FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

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Rosenthal's subsequence splitting lemma

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Study programme: General Mathematics (B0541A170011)
Study branch: MOMP (0541RA170011)

Prague 2024

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I am extremely grateful to my supervisor prof. RNDr. Jiří Spurný, Ph.D., DSc. for his invaluable expertise, guidance and patience.

I would also like to acknowledge my family for their endless support.

Title: Rosenthal's subsequence splitting lemma

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Abstract:

The aim of the thesis is to give complete and thorough proofs of some well-known results from the measure theory. Oftentimes, arguments from functional analysis will be used to prove these results. For example, we will use an enhanced version of Schur's theorem to prove the Nikodym theorem and the Vitali-Hahn-Saks theorem. Then we will focus on the weak compactness in L_1 and we will present a proof of the Biting Lemma and its corollary Rosenthal's subsequence splitting lemma.

Keywords: Nikodym, Vitali-Hahn-Saks, Biting Lemma, Rosenthal

Název práce: Rosenthalovo lemma o rozkladu posloupností v L_1

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Abstrakt:

Hlavním cílem této práce je představit kompletní a detailně zpracované důkazy některých známých tvrzení z teorie míry. K tomuto účelu budeme častokrát využívat poznatky z oblasti funkcionální analýzy. Například použijeme silnější verzi Schurovy věty, abychom dokázali Nikodýmovu větu a Vitali-Hahn-Saksovu větu. Následně se budeme zabývat slabou kompaktností v L_1 a předvedeme důkaz Biting lemmatu. Nakonec dokážeme Rosenthalovo lemma o rozkladu posloupností v L_1, což bude důsledek již zmíněného Biting lemmatu.

Klíčová slova: Nikodým, Vitali-Hahn-Saks, Biting lemma, Rosenthal

Contents

Introduction

The Nikodym theorem, the Vitali-Hahn-Saks theorem, the Biting Lemma and Rosenthal's subsequence splitting lemma are all classical yet not trivial results from the measure theory. However, their proofs are sometimes very brief and for an untrained eye it might not be clear what is going on. The aim of this thesis is to present thorough and more detailed proofs of those results.

Now we would like to present the content of the thesis.

The first chapter contains some preliminary results that we will need later.

In the second chapter we will prove two theorems concerning setwise convergent sequences of measures: the Nikodym and the Vitali-Hahn-Saks theorems. The proof of the Nikodym theorem will be non-standard. Usually it uses the Baire theorem. However, we would like to present a more recent version of the proof using an enhanced version of Schur's theorem.

In the third chapter, we will show a proof of the Biting Lemma and Rosenthal's subsequence splitting lemma. Both of these theorems are about extracting a convergent subsequence from a bounded sequence of functions in L_1 (the sense of convergence will be precisely defined in the third chapter; roughly speaking, it will be even weaker convergence then the weak convergence).

At first, it might seem that the second and the third chapter are completely unrelated: one is about measures, the other is about functions. However, there is a connection between them, given by the Radon-Nikodym theorem. As we will see, the Vitali-Hahn-Saks theorem states, loosely speaking, that for every setwise convergent sequence of bounded measures, all of which are absolutely continuous with respect to one common measure, the limit set function is also an absolutely continuous measure (with respect to the same measure) and moreover, the original sequence is *uniformly* absolutely continuous. Now, what happens if we replace the assumption of the setwise convergence by the assumption of boundedness of that sequence of absolutely continuous measures (in a proper Banach space)? Sure the theorem will no longer hold (for example because the limit set function might not exist at all), but the question we might ask ourselves is if there exists at least a setwise convergent subsequence. We have thus arrived to a new problem: when does a bounded sequence of absolutely continuous finite measures admit a setwise convergent subsequence?

This is the point when we will use the Radon-Nikodym theorem. Since all the measures are finite, they can be represented by functions from L_1 in a standard manner. This way, we will translate the problem of convergence of measures into a problem of convergence of functions from L_1. And in the third chapter we will see how and when we can extract a convergent subsequence from a bounded sequence in L_1.

We will come back to this connection between the absolutely continuous measures and functions from L_1 at the very end of the third chapter, when we will talk about it in more detail by using the theorems from this thesis. For now, the main takeaway is that the two main parts of the thesis (Chapter 2 and Chapter 3) are connected by the Radon-Nikodym theorem.

Finally, let us mention that our contribution is the collection of more recent proofs of the above theorems and their detailed presentation.

1. Preliminaries

In this chapter we will introduce basic definitions, notation and theorems that will be used throughout the thesis.

Let us start with the definition of measure. In some articles, measure may be only finitely additive. However, that will not be our case.

Definition 1 (signed measure and measure)**.** *Let $(X, \mathcal{A})$ be a measurable space and let $\lambda\colon \mathcal{A} \to [-\infty,\infty]$. Then λ is said to be a* signed measure *if*

(i) $\lambda(\emptyset) = 0$,

(ii) for every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ of pairwise disjoint sets $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda(A_n)$,

(iii) λ assumes at most one of the values $-\infty, \infty$.

If a set function $\lambda\colon \mathcal{A} \to [0,\infty]$ satisfies (i) and (ii), then λ is called a (positive) measure.

In the thesis, we will use the following notation:

- $\mathbb{R}^*$ denotes the extended number line, i.e. $\mathbb{R}^* = [-\infty,\infty]$,
- $ca(\mathcal{A})$, for $(X, \mathcal{A})$ a measurable space, denotes the vector space of real-valued signed measures on $\mathcal{A}$ (under the usual addition and scalar multiplication),
- $ca^+(\mathcal{A})$, for $(X, \mathcal{A})$ a measurable space, denotes the space of real-valued positive measures on $\mathcal{A}$,
- for $(X, \|\cdot\|)$ a normed vector space, $x \in X$, $r > 0$, let us denote $B(x, r) = \{y \in X : \| y - x \| \leq r\}$ and $B_X = B(0, 1)$,
- for a set X and for $A \subseteq X$, $\chi_A\colon X \to \{0, 1\}$ denotes the characteristic function of the set A.

Now for a measure space $(X, \mathcal{A}, \mu)$ with a signed measure μ we would like to find a partition of X into subsets P and N, satisfying that μ is non-negative on every measurable subset of P and non-positive on every measurable subset of N. First, let us introduce the following terminology.

Definition 2 (λ-positive set, λ-negative set)**.** *Let $\lambda\colon \mathcal{A} \to \mathbb{R}^*$ be a signed measure and $A \in \mathcal{A}$. Then A is said to be*

- λ-positive *if, for every $B \in \mathcal{A}$, $\lambda(A \cap B) \geq 0$,*
- λ-negative *if, for every $B \in \mathcal{A}$, $\lambda(A \cap B) \leq 0$ and*
- λ-null *if, for every $B \in \mathcal{A}$, $\lambda(A \cap B) = 0$.*

Definition 3 (Hahn decomposition)**.** *Let λ be a signed measure on a measurable space $(X, \mathcal{A})$ and let $P, N \in \mathcal{A}$. Then the pair (P, N) is said to be a* Hahn decomposition of X relatively to λ *if*

- *$P \cup N = X$,*

- *$P \cap N = \emptyset$,*

- *P is λ-positive and N is λ-negative.*

Now let us present the Hahn decomposition theorem, which states that we can always find the desired partition from above. Let us recall that if A and B are sets, their *symmetric difference* is defined to be $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Theorem 1 (Hahn decomposition theorem)**.** *Let $(X, \mathcal{A})$ be a measurable space and let $\lambda \colon \mathcal{A} \to \mathbb{R}^*$ be a signed measure. Then there exists (P, N) a Hahn decomposition of X relatively to λ. For any other (P_1, N_1) a Hahn decomposition of X relatively to λ, $P \triangle P_1$ and $N \triangle N_1$ are λ-null sets.*

Proof. See Florescu and Godet-Thobie [2012, Theorem 1.6] for a proof. ☐

From Theorem 1 we obtain the following corollary.

Corollary 2. *Every signed measure $\lambda \in ca(\mathcal{A})$ is bounded.*

Proof. See Florescu and Godet-Thobie [2012, Corollary 1.14] for a proof. ☐

From Theorem 1 we also get that the following definition is correct.

Definition 4 (Jordan decomposition)**.** *Let λ be a signed measure on a measurable space $(X, \mathcal{A})$ and let (P, N) be a Hahn decomposition of X relatively to λ. Then we define*

- *$\lambda^+(A) = \lambda(A \cap P)$, for every $A \in \mathcal{A}$,*

- *$\lambda^-(A) = -\lambda(A \cap N)$, for every $A \in \mathcal{A}$, and*

- *$|\lambda|(A) = \lambda^+(A) + \lambda^-(A)$, for every $A \in \mathcal{A}$.*

The pair (λ^+, λ^-) is called the Jordan decomposition *of λ. The set function $|\lambda|$ is called the* total variation *of λ.*

Note that, by definition, λ^+ and λ^- are measures and $\lambda = \lambda^+ - \lambda^-$ since we can write

$$\lambda(A) = \lambda(A \cap (P \cup N)) = \lambda(A \cap P) + \lambda(A \cap N) = \lambda^+(A) - \lambda^-(A), \ A \in \mathcal{A}.$$

Also $|\lambda|$ is a measure on $\mathcal{A}$, because it is a sum of two measures on $\mathcal{A}$. Notice that if $\lambda \in ca(\mathcal{A})$, then (by Corollary 2) λ is bounded and since $\lambda = \lambda^+ - \lambda^-$, both measures λ^+ and λ^- must be bounded, and therefore $|\lambda|$ is also bounded, which means that $|\lambda| \in ca(\mathcal{A})$.

Sometimes $|\lambda|$ is defined in this manner:

$$|\lambda|(A) = \sup \left\{ \sum_{i=1}^{n} |\lambda(A_i)| : A_1, \ A_2, \ldots, \ A_n \in \mathcal{A}, \right. \\ \left. A_i \text{ are pairwise disjoint}, \ A = \bigcup_{i=1}^{n} A_i, \ n \in \mathbb{N} \right\}.$$

However, the following lemma shows that these definitions are equivalent.

Lemma 3. *Let λ be a signed measure on a measurable space $(X, \mathcal{A})$. Then*

$$|\lambda|(A) = \sup\left\{\sum_{i=1}^{n} |\lambda(A_i)| : A_1,\ A_2, \dots,\ A_n \in \mathcal{A},\right.$$
$$\left. A_i \text{ are pairwise disjoint},\ A = \bigcup_{i=1}^{n} A_i,\ n \in \mathbb{N}\right\}.$$

Proof. Let $A \in \mathcal{A}$, $n \in \mathbb{N}$, $(A_i)_{i=1}^n \subseteq \mathcal{A}$ be a sequence of pairwise disjoint sets such that $A = \bigcup_{i=1}^n A_i$. Then

$$\sum_{i=1}^{n} |\lambda(A_i)| = \sum_{i=1}^{n} \left|\lambda^+(A_i) - \lambda^-(A_i)\right| \leq \sum_{i=1}^{n} \left|\lambda^+(A_i)\right| + \left|\lambda^-(A_i)\right| =$$
$$= \sum_{i=1}^{n} \left(\lambda^+(A_i) + \lambda^-(A_i)\right) = \sum_{i=1}^{n} |\lambda|(A_i) = |\lambda| \left(\bigcup_{i=1}^{n} A_i\right) = |\lambda|(A).$$

Conversely, let $A \in \mathcal{A}$ and let (P, N) be a Hahn decomposition of X relatively to λ. Then for $A_1 = A \cap P$ and $A_2 = A \cap N$ we have that $A_1 \cap A_2 = \emptyset$ (because $P \cap N = \emptyset$), $A = A_1 \cup A_2$, and

$$\sum_{i=1}^{2} |\lambda(A_i)| = |\lambda(A_1)| + |\lambda(A_2)| = |\lambda(A \cap P)| + |\lambda(A \cap N)| =$$
$$= \left|\lambda^+(A)\right| + \left|-\lambda^-(A)\right| = \lambda^+(A) + \lambda^-(A) = |\lambda|(A),$$

which completes the proof.

□

Now we want to assert that the set $ca(\mathcal{A})$ with a correct norm is a Banach space. For that, we will need to define a mapping that will be our norm.

Definition 5. *Let $(X, \mathcal{A})$ be a measurable space. Then we define the mappings $\|\cdot\|,\ \|\cdot\|_\infty \colon ca(\mathcal{A}) \to [0,\infty)$ as follows:*

$$\|\lambda\| = |\lambda|(X), \qquad \|\lambda\|_\infty = \sup_{A \in \mathcal{A}}(|\lambda(A)|).$$

Theorem 4 (Florescu and Godet-Thobie, 2012, Theorem 1.23)**.** *Let $(X, \mathcal{A})$ be a measurable space. Then the mappings $\|\cdot\|,\ \|\cdot\|_\infty \colon ca(\mathcal{A}) \to [0,\infty)$ from the definition above are two equivalent norms on $ca(\mathcal{A})$.*

The spaces $(ca(\mathcal{A}), \|\cdot\|)$ and $(ca(\mathcal{A}), \|\cdot\|_\infty)$ are Banach spaces.

Proof. In Florescu and Godet-Thobie [2012, Theorem 1.23], the theorem is proved for spaces $ba(\mathcal{A})$ of real-valued bounded additive (i.e. not necessarily σ-additive) measures on $\mathcal{A}$. However, in the same theorem it is proved that the spaces $(ca(\mathcal{A}), \|\cdot\|)$, respectively $(ca(\mathcal{A}), \|\cdot\|_\infty)$ are closed subspaces of $(ba(\mathcal{A}), \|\cdot\|)$, respectively $(ba(\mathcal{A}), \|\cdot\|_\infty)$ (the fact that $ca(\mathcal{A}) \subseteq ba(\mathcal{A})$ is stated in Florescu and Godet-Thobie [2012, corollary 1.14]). And since a closed subspace of a Banach space is also a Banach space, our weaker form of the theorem is proved.

□

Finally, let us recall the definition of an absolutely continuous measure.

Definition 6. *Let $(X, \mathcal{A})$ be a measurable space, let $\lambda\colon \mathcal{A} \to [0,\infty]$ be a positive measure and $\mu\colon \mathcal{A} \to \mathbb{R}^*$ be a signed measure. Then μ is said to be* absolutely continuous with respect to λ *if*

$$\forall A \in \mathcal{A}: \ \lambda(A) = 0 \Rightarrow \mu(A) = 0.$$

We denote this by $\mu \ll \lambda$.

By Florescu and Godet-Thobie [2012, Remark 1.29], we have

$$\mu \ll \lambda \Leftrightarrow |\mu| \ll \lambda \Leftrightarrow (\mu^+ \ll \lambda \text{ and } \mu^- \ll \lambda).$$

Therefore μ is absolutely continuous with respect to λ if and only if $|\mu|$ is absolutely continuous with respect to λ. The following proposition shows that for $\mu \in ca(\mathcal{A})$ we could use an ε-δ definition to define the absolute continuity of μ (and thus $|\mu|$) with respect to λ.

Proposition 5 (Florescu and Godet-Thobie, 2012, Proposition 1.30)**.** *Let $(X, \mathcal{A})$ be a measurable space, λ be a positive measure on $\mathcal{A}$ and let $\mu \in ca(\mathcal{A})$. Then the following properties are equivalent:*

(i) μ is absolutely continuous with respect to λ,

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{A}$ satisfying $\lambda(A) < \delta$ we have $|\mu|(A) < \varepsilon$.

Proof. See Florescu and Godet-Thobie [2012, Proposition 1.30] for a proof. □

Remark 1. Notice that, by Florescu and Godet-Thobie [2012, Theorem 1.17 i)], in (ii) we can use $|\mu(A)|$ instead of $|\mu|(A)$.

Finally, let us mention that the equivalence does not hold if μ is not bounded. For example, let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable sets on $\mathbb{R}$ and let $\mu(A) = \int_A |x| \, d\lambda$, $A \in \mathcal{A}$. Then $\mu \ll \lambda$, but for arbitrarily small $\delta > 0$ we have

$$\mu\Big(\big(\frac{1}{\delta}, \frac{1}{\delta} + \delta\big)\Big) = \int_{\left(\frac{1}{\delta}, \frac{1}{\delta}+\delta\right)} |x| \, d\lambda \geq \int_{\left(\frac{1}{\delta}, \frac{1}{\delta}+\delta\right)} \frac{1}{\delta} \, d\lambda = 1,$$

therefore (ii) does not hold.

2. Vitali-Hahn-Saks and Nikodym Theorems

Let us have $(X, \mathcal{A})$ a measurable space and $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ a setwise convergent sequence of bounded signed measures. Let $\mu\colon \mathcal{A} \to \mathbb{R}^*$ be a set function such that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$. Then the Nikodym theorem states that $\mu \in ca(\mathcal{A})$ and that the family $\{\mu_n : n \in \mathbb{N}\}$ "is nice". If in addition there exists a positive measure λ such that $\mu_n \ll \lambda$ for every $n \in \mathbb{N}$, then the Vitali-Hahn-Saks theorem states that also $\mu \ll \lambda$ and that the family $\{\mu_n : n \in \mathbb{N}\}$ "is even better". We will need to define what we mean by "being nice" and "being even better" before stating and proving the theorems. Both of these refer to some sort of uniformity in n. The precise definitions are stated in the following section.

2.1 Uniformity

Definition 7 (uniform σ-additivity)**.** *Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{K} \subseteq ca(\mathcal{A})$. Then $\mathcal{K}$ is said to be* uniformly σ-additive *if for every $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$, A_i pairwise disjoint and for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N},\ n \geq n_0$ and for every $\mu \in \mathcal{K}$ it holds that $\left|\mu\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mu(A_i)\right| < \varepsilon$.*

Definition 8 (uniform absolute continuity)**.** *Let $(X, \mathcal{A})$ be a measurable space, λ be a positive measure on $\mathcal{A}$ and let $\mathcal{K} \subseteq ca(\mathcal{A})$ be a family of signed measures on $\mathcal{A}$. Then $\mathcal{K}$ is said to be* uniformly absolutely continuous with respect to λ *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{A}$ with $\lambda(A) < \delta$ and for every $\mu \in \mathcal{K}$ we have $|\mu(A)| < \varepsilon$.*

Notice that if we used (ii) of Proposition 5 to define the absolute continuity of the family $\{\mu_n : n \in \mathbb{N}\} \subseteq ca(\mathcal{A})$ with respect to (a positive measure) λ, then we would get that the family $\{\mu_n : n \in \mathbb{N}\} \subseteq ca(\mathcal{A})$ is absolutely continuous with respect to λ if and only if

$$\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall A \in \mathcal{A},\ \lambda(A) < \delta : \ \ |\mu_n|(A) < \varepsilon. \tag{2.1}$$

According to Remark 1, we can equivalently rewrite (2.1) as

$$\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall A \in \mathcal{A},\ \lambda(A) < \delta : \ \ |\mu_n(A)| < \varepsilon. \tag{2.2}$$

Now we see that we can obtain Definition 8 from (2.2) only by shuffling a few quantifiers.

2.2 Nikodym and Vitali-Hahn-Saks Theorems

In this section we will want to prove two theorems about sequences of measures: the Nikodym theorem and the Vitali-Hahn-Saks theorem. Note that, in Brooks [1969], the theorems are proved simultaneously.

We will need the following lemmata to prove the Nikodym theorem.

Lemma 6 (Walter Rudin, 1987, Lemma 6.3)**.** *If $z_1, z_2, \dots, z_N$ are complex numbers, then there exists a subset S of $\{1, 2, \dots, N\}$ for which*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k| \, .$$

Proof. See Walter Rudin [1987, Lemma 6.3] for a proof. □

Let us recall Schur's theorem, which will be needed in the following lemma.

Theorem 7 (Schur's theorem)**.** *Let $(x_n)_{n \in \mathbb{N}}$ be a weakly convergent sequence in $(\ell_1, \| \cdot \|)$. Then $(x_n)_{n \in \mathbb{N}}$ is a (strongly) convergent sequence in $(\ell_1, \| \cdot \|)$.*

The following lemma asserts that we can enhance the above theorem. To prove the Nikodym theorem, we will need this enhanced version. The proof of the lemma follows the one in Michal Johanis and Jiří Spurný [2022, Lemma 101, pages 359-361].

Lemma 8. *Let $A = \{\chi_N : N \subseteq \mathbb{N}\}$ be considered as a set in ℓ_∞ and let $f_a \in (\ell_1)^*$ denote the functional represented by $a \in A$. Let $(x_n)_{n \in \mathbb{N}} \subseteq \ell_1$ be a sequence in ℓ_1 such that $(f_a(x_n))_{n \in \mathbb{N}}$ converges for every $a \in A$. Then there exists $x \in \ell_1$ satisfying* $\lim_{n \to \infty} x_n = x$.

Proof. Let $\| \cdot \|$ denote the standard norm on ℓ_1 and for $y \in \ell_1$ and a set $N \subseteq \mathbb{N}$ let $y\chi_N$ denote the vector $y(i)\chi_N(i),\ i \in \mathbb{N}$.

Step 1. We will show that the sequence $(x_n)_{n \in \mathbb{N}} \subseteq \ell_1$ is bounded in $(\ell_1, \| \cdot \|)$.

By way of contradiction, let us suppose that $(x_n)_{n \in \mathbb{N}}$ is unbounded in $(\ell_1, \| \cdot \|)$. Then without loss of generality we can suppose that $\lim_{n \to \infty} \| x_n \| = \infty$ (because otherwise there exists at least a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ extracted from $(x_n)_{n \in \mathbb{N}}$ such that $\lim_{k \to \infty} \| x_{n_k} \| = \infty$ and we could make an analogical proof for it). Let us set $n_1 = 1$ and let us inductively construct a strictly increasing sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ satisfying $\| x_{n_{k+1}} \| > k + \| x_{n_k} \|$ for every $k \in \mathbb{N}$ (the induction step is trivial and follows from the definition of the limit). Let us set $y_k = x_{n_{k+1}} - x_{n_k},\ k \in \mathbb{N}$. Then the sequence $(y_k)_{k \in \mathbb{N}}$ satisfies

$$\| y_k \| = \| x_{n_{k+1}} - x_{n_k} \| \geq \Big| \| x_{n_{k+1}} \| - \| x_{n_k} \| \Big| > k,\ k \in \mathbb{N}.$$

Furthermore, if for $a \in A$ we denote $L_a = \lim_{n \to \infty} f_a(x_n)$ the limit from the assumption, we obtain

$$\lim_{k \to \infty} f_a(y_k) = \lim_{k \to \infty} f_a(x_{n_{k+1}} - x_{n_k}) = \lim_{k \to \infty} f_a(x_{n_{k+1}}) - f_a(x_{n_k}) = L_a - L_a = 0. \quad (2.3)$$

Thus $(y_k)_{k \in \mathbb{N}}$ satisfies $\lim_{k \to \infty} \| y_k \| = \infty$ and $\lim_{k \to \infty} f_a(y_k) = 0$ for every $a \in A$.

Let us notice that for every finite $N \subseteq \mathbb{N}$ it holds $\sup_{k \in \mathbb{N}} \| y_k \chi_N \| < \infty$. That is because we can choose (for arbitrary $i \in \mathbb{N}$) $a = \chi_{\{i\}}$ and obtain from (2.3)

$$\lim_{k \to \infty} y_k(i) = \lim_{k \to \infty} f_a(y_k) = 0, \quad (2.4)$$

which implies that every sequence $(y_k(i))_{k \in \mathbb{N}} \subseteq \mathbb{F}$ is bounded. In particular, if we denote $N = \{i_1, i_2, \dots, i_p\}$, we obtain that there exist $K_1, K_2, \dots, K_p$ such that for every $j \in \{1, 2, \dots, p\}$ and for every $k \in \mathbb{N}$ it holds that $|y_k(i_j)| < K_j$. Therefore by setting $M = \max\{K_1, K_2, \dots, K_p\} < \infty$ we obtain

$$\sup_{k\in\mathbb{N}} \| y_k \chi_N \| = \sup_{k\in\mathbb{N}} \sum_{i\in N} |y_k(i)| \leq M\,|N|\,,$$

where $|N|$ denotes the number of elements of N.

Now let us inductively construct a sequence of finite sets $(N_j)_{j\in\mathbb{N}} \subseteq \mathcal{P}(\mathbb{N})$ and two sequences of natural numbers $(k_j)_{j\in\mathbb{N}}$, $(m_j)_{j\in\mathbb{N}}$ such that for every $j \in \mathbb{N}$ we have:

(i) $\max N_j < m_j < \min N_{j+1}$ and $k_j < k_{j+1}$,

(ii) $\left|\sum_{i\in N_j} y_{k_j}(i)\right| > j$,

(iii) $\| y_{k_{j+1}} \chi_{\{1,2,\dots,\max N_j\}} \| < 1$,

(iv) $\| y_{k_j} \chi_{\{m_j, m_j+1,\dots\}} \| < 1$.

To start the induction, let us find $k_1 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} |y_{k_1}(i)| = \| y_{k_1} \| > 1{\cdot}\pi$ (note that such k_1 exists since $\lim_{k\to\infty} \| y_k \| = \infty$). Then there exists $M_1 \subseteq \mathbb{N}$ finite such that $\| y_{k_1}\chi_{M_1} \| = \sum_{i\in M_1} |y_{k_1}(i)| > \pi$. From Lemma 6 we obtain that there exists $N_1 \subseteq M_1$ for which

$$\pi < \| y_{k_1}\chi_{M_1} \| = \sum_{i\in M_1} |y_{k_1}(i)| \leq \pi \left| \sum_{i\in N_1} y_{k_1}(i) \right|.$$

Let $m_1 > \max N_1$ satisfy $\| y_{k_1}\chi_{\{m_1,\dots\}} \| = \sum_{i=m_1}^{\infty} |y_{k_1}(i)| < 1$. Then N_1, k_1 and m_1 satisfy (i), (ii) and (iv) by construction and (iii) is satisfied trivially.

Now let us assume that $j \in \mathbb{N}$ and that we have $N_1, N_2, \dots, N_j$, $k_1, k_2, \dots, k_j$ and $m_1, m_2, \dots, m_j$ satisfying (i)-(iv). Then for every $i \in M = \{1, 2, \dots, m_j\}$ we have $\lim_{k\to\infty} |y_k(i)| = 0$ by (2.4). Therefore there exists $k_{j+1}^1 > k_j$ such that for every $k \geq k_{j+1}^1$ it holds that $\| y_k\chi_M \| = \sum_{i\in M} |y_k(i)| < 1$. On the other hand, we have $\lim_{k\to\infty} \| y_k \| = \infty$, hence there exists $k_{j+1}^2 > k_j$ such that for every $k \geq k_{j+1}^2$ it holds that $\| y_k \| > (j+1)\cdot\pi + 2$. If we take $k_{j+1} = \max\{k_{j+1}^1,\ k_{j+1}^2\} > k_j$, we obtain the vector $y_{k_{j+1}}$ satisfying $\| y_{k_{j+1}}\chi_{\{1,2,\dots,\max N_j\}} \| \leq \| y_{k_{j+1}}\chi_M \| < 1$ and $(j+1)\cdot\pi + 2 < \| y_{k_{j+1}} \| = \| y_{k_{j+1}}\chi_M \| + \| y_{k_{j+1}}\chi_{\mathbb{N}\setminus M} \| < 1 + \| y_{k_{j+1}}\chi_{\mathbb{N}\setminus M} \|$, thus $(j+1)\cdot\pi < \| y_{k_{j+1}}\chi_{\mathbb{N}\setminus M} \|$. Let us find a finite set $M_{j+1} \subseteq \mathbb{N}\setminus M$ satisfying $\| y_{k_{j+1}}\chi_{M_{j+1}} \| > (j+1)\cdot\pi$. By Lemma 6 there exists $N_{j+1} \subseteq M_{j+1}$ such that

$$(j+1)\cdot\pi < \| y_{k_{j+1}}\chi_{M_{j+1}} \| = \sum_{i\in M_{j+1}} \left| y_{k_{j+1}}(i) \right| \leq \pi \left| \sum_{i\in N_{j+1}} y_{k_{j+1}}(i) \right|.$$

Finally, let us take $m_{j+1} > \max N_{j+1}$ satisfying $\| y_{k_{j+1}}\chi_{\{m_{j+1},\dots\}} \| < 1$. Then (i)-(iv) are clearly satisfied and the construction is complete.

Let us define $N = \bigcup_{j=1}^{\infty} N_j$ and $a = \chi_N$. Then the corresponding element

$f_a \in (\ell_1)^*$ satisfies (for every $j \in \mathbb{N}$, $j \geq 2$) the following inequality:

$$\begin{aligned}
\left|f_a(y_{k_j})\right| &= \left|\sum_{i=1}^{\infty} y_{k_j}(i)\chi_N(i)\right| = \\
&= \left|\sum_{i\in N_1\cup\cdots\cup N_{j-1}} y_{k_j}(i) + \sum_{i\in N_j} y_{k_j}(i) + \sum_{i\in\bigcup_{t=j+1}^{\infty} N_t} y_{k_j}(i)\right| \geq \\
&\geq \left|\sum_{i\in N_j} y_{k_j}(i)\right| - \left|\sum_{i\in N_1\cup\cdots\cup N_{j-1}} y_{k_j}(i)\right| - \left|\sum_{i\in\bigcup_{t=j+1}^{\infty} N_t} y_{k_j}(i)\right| \geq \\
&\geq j - \| y_{k_j}\chi_{\{1,2,\ldots,\max N_{j-1}\}} \| - \| y_{k_j}\chi_{\{m_j,\ldots\}} \| > j - 2.
\end{aligned}$$

Thus $\lim_{k\to\infty} f_a(y_k) \neq 0$, which contradicts (2.3). Therefore $(x_n)_{n\in\mathbb{N}}$ has to be bounded.

Step 2. We want to show that for every $a \in \ell_\infty$ it holds that the sequence $(f_a(x_n))_{n\in\mathbb{N}} \subseteq \mathbb{F}$ converges.

Let $a \in \ell_\infty$ be given. Let $\varepsilon > 0$ be given. Note that without loss of generality, we can assume that $a \in B_{\ell_\infty}$ (if $a = 0$, the statement is obvious. Otherwise, let us assume the vector $b = a/\| a \|_\infty \in B_{\ell_\infty}$. If $(f_b(x_n))_{n\in\mathbb{N}}$ converges to $L \in \mathbb{F}$, then $(f_a(x_n))_{n\in\mathbb{N}}$ converges to $\| a \|_\infty \cdot L$ and the statement holds). That means that for every $i \in \mathbb{N}$ we have $a(i) \in B_{\mathbb{F}}$. Since $B_{\mathbb{F}}$ is totally bounded, there exists a finite $\frac{\varepsilon}{2}$-net for $B_{\mathbb{F}}$ that consists of nonzero $z_1, z_2, \ldots, z_m \in B_{\mathbb{F}}$. Set $M_j = \{i \in \mathbb{N} : a(i) \in B(z_j, \varepsilon/2)\}$, $j \in \{1, 2, \ldots, m\}$. Let us define $N_1 = M_1$ and $N_j = M_j \setminus (N_1 \cup \cdots \cup N_{j-1})$, $j \in \{2, \ldots, m\}$. By construction, N_j are pairwise disjoint and furthermore $\mathbb{N} = \bigcup_{j=1}^m N_j$ (because $B_{\mathbb{F}} = \bigcup_{j=1}^m B(z_j, \varepsilon/2)$). Therefore we can define $b = \sum_{j=1}^m z_j\chi_{N_j}$. Then $\| a - b \|_\infty \leq \varepsilon$, because for every $i \in \mathbb{N}$ there exists $j \in \{1, 2, \ldots, m\}$ such that $i \in N_j \subseteq M_j$, which means $|a(i) - b(i)| = |a(i) - z_j| \leq \varepsilon/2 < \varepsilon$.

Let $M = \sup_{n\in\mathbb{N}}\| x_n \|$. By *Step 1*, $M < \infty$. By the assumption applied on the sets $N_1, N_2, \ldots, N_m$ we have that the sequences $(f_{\chi_{N_j}}(x_n))_{n\in\mathbb{N}}$, $j \in \{1, 2, \ldots, m\}$ are all convergent, and thus they are Cauchy. Therefore there exist indices $n_1, n_2, \ldots, n_m \in \mathbb{N}$ such that for every $j \in \{1, 2, \ldots, m\}$ and for every $n, n' \geq n_j$ it holds $\left|f_{\chi_{N_j}}(x_n) - f_{\chi_{N_j}}(x_{n'})\right| < \frac{\varepsilon}{\sum_{j=1}^m |z_j|}$. Let us set $n_0 = \max\{n_1, n_2, \ldots, n_m\}$ and let $n, n' \geq n_0$. Then we have

$$\begin{aligned}
|f_a(x_n) - f_a(x_{n'})| = |f_a(x_n - x_{n'})| &\leq |(f_a - f_b)(x_n - x_{n'})| + |f_b(x_n - x_{n'})| \leq \\
&\leq \| f_a - f_b \|_{\ell_1^*} \| x_n - x_{n'} \| + \left|\sum_{j=1}^m z_j f_{\chi_{N_j}}(x_n - x_{n'})\right| = \\
&= \| a - b \|_\infty (\| x_n \| + \| x_{n'} \|) + \left|\sum_{j=1}^m z_j f_{\chi_{N_j}}(x_n - x_{n'})\right| \leq \\
&\leq 2M\varepsilon + \sum_{j=1}^m |z_j| \left|f_{\chi_{N_j}}(x_n) - f_{\chi_{N_j}}(x_{n'})\right| \leq \\
&\leq 2M\varepsilon + \frac{\varepsilon}{\sum_{j=1}^m |z_j|} \sum_{j=1}^m |z_j| = \varepsilon(2M + 1).
\end{aligned}$$

Hence $(f_a(x_n))_{n\in\mathbb{N}} \subseteq \mathbb{F}$ is a Cauchy sequence, and thus it is convergent.

Step 3. Finally, let us show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (in $(\ell_1, \|\cdot\|)$).

By way of contradiction, if $(x_n)_{n\in\mathbb{N}}$ is not Cauchy, there exists $\varepsilon > 0$ and two strictly increasing sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\| x_{n_k} - x_{m_k} \| \geq \varepsilon$ for every $k \in \mathbb{N}$. Let us set $y_k = x_{n_k} - x_{m_k}$. Let $a \in \ell_\infty$ be given. By *Step 2* there exists $L_a = \lim_{n\to\infty} f_a(x_n)$, and thus

$$\lim_{k\to\infty} f_a(y_k) = \lim_{k\to\infty} (f_a(x_{n_k}) - f_a(x_{m_k})) = L_a - L_a = 0.$$

Since $a \in \ell_\infty \cong (\ell_1)^*$ was arbitrary, we can apply Schur's theorem (Theorem 7) and obtain that $\lim_{k\to\infty}\| y_k \| = 0$, which contradicts $\| y_k \| = \| x_{n_k} - x_{m_k} \| \geq \varepsilon$ for every $k \in \mathbb{N}$. Since $(\ell_1, \|\cdot\|)$ is complete, there exists $x \in \ell_1$ such that $\lim_{n\to\infty} x_n = x$, which completes the proof.

□

Now we are ready to prove the Nikodym theorem (see Brooks [1969]).

Theorem 9 (Nikodym). *Let $(X, \mathcal{A})$ be a measurable space, let $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ be a sequence of measures on $\mathcal{A}$ such that there exists a set function $\mu\colon \mathcal{A} \to \mathbb{R}$ satisfying $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$. Then μ is a measure on $\mathcal{A}$ and the family $\{\mu_n : n \in \mathbb{N}\}$ is uniformly σ-additive.*

Proof. Let $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ for every $A \in \mathcal{A}$ and let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ a sequence of pairwise disjoint sets be given.

First of all, μ is (finitely) additive. Indeed, for every $k \in \mathbb{N}$ we have

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{k} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{k} \mu_n(A_i) = \\ = \sum_{i=1}^{k} \lim_{n\to\infty} \mu_n(A_i) = \sum_{i=1}^{k} \mu(A_i).$$

Set $A = \bigcup_{i=1}^{\infty} A_i$, $A_0 = \emptyset$ and $E_k = A \setminus \bigcup_{i=1}^{k-1} A_i$ for every $k \in \mathbb{N}$. We want to show that $\lim_{k\to\infty} \mu(E_k) = 0$, because then we would also have

$$\sum_{i=1}^{\infty} \mu(A_i) = \lim_{k\to\infty} \sum_{i=1}^{k} \mu(A_i) = \lim_{k\to\infty} \mu\left(\bigcup_{i=1}^{k} A_i\right) = \\ = \lim_{k\to\infty} \mu\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \setminus E_{k+1}\right) = \lim_{k\to\infty} \mu\left(\bigcup_{i=1}^{\infty} A_i\right) - \mu(E_{k+1}) = \mu(A) - 0,$$

where the fifth equality follows from $\lim_{m\to\infty} \mu(E_m) = 0$.

Note that $A_i = E_i \setminus E_{i+1}$, $i \in \mathbb{N}$, because the sets A_i are pairwise disjoint.

Let us denote $x_n(i) = \mu_n(A_i)$. Then $x_n \in \ell_1$ for every $n \in \mathbb{N}$. Indeed, let $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{\infty} |x_n(i)| = \sum_{i=1}^{\infty} |\mu_n(A_i)| \leq \sum_{i=1}^{\infty} |\mu_n|(A_i) = |\mu_n|(A) < \infty,$$

which means $x_n \in \ell_1$. Furthermore, let $N \subseteq \mathbb{N}$, then obviously $a = \chi_N \in \ell_\infty$ and for the functional $f_a \in (\ell_1)^*$ represented by a we have

$$\lim_{n\to\infty} f_a(x_n) = \lim_{n\to\infty} \sum_{i\in N} x_n(i) = \lim_{n\to\infty} \sum_{i\in N} \mu_n(A_i) = \\ = \lim_{n\to\infty} \mu_n\left(\bigcup_{i\in N} A_i\right) = \mu\left(\bigcup_{i\in N} A_i\right) \in \mathbb{R}.$$

Therefore by Lemma 8 there exists $x \in \ell_1$ such that $\lim_{n\to\infty} x_n = x$. Then for every $i \in \mathbb{N}$ we have

$$x(i) = \lim_{n\to\infty} x_n(i) = \lim_{n\to\infty} \mu_n(A_i) = \mu(A_i).$$

Hence

$$0 = \lim_{n\to\infty} \| x_n - x \| = \lim_{n\to\infty} \sum_{i=1}^{\infty} |x_n(i) - x(i)| = \lim_{n\to\infty} \sum_{i=1}^{\infty} |\mu_n(A_i) - \mu(A_i)| \,. \tag{2.5}$$

Note that since $x \in \ell_1$, it holds

$$0 = \lim_{k\to\infty} \sum_{i=k}^{\infty} |x(i)| = \lim_{k\to\infty} \sum_{i=k}^{\infty} |\mu(A_i)| \,,$$

which implies

$$0 = \lim_{k\to\infty} \sum_{i=k}^{\infty} \mu(A_i). \tag{2.6}$$

Let us set $s_{n,k} = \mu_n(E_k) - \sum_{i=k}^{\infty} \mu(A_i)$. Then

$$\lim_{k\to\infty} s_{n,k} = \lim_{k\to\infty} (\mu_n(E_k) - \sum_{i=k}^{\infty} \mu(A_i)) = 0 \text{ for every } n \in \mathbb{N}. \tag{2.7}$$

That follows from (2.6) and from the fact that $E_1 \supseteq E_2 \supseteq \dots$, $\bigcap_{k=1}^{\infty} E_k = \emptyset$ and $|\mu_n|(E_1) < \infty$, $n \in \mathbb{N}$ (measures μ_n are bounded by Corollary 2). By assumption, we also have

$$\lim_{n\to\infty} s_{n,k} = \lim_{n\to\infty} \mu_n(E_k) - \sum_{i=k}^{\infty} \mu(A_i) = \mu(E_k) - \sum_{i=k}^{\infty} \mu(A_i). \tag{2.8}$$

Let $\varepsilon > 0$ be given. By (2.5), there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$ it holds $\sum_{i=1}^{\infty} |\mu_n(A_i) - \mu(A_i)| < \varepsilon$. Since $x, x_1, x_2, \dots \in \ell_1$, there exists $k_0 \in \mathbb{N}$ such that for every $n \in \{1, 2, \dots, n_0\}$ and for every $k \geq k_0$ we have $\varepsilon > \sum_{i=k}^{\infty} |x_n(i) - x(i)| = \sum_{i=k}^{\infty} |\mu_n(A_i) - \mu(A_i)|$. Therefore for every $k \in \mathbb{N}$, $k \geq k_0$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
|s_{n,k}| &= \left| \mu_n(E_k) - \sum_{i=k}^{\infty} \mu(A_i) \right| = \left| \mu_n\left(A \setminus \bigcup_{i=1}^{k-1} A_i \right) - \sum_{i=k}^{\infty} \mu(A_i) \right| = \\
&= \left| \mu_n\left(\bigcup_{i=k}^{\infty} A_i \right) - \sum_{i=k}^{\infty} \mu(A_i) \right| = \left| \sum_{i=k}^{\infty} \Big(\mu_n(A_i) - \mu(A_i) \Big) \right| \leq \\
&\leq \sum_{i=k}^{\infty} |\mu_n(A_i) - \mu(A_i)| < \varepsilon.
\end{aligned} \tag{2.9}$$

That means, by definition, that the functions $g_k \colon \mathbb{N} \to \mathbb{R}$ defined by $g_k(n) = s_{n,k}$ converge uniformly to 0 on $\mathbb{N}$. Therefore we can interchange the limits and (2.7) and (2.8) yield

$$\begin{aligned}
\lim_{k\to\infty} \Big(\mu(E_k) - \sum_{i=k}^{\infty} \mu(A_i) \Big) &= \lim_{k\to\infty} \lim_{n\to\infty} s_{n,k} = \lim_{k\to\infty} \lim_{n\to\infty} g_k(n) = \\
&= \lim_{n\to\infty} \lim_{k\to\infty} g_k(n) = \lim_{n\to\infty} 0 = 0.
\end{aligned}$$

Then the previous equation and equation (2.6) imply that

$$\lim_{k\to\infty}\mu(E_k)=\lim_{k\to\infty}(\mu(E_k)-\sum_{i=k}^{\infty}\mu(A_i)+\sum_{i=k}^{\infty}\mu(A_i))=0+0=0,$$

which is the desired conclusion.

The uniform σ-additivity is now easy to verify since we know that μ itself is σ-additive. In fact, $A_i \in \mathcal{A}$ pairwise disjoint and $\varepsilon > 0$ were given and we have already found $k_0 \in \mathbb{N}$ such that the equation (2.9) holds. Since we know $\lim_{k\to\infty}\mu(E_k) = 0$, we can find $k_1 \geq k_0$ such that for every $k \geq k_1$ the inequality $|\mu(E_k)| < \varepsilon$ is satisfied. Then for every $k \in \mathbb{N}$, $k \geq k_1$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\left|\mu_n\left(\bigcup_{i=1}^{\infty}A_i\right)-\sum_{i=1}^{k}\mu_n(A_i)\right| &= \left|\sum_{i=1}^{\infty}\mu_n(A_i)-\sum_{i=1}^{k}\mu_n(A_i)\right| = \\ &= \left|\sum_{i=k+1}^{\infty}\mu_n(A_i)\right| = \left|\sum_{i=k+1}^{\infty}\Big(\mu_n(A_i)-\mu(A_i)+\mu(A_i)\Big)\right| \leq \\ &\leq \left|\sum_{i=k+1}^{\infty}(\mu_n(A_i)-\mu(A_i))\right| + \left|\sum_{i=k+1}^{\infty}\mu(A_i)\right| \leq \\ &\leq \sum_{i=k+1}^{\infty}|\mu_n(A_i)-\mu(A_i)| + \left|\bigcup_{i=k+1}^{\infty}\mu(A_i)\right| = \\ &= \sum_{i=k+1}^{\infty}|\mu_n(A_i)-\mu(A_i)| + |\mu(E_{k+1})| < 2\varepsilon,\end{aligned}$$

which completes the proof.

□

In the Vitali-Hahn-Saks theorem we put an extra assumption on the sequence $(\mu_n)_{n\in\mathbb{N}}$ from the previous theorem: the absolute continuity with respect to one universal measure λ. Then we will mainly want to obtain the uniform absolute continuity with respect to λ. This result is proved as the second part of the theorem in Brooks [1969]. The result *2.* is stated in Florescu and Godet-Thobie [2012] as Theorem 1.36 iii).

Theorem 10 (Vitali-Hahn-Saks)**.** *Let $(X,\mathcal{A})$ be a measurable space and suppose that $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ is a sequence of measures on $\mathcal{A}$ such that there exists a set function $\mu\colon \mathcal{A}\to\mathbb{R}$ satisfying $\lim_{n\to\infty}\mu_n(A)=\mu(A)$ for every $A\in\mathcal{A}$. Let $\lambda\colon\mathcal{A}\to[0,\infty]$ be a measure such that for every $n\in\mathbb{N}$ it holds $\mu_n \ll \lambda$. Then*

1. *μ is a measure on $\mathcal{A}$,*
2. *$\mu \ll \lambda$ and*
3. *$(\mu_n)_{n\in\mathbb{N}}$ is uniformly absolutely continuous with respect to λ.*

Proof. *1.* Follows from the Nikodym theorem (Theorem 9).

2. This is a simple observation. Let $A\in\mathcal{A}$ such that $\lambda(A)=0$ be given. By definition, $\mu_n(A)=0$ for every $n\in\mathbb{N}$, thus $\mu(A)=\lim_{n\to\infty}\mu_n(A)=0$. Therefore $\mu\ll\lambda$.

3. By way of contradiction, let us assume that $(\mu_n)_{n\in\mathbb{N}}$ is not uniformly absolutely continuous with respect to λ. Then, by Definition 8, there exist $\varepsilon > 0$, a strictly increasing sequence $(n_m)_{m\in\mathbb{N}} \subseteq \mathbb{N}$ of indices and a sequence of measurable sets $(A_m)_{m\in\mathbb{N}} \subseteq \mathcal{A}$ such that for every $m \in \mathbb{N}$ it holds $\lambda(A_m) < 1/2^m$ and $|\mu_{n_m}(A_m)| \geq \varepsilon$.

Without loss of generality we can assume that $\mu_{n_m} = \mu_m$ (otherwise we set $\nu_m = \mu_{n_m}$ and we prove the theorem for the sequence $(\nu_m)_{m\in\mathbb{N}}$). Therefore we have

$$\lambda(A_m) < 1/2^m \tag{2.10}$$

and

$$|\mu_m(A_m)| \geq \varepsilon \tag{2.11}$$

for every $m \in \mathbb{N}$.

Now we assert that for every $k \in \mathbb{N}$ there exists a subsequence of natural numbers $(n_i^{(k)})_{i\in\mathbb{N}}$ such that, if we set $(n_i^{(0)})_{i\in\mathbb{N}} = (i)_{i\in\mathbb{N}}$, it holds

(i) $(n_i^{(k+1)})_{i\in\mathbb{N}}$ is a subsequence of $(n_i^{(k)})_{i\in\mathbb{N}}$ for every $k \in \mathbb{N} \cup \{0\}$,

(ii) $\sum_{i=1}^{\infty} \left|\mu_{n_1^{(k)}}\right| \left(A_{n_i^{(k+1)}}\right) < \varepsilon/2$ for every $k \in \mathbb{N} \cup \{0\}$,

(iii) $n_1^{(k)} < n_1^{(k+1)}$, $k \in \mathbb{N} \cup \{0\}$.

The sequences $(n_i^{(k)})_{i\in\mathbb{N}}$ will be constructed inductively.

To start the induction, let us notice that $\lim_{n\to\infty} |\mu_1| (A_n) = 0$. That follows from the assumption $\mu_1 \ll \lambda$ and Proposition 5. Therefore, by definition, there exists a subsequence $(n_i^{(1)})_{i\in\mathbb{N}}$ of natural numbers (or in other words, $(n_i^{(1)})_{i\in\mathbb{N}}$ is a subsequence of $(n_i^{(0)})_{i\in\mathbb{N}}$) such that $|\mu_1| (A_{n_i^{(1)}}) < \varepsilon/2^{i+1}$ for every $i \in \mathbb{N}$. Furthermore, by definition of the limit, we can assume that $n_1^{(1)}$ satisfies $n_1^{(1)} > n_1^{(0)} = 1$. Consequently,

$$\sum_{i=1}^{\infty} \left|\mu_{n_1^{(0)}}\right| (A_{n_i^{(1)}}) = \sum_{i=1}^{\infty} |\mu_1| (A_{n_i^{(1)}}) < \sum_{i=1}^{\infty} \varepsilon/2^{i+1} = \varepsilon/2.$$

Therefore, by construction, (i)-(iii) are satisfied.

Now let us assume that we have $(n_i^{(1)})_{i\in\mathbb{N}}, \ldots, (n_i^{(k)})_{i\in\mathbb{N}}$ satisfying (i)-(iii). By assumption, $\mu_{n_1^{(k)}} \ll \lambda$ and thus by Proposition 5 we have $\lim_{n\to\infty} \left|\mu_{n_1^{(k)}}\right| (A_n) = 0$, which implies $\lim_{i\to\infty} \left|\mu_{n_1^{(k)}}\right| (A_{n_i^{(k)}}) = 0$. Therefore there exists a subsequence $(n_i^{(k+1)})_{i\in\mathbb{N}}$ extracted from $(n_i^{(k)})_{i\in\mathbb{N}}$ satisfying $\left|\mu_{n_1^{(k)}}\right| (A_{n_i^{(k+1)}}) < \varepsilon/2^{i+1}$ for every $i \in \mathbb{N}$. Furthermore, by definition of the limit, we can assume that $n_1^{(k+1)}$ satisfies $n_1^{(k+1)} > n_1^{(k)}$. It holds that

$$\sum_{i=1}^{\infty} \left|\mu_{n_1^{(k)}}\right| (A_{n_i^{(k+1)}}) < \sum_{i=1}^{\infty} \varepsilon/2^{i+1} = \varepsilon/2.$$

Then, by construction, (i)-(iii) are satisfied and the construction is complete.

Let $\nu_i = \mu_{n_1^{(i)}}$ and $B_i = A_{n_1^{(i)}}$. Then from (2.10) and from (iii) we obtain

$$\lambda(B_i) = \lambda(A_{n_1^{(i)}}) < 1/2^{n_1^{(i)}} \leq 1/2^i. \tag{2.12}$$

We also have that for every $k \in \mathbb{N}$

$$\begin{aligned}\sum_{j=k+1}^{\infty} |\nu_k|(B_j) = \sum_{j=k+1}^{\infty} |\nu_k|(A_{n_1^{(j)}}) \leq \sum_{i=1}^{\infty} |\nu_k|(A_{n_i^{(k+1)}}) = \\ = \sum_{i=1}^{\infty} \left|\mu_{n_1^{(k)}}\right|(A_{n_i^{(k+1)}}) < \varepsilon/2,\end{aligned} \tag{2.13}$$

where the equalities follow from definitions, the first inequality follows from (i) and (iii) and the second inequality follows from (ii).

Let $Q = \limsup B_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i$ and $C_n = \bigcup_{i=n}^{\infty} B_i$, $n \in \mathbb{N}$. Then $C_n \searrow Q$. Let $T_n = C_n \setminus Q$. Then $T_n \searrow \emptyset$. By (2.12) and by the assumption that λ is a positive measure, it holds for every $n \in \mathbb{N}$ that

$$\lambda(Q) \leq \lambda(C_n) \leq \sum_{i=n}^{\infty} \lambda(B_i) \leq \sum_{i=n}^{\infty} 1/2^i = 1/2^{n-1},$$

which implies $\lambda(Q) = 0$. By assumption, $\nu_k = \mu_{n_1^{(k)}} \ll \lambda$, $k \in \mathbb{N}$, hence $\nu_k(Q) = 0$ for every $k \in \mathbb{N}$. Therefore, for every $k, n \in \mathbb{N}$, we have

$$\nu_k(T_n) = \nu_k(C_n \setminus Q) = \nu_k(C_n) - \nu_k(Q) = \nu_k(C_n). \tag{2.14}$$

Let us define $g_n \colon \mathbb{N} \to \mathbb{R}$, $g_n(k) = \nu_k(T_n)$. We claim that $g_n \rightrightarrows 0$ on $\mathbb{N}$. To prove this, let $\eta > 0$ be given. By assumption,

$$\forall A \in \mathcal{A} : \lim_{n \to \infty} \mu_n(A) = \mu(A),$$

thus

$$\forall A \in \mathcal{A} : \lim_{k \to \infty} \nu_k(A) = \mu(A).$$

Since ν_k are also real-valued, we can apply the Nikodym theorem (Theorem 9) and obtain that the family $\{\nu_k : k \in \mathbb{N}\}$ is uniformly σ-additive. Let us set $S_n = T_n \setminus T_{n+1}$. Since $T_n \searrow \emptyset$, we obtain that $\bigcup_{i=1}^{n} S_i \nearrow T_1$. Moreover, S_n are pairwise disjoint. Therefore, by Definition 7, there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$ and for every $k \in \mathbb{N}$ it holds that $|\nu_k(T_1) - \sum_{i=1}^{n} \nu_k(S_i)| < \eta$. Then for every $n \in \mathbb{N}$, $n \geq n_0$ and for every $k \in \mathbb{N}$ it holds that

$$|g_n(k)| = |\nu_k(T_{n+1})| = \left|\nu_k(T_1 \setminus (\bigcup_{i=1}^{n} S_i))\right| = \left|\nu_k(T_1) - \sum_{i=1}^{n} \nu_k(S_i)\right| < \eta,$$

which establishes the claim. (Note that the third equality follows from Corollary 2.)

Since $g_n \rightrightarrows 0$ on $\mathbb{N}$, we can find $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq N$ it holds

$$\forall k \in \mathbb{N} : \quad |g_n(k)| < \varepsilon/2.$$

Then (2.14) yields

$$|\nu_k(C_n)| = |\nu_k(T_n)| = |g_n(k)| < \varepsilon/2 \tag{2.15}$$

for every $n \geq N$ and for every $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
\nu_k(B_k) &= \nu_k\left(\bigcup_{i=k}^{\infty} B_i \setminus \left(\bigcup_{i=k+1}^{\infty} B_i \setminus B_k\right)\right) = \\
&= \nu_k\left(C_k \setminus \left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right)\right) = \\
&= \nu_k(C_k) - \nu_k\left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right).
\end{aligned}$$

Therefore

$$|\nu_k(B_k)| \leq |\nu_k(C_k)| + \left|\nu_k\left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right)\right|. \tag{2.16}$$

It also holds that

$$\begin{aligned}
\left|\nu_k\left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right)\right| &\leq |\nu_k|\left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right) \leq \\
&\leq |\nu_k|\left(\bigcup_{i=k+1}^{\infty} B_i\right).
\end{aligned} \tag{2.17}$$

Therefore, if we set $n = k > N$, we obtain by (2.11), (2.16), (2.15) and (2.17), by σ-subadditivity and by (2.13) the following inequality:

$$\begin{aligned}
\varepsilon &\leq \left|\mu_{n_1^{(k)}}\left(A_{n_1^{(k)}}\right)\right| = |\nu_k(B_k)| \leq |\nu_k(C_k)| + \left|\nu_k\left(\left(\bigcup_{i=k+1}^{\infty} B_i\right) \cap (X \setminus B_k)\right)\right| < \\
&< \varepsilon/2 + |\nu_k|\left(\bigcup_{i=k+1}^{\infty} B_i\right) \leq \varepsilon/2 + \sum_{i=k+1}^{\infty} |\nu_k|(B_i) < \varepsilon,
\end{aligned}$$

which yields the desired contradiction.

□

Finally, we would like to show that if we have $(X, \mathcal{A})$ a measurable space and a sequence of measures $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ such that for each set $A \in \mathcal{A}$ the set $\{\mu_n(A) : n \in \mathbb{N}\}$ is bounded, then the set $\{\mu_n : n \in \mathbb{N}\}$ is bounded in $(ca(\mathcal{A}), \|\cdot\|)$. The proof of this theorem follows the one in Florescu and Godet-Thobie [2012, Theorem 1.38].

Theorem 11 (Uniform boundedness Nikodym's theorem)**.** *Let $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ be a sequence of measures such that*

$$\sup_{n\in\mathbb{N}} |\mu_n(A)| < \infty \text{ for every } A \in \mathcal{A}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is bounded in the space $(ca(\mathcal{A}), \|\cdot\|)$.

Proof. We suppose that $\{\mu_n : n \in \mathbb{N}\}$ is unbounded in $(ca(\mathcal{A}), \|\cdot\|)$. By Theorem 4, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent in $ca(\mathcal{A})$, which means that

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}}|\mu_n(A)|) = \infty. \tag{2.18}$$

By hypothesis

$$\sup_{n\in\mathbb{N}}|\mu_n(X)| < \infty. \tag{2.19}$$

Let us construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and two sequences $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$, $(B_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ such that

(i) $\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}}|\mu_n(A\cap A_k)|) = \infty$ for every $k \in \mathbb{N}$,

(ii) $|\mu_{n_k}(A_k)| \geq \sum_{l=1}^{k-1}|\mu_{n_k}(B_l)| + k + 1$ and $|\mu_{n_k}(B_k)| \geq \sum_{l=1}^{k-1}|\mu_{n_k}(B_l)| + k + 1$ for every $k \in \mathbb{N}$,

(iii) $A_1 \supseteq A_2 \supseteq \dots$ and

(iv) $B_k = A_{k-1} \setminus A_k$ for every $k \in \mathbb{N}$,

where we shall define $A_0 = X, B_0 = \emptyset$ and $\sum_{l=1}^{0}|\mu_{n_k}(B_l)| = 0$.

All objects will be constructed inductively.

From (2.19) it follows that $\sup_{n\in\mathbb{N}}|\mu_n(X)|+2 \in \mathbb{R}$. Therefore, by (2.18), there exists $n_1 \in \mathbb{N}$ and $M_1 \in \mathcal{A}$ such that $|\mu_{n_1}(M_1)| > \sup_{n\in\mathbb{N}}|\mu_n(X)| + 2 \geq 2$. Then, since μ_{n_1} is bounded by hypothesis and by Corollary 2, we have that

$$\begin{aligned}|\mu_{n_1}(X\setminus M_1)| &= |\mu_{n_1}(X) - \mu_{n_1}(M_1)| \geq |\mu_{n_1}(M_1)| - |\mu_{n_1}(X)| \geq \\ &\geq |\mu_{n_1}(M_1)| - \sup_{n\in\mathbb{N}}|\mu_n(X)| \geq \sup_{n\in\mathbb{N}}|\mu_n(X)| + 2 - \sup_{n\in\mathbb{N}}|\mu_n(X)| = 2.\end{aligned}$$

Therefore we have

$$|\mu_{n_1}(M_1)| \geq 2 \quad \text{and} \quad |\mu_{n_1}(X\setminus M_1)| \geq 2. \tag{2.20}$$

According to (2.18), one of the following equalities must be satisfied:

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}}|\mu_n(A\cap M_1)|) = \infty \tag{2.21}$$

or

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}}|\mu_n(A\cap (X\setminus M_1))|) = \infty. \tag{2.22}$$

If (2.21) is satisfied, let us denote $A_1 = M_1$ and $B_1 = X \setminus M_1$. Otherwise, we denote $B_1 = M_1$ and $A_1 = X\setminus M_1$. Then, by (2.20) and by (2.21) (or by (2.22)), we have that (i) and (ii) are satisfied and, by construction, (iii) and (iv) are satisfied as well.

Now let us assume that $k \geq 2$ and that we have $A_1, A_2, \dots, A_k$, $B_1, B_2, \dots, B_k$ and $\mu_{n_1}, \mu_{n_2}, \dots, \mu_{n_k}$ satisfying (i)-(iv). By hypothesis

$$\sup_{n\in\mathbb{N}}|\mu_n(A_k)| < \infty,$$

thus $\sup_{n\in\mathbb{N}} |\mu_n(A_k)| + \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2 < \infty$. From (i) it follows that there exist $n_{k+1} \in \mathbb{N}$ and $M_{k+1} \in \mathcal{A}$, $M_{k+1} \subseteq A_k$, such that

$$\left|\mu_{n_{k+1}}(M_{k+1})\right| > \sup_{n\in\mathbb{N}} |\mu_n(A_k)| + \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2 \geq \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2.$$

We may choose $n_{k+1} > n_k$ because, if no such n_{k+1} exists, we have that

$$\sup_{n\in\mathbb{N},\ n\geq n_k} (\sup_{A\in\mathcal{A}} |\mu_n(A \cap A_k)|) \leq \sup_{n\in\mathbb{N}} |\mu_n(A_k)| + \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2 < \infty.$$

Then, since μ_n is bounded for every $n \in \mathbb{N}$ by Corollary 2, we obtain that there exist $K_1, K_2, \ldots, K_{n_k} \in (0,\infty)$ satisfying $\sup_{A\in\mathcal{A}} |\mu_{n_i}(A)| \leq K_i$ for each $i \leq n_k$. This implies

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A \cap A_k)|) \leq \\
&\qquad \leq \max\left\{K_1, K_2, \ldots, K_{n_k},\ \sup_{n\in\mathbb{N}} |\mu_n(A_k)| + \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2\right\} < \infty,
\end{aligned}$$

which contradicts (i).

It holds that

$$\begin{aligned}
\left|\mu_{n_{k+1}}(A_k \setminus M_{k+1})\right| &= \left|\mu_{n_{k+1}}(A_k) - \mu_{n_{k+1}}(M_{k+1})\right| \geq \\
&\geq \left|\mu_{n_{k+1}}(M_{k+1})\right| - \left|\mu_{n_{k+1}}(A_k)\right| \geq \\
&\geq \left|\mu_{n_{k+1}}(M_{k+1})\right| - \sup_{n\in\mathbb{N}} |\mu_n(A_k)| \geq \\
&\geq \sup_{n\in\mathbb{N}} |\mu_n(A_k)| + \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2 - \sup_{n\in\mathbb{N}} |\mu_n(A_k)| = \\
&= \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2.
\end{aligned}$$

Therefore we have

$$\left|\mu_{n_{k+1}}(M_{k+1})\right| \geq \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2 \tag{2.23}$$

and

$$\left|\mu_{n_{k+1}}(A_k \setminus M_{k+1})\right| \geq \sum_{l=1}^{k} |\mu_{n_k}(B_l)| + k + 2. \tag{2.24}$$

According to (i), one of the following equalities must be satisfied:

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A \cap M_{k+1})|) = \infty \tag{2.25}$$

or

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A \cap (A_k \setminus M_{k+1}))|) = \infty. \tag{2.26}$$

If (2.25) is satisfied, let us denote $A_{k+1} = M_{k+1}$ and $B_{k+1} = A_k \setminus M_{k+1}$. Otherwise, we denote $B_{k+1} = M_{k+1}$ and $A_{k+1} = A_k \setminus M_{k+1}$. Then, by (2.23), (2.24) and by (2.25) (or by (2.26)), we have that (i) and (ii) are satisfied and, by construction, (iii) and (iv) are satisfied as well. That completes the inductive construction.

Notice that from (iii) and (iv) it follows that the sets B_k, $k \in \mathbb{N}$, are pairwise disjoint.

Let $E = \{n_k : k \in \mathbb{N}\}$. Notice that E is infinite since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing. We assert that there exist $(k_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ a strictly increasing sequence of natural numbers, $(N_j)_{j\in\mathbb{N}} \subseteq \mathcal{P}(E)$ and $(C_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}$ such that

(a) $(N_j)_{j\in\mathbb{N}}$ consists of infinite subsets of E and $N_j \supseteq N_{j+1}$ for every $j \in \mathbb{N}$,

(b) k_j is the smallest element of N_j and $k_{j-1} < k_j$ for every $j \in \mathbb{N}$,

(c) $C_j = B_{k_j}$ for every $j \in \mathbb{N}$ and

(d) $\left|\mu_{n_{k_{j-1}}}\right| \left(\bigcup_{l\in N_j} B_l\right) < 1$ for every $j \in \mathbb{N}$,

where we shall define $k_0 = 1$.

The objects will be constructed inductively with respect to j. For $j = 1$, let $\{M^1_m : m \in \mathbb{N}\}$ be an infinite countable partition into infinite sets of the set E. Since

$$\sum_{m=1}^{\infty} |\mu_{n_1}| \left(\bigcup_{l\in M^1_m} B_l\right) = |\mu_{n_1}| \left(\bigcup_{l\in E} B_l\right) \leq |\mu_{n_1}| (X) < \infty,$$

there exists $m_0 \in \mathbb{N}$ such that $|\mu_{n_1}| \left(\bigcup_{l\in M^1_{m_0}} B_l\right) < 1$. Set $N_1 = M^1_{m_0}$, let k_1 be the smallest element of N_1 and let $C_1 = B_{k_1}$. Then indeed $k_1 > k_0 = 1$, because if not, we have $k_1 = 1$, thus

$$1 > |\mu_{n_1}| \left(\bigcup_{l\in N_1} B_l\right) \geq |\mu_{n_1}| (B_{k_1}) = |\mu_{n_1}| (B_1),$$

which contradicts (ii). Therefore $k_1 > k_0 = 1$ and (a)-(d) are satisfied.

Now let us assume that $j \geq 2$ and that we have $k_1, k_2, \ldots, k_j$, $N_1, N_2, \ldots, N_j$ and $C_1, C_2, \ldots, C_j$ satisfying (a)-(d). Let $\left\{M^{(k+1)}_m : m \in \mathbb{N}\right\}$ be an infinite countable partition into infinite sets (of the infinite set) N_j. Since

$$\sum_{m=1}^{\infty} \left|\mu_{n_{k_j}}\right| \left(\bigcup_{l\in M^{(k+1)}_m} B_l\right) = \left|\mu_{n_{k_j}}\right| \left(\bigcup_{l\in N_j} B_l\right) \leq \left|\mu_{n_{k_j}}\right| (X) < \infty,$$

there exists $m_0^* \in \mathbb{N}$ such that $\left|\mu_{n_{k_j}}\right| \left(\bigcup_{l\in M^{(k+1)}_{m_0^*}} B_l\right) < 1$. Set $N_{j+1} = M^{(k+1)}_{m_0^*}$, let k_{j+1} be the smallest element of N_{j+1} and let $C_{j+1} = B_{k_{j+1}}$. Then indeed $k_{j+1} > k_j$, because if not, we have $k_{j+1} = k_j$ (since k_j is the smallest element of N_j and $N_{j+1} \subseteq N_j$, we know $k_{j+1} \geq k_j$), thus

$$1 > \left|\mu_{n_{k_j}}\right| \left(\bigcup_{l\in N_{j+1}} B_l\right) \geq \left|\mu_{n_{k_j}}\right| (B_{k_{j+1}}) = \left|\mu_{n_{k_j}}\right| (B_{k_j}),$$

which contradicts (ii). Therefore $k_{j+1} > k_j$ and (a)-(d) are satisfied, which completes the inductive construction.

Notice that, because the sets $B_k, k \in \mathbb{N}$, are pairwise disjoint, (b) and (c) immediately imply that $C_j, j \in \mathbb{N}$, are pairwise disjoint.

Let us denote $C = \bigcup_{j \in \mathbb{N}} C_j$. Then from (a), (b), (c) and (d) we have, for every $j \in \mathbb{N}$,

$$\left|\mu_{n_{k_{j-1}}}\right| \left(\bigcup_{h=j}^{\infty} C_h \right) \leq \left|\mu_{n_{k_{j-1}}}\right| \left(\bigcup_{l \in N_j} B_l \right) < 1, \tag{2.27}$$

and by (ii) and the obvious inequality $j - 1 \leq k_{j-1} \leq k_j - 1$, holding for every $j \in \mathbb{N}$, we have

$$\left|\mu_{n_{k_j}}(C_j)\right| = \left|\mu_{n_{k_j}}(B_{k_j})\right| \geq \sum_{l=1}^{k_j - 1} \left|\mu_{n_{k_j}}(B_l)\right| + k_j + 1 \geq \sum_{h=1}^{j-1} \left|\mu_{n_{k_j}}(C_h)\right| + k_j + 1,$$

from where

$$\left|\mu_{n_{k_j}}(C_j)\right| - \sum_{h=1}^{j-1} \left|\mu_{n_{k_j}}(C_h)\right| \geq k_j + 1 \geq j + 1. \tag{2.28}$$

Therefore, by (2.27) and (2.28), for every $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\left|\mu_{n_{k_j}}(C)\right| &= \left|\mu_{n_{k_j}}\left(\bigcup_{h=1}^{\infty} C_h \right)\right| \geq \\
&\geq \left|\mu_{n_{k_j}}(C_j)\right| - \left|\mu_{n_{k_j}}\left(\bigcup_{h=1}^{j-1} C_h \right)\right| - \left|\mu_{n_{k_j}}\left(\bigcup_{h=j+1}^{\infty} C_h \right)\right| \geq \\
&\geq \left|\mu_{n_{k_j}}(C_j)\right| - \left|\mu_{n_{k_j}}\left(\bigcup_{h=1}^{j-1} C_h \right)\right| - \left|\mu_{n_{k_j}}\right| \left(\bigcup_{h=j+1}^{\infty} C_h \right) \geq \\
&\geq \left|\mu_{n_{k_j}}(C_j)\right| - \sum_{h=1}^{j-1} \left|\mu_{n_{k_j}}(C_h)\right| - \left|\mu_{n_{k_j}}\right| \left(\bigcup_{h=j+1}^{\infty} C_h \right) \geq \\
&\geq j + 1 - 1 = j
\end{aligned}$$

(where we define $\bigcup_{h=1}^{0} C_h = \emptyset$), which implies that $\sup_{j \in \mathbb{N}} \left|\mu_{n_{k_j}}(C)\right| = \infty$, hence $\sup_{n \in \mathbb{N}} |\mu_n(C)| = \infty$, which contradicts the hypothesis of the theorem.

$\square$

3. Biting Lemma, Rosenthal's Subsequence Splitting Lemma

In this chapter we will look at the following problem. From topology we know that a topological space X is compact if and only if for every net in X there exists its convergent subnet. Sometimes, for example in metric spaces, it is sufficient to work with sequences instead of nets. In particular, from real analysis we know that every closed and bounded subset of $\mathbb{R}$ is compact. The question is whether a similar statement holds in L_p spaces, $p \in [1,\infty]$; i.e. is every bounded closed subset of L_p, $p \in [1,\infty]$, compact? The answer is no, however, since for $p \in (1,\infty)$ the spaces L_p are reflexive, we get at least the weak compactness. In L_1, the situation is much more complicated since we do not have the reflexivity nor weak compactness. However, we would still want to extract somehow convergent subsequence from any bounded sequence. And the way to do it is described in the Biting Lemma - essentially we need to consider even weaker form of convergence.

3.1 Uniform Integrability

In this chapter, our object of interest will be sequences of functions in L_1. Some of them "behave nicely" with respect to integral in the following way.

Definition 9 (Uniform integrability)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space with a positive measure μ and let $\mathcal{F} \subseteq L_1(X, \mathcal{A}, \mu)$. Then $\mathcal{F}$ is called* uniformly integrable *if*

$$\lim_{C\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>C\}} |f| \, d\mu = 0.$$

It should be clear that every finite set $\{f_1, \ldots, f_n\} \subseteq L_1$ is uniformly integrable. This follows from the continuity of finitely many finite measures $\nu_k(A) = \int_A |f_k| \, d\mu$, $A \in \mathcal{A}$, $k \in \{1, \ldots, n\}$.

We will also need to work with the following types of compactness.

Definition 10. *Let (X, τ) be a topological space and let $A \subseteq X$. Then A is said to be*

- relatively compact *if $\overline{A}$ is compact,*
- sequentially compact *if for every $(x_n)_{n\in\mathbb{N}} \subseteq A$ there exists a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}} \subseteq A$ (with the limit in A),*
- relatively sequentially compact *if for every $(x_n)_{n\in\mathbb{N}} \subseteq A$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ convergent in X and*
- countably compact *if every countable open cover of A has a finite subcover.*

The following theorem uses the uniform integrability to characterize the weak compactness in L_1. You can find it in Vladimir I. Bogachev [2007, page 285, Theorem 4.7.18] with its proof.

Theorem 12 (Dunford-Pettis theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite positive measure μ. Let $\mathcal{F} \subseteq L_1(X, \mathcal{A}, \mu)$. Then the set $\mathcal{F}$ is relatively compact in the weak topology of $L_1(X, \mathcal{A}, \mu)$ if and only if it is uniformly integrable.*

Notice that if μ is a finite positive measure and $\mathcal{F}$ is a uniformly integrable set, then $\mathcal{F}$ is automatically bounded. Indeed, by definition, there exists $C > 0$ such that $\sup_{f\in\mathcal{F}} \int_{\{|f|>C\}} |f| \; d\mu < 1$. Then for every $f \in \mathcal{F}$ we have

$$\int_X |f| \; d\mu = \int_{\{|f|\leq C\}} |f| \; d\mu + \int_{\{|f|>C\}} |f| \; d\mu \leq C\mu(X) + 1 < \infty.$$

From Theorem 12 it follows that if $(f_i)_{i\in I}$ is a net in L_1 and the set $\{f_i : i \in I\}$ is uniformly integrable, then there exists a subnet $(f_j)_{j\in J}$ weakly convergent in L_1. However, we would like that property to hold even for sequences and subsequences. The following theorem, proved for example in Whitley [1967, pages 116-118], asserts that it works.

Theorem 13 (Eberlain-Šmulian). *Let A be a subset of a Banach space X. Then for the weak topology of X the following assertions are equivalent:*

1. *A is relatively compact,*
2. *A is relatively sequentially compact and*
3. *A is relatively countably compact.*

3.2 Modulus of Uniform Integrability

From the previous section it follows that if we have a uniformly integrable sequence $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X, \mathcal{A}, \mu)$ and if μ is a finite positive measure, then there exists a weakly convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ (Theorems 12 and 13). However, not every bounded sequence in L_1 is uniformly integrable. In this section, we will work with a number that tells us "how much a family in L_1 is not uniformly integrable". Then we will want to show that for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $L_1(X, \mathcal{A}, \mu)$ with a finite measure μ there exists a subsequence $(g_n)_{n\in\mathbb{N}}$ that we can split into 2 parts: one part will be uniformly integrable (i.e. its "measure of non-uniform integrability" is 0) and the other part will tend to this "measure of non-uniform integrability".

Definition 11 (Modulus of uniform integrability). *Let $(X, \mathcal{A}, \lambda)$ be a measure space with a positive measure λ and let $\mathcal{F} \subseteq L_1(X, \mathcal{A}, \lambda)$. Then we define*

$$\eta(\mathcal{F}) = \inf_{\delta>0} \left[\sup \left\{ \int_A |f| \; d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, f \in \mathcal{F} \right\} \right] \in [0,\infty].$$

We say that $\eta(\mathcal{F})$ is the modulus of uniform integrability of *$\mathcal{F}$.*

Let us notice that

(i) it holds

$$\eta(\mathcal{F}) = \lim_{\delta \to 0_+} \left[\sup \left\{ \int_A |f| \, d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, f \in \mathcal{F} \right\} \right].$$

This is because the function

$$\delta \mapsto \sup \left\{ \int_A |f| \, d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, f \in \mathcal{F} \right\}, \ \delta \in (0,\infty),$$

is nonincreasing.

(ii) $\eta(\mathcal{F}) = 0$ if and only if the family $\{\mu_f : f \in \mathcal{F}\}$ is uniformly absolutely continuous with respect to λ, where $\mu_f(A) = \int_A f \, d\lambda$, $A \in \mathcal{A}$. That is because from (i) we have

$$\begin{aligned} \eta(\mathcal{F}) = 0 \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \in (0,\delta) : \\ \sup \left\{ \int_A |f| \, d\lambda : A \in \mathcal{A}, \lambda(A) < t, f \in \mathcal{F} \right\} < \varepsilon \Leftrightarrow \\ \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A \in \mathcal{A}, \ \lambda(A) < \delta \ \forall f \in \mathcal{F} : \int_A |f| \, d\lambda < \varepsilon. \end{aligned} \tag{3.1}$$

From Florescu and Godet-Thobie [2012, Remark 1.53 ii)] it follows that the right-hand side holds if and only if the family $\{\mu_f : f \in \mathcal{F}\}$ is uniformly absolutely continuous with respect to λ.

The following proposition describes the modulus of uniform integrability via the limit used in Definition 9. For a proof see Florescu and Godet-Thobie [2012, Proposition 1.83].

Proposition 14. *Let $(X, \mathcal{A}, \lambda)$ be a measure space with a finite positive measure λ and let $\mathcal{F} \subseteq L_1(X, \mathcal{A}, \lambda)$ be a bounded set. Then*

$$\eta(\mathcal{F}) = \lim_{t \to \infty} \left[\sup_{f \in \mathcal{F}} \int_{\{|f| > t\}} |f| \, d\lambda \right] = \lim_{t \to \infty} \left[\sup_{f \in \mathcal{F}} \int_{\{|f| \geq t\}} |f| \, d\lambda \right]$$

.

Remark 2. From Proposition 14 we have that every bounded set $\mathcal{F} \subseteq L_1(X, \mathcal{A}, \lambda)$, where λ is a finite positive measure, is uniformly integrable if and only if $\eta(\mathcal{F}) = 0$.

For the following theorems, we will need to define another type of convergence in L_1.

Definition 12. *Let $(X, \mathcal{A}, \mu)$ be a measure space with a positive and σ-finite measure μ. Then a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L_1(X, \mathcal{A}, \mu)$ is said to be w^2-*convergent *to $f \in L_1(X, \mathcal{A}, \mu)$ if the following two conditions are satisfied:*

(a) there exists a decreasing sequence $(B_p)_{p \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\lim_{p \to \infty} \mu(B_p) = 0$,

(b) for every fixed $p \in \mathbb{N}$, the sequence $(f_n \restriction_{X \setminus B_p})_{n \in \mathbb{N}}$ converges weakly to the function $f \restriction_{X \setminus B_p}$ in $L_1(X \setminus B_p, \mu)$.

We say that the sequence $(B_p)_{p \in \mathbb{N}}$ satisfying the above conditions (a) and (b) localizes the concentration of mass *of the w^2-convergent sequence $(f_n)_{n \in \mathbb{N}}$.*

According to Florescu and Godet-Thobie [2012, Remark 1.96 i)], if a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X,\mathcal{A},\mu)$ is w^2-convergent, then the limit function $f \in L_1(X,\mathcal{A},\mu)$ is uniquely determined (modulo almost everywhere). Furthermore, according to Florescu and Godet-Thobie [2012, Remark 1.96 vi)], a w^2-convergent sequence $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X,\mathcal{A},\mu)$ does not have to be bounded in $L_1(X,\mathcal{A},\mu)$.

The following theorem is technical and we will need it in the next proof. It can be found in Florescu and Godet-Thobie [2012, Proposition 1.93] with its proof.

Theorem 15. *Let $(X,\mathcal{A},\lambda)$ be a measure space with a finite positive measure λ. Let $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X,\mathcal{A},\lambda)$ be a w^2-convergent sequence and let $(B_p)_{p\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence which localizes the concentration of mass of $(f_n)_{n\in\mathbb{N}}$. Then*

$$\eta(\{f_n : n \in \mathbb{N}\}) = \lim_{p\to\infty} \limsup_{n\to\infty} \int_{B_p} |f_n| \, d\lambda.$$

The following theorem (Theorem 1.100 in Florescu and Godet-Thobie [2012]) asserts that if we have a w^2-convergent and bounded sequence in L_1, we can find its subsequence that we can split into a weakly convergent part and "an unpleasant part". Later we would like to show that this theorem holds even if $(f_n)_{n\in\mathbb{N}}$ is not w^2-convergent (see Corollary 19).

Theorem 16. *Let $(X,\mathcal{A},\lambda)$ be a measure space with a finite positive measure λ. Let $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X,\mathcal{A},\lambda)$ be a bounded (in $L_1(X,\mathcal{A},\lambda)$) and w^2-convergent sequence in $L_1(X,\mathcal{A},\lambda)$. Then there exists a subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that:*

(i) $\eta(\{f_n : n \in \mathbb{N}\}) = \lim_{n\to\infty} \int_{A_n} |g_n| \, d\lambda$ and

(ii) $(\chi_{X\setminus A_n} \cdot g_n)_{n\in\mathbb{N}}$ is weakly convergent in $L_1(X,\mathcal{A},\lambda)$.

Proof. Let $(B_p)_{p\in\mathbb{N}} \subseteq \mathcal{A}$ be a decreasing sequence which localizes the concentration of mass of $(f_n)_{n\in\mathbb{N}}$. By Theorem 15 we have

$$\eta = \eta(\{f_n : n \in \mathbb{N}\}) = \lim_{p\to\infty} \limsup_{n\to\infty} \int_{B_p} |f_n| \, d\lambda.$$

Notice that η is finite since the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L_1(X,\mathcal{A},\lambda)$. Therefore, by the definition of η, for every $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ and $k_n \in \mathbb{N}$ such that

(I) for every $n \in \mathbb{N}$,
$$\eta - \frac{1}{n} < \int_{B_{p_n}} |f_{k_n}| \, d\lambda < \eta + \frac{1}{n}, \tag{3.2}$$

(II) $p_{n+1} > p_n$ for every $n \in \mathbb{N}$,

(III) $k_{n+1} > k_n$ for every $n \in \mathbb{N}$.

Indeed, for $n = 1$ there exists $p_1 \in \mathbb{N}$ such that for all $p \geq p_1$ we have

$$\limsup_{n\to\infty} \int_{B_p} |f_n| \, d\lambda \in (\eta - 1, \eta + 1).$$

In particular, for $p = p_1$ there exists $k_1 \in \mathbb{N}$ such that $\eta - 1 < \int_{B_{p_1}} |f_{k_1}| \, d\lambda < \eta + 1$.

Now let us assume that $n \geq 2$ and that $p_1, p_2, \ldots, p_n$ and $k_1, k_2, \ldots, k_n$ satisfy (I)-(III). Since there exists $p^1_{n+1} \in \mathbb{N}$ such that for all $p \geq p^1_{n+1}$ we have $\limsup_{n\to\infty} \int_{B_p} |f_n| \, d\lambda \in (\eta - \frac{1}{n+1}, \eta + \frac{1}{n+1})$, we can set $p_{n+1} = \max\{p_n + 1, p^1_{n+1}\}$. Then (II) holds and for $p = p_{n+1}$ we can find $k_{n+1} > k_n$ such that

$$\eta - \frac{1}{n+1} < \int_{B_{p_{n+1}}} \left| f_{k_{n+1}} \right| \, d\lambda < \eta + \frac{1}{n+1}.$$

Hence (I) and (III) hold as well.

Now we assert that there exists a strictly increasing sequence $(i_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that, for every $n \in \mathbb{N} \cup \{0\}$,

$$\eta - \frac{1}{i_n} - \frac{1}{n+1} < \int_{B_{p_{i_n}} \setminus B_{p_{i_{n+1}}}} \left| f_{k_{i_n}} \right| \, d\lambda < \eta + \frac{1}{i_n}, \tag{3.3}$$

where we define $i_0 = 1$. Notice that, by (3.2), we have

$$\eta - 1 < \int_{B_{p_{i_0}}} \left| f_{k_{i_0}} \right| \, d\lambda < \eta + 1. \tag{3.4}$$

The sequence (i_n) will be constructed inductively. To demonstrate the base case, we will use the fact that the measure $\nu_{k_1}(A) = \int_A |f_{k_1}| \, d\lambda$, $A \in \mathcal{A}$, is absolutely continuous with respect to λ. Since $f_{k_1} \in L_1(X, \mathcal{A}, \lambda)$, ν_{k_1} is bounded. Hence we can use Proposition 5 and obtain that for $\varepsilon_1 = 1$ there exists $\delta_1 > 0$ such that

$$\nu_{k_1}(A) = \int_A |f_{k_1}| \, d\lambda < \varepsilon_1 = 1 \text{ for every } A \in \mathcal{A} \text{ with } \lambda(A) < \delta_1. \tag{3.5}$$

Since $\lim_{p\to\infty} \lambda(B_p) = 0$, we can find $i_1 > i_0$ such that $\lambda(B_{p_{i_1}}) < \delta_1$. Then by (3.5) we have $\int_{B_{p_{i_1}}} |f_{k_1}| \, d\lambda < 1$, which together with (3.4) implies

$$\eta - 1 - 1 < \int_{B_{p_{i_0}}} \left| f_{k_{i_0}} \right| \, d\lambda - \int_{B_{p_{i_1}}} \left| f_{k_{i_0}} \right| \, d\lambda \leq \int_{B_{p_{i_0}}} \left| f_{k_{i_0}} \right| \, d\lambda < \eta + 1,$$

therefore

$$\eta - 1 - 1 < \int_{B_{p_{i_0}} \setminus B_{p_{i_1}}} \left| f_{k_{i_0}} \right| \, d\lambda < \eta + 1.$$

Thus (3.3) holds for our i_1.

Now let us assume that $n \geq 2$ and that we have $i_0 < i_1 < \cdots < i_n$ satisfying (3.3). By (3.2), there exists p_{i_n} such that

$$\eta - \frac{1}{i_n} < \int_{B_{p_{i_n}}} \left| f_{k_{i_n}} \right| \, d\lambda < \eta + \frac{1}{i_n}. \tag{3.6}$$

As above, for $\varepsilon_{n+1} = \frac{1}{n+1}$ there exists $\delta_{n+1} > 0$ such that

$$\int_A \left| f_{k_{i_n}} \right| \, d\lambda < \varepsilon_{n+1} = \frac{1}{n+1} \text{ for every } A \in \mathcal{A} \text{ with } \lambda(A) < \delta_{n+1}. \tag{3.7}$$

Since $\lim_{p\to\infty} \lambda(B_p) = 0$, we can find $i_{n+1} > i_n$ such that $\lambda(B_{p_{i_{n+1}}}) < \delta_{n+1}$. Then by (3.7) we have $\int_{B_{p_{i_{n+1}}}} \left| f_{k_{i_n}} \right| \, d\lambda < \frac{1}{n+1}$, which together with (3.6) implies

$$\eta - \frac{1}{i_n} - \frac{1}{n+1} < \int_{B_{p_{i_n}}} \left| f_{k_{i_n}} \right| \, d\lambda - \int_{B_{p_{i_{n+1}}}} \left| f_{k_{i_n}} \right| \, d\lambda \leq \int_{B_{p_{i_n}}} \left| f_{k_{i_n}} \right| \, d\lambda < \eta + \frac{1}{i_n},$$

therefore

$$\eta - \frac{1}{i_n} - \frac{1}{n+1} < \int_{B_{p_{i_n}} \setminus B_{p_{i_{n+1}}}} \left| f_{k_{i_n}} \right| \, d\lambda < \eta + \frac{1}{i_n}.$$

This completes the construction.

For every $n \in \mathbb{N}$, let us define $A_n = B_{p_{i_n}} \setminus B_{p_{i_{n+1}}}$ and $h_n = f_{k_{i_n}}$. Then $(h_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets and, by (3.3),

$$\lim_{n\to\infty} \int_{A_n} |h_n| \, d\lambda = \eta = \eta(\{f_n : n \in \mathbb{N}\}). \tag{3.8}$$

Let us set

$$\begin{aligned}\eta_0 &= \eta(\{\chi_{X\setminus A_n} \cdot h_n : n \in \mathbb{N}\}) = \\ &= \inf_{\delta>0} \left[\sup \left\{ \int_E \chi_{X\setminus A_n} \cdot |h_n| \, d\lambda : E \in \mathcal{A}, \lambda(E) < \delta, n \in \mathbb{N} \right\} \right] = \\ &= \lim_{\delta\to 0_+} \left[\sup \left\{ \int_E \chi_{X\setminus A_n} \cdot |h_n| \, d\lambda : E \in \mathcal{A}, \lambda(E) < \delta, n \in \mathbb{N} \right\} \right].\end{aligned}$$

We want to show that $\eta_0 = 0$.

By way of contradiction, let $\eta_0 > 0$ and let $\alpha \in (0,\eta_0)$. Then for every $p \in \mathbb{N}$ we have

$$\alpha < \sup \left\{ \int_E \chi_{X\setminus A_n} \cdot |h_n| \, d\lambda : E \in \mathcal{A}, \lambda(E) < \frac{1}{p}, n \in \mathbb{N} \right\},$$

thus there exists $E_p \in \mathcal{A}$ with $\lambda(E_p) < \frac{1}{p}$ and $n_p \in \mathbb{N}$ such that

$$\int_{E_p\setminus A_{n_p}} \left| h_{n_p} \right| \, d\lambda > \alpha. \tag{3.9}$$

Moreover, the indices n_p can be chosen in such a way that for every $p \in \mathbb{N}$ we have $n_{p+1} > n_p$. Because if not, then there exists $p \in \mathbb{N}$ such that for every $E \in \mathcal{A}$ with $\lambda(E) < \frac{1}{p+1}$ and for every $l > n_p$ we have $\int_{E\setminus A_l} |h_l| \, d\lambda \leq \alpha$. We also have that the set $\{h_1, \ldots, h_{n_p}\}$ is uniformly integrable (it is finite), and thus by Proposition 14 we have $\eta(\{h_1, \ldots, h_{n_p}\}) = 0$. Thus by i) after Definition 9 we can find $\delta_1 > 0$ such that for every $\delta \in (0,\delta_1)$ we have

$$\sup \left\{ \int_A |h_k| \, d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, k \in \{1, \ldots, n_p\} \right\} < \alpha.$$

By setting $\delta_0 = \min\{\delta_1, \frac{1}{p+1}\}$, we obtain that for each $\delta \in (0,\delta_0)$ it holds

$$\sup \left\{ \int_E \chi_{X\setminus A_n} \cdot |h_n| \, d\lambda : E \in \mathcal{A}, \lambda(E) < \delta, n \in \mathbb{N} \right\} \leq \alpha.$$

As a result,

$$\eta_0 = \lim_{\delta\to 0_+} \left[\sup \left\{ \int_E \chi_{X\setminus A_n} \cdot |h_n| \, d\lambda : E \in \mathcal{A}, \lambda(E) < \delta, n \in \mathbb{N} \right\} \right] \leq \alpha.$$

This is a contradiction since $\eta_0 > \alpha$.

Notice that $E_p \setminus A_{n_p} = (E_p \cup A_{n_p}) \setminus A_{n_p}$, $p \in \mathbb{N}$ and since $\left| h_{n_p} \right| \in L_1(X, \mathcal{A}, \lambda)$, we obtain from (3.9)

$$\int_{E_p\cup A_{n_p}} \left| h_{n_p} \right| \, d\lambda - \int_{A_{n_p}} \left| h_{n_p} \right| \, d\lambda = \int_{E_p\setminus A_{n_p}} \left| h_{n_p} \right| \, d\lambda > \alpha.$$

Then for every $p \in \mathbb{N}$, we obtain from (3.3) that

$$\eta - \frac{2}{n_p} \leq \eta - \frac{1}{i_{n_p}} - \frac{1}{n_p + 1} < \int_{A_{n_p}} \left|h_{n_p}\right| \, d\lambda < \int_{E_p \cup A_{n_p}} \left|h_{n_p}\right| \, d\lambda - \alpha. \tag{3.10}$$

Since $(h_{n_p})_{p\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, it holds $\eta(\{h_{n_p} : p \in \mathbb{N}\}) \leq \eta < \eta + \frac{1}{2} \cdot \alpha$. Therefore by Definition 11 there exists $\delta > 0$ such that

$$\forall E \in \mathcal{A},\ \lambda(E) < \delta \ \forall p \in \mathbb{N} : \int_E \left|h_{n_p}\right| \, d\lambda < \eta + \frac{1}{2} \cdot \alpha. \tag{3.11}$$

By definition, $\lim_{p\to\infty} \lambda(A_{n_p}) = 0$, and for every $p \in \mathbb{N}$ we have $\lambda(E_p \cup A_{n_p}) < \frac{1}{p} + \lambda(A_{n_p})$. That implies $\lim_{p\to\infty} \lambda(E_p \cup A_{n_p}) = 0$. Therefore there exists $p_0 \in \mathbb{N}$ such that for every $p \geq p_0$ it holds $\lambda(E_p \cup A_{n_p}) < \delta$.

Therefore, for all $p \geq p_0$, we obtain from (3.10) and (3.11) that

$$\eta - \frac{2}{n_p} < \int_{A_{n_p}} \left|h_{n_p}\right| \, d\lambda < \eta + \frac{1}{2} \cdot \alpha - \alpha = \eta - \frac{1}{2} \cdot \alpha,$$

thus $\eta - \frac{2}{n_p} \leq \eta - \frac{1}{2} \cdot \alpha$ for every $p \geq p_0$. Then we have $\eta \leq \eta - \frac{1}{2} \cdot \alpha$, because $\lim_{p\to\infty} \frac{2}{n_p} = 0$. However, $\alpha > 0$ and $\eta \leq \eta - \frac{1}{2} \cdot \alpha$, which yields the desired contradiction.

Therefore $\eta_0 = 0$ and the family $\{\chi_{X\setminus A_n} \cdot h_n\}$ is bounded. By (3.1), the corresponding family of measures is uniformly absolutely continuous with respect to λ. Therefore from Theorem 1.65 or Theorem 1.84 in Florescu and Godet-Thobie [2012] it follows that the set $\{\chi_{X\setminus A_n} \cdot h_n\}$ is weakly relatively compact in $L_1(X, \mathcal{A}, \lambda)$. By Theorem 13, the set $\{\chi_{X\setminus A_n} \cdot h_n\}$ is relatively weakly sequentially compact. Therefore there exists a subsequence $\chi_{X\setminus A_{n_k}} \cdot h_{n_k}$ of $\chi_{X\setminus A_n} \cdot h_n$ weakly convergent in $L_1(X, \mathcal{A}, \lambda)$. Thus, if we set $g_k = h_{n_k}$ and $\widetilde{A}_k = A_{n_k}$, we obtain that $\widetilde{A}_k$ is a sequence of pairwise disjoint sets and that $(g_k)_{k\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$ such that $(\chi_{X\setminus\widetilde{A}_k \cdot g_k})$ is weakly convergent in $L_1(X, \mathcal{A}, \lambda)$. Furthermore, since $(\int_{\widetilde{A}_k} |g_k| \, d\lambda)$ is a subsequence of $(\int_{A_n} |h_n| \, d\lambda)$, (3.8) yields

$$\lim_{k\to\infty} \int_{\widetilde{A}_k} |g_k| \, d\lambda = \eta = \eta(\{f_n : n \in \mathbb{N}\}),$$

which completes the proof.

□

3.3 Main Results

Before we get to the Biting Lemma, we will need the following definition and theorem.

Definition 13. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Then f is said to be* upper semicontinuous at the point x_0 *if*

$$\forall y \in \mathbb{R}^*,\ y > f(x_0)\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow f(x) < y.$$

Let $M \subseteq \mathbb{R}$ be an open set. Then the function f is said to be upper semicontinuous on M *if it is upper semicontinuous at each point $x_0 \in M$.*

Note that f is upper semicontinuous at x_0 if and only if $\limsup_{x\to x_0} f(x) \le f(x_0)$. Indeed, if f is upper semicontinuous at x_0, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0 + \delta)$ it holds $f(x) < f(x_0) + \varepsilon$. On the other hand, if $\limsup_{x\to x_0} f(x) \le f(x_0)$, then the upper semicontinuity at x_0 follows from the definition.

For the following theorem, see Walter Rudin [1976, page 167].

Theorem 17 (Helly's Selection Theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonincreasing functions from $\mathbb{R}$ to $\mathbb{R}$ and let us assume that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded, i.e., there exist $a, b \in \mathbb{R}$ such that $a \le f_n \le b$ for every $n \in \mathbb{N}$. Then there exists a function f from $\mathbb{R}$ to $\mathbb{R}$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that*

$$\forall x \in \mathbb{R} : \lim_{k\to\infty} f_{n_k}(x) = f(x).$$

Now we are ready to prove the Biting Lemma, which is really important whenever we need to extract a (w^2-)convergent subsequence from a bounded sequence in L_1. In particular it is used in the theory of partial differential equations. This proof follows the one in John M. Ball and François Murat [1989]. The "besides" part is proved in Florescu and Godet-Thobie [2012, Theorem 1.103].

Lemma 18 (Biting Lemma)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite positive measure μ and let $f_n\colon X \to \mathbb{R}$, $n \in \mathbb{N}$ be a bounded sequence in $L_1(X, \mathcal{A}, \mu)$, i.e.*

$$\sup_{n\in\mathbb{N}} \| f_n \|_{L_1} = \sup_{n\in\mathbb{N}} \int_X |f_n| \; d\mu = C_0 < \infty. \tag{3.12}$$

Then there exists a function $f\colon X \to \mathbb{R}$, $f \in L_1(X, \mathcal{A}, \mu)$, a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a nonincreasing sequence of sets $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ with $\lim_{k\to\infty} \mu(E_k) = 0$, such that

$$f_{n_j} \to f \qquad \textit{weakly in } L_1(X \setminus E_k, \mu)$$

as $j \to \infty$ for every fixed $k \in \mathbb{N}$.

Besides, for every subsequence $(f_{n_{j_p}})_{p\in\mathbb{N}}$ of $(f_{n_j})_{j\in\mathbb{N}}$ it holds

$$\eta(\{f_{n_{j_p}} : p \in \mathbb{N}\}) = \eta(\{f_n : n \in \mathbb{N}\}).$$

Proof. Let $f_n\colon X \to \mathbb{R}$, $n \in \mathbb{N}$ be a bounded sequence in $L_1(X, \mathcal{A}, \mu)$, i.e.

$$\sup_{n\in\mathbb{N}} \int_X |f_n| \; d\mu = C_0 < \infty.$$

.

Let us denote $\eta = \eta(\{f_n : n \in \mathbb{N}\})$. Notice that by the definition of η it follows that $\eta \in [0,\infty)$.

For every $l \ge 0$, let us define

$$\varphi_n(l) = \int_{\{|f_n|\ge l\}} |f_n| \; d\mu.$$

Then

(i) $\varphi_n(0) = \| f_n \|_{L_1} \le C_0$ by (3.12),

(ii) for each $n \in \mathbb{N}$, the function φ_n is nonincreasing (because if $m \geq l$, then $\varphi_n(l) = \int_{\{|f_n| \geq l\}} |f_n| \; d\mu \geq \int_{\{|f_n| \geq m\}} |f_n| \; d\mu = \varphi_n(m)$),

(iii) for each $n \in \mathbb{N}$, the function φ_n is upper semicontinuous on $(0,\infty)$ (right upper semicontinuous at 0). Indeed, fix $n \in \mathbb{N}$ and let $l_0 \in [0,\infty)$ be given. Let $(l_m)_{m\in\mathbb{N}} \subseteq [0,\infty)$ satisfy $\lim_{m\to\infty} l_m = l_0$. We want to show that $\limsup_{m\to\infty} \varphi_n(l_m) \leq \varphi_n(l_0)$. If $l_0 = 0$, we are done since φ_n is nonincreasing. Now let $l_0 > 0$. By way of contradiction, let

$$\limsup_{m\to\infty} \varphi_n(l_m) > \alpha > \varphi_n(l_0).$$

By definition, we have

$$\varphi_n(l_0) = \int_{\{|f_n| \geq l_0\}} |f_n| \; d\mu \quad \text{and} \quad \varphi_n(l_m) = \int_{\{|f_n| \geq l_m\}} |f_n| \; d\mu, \; m \in \mathbb{N}.$$

Let $(m_p)_{p\in\mathbb{N}} \subseteq \mathbb{N}$ be a strictly increasing sequence of indices such that $\varphi_n(l_{m_p}) > \alpha$ for every $p \in \mathbb{N}$. Then for every $p \in \mathbb{N}$ we have $l_{m_p} < l_0$ because $\varphi_n(l_{m_p}) > \alpha > \varphi_n(l_0)$ and φ_n is nonincreasing. Then $\lim_{p\to\infty} l_{m_p} = l_0$ and

$$\alpha < \varphi_n(l_{m_p}) = \int_{\{|f_n| \geq l_{m_p}\}} |f_n| \; d\mu \quad \text{and} \quad \alpha > \varphi_n(l_0) = \int_{\{|f_n| \geq l_0\}} |f_n| \; d\mu.$$

Furthermore, for every $p \in \mathbb{N}$, we have

$$\int_X |f_n| \, \chi_{\{|f_n| \geq l_{m_p}\}} \, d\mu \leq \int_X |f_n| \; d\mu = C_0,$$

thus Lebesgue's Dominated Convergence Theorem gives us

$$\begin{aligned} \lim_{p\to\infty} \varphi_n(l_{m_p}) &= \lim_{p\to\infty} \int_{\{|f_n| \geq l_{m_p}\}} |f_n| \; d\mu = \\ &= \lim_{p\to\infty} \int_X |f_n| \, \chi_{\{|f_n| \geq l_{m_p}\}} \, d\mu = \int_X |f_n| \, \chi_{\{|f_n| \geq l_0\}} \, d\mu = \varphi_n(l_0). \end{aligned}$$

However, that is not possible since

$$\alpha \leq \lim_{p\to\infty} \varphi_n(l_{m_p}) = \varphi_n(l_0) < \alpha.$$

(iv) Finally, for each fixed $n \in \mathbb{N}$, $\lim_{l\to\infty} \varphi_n(l) = 0$. That is because for any given $n \in \mathbb{N}$, the set function $\nu(A) = \int_A |f_n| \; d\mu$, $A \in \mathcal{A}$, is a measure on $\mathcal{A}$ and since $f_n \in L_1(X, \mathcal{A}, \mu)$, ν is finite. Now since φ_n is nonincreasing, $\lim_{l\to\infty} \varphi_n(l)$ exists and it holds

$$\lim_{l\to\infty} \varphi_n(l) = \lim_{k\to\infty} \varphi_n(k) = \lim_{k\to\infty} \nu(\{|f_n| \geq k\}) = \nu\left(\bigcap_{k=1}^{\infty} \{|f_n| \geq k\}\right) = 0.$$

Now we will find a subsequence $(f_{n_q})_{q\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that for every subsequence $(f_{n_{q_s}})_{s\in\mathbb{N}}$ of $(f_{n_q})_{q\in\mathbb{N}}$ it holds $\eta(\{f_{n_{q_s}} : s \in \mathbb{N}\}) = \eta$.

For each $n \in \mathbb{N}$, let us define the functions $F_n \colon [0,\infty) \to [0,\infty)$ by $F_n(t) = \sup_{i\geq n} \varphi_i(t)$. Then, for every $n \in \mathbb{N}$, F_n is nonincreasing by (ii). Furthermore, by Proposition 14, $\lim_{t\to\infty} F_n(t) = \eta(\{f_i : i \geq n\})$ for every fixed $n \in \mathbb{N}$. By Proposition 14 and by (iv), $\eta(\{f_i : i \leq n\}) = \lim_{t\to\infty} \sup_{i\leq n} \varphi_i(t) = 0$. Then for all $n \in \mathbb{N}$

and for all $t \in [0,\infty)$ we have $\sup_{i\in\mathbb{N}} \varphi_i(t) \geq \sup_{i\geq n} \varphi_i(t)$ and conversely, since the functions φ_i are nonnegative, $\sup_{i\in\mathbb{N}} \varphi_i(t) \leq \sup_{i\leq n} \varphi_i(t) + \sup_{i\geq n} \varphi_i(t)$ for every $n \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$ we have $\lim_{t\to\infty} F_n(t) \leq \lim_{t\to\infty}[\sup_{i\in\mathbb{N}} \varphi_i(t)] = \eta$, and on the other hand, $\lim_{t\to\infty} F_n(t) \geq \lim_{t\to\infty}(\sup_{i\in\mathbb{N}} \varphi_i(t) - \sup_{i\leq n} \varphi_i(t)) = \eta - 0 = \eta$. Thus

$$\forall n \in \mathbb{N}: \quad \lim_{t\to\infty} F_n(t) = \eta. \tag{3.13}$$

Then, if we fix $n = 1$, we obtain from (3.13) that there exists $(t_q) \subseteq [0,\infty)$ a strictly increasing sequence such that $\lim_{q\to\infty} t_q = \infty$ and

$$\forall q \in \mathbb{N}: \quad F_1(t_q) \in \left(\eta - \frac{1}{q}, \eta + \frac{1}{q}\right) \tag{3.14}$$

From the definition of the functions F_n it follows that for every fixed $t \in [0,\infty)$ we have $F_n(t) \leq F_m(t)$ whenever $n \geq m$. Then, for every $q \in \mathbb{N}$, we obtain from (3.13) and from (3.14)

$$\begin{aligned} \eta - \frac{1}{q} < \eta = \lim_{t\to\infty} F_q(t) &= \inf_{t\in[0,\infty)} F_q(t) \leq \\ &\leq F_q(t_q) = \sup_{i\geq q} \varphi_i(t_q) = \sup_{i\geq q} \int_{\{|f_i|\geq t_q\}} |f_i| \, d\mu = \\ &= F_q(t_q) \leq F_1(t_q) < \eta + \frac{1}{q}. \end{aligned} \tag{3.15}$$

In particular, from (3.15) it follows that

$$\forall q \in \mathbb{N}: \eta - \frac{1}{q} < \sup_{i\geq q} \int_{\{|f_i|\geq t_q\}} |f_i| \, d\mu,$$

and thus for each $q \in \mathbb{N}$ there exists $n_q \geq q$ such that

$$\eta - \frac{1}{q} < \int_{\{|f_{n_q}|\geq t_q\}} \left|f_{n_q}\right| \, d\mu. \tag{3.16}$$

Without loss of generality we can assume that $(f_{n_q})_{q\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, i.e. n_q is strictly increasing (otherwise we find a strictly increasing subsequence $(n_{q_s})_{s\in\mathbb{N}}$, which is possible since $n_q \geq q$ for each $q \in \mathbb{N}$. Then the rest of the proof will be the same, there will only be an extra subscript).

Now let $(f_{n_{q_s}})_{s\in\mathbb{N}}$ be an arbitrary subsequence of $(f_{n_q})_{q\in\mathbb{N}}$. Then by Proposition 14 we have

$$\begin{aligned} \eta(\{f_{n_{q_s}} : s \in \mathbb{N}\}) &= \lim_{t\to\infty} \left[\sup_{s\in\mathbb{N}} \int_{\{|f_{n_{q_s}}|\geq t\}} \left|f_{n_{q_s}}\right| \, d\mu\right] \leq \\ &\leq \lim_{t\to\infty} \left[\sup_{n\in\mathbb{N}} \int_{\{|f_n|\geq t\}} |f_n| \, d\mu\right] = \eta(\{f_n : n \in \mathbb{N}\}). \end{aligned}$$

On the other hand, from Proposition 14 and from (3.16) we have

$$\begin{aligned} \eta &= \lim_{s\to\infty} (\eta - \frac{1}{q_s}) \leq \lim_{s\to\infty} \int_{\{|f_{n_{q_s}}|\geq t_{q_s}\}} \left|f_{n_{q_s}}\right| \, d\mu \leq \\ &\leq \lim_{s\to\infty} \sup_{r\in\mathbb{N}} \int_{\{|f_{n_{q_r}}|\geq t_{q_s}\}} \left|f_{n_{q_r}}\right| \, d\mu \leq \\ &\leq \lim_{t\to\infty} \sup_{r\in\mathbb{N}} \int_{\{|f_{n_{q_r}}|\geq t\}} \left|f_{n_{q_r}}\right| \, d\mu = \eta(\{f_{n_{q_s}} : s \in \mathbb{N}\}), \end{aligned}$$

hence $\eta(\{f_{n_{q_s}} : s \in \mathbb{N}\}) = \eta$.

Therefore it now suffices to find a w^2-convergent subsequence of $(f_{n_q})_{q\in\mathbb{N}}$ (and thus a w^2-convergent subsequence of $(f_n)_{n\in\mathbb{N}}$). That is because by Definition 12 such subsequence satisfies the desired conclusion of the lemma and by the above it also satisfies the "besides" part.

Therefore, we are finding a w^2-convergent subsequence of $(f_{n_q})_{q\in\mathbb{N}}$. By (i) and by (ii), $(\varphi_{n_q})_{q\in\mathbb{N}}$ is a uniformly bounded sequence of nonincreasing functions, thus by Theorem 17 there exists $(\varphi_{n_{q_j}})_{j\in\mathbb{N}}$ a subsequence of $(\varphi_{n_q})_{q\in\mathbb{N}}$ and a function $\alpha\colon [0,\infty) \to \mathbb{R}$ such that

$$\forall l \in [0,\infty) : \alpha(l) = \lim_{j\to\infty} \varphi_{n_{q_j}}(l). \tag{3.17}$$

Since for every $l \geq 0$ we have $\varphi_{n_{q_j}}(l) \geq \varphi_{n_{q_j}}(t)$ for each $t \geq l$ and for all $j \in \mathbb{N}$, the function α is nonincreasing. Hence we can set $\lim_{l\to\infty} \alpha(l) = L \in [0,\infty)$.

Case 1. $L = 0$. We want to show that the set $\{f_{n_{q_j}} : j \in \mathbb{N}\}$ is sequentially weakly relatively compact in $L_1(X, \mathcal{A}, \mu)$. Let $\varepsilon > 0$ be given. Since $L = 0$, we can find $l_0 > 0$ such that for every $l \geq l_0$ we have $\alpha(l) < \varepsilon$. By (3.17) there exists j_0 such that for every $j \geq j_0$ it holds $\varphi_{n_{q_j}}(l_0) < \varepsilon$. Then we can find $l_1 > l_0$ such that $\varphi_{n_{q_j}}(l_1) < \varepsilon$ for all $j \leq j_0$. Thus, by (ii), $\varphi_{n_{q_j}}(l_1) < \varepsilon$ for all $j \in \mathbb{N}$. Thus for every $l > l_1$ and for every $j \in \mathbb{N}$ we have

$$0 \leq \int_{\left\{\left|f_{n_{q_j}}\right|>l\right\}} \left|f_{n_{q_j}}\right| \, d\mu \leq \int_{\left\{\left|f_{n_{q_j}}\right|>l_1\right\}} \left|f_{n_{q_j}}\right| \, d\mu = \varphi_{n_{q_j}}(l_1) < \varepsilon,$$

which implies

$$\lim_{l\to\infty} \sup_{j\in\mathbb{N}} \int_{\left\{\left|f_{n_{q_j}}\right|>l\right\}} \left|f_{n_{q_j}}\right| \, d\mu = 0.$$

Thus the set $\{f_{n_{q_j}} : j \in \mathbb{N}\}$ is uniformly integrable. Then Theorem 12 yields that the set $\{f_{n_{q_j}} : j \in \mathbb{N}\}$ is weakly relatively compact. From Theorem 13 we obtain that the set $\{f_{n_{q_j}} : j \in \mathbb{N}\}$ is sequentially weakly relatively compact in $L_1(X, \mathcal{A}, \mu)$, which is what we wanted.

Therefore, by Definition 10 there exists a subsequence $(f_{n_{q_{j_i}}})_{i\in\mathbb{N}}$ of $(f_{n_{q_j}})_{j\in\mathbb{N}}$ which converges weakly in $L_1(X, \mathcal{A}, \mu)$ to some $f \in L_1(X, \mathcal{A}, \mu)$, hence the conclusion of the lemma holds with all the sets E_k empty.

Case 2. $L > 0$.

Step 1. We claim that there exists a sequence $(l_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\lim_{j\to\infty} l_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \varphi_{n_{q_j}}(l_j) = L.$$

To prove this, let us define $l_j = \sup\{l > 0 : \varphi_{n_{q_j}}(l) \geq L - 1/l\}$. Then $l_j \in \mathbb{R}$ for every $j \in \mathbb{N}$, because for all $j \in \mathbb{N}$, the set $\{l > 0 : \varphi_{n_{q_j}}(l) \geq L - 1/l\}$ is nonempty (because by (ii) and (iii), the functions $\varphi_{n_{q_j}}$ are all nonnegative whereas $\lim_{l\to 0_+} L - 1/l = -\infty$) and bounded from above (by (iv) and because $L > 0$). Moreover, the supremum is attained, because if not, we have $\varphi_{n_{q_j}}(l_j) < L - 1/l_j$ and by (iii) and by Definition 13 there exists $\delta > 0$ such that $\varphi_{n_{q_j}}(l) < L - 1/l_j$ for all $l \in (l_j - \delta, l_j + \delta)$, which contradicts the definition of l_j. Notice that $(l_j)_{j\in\mathbb{N}}$ cannot contain a bounded subsequence $(l_{j_\gamma})_{\gamma\in\mathbb{N}}$, because if $(l_{j_\gamma})_{\gamma\in\mathbb{N}}$ is a bounded subsequence of $(l_j)_{j\in\mathbb{N}}$, then any $l' > \sup_{\gamma\in\mathbb{N}} l_{j_\gamma}$ satisfies $\varphi_{n_{q_{j_\gamma}}}(l') < L - 1/l'$ for every $\gamma \in \mathbb{N}$. Then

$$\alpha(l') = \lim_{j\to\infty} \varphi_{n_{q_j}}(l') = \lim_{\gamma\to\infty} \varphi_{n_{q_{j_\gamma}}}(l') \leq L - 1/l',$$

which is not possible since the function α is nonincreasing and L is its limit as $l \to \infty$. Hence $\lim_{j\to\infty} l_j = \infty$. Therefore, given any $m \geq 0$, we can find $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ it holds $l_j \geq m$. Then for every $j \geq j_0$ we have $L - 1/l_j \leq \varphi_{n_{q_j}}(l_j) \leq \varphi_{n_{q_j}}(m)$, where the first inequality follows from the fact that the supremum l_j is attained and the second inequality follows from (ii). Hence

$$\begin{aligned} L = \liminf_{j\to\infty}(L - 1/l_j) &\leq \liminf_{j\to\infty} \varphi_{n_{q_j}}(l_j) \leq \\ &\leq \limsup_{j\to\infty} \varphi_{n_{q_j}}(l_j) \leq \limsup_{j\to\infty} \varphi_{n_{q_j}}(m) = \alpha(m), \end{aligned}$$

therefore by letting $m \to \infty$ we get $\lim_{j\to\infty} \varphi_{n_{q_j}}(l_j) = L$, which proves our claim.

Step 2. We claim that

$$\lim_{m\to\infty} \sup_{j\in\mathbb{N}} \int_{\left\{m\leq\left|f_{n_{q_j}}\right|<l_j\right\}} \left|f_{n_{q_j}}\right| \, d\mu = 0.$$

To prove this, let us define

$$S(m) = \sup_{j\in\mathbb{N}} \int_{\left\{m\leq\left|f_{n_{q_j}}\right|<l_j\right\}} \left|f_{n_{q_j}}\right| \, d\mu.$$

Notice that S is nonicreasing and also, by definition of $\varphi_{n_{q_j}}$, we have that

$$S(m) = \sup_{j\in\mathbb{N},\ l_j>m} \Big(\varphi_{n_{q_j}}(m) - \varphi_{n_{q_j}}(l_j)\Big).$$

Let $\varepsilon > 0$ be given. Then there exists $m_1 \in (0,\infty)$ such that for every $m \geq m_1$ it holds $\alpha(m) \in (L-\varepsilon, L+\varepsilon)$. Then, by (3.17) and by *Step 1*, there exists j_0 such that for every $j \geq j_0$ it holds both $\varphi_{n_{q_j}}(m_1) \in (\alpha(m_1)-\varepsilon, \alpha(m_1)+\varepsilon)$ and $\varphi_{n_{q_j}}(l_j) \geq L - \varepsilon$. Therefore for every $j \geq j_0$ we have

$$\varphi_{n_{q_j}}(m_1) - \varphi_{n_{q_j}}(l_j) \leq \alpha(m_1) + \varepsilon - L + \varepsilon \leq 3\varepsilon.$$

Notice that for every $m \geq m_1$ and for every $j \in \mathbb{N}$ we have $\varphi_{n_{q_j}}(m) \leq \varphi_{n_{q_j}}(m_1)$. That is because the functions $\varphi_{n_{q_j}}$ are nonincreasing by (ii). Therefore, if we set $M = \max\{m_1, l_1, \ldots, l_{j_0}\}$, then for any given $m \geq M$ and for any $j \geq j_0$ it holds

$$\varphi_{n_{q_j}}(m) - \varphi_{n_{q_j}}(l_j) \leq \varphi_{n_{q_j}}(m_1) - \varphi_{n_{q_j}}(l_j) \leq 3\varepsilon. \tag{3.18}$$

Let us notice that if $m \geq M$ and $l_j > m$, then $l_j > m \geq l_1, l_2, \ldots, l_{j_0}$, thus $j > j_0$. Hence for every $m \geq M$, inequality (3.18) yields

$$0 \leq S(m) = \sup_{j\in\mathbb{N},\ l_j>m} (\varphi_{n_{q_j}}(m) - \varphi_{n_{q_j}}(l_j)) \leq 3\varepsilon,$$

which proves the claim.

Step 3. Now we assert that for every $k \in \mathbb{N}$ there exist sets $E_k \in \mathcal{A}$ and $N_k, M_k \subseteq \mathbb{N}$ (for the rest of the proof we denote $E_0 = X$ and $M_0 = N_0 = \mathbb{N}$) satisfying

(a) the sets N_k and M_k are infinite,

(b) $M_k \supseteq N_k \supseteq M_{k+1} \supseteq N_{k+1}$ for every $k \in \mathbb{N} \cup \{0\}$,

(c) $\min N_k < \min N_{k+1}$ for every $k \in \mathbb{N} \cup \{0\}$,

(d) $\mu(E_k) < \frac{1}{k}$ for every $k \in \mathbb{N}$,

(e) $E_k = \bigcup_{j \in M_k} \left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\}$ for every $k \in \mathbb{N}$ and

(f) for every $k \in \mathbb{N}$, there exists $g_k \in L_1(X \setminus E_k, \mu)$ such that the sequence $(f_{n_{q_j}})_{j \in N_k}$ converges weakly to g_k in $L_1(X \setminus E_k, \mu)$.

The objects will be constructed inductively.

To start the induction, let us find an infinite set $M_1 \subseteq \mathbb{N}$ satisfying $\min M_1 > 1$ and $\left(\sum_{j \in M_1} \frac{1}{l_j}\right) \cdot C_0 < 1$ (that can be done since $\lim_{j \to \infty} l_j = \infty$, hence it suffices to choose M_1 such that $l_j > 2^j \cdot C_0$ for all $j \in M_1$). Let $E_1 = \bigcup_{j \in M_1} \left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\}$. Then Chebyshev's inequality implies

$$\mu(E_1) = \mu\left(\bigcup_{j \in M_1} \left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\} \right) \leq \sum_{j \in M_1} \mu\left(\left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\} \right) \leq$$
$$\leq \sum_{j \in M_1} \frac{1}{l_j} \int_{\left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\}} \left| f_{n_{q_j}} \right| \, d\mu \leq \sum_{j \in M_1} \left(\frac{1}{l_j} \cdot C_0 \right) < 1.$$

By *Step 2* we also have

$$\lim_{m \to \infty} \sup_{j \in M_1} \int_{\left\{ \left| f_{n_{q_j}} \right| \geq m \right\} \setminus E_1} \left| f_{n_{q_j}} \right| \, d\mu \leq \lim_{m \to \infty} \sup_{j \in M_1} \int_{\left\{ m \leq \left| f_{n_{q_j}} \right| < l_j \right\}} \left| f_{n_{q_j}} \right| \, d\mu = 0.$$

Therefore by Definition 9 the set $\{f_{n_{q_j}} : j \in M_1\}$ is uniformly integrable, and thus weakly relatively compact in $L_1(X \setminus E_1, \mu)$ by Theorem 12. Then Theorem 13 implies that the set $\{f_{n_{q_j}} : j \in M_1\}$ is sequentially weakly relatively compact, hence there exists $N_1 \subseteq M_1$ infinite such that the sequence $(f_{n_{q_j}})_{j \in N_1}$ converges weakly to a function $g_1 \in L_1(X \setminus E_1, \mu)$ in $L_1(X \setminus E_1, \mu)$. Then (a)-(f) are satisfied.

Now let us assume that $k \geq 2$ and that we have the sets $E_1, \ldots, E_k$, $N_1, \ldots, N_k$ and $M_1, \ldots, M_k$ satisfying (a)-(f). Let us find an infinite set $M_{k+1} \subseteq N_k$ satisfying $\min M_{k+1} > \min N_k$ as well as $\left(\sum_{j \in M_{k+1}} \frac{1}{l_j}\right) \cdot C_0 < \frac{1}{k+1}$. Let us define $E_{k+1} = \bigcup_{j \in M_{k+1}} \left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\}$. Then Chebyshev's inequality implies

$$\mu(E_{k+1}) = \mu\left(\bigcup_{j \in M_{k+1}} \left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\} \right) \leq \sum_{j \in M_{k+1}} \mu\left(\left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\} \right) \leq$$
$$\leq \sum_{j \in M_{k+1}} \frac{1}{l_j} \int_{\left\{ \left| f_{n_{q_j}} \right| \geq l_j \right\}} \left| f_{n_{q_j}} \right| \, d\mu \leq \sum_{j \in M_{k+1}} \left(\frac{1}{l_j} \cdot C_0 \right) < \frac{1}{k+1}.$$

By *Step 2* we also have

$$\lim_{m \to \infty} \sup_{j \in M_{k+1}} \int_{\left\{ \left| f_{n_{q_j}} \right| \geq m \right\} \setminus E_{k+1}} \left| f_{n_{q_j}} \right| \, d\mu \leq \lim_{m \to \infty} \sup_{j \in M_{k+1}} \int_{\left\{ m \leq \left| f_{n_{q_j}} \right| < l_j \right\}} \left| f_{n_{q_j}} \right| \, d\mu = 0.$$

Therefore by Definition 9 the set $\{f_{n_{q_j}} : j \in M_{k+1}\}$ is uniformly integrable, and thus weakly relatively compact in $L_1(X \setminus E_{k+1}, \mu)$ by Theorem 12. Then Theorem 13 implies that the set $\{f_{n_{q_j}} : j \in M_{k+1}\}$ is sequentially weakly relatively compact, hence there exists $N_{k+1} \subseteq M_{k+1}$ infinite such that the sequence $(f_{n_{q_j}})_{j\in N_{k+1}}$ converges weakly to a function $g_{k+1} \in L_1(X \setminus E_{k+1}, \mu)$ in $L_1(X \setminus E_{k+1}, \mu)$. Then (a)-(f) are satisfied and the inductive construction is complete.

Step 4. We want to show that everything works.

First of all, the sequence $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ is nonincreasing by (b) and (e) and also $\lim_{k\to\infty} \mu(E_k) = 0$ by (d).

For every $k \in \mathbb{N}$, let us set

$$\widetilde{g}_k(x) = \begin{cases} g_k(x), & x \in X \setminus E_k, \\ 0, & x \in E_k, \end{cases}$$

where the functions g_k come from (f). Then the functions $\widetilde{g}_k$ are measurable, because the sets E_k are measurable and $g_k \in L_1(X \setminus E_k, \mu)$ for every $k \in \mathbb{N}$. Let us set $f(x) = \lim_{k\to\infty} \widetilde{g}_k(x)$ for those $x \in X$ for which the limit exists. Notice that the limit exists for μ-almost every $x \in X$, because it holds

$$\forall n \in \mathbb{N}\ \forall k \in \mathbb{N},\ k \geq n : \widetilde{g}_k(x) = g_n(x) \quad \text{for almost every } x \in X \setminus E_n. \tag{3.19}$$

That follows from (e), (f) and uniqueness of the weak limit. Therefore the limit exists for μ-almost every $x \in \bigcup_{n=1}^{\infty}(X \setminus E_n) = X \setminus \bigcap_{n=1}^{\infty} E_n$. Then from the assumption that the measure μ is finite and from (d) we obtain that the limit exists for μ-almost every $x \in X$. Therefore f is measurable as a pointwise (for μ-almost every $x \in X$) limit of measurable functions.

Let us denote $p_i = \min N_i$. We want to show that

$$(f_{n_{q_{p_i}}}) \to f \quad \text{weakly in } L_1(X \setminus E_k, \mu)$$

as $i \to \infty$ for every fixed $k \in \mathbb{N}$. However, this follows immediately from (b), (c), (f) and (3.19).

Finally we need to show that $f \in L_1(X, \mathcal{A}, \mu)$. Let $k \in \mathbb{N}$. Let us consider a function $h(x) = 2\chi_{\{f\geq 0\}}(x) - 1$, $x \in X \setminus E_k$. Then $h \in L_\infty(X \setminus E_k, \mu)$, and thus h represents a functional $\phi \in L_1^*(X \setminus E_k, \mu)$. By definition of the weak convergence, $\lim_{i\to\infty} \phi(f_{n_{q_{p_i}}}) = \phi(f)$. That implies

$$\lim_{i\to\infty} \int_{X\setminus E_k} (f_{n_{q_{p_i}}} h)\, d\mu = \lim_{i\to\infty} \phi(f_{n_{q_{p_i}}}) = \phi(f) = \int_{X\setminus E_k} (fh)\, d\mu = \int_{X\setminus E_k} |f|\, d\mu.$$

Since $h(X) \subseteq [-1,1]$, we obtain

$$\int_{X\setminus E_k} \left|f_{n_{q_{p_i}}}\right|\, d\mu \geq \int_{X\setminus E_k} (f_{n_{q_{p_i}}} h)\, d\mu$$

for every $i \in \mathbb{N}$, which implies

$$\int_{X\setminus E_k} |f|\, d\mu \leq \liminf_{i\to\infty} \int_{X\setminus E_k} \left|f_{n_{q_{p_i}}}\right|\, d\mu \leq C_0.$$

Then Levi's theorem yields

$$\int_X |f|\, d\mu = \int_X \lim_{k\to\infty} |f|\, \chi_{X\setminus E_k}\, d\mu = \lim_{k\to\infty} \int_X |f|\, \chi_{X\setminus E_k}\, d\mu =$$
$$= \lim_{k\to\infty} \int_{X\setminus E_k} |f|\, d\mu \le \lim_{k\to\infty} C_0 = C_0,$$

which means $f \in L_1(X, \mathcal{A}, \mu)$. This completes the proof.

□

As promised above, for any bounded sequence in L_1 we would like to find a subsequence, which we can split into a weakly convergent part and an "unpleasant" part. Or, more precisely, if we use the modulus of uniform integrability, we would like to find a subsequence, which we can split into 2 parts: one with the modulus of uniform integrability equal to 0 and the other with the same modulus of uniform integrability as the original sequence. However, that is a corollary of Theorem 16 and Lemma 18. The proof of this corollary follows the one in Florescu and Godet-Thobie [2012, Corollary 1.106].

Corollary 19 (Rosenthal's subsequence splitting lemma)**.** *Let $(X, \mathcal{A}, \lambda)$ be a measurable space with a finite positive measure λ. Let $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X, \mathcal{A}, \lambda)$ be a bounded sequence (in $L_1(X, \mathcal{A}, \lambda)$). Then there exist a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that*

(i) $\eta(\{f_n : n \in \mathbb{N}\}) = \lim_{n\to\infty} \int_{A_n} |h_n|\, d\lambda$ and

(ii) $(\chi_{X\setminus A_n} \cdot h_n)_{n\in\mathbb{N}}$ is weakly convergent in $L_1(X, \mathcal{A}, \lambda)$.

Proof. According to Theorem 18 there exists a w^2-convergent subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that for every subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ it holds $\eta(\{h_n : n \in \mathbb{N}\}) = \eta(\{f_n : n \in \mathbb{N}\})$.

According to Theorem 16 there exists a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ and a sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\lim_{n\to\infty} \int_{A_n} |h_n|\, d\lambda = \eta(\{g_n : n \in \mathbb{N}\}) = \eta(\{f_n : n \in \mathbb{N}\})$ and $(\chi_{X\setminus A_n} \cdot h_n)_{n\in\mathbb{N}}$ is weakly convergent in $L_1(X, \mathcal{A}, \lambda)$.

□

Now we would like to finish what we have started in the introduction, where we were trying to extract a convergent subsequence from a bounded sequence of finite absolutely continuous measures. Let us consider a bounded sequence of finite measures $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ such that for each $n \in \mathbb{N}$ it holds $\mu_n \ll \lambda$ for a finite positive measure λ. By the Radon-Nikodym theorem, we can find for every $n \in \mathbb{N}$ a function $f_n \in L_1(X, \mathcal{A}, \lambda)$ such that for all $A \in \mathcal{A}$ it holds $\mu_n(A) = \int_A f_n\, d\lambda$. Then, because the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(ca(\mathcal{A}), \|\cdot\|)$, we get that the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L_1(X, \mathcal{A}, \lambda)$. That is because the subspace $ca_\lambda(\mathcal{A}) = \{\mu \in ca(\mathcal{A}) : \mu \ll \lambda\}$ of $(ca(\mathcal{A}), \|\cdot\|)$ is linearly isometric to $L_1(X, \mathcal{A}, \lambda)$, where the linear isometry is $\mu \mapsto f$, where $f = \frac{d\mu}{d\lambda}$ is the Radon-Nikodym derivative (the fact that it is a bijection follows from the Radon-Nikodym theorem). To see that it is an isometry, let $\mu \in ca_\lambda(\mathcal{A})$ be given and let $f = \frac{d\mu}{d\lambda}$. Then we have

$$\|\, \mu \,\| = |\mu|\,(X) = \int_X |f|\, d\lambda = \|\, f \,\|,$$

which is what we wanted. We are thus in a situation when we have a bounded sequence $(f_n)_{n\in\mathbb{N}} \subseteq L_1(X, \mathcal{A}, \lambda)$. Then if $(f_n)_{n\in\mathbb{N}}$ converges weakly to a function $f \in L_1(X, \mathcal{A}, \lambda)$, we can see that in particular

$$\forall A \in \mathcal{A}: \lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \int_X \chi_A \cdot f_n \, d\lambda = \int_X \chi_A \cdot f \, d\lambda = \int_A f \, d\lambda.$$

Or in other words, the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges setwise to an absolutely continuous measure μ with respect to λ where $\mu(A) = \int_A f \, d\lambda$ for all $A \in \mathcal{A}$. Therefore, finding a convergent subsequence of $(\mu_n)_{n\in\mathbb{N}}$ is easy, because we just take the original sequence. Note that this satisfies the last assumption of Theorem 10 (Vitali-Hahn-Saks theorem), and thus the measures μ_n are automatically uniformly absolutely continuous with respect to λ.

For the sake of completeness, let us note that the converse is also true: i.e. if the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to $\mu \in ca_\lambda(\mathcal{A})$ setwise, then if we denote $f = \frac{d\mu}{d\lambda}$, we obtain that the functions f_n converge to f weakly in $L_1(X, \mathcal{A}, \lambda)$. That is because

$$\forall A \in \mathcal{A}: \lim_{n\to\infty} \int_A f_n \, d\lambda = \lim_{n\to\infty} \mu_n(A) = \mu(A) = \int_A f \, d\lambda,$$

which is according to Florescu and Godet-Thobie [2012, Theorem 1.57] equivalent to a statement that the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L_1(X, \mathcal{A}, \lambda)$. Notice that this is very similar to what we have proved in Lemma 8, except there we could improve it to a strong convergence by using Theorem 7.

Similarly, if the sequence $(f_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence, then the sequence $(\mu_n)_{n\in\mathbb{N}}$ admits a setwise convergent subsequence. And conversely, if the sequence $(\mu_n)_{n\in\mathbb{N}}$ admits a setwise convergent subsequence, then the sequence $(f_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence.

Now the more interesting situation is when the sequence $(f_n)_{n\in\mathbb{N}}$ does not admit a weakly convergent subsequence in $L_1(X, \mathcal{A}, \lambda)$, which is by Theorem 13, Theorem 12 and by Remark 2 equivalent to $\eta = \eta(\{f_n : n \in \mathbb{N}\}) > 0$. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded and λ is a finite positive measure, we can apply the Biting Lemma (Theorem 18) and extract a w^2-convergent subsequence $(f_{n_j})_{j\in\mathbb{N}}$. Or in other words, we can find a nonincreasing sequence $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\lim_{k\to\infty} \lambda(E_k) = 0$, a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ and a function $f \in L_1(X, \mathcal{A}, \lambda)$ such that $f_{n_j} \to f$ weakly in $L_1(X \setminus E_k, \mathcal{A}, \lambda)$ for every fixed $k \in \mathbb{N}$. Therefore, for every fixed $k \in \mathbb{N}$, we obtain that

(i) by the above, the sequence $(\mu_{n_j})_{j\in\mathbb{N}}$ converges setwise (to the measure μ which has f as its Radon-Nikodym derivative) on $X \setminus E_k$ and

(ii) $\eta(\{f_{n_j} \cdot \chi_{X\setminus E_k}\}) = 0$ by Theorem 12.

Now, from (ii) we obtain the following:

$$
\begin{aligned}
\eta = \eta(\{f_n : n \in \mathbb{N}\}) &= \lim_{\delta \to 0_+} \left[\sup \left\{ \int_A \left| f_{n_j} \right| d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, j \in \mathbb{N} \right\} \right] = \\
&= \lim_{\delta \to 0_+} \left[\sup \left\{ \int_{A \cap E_k} \left| f_{n_j} \right| d\lambda + \int_{A \setminus E_k} \left| f_{n_j} \right| d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, j \in \mathbb{N} \right\} \right] \leq \\
&\leq \lim_{\delta \to 0_+} \left[\sup \left\{ \int_{A \cap E_k} \left| f_{n_j} \right| d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, j \in \mathbb{N} \right\} + \right. \\
&\left. + \sup \left\{ \int_{A \setminus E_k} \left| f_{n_j} \right| d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, j \in \mathbb{N} \right\} \right] = \\
&= \lim_{\delta \to 0_+} \left[\sup \left\{ \int_{A \cap E_k} \left| f_{n_j} \right| d\lambda : A \in \mathcal{A}, \lambda(A) < \delta, j \in \mathbb{N} \right\} \right] = \\
&= \eta(\{f_{n_j} \cdot \chi_{E_k} : n \in \mathbb{N}\}),
\end{aligned}
\tag{3.20}
$$

where the second equality follows from the "besides" part of the Biting Lemma (Theorem 18). By Definition 11 we have $\eta(\{f_n : n \in \mathbb{N}\}) \geq \eta(\{f_{n_j} \cdot \chi_{E_k} : n \in \mathbb{N}\})$, and thus from (3.20) it follows

$$
\eta = \eta(\{f_{n_j} \cdot \chi_{E_k} : n \in \mathbb{N}\}). \tag{3.21}
$$

Now by Definition 11 and by the correspondence between the measures μ_{n_j} and the functions f_{n_j}, we can rewrite (3.21) as follows:

$$
\eta = \lim_{\delta \to 0_+} \left[\sup \left\{ \left| \mu_{n_j} \right| (A) : A \in \mathcal{A}, A \subseteq E_k, \lambda(A) < \delta, j \in \mathbb{N} \right\} \right].
$$

Now we can see that in a sense, the measures μ_{n_j} *concentrate their masses* on the sets E_k.

This gets very interesting if X in our setup is a compact Hausdorff topological space. For instance, let us suppose that $X = [0,1]$, $\mathcal{A}$ is the σ-algebra of Borel sets (with respect to the standard Euclidean topology) and λ is the Lebesgue measure. Then $\mathcal{M}(X)$, the space of all regular Borel measures on X, is linearly isometric to the dual space of $\mathcal{C}(X)$ (the space of all continuous functions on X). Now let us consider the sequence $f_n = n\chi_{[0,\frac{1}{n}]}$. This is a bounded sequence in $L_1(X, \mathcal{A}, \lambda)$; for each $n \in \mathbb{N}$ we have $\| f_n \| = 1$. Now, for all $n \in \mathbb{N}$, let μ_n denote the measure such that $f_n = \frac{d\mu_n}{d\lambda}$. Then for every $n \in \mathbb{N}$ it holds $\mu_n \in \mathcal{M}(X)$: obviously the measures μ_n are all Borel and by Donald L. Cohn [2013, Proposition 7.2.3] they are also regular. Moreover, it also holds $\mu_n \ll \lambda$ for each $n \in \mathbb{N}$. Now let us observe that the sequence $(f_n)_{n \in \mathbb{N}}$ does not converge weakly in $L_1(X, \mathcal{A}, \lambda)$. Indeed, for each $C \in \mathbb{R}$ there exists $n_C \in \mathbb{N}$ satisfying $n_C > C$, and therefore

$$
\sup_{n \in \mathbb{N}} \int_{\{f_n > C\}} |f_n| \, d\lambda \geq \int_{\{f_{n_C} > C\}} |f_{n_C}| \, d\lambda = \int_0^1 n_C \chi_{[0,\frac{1}{n_C}]} \, d\lambda = 1,
$$

which implies $\lim_{C \to \infty} \sup_{n \in \mathbb{N}} \int_{\{f_n > C\}} |f_n| \, d\lambda \geq 1$. Furthermore, since for each $n \in \mathbb{N}$ we have $\| f_n \| = 1$, we have $\lim_{C \to \infty} \sup_{n \in \mathbb{N}} \int_{\{f_n > C\}} |f_n| \, d\lambda = 1$. This by Definition 9 means that the family $\{f_n : n \in \mathbb{N}\}$ is not uniformly integrable. Thus by Theorem 12 the sequence $(f_n)_{n \in \mathbb{N}}$ does not converge weakly in $L_1(X, \mathcal{A}, \mu)$. However, it is easy to see that the sequence $(f_n)_{n \in \mathbb{N}}$ is w^2-convergent to 0. Indeed, the sets $E_k = [0, \frac{1}{k}]$ localize the concentration of mass of (f_n). So again, we have that for each $k \in \mathbb{N}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly on $X \setminus E_k$, but this time we can say something more about what happens on the whole set X.

We already know that the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge weakly on the whole X, and thus by the above, the sequence $(\mu_n)_{n\in\mathbb{N}}$ does not converge setwise on X. However, since we already know that for each $n \in \mathbb{N}$ it holds $\mu_n \in \mathcal{M}(X)$, we could at least try for the weak* convergence. Let $f \in \mathcal{C}(X)$, then

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 f\, d\mu_n &= \lim_{n\to\infty} \int_0^1 f \cdot n\chi_{[0,\frac{1}{n}]}\, d\lambda = \\ &= \lim_{n\to\infty} n \cdot \int_0^{\frac{1}{n}} f\, d\lambda = f(0) = \int_0^1 f d\delta_0,\end{aligned}$$

where δ_0 denotes the Dirac measure at 0. Therefore the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to δ_0 weakly* in $\mathcal{M}(X)$ (note that $\delta_0 \in \mathcal{M}(X)$: obviously it is a Borel measure, by definition it is outer regular and since $\{0\}$ is a closed set, it is also inner regular). So, each set E_k splits our sequence (f_n) into two sequences. One of them is $(f_n \cdot \chi_{X\setminus E_k})_{n\in\mathbb{N}}$, which converges weakly in $L_1(X, \mathcal{A}, \lambda)$ and the corresponding sequence of measures (on $X \setminus E_k$) converges setwise to an absolutely continuous measure with respect to λ. The other sequence is $(f_n \cdot \chi_{E_k})_{n\in\mathbb{N}}$, which does not converge weakly and the corresponding sequence of measures (on E_k) does not converge setwise (the example of this is the set $\{0\}$). However, we have at least the weak* convergence of the corresponding measures to δ_0, which is not absolutely continuous with respect to λ.

Therefore, to summarize, we can see that the two parts of this thesis – one about measures, the other about functions – are indeed deeply connected by the Radon-Nikodym theorem.

Conclusion

In Chapter 2, we have proved two theorems concerning sequences of bounded measures and their limits. We have used a stronger version of Schur's theorem to prove that the setwise limit of a sequence of bounded measure is a measure and moreover, such a sequence is uniformly σ-additive (Nikodym theorem). Then we have shown that the setwise limit of a sequence of bounded measures that are absolutely continuous with respect to one universal measure λ is an absolutely continuous measure with respect to λ (Vitali-Hahn-Saks theorem).

In Chapter 3, we have shown that even in L_1, a non-reflexive space, it is possible to extract a (w^2-)convergent subsequence from any bounded sequence (Biting Lemma). Then we used this fact to prove that every bounded sequence in L_1 contains a subsequence, which we can split into two parts in such a way that the modulus of uniform integrability of one part is equal to 0 and the modulus of uniform integrability of the other part is the same as the modulus of uniform integrability of the original sequence (Rosenthal's subsequence splitting lemma).

Finally, at the end of the third chapter, we have discussed how the Radon-Nikodym theorem connects these two parts of the thesis together.

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