Review by the supervisor/opponent of the bachelor thesis	
Thesis title:	Conway's topograph
Thesis author:	Emma Pěchoučková
Review by:	Nicolas Daans

Summary of the thesis content

This thesis discusses the so-called *Topograph* as introduced by John Conway, which is a planar graph (in fact a tree) whose edges correspond roughly to bases of the $\mathbb{Z}$-module $\mathbb{Z}^2$, and whose vertices express a type of base change. In this thesis paths in the Topograph are related to continued fractions of quadratic irrational numbers.

The thesis thus starts by introducing with precision first the basics of continued fractions (in Chapter 1), doing some preparatory technical computations with them, and then (in Chapter 2) defining precisely the Topograph and describing its structure.

In Chapter 3 the so-called Farey tree is introduced, which provides an alternative description of the topograph in such a way that its edges correspond (roughly) to pairs of rational numbers, and where adjacent edges share a rational number. This then opens the way to describing paths in the Topograph via sequences of rational numbers, as discussed in Chapter 4 and subsequently related to continued fraction representations.

Whereas Chapters 1 and 2 are written in a mostly self-contained and very rigorous manner, Chapters 3 and 4 have a more expository style, and proofs of many of the more delicate results in these last chapters are omitted or sketched, but many examples and illustrations are provided.

Overall evaluation of the thesis

Thesis topic. The topic is interesting and appropriate for a bachelor thesis, and the choice to start from research articles and books and elaborate on their details, is very good. I understood that the student had initially planned to discuss the more advanced topics from the second half of the thesis (Chapters 3-4) in more detail, but that due to time constraints, it was chosen to work out the mathematical details primarily in the first two chapters; this is an understandable choice.

Author's contribution. The student has studied multiple sources, several of which are written in a very summary style, and has made her own synthesis and presentation of the relevant mathematics. The student has written her own version of several propositions and proofs, has introduced new helpful notations and terminology to help clarify complicated computations, and has provided original examples and own visual illustrations. In general, I think the level of originality and own contribution is high and the contributions add value.

Mathematical level. The thesis exhibits a solid understanding of mathematical writing, with for the most part clearly written text and well-structured proofs, which the student has written herself. The difficulty of the worked out mathematics is of the expected level for a bachelor's thesis. There are an average amount of mathematical mistakes and gaps, which generally are small and not distracting, although on a few occassions they are more significant. Near the very end of the thesis (in particular sections 3.4 and 4.3, and part of 4.2), some of the statements are made insufficiently rigorously to verify them.

Work with sources. Sources are always cited correctly and clearly. All of the student's writing seems to be original, and she clearly indicate where she takes inspiration for each part of the thesis. On a few occasions, slightly more could be said to clearly bridge differences in definitions and viewpoints of different sources.

Comments and questions

Below are two questions which the student could be asked to *address during the defense*:

1. If (e_1, e_2) is a strict base, as defined in Definition 2.13, are e_1 and e_2 automatically primitive? This seems to be implicitly used in the proof of Lemma 2.21 but is not said or explained.

2. Regarding the "mediant rule": note that even when $\frac{a}{c}$ and $\frac{b}{d}$ are fractions in reduced from, then $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$ might not be in reduced form (think of $a = b = c = d = 1$ for example). With this in mind, the argument that this operation is associative on page 32 breaks down. Is $\oplus$ nevertheless associative? If so, why? And if not, does it create problems in Chapter 4 when one wants to apply $\oplus$ to more than 2 elements?
   In Proposition 3.7 a reducedness statement is shown for $\frac{a+c}{b+d}$, but directly afterwards one needs this for $\frac{a+b}{c+d}$; is this the same? And does Proposition 3.7 not require $\frac{a}{c}$ and $\frac{b}{d}$ to be adjacent?

A few further comments on issues around clarity or accuracy of statements, not necessarily to be addressed during the defense:

3. Lemma 1.9 says that two very different looking definitions are "equivalent", but it is not clear what is mathematically being claimed and how the reader should translate between the definitions. Possibly the student used different sources here; this is all the more reason to help the reader bridge the gap.

4. Sometimes wrong/idiosyncratic usage of set theory notation (ordered vs. unordered pairs, unions, power sets) makes proofs hard to parse, e.g. centered equations in proof of Lemma 2.32.

5. Definition 3.12 is not just a definition but also a claim, and nothing is said to help the reader verify the claim (and probably there is a typo in the claim).

Conclusion

I consider the thesis to be very good and I recommend that it be accepted as a bachelor's thesis with grade 1 or 2.

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May 23, 2024