times K^1. \tag{44}$$

Since the mapping ϱ_i^1 is bounded on the sets $G_{1,i}$ and the sets $K^1, G_{1,i}$ are bounded, we have that the right hand side of (44) can be estimated from below by some $\alpha < 0$. Hahn Banach theorem (applied on the compact convex sets K^1 and $\overline{G_i^1}$) enables to find $d_i \in \mathbb{R}^d$ and $\epsilon > 0$ such that

$$\langle d_i; x - z\rangle \le -\epsilon,\ (x, z) \in G_{1,i} \times K^1.$$

A multiple $-\frac{\alpha}{\epsilon} d_i$ is the desired vector c_i.

Let define the mappings $\rho_{1,i} : \mathbb{R}^d \to \mathbb{R}^d$ as monotone extensions of $\varrho_{1,i} : G_i^1 \to \mathbb{R}^d$ (see Lemma 28). Put

$$\rho_1 := \sum_{i=1}^{m(1)} \rho_{1,i}$$

and finally $k^1 := \rho_1 + k_1$ is the desired mapping satisfying 1-3. □

Corollary 97 *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz mapping. Then A is a DM mapping.*

Proof. Proof is a direct combination of Theorem 96 and Proposition 88. □

Lemma 98 *The mapping $A : X \to X^*$ is DM if and only if there exists a monotone mapping $M : X \to X^*$ such that for each $y, h \in X$ the inequality*

$$\varphi(y, h, M; r) \ge \varphi(y, h, A; r)$$

is satisfied for $r \in [0; 1]$. The function φ was defined in Lemma 44.

Proof. Assume the existence of a mapping M. Define the mapping $B := M - A$. We claim that B is monotone. By Lemma 44 it is sufficient to show that the function $\varphi(x, h, B; \cdot)$ is nondecreasing for each $x, h \in X$. We have for arbitrary $0 \le s < t \le 1$

$$\varphi(x, h, B; t) - \varphi(x, h, B; s) = \langle M(x+th) - M(x+sh); h\rangle - \langle A(x+th) - A(x+sh); h\rangle$$

$$= \varphi(x + sh, h, M; t - s) - \varphi(x + sh, h, A; t - s) \ge 0.$$

The proof of the reverse implication is similar. If $A(x) = A^{\uparrow}(x) - A^{\downarrow}(x)$ for every $x \in X$ and $A^{\uparrow}$, $A^{\downarrow}$ are monotone we define the mapping $M := A^{\uparrow} + A^{\downarrow}$. Verifying of the inequality

$$\varphi(y, h, M; r) \ge \varphi(y, h, A; r),$$

where $y, h \in X$ are arbitrary, is easy

$$\langle M(y+rh) - M(y); h\rangle = \langle A^{\uparrow}(y+rh) - A^{\uparrow}(y); h\rangle + \langle A^{\downarrow}(y+rh) - A^{\downarrow}(y); h\rangle \geq$$

$$\langle A^{\uparrow}(y+rh) - A^{\uparrow}(y); h\rangle - \langle A^{\downarrow}(y+rh) - A^{\downarrow}(y); h\rangle = \langle A(y+rh) - A(y); h\rangle.$$

We have used that the term $A^{\downarrow}(y+rh) - A^{\downarrow}(y); h\rangle$ is non-negative. □

THEOREM 99 *Let $D \subset X$ be an open convex set. Then the mapping $T : D \to X^*$ is DM if and only if there is a monotone mapping $A : D \to X^*$ such that for every line segment $L = [L_0, L_0 + L_1]$*

$$\bigvee_0^1 T_L^\star \leq \bigvee_0^1 A_L^\star, \tag{45}$$

where $A_L^\star(t) := \langle A(L_0 + tL_1); L_1\rangle$ and $T_L^\star$ is defined analogously.

Proof. Lemma 44 implies that the function $A_L^\star$ is nondecreasing. We put $B := A - T$ and show that B is monotone. Then it suffices to write $T = A - (A - T)$ for the proof of the DM property of T.

The monotonicity of B will be proved by showing that the function $B_L^\star$ is nondecreasing. So let $0 \leq t_1 < t_2 \leq 1$ be chosen, we have by the monotonicity of A, the assumption (45) and the definition of the variation

$$B_L^\star(t_2) - B_L^\star(t_1) = A_L^\star(t_2) - A_L^\star(t_1) - (T_L^\star(t_2) - T_L^\star(t_1)) \geq \bigvee_{t_1}^{t_2} A_L^\star - \bigvee_{t_1}^{t_2} T_L^\star \geq 0.$$

Actually, we have used the relation (45) for the line segment $[L_0 + t_1 L_1; L_0 + t_2 L_1]$. This gives the monotonicity of B.

For the reverse implication we put $A := T^{\uparrow} + T^{\downarrow}$, where $T = T^{\uparrow} - T^{\downarrow}$ is supposed to be DM. For arbitrary line segment L we have

$$\bigvee_0^1 T_L^\star = \bigvee_0^1 (T_L^{\uparrow\star} - T_L^{\downarrow\star}) \leq \bigvee_0^1 T_L^{\uparrow\star} + \bigvee_0^1 T_L^{\downarrow\star} = \bigvee_0^1 (T_L^{\uparrow\star} + T_L^{\downarrow\star}) = \bigvee_0^1 A_L^\star.$$

We have used that the functions $T_L^{\uparrow\star}$, $T_L^{\downarrow\star}$ are nondecreasing. □

REMARK 100 The similar result can be obtained for accretive mappings in smooth Banach spaces, the proof is an easy modification of the previous one where we use the fact that the duality mapping in smooth Banach space is single-valued.

PROPOSITION 101 *Let $T : X \to X$ be a mapping from a Hilbert space to itself. Assume that there exists a monotone Gateaux differentiable mapping $A : X \to X$ such that for each line segment $L = [L_0, L_0 + L_1]$*

$$\limsup_{\lambda \searrow 0} \bigvee_0^\lambda \left(t \mapsto \left\langle \frac{T(L_0 + tL_1) - T(L_0)}{\lambda}; L_1 \right\rangle \right) < \langle \delta A(L_0) L_1; L_1\rangle. \tag{46}$$

Then T is a DM mapping.

Proof. We need to verify the inequality (45).

We have guaranteed the existence of a positive number λ_0 such that for each $\lambda \le \lambda_0$

$$\bigvee_0^\lambda \left(t \mapsto \left\langle \frac{T(L_0+tL_1)-T(L_0)}{\lambda}; L_1 \right\rangle\right) < \langle \delta A(L_0)L_1; L_1\rangle.$$

Since A is Gateaux differentiable, we have

$$\bigvee_0^\lambda \left(t \mapsto \left\langle \frac{T(L_0+tL_1)-T(L_0)}{\lambda}; L_1 \right\rangle\right) < \lim_{s\searrow 0} \left\langle \frac{A(L_0+sL_1)-A(L_0)}{s}; L_1 \right\rangle.$$

This gives the existence of a number $s_0 > 0$ such that for $\lambda \le \min\{s_0, \lambda_0\}$

$$\bigvee_0^\lambda \left(t \mapsto \left\langle \frac{T(L_0+tL_1)-T(L_0)}{\lambda}; L_1 \right\rangle\right) \le \left\langle \frac{A(L_0+\lambda L_1)-A(L_0)}{\lambda}; L_1 \right\rangle.$$

We multiply both sides by the positive number λ, we use that the function

$$A^*_L(t) = \langle A(L_0+tL_1; L_1\rangle$$

is nondecreasing and by a simple manipulation we get the inequality (45) for line segments with the length bounded by $\min\{\lambda_0, s_0\}$, but this is sufficient for the proof that $A-T$ is monotone at each point and thus it is monotone (see Lemma 46). □

3.2 UDM mappings

Definition 102 Let X, Y be Banach spaces, $C \subset X$ be an open convex set. We say that a mapping $F : C \to Y$ is the *UDM mapping* if there is a monotone operator $f : C \to X^*$ such that for every $Q \in B_{\mathcal{L}(Y,X^*)}$ the mapping

$$Q \circ F + f : C \to X^*$$

is monotone. The monotone operator f is called the *control mapping for F*.

Remark 103 It is evident that the definition has a good sense for an arbitrary set C not necessarily open and convex and we will sometime use it, but for the differentiability properties the assumption of the openness is natural and sometimes the convexity assumption is also needed.

It is an easy observation that if $|a| \le b$ and F is controlled by f then aF is controlled by bf. Thus if we consider in Definition 102 the operators Q with the norm bounded by some $c > 0$ we obtain an equivalent definition and also it is obvious that the class of UDM mappings forms a linear space. Further, it is easy to see that if $F : C \subset X \to Y$ is an UDM mapping controlled by f and $L : Y \to Z$ is a continuous affine mapping then $L \circ F : C \to Z$ is an UDM mapping controlled by $\text{lip}(L)f$. Finally notice that for the case $Y = X^*$ every UDM mapping is a DM mapping. Indeed, we can write

$$F = \frac{1}{2}(f + F) - \frac{1}{2}(f - F).$$

DEFINITION 104 Let $\Omega \subset X$ be an open convex subset of a Hilbert space X. Let $u : \Omega \to X$ be a mapping. We say that u is the *δ-monotone mapping* if there is a number $\delta > 0$ such that for all $x, y \in \Omega$

$$\langle u(x) - u(y); x - y \rangle \geq \delta |u(x) - u(y)||x - y|. \tag{47}$$

REMARK 105 The class of the δ-monotone mappings is in detail studied in papers by L. Kovalev, see for instance [12].

PROPOSITION 106 *Every δ-monotone mapping is an UDM mapping. Consequently each linear combination of δ-monotone mappings is an UDM mapping.*

Proof. Let $Q \in B_{\mathcal{L}(X)}$ be arbitrary. We use (47) to estimate

$$\langle Q \circ u(x) - Q \circ u(y); x - y \rangle \leq |u(x) - u(y)||x - y| \leq \left\langle \frac{u}{\delta}(x) - \frac{u}{\delta}(y); x - y \right\rangle,$$

hence u/δ is a control mapping for u, consequently u is an UDM mapping. □

The following lemma will be useful.

LEMMA 107 *Let X, Y be Banach spaces and let $x \in X$ and $y \in Y$ be fixed. Then*

$$|x|_X |y|_Y = \sup\{\langle Qy; x \rangle; Q \in B_{\mathcal{L}(Y, X^*)}\}.$$

Proof. We can suppose that $|x|_X = |y|_Y = 1$. Hahn-Banach theorem gives functionals $x^* \in X^*, y^* \in Y^*$ with the norms equal to one such that $\langle x; x^* \rangle = \langle y; y^* \rangle = 1$.

We define the operator $Q_{x,y} z := \langle y^*; z \rangle x^*$. It is evident that $\|Q\| \leq 1$. We have

$$\langle Q_{x,y} y; x \rangle = \langle y^*; y \rangle \langle x^*; x \rangle = 1.$$

This concludes the proof. □

COROLLARY 108 *Let $C \subset X$ be an open convex set and let $F : C \to Y$ be an UDM mapping and $f : C \to X^*$ be its control mapping. Then for each $x_1, x_2 \in C$ the estimate*

$$|F(x_1) - F(x_2)|_Y |x_1 - x_2|_X \leq \langle f(x_1) - f(x_2); x_1 - x_2 \rangle \tag{48}$$

is satisfied. Moreover if F satisfies the inequality (48) then F is an UDM mapping with the control mapping f. We will call the inequality (48) the control inequality.

Proof. The corollary is an immediate consequence of Lemma 107. □

REMARK 109 This corollary immediately implies that if the control mapping f is Lipschitz continuous then F is Lipschitz continuous as well. The results of this type for d.c. mappings are nontrivial, see [8]. Further, if the control mapping f is continuous at a point x then the mapping F is continuous at the point x too.

COROLLARY 110 *Let $F_n : C \subset X \to Y$ and $f_n : C \to X^*$, $n \in \mathbb{N}$ be sequences of mappings. Assume that each F_n is an UDM mapping with a control mapping f_n and that for every $x \in C$ it is $F_n(x) \to F(x)$, $f_n(x) \to f(x)$ as $n \to \infty$. Then F is an UDM mapping with the control mapping f.*

Proof. We pass directly to the limit in the inequality

$$|F_n(x) - F_n(y)|_Y |x - y|_X \leq \langle f_n(x) - f_n(y); x - y \rangle.$$

□

COROLLARY 111 *Let $F : C \subset \mathbb{R}^n \to \mathbb{R}^d$, where C is an open convex set be an UDM mapping. Then $F \in RR^1_*(C; \mathbb{R}^d)$. Consequently $F \in BV(C; \mathbb{R}^d)$*

Proof. The proof is an easy combination of Lemma 108 and the definition of the class RR^1_* (Definition 78). Let f be a control mapping for F. Let $\mu \in \mathcal{M}^+(C)$ be a weight from the RR^1_* property of the mapping f. Let $B(x_0, r) \subset B(x_0, 2r) \subset C$ be balls. We have for $x, y \in B(x_0, r)$

$$\frac{|F(x) - F(y)|}{r} \leq \frac{|f(x) - f(y)|}{r} \leq \frac{\operatorname{osc}(f, B(x_0, r))}{r} \leq \fint_{B(x_0,2r)} d\mu,$$

we pass to the supremum for $x, y \in B(x_0, r)$. This implies the desired. The fact that $F \in BV(C; \mathbb{R}^d)$ follows by the same way from the well known characterization of the functions of bounded variation using one-dimensional sections, see [3]. Alternatively we can use Proposition 80. □

PROPOSITION 112 *Let $F : (a; b) \to Y$, where Y is a Banach space, be a mapping. Then F is an UDM mapping if and only if F has the locally finite variation.*

Proof. Let F be an UDM mapping, $f : (a; b) :\to \mathbb{R}$ be its nondecreasing control mapping and let $a < c < d < b$ be arbitrary. By Hahn-Banach theorem and by the control property we obtain

$$\bigvee_c^d F = \sup\{\sum_{i=1}^k |F(t_i) - F(t_{i-1})|; c = t_0 < t_1 < \ldots t_k = d\}$$

$$= \sup\{\sum_{i=1}^{k} \sup\{\langle F(t_i) - F(t_{i-1}); y^*\rangle; |y^*|_{Y^*} \leq 1\}; c = t_0 < t_1 < \dots t_k = d\}$$

$$\leq \sup\{\sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|; c = t_0 < t_1 < \dots t_k = d\} = \bigvee_c^d f < \infty.$$

Conversely let $F : (a; b) \to Y$ be an arbitrary mapping of the locally finite variation. Choose arbitrary $c \in (a; b)$ and define

$$f(t) = \begin{cases} \bigvee_c^t F,\ t \geq c \\ -\bigvee_t^c F,\ t < c. \end{cases}$$

If $y^* \in B_{Y^*}$, $a < s < t < b$ are given we have

$$\langle F(t) - F(s); y^*\rangle + f(t) - f(s) \geq f(t) - f(s) - |F(t) - F(s)| \geq 0.$$

Thus f is a control mapping for F. $\square$

The following two theorems present simple results about compositions of UDM mappings. We briefly recall one concept of linear functional analysis. Let V, X be Banach spaces and let $L \in \mathcal{L}(V, X)$ be a bounded linear operator. We define the operator $L_* \in \mathcal{L}(X^*, V^*)$ by he formula

$$\langle L_* x^*; v\rangle := \langle x^*; Lv\rangle,\ v \in V,\ x^* \in X^*.$$

Then the operator L_* is called *Banach adjoint operator*.

The operator $L \in \mathcal{L}(V, X)$ is called *bounded from bellow* if there is $\epsilon > 0$ so that for every $v \in V$ it is

$$|Lv|_X \geq \epsilon |v|_V.$$

THEOREM 113 *Let V, X, Y be Banach spaces and let $D \subset V$ and $C \subset X$ be open convex sets. Let $L \in \mathcal{L}(V, X)$ be a bounded linear operator which is bounded from bellow and which fulfills $LD \subset C$. Let $F : C \to Y$ be an UDM mapping. Then the composition $F \circ L : D \to Y$ is an UDM mapping.*

Proof. Let $\epsilon > 0$ be a non-negative number from the boundedness from bellow of the operator L, let $f : C \to X^*$ be a monotone control mapping for F, and let $u, v \in D$ be arbitrary. We have

$$\begin{aligned} |F \circ L(u) - F \circ L(v)|_Y |u - v|_V &\leq \frac{1}{\epsilon} |F \circ L(u) - F \circ L(v)|_Y |Lu - Lv|_X \\ &\leq \frac{1}{\epsilon} \langle f \circ L(u) - f \circ L(v); Lu - Lv\rangle \\ &= \frac{1}{\epsilon} \langle L_* \circ f \circ L(u) - L_* \circ f \circ L(v); u - v\rangle. \end{aligned}$$

Thus we see by Corollary 108 that

$$\frac{1}{\epsilon} L_* \circ f \circ L : D \to V^*$$

is a monotone control mapping for $F \circ L$. □

THEOREM 114 *Let X, Y, Z be Banach spaces, let C resp. D be an open convex set of X resp. Y. Assume that $F : C \to D$ be an UDM mapping with a control mapping f and let $G : D \to Z$ be Lipschitz continuous with a constant ℓ_G. Then the composition mapping $G \circ F : C \to Z$ is an UDM mapping with the control mapping $\ell_G f$.*

Proof. Let $Q \in B_{\mathcal{L}(Z,X^*)}$ and $x, y \in C$ be arbitrary. We use Corollary 108 and the Lipschitz continuity of G to estimate

$$\begin{aligned}\langle Q \circ G \circ F(x) - Q \circ G \circ F(y); x - y\rangle &\le \ell_G |F(x) - F(y)|_Y |x - y|_X \\ &\le \langle \ell_G f(x) - \ell_G f(y); x - y\rangle.\end{aligned}$$

Thus $\ell_G f$ is a control mapping for $G \circ F$ and hence $G \circ F$ is an UDM mapping.□

It can be suitable to slightly generalize Definition 102.

DEFINITION 115 Let X and Y be Banach spaces, let C be an open convex subset of X and let $\Phi : C \to 2^Y$ be a multi-mapping. We say that Φ is the *UDM multi-mapping* if there is a monotone multi operator $\phi : C \to 2^{X^*}$ such that for arbitrary two pairs $(x, y), (x', y') \in \mathrm{Gr}(\Phi)$

$$|y - y'|_Y |x - x'|_X \le \sup\{\langle \xi - \xi'; x - x'\rangle; \xi \in \phi(x), \xi' \in \phi(x')\} \tag{49}$$

and for every $x \in C$ we have

$$\mathrm{diam}\Phi(x) \le \mathrm{diam}\phi(x). \tag{50}$$

The multi-operator ϕ is called the *control multi-mapping* and the inequality (49) is again called the *control inequality.*

REMARK 116 It is easily seen (Corollary 108) that for a single-valued UDM mapping F with a single-valued monotone mapping f Definition 115 is equivalent to Definition 102. The seemingly artificial condition (50) is evidently fulfilled in the single valued case and ensures that $\Phi(x)$ is a singleton whenever $\phi(x)$ is a singleton.

The following proposition we formulate for $X = \mathbb{R}^d$, it is possible that it is valid in higher generality too.

Proposition 117 *Let C be an open convex subset of $\mathbb{R}^d$, let Y be a Banach space and let $F : C \to 2^Y$ be an UDM multi-mapping. Denote by $L := \overline{\mathrm{Gr}(F)}^{C\times Y}$ (the relative closure in $C \times Y$ of the graph of F with respect to the euclidean topology on C and the norm topology on Y). Then L is the graph of an UDM multi-mapping $\overline{F} : C \to 2^Y$.*

Proof. Let $f : C \to 2^{\mathbb{R}^d}$ be a monotone control multi-mapping for F. Let $\overline{f} : C \to 2^{\mathbb{R}^d}$ be the restriction on the set C of a maximal monotone extension of f (see Remark 19). We show that $\overline{f}$ is a control multi-mapping for $\overline{F}$.

Choose arbitrary two distinct pairs $(x, y), (x', y') \in \mathrm{Gr}(\overline{F}) \subset C \times Y$. We can find the sequences $(x_n, y_n)_{n\in\mathbb{N}}, (x'_n, y'_n)_{n\in\mathbb{N}} \subset \mathrm{Gr}(F) \subset C \times Y$ such that

$$|x - x_n| \to 0,\ |x' - x'_n| \to 0,\ |y - y_n|_Y \to 0,\ |y' - y'_n|_Y \to 0$$

as $n \to \infty$. We can suppose $x_n \neq x'_n$, $n \in \mathbb{N}$. Since F is the UDM multi-mapping we have for $n \in \mathbb{N}$

$$|y_n - y'_n|_Y \leq \frac{\sup\{\langle \xi_n - \xi'_n; x_n - x'_n\rangle; \xi_n \in \overline{f}(x_n),\ \xi'_n \in \overline{f}(x'_n)\}}{|x_n - x'_n|}.$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given. We can find $\tilde{\xi}_n \in \overline{f}(x_n)$ and $\tilde{\xi}'_n \in \overline{f}(x'_n)$ such that

$$|y_n - y'_n| - \varepsilon \leq \frac{\langle \tilde{\xi}_n - \tilde{\xi}'_n; x_n - x'_n\rangle}{|x_n - x'_n|}. \tag{51}$$

Since $x, x' \in \mathrm{intDom}(\overline{f})$ we can suppose that $x_n, x'_n \in \mathrm{intDom}(\overline{f})$. Theorem 27 implies that the sequences $(\tilde{\xi}_n)_{n\in\mathbb{N}}$, $(\tilde{\xi}'_n)_{n\in\mathbb{N}}$ are bounded. Thus we can suppose (after passing to a subsequence if necessary) that $\tilde{\xi}_n \to \tilde{\xi}$, $\tilde{\xi}'_n \to \tilde{\xi}'$ as $n \to \infty$. Proposition 26 gives that $\tilde{\xi} \in \overline{f}(x)$ and $\tilde{\xi}' \in \overline{f}(x')$.

The limit passage in (51) gives

$$\begin{aligned}|y - y'|_Y - \varepsilon &\leq \frac{\langle \tilde{\xi} - \tilde{\xi}'; x - x'\rangle}{|x - x'|}\\ &\leq \frac{\sup\{\langle \xi - \xi'; x - x'\rangle; \xi \in \overline{f}(x), \xi' \in \overline{f}(x')\}}{|x - x'|}.\end{aligned}$$

Since ε was arbitrary the inequality (49) is proved.

The proof of condition (50) is easy too. Let $(x, y), (x, y') \in \mathrm{Gr}(\overline{F})$ be chosen arbitrarily. We find the sequences $(x_n, y_n)_{n\in\mathbb{N}}, (x'_n, y'_n)_{n\in\mathbb{N}} \subset \mathrm{Gr}(F)$ which converge to $(x, y), (x, y')$ respectively. Distinguishing the cases $x_n = x'_n$ and $x_n \neq x'_n$ and using the conditions (49) and (50) for F and its control multi-mapping $\overline{f}$ we infer

$$|y_n - y'_n|_Y \leq \sup\{|\xi_n - \xi'_n|; \xi_n \in \overline{f}(x_n), \xi'_n \in \overline{f}(x'_n)\}.$$

For $\varepsilon > 0$ and $n \in \mathbb{N}$ we find $\tilde{\xi}_n \in \overline{f}(x_n)$ and $\tilde{\xi}'_n \in \overline{f}(x'_n)$ such that

$$|y_n - y'_n|_Y - \varepsilon \leq |\tilde{\xi}_n - \tilde{\xi}'_n|. \tag{52}$$

After passing to a subsequence we can again suppose that $\tilde{\xi}_n \to \tilde{\xi}$, $\tilde{\xi}'_n \to \tilde{\xi}'$ as $n \to \infty$ and $\tilde{\xi} \in \overline{f}(x)$ and $\tilde{\xi}' \in \overline{f}(x)$. Hence

$$|y - y'| - \varepsilon \leq |\tilde{\xi} - \tilde{\xi}'| \leq \mathrm{diam}\overline{f}(x).$$

This concludes the proof. □

The following simple lemma may be sometime useful.

LEMMA 118 *Let C be an open convex subset of a Banach space X and let $F : C \to Y$ be an UDM mapping with a monotone control mapping $f : C \to X^*$. Assume that $f(C)$ is a separable subset of X^*. Then $F(C)$ is a separable subset of Y. In particular if X^* is separable than $F(C)$ is a separable subset of Y.*

Proof. Let $(x_n^*)_{n\in\mathbb{N}}$ be a countable dense subset of $f(C)$ and let $x_n \in C$ be such that $x_n^* = f(x_n)$, $n \in \mathbb{N}$. Let $y_n = F(x_n)$, $n \in \mathbb{N}$. If $y \in F(C)$ and $\varepsilon > 0$ are arbitrary we find $x \in C$ such that $y = F(x)$ and $x_j^* \in f(C)$ such that $|f(x) - x_j^*|_X \leq \varepsilon$. Corollary 108 implies

$$|y - y_j|_Y \leq |f(x) - f(x_j)|_X = |f(x) - x_j^*|_X \leq \varepsilon.$$

Thus $(y_n)_{n\in\mathbb{N}}$ is a countable dense subset of $F(C)$. □

LEMMA 119 *Let $C \subset \mathbb{R}^n$ be an open convex set and let $F : C \to Y$, where Y is a Banach space, be an UDM mapping. Let $\overline{F} : C \to 2^Y$ be a multi-mapping whose graph is the relative closure of the graph of F in $C \times Y$ with respect to the norm topology. Then $\overline{F}(C)$ is a separable subset of Y.*

Proof. Since the graph $\mathrm{Gr}(F)$ is a subset of $C \times F(C)$, which is separable by Lemma 118, we have that $\mathrm{Gr}(F)$ is separable. Let $\overline{F}$ be a mapping whose graph is the closure of the graph of F. Thus $\mathrm{Gr}(\overline{F})$ is separable too. Since $\overline{F}(C)$ is the image under continuous projection of $\mathrm{Gr}(\overline{F})$ it is separable as well. □

The inspiration for the following theorem is a well known fact that in Banach spaces with the Radon-Nikodým property the curves with the locally finite variation are Fréchet differentiable almost everywhere. More precisely, a nontrivial result says that a Banach space Y has the Radon-Nikodým property if and only if every $\phi : (a; b) \to Y$ such that $\bigvee_c^d \phi < \infty$ for all $a < c < d < b$ is Fréchet differentiable almost everywhere in $(a; b)$. This characterization seems to be more transparent than the classical definition of the Radon-Nikodým property, thus we take it as the definition, see [14] and references therein.

THEOREM 120 *Let $C \subset \mathbb{R}^n$ be an open convex set and let Y be a Banach space having the Radon-Nikodým property and suppose that there exists a countable* *total subset $\mathcal{Q} \subset B_{\mathcal{L}(Y,\mathbb{R}^n)}$ (i.e. for every $y \neq 0$ there is $Q \in \mathcal{Q}$ such that $Qy \neq 0$.) If $F : C \to Y$ is an UDM mapping, which is Borel measurable, then F is Fréchet differentiable almost everywhere in C.*

Proof. Let $f : C \to \mathbb{R}^n$ be a monotone control mapping for F. We infer from Mignot theorem 59 that the control mapping f has the Fréchet derivative $f'(c)$ and that $Q \circ F$ has the Fréchet derivative $A_{Q,c}$ for almost every $c \in C$.

Let $\tilde{E} := \{\mathbf{e}_i; i = 1, \dots, n\}$, where $\mathbf{e}_i$ is the i-th canonical vector and let $E \supset \tilde{E}$ be a countable dense subset of $S^{n-1} := \{|\cdot| = 1\}$. For $v \in E$ denote by Δ_v the set of all points $c \in C$ such that the directional derivative $\delta F(c; v)$ of F at the point c in the direction v exists. We denote

$$A_{c,v} := \delta F(c; v),\ c \in \Delta_v,\ v \in E.$$

Let $E^v_{i,j,k}$ be the set of all such points of the form (x, t, s), where $x \in C$, $t, s \in \mathbb{R}$, $\frac{1}{i} \leq |t| \leq \frac{1}{j}$, $\frac{1}{i} \leq |s| \leq \frac{1}{j}$,

$$\left| \frac{F(x + tv) - F(x)}{t} - \frac{F(x + sv) - F(x)}{s} \right|_Y \leq \frac{1}{k}$$

and $\overline{B(x, 1/j)} \subset C$. Since $F : C \to Y$ is a Borel measurable mapping the set$E^v_{i,j,k}$ is a Borel subset of

$$Z := \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}.$$

Z is a separable complete metric space. Let π_1 be the projection of Z on the first component i.e.

$$\pi_1 : (x, t, s) \mapsto x.$$

The set $\pi_1 E^v_{i,j,k}$ is an analytic subset of $\mathbb{R}^n$ by Proposition 15, thus it is a measurable set. Since

$$\Delta_v = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{i=j+1}^{\infty} \pi_1 E^v_{i,j,k}$$

we have that the set Δ_v is measurable.

Take an arbitrary point $c \in C$. Define the mappings $F_{c,v} : (a_{c,v}; b_{c,v}) \to Y$ by the formula

$$F_{c,v}(t) := F(c + tv).$$

The functions

$$\varphi_{c,v}(t) := \langle f(c + tv); v \rangle$$

are defined and nondecreasing (see Lemma 44) on intervals $(a_{c,v}; b_{c,v})$.

Let $a_{c,v} < s < t < b_{c,v}$ be arbitrary. Since $|v| = 1$, Corollary 108 implies

$$\begin{aligned} |F_{c,v}(t) - F_{c,v}(s)|_Y |t - s| &= |F(c + tv) - F(c + sv)|_Y |t - s||v| \\ &\leq \langle f(c + tv) - f(c + sv); (t - s)v \rangle \\ &= (\varphi_{c,v}(t) - \varphi_{c,v}(s))(t - s). \end{aligned}$$

Thus $F_{c,v}$ is an UDM mapping. Proposition 112, the Radon-Nikodým property and Fubini theorem give that $|C \setminus \Delta_v| = 0$ for all $v \in E$.

It is sufficient to prove that there is the Fréchet derivative $F'(a)$ for all $a \in C$ satisfying:
i) there exists the Fréchet derivative $f'(a)$,
ii) $a \in \bigcap_{v \in E} \Delta_v$,
iii) there exists the Fréchet derivative $(Q \circ F)'(a) =: A_{Q,a}$ for all $Q \in \mathcal{Q}$.

Choose such a point a, it is not restrictive to assume that

$$a = f(a) = 0,\ F(a) = 0.$$

We identify an element $A_{c,v} \in Y$ with the linear mapping

$$\mathbb{R} \ni t \mapsto tA_{c,v}.$$

At first we realize, that there is a unique bounded linear operator $A \in \mathcal{L}(\mathbb{R}^n, Y)$ such that

$$A(tv) = A_{a,v}(t),\ v \in \tilde{E}. \tag{53}$$

Indeed, we consider the vectors $A_{a,v} \in Y$, $v \in \tilde{E}$ and for $\mathbb{R}^n \ni h = \sum_{i=1}^n t_i \mathbf{e}_i$ define

$$A(h) := \sum_{i=1}^n t_i A_{a,\mathbf{e}_i}.$$

Then A is the desired operator satisfying (53). We show that A is the desired Fréchet derivative $F'(a)$. By the uniqueness of the Fréchet derivative we have

$$A_{Q,a}(tv) = Q \circ A_{a,v}(t),\ v \in E,\ Q \in \mathcal{Q}. \tag{54}$$

This together with (53) gives

$$A_{Q,a}(tv) = Q \circ A(tv),\ v \in \tilde{E},\ Q \in \mathcal{Q}$$

thus

$$A_{Q,a} = Q \circ A,\ Q \in \mathcal{Q}. \tag{55}$$

Now by (55) and by (54) we obtain for all $v \in E$, $Q \in \mathcal{Q}$

$$Q \circ A(tv) = A_{Q,a}(tv) = Q \circ A_{a,v}(t).$$

The totality of $\mathcal{Q}$ implies $A(tv) = A_{a,v}(t)$ for $v \in E$.

Let $\varepsilon > 0$ be given. We can find a finite set $K \subset E$ such that for every $v \in S^{n-1}$ there is $\bar{z} = \bar{z}(v) \in K$ satisfying $|v - \bar{z}| < \varepsilon$.

Let $\delta > 0$ be such that

$$(|t| < \delta\,\&\, v \in K) \Rightarrow (|F(tv) - A(tv)|_Y \leq \varepsilon t), \tag{56}$$

$$(|h| < \delta) \Rightarrow (|f(h) - f'(a)h| \leq \varepsilon|h|). \quad (57)$$

Choose arbitrary $h \in \mathbb{R}^n$, $|h| < \delta$. Consider $\bar{z} := \bar{z}(v)$, where $v := \frac{h}{|h|}$ and put $z := |h|\bar{z}$, thus we have

$$|z| \leq |h|,\ |z - h| \leq |h||\bar{z} - v| \leq \varepsilon|h|. \quad (58)$$

Lemma 107 implies $|F(h)-F(z)|_Y \leq |f(h)-f(z)|$, using this fact we estimate

$$|F(h) - Ah|_Y \leq |F(h) - F(z)|_Y + |F(z) - Ah|_Y$$

$$\leq |f(h) - f(z)| + |F(z) - Az|_Y + |Az - Ah|_Y =: T_1 + T_2 + T_3.$$

The term T_3 can be estimated using (58) as $T_3 \leq M\varepsilon|h|$, where M is the norm of A. The term $T_2 \leq \varepsilon|h|$ thanks to (56). Finally the term T_1 we treat as

$$T_1 = |f(h) - f(z)| \leq |f(h) - f'(a)h| + |f'(a)h - f'(a)z| + |f'(a)z - f(z)|.$$

This can be similarly estimated with help of (57), (58) and denoting the norm of $f'(a)$ by L as

$$T_1 \leq \varepsilon|h| + \varepsilon|h|L + \varepsilon|h|.$$

Hence $T_1 + T_2 + T_3 \leq N\varepsilon|h|$, where N does not depend on ε and h, which immediately gives $A = F'(a)$. □

Remark 121 The straightforward generalization enables to consider not necessarily UDM mappings but only mappings which are locally UDM, i.e. for every $x \in C$ there is a ball $U(x)$ such that $F_{|U(x)}$ is an UDM mapping.

Definition 122 A mapping $A : X \to 2^{X^*}$, where X is a Banach space is called *strongly monotone* if there is a constant $\beta > 0$ such that for every $x_1, x_2 \in X$, $x_1^* \in Ax_1, x_2^* \in Ax_2$ the inequality

$$\langle x_1^* - x_2^*; x_1 - x_2\rangle \geq \beta\|x_1 - x_2\|^2 \quad (59)$$

is satisfied.

Definition 123 Let X be a Banach space. The function $\delta_X : [0; 2] \to \mathbb{R}$, defined by the formula

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\|; \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\right\},$$

is called the *modulus of convexity of the Banach space* X. We say that the modulus of convexity is of the power type 2 if there is a constant $b > 0$ such that for every $\varepsilon \in [0; 2]$

$$\delta_X(\varepsilon) \geq b\varepsilon^2.$$

Proposition 124 (see [28], [5]) *For a Banach space $(X, |\cdot|)$ the following conditions are equivalent*
(i) the duality mapping is strongly monotone,
(ii) the modulus of convexity is of the power type 2,
(iii) X satisfies the weak parallelogram law*, i.e. there is $c > 0$ such that*

$$2|x|^2 + 2|y|^2 \geq |x + y|^2 + c|x - y|^2, x, y \in X.$$

In the following proposition it will be suitable to extend slightly Definition 102. We will admit the monotone control mapping f in that definition to be multivalued, otherwise we would have to restrict ourselves only to smooth Banach spaces.

Definition 125 We say that the mapping $F : C \to Y$ is the mapping of the class UDM* if there is a monotone multi-operator $f : C \to 2^{X^*}$ such that for each $Q \in B_{\mathcal{L}(Y,X^*)}$ the multi-operator

$$Q \circ F + f : C \to 2^{X^*}$$

is a monotone multi-operator.

Let us note that this definition is more restrictive than to say that F is a single-valued operator which is an UDM multi-mapping in the sense of Definition 115.

Recall some basic facts from the linear functional analysis. Consider a continuous bilinear mapping

$$b : X \times X \to Y,$$

where X and Y are arbitrary Banach spaces. We associate with the continuous bilinear mapping b the continuous linear mapping $\tilde{b} : X \to \mathcal{L}(X, Y)$ defined by the formula

$$\tilde{b}x(y) := b(x, y),$$

where $x \in X$, $y \in Y$. It can be easily seen that $\tilde{b}$ is Lipschitz continuous with the constant $\|b\|$ where $\|b\|$ is defined as the smallest constant C such that

$$|b(x, y)|_Y \leq C|x|_X|y|_X.$$

The reverse correspondence between $\tilde{b}$ and b is obvious. For the details see [7].

Definition 126 Let $(X, |\cdot|_X)$ be a Banach space. We say that a maximal cyclically monotone multi-mapping $k : X \to 2^{X^*}$ is the *equivalent duality mapping* if for each $x \in X$ the number $\langle k_x; x\rangle$ is independent on the choice of $k_x \in k(x)$ and $\|x\|_X^2 := \langle k_x; x\rangle$ defines a norm which is equivalent to the original norm $|\cdot|_X$, and $\|x\|_X^2 = \|k_x\|_*^2$, where $\|\cdot\|_*$ is a canonical dual norm to the norm $\|\cdot\|_X$.

THEOREM 127 *For a Banach space X the following conditions are equivalent:*
(i) X admits an equivalent norm such that the duality mapping associated with this new norm is strongly monotone.
(i') X admits an equivalent norm with the modulus of convexity of power type 2.
(ii) There exists a cyclically monotone multi-mapping $k : X \to 2^{X^*}$, which is an equivalent duality mapping, such that for every open convex set $C \subset X$ and every Banach space Y every Lipschitz continuous mapping $F : C \to Y$ is an UDM mapping with the monotone control multi-mapping* $\mathrm{lip}(F)k$.
(iii) There exists a cyclically monotone multi-mapping $k : X \to 2^{X^*}$ which is an equivalent duality mapping such that every continuous linear mapping $\tilde{b} : X \to X^*$ is an UDM mapping with a control multi-mapping* $\mathrm{lip}(\tilde{b})k$.
(iv) For every open convex set $C \subset X$ and every Banach space Y, each Lipschitz continuous mapping $F : C \to Y$ is an UDM mapping with a control multi-mapping k^F, which is an equivalent duality mapping .*
(v) For every Banach space Y and every continuous bilinear mapping $b : X \times X \to Y$ the associated linear mapping $\tilde{b} : X \to \mathcal{L}(X, Y)$ is an UDM mapping with a monotone control multi-mapping k^b, which is an equivalent duality mapping.*

Proof.
(i)⇔(i') follows from Proposition 124.
(i)⇒(ii). Let $j : X \to 2^{X^*}$ be the duality mapping and define $k := \frac{j}{\beta}$, where β is the number from the definition of strong monotonicity (Definition 122). Let $F : C \to Y$ be a Lipschitz continuous mapping, denote $L := \mathrm{lip}(F)$ and let $Q \in B_{\mathcal{L}(Y,X^*)}$ be arbitrary. We have for $x, y \in C$ and $k_x \in k(x), k_y \in k(y)$

$$\langle Q \circ F(x) + Lk_x - Q \circ F(y) - Lk_y ; x - y\rangle \geq -|F(x) - F(y)|_Y |x - y|_X + L|x - y|_X^2 \geq 0.$$

Thus Lk is the desired monotone control multi-mapping and the UDM* property of F is proved. The maximality and the cyclical monotonicity of k follows from Proposition 37.
(ii)⇒(iii). Using the properties of bilinear form b and its associated linear mapping $\tilde{b}$ and the basic properties of duality mapping the proof of the implication (ii)⇒(iii) is easy.
The implications (ii)⇒(iv) and (iv)⇒(v) are trivial.
(iii)⇒(i). Let $b : X \times X \to \mathbb{R}$ be a continuous bilinear form, we know that the associated linear mapping $\tilde{b} : X \to X^*$ is Lipschitz continuous. Let $x, y \in X$ be given, we can find $\varrho^* \in S_{X^*}$ such that $\langle \varrho^*; x - y\rangle = |x - y|_X$. Consider the continuous symmetric bilinear form

$$b_{x,y}(u, v) := \langle \varrho^*; u\rangle\langle \varrho^*; v\rangle$$

and the corresponding linear mapping

$$\langle \tilde{b}_{x,y}(u); v\rangle := b_{x,y}(u, v).$$

We have for all linear mappings $\tilde{b}_{x,y}$, $x, y \in X$ the existence of a multi-mapping k with the postulated properties such that for all $x, y \in X$, $k_x \in k(x), k_y \in k(y)$

$$\begin{aligned}\langle k_x - k_y; x - y\rangle &\geq \langle \tilde{b}_{x,y}(x) - \tilde{b}_{x,y}(y); x - y\rangle \\ &= \langle \varrho^*; x\rangle\langle \varrho^*; x - y\rangle - \langle \varrho^*; y\rangle\langle \varrho^*; x - y\rangle \\ &= |x - y|_X^2.\end{aligned}$$

Thus defining the norm $\|x\|_X^2 := \langle k_x; x\rangle$ we obtain a norm which is equivalent to the norm $|\cdot|_X$. Let l be the duality mapping associated to the norm $\|\cdot\|_X$. We have to realize that $l = k$. Since for every $x \in X$ and every $g \in k(x)$ we have $\langle g; x\rangle = \|g\|_*^2 = \|x\|_X^2$ we conclude $k \subset l$. The maximality of k implies $k = l$.

Using the equivalence of the norms $|\cdot|_X$ and $\|\cdot\|_X$ we see that the duality mapping associated with the norm $\|\cdot\|_X$ satisfies the condition of the strong monotonicity.
(v)⇒(i). Let $\Xi \subset S_{X^*}$ be an arbitrary norming set i.e. for each $x \in X$

$$|x|_X = \sup\{\langle \xi; x\rangle; \xi \in \Xi\}.$$

Consider the continuous bilinear mapping

$$b : X \times X \to \ell^\infty(\Xi \times \Xi)$$

defined by the formula

$$b(x_1, x_2) := (\langle \xi_1; x_1\rangle\langle \xi_2; x_2\rangle)_{(\xi_1,\xi_2)\in\Xi\times\Xi}$$

and the corresponding linear mapping

$$\tilde{b} : X \to \mathcal{L}(X, \ell^\infty(\Xi \times \Xi)).$$

Thus we have $\tilde{b} : X \to \mathcal{L}(X, Y)$ is a bounded linear operator, where

$$Y := \ell^\infty(\Xi \times \Xi).$$

The assumption gives the existence of a multi-mapping k^b such that k^b is the equivalent duality mapping, and k^b satisfies

$$\langle k_x^b - k_y^b; x - y\rangle \geq \langle Q \circ \tilde{b}(x) - Q \circ \tilde{b}(y); x - y\rangle,$$

for $x, y \in X$, $k_x^b \in k^b(x), k_y^b \in k^b(y)$ and $Q \in \mathcal{L}(\mathcal{L}(X, Y), X^*)$, $\|Q\| \leq 1$ (we use the standard operator norm). We claim that k^b is strongly monotone. Using Lemma 107, the expression of the operator norm, the relation between b and $\tilde{b}$

and the norming property of the set Ξ, we can write

$$\begin{aligned}\langle k_x^b - k_y^b; x-y\rangle &\geq |x-y|_X ||\tilde{b}x - \tilde{b}y||_{\mathcal{L}(X,Y)} \\ &= |x-y|_X \sup_{|h|_X \leq 1} |\tilde{b}(x-y)h|_Y \\ &= |x-y|_X \sup_{|h|_X \leq 1} \Big| (\langle \xi_1; x-y\rangle \langle \xi_2; h\rangle)_{(\xi_1,\xi_2)\in \Xi\times\Xi} \Big|_Y \\ &= |x-y|_X \sup_{|h|_X \leq 1} (\sup_{(\xi_1,\xi_2)\in\Xi\times\Xi} |\langle \xi_1; x-y\rangle\langle \xi_2; h\rangle|) \\ &= |x-y|_X^2 \sup_{|h|_X\leq 1} |h|_X = |x-y|_X^2.\end{aligned}$$

We again infer that k^b is a duality mapping associated with the norm $\|\cdot\|_X$. This together with the equivalence of the norms $\|\cdot\|_X$ and $|\cdot|_X$ implies the strong monotonicity of the mapping k^b. □

Remark 128 We can show the existence of a nontrivial Banach space which satisfies the condition i) from the theorem 127. Consider the space

$$X := L^p(S, \mathcal{S}, \mu),\ 1 < p < 2,$$

where $(S, \mathcal{S}, \mu)$ is arbitrary measure space. It is sufficient to prove that the modulus of convexity of X satisfies the condition (iii) from proposition 124. Recall the well known Clarkson's inequality

$$\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \leq 2(\|f\|_p^p + \|g\|_p^p)^{p'-1},$$

which holds for arbitrary $f, g \in X$ for $p \in (1;2)$ and $p' = \frac{p}{p-1}$. Using this inequality and the definition of modulus of convexity, we obtain

$$\delta_X(\varepsilon) \geq 1 - \frac{1}{2}(2^{p'} - \varepsilon^{p'})^{\frac{1}{p'}}.$$

Thus by concavity of the function $t \mapsto t^{\frac{1}{p'}}$ we have

$$1 - \delta_X(\varepsilon) \leq \left[1 - \left(\frac{\varepsilon}{2}\right)^{p'}\right]^{\frac{1}{p'}} \leq 1 - \frac{1}{p'}\left(\frac{\varepsilon}{2}\right)^{p'}.$$

Thus the inequality $\delta_X(\varepsilon) \geq b\varepsilon^2$ is satisfied for suitable constant $b > 0$.

Remark 129 The introducing of UDM mappings provides a possibility to define a similar concept of DM mappings between two arbitrary Banach spaces. This generalization lacks some good properties posed by the generalization of d.c. functions, which is due to L. Veselý and L. Zajíček.

We can modify the definition of DM mapping by taking formally the derivative of a d.c. mapping and remove the assumption of the potentiality. Thus we obtain the following definition.

Let X, V be Banach spaces and let $C \subset X$ be an open convex set. Let $F : C \to \mathcal{L}(X, V)$ be a mapping. For $g : C \to X^*$ and $v^* \in B_{V^*}$ consider the mapping

$$k_{v^*}^g : C \to X^*,$$

defined by

$$\langle k_{v^*}^g(x); \tilde{x} \rangle = \langle g(x); \tilde{x} \rangle + \langle v^*; F(x)\tilde{x} \rangle, \ \tilde{x} \in X.$$

We say that F is the *generalized DM mapping* if there is a monotone operator $f : C \to X^*$ such that for every $v^* \in B_{V^*}$ the mapping $k_{v^*}^f : C \to X^*$ is monotone. For $V = \mathbb{R}$ this definition coincides with the definition of DM mapping.

This definition is quite complicated and it is not more studied in this text but it seems that some interesting results could be obtained in this direction.

3.3 Examples and applications of DM and UDM mappings

Example 130 We show that in the contrast to the one-dimensional case there is a function $u \in BV(\Omega; \mathbb{R}^n)$ which is not the DM mapping. At first notice that if $v : \Omega \to \mathbb{R}^n$ is a DM mapping then, by Lemma 44, we have for all closed line segments $L = [L_0; L_0 + L_1] \subset \Omega$

$$\bigvee_0^1 v_L^\star < \infty$$

where $v_L^\star(t) = \langle v(L_0 + tL_1); L_1 \rangle$. Suppose that we have a function $u_1 \in BV(\Omega)$ for which holds

$$\bigvee (u_1, L) = \infty,$$

where $L := [0, \mathbf{e}_1]$. Then we put $u := (u_1, 0, \ldots, 0)$. Thus we have $u_L^\star(t) = u_1(t\mathbf{e}_1)$ which gives

$$\bigvee_0^1 u_L^\star = \infty,$$

Thus u can not be DM. It is well known that the mapping defined on an open subset Ω of $\mathbb{R}^n$ is a mapping of bounded variation if and only if it has a representative which has the bounded variation over almost all line sections of Ω by the lines parallel with the coordinate axis. Examples of functions of the bounded variations which does not have the finite variation on all line segments are known. We construct such an example using the results about monotone mappings.

For the transparency we will work only in the two-dimensional space but it will be clear that a similar construction can be done in a space of an arbitrary finite dimension. Let $[a;b] \subset \mathbb{R}$ be a compact interval and let $f : [a;b] \to \mathbb{R}$ be an arbitrary bounded function which does not have the finite variation $\bigvee_a^b f$. Consider the function $u : D := [a;b] \times \{0\} \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula

$$u(x_1, x_2) = (x_1, f(x_1)).$$

It is easily seen that this function $u : D \to \mathbb{R}^2$ is monotone and bounded. Lemma 28 enables to find a mapping $v : \mathbb{R}^2 \to \mathbb{R}^2$ which is a monotone extension of the mapping u. Using Theorem 72 we have that v is a mapping of the locally bounded variation. Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping given by the matrix

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This linear mapping, which is in fact the anti-clock wise rotation with the angle $\pi/2$, is obviously Lipschitz continuous, thus the composition

$$w(x) := Q \circ v(x) = (-f(x_1), x_1)$$

is a mapping of bounded variation (see [3]). Since $\langle w(t\mathbf{e}_1); \mathbf{e}_1 \rangle = -f(t)$ the mapping w can not be DM again by Lemma 44.

Let us realize that this example gives also a counterexample of a DM mapping which is not an UDM mapping and it also demonstrates that the satisfying of the Radó-Reichelderfer condition by the mapping v and the Morrey condition by the measure Dv is not sufficient for the posing the DM property by the mapping v. Indeed suppose at first that v is an UDM mapping. Theorem 114 asserts that $w = Q \circ v$ is an UDM mapping as well. But this is a contradiction since w is not DM as we have ensured. Further, the mapping v fulfills the Radó-Reichelderfer condition by Corollary 79 and the measure Dv fulfills the Morrey estimate by Corollary 76. Propositions 82 and 83 imply that the same conclusion holds for the mapping $w = Q \circ v$ and the measure Dw. The absence of the DM property for the mapping w was already discussed.

Finally if we consider the mapping

$$z(x) := v \circ Q(x)$$

we obtain for $-t \in [a;b]$

$$\langle z(t\mathbf{e}_2; \mathbf{e}_2 \rangle = \langle u(-t, 0); \mathbf{e}_2 \rangle = f(-t).$$

This demonstrates the non-stability of DM mappings with respect to inner compositions. ♣

Example 131 Another possibility how to construct an example of a mapping of bounded variation which is not DM is to use the Proposition 84 which implies the Fréchet differentiability almost everywhere of DM mappings. Let $n \geq 2$, it is possible to find a function $u \in W^{1,1}(\mathbb{R}^n) \subset BV(\mathbb{R}^n)$ which is discontinuous in every point. This function can not be Fréchet differentiable in any point. The function $v : \mathbb{R}^n \to \mathbb{R}^n$ defined by $v = (u, 0, \dots, 0)$ presents the desired example of a BV mapping which is not DM. The following construction belongs to the standard techniques in the theory of Sobolev functions thus we describe it briefly and omit some details.

Let $\alpha, c > 0$ and $d \in \mathbb{R}^n$. In this example we will say that the function $f : \mathbb{R}^n \to \mathbb{R}$ is of the type (α, c, d) if

$$f(x) = \max\{|x - d|^{-\alpha}, c\} - c.$$

It is a standard calculation to show that for $\alpha \leq n - 1$ the function

$$x \mapsto |x|^{-\alpha}$$

belongs to the space $W^{1,1}_{loc}(\mathbb{R}^n)$. It is well known that the Sobolev functions are stable with respect to the truncation. Since every function of the type (α, c, d) has a compact support we have for $\alpha < n - 1$ that it belongs to the space $W^{1,1}(\mathbb{R}^n)$. It is easily seen that for every ball B and for every $\epsilon > 0$ we can find $d \in \mathbb{R}^n$, $c > 0$ and $\alpha < n - 1$ such that the function of the type (α, c, d) has the support contained in B and the $W^{1,1}$ norm not exceeding ϵ.

Roughly speaking the basic idea of the construction is to find a sequence of balls $B(x_j, r_j)$ such that for every open set $U \subset \mathbb{R}^n$ and for every $k \in \mathbb{N}$ there is an integer $j \geq k$ such that $B(x_j, r_j) \subset U$. Further, we find balls $B_j^{\pm}$ in every $B(x_j, r_j)$ such that $u = \pm 1$ on the set whose Lebesgue measure is large with respect to the Lebesgue measure of $B_j^{\pm}$.

The construction is done by induction. We take $u_0 := 0$. We consider in the j-th step the ball $B(x_j, r_j)$ and find disjoint balls $\tilde{B}_j^{\pm}$ such that for every $i < j$ it is

$$\frac{|\tilde{B}_j^{+}|}{|B_i^{-}|} \leq 2^{-j}$$

and

$$\frac{|\tilde{B}_j^{-}|}{|B_i^{+}|} \leq 2^{-j}.$$

We put $\tilde{u}_{j+1} := u_j + u_{j+} - u_{j-}$, where $u_{j\pm}$ are the functions of type (α, c, d) with the support contained in $\tilde{B}_j^{\pm}$ and the norm $\|u_{j\pm}\|_{W^{1,1}} \leq 2^{-j}$. The function u_j is defined as a truncation of the function $\tilde{u}$ i.e.

$$u_{j+1} := \max\{-1; \min\{\tilde{u}_j; 1\}\}.$$

The stability with respect to the truncation implies that the function u_{j+1} remains in the space $W^{1,1}(\mathbb{R}^n)$. Every function of the sequence $(u_j)_{j \in \mathbb{N}}$ is bounded from below by -1 and from above by 1. The construction easily implies that the sequence $(u_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the space $W^{1,1}(\mathbb{R}^n)$. Thus there is a function $u \in W^{1,1}(\mathbb{R}^n)$ such that $\|u-u_j\|_{W^{1,1}} \to 0$. Thus $u_j \to u$ almost everywhere. For every representative of u and for every ball $B(x_j,r_j)$ there are points $x_\pm \in B(x_j,r_j)$ such that $u(x_\pm)=\pm 1$. This concludes the construction. ♣

Example 132 This example demonstrates some effects for monotone and UDM mappings which can not occur in the situation of convex and d.c. functions.

It is easily seen that if $\Psi : X \to \mathbb{R}$ is a convex function then Ψ is a d.c. function with the control function Ψ. We show that there is a monotone mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose no multiple can be control mapping for F in the sense of Definition 102.

Let define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$F(x) := \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

where $|\cdot|$ stands for Euclidean norm. Let us notice that the mapping

$$x \mapsto \begin{cases} \frac{x}{|x|^\lambda} & x \neq 0 \\ 0, & x = 0, \end{cases}$$

where $\lambda < 1$, is shown to be δ-monotone (consequently by Proposition 106 UDM) in [12] (in a bit tricky way).

At first we realize that F is a monotone mapping. This can be verified by a geometric argument. Alternatively we realize that F is a selection of $\partial|\cdot|$ thus the monotonicity easily follows.

We show that there is no $K>0$ such that

$$\left|\frac{x}{|x|}-\frac{y}{|y|}\right| |x-y| \le K \left\langle \frac{x}{|x|}-\frac{y}{|y|}; x-y \right\rangle \tag{60}$$

is fulfilled for all $x, y \in \mathbb{R}^2$.

We can do an analytic computation but we can proceed in more geometric way. The inequality (60) in fact means that the angle between the vectors $x-y$ and $F(x)-F(y)$ is less or equal than $\pi/2-\epsilon$, where $\epsilon=\epsilon(K)>0$. For $x_1, y_1 \in S^1$ set $x_t := tx_1$ and $y_s := sy_1$. For fixed $t>1$ the angle between the vectors $x_t - y_s$ and x_1-y_1 can be arbitrary close to $\pi/2$ by taking $s>0$ and $|x_1-y_1|$ sufficiently small . Thus the inequality (60) can not be fulfilled for any $K>0$. ♣

Remark 133 Let us note that the previous example is not too surprising. Since if X is a Hilbert space and $\Omega \subset X$ an open convex set and a monotone mapping $F : \Omega \to X$ is an UDM mapping with a suitable multiple of F as a control mapping then F is necessarily δ-monotone mapping. It is proved in [12] (the proof is not obvious) that every δ-monotone mapping is locally Hölder continuous with an exponent λ depending only on δ. This is the reason why in Example 132 the discontinuous function can not be controlled by its multiple.

EXAMPLE 134 We show an example of an UDM mapping which is not continuous in an interior point of its domain. Let define

$$F(x) := \mathbf{1}_{\{0\}},$$

i.e. $F(0) = 1$ and $F(x) = 0$ for $\mathbb{R}^2 \ni x \neq 0$. Let $f := x/|x|$ for $x \neq 0$ and $f(0) := 0$. We have already realized in Example 132 that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a monotone mapping.

We prove that F is an UDM mapping with a control mapping f. Since for $x, y \neq 0$ the control inequality

$$|F(x) - F(y)||x - y| \leq \langle f(x) - f(y); x - y\rangle$$

is trivial, we need to check this inequality for $x \neq 0 = y$. But this is easy since we have

$$|F(x) - F(0)||x| = |x| \leq \left\langle \frac{x}{|x|}; x \right\rangle.$$

This gives the desired.

This example can be easily modified. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ be a countable dense subset of $\mathbb{R}^2$ and let

$$F_n(x) := \frac{1}{n^2}\mathbf{1}_{\{a_n\}}$$

and

$$f_n(x) := \begin{cases} \frac{x-a_n}{n^2|x-a_n|}, & x \neq a_n \\ 0, & x = a_n. \end{cases}$$

We can show as in the first part of the example that F_n is an UDM mapping with a control mapping f_n. The series

$$f := \sum_{n=1}^{\infty} f_n,$$

converge by the Weierstrass criterion and the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a monotone mapping. The mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F := \sum_{n=1}^{\infty} F_n$$

is also correctly defined. We conclude by Proposition 51 that F is an UDM mapping with control mapping f. Thus we have found an UDM mapping which is discontinuous on the dense set. ♣

In the following we will present two very simple examples of a DM mapping resp. a difference of two accretive mappings between infinite dimensional spaces.

Recall the definition and the basic properties of the Nemytskii mapping in spaces of integrable functions. For the details about Nemytskii mappings see [2] and [22].

Definition 135 Let Ω be an open subset of $\mathbb{R}^n$. The mapping

$$a : \Omega \times \mathbb{R}^{d_1} \times \dots \mathbb{R}^{d_m} \to \mathbb{R}^d$$

is said to be the *Carathéodory integrand* if $a(\cdot, y_1, \dots, y_m) : \Omega \to \mathbb{R}^d$ is measurable for all

$$(y_1, \dots, y_m) \in \mathbb{R}^{d_1} \times \dots \mathbb{R}^{d_m}$$

and $a(x, \cdot) : \mathbb{R}^{d_1} \times \dots \mathbb{R}^{d_m} \to \mathbb{R}^d$, is continuous for a.e. $x \in \Omega$. The *Nemytskii mapping* $\mathcal{N}_a$ is defined for functions $u_i : \Omega \to \mathbb{R}^{d_i}$, $i = 1, \dots, m$ by the formula

$$\mathcal{N}_a(u_1, \dots, u_m)(x) := a(x, u_1(x), \dots, u_m(x)).$$

Theorem 136 *Let $a : \Omega \times \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$ be an Carathéodory integrand and the functions $u_i : \Omega \to \mathbb{R}^{d_i}$, $i = 1, \dots, m$ be measurable. Then $\mathcal{N}_a : \Omega \to \mathbb{R}^d$ is measurable. Moreover, if a satisfies the growth condition*

$$|a(x, y_1, \dots, y_m)| \leq \gamma(x) + c \sum_{i=1}^{m} |y_i|^{\frac{p_i}{p}}$$

for some $\gamma \in L^p(\Omega)$, then $\mathcal{N}_a$ is the bounded continuous mapping
$L^{p_1}(\Omega; \mathbb{R}^{d_1}) \times \dots \times L^{p_m}(\Omega; \mathbb{R}^{d_m}) \to L^p(\Omega; \mathbb{R}^d)$.

Proposition 137 (DM property of the Nemyckii mapping) *Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory integrand and let for a.e. $x \in \Omega$ be $a(x, \cdot)$ DM with a control function $k(x, \cdot)$. Further, let the functions a and k satisfy the growth conditions*

$$|a(x, y)| \leq \gamma_a(x) + c|y|^{\frac{r}{p}}$$

and

$$|k(x, y)| \leq \gamma_k(x) + c|y|^{\frac{r}{p}},$$

where $\gamma_a, \gamma_k \in L^p(\Omega)$ and $r \leq p'$. Then the Nemyckii mapping $\mathcal{N}_a : u(\cdot) \mapsto a(\cdot, u(\cdot))$ is DM as a mapping $\mathcal{N}_a : L^p(\Omega; \mathbb{R}^n) \to L^{p'}(\Omega; \mathbb{R}^n)$ with the control mapping $\mathcal{N}_k$.

Proof. By Theorem 136 the Nemyckii mappings $\mathcal{N}_a, \mathcal{N}_k$ are well defined as mappings $L^p(\Omega; \mathbb{R}^n) \to L^r(\Omega; \mathbb{R}^n) \subset L^{p'}(\Omega; \mathbb{R}^n) = (L^p(\Omega; \mathbb{R}^n))^*$. Choose $u, v \in L^p(\Omega; \mathbb{R}^n)$, we have

$$\begin{aligned}
|\langle \mathcal{N}_a u - \mathcal{N}_a v; u - v \rangle| &= \left| \int_\Omega \langle a(x, u(x)) - a(x, v(x)); u(x) - v(x) \rangle \, dx \right| \\
&\leq \int_\Omega |\langle a(x, u(x)) - a(x, v(x)); u(x) - v(x) \rangle| \, dx \\
&\leq \int_\Omega |\langle k(x, u(x)) - k(x, v(x)); u(x) - v(x) \rangle| \, dx \\
&= \langle \mathcal{N}_k u - \mathcal{N}_k v; u - v \rangle.
\end{aligned}$$

This by Lemma 92 completes the proof. □

In the following lemma we consider the space $\mathbb{R}^d$ endowed with the Euclidean norm.

Lemma 138 (see [22]) *Let* $1 < p < \infty$. *Then the duality mapping*

$$j : L^p(\Omega; \mathbb{R}^d) \to L^{p'}(\Omega; \mathbb{R}^d)$$

is given by the formula

$$j(u)(x) = \frac{u(x)|u(x)|^{p-2}}{\|u\|_{L^p}^{p-2}}.$$

Proposition 139 *Let* $1 < p < \infty$ *and let* $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *be a Carathéodory integrand which is DM with a monotone control mapping* $k : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *in the second variable, further suppose that a resp k satisfy the linear growth condition* $|a(x, y)| \leq \gamma(x) + c|y|$ *resp.* $|k(x, y)| \leq \gamma(x) + c|y|$ *with some* $\gamma \in L^p(\Omega)$. *(Especially the linear growth condition and DM property are satisfied for a being Lipschitz continuous with* $a(0) = 0$.*) Then the Nemyckii mapping* $\mathcal{N}_a : L^p(\Omega; \mathbb{R}^d) \to L^p(\Omega; \mathbb{R}^d)$ *is a difference of two accretive mappings with accretive control mapping* $\mathcal{N}_k$.

Proof. By Theorem 136 the operators $\mathcal{N}_a$ resp. $\mathcal{N}_k$ are well defined mappings of Lebesgue space $L^p(\Omega, \mathbb{R}^d)$ to itself and $\mathcal{N}_k$ is an accretive mapping. Using the formula for the duality mapping in Lebesgue spaces (see Lemma 138) we obtain

$$\begin{aligned}
&|\langle \mathcal{N}_a u - \mathcal{N}_a v; j(u - v)\rangle| \\
&\leq \int_\Omega \left| (a(u(x)) - a(v(x))) \frac{(u(x) - v(x))|u(x) - v(x)|^{p-2}}{||u - v||_p^{p-2}} \right| dx \\
&\leq \int_\Omega (k(u(x)) - k(v(x))) \frac{(u(x) - v(x))|u(x) - v(x)|^{p-2}}{||u - v||_p^{p-2}} dx \\
&= \langle \mathcal{N}_k u - \mathcal{N}_k v; j(u - v)\rangle.
\end{aligned}$$

Thus $\mathcal{N}_k - \mathcal{N}_a$ is an accretive mapping. This gives the assertion. □

Remark 140 Let us add one brief comment. The Nemyckii operator provides a counter example for the validity of a variant of Theorem 59 in infinite-dimensional spaces. Assume that

$$a : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory integrand such that the Nemyckii operator $\mathcal{N}_a$ is a bounded continuous mapping

$$\mathcal{N}_a : L^2(\Omega) \to L^2(\Omega).$$

If a is a nondecreasing in the second variable then the operator $\mathcal{N}_a$ is a monotone operator. It is proved in the monograph [2] that if $\mathcal{N}_a$ is Fréchet differentiable at least at one point $u_0 \in L^2(\Omega)$ then the function a is of the form

$$a(x,r) = a_1(x)r + a_2(x).$$

Thus the Nemyckii mapping

$$\mathcal{N}_a : L^2(\Omega) \to L^2(\Omega)$$

which is induced by Carathéodory integrand, which is not linear in second variable, is nowhere Fréchet differentiable.

3.4 Problems and future projects

Since this thesis is, as we know, the first text devoted to the study of differences of monotone mappings, many interesting questions were not answered here. We bring up some of them in this last section. It is possible that some of these problems are easy, but they appeared to late to be solved here. Some of these problems are formulated rather vaguely since it is not a-priori clear which exact assumptions are reasonable. It can be seen during solving these problems.

PROBLEM 141 Does every DM mapping have a potential control mapping? More precisely, let X be a Banach space, let $\Omega \subset X$ be an open convex set and let $A : \Omega \to X^*$ be a DM mapping. Does there exist a Gateaux differentiable convex function $g : X \to \mathbb{R}$ such that the mappings $\delta g \pm A$ (or at least one of them) are monotone? ♠

PROBLEM 142 In this problem we are asking, roughly speaking, if by splitting a mapping into a difference of two monotone mappings, we obtain mappings of the similar quality.

At first we ask whether every continuous DM mapping can be written as a difference of two continuous monotone mappings?

Similar question to Problem 141 is whether every potential DM mapping can be written as a difference of two potential monotone mappings.

We have realized in Corollary 97 that locally Lipschitz continuous mappings $\mathbb{R}^d \to \mathbb{R}^d$ are DM. Assume that $A : \mathbb{R}^d \to \mathbb{R}^d$ is a polynomial. Thus the mapping of the form $A(x) = (A_1(x), \dots, A_d(x))$, where

$$A_j(x) = \sum_{|\alpha(j)|=0}^{n(j)} a_{\alpha(j)} x^{\alpha(j)},\ j = 1, \dots, d, \tag{61}$$

where $\alpha(j) \in \mathbb{N}_0^d$, $j = 1, \dots, d$ are multiindices with $|\alpha(j)| := \sum_i^d \alpha_i(j)$ and

$$x^{\alpha(j)} := x_1^{\alpha_1(j)} x_2^{\alpha_2(j)} \dots x_d^{\alpha_d(j)}.$$

Is it true that each such polynomial can be written as a difference of two monotone polynomials?

Let us note that for $d = 1$ the answer is positive as can be easily observed by analyzing the monomials of odd and even degree.

We can ask more generally if every real analytic mapping can be written as a difference of two real analytic monotone mappings. (We can imagine real analytic mappings by setting formally $n(j) = \infty$ in (61).)

Let us only note that the similar question can be studied for holomorphic functions (after identification $\mathbb{C}$ with $\mathbb{R}^2$), for harmonic mappings, caloric mappings. ♠

Problem 143 The fact that the class of the DM mappings is not stable with respect to compositions even with linear mappings and since the requirement of the UDM property seems to be sometimes too strong we are led to an idea of a modification of the definition of the class of DM mappings. Let X, Y be Banach spaces and let $\Omega \subset X$ be an open convex set. We consider a class of mappings $u : \Omega \to Y$, which belong to the linear span of the mappings of the form

$$R \circ v,$$

where $v : \Omega \to X^*$ is a monotone mapping and $R : X^* \to Y$ is a linear mapping. Let call this class of mappings IDM mappings. If $Y = X^*$ then the class of IDM mappings contains all DM mappings and their compositions with continuous linear mappings. Let us note that for $X = \mathbb{R}$ every IDM mapping is an UDM mapping and in the case $Y = X^*$ we have that UDM $\Rightarrow$ DM $\Rightarrow$ IDM. Since for X being finite dimensional the range of IDM mapping is finite dimensional there exists an UDM mapping which is not an IDM mapping. In the case of $X = \mathbb{R}^n$ every IDM mappings is Fréchet differentiable almost everywhere and no Radon-Nikodým property is needed. Otherwise in the case $X = Y = \mathbb{R}^n$ there are IDM mappings which are not UDM mappings as we have realized in Example 130.

The definition of d.c. mappings can be modified in a similar way. It is again easilily seen that for $Y = \mathbb{R}$ this new definition coincides with the standard definition of d.c. mappings.

We would like to study the properties of these classes of mappings. ♠

Problem 144 Korn's inequality is an important tool in partial differential equations. It asserts that for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $p > 1$ the L^p-norm of the gradient can be estimated by the L^p-norm of the symmetric gradient, i.e.

$$\|\nabla u\|_{L^p} \leq C\|Eu\|_{L^p},$$

where the constant C does not depend on the function u and

$$2Eu := \nabla u + \nabla^\top u.$$

The examples of $W^{1,1}$ functions for which Korn's inequality fails are known. There is, in fact, proved in the paper [1] that Korn's inequality remains valid for monotone functions even in the case $u \in BV(\Omega; \mathbb{R}^n)$ only. The question is whether it is possible to prove a variant of Korn type inequality for DM or UDM mappings. ♠

PROBLEM 145 Is it possible to relax the assumption of the Borel measurability of the UDM mapping F in Theorem 120? This assumption was needed only for the proof of the measurability of the set Δ_v. ♠

PROBLEM 146 Is it possible to construct an UDM mapping on $\mathbb{R}^n$ which is not continuous on the larger set than countable? Which types of discontinuities can occur? (We have constructed only the removable singularities.)

A characterization of the points of discontinuity of UDM mappings would be interesting too. It could be also interesting to investigate relations between UDM and quasiconformal mappings.

Similar questions is whether we can get an estimate on the dimension of $F(x)$ for F an UDM multi-mapping if we know the dimension of $f(x)$, where f is a monotone control multi-mapping for F. ♠

PROBLEM 147 We have noticed in Remark 140 that the nonlinear Nemyckii mapping is not Fréchet differentiable in any point as a mapping from $L^2 \to L^2$. However the situation is not so bad in general. It can be proved that under some technical assumptions the Nemyckii operator is Fréchet differentiable as a mapping $L^p \to L^{p'}$, where $p > 2$ and $p + p' = pp'$ (for details see [2]). Thus we can still whether if it can be established some differentiability properties of monotone operators in a suitable class of Banach spaces. ♠

PROBLEM 148 It is not difficult to realize that Theorem 96 can be reformulated for UDM mappings $C \to Y$, where $C \subset \mathbb{R}^d$ is an open convex set. But it is shown in [11] that there is a d.c. function $\ell^2 \to \mathbb{R}$ which is locally d.c. but which is not globally d.c.. We are asking whether a similar effect can occur for UDM mappings. ♠

PROBLEM 149 Theorems 113 and 114, without providing counterexamples, are not sure to be optimal. The assumptions of Lipschitz continuity or linearity and boundedness from bellow of the mappings from these theorems seems to be very restrictive.

Thus we would like to study under which assumptions it is possible to prove similar composition theorems for more general UDM mappings or provide appropriate counterexamples. Let us note that the similar problem for d.c. mappings is investigated in the up till now unpublished paper of L. Veselý and L. Zajíček.

Further we would like to ensure if it possible to show the DM or the UDM property of the inversion of a DM or UDM mapping.

The problem of posing of the DM or the UDM properties of implicit mappings seems to be interesting too. ♠

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