FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

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Observational aspects of a massive graviton

Institute of Theoretical Physics

Supervisor of the bachelor thesis: Dr. Federico Urban
Study programme: Physics
Study branch: Physics FP

Prague 2024

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At this place I would like to thank all those who contributed to the creation of this thesis. First of all, of course, to my supervisor Federico Urban for his professional guidance throughout the work and for introducing this interesting topic to me. I would also like to say big thanks to Pavel Kůs and Jakub Vošmera for their valuable comments, discussions and support. Last but not least, I am grateful to all my friends who supported me while working on this thesis.

Title: Observational aspects of a massive graviton

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Abstract: The possible existence of a massive spin-2 field alongside the standard massless graviton of General Relativity is a recent theoretical development in gravitational physics. If such an additional tensor field exists, it could leave potentially observable imprints in several astrophysical, cosmological and laboratory settings. This thesis studies the phenomenon of flavour oscillations in a nearly degenerate coupled system of photons with massive and massless spin-2 particles propagating in an external magnetic field. The framework for our calculations is provided by the ghost-free bimetric theory of gravity coupled to electromagnetism. We discuss several potentially observable manifestations of such oscillations and make numerical predictions for some relevant laboratory setups and astrophysical scenarios.

Keywords: massive spin-2 field, dark matter, mixing of photons with other particles, bigravity

Název práce: Pozorovací aspekty hmotného gravitonu

Autor: Radka Křížová

Katedra: Ústav teoretické fyziky

Vedoucí bakalářské práce: Dr. Federico Urban, CEICO, Fyzikální ústav AV ČR, v.v.i.

Abstrakt: Možná existence hmotného spin-2 pole vedle standardního nehmotného gravitonu obecné relativity představuje nedávný teoretický vývoj v gravitační fyzice. Pokud takové dodatečné tenzorové pole existuje, mohlo by zanechat potenciálně pozorovatelné stopy v různých astrofyzikálních, kosmologických a laboratorních podmínkách. Tato práce studuje flavourové oscilace v téměř degenerovaném interagujícím systému fotonů s hmotnými a nehmotnými částicemi se spinem 2 šířícími se v magnetickém poli. Rámec pro naše výpočty je poskytnut bimetrickou teorií gravitace bez duchů, která interaguje s elektromagnetismem. Diskutujeme několik potenciálně pozorovatelných projevů zmíněných oscilací a provádíme číselné předpovědi pro některá relevantní laboratorní uspořádání a astrofyzikální scénáře.

Klíčová slova: hmotné pole se spinem 2, temná hmota, mixing fotonů s jinými částicemi, bigravitace

Contents

Introduction

In 2022, the LHAASO experiment reported the observation of a gamma-ray burst with high-energy photons, specifically with energies up to 18 TeV, whose redshift was estimated to be $z = 0{,}151$ [1]. In addition to this, a shower corresponding to a photon with an energy of 251 TeV originating from the same gamma-ray burst was observed shortly afterwards by the Carpet-2 experiment at the Baksan Neutrino Observatory [2]. However, this creates a puzzle, because such high-energy photons should be significantly inhibited through interactions with photons from the cosmic microwave background radiation (CMB) and extragalactic background light (EBL) and thus should not be able to reach us from the distance corresponding to the estimated redshift. That we observe them nonetheless could therefore very well be a signature of phenomena beyond the standard model of particle physics. Most often discussed in this context are axions, hypothetical spin-0 particles with pseudoscalar coupling to the EM field that, among other things, are considered as possible candidates for dark matter particles. The significance which axions bear in relation to the problem of observing energetic photons lies in the fact that, unlike photons, they are assumed to interact only very weakly with ordinary matter. Moreover, in the presence of an external magnetic field, coupling between two photons and an axion should lead to oscillations between the two particle species, thus allowing the observation of the aforementioned high-energy photons: having propagated for most of its journey to the Earth in the guise of an axion, the photon would escape the loss of energy through interactions with background photons and the subsequent e^+e^- production [3].

Several specific astrophysical setups where the axions should be observable have already been proposed [4]. Experimentally, particles such as axions could be detectable in "light-shining-through-wall" experiments ([5, 6], see [7] for a review). In these, a laser beam is propagated through a magnetic field into which an obstacle (wall) is inserted. While photons cannot pass through the wall, particles like axions have no problem penetrating even very dense materials due to the their very weak interaction with ordinary matter. Having passed through the obstacle, they can then be converted back into photons through another magnetic field, which, in turn, can be detected on the other side of the barrier. An outside observer would therefore conclude that some light has been shone through the wall (albeit in the form of axions). So far, however, the existence of particles that would allow for this phenomenon has not been demonstrated from these experiments.

Many studies which consider the problem of particles mixing with the EM field focus on the case of the axion, because it also provides a possible resolution of the strong CP problem [8, 9]. However, one has to bear in mind that a similar effect can be mediated by any other light particle with suitable coupling to the EM field. In particular, one does not have to restrict to spin-0 and consider, for instance, coupling the photons to spin-2 particles [10] (mixing of photons with spin-1 particles is forbidden by the Landau-Yang theorem [11, 12]). This is closely related to (linearized) theory of general relativity, which can be understood as a theory of a single massless particle with spin 2, called the massless graviton.

In addition to this massless graviton of general relativity, the existence of its massive variant, called the massive graviton, has recently been considered. Indeed, one could envisage that there could be a natural extension of general relativity, where one would add (in a well-defined and consistent manner) a second metric tensor to the metric tensor of general relativity so as to end up with one massless and one massive dynamical spin-two field. Such a setup is known as *bigravity*. Note that this theory is relatively new, because for a long time, it was believed that no consistent theory for gravitating massive spin-2 fields can be formulated owing to the unavoidable presence of a fatal ghost instability ([13], see [14] for a review). While the first linearized massive spin-2 field theory was proposed by Fierz and Pauli [15], it was found that upon taking the massless limit, the theory exhibits the so-called *van Dam – Veltman – Zakharov (vDVZ) discontinuity* [16, 17]. In particular, the massless limit did not seem to provide a theory for a massless spin-2 particle on its own, but only in combination with a propagating scalar field as a remnant of the spin-0 polarization of the massive spin-2 field. Nevertheless, the so-called *Vainshtein mechanism* [18] (see [19] for a review) soon explained that up to a certain distance from a source (the Vainshtein radius), this scalar is so strongly coupled that a linearized analysis breaks down. As a consequence, inside the Vainshtein sphere the scalar becomes hidden by non-linear effects and its presence is felt only at large distances. At the same time, the problem of consistency of such non-linear extensions of massive gravity was addressed by Boulware and Deser, who pointed out the necessity of the presence of a ghost instability (the so-called *Boulware-Deser ghost*), a scalar field with negative kinetic energy that would cause fatal instabilities [13]. Decades later, however, it turned out that their analysis was not general enough and the ghost instability can be systematically eliminated, a development which culminated in the de-Rham-Gabadadze-Tolley (dRGT) theory [20] and its later generalization, the Hassan-Rosen bimetric theory [21].

If the above mentioned additional tensor field exists, one could in principle observe its signatures in various astrophysical, cosmological and laboratory settings. For example, the massive graviton, similarly to the axion, is a seriously-considered candidate for a particle which would account for the observed abundance of dark matter in present universe: both in the regime where its mass is large [22], thus working in the weakly interacting massive particle (WIMP) / ΛCDM (Λ cold dark matter) paradigm (see [23] for a review), as well as for very small massive spin-2 masses [24] (the ULDM – ultra light dark matter paradigm [25, 26, 27, 28, 29, 30]).

So far, we know about the existence of dark matter only from its gravitational effects. One possible explanation is that it is a manifestation of gravity itself, naturally extended by a massive spin-2 field [31]. Due to its weak coupling, the presence of this field would have significant effects only at larger distances, so as to retain consistency with observations. It appears that using a combination of the standard Newtonian potential and the Yukawa potential induced by the massive graviton, the rotation curves of galaxies can be fit quite successfully [32]. The existence of ultralight dark matter could also be revealed by pulsars, as the oscillations of the spin-2 field could affect the dynamics of binary pulsar systems, causing changes in the arrival times of the pulses [24, 33]. Moreover, signatures in fluctuations of the cosmic microwave background may also be associated with gravitons [34, 35] as the thermal CMB photons could have been converted into gravitons when interacting with the primordial magnetic field during the recombination epoch. Therefore, the observations of CMB fluctuations can be used to detect high frequency gravitational waves [36, 37] and provide constraints on the strength of the primordial magnetic field. This could possibly be an example of a magnetic field strong enough also for the massive gravitons to undergo conversion to photons with a non-trivial rate. In turn, the same mechanism also allows for the conversion of relic gravitons into photons, which could be a possibility in testing various cosmological theories [34]. This should be enough of a motivation for a deeper study of these particles and their interactions.

In this thesis, we will mainly focus on the aspects of eigenstate mixing between the EM fluctuations and other particles in an external magnetic field. This phenomenon arises whenever there is a (near) degeneracy of masses of mutually interacting particles and gives rise to flavour oscillations. As a consequence, we will work in the regime, where the particles interacting with the EM field are ultrarelativistic and thus approximately massless (as is the photon). In particular, as we have argued above, these oscillations may leave potentially observable imprints: among other things, they could provide explanation for the recent observations of high energy photons. In Chapter 1, we study (mostly as a toy example) a spin-0 particle interacting with electromagnetic field through a scalar coupling: step by step, we derive the equations of motion from the lagrangian, linearize them and find the solution for the oscillations of photons and scalars. We also discuss possible signatures of these oscillations in various experiments and observations. In chapter 2, we then turn to a more complicated situation that will be of more interest to us later in this thesis: the interaction of the EM field with a massive spin-2 field. Starting with the linearized Fierz-Pauli lagrangian coupled to the EM stress-energy tensor and going in detail through all derivations, we emphasize various similarities and differences compared with the story of the scalar. Noticing that upon taking the massless limit, the spin-0 polarization of the massive spin-2 field does not decouple from the dynamics (a direct manifestation of the vDVZ discontinuity), we realize that the massless spin-2 case deserves separate treatment, which we then outline in Chapter 3. Starting from the Einstein-Hilbert action minimally coupled to the Maxwell action, we derive a linearized action for the metric and the EM fluctuations which, as in the massive spin-2 case, we find to be coupled together through the EM stress-energy tensor. Since this is an example of a theory where all parameters are explicitly known, we are able to make quantitative predictions for the photon-graviton conversion rates (the so-called *Gertsenshtein-Zel'dovich effect*) and other observables. In particular, we conclude that a photon double-conversion mediated by the massless graviton alone is unlikely to explain the observations of high-energy photons from GRB221009A. Finally, in Chapter 4, we discuss the combined interaction of the EM field with both the massive and the massless spin-2 fluctuations within the unifying framework of the bimetric theory of gravity. Building heavily upon the results derived in the previous chapters, we are able to reduce the coupled dynamics of the EM field and the two gravitons into two separate systems, where each exhibits 3-flavour mixing. Finding the mass eigenstates perturbatively in the couplings, we derive general formulae for various observables effects which arise as a consequence of oscillations between various polarizations of the fields involved. We also evaluate some of these quantities in the two regimes in the bimetric parameter space for which the massive spin-2 field has so far been considered as a candidate for dark matter: the heavy CDM regime (relative coupling strength of the two spin-2 fields $\alpha \simeq 10^{-11} - 10^{-13}$, Fierz-Pauli mass $m_{\mathrm{FP}} \simeq 1 - 100\,\mathrm{TeV}$) and the ULDM regime ($\alpha \simeq 10^{-5}$ and $m_{\mathrm{FP}} \simeq 10^{-23} - 10^{-17}\,\mathrm{eV}$). At the end, we conclude with a summary and discussion of the main points.

Conventions

Throughout the thesis, we will use the mostly-minus (West-Coast) Minkowski metric

$$\eta = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1}$$

Unless we say otherwise, we will also use the system of natural units common to the high-energy physics literature, that is

$$c = \hbar = \varepsilon_0 = 1 \, . \tag{2}$$

This also implies

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} = 1 \tag{3}$$

for the vacuum permeability μ_0 as well as

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{e^2}{4\pi} \tag{4}$$

for the fine structure constant in terms of the elementary charge e.

1. Mixing of photons with scalars

Let us first explore the problem of eigenstate mixing in the simple case of a massive scalar particle interacting with electromagnetic field. In order to achieve efficient mixing, we will work in the ultrarelativistic regime $m_0 \gg \omega$ (denoting by m_0 is the mass of the scalar field) where the masses of the scalars and photons become effectively degenerate. We will mainly follow the presentation of [5] and [10].

1.1 Derivation of the mixing equations

We first focus on deriving the evolution of coupled EM and scalar (spin-0) modes as they propagate through a background magnetic field.

1.1.1 Action and its variation

Let us first write down the lagrangian (density) describing a scalar field ϕ coupled to the electromagnetic field A_μ through a cubic vertex [15]. That is, we write

$$\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{KG}} + \frac{1}{4} g_0 F^{\mu\nu} F_{\mu\nu} \phi \,, \tag{1.1}$$

where $\mathcal{L}_{\text{EM}}$ is the usual free electromagnetic (Maxwell) lagrangian, which is written in terms of the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{1.2}$$

and $\mathcal{L}_{\text{KG}}$ is the lagrangian describing a free massive scalar field (with mass m_0) [38]

$$\mathcal{L}_{\text{KG}} = \frac{1}{2} (\partial_\alpha \phi)(\partial^\alpha \phi) - \frac{1}{2} m_0^2 \phi^2 \,. \tag{1.3}$$

Finally, g_0 is a coupling constant with dimensions mass^{-1}. Note that having in mind our later applications to deriving the mixing equations, we chose to ignore any potential $\mathcal{O}(\phi^3)$ self-interactions of the scalar, as these would have been dropped during the process of linearization around a background with $\phi = 0$. Also note that the interaction term is invariant under the EM gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{1.4}$$

for some gauge parameter λ because $F_{\mu\nu}$ does not transform under (1.4).

We will soon convince ourselves that the lagrangian (1.1) describes well the oscillations between the EM and scalar fluctuations propagating on a constant magnetic background, at least in pure vacuum and at a classical field theory level. However, for practical applications, it will be important to also consider the following two effects.

First, one has to bear in mind that perfect vacuum does not exist. Therefore, as they propagate through the medium the particles, mainly the photons,[1] will

[1] Mainly the photons, the scalars and other highr-spin fields considered in the remainder of this thesis, will be treated as dark matter, which interacts very weakly with ordinary matter.

interact with external matter, whose presence, in principle, should be reflected through the inclusion of other couplings in the lagrangian (1.1). In practice, we will later on bypass this by parametrizing this interaction with effective refractive indices for the EM modes.

Second, even in a (hypothetical) perfect vacuum, a background EM field may effectively induce different refractive indices for different EM polarizations through loop diagrams in QED [39]. This is a consequence of the fact that QED is not purely a theory EM field fluctuations, but it couples photons to charged matter (electrons), which may mediate 2-2 photon-photon scattering through higher loop Feynman diagrams. At leading order, such a scattering would be mediated by one electron loop with four vertices, to which the external photons are attached (see figure 1.1). This gives rise to the effective Euler-Heisenberg coupling [40]

$$\mathcal{L}_{\mathrm{EH}} = \frac{\alpha^2}{90 m_{\mathrm{e}}^4}\left[\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right], \tag{1.5}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant and m_{e} is the electron mass. Also, $\tilde{F}^{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$, namely

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}\,. \tag{1.6}$$

We will postpone explicit inclusion of effects due to $\mathcal{L}_{\mathrm{EH}}$ up until section 1.1.5 below, where we will observe that upon turning on a background EM field, it effectively gives rise to a birefringent medium, whose refractive indices we will compute.

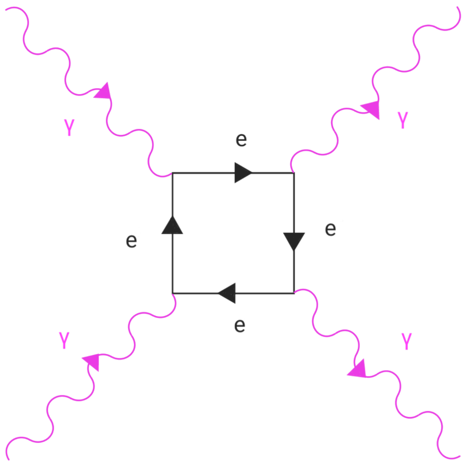

Figure 1.1: Leading order contribution to 2-2 photon-photon scattering in QED.

For the moment, let us come back to considering just the classical lagrangian (1.1). In order to derive the equations of motion, we can either vary the action directly or substitute into the the Euler-Lagrange equations. Let us adopt the second approach here. For the scalar equation of motion we will thus be substituting into

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{1.7}$$

that is, in detail,

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}_{\text{EM}}}{\partial \phi} + \frac{\partial \mathcal{L}_{\text{KG}}}{\partial \phi} + \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial \phi} + \\
&\quad - \partial_\mu \frac{\partial \mathcal{L}_{\text{EM}}}{\partial(\partial_\mu \phi)} - \partial_\mu \frac{\partial \mathcal{L}_{\text{KG}}}{\partial(\partial_\mu \phi)} - \partial_\mu \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial(\partial_\mu \phi)}\,, && (1.8a) \\
0 &= \underbrace{\frac{\partial(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})}{\partial \phi}}_{=0} + \frac{\partial\left[\frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) - \frac{1}{2}m_0^2\phi^2\right]}{\partial \phi} + \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial \phi} && (1.8b) \\
&\quad - \underbrace{\partial_\mu \frac{\partial(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})}{\partial(\partial_\mu \phi)}}_{=0} - \partial_\mu \frac{\partial\left[\frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) - \frac{1}{2}m_0^2\phi^2\right]}{\partial(\partial_\mu \phi)} - \underbrace{\partial_\mu \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial(\partial_\mu \phi)}}_{=0}\,, && (1.8c) \\
0 &= -m_0^2\phi + \frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu} - \partial_\mu(\partial^\mu \phi)\,, && (1.8d)
\end{aligned}
$$

so that finally we obtain

$$
0 = (\Box + m_0^2)\phi - \frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\,. \tag{1.9}
$$

As for the equation of motion for the photon, we need to substitute into the Euler-Lagrange equation

$$
0 = \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \tag{1.10}
$$

and, again in detail,

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}_{\text{EM}}}{\partial A_\nu} + \frac{\partial \mathcal{L}_{\text{KG}}}{\partial A_\nu} + \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial A_\nu} + \\
&\quad - \partial_\mu \frac{\partial \mathcal{L}_{\text{EM}}}{\partial(\partial_\mu A_\nu)} - \partial_\mu \frac{\partial \mathcal{L}_{\text{KG}}}{\partial(\partial_\mu A_\nu)} - \partial_\mu \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial(\partial_\mu A_\nu)}\,, && (1.11a) \\
0 &= \underbrace{\frac{\partial(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})}{\partial A_\nu}}_{=0} + \underbrace{\frac{\partial\left[\frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) - \frac{1}{2}m_0^2\phi^2\right]}{\partial A_\nu}}_{=0} + \underbrace{\frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial A_\nu}}_{=0} + \\
&\quad - \partial_\mu \frac{\partial(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})}{\partial(\partial_\mu A_\nu)} - \underbrace{\partial_\mu \frac{\partial\left[\frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) - \frac{1}{2}m_0^2\phi^2\right]}{\partial(\partial_\mu A_\nu)}}_{=0} - \partial_\mu \frac{\partial(\frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\phi)}{\partial(\partial_\mu A_\nu)}\,, \\
& && (1.11b) \\
0 &= \partial_\mu F^{\mu\nu} - g_0(\phi \partial_\mu F^{\mu\nu} + F^{\mu\nu}\partial_\mu \phi)\,, && (1.11c)
\end{aligned}
$$

so that we end up with the equation of motion

$$
0 = \partial_\mu F^{\mu\nu} - g_0(\phi \partial_\mu F^{\mu\nu} + F^{\mu\nu}\partial_\mu \phi)\,. \tag{1.12}
$$

In total, we will therefore consider the equations of motion

$$
\begin{aligned}
0 &= (\Box + m_0^2)\phi - \frac{1}{4}g_0 F^{\mu\nu}F_{\mu\nu}\,, && (1.13a) \\
0 &= \partial_\mu F^{\mu\nu} - g_0(\phi \partial_\mu F^{\mu\nu} + F^{\mu\nu}\partial_\mu \phi)\,. && (1.13b)
\end{aligned}
$$

1.1.2 Propagation on a magnetic background

We would like to expand the fields in small fluctuations around a background, which is defined by the constant magnetic field. Let us assume that the particle propagates along the z-axis. The background can be divided into two components: the component parallel to the direction of propagation, denoted by B_{L} and the component perpendicular to it, denoted by B_{T}. Without loss of generality, we can consider this second component only in the x-axis direction, which gives us the field-strength tensor in the form

$$F^{\mathrm{ext}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_{\mathrm{L}} & 0 \\ 0 & -B_{\mathrm{L}} & 0 & B_{\mathrm{T}} \\ 0 & 0 & -B_{\mathrm{T}} & 0 \end{pmatrix}, \tag{1.14}$$

which is assumed to be turned on in a region of space $|r| \lesssim L$ and vanishes otherwise. Here L characterizes the size of the region. In order to describe the propagation of photons in such a background, we want to expand the vector potential as

$$A_\nu \to A^{\mathrm{ext}}_\nu + A_\nu\,. \tag{1.15}$$

and correspondingly the EM tensor as

$$F_{\mu\nu} \to F^{\mathrm{ext}}_{\mu\nu} + F_{\mu\nu}\,. \tag{1.16}$$

However, note that turning on a non-trivial background $F^{\mathrm{ext}}_{\mu\nu}$ for the EM field *only* is apparently inconsistent with the equations of motion (1.13a), because non-zero $F^{\mathrm{ext}}_{\mu\nu}$ clearly becomes a source for the scalar. Hence, expanding the equations of motion (1.13a) and (1.13b) in small fluctuations around the background with $F^{\mathrm{ext}}_{\mu\nu} \neq 0$ while $\phi^{\mathrm{ext}}_{\mu\nu} = 0$ should be expected to lead to contradictory results. Although it is true that formally speaking, this necessitates turning on a non-trivial classical background for the scalar, we will now argue that this can be neglected on the grounds of perturbative consistency of the lagrangian (1.1).

Neglecting the induced scalar background

Indeed, starting with the scalar equation of motion (1.13a), we obtain

$$0 = (\Box + m_0^2)\phi - \frac{1}{4} g_0 \Big[F^{\mu\nu}F_{\mu\nu} + F^{\mu\nu}F^{\mathrm{ext}}_{\mu\nu} + (F^{\mathrm{ext}})^{\mu\nu}F_{\mu\nu} + (F^{\mathrm{ext}})^{\mu\nu}F^{\mathrm{ext}}_{\mu\nu} \Big]\,. \tag{1.17}$$

The first term in the parentheses is quadratic in small fluctuations, meaning that we will drop it. The second and third are identical and together equal to

$$2F^{\mathrm{ext}}_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) = 4F^{\mathrm{ext}}_{\mu\nu}\partial^\mu A^\nu\,, \tag{1.18}$$

while the last term can be evaluated as

$$(F^{\mathrm{ext}})^{\mu\nu}F^{\mathrm{ext}}_{\mu\nu} = 2(B_{\mathrm{L}}^2 + B_{\mathrm{T}}^2)\,. \tag{1.19}$$

Hence, equation (1.17) can be rewritten as

$$0 = (\Box + m_0^2)\phi - g_0(F^{\mathrm{ext}})^{\mu\nu}\partial^\mu A^\nu - \frac{1}{2} g_0 (B_{\mathrm{L}}^2 + B_{\mathrm{T}}^2)\,. \tag{1.20}$$

Note that since the last term is a constant, equation (1.20) cannot be balanced for small enough fluctuations ϕ and A^ν

To correct this problem, let us now also consider turning on a background $\phi^{\text{ext}} \neq 0$ for the scalar. To determine the value of ϕ^{ext}, let us again start with the scalar equation of motion, but now let us try to solve it for the background fields ϕ^{ext} and A^{ext}_μ, as those need to satisfy the equations of motion (1.13a) and (1.13b) as well. Assuming first that $m_0 \gg 1/L$ in natural units, the mass term in the equation of motion dominates over the derivative terms and we can therefore assume that $\phi^{\text{ext}} = \text{const.}$ inside $|r| \leq L$ (and $\phi^{\text{ext}} = 0$ outside the region, with an $\exp(-m_0 r)$-suppression near $|r| = L$). We then have

$$0 = (\Box + m_0^2)\phi^{\text{ext}} - \frac{1}{4} g_0 (F^{\text{ext}})^{\mu\nu} F^{\text{ext}}_{\mu\nu}\,, \tag{1.21a}$$

$$0 = (\Box + m_0^2)\phi^{\text{ext}} - \frac{1}{2} g_0 (B_{\text{L}}^2 + B_{\text{T}}^2)\,, \tag{1.21b}$$

$$0 = m_0^2 \phi^{\text{ext}} - \frac{1}{2} g_0 (B_{\text{L}}^2 + B_{\text{T}}^2)\,. \tag{1.21c}$$

This gives

$$g_0 \phi^{\text{ext}} = \frac{g_0^2 (B_{\text{L}}^2 + B_{\text{T}}^2)}{2 m_0^2} \neq 0 \qquad \text{inside } |r| \leq L\,. \tag{1.22}$$

On the other hand, in the regime $m_0 \leq 1/L$, the derivative terms in the equation of motion start to dominate the mass term and the profile of ϕ^{ext} can no longer assumed to be constant. Assuming the fall-off condition $\phi^{\text{ext}} \to 0$ as $r \to \infty$, one can derive that the profile will have characteristic size $g_0 \phi^{\text{ext}} \propto g_0^2 B^2 L^2$. (Correspondingly, this would also induce a non-constant $\mathcal{O}(g_0^2 B^2 L^2)$ correction to F^{ext} through the equation of motion for the EM background.)

However, here one has to note that perturbative consistency of the lagrangian (1.1) (which we have to demand) remains in place only if the interaction term $\frac{1}{4} g_0 F^{\mu\nu} F_{\mu\nu} \phi$ is small compared to $\mathcal{L}_{\text{EM}}$ and $\mathcal{L}_{\text{KG}}$. This means that $g_0 \phi$ has to be small and so $g_0 \phi^{\text{ext}}$ has to be small too. If this condition were not satisfied, we would escape the regime of validity of our analysis and we would have to add higher order interaction terms into the lagrangian (1.1) to restore consistency. In other words, in order to be inline with perturbative consistency of the lagrangian (1.1) (which in turn guarantees that one can neglect the scalar background ϕ^{ext} in the ensuing analysis), one needs to assume that the external magnetic field is weak compared to the scales given by the mass m_0 of the scalar, or, by the size L of the region, inside which it is being turned on. In particular, one needs to demand

$$\frac{g_0^2 B_{\text{ext}}^2}{m_0^2} \ll 1 \tag{1.23}$$

in the regime $m_0 \gg 1/L$ and

$$g_0^2 B_{\text{ext}}^2 L^2 \ll 1 \tag{1.24}$$

for $m_0 \leq 1/L$.

Also note that had we considered coupling the EM field to a pseudoscalar particle instead (that is a coupling $a F_{\mu\nu} \tilde{F}^{\mu\nu}$ [5] for an axion-like particle a instead of $\phi F_{\mu\nu}F^{\mu\nu}$), we would be automatically getting $\phi^{\mathrm{ext}} = 0$. Indeed, schematically written, we have

$$F = \begin{pmatrix} 0 & +E \\ -E & B \end{pmatrix}, \qquad \tilde{F} = \begin{pmatrix} 0 & B \\ -B & E \end{pmatrix}, \tag{1.25}$$

so that in an analogous way as in the case of a scalar particle, the case of a pseudoscalar would yield

$$a^{\mathrm{ext}} = \frac{g_0}{4m_0^2}(F^{\mathrm{ext}})^{\mu\nu}\tilde{F}^{\mathrm{ext}}_{\mu\nu}, \tag{1.26}$$

that is

$$a^{\mathrm{ext}} = \frac{g_0}{4m_0^2}(\mathbf{E}^{\mathrm{ext}} \cdot \mathbf{B}^{\mathrm{ext}}). \tag{1.27}$$

Since in our case the electric field is zero, we would have got $a^{\mathrm{ext}} = 0$ exactly.

Linearization around the EM background

Going back to the case of a scalar particle, expanding the equation of motion (1.13a) using $\phi^{\mathrm{ext}} \neq 0$ and keeping only terms which are linear in fluctuations, we get

$$0 = (\Box + m_0^2)(\phi + \phi^{\mathrm{ext}}) - \frac{1}{4}g_0(F^{\mu\nu}F^{\mathrm{ext}}_{\mu\nu} + F^{\mu\nu}_{\mathrm{ext}}F_{\mu\nu} + F^{\mu\nu}_{\mathrm{ext}}F^{\mathrm{ext}}_{\mu\nu}), \tag{1.28a}$$

$$0 = (\Box + m_0^2)\phi - \frac{1}{2}g_0F^{\mu\nu}_{\mathrm{ext}}F_{\mu\nu} + (\Box + m_0^2)\phi^{\mathrm{ext}} - \frac{1}{4}g_0F^{\mu\nu}_{\mathrm{ext}}F^{\mathrm{ext}}_{\mu\nu}. \tag{1.28b}$$

Using the equation of motion (1.21a) for ϕ^{ext}, the last two terms will go away. Hence, we get the scalar equation of motion in the form

$$0 = (\Box + m_0^2)\phi - \frac{1}{2}g_0F^{\mu\nu}_{\mathrm{ext}}F_{\mu\nu}. \tag{1.29}$$

Substituting the expansions into the photon equation of motion (1.13b) we get

$$0 = \partial_\mu(F^{\mu\nu} + F^{\mu\nu}_{\mathrm{ext}}) - g_0\Big[(\phi + \phi^{\mathrm{ext}})\partial_\mu(F^{\mu\nu}_{\mathrm{ext}} + F^{\mu\nu}) + (F^{\mu\nu}_{\mathrm{ext}} + F^{\mu\nu})\partial_\mu(\phi^{\mathrm{ext}} + \phi)\Big]. \tag{1.30}$$

Subtracting the equation of motion satisfied by the background, we get

$$0 = \partial_\mu F^{\mu\nu} - g_0(\phi^{\mathrm{ext}}\partial_\mu F^{\mu\nu} + \phi\partial_\mu F^{\mu\nu} + F^{\mu\nu}_{\mathrm{ext}}\partial_\mu\phi + F^{\mu\nu}\partial_\mu\phi) \tag{1.31}$$

and after we leave just the terms of first order in fluctuations, we get

$$0 = \partial_\mu F^{\mu\nu} - g_0(\phi^{\mathrm{ext}}\partial_\mu F^{\mu\nu} + F^{\mu\nu}_{\mathrm{ext}}\partial_\mu\phi) \tag{1.32a}$$

$$0 = (1 - g_0\phi^{\mathrm{ext}})\partial_\mu F^{\mu\nu} - g_0F^{\mu\nu}_{\mathrm{ext}}\partial_\mu\phi. \tag{1.32b}$$

Finally, using the above-discussed assumption $g_0\phi^{\mathrm{ext}} \ll 1$, we obtain

$$0 = \partial_\mu F^{\mu\nu} - g_0F^{\mu\nu}_{\mathrm{ext}}\partial_\mu\phi. \tag{1.33}$$

To summarize, so far we have managed to put the linearized equations of motion into the form

$$0 = (\Box + m_0^2)\phi - \frac{1}{2}g_0F^{\mu\nu}_{\mathrm{ext}}F_{\mu\nu}, \tag{1.34a}$$

$$0 = \partial_\mu F^{\mu\nu} - g_0F^{\mu\nu}_{\mathrm{ext}}\partial_\mu\phi. \tag{1.34b}$$

1.1.3 Plane-wave solution and gauge fixing

Far away from the source, the wave can be treated locally as a plane wave. So from now on we will be searching for a solution of the equations (1.34) in the form of a plane wave with wave vector

$$k^\mu = (\omega(p), 0, 0, p)\,, \tag{1.35}$$

where $p > 0$ is some momentum and $\omega(p)$ the corresponding energy. In particular, we will write

$$A_\mu(t, z) = A_\mu(p)e^{i(\omega t - pz)}\,, \tag{1.36a}$$
$$\phi(t, z) = \phi(p)e^{i(\omega t - pz)}\,. \tag{1.36b}$$

In order to simplify the analysis, we would now like to completely fix the gauge symmetry (1.4) associated with the EM field. Let us start by fixing the Lorentz gauge

$$\partial_\mu A^\mu = 0\,. \tag{1.37}$$

Using this condition, the equations of motion (1.34) reduce to

$$0 = (\Box + m_0^2)\phi - g_0 F_{\text{ext}}^{\mu\nu}\partial_\mu A_\nu\,, \tag{1.38a}$$
$$0 = \Box A^\nu - g_0 F_{\text{ext}}^{\mu\nu}\partial_\mu \phi\,. \tag{1.38b}$$

In particular, since $F_{\text{ext}}^{\mu 0} = 0$, we observe that

$$\Box A^0 = 0\,. \tag{1.39}$$

This enables us to fix the residual gauge symmetry by putting

$$A^0 = 0\,. \tag{1.40}$$

Indeed, the residual gauge symmetry is a transformation of the form (1.4), which does not violate the Lorentz condition (1.37), i.e. the one given by a gauge parameter that satisfies

$$\Box\lambda = -\partial_\mu A^\mu = 0\,. \tag{1.41}$$

Hence, for a generic $A^0 \neq 0$, we can put

$$\lambda = -\int^t d\tau\, A^0\,, \tag{1.42}$$

so that after the gauge transformation, we indeed have

$$(A')^0 = A^0 + \partial^0\lambda = 0 \tag{1.43}$$

and, at the same time

$$\Box\lambda = \partial_t^2\lambda - \partial_i^2\lambda \tag{1.44a}$$
$$= -\partial_t^2 \int^t d\tau\, A^0 + \int^t d\tau\, \partial_i^2 A^0 \tag{1.44b}$$

$$= -\dot{A}^0 + \int^t d\tau\, \partial_\tau^2 A^0 - \int^t d\tau\, \Box A^0 \tag{1.44c}$$

$$= -\dot{A}^0 + \dot{A}^0 \tag{1.44d}$$

$$= 0\,, \tag{1.44e}$$

where we have used (1.39) in the process. Substituting the constraint (1.40) into the gauge condition (1.37) this yields

$$0 = \partial_t A^0 - \partial_z A^3 = -\partial_z A^3\,. \tag{1.45}$$

Therefore, for the plane-wave solution (1.36), we can simply put

$$A^3 = 0\,. \tag{1.46}$$

This is clearly consistent with the equation of motion for A^3, namely

$$\Box A^3 = 0\,, \tag{1.47}$$

where we have noted that $F^{03}_{\text{ext}} = F^{33}_{\text{ext}} = 0$. Hence, the EM field will have only two independent degrees of freedom, namely

$$A_1 = A_\parallel\,, \tag{1.48a}$$

$$A_2 = A_\perp\,. \tag{1.48b}$$

Summarizing the discussion so far, the plane wave solution for both the EM field and the scalar will be assumed to propagate along the z-axis. In the region where the background EM field is turned on, we will assume pure magnetic field $B^i = (B_{\text{T}}, 0, B_{\text{L}}) = (F^{\text{ext}}_{23}, 0, F^{\text{ext}}_{12})$. The two independent photon polarizations of the EM field are then assembled into the 4-potential as $A_\mu = (0, A_\parallel, A_\perp, 0)$, where $A_\parallel$ is the polarization parallel to B_{T} and $A_\perp$ is the polarization perpendicular to B_{T}. See also figure 1.2.

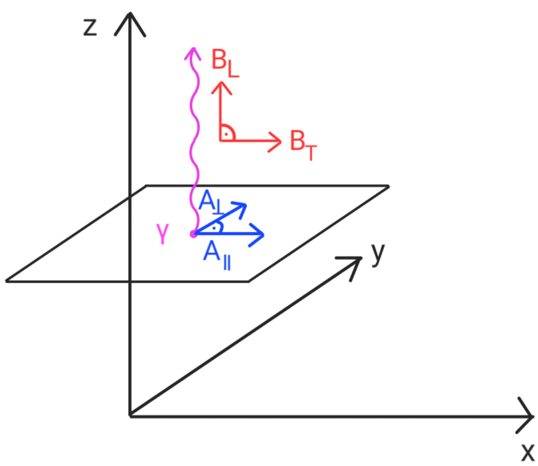

Figure 1.2: The relative configuration of the magnetic field and the propagating beam.

1.1.4 Ultrarelativistic approximation

In this chapter, we will be interested in considering mixing between the propagating EM excitations (which are massless) and the excitations of the spin-0 field. Since flavour mixing can only efficiently take place when the two respective particles have degenerate masses, we are motivated to focus on the regime where the scalar excitations are ultrarelativistic and thus effectively massless. In such a case, that is when the rest mass is negligible compared to the total energy, i.e. when

$$\frac{m_0^2}{2\omega^2} \ll 1\,, \tag{1.49}$$

(where $\omega(p)$ is the energy) we expect $p \approx \omega$, so that we can linearize the differential operators appearing in the equations of motion. In particular, we can linearize the operator $-\Box$ (in the frequency space) as

$$\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z) = (\omega + p)(\omega - i\partial_z) \approx 2\omega(\omega - i\partial_z)\,. \tag{1.50}$$

Considering the conditions above, the field solutions to the equations of motion will be plane waves with an energy dependence $e^{i\omega t}$ and a spatial dependence e^{-ipz}. Isolating the time-dependence as

$$A_\mu(t,z) = A_\mu(z)e^{i\omega t}\,, \tag{1.51a}$$

$$\phi(t,z) = \phi(z)e^{i\omega t}\,. \tag{1.51b}$$

and plugging the linearized operator above into the equations of motion, we have for the scalar equation of motion

$$[-2\omega(\omega - i\partial_z) + m_0^2]\phi = \frac{1}{2} g_0 F^{\text{ext}}_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1.52a}$$

$$\Big(-\omega + i\partial_z + \frac{m_0^2}{2\omega}\Big)\phi = \frac{g_0}{4\omega} F^{\mu\nu}_{\text{ext}}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{1.52b}$$

$$\Big(-\omega + i\partial_z + \frac{m_0^2}{2\omega}\Big)\phi = 2\frac{g_0}{4\omega} F^{\mu\nu}_{\text{ext}}\partial_\mu A_\nu \tag{1.52c}$$

$$\Big(-\omega + i\partial_z + \frac{m_0^2}{2\omega}\Big)\phi = -\frac{g_0}{2\omega} B_{\text{T}}\partial_z A_\perp\,, \tag{1.52d}$$

(where, in the last step, we have used that $\partial_1 A_2 = 0$ because there is no x-dependence) that is

$$0 = \Big(-\omega + i\partial_z + \frac{m_0^2}{2\omega}\Big)\phi + \frac{g_0}{2\omega} B_{\text{T}}\partial_z A_\perp\,. \tag{1.53}$$

For the photon equation of motion we have

$$0 = \Box A^\nu - g_0 F^{\mu\nu}_{\text{ext}}\partial_\mu\phi \tag{1.54a}$$

$$0 = \Box A^\nu - g_0(F^{0\nu}_{\text{ext}}\partial_t\phi + F^{3\nu}_{\text{ext}}\partial_z\phi)\,. \tag{1.54b}$$

From this, we can get four equations, one for each ν. With $\nu = 0, 3$ as we have extensively discussed in the previous section, there are no associated propagating degrees of freedom. On the other hand, for $\nu = 1$ ($\|$) we get

$$0 = \Box A_\| \tag{1.55a}$$

$$0 = 2\omega(\omega - i\partial_z)A_\parallel \tag{1.55b}$$
$$0 = (\omega - i\partial_z)A_\parallel\,, \tag{1.55c}$$

while for the polarization $\nu = 2$ (that is, $\perp$), we similarly get

$$0 = -\Box A_\perp - g_0 F^{32}_{\mathrm{ext}}\partial_z\phi \tag{1.56a}$$
$$0 = 2\omega(\omega - i\partial_z)A_\perp + g_0 B_\mathrm{T}\partial_z\phi \tag{1.56b}$$
$$0 = (\omega - i\partial_z)A_\perp + \frac{g_0 B_\mathrm{T}}{2\omega}\partial_z\phi\,. \tag{1.56c}$$

In summary, at this stage, the evolution of the coupled system of the EM and scalar fluctuations is dictated by the equations

$$0 = \Big(-\omega + i\partial_z + \Delta_0\Big)\phi + \frac{g_0}{2\omega}B_\mathrm{T}\partial_z A_\perp\,, \tag{1.57a}$$
$$0 = (\omega - i\partial_z)A_\perp + \frac{g_0 B_\mathrm{T}}{2\omega}\partial_z\phi\,, \tag{1.57b}$$

as well as by the equation

$$0 = (\omega - i\partial_z)A_\parallel \tag{1.58}$$

for the (decoupled) polarization $A_\parallel$. Note that we have introduced the notation

$$\Delta_0 = \frac{m_0^2}{2\omega} \tag{1.59}$$

for the mass term in the scalar equation of motion.

1.1.5 Environmental effects and vacuum polarization

In general, as we have already discussed above, we should allow for the medium, in which the EM modes $A_\parallel$ and $A_\perp$ propagate, to have a non-trivial refractive index due to interaction with matter. Moreover, upon turning on a background magnetic field, it turns out, that the two modes $A_\parallel$ and $A_\perp$ will be subject to slightly different values of this refractive index, which we will denote by $n_\parallel$ and $n_\perp$.

The contribution to this birefringence due to the interaction with matter is called the *Cotton-Mouton effect* and is sensitive to a non-zero transverse external magnetic field B_T. Furthermore, even in the case, when we could completely evacuate the region around the beam so that it does not interact with any external matter, the non-zero background magnetic field will polarize the vacuum through the (1-loop effective) Euler-Heisenberg lagrangian (1.5) so that one obtains differing contributions to both $n_\parallel$ and $n_\perp$. In other words, in the presence of an external magnetic field, the vacuum itself becomes a birefringent medium [39, 41].

For the sake of completeness, let us also mention that a non-zero longitudinal magnetic field B_L would have given rise to off-diagonal terms mixing the two transverse polarizations $A_\parallel$ and $A_\perp$ through the *Faraday effect* [42]. However, we have to bear in mind the kind of applications we will consider later on: 1. in laboratory experiments, we can always arrange for $B_\mathrm{L} = 0$ exactly, so that the Faraday effect does not contaminate the measurements, 2. in astrophysical environments, we mostly consider propagation of high-energetic photons, where one would have found this effect to contribute negligibly, as it is significantly suppressed at high values of ω. We will therefore neglect contributions due to the Faraday effect from now on.

Let us now quantify the above-described effects. Assuming both $n_\parallel$ and $n_\perp$ to be close to unity, it will be convenient to introduce the parametrization

$$n_\parallel = 1 + \frac{\Delta_\parallel}{\omega}\,, \tag{1.60a}$$

$$n_\perp = 1 + \frac{\Delta_\perp}{\omega}\,, \tag{1.60b}$$

where $\Delta_\parallel \ll \omega$, as well as $\Delta_\perp \ll \omega$. The Δ-parameters will then enter the EM equations of motion in a similar manner as the mass parameter Δ_0 in the scalar equation of motion. In particular, we will end up with the coupled system of equations of motion

$$\phi: \qquad 0 = (\omega - i\partial_z - \Delta_0)\phi - \frac{g_0}{2\omega} B_\mathrm{T} \partial_z A_\perp\,, \tag{1.61a}$$

$$\parallel: \qquad 0 = (\omega - i\partial_z + \Delta_\parallel) A_\parallel\,, \tag{1.61b}$$

$$\perp: \qquad 0 = (\omega - i\partial_z + \Delta_\perp) A_\perp + \frac{g_0 B_\mathrm{T}}{2\omega} \partial_z \phi\,. \tag{1.61c}$$

As suggested above, we will take into account two main contributions to the Δ-parameters for the EM fluctuations: due to interactions with matter (typically a near-vacuum gas) and due to vacuum birefringence. Namely, we will write

$$\Delta_\parallel = \Delta_\parallel^{(\mathrm{gas})} + \Delta_\parallel^{(\mathrm{vac})}\,, \tag{1.62a}$$

$$\Delta_\perp = \Delta_\perp^{(\mathrm{gas})} + \Delta_\perp^{(\mathrm{vac})}\,. \tag{1.62b}$$

Mean refractive indices

First the interaction with matter will give rise to a mean refractive index, which will of course depend on the precise nature and state of the gas the photons interact with as they propagate through the magnetic field. For instance, in the case of a laboratory experiment, where the beam passes through an evacuated chamber with air at pressure $10^{-9}\,\mathrm{Pa}$, one would find [5] a mean refractive index n with $n^{(\mathrm{gas})} - 1 \simeq 10^{-17}$. In astrophysical environments, the refractive index would arise due to the interaction of EM waves with free electrons. One would then obtain [5]

$$n^{(\mathrm{gas})} - 1 = -\frac{\omega_\mathrm{p}^2}{2\omega^2}\,, \tag{1.63}$$

where

$$\omega_\mathrm{p} = \sqrt{\frac{N_\mathrm{e}}{\varepsilon_0}\frac{e^2}{m_\mathrm{e}}} \tag{1.64}$$

is the plasma frequency. (By N_e we have denoted the number density of the free electrons.) Note that for a plasma, one is faced with a refractive index which is smaller than one, meaning $\Delta^{(\mathrm{gas})} < 0$ and a phase velocity of EM propagation greater than c. The group velocity of course needs to be smaller then c.

Cotton-Mouton effect

Introducing a constant transverse background magnetic field B_T will generally give rise to a non-zero difference $n_\parallel^{(\mathrm{gas})} - n_\perp^{(\mathrm{gas})}$. In the case of neutral gases, this satisfies [5]

$$n_\parallel^{(\mathrm{gas})} - n_\perp^{(\mathrm{gas})} = \frac{2\pi c}{\omega} C B_\mathrm{T}^2 , \tag{1.65}$$

where C is the material-dependent Cotton-Mouton constant. For example, in the case of air evacuated down to a pressure of $10^{-9}\,\mathrm{Pa}$, one would find [5] the value $C = -5 \times 10^{-20}\,\mathrm{T}^{-2}\,\mathrm{m}^{-1}$. For a laser beam at energy $\hbar\omega = 2.4\,\mathrm{eV}$ (about $520\,\mathrm{nm}$) and a magnetic field $B_\mathrm{T} = 10\,\mathrm{T}$, one would obtain $n_\parallel^{(\mathrm{gas})} - n_\perp^{(\mathrm{gas})}$ of the order 10^{-24}, a very small number indeed.

Birefringence in plasma

On the other hand, in media where the EM waves interact with free electrons, one obtains [43] (in the limit $\omega \gg \omega_\mathrm{p}$ and $\omega \gg \omega_\mathrm{c}$)

$$n_\parallel^{(\mathrm{gas})} - n_\perp^{(\mathrm{gas})} = \frac{\omega_\mathrm{p}^2}{2\omega^2} \frac{\omega_\mathrm{c}^2}{\omega^2} , \tag{1.66}$$

where

$$\omega_\mathrm{c} = \frac{e B_\mathrm{T}}{m_\mathrm{e}} \tag{1.67}$$

is the cyclotron frequency of the background transverse magnetic field.

Vacuum birefringence

Finally, as we have advertised, the refractive indices $n_\parallel$ and $n_\perp$ receive non-zero contributions even in pure vacuum. These are due to loop effects in QED. Let us now discuss deriving these contributions starting from the Euler-Heisenberg lagrangian (1.5) [5, 40]. First, varying the corresponding term in the action with respect to the EM potential A_μ, we obtain

$$\delta \int d^4x\, \mathcal{L}_\mathrm{EH} = \frac{\alpha^2}{90 m_\mathrm{e}^4} \delta \int d^4x \left[\left(F_{\mu\nu} F^{\mu\nu} \right)^2 + \frac{7}{4} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \right] \tag{1.68a}$$

$$= \frac{\alpha^2}{90 m_\mathrm{e}^4} \int d^4x \left[4 F_{\mu\nu} F^{\mu\nu} F^{\alpha\beta} \delta F_{\alpha\beta} + \right.$$
$$\left. + \frac{7}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} \epsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma} \delta F_{\alpha\beta} \right] \tag{1.68b}$$

$$= -\frac{\alpha^2}{90 m_\mathrm{e}^4} \int d^4x\, \delta A_\beta\, \partial_\alpha \left[8 F_{\mu\nu} F^{\mu\nu} F^{\alpha\beta} + \right.$$
$$\left. + 7 F_{\mu\nu} \tilde{F}^{\mu\nu} \epsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma} \right] \tag{1.68c}$$

The Euler-Heisenberg lagrangian therefore contributes with a total divergence term

$$-\frac{\alpha^2}{90 m_\mathrm{e}^4} \partial_\alpha \left[8 F_{\mu\nu} F^{\mu\nu} F^{\alpha\beta} + 14 F_{\mu\nu} \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta} \right] \tag{1.69}$$

into the EM equations of motion. We will now linearize this term around the constant magnetic background $(F_{\rm ext})_{\mu\nu}$. First, we obtain

$$
\begin{aligned}
-\frac{\alpha^2}{90m_{\rm e}^4}\partial_\alpha\Big[16F_{\mu\nu}(F_{\rm ext})^{\mu\nu}(F_{\rm ext})^{\alpha\beta} + 8(F_{\rm ext})_{\mu\nu}(F_{\rm ext})^{\mu\nu}F^{\alpha\beta} + \\
+ 28F_{\mu\nu}(\tilde{F}_{\rm ext})^{\mu\nu}(\tilde{F}_{\rm ext})^{\alpha\beta} + 14(F_{\rm ext})_{\mu\nu}(\tilde{F}_{\rm ext})^{\mu\nu}\tilde{F}^{\alpha\beta}\Big]. \quad (1.70)
\end{aligned}
$$

We then recall that in the absence of the background electric field, we have already shown in (1.27) that $(\tilde{F}_{\rm ext})_{\mu\nu}(F_{\rm ext})^{\mu\nu} = 0$. Substituting

$$
(F^{\rm ext})^{\mu\nu}F^{\rm ext}_{\mu\nu} = 2(B_{\rm L}^2 + B_{\rm T}^2)\,, \quad (1.71)
$$

and using the Lorentz gauge condition (1.37) we can further manipulate the linearized Euler-Heisenberg contribution (1.70) to the the EM equations of motion into

$$
\begin{aligned}
-\frac{\alpha^2}{90m_{\rm e}^4}\Big[\Big(32(F_{\rm ext})^{\mu\nu}(F_{\rm ext})^{\alpha\beta} + 56(\tilde{F}_{\rm ext})^{\mu\nu}(\tilde{F}_{\rm ext})^{\alpha\beta}\Big)\partial_\alpha\partial_\mu A_\nu + \\
+ 16(B_{\rm L}^2 + B_{\rm T}^2)\Box A^\beta\Big]. \quad (1.72)
\end{aligned}
$$

Since the Euler-Heisenberg contribution represents already a small effect, we can safely put $\Box A^\beta \approx 0$. The remaining two terms can be explicitly evaluated case by case for $\beta = 1$ and $\beta = 2$ using the explicit form (1.14) of $(F_{\rm ext})_{\mu\nu}$ and the fact that there is no x and y dependence in the plane wave solution (1.36). For $\beta = 1$, we eventually obtain the contribution

$$
-\frac{28\alpha^2}{45m_{\rm e}^4}(\tilde{F}_{\rm ext})^{01}(\tilde{F}_{\rm ext})^{01}\partial_0\partial_0 A_1 = \frac{28\alpha^2}{45m_{\rm e}^4}(B_{\rm T})^2\omega^2 A_\parallel \quad (1.73)
$$

into the $A_\parallel$ equation of motion. Similarly, for $\beta = 2$, one obtains that the EH effective term contributes into the $A_\perp$ equation of motion with

$$
-\frac{16\alpha^2}{45m_{\rm e}^4}(F_{\rm ext})^{32}(F_{\rm ext})^{32}\partial_3\partial_3 A_2 = \frac{16\alpha^2}{45m_{\rm e}^4}(B_{\rm T})^2\omega^2 A_\perp\,. \quad (1.74)
$$

Hence, we can identify

$$
n_\parallel^{\rm (vac)} - 1 = \frac{\Delta_\parallel^{\rm (vac)}}{\omega} = \frac{14\alpha^2}{45m_{\rm e}^4}(B_{\rm T})^2 = \frac{7}{2}\xi(B_{\rm T})\,, \quad (1.75a)
$$

$$
n_\perp^{\rm (vac)} - 1 = \frac{\Delta_\perp^{\rm (vac)}}{\omega} = \frac{8\alpha^2}{45m_{\rm e}^4}(B_{\rm T})^2 = \frac{4}{2}\xi(B_{\rm T})\,, \quad (1.75b)
$$

where the parameter

$$
\xi(B_{\rm T}) = \frac{\alpha}{45\pi}\left(\frac{B_{\rm T}}{B_{\rm crit}}\right)^2 \quad (1.76)
$$

is defined in terms of the critical magnetic field [5]

$$
B_{\rm crit} = \frac{m_{\rm e}^2c^2}{e\hbar} \simeq 4.42\times10^9\,{\rm T}\,. \quad (1.77)
$$

In particular, we confirm that vacuum "polarized" by an external transverse magnetic field $B_{\rm T}$ behaves as a birefringent medium, as the difference

$$n_{\parallel}^{(\rm vac)} - n_{\perp}^{(\rm vac)} = \frac{3}{2}\xi(B_{\rm T}) = \frac{\alpha}{30\pi}\left(\frac{B_{\rm T}}{B_{\rm crit}}\right)^2 \tag{1.78}$$

is clearly non-zero. In a possibly-achievable laboratory setup with $B_{\rm T} \simeq 10\,{\rm T}$, one would obtain a very small refractive-index difference $n_{\parallel}^{(\rm vac)} - n_{\perp}^{(\rm vac)}$ of the order 10^{-22}. While this is still an extremely small number, we note that it represents an effect which is about 100 times larger than the matter Cotton-Mouton effect computed above for an evacuated air chamber and $\hbar\omega = 2.4\,{\rm eV}$. At the same time, for neutron stars, whose magnetic fields are typically comparable with $B_{\rm crit}$, we can see that the vacuum birefringence becomes a significant effect.

1.1.6 Equations of motion in momentum space

Let us finally substitute for the momentum dependence of the plane wave solution. We will therefore substitute

$$A_{\parallel}(z) = A_{\parallel}(p)e^{-ipz}\,, \tag{1.79a}$$
$$A_{\perp}(z) = A_{\perp}(p)e^{-ipz}\,, \tag{1.79b}$$
$$\phi(z) = \phi(p)e^{-ipz}\,. \tag{1.79c}$$

We will first focus on the $\parallel$ photon equation of motion, which can be written as

$$0 = \omega e^{-ipz}A_{\parallel}(p) - pe^{-ipz}A_{\parallel}(p) + \Delta_{\parallel}e^{-ipz}A_{\parallel}(p)\,, \tag{1.80a}$$
$$0 = (\omega - p + \Delta_{\parallel})e^{-ipz}A_{\parallel}(p)\,, \tag{1.80b}$$

that is

$$0 = (\omega - p + \Delta_{\parallel})A_{\parallel}(p)\,. \tag{1.81}$$

Let us now repeat the process also for the other two equations. The equation for ϕ gives

$$0 = \left[(-\omega + p + \frac{m_0^2}{2\omega})\phi(p) - i\frac{g_0 B_{\rm T}}{2\omega}pA_{\perp}(p)\right]e^{-ipz}\,, \tag{1.82}$$

that is

$$0 = \left(-\omega + p + \frac{m_0^2}{2\omega}\right)\phi(p) - i\frac{g_0 B_{\rm T}}{2\omega}pA_{\perp}(p)\,. \tag{1.83}$$

Finally, for $A_{\perp}$ we get

$$0 = (\omega - p + \Delta_{\perp})A_{\perp}(p) - i\frac{g_0 B_{\rm T}}{2\omega}p\phi(p)\,. \tag{1.84}$$

Note that the equations for $\perp$ and ϕ can be written in matrix form as

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}\omega - p - \Delta_0 & +iap\\ -iap & \omega - p + \Delta_{\perp}\end{pmatrix}\begin{pmatrix}\phi(p)\\ A_{\perp}(p)\end{pmatrix} \tag{1.85a}$$
$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}\omega - p - \Delta_0 & ap\\ ap & \omega - p + \Delta_{\perp}\end{pmatrix}\begin{pmatrix}\phi(p)\\ iA_{\perp}(p)\end{pmatrix}\,, \tag{1.85b}$$

where we have denoted

$$a = \frac{g_0 B_{\mathrm{T}}}{2\omega}\,. \tag{1.86}$$

Relabelling $iA_\perp \to A_\perp$ and recalling also the equation for $A_\parallel$ this finally gives the mixing equations

$$0 = (\omega - p + \Delta_\parallel) A_\parallel(p)\,, \tag{1.87a}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega - p - \Delta_0 & ap \\ ap & \omega - p + \Delta_\perp \end{pmatrix} \begin{pmatrix} \phi(p) \\ A_\perp(p) \end{pmatrix}. \tag{1.87b}$$

We observe that the parallel polarization $A_\parallel$ of the photon decouples and propagates independently without getting mixed with the scalar and the perpendicular photon polarization $A_\perp$.

1.2 Searching for mass eigenstates

To determine the possible values of p of the remaining mixture of ϕ and $A_\perp$ in terms of the energy ω, we use the condition that the determinant of the matrix

$$M(p) = \begin{pmatrix} \omega - p - \Delta_0 & ap \\ ap & \omega - p + \Delta_\perp \end{pmatrix} \tag{1.88}$$

appearing in (1.87b) has to be zero in order for the equation (1.87b) to have solutions. This will give us some relations of the form $p_1(\omega)$ and $p_2(\omega)$. These should then be interpreted as the *mass-shell (dispersion) relations* for two independently-propagating particle species which arise from the mixture of the scalar and the perpendicular photon polarization $A_\perp$. On the other hand the $A_\parallel$ polarization already propagates as a mass eigenstate with dispersion relation

$$p_\parallel(\omega) = \omega + \Delta_\parallel\,, \tag{1.89}$$

since it decouples from the rest and does not participate in the oscillations. As a result, the evolution of the $A_\parallel$ mode as it propagates through the magnetic field, is simply given as

$$A_\parallel(z) = A_\parallel(0) e^{-i(\omega+\Delta_\parallel)z} = A_\parallel(0) e^{-i\omega n_\parallel z}\,. \tag{1.90}$$

1.2.1 Diagonalization of $M(p)$

First, note that for the determinant of $M(p)$, we can write

$$\begin{aligned} \det M &= (\omega - p - \Delta_0)(\omega - p + \Delta_\perp) - a^2 p^2 && (1.91a) \\ &= (\omega - p)^2 + p(\Delta_0 - \Delta_\perp) - \Delta_0 \Delta_\perp - a^2 p^2 && (1.91b) \\ &= (1 - a^2) p^2 + p(\Delta_0 - \Delta_\perp - 2\omega) - \Delta_0 \Delta_\perp + \omega^2 + \omega(\Delta_\perp - \Delta_0)\,. && (1.91c) \end{aligned}$$

Let $p_1(\omega)$ and $p_2(\omega)$ be the two solutions of the equation

$$\det M = 0\,. \tag{1.92}$$

These give the dispersion relations

$$2(1-a^2)p_1(\omega) = 2\omega + \Delta_\perp - \Delta_0 - \sqrt{D}\,, \tag{1.93a}$$
$$2(1-a^2)p_2(\omega) = 2\omega + \Delta_\perp - \Delta_0 + \sqrt{D}\,, \tag{1.93b}$$

where D is the discriminant. We will shortly see that it will prove useful to note the corresponding Vieta's formulae

$$(1-a^2)p_1p_2 = \omega^2 + \omega(\Delta_\perp - \Delta_0) - \Delta_0\Delta_\perp\,, \tag{1.94a}$$
$$(1-a^2)(p_1+p_2) = 2\omega + \Delta_\perp - \Delta_0\,. \tag{1.94b}$$

It is then straightforward to see that the solutions $(\phi^{(1)}, A_\perp^{(1)})$ and $(\phi^{(2)}, A_\perp^{(2)})$ of (1.87b) which correspond to p_1 and p_2 satisfy

$$\frac{A_\perp^{(1)}}{\phi^{(1)}} = \frac{-ap_1}{\omega - p_1 + \Delta_\perp} \equiv -\tan\Theta_1\,, \tag{1.95a}$$
$$\frac{\phi^{(2)}}{A_\perp^{(2)}} = \frac{-ap_2}{\omega - p_2 - \Delta_0} \equiv +\tan\Theta_2\,. \tag{1.95b}$$

The definition of the angles Θ_1 and Θ_2 is such that they measure the angular distance (in the $\phi - A_\perp$ flavour space) of the solution 1 to the pure scalar state $(1,0)$ and of the solution 2 to the pure photon state $(0,1)$. In other words, the directions in the flavour space corresponding to the two mass eigenstates with the dispersion relations (1.93) are given by the normalized vectors

$$e_1 = \frac{1}{\sqrt{1+\tan^2\Theta_1}}\begin{pmatrix}1\\ -\tan\Theta_1\end{pmatrix}\,, \qquad e_2 = \frac{1}{\sqrt{1+\tan^2\Theta_2}}\begin{pmatrix}\tan\Theta_2\\ 1\end{pmatrix}\,. \tag{1.96}$$

The z-dependence of a generic plane-wave solution involving the scalar and $\perp$-photon polarization can then be written as a linear combination of the two mass-eigenstates, namely

$$\begin{pmatrix}\phi(z)\\ A_\perp(z)\end{pmatrix} = C_1e_1e^{-ip_1z} + C_2e_2e^{-ip_2z}\,, \tag{1.97a}$$
$$= \frac{C_1}{\sqrt{1+\tan^2\Theta_1}}\begin{pmatrix}1\\ -\tan\Theta_1\end{pmatrix}e^{-ip_1z} + {}$$
$$+ \frac{C_2}{\sqrt{1+\tan^2\Theta_2}}\begin{pmatrix}\tan\Theta_2\\ 1\end{pmatrix}e^{-ip_2z} \tag{1.97b}$$

for some constants C_1, C_2. Assuming that the system was prepared in an initial state

$$\begin{pmatrix}\phi(0)\\ A_\perp(0)\end{pmatrix} \tag{1.98}$$

at $z=0$, we obtain conditions

$$\phi(0) = \frac{C_1}{\sqrt{1+\tan^2\Theta_1}} + \frac{C_2\tan\Theta_2}{\sqrt{1+\tan^2\Theta_2}}\,, \tag{1.99a}$$

$$A_\perp(0) = -\frac{C_1 \tan\Theta_1}{\sqrt{1+\tan^2\Theta_1}} + \frac{C_2}{\sqrt{1+\tan^2\Theta_2}}\,, \tag{1.99b}$$

that is

$$C_1 = \frac{\sqrt{1+\tan^2\Theta_1}}{1+\tan\Theta_1\tan\Theta_2}\Big[\phi(0) - A_\perp(0)\tan\Theta_2\Big]\,, \tag{1.100}$$

$$C_2 = +\frac{\sqrt{1+\tan^2\Theta_2}}{1+\tan\Theta_1\tan\Theta_2}\Big[A_\perp(0) + \phi(0)\tan\Theta_1\Big]\,. \tag{1.101}$$

Altogether, we therefore find that the evolution of the ϕ–$A_\perp$ system with generic initial conditions $\phi(0)$ and $A_\perp(0)$ is described by the solution

$$\begin{aligned}\begin{pmatrix}\phi(z)\\ A_\perp(z)\end{pmatrix} &= \\ &= \frac{1}{1+\tan\Theta_1\tan\Theta_2}\begin{pmatrix}1\\ -\tan\Theta_1\end{pmatrix}\Big[\phi(0) - A_\perp(0)\tan\Theta_2\Big]e^{-ip_1(\omega)z} + \\ &\quad + \frac{1}{1+\tan\Theta_1\tan\Theta_2}\begin{pmatrix}\tan\Theta_2\\ 1\end{pmatrix}\Big[A_\perp(0) + \phi(0)\tan\Theta_1\Big]e^{-ip_2(\omega)z}\,.\end{aligned} \tag{1.102}$$

At the same time, we recall that for the decoupled polarization $A_\parallel(z)$, we simply obtain

$$A_\parallel(z) = A_\parallel(0)e^{-i\omega n_\parallel z}\,. \tag{1.103}$$

1.2.2 Transfer matrix

We can notice that the evolution equations (1.102) and (1.103) can be recast in a form familiar from quantum mechanics. Introducing a combined amplitude (“wave function” or “state”) in the three-dimensional $A_\parallel$–ϕ–$A_\perp$ flavour space as

$$\Psi(z) = \begin{pmatrix}A_\parallel(z)\\ \phi(z)\\ A_\perp(z)\end{pmatrix}\,, \tag{1.104}$$

the evolution of the system, as it propagates in the external magnetic field, can be succinctly expressed as

$$\Psi(z) = \mathsf{U}(z,0)\Psi(0)\,, \tag{1.105}$$

where we have introduced the unitary *transfer matrix* (or operator)

$$\mathsf{U}(z,0) = \mathsf{U}_\parallel(z,0)\mathsf{U}_\perp(z,0)\,. \tag{1.106}$$

The matrix $\mathsf{U}_\parallel(z,0)$ acts only on the $A_\parallel$ subsystem and is simply identified as

$$\mathsf{U}_\parallel(z,0) = \begin{pmatrix}e^{-ip_\parallel(\omega)z} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\,, \tag{1.107}$$

while for $\mathsf{U}_\perp(z,0)$, we have to write

$$\mathsf{U}_\perp(z,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$U(z,0)$}} \\ 0 & & \end{pmatrix}, \tag{1.108}$$

where the 2 by 2 submatrix $U(z,0)$ reads

$$\begin{aligned} U(z,0) = {} & \frac{1}{1+\tan\Theta_1\tan\Theta_2}\times \\ & \times \begin{pmatrix} e^{-ip_1 z} + e^{-ip_2 z}\tan\Theta_1\tan\Theta_2 & (e^{-ip_2 z} - e^{-ip_1 z})\tan\Theta_2 \\ (e^{-ip_2 z} - e^{-ip_1 z})\tan\Theta_1 & e^{-ip_2 z} + e^{-ip_1 z}\tan\Theta_1\tan\Theta_2 \end{pmatrix}. \end{aligned} \tag{1.109}$$

The unitarity of $U(z,0)$ (and hence the unitarity of $\mathsf{U}(z,0)$) can be checked to hold as the consequence of relations satisfied by $p_1(\omega)$, $p_2(\omega)$, as well as Θ_1, Θ_2.

1.2.3 $A_\perp$–ϕ oscillations

It follows that given the system (beam) was prepared at $z=0$ in a state $\Psi(0)\equiv\Psi_\mathrm{i}$, then the probability $P(\Psi_\mathrm{i}\to\Psi_\mathrm{f})$, that we will measure a state $\Psi(z)\equiv\Psi_\mathrm{f}$ in the beam after the beam has travelled over a distance z through the magnetic field, can be computed as

$$P(\Psi_\mathrm{i}\to\Psi_\mathrm{f}) = \frac{|\Psi_\mathrm{f}^\dagger \mathsf{U}(z,0)\Psi_\mathrm{i}|^2}{|\Psi_\mathrm{f}|^2|\Psi_\mathrm{i}|^2}. \tag{1.110}$$

For instance, one could be interested in the (transition) probability that one will find a scalar mode

$$\Psi_\phi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{1.111}$$

in a beam, which was initially prepared in a (normalized) pure-photon state

$$\Psi_A = \frac{1}{\sqrt{|A_\parallel(0)|^2 + |A_\perp(0)|^2}} \begin{pmatrix} A_\parallel(0) \\ 0 \\ A_\perp(0) \end{pmatrix}. \tag{1.112}$$

We obtain

$$(\Psi_\phi)^\dagger \mathsf{U}(z,0)\Psi_A =$$

$$= \frac{1}{\sqrt{|A_\parallel(0)|^2 + |A_\perp(0)|^2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \mathsf{U}(z,0) \begin{pmatrix} A_\parallel(0) \\ 0 \\ A_\perp(0) \end{pmatrix} \tag{1.113a}$$

$$\begin{aligned} = {} & \frac{1}{\sqrt{|A_\parallel(0)|^2 + |A_\perp(0)|^2}} \frac{1}{1+\tan\Theta_1\tan\Theta_2}\times \\ & \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} A_\parallel(0) e^{-ip_\parallel(\omega) z}(1+\tan\Theta_1\tan\Theta_2) \\ A_\perp(0)(e^{-ip_2 z} - e^{-ip_1 z})\tan\Theta_2 \\ A_\perp(0)(e^{-ip_2 z} + e^{-ip_1 z}\tan\Theta_1\tan\Theta_2) \end{pmatrix} \end{aligned} \tag{1.113b}$$

$$= \frac{A_\perp(0)}{\sqrt{|A_\parallel(0)|^2 + |A_\perp(0)|^2}} \frac{\tan\Theta_2}{1+\tan\Theta_1\tan\Theta_2} (e^{-ip_2 z} - e^{-ip_1 z}) \,, \qquad (1.113c)$$

that is, using the fact that the two states Ψ_A and Ψ_ϕ defined in (1.112) and (1.111) were already normalized, we get the transition probability

$$P(\Psi_A \to \Psi_\phi; z) =$$

$$= |(\Psi_\phi)^\dagger \mathsf{U}(z,0)\Psi_A|^2 \qquad (1.114a)$$

$$= \frac{(A_\perp(0))^2}{(A_\parallel(0))^2 + (A_\perp(0))^2} \frac{4\tan^2\Theta_2}{(1+\tan\Theta_1\tan\Theta_2)^2} \sin^2\frac{(p_1-p_2)z}{2} \,. \qquad (1.114b)$$

This can be directly interpreted as a (measurable) relative decrease in the intensity of the photon component of the beam. We can see that $P(\Psi_A \to \Psi_\phi; z)$ oscillates with length $l_{\text{osc}} = \frac{2\pi}{p_1-p_2}$: indeed, while the transition probability increases from 0 as we increase z to reach a maximum at $z = l_{\text{osc}}/2$, it will again fall to zero when z is an integer multiple of l_{osc}. This behaviour is a direct consequence of having initially prepared the system in a state which was not a pure mass eigenstate of the system in a background magnetic field – situation which is quite analogous to the well known case of neutrino oscillations. As expected, the amplitude of these oscillations is maximized when $A_\parallel(0) = 0$, because the parallel polarization does not mix with the scalar. Also, since the transfer matrix $\mathsf{U}(z,0)$ is unitary, the probability of measuring *any* photon state in the beam is simply equal to $1 - P(\Psi_A \to \Psi_\phi)$.

1.2.4 Simplification of the transfer matrix

Our objective now will be to simplify the above derived expressions for the mixing angles and the dispersion relations $p_1(\omega)$, $p_2(\omega)$ given that most efficient mixing between the photon and the scalar should take place in the ultrarelativistic regime. Based on the simplified formulae derived in this section, we will then work through two interesting limits of the mixing angle.

To this end, let us first denote

$$\epsilon = \frac{g_0 B_{\mathrm{T}}}{m_0} \,. \qquad (1.115)$$

where we recall that for large enough m_0 (grater than $1/L$), we need to assume that $\epsilon \ll 1$ in order for $g_0\phi^{\text{ext}} \ll 1$ and hence in order for the perturbative lagrangian description to be valid. On the other hand, in the regime $m_0 \leq 1/L$ (which is relevant for the limit $m_0 \to 0$), we would need to assume $g_0 B_{\mathrm{T}} L \ll 1$.

Let us then consider the discriminant D of the quadratic equation

$$0 = (1-a^2)p^2 + p(\Delta_0 - \Delta_\perp - 2\omega) - \Delta_0\Delta_\perp + \omega^2 + \omega(\Delta_\perp - \Delta_0) \qquad (1.116)$$

resulting from the condition that the determinant of $M(p)$ needs to vanish. We can first manipulate it as

$$D = (\Delta_0 - \Delta_\perp - 2\omega)^2 - 4(1-a^2)(\omega(\Delta_\perp - \Delta_0) + \omega^2 - \Delta_0\Delta_\perp) \qquad (1.117a)$$

$$= 4\omega^2 + (\Delta_0 - \Delta_\perp)^2 - 4\omega(\Delta_0 - \Delta_\perp) - 4\omega^2 + 4\Delta_0\Delta_\perp +$$

$$
\begin{aligned}
&\quad - 4\omega(\Delta_\perp - \Delta_0) + 4a^2(\omega(\Delta_\perp - \Delta_0) + \omega^2 - \Delta_0\Delta_\perp) && (1.117\text{b}) \\
&= (\Delta_0 - \Delta_\perp)^2 + 4\Delta_0\Delta_\perp + 4a^2(\omega(\Delta_\perp - \Delta_0) + \omega^2 - \Delta_0\Delta_\perp) && (1.117\text{c}) \\
&= (\Delta_0 + \Delta_\perp)^2 + 4\omega^2 a^2\Big(1 + \frac{\Delta_\perp - \Delta_0}{\omega} - \frac{\Delta_0\Delta_\perp}{\omega^2}\Big) . && (1.117\text{d})
\end{aligned}
$$

We will now analyze the magnitudes of the individual terms. In particular, we have

$$
\begin{aligned}
(\Delta_0 + \Delta_\perp)^2 &\sim \frac{\Delta^2}{\omega^2}\omega^2 , && (1.118\text{a}) \\
4a^2\omega^2 &\sim \epsilon^2 \frac{\Delta}{\omega}\omega^2 , && (1.118\text{b}) \\
4a^2\omega^2 \frac{\Delta_\perp - \Delta_0}{\omega} &\sim \frac{\Delta^2}{\omega^2}\epsilon^2\omega^2 , && (1.118\text{c}) \\
4a^2\omega^2 \frac{\Delta_0\Delta_\perp}{\omega^2} &\sim \frac{\Delta^2}{\omega^2}\epsilon^2\frac{\Delta}{\omega}\omega^2 . && (1.118\text{d})
\end{aligned}
$$

We can therefore see that in the ultrarelativistic limit (1.49), which we now write as

$$
\frac{\Delta}{\omega} \ll 1 , \tag{1.119}
$$

(where Δ can be any of $\Delta_0, \Delta_\perp, \Delta_\parallel$) we can safely neglect the third and the fourth term in (1.118) compared to the first two (whose relative size will depend on the ratio of (Δ/ω) and ϵ^2 and will be discussed below). Hence, we can approximately write

$$
\begin{aligned}
D &\approx (\Delta_0 + \Delta_\perp)^2 + 4\omega^2 a^2 && (1.120\text{a}) \\
&= \omega^2\Big[4a^2 + \Big(\frac{\Delta_0 + \Delta_\perp}{\omega}\Big)^2\Big] , && (1.120\text{b})
\end{aligned}
$$

that is

$$
\sqrt{D} \approx \omega\sqrt{4a^2 + \Big(\frac{\Delta_0 + \Delta_\perp}{\omega}\Big)^2} . \tag{1.121}
$$

Also note that

$$
a = \frac{g_0 B_\mathrm{T}}{2\omega} = \epsilon\frac{m_0}{2\omega} , \tag{1.122}
$$

meaning that we automatically get $a \ll 1$ in the regime $m_0 \gg 1/L$ due to the relativistic approximation and the consistency condition $\epsilon \ll 1$. In the regime $m_0 \leq 1/L$, it is useful to note that

$$
a = g_0 B_\mathrm{T} L \frac{1}{2\omega L} , \tag{1.123}
$$

where $g_0 B_\mathrm{T} L \ll 1$ due to perturbative consistency of the lagrangian and $1/(\omega L) = \lambda/L \ll 1$, so that a large number of waves can fit inside the region with non-zero background magnetic field. Hence, in both regimes we get

$$
a \ll 1 . \tag{1.124}
$$

For the solutions p_1 and p_2, we therefore have

$$p_1 \approx \left(\omega + \frac{\Delta_\perp - \Delta_0 - \sqrt{D}}{2}\right)(1 + a^2) \tag{1.125a}$$

$$\approx \omega + \frac{\Delta_\perp - \Delta_0 - \sqrt{D}}{2} + a^2\omega\,, \tag{1.125b}$$

where we have neglected products of small quantities. Similarly we have

$$p_2 = \omega + \frac{\Delta_\perp - \Delta_0 + \sqrt{D}}{2} + a^2\omega\,. \tag{1.126}$$

Multiplying by a, we can therefore drop everything except for the leading contribution, namely

$$ap_1 \approx a\omega\,, \tag{1.127a}$$

$$ap_2 \approx a\omega\,. \tag{1.127b}$$

Finally, we can establish

$$-(\omega - p_1 + \Delta_\perp) = a^2\omega + \frac{\Delta_\perp - \Delta_0 - \sqrt{D}}{2} - \Delta_\perp \tag{1.128a}$$

$$= a^2\omega - \frac{\Delta_0 + \Delta_\perp}{2} - \frac{\sqrt{D}}{2} \tag{1.128b}$$

$$\approx \omega\left[-\frac{1}{2}\sqrt{4a^2 + \left(\frac{\Delta_0 + \Delta_\perp}{\omega}\right)^2} - \frac{1}{2}\frac{\Delta_0 + \Delta_\perp}{\omega} + a^2\right] \tag{1.128c}$$

$$= \frac{1}{2}(\Delta_0 + \Delta_\perp)\left[-\sqrt{1 + \frac{4a^2\omega^2}{(\Delta_0 + \Delta_\perp)^2}} - 1 + a\frac{2a\omega}{\Delta_0 + \Delta_\perp}\right] \tag{1.128d}$$

and similarly

$$(\omega - p_2 - \Delta_0) \approx \frac{1}{2}(\Delta_0 + \Delta_\perp)\left[-\sqrt{1 + \frac{4a^2\omega^2}{(\Delta_0 + \Delta_\perp)^2}} - 1 - a\frac{2a\omega}{\Delta_0 + \Delta_\perp}\right]. \tag{1.129}$$

Recalling that we have defined

$$\tan\Theta_1 = -\frac{ap_1}{-(\omega - p_1 + \Delta_\perp)}\,, \tag{1.130a}$$

$$\tan\Theta_2 = -\frac{ap_2}{\omega - p_2 - \Delta_0}\,, \tag{1.130b}$$

we therefore obtain

$$\tan\Theta_1 \approx -\frac{2a\omega}{\Delta_0 + \Delta_\perp}\frac{1}{-\sqrt{1 + \frac{4a^2\omega^2}{(\Delta_0+\Delta_\perp)^2}} - 1 + a\frac{2a\omega}{\Delta_0+\Delta_\perp}}\,, \tag{1.131a}$$

$$\tan\Theta_2 \approx -\frac{2a\omega}{\Delta_0 + \Delta_\perp}\frac{1}{-\sqrt{1 + \frac{4a^2\omega^2}{(\Delta_0+\Delta_\perp)^2}} - 1 - a\frac{2a\omega}{\Delta_0+\Delta_\perp}}\,. \tag{1.131b}$$

Hence, denoting

$$y \equiv \frac{2a\omega}{\Delta_0 + \Delta_\perp} = \frac{g_0 B_\mathrm{T}}{\Delta_0 + \Delta_\perp}\,, \tag{1.132}$$

we can rewrite (1.131) as

$$\tan\Theta_1 \approx \frac{y}{\sqrt{1+y^2}+1-ay}\,, \tag{1.133a}$$

$$\tan\Theta_2 \approx \frac{y}{\sqrt{1+y^2}+1+ay}\,. \tag{1.133b}$$

1.2.5 Mixing limits

We will now distinguish two limits depending on the value of the parameter y: the small mixing scenario, which arises for $y \ll 1$ and the large mixing scenario $y \gg 1$.

Small mixing

Let us first assume $y \ll 1$, that is $g_0 B_\mathrm{T} \ll \Delta_0 + \Delta_\perp$. This gives

$$\tan\Theta_1 \approx \tan\Theta_2 \approx \frac{y}{2} = \frac{1}{2}\frac{g_0 B_\mathrm{T}}{\Delta_0 + \Delta_\perp} \ll 1\,, \tag{1.134}$$

meaning that this lands us at the regime of small mixing between the scalar and the $A_\perp$ polarization of the photon. The square-root of the discriminant can now be manipulated as

$$\sqrt{D} \approx \omega\sqrt{4a^2 + \left(\frac{\Delta_0+\Delta_\perp}{\omega}\right)^2} \tag{1.135a}$$

$$= (\Delta_0 + \Delta_\perp)\sqrt{1+y^2} \tag{1.135b}$$

$$\approx (\Delta_0 + \Delta_\perp)\left(1+\frac{y^2}{2}\right). \tag{1.135c}$$

For the solution p_1, we therefore obtain

$$p_1 = \omega + \frac{\Delta_\perp - \Delta_0 - \sqrt{D}}{2} + a^2\omega \tag{1.136a}$$

$$\approx \omega + \frac{\Delta_\perp - \Delta_0 - (\Delta_0+\Delta_\perp)(1+\frac{y^2}{2})}{2} + a^2\omega \tag{1.136b}$$

$$= \omega - \Delta_0 - \frac{y^2}{4}(\Delta_0+\Delta_\perp) + a^2\omega\,. \tag{1.136c}$$

Finally, substituting $a^2 = \frac{g_0^2 B_\mathrm{T}^2}{4\omega^2}$ and using the relativistic approximation (1.119), we end up with

$$p_1 = \omega - \Delta_0 - \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0+\Delta_\perp} + \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\omega} \tag{1.137a}$$

$$\approx \omega - \Delta_0 - \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0+\Delta_\perp}\,. \tag{1.137b}$$

Similarly, we have

$$p_2 = \omega + \frac{\Delta_\perp - \Delta_0 + \sqrt{D}}{2} + a^2\omega \tag{1.138a}$$

$$\approx \omega + \frac{\Delta_\perp - \Delta_0 + (\Delta_0 + \Delta_\perp)(1 + \frac{y^2}{2})}{2} + a^2\omega \tag{1.138b}$$

$$= \omega + \Delta_\perp + \frac{y^2}{4}(\Delta_0 + \Delta_\perp) + a^2\omega \tag{1.138c}$$

$$= \omega + \Delta_\perp + \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0 + \Delta_\perp} + \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\omega} \tag{1.138d}$$

$$\approx \omega + \Delta_\perp + \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0 + \Delta_\perp}\,. \tag{1.138e}$$

In total, in the small-mixing scenario we obtain the dispersion relations

$$p_1(\omega) = \omega - \Delta_0 - \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0 + \Delta_\perp}\,, \tag{1.139a}$$

$$p_2(\omega) = \omega + \Delta_\perp + \frac{1}{4}\frac{g_0^2 B_\mathrm{T}^2}{\Delta_0 + \Delta_\perp} \tag{1.139b}$$

for the scalar-like and the photon-like particles, with the mixing angle

$$\Theta_1 \approx \Theta_2 \approx \Theta = \frac{1}{2}\frac{g_0 B_\mathrm{T}}{\Delta_0 + \Delta_\perp}\,. \tag{1.140}$$

Note that by introducing the mass parameter

$$b \equiv \Delta_0 + \Delta_\perp\,, \tag{1.141}$$

we can rewrite the dispersion relations (1.139) using the refractive index $n_\perp$ (which was defined by (1.60b)) as

$$p_1(\omega) = \omega n_\perp - b(1 + \Theta^2)\,, \tag{1.142a}$$

$$p_2(\omega) = \omega n_\perp + b\Theta^2\,. \tag{1.142b}$$

The directions in the flavour space corresponding to the two mass eigenstates are specified by the normalized vectors

$$e_1 = \frac{1}{\sqrt{1+\Theta^2}}\begin{pmatrix}1\\ -\Theta\end{pmatrix}\,, \qquad e_2 = \frac{1}{\sqrt{1+\Theta^2}}\begin{pmatrix}\Theta\\ 1\end{pmatrix}\,, \tag{1.143}$$

respectively. These can be used to write down the general solution for the ϕ–$A_\perp$ oscillations in the small-mixing scenario

$$e^{i\omega n_\perp z}\begin{pmatrix}\phi(z)\\ A_\perp(z)\end{pmatrix} = \frac{1}{1+\Theta^2}\begin{pmatrix}1\\ -\Theta\end{pmatrix}\Big[\phi(0) - A_\perp(0)\Theta\Big]e^{ib(1+\Theta^2)z} + \\ + \frac{1}{1+\Theta^2}\begin{pmatrix}\Theta\\ 1\end{pmatrix}\Big[A_\perp(0) + \phi(0)\Theta\Big]e^{-ib\Theta^2 z}\,. \tag{1.144}$$

The corresponding sub-matrix $U(z,0)$ of the full transfer matrix $\mathsf{U}(z,0)$ for the combined $A_\parallel$–ϕ–$A_\perp$ system (see (1.108) for definitions and discussion) therefore reads

$$U(z,0) = \frac{e^{-i\omega n_\perp z}}{1+\Theta^2}\begin{pmatrix} e^{ib(1+\Theta^2)z} + \Theta^2 e^{-ib\Theta^2 z} & \Theta(e^{-ib\Theta^2 z} - e^{ib(1+\Theta^2)z}) \\ \Theta(e^{-ib\Theta^2 z} - e^{ib(1+\Theta^2)z}) & \Theta^2 e^{ib(1+\Theta^2)z} + e^{-ib\Theta^2 z}\end{pmatrix}\,, \tag{1.145}$$

whose unitarity can be explicitly checked. This enables to rewrite (1.144) as

$$\begin{pmatrix} \phi(z) \\ A_\perp(z) \end{pmatrix} = U(z,0) \begin{pmatrix} \phi(0) \\ A_\perp(0) \end{pmatrix} . \tag{1.146}$$

Substituting this into the expressions (1.108) and (1.106) for the transition matrix, one then obtains the transition probability $P(\Psi_A \to \Psi_\phi; z)$ as

$$\begin{aligned} P(\Psi_A \to \Psi_\phi; z) &= \\ &= \frac{1}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \mathsf{U}(z,0) \begin{pmatrix} A_\parallel(0) \\ 0 \\ A_\perp(0) \end{pmatrix} \right|^2 && \text{(1.147a)} \\ &= \frac{1}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \frac{1}{(1+\Theta^2)^2} \times \\ &\quad \times \left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} e^{-i\omega n_\parallel z} A_\parallel(0)(1+\Theta^2) \\ e^{-i\omega n_\perp z} \Theta (e^{-ib\Theta^2 z} - e^{ib(1+\Theta^2)z}) A_\perp(0) \\ e^{-i\omega n_\perp z} (\Theta^2 e^{ib(1+\Theta^2)z} + e^{-ib\Theta^2 z}) A_\perp(0) \end{pmatrix} \right|^2 && \text{(1.147b)} \\ &\approx \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \times 4\Theta^2 \sin^2 \frac{bz}{2} && \text{(1.147c)} \end{aligned}$$

where we have kept only leading terms in the (small) mixing angle Θ. (This also corresponds to a relative decrease in the intensity of the photon component of the beam.) Again, we observe that the probability $P(\Psi_A \to \Psi_\phi; z)$ (which is the same as the relative intensity of the scalar component of the beam) oscillates with length

$$l_{\text{osc}} = \frac{2\pi}{b} = \frac{2\pi}{\Delta_0 + \Delta_\perp} . \tag{1.148}$$

The amplitude of these oscillations is maximized when $A_\parallel(0) = 0$ (i.e. no admixture of the parallel polarization which does not contribute to the mixing). This maximal value first occurs for $z = l_{\text{osc}}/2$ and is simply given by the mixing angle as

$$P_{\max}(\Psi_A \to \Psi_\phi) = 4\Theta^2 = \left(\frac{g_0 B_{\mathrm{T}}}{\Delta_0 + \Delta_\perp} \right)^2 . \tag{1.149}$$

Focusing on the case when $A_\parallel(0) = 0$, if our measurements are carried out over a sufficiently large length scale $z \gg l_{\text{osc}}$, one will find a mean transition probability (relative decrease of photon intensity)

$$\langle P(\Psi_{A_\perp} \to \Psi_\phi; z) \rangle = 2\Theta^2 = \frac{1}{2} \left(\frac{g_0 B_{\mathrm{T}}}{\Delta_0 + \Delta_\perp} \right)^2 . \tag{1.150}$$

On the other hand, for $z \ll l_{\text{osc}}$, the transition probability can be well approximated as

$$P(\Psi_{A_\perp} \to \Psi_\phi; z) \approx \Theta^2 b^2 z^2 = \frac{1}{4} g_0^2 B_{\mathrm{T}}^2 z^2 . \tag{1.151}$$

In this limit, the transition probability therefore becomes insensitive to the mass m_0 of the scalar and the refractive index $n_\perp$. It only depends on the coupling g_0.

Large mixing

We will now on the other hand assume $y \gg 1$, that is $g_0 B_\mathrm{T} \gg \Delta_0 + \Delta_\perp$. Note that in principle, this could be achievable by experimentally tuning

$$\Delta_\perp \approx -\Delta_0 = -\frac{m_0^2}{2\omega} \tag{1.152}$$

in materials (such as plasma), where one can arrange for $\Delta_\perp < 0$. Alternatively, one could arrange for separately having $\Delta_0 \ll g_0 B_\mathrm{T}$, as well as $|\Delta_\perp| \ll g_0 B_\mathrm{T}$.

Reflecting the limit $y \gg 1$ on the mixing angles, we obtain

$$\tan\Theta_1 \approx -y\frac{1}{-y-1+ay} \approx 1\,, \tag{1.153a}$$
$$\tan\Theta_2 \approx -y\frac{1}{-y-1-ay} \approx 1\,, \tag{1.153b}$$

meaning that we have

$$\Theta_1 \approx \Theta_2 = \frac{\pi}{4}\,, \tag{1.154}$$

namely the case of the maximum mixing. For $y \gg 1$, the ϕ-$A_\perp$ mixing problem therefore becomes precisely analogous to the resonant regime of the *Mikheyev–Smirnov–Wolfenstein effect* in the context of neutrino oscillations in matter. [44, 45] We then also have

$$\frac{\sqrt{D}}{\omega} \approx 2a\sqrt{1+\frac{1}{y^2}} \approx 2a\left(1+\frac{1}{2y^2}\right)\,, \tag{1.155}$$

so that the dispersion relations can be rewritten as

$$p_1 = \omega + \frac{\Delta_\perp - \Delta_0 - \sqrt{D}}{2} + a^2\omega \tag{1.156a}$$
$$= \omega + \frac{\Delta_\perp - \Delta_0}{2} - a\omega + a^2\omega \tag{1.156b}$$
$$\approx \omega(1 + \frac{\Delta_\perp - \Delta_0}{2\omega} - a) \tag{1.156c}$$
$$\approx \omega(1-a)\,, \tag{1.156d}$$

and similarly

$$p_2 \approx \omega(1+a)\,. \tag{1.157}$$

In total, the maximum mixing is therefore associated with the mass-shell (dispersion) relations

$$p_1(\omega) = \omega(1-a)\,, \tag{1.158a}$$
$$p_2(\omega) = \omega(1+a)\,, \tag{1.158b}$$

which correspond to the normalized states

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\,, \qquad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\,, \tag{1.159}$$

in the $\phi - A_\perp$ flavour space. As we have noted above, this large mixing scenario arises as a consequence of the resonant condition

$$\Delta_0 + \Delta_\perp \approx 0\,. \tag{1.160}$$

One of the ways through which this may arise is when we take the separate limits $m_0 \to 0$ and $\Delta_\perp \to 0$. We can actually check that performing these limits at the level of the original momentum-space equations of motion (1.87b), one would directly obtain the solutions (1.158) for $p_1(\omega)$ and $p_2(\omega)$.

Oscillations in the ϕ–$A_\perp$ flavour space are therefore described by the solution

$$e^{i\omega z}\begin{pmatrix}\phi(z)\\ A_\perp(z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}\Big[\phi(0) - A_\perp(0)\Big]e^{\frac{1}{2}ig_0B_\mathrm{T}z} + \\ + \frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\Big[A_\perp(0) + \phi(0)\Big]e^{-\frac{1}{2}ig_0B_\mathrm{T}z}\,. \tag{1.161}$$

which can be again conveniently rewritten as

$$\begin{pmatrix}\phi(z)\\ A_\perp(z)\end{pmatrix} = U(z,0)\begin{pmatrix}\phi(0)\\ A_\perp(0)\end{pmatrix}\,, \tag{1.162}$$

where the unitary sub-matrix $U(z,0)$ of the full transfer matrix $\mathsf{U}(z,0)$ reads

$$U(z,0) = e^{-i\omega z}\begin{pmatrix}\cos g_0B_\mathrm{T}z & -i\sin g_0B_\mathrm{T}z\\ -i\sin g_0B_\mathrm{T}z & \cos g_0B_\mathrm{T}z\end{pmatrix}\,. \tag{1.163}$$

Similarly as in the case of large mixing, one can calculate the transition probability $P(\Psi_A \to \Psi_\phi; z)$ as (assuming that the photon beam has no parallel component, that is $A_\parallel(0) = 0$, to maximize the transition probability)

$$P(\Psi_{A_\perp} \to \Psi_\phi; z) = \left|\begin{pmatrix}0\\1\\0\end{pmatrix}^\dagger \mathsf{U}(z,0)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \sin^2 g_0B_\mathrm{T}z\,. \tag{1.164}$$

In contrast to the small-mixing scenario, this time the probability oscillates with oscillation length

$$l_\mathrm{osc} = \frac{\pi}{g_0B_\mathrm{T}}\,, \tag{1.165}$$

with amplitude equal to 1 and is completely insensitive to the mass scale m_0 and the refractive indices of the medium. This means that after the beam travels through the magnetic field a distance $z = \frac{l_\mathrm{osc}}{2}$, the photon component $A_\perp$ of the beam should completely die out. Provided that the scale of l_osc is much less than the extent of the magnetic field, we can average over the oscillations and find that the mean transition probability is simply $\frac{1}{2}$. However, note that one should be cautious about this regime, as it fails to satisfy the perturbativity condition $g_0B_\mathrm{T}L \ll 1$ on the validity of the lagrangian (1.1). On the other hand, for $z \ll l_\mathrm{osc}$, we can approximate

$$P(\Psi_{A_\perp} \to \Psi_\phi; z) \approx g_0^2B_\mathrm{T}^2z^2\,, \tag{1.166}$$

which is large by a factor of four than the corresponding result obtained for large mixing.

1.3 Observable effects

In this section we will use the results derived up to this point in order to discuss certain (potentially) observable effects due to the mixing between photons and ultrarelativistic massive spin-0 particles in an external EM field. We will focus on the minimal-mixing scenario, as this should be (comparably) less difficult to achieve in practice.

1.3.1 Effects on photon polarization

Let us start by considering the laboratory setup, where we take a source of linearly polarized photons (laser beam) and shine them through a magnetic field. We will now try to exploit the fact that the two photon polarizations $A_\parallel$ and $A_\perp$ are affected differently by the mixing with the massive scalar field ϕ.

Let us first recall the experimental configuration, which will be taken to be the same as we have considered throughout this chapter: we consider a beam photons propagating along the z axis with the magnetic field having components B_T perpendicular to the z-axis and B_L parallel to it. The EM four-potential will have spatial components lying in the plane perpendicular to the line of propagation: $A_1 \equiv A_\parallel$ which is parallel to B_T and $A_2 \equiv A_\perp$, which is perpendicular to B_T. We will assume that the environment in which the photons propagate, has refractive indices $n_\parallel$ and $n_\perp$ for the polarizations $A_\parallel$ and $A_\perp$, respectively. These can be thought of as arising due to both the presence of free electrons, as well as due to the effect of vacuum birefringence.

Relative amplitude decrease $\eta(z)$ and phase delay $\varphi(z)$

Clearly, in such a setup, the system is initially prepared in a pure photon state $A_\parallel(0)$, $A_\perp(0)$ and $\phi(0) = 0$. After entering the region with the magnetic field at $z = 0$, the solution for the ϕ–$A_\perp$ oscillations, as well as for the decoupled propagation of the $A_\parallel$ polarization, will therefore read

$$e^{i\omega n_\parallel z} A_\parallel(z) = A_\parallel(0)\,, \tag{1.167a}$$

$$e^{i\omega n_\perp z} A_\perp(z) = \frac{1}{1+\Theta^2}\Big[\Theta^2 e^{ib(1+\Theta^2)z} + e^{-ib\Theta^2 z}\Big] A_\perp(0)\,, \tag{1.167b}$$

$$e^{i\omega n_\perp z} \phi(z) = -\frac{\Theta}{1+\Theta^2}\Big[e^{-ib\Theta^2 z} - e^{ib(1+\Theta^2)z}\Big] A_\perp(0)\,. \tag{1.167c}$$

We observe that as a consequence of the mixing between the $A_\perp$ polarization and the scalar field ϕ in the magnetic field (as well as due to the refractive indices $\Delta_\parallel$, $\Delta_\perp$), the two EM polarizations behave differently as they propagate throught the region with non-zero B_T. In particular, for the ratio of the two EM modes we obtain

$$\frac{A_\perp(z)}{A_\parallel(z)} = e^{-i\omega(n_\perp - n_\parallel)z} \frac{1}{1+\Theta^2}\Big[\Theta^2 e^{ib(1+\Theta^2)z} + e^{-ib\Theta^2 z}\Big] \frac{A_\perp(0)}{A_\parallel(0)}\,. \tag{1.168}$$

Since we expect the change in this ratio as a function of z to be very small, let us introduce the parametrization

$$\frac{A_\perp(z)}{A_\parallel(z)} = \frac{A_\perp(0)}{A_\parallel(0)}\Big[1 - \eta(z)\Big] e^{-i\varphi(z)}\,, \tag{1.169}$$

where $\eta(z)$ measures the decrease in the amplitude of the $A_\perp$ mode relative to the $A_\parallel$ mode while $\varphi(z)$ represents a phase delay. Assuming both η and φ to be small this can be rewritten as

$$\frac{A_\perp(z)/A_\perp(0)}{A_\parallel(z)/A_\parallel(0)} = 1 - \eta(z) - i\varphi(z)\,. \tag{1.170}$$

In order for $\eta, \varphi \ll 1$ to be indeed true, we will first assume that the experiment is set up so that

$$\omega(n_\perp - n_\parallel)z \ll 1\,. \tag{1.171}$$

This is crucial in order for the refractive indices of the environment not to cause decoherence of the two EM polarizations which would obscure the effects due to the mixing with the scalar field which are to be expected as very small, namely

$$b\Theta^2 z \ll 1\,. \tag{1.172}$$

In practice, assuming a conceivable laboratory setup as in section 1.1.5 (with laser frequency $\omega = 2.4\,\mathrm{eV}$, transverse magnetic field $10\,\mathrm{T}$ and a chamber evacuated down to a pressure of $10^{-9}\,\mathrm{Pa}$), where we have shown that the vacuum contribution to birefringence dominates over the matter Cotton-Mouton effect, we would obtain

$$\frac{\omega}{c}(n_\parallel - n_\perp) \simeq 10^{-15}\,\mathrm{m}^{-1}\,, \tag{1.173}$$

so we are safe to assume that (1.171) is valid for any practical value of z. Expanding the r.h.s. of (1.168) using these assumptions, one obtains

$$\frac{A_\perp(z)/A_\perp(0)}{A_\parallel(z)/A_\parallel(0)} = 1 - 2\Theta^2 \sin^2\frac{bz}{2} - i\omega(n_\perp - n_\parallel)z + i\Theta^2(\sin bz - bz)\,, \tag{1.174}$$

so that one can identify the relative decrease in the amplitude $\eta(z)$ and the phase delay $\varphi(z)$ to be

$$\eta(z) = 2\Theta^2 \sin^2 \tfrac{bz}{2}\,, \tag{1.175a}$$
$$\varphi(z) = \omega(n_\perp - n_\parallel)z + \Theta^2\Big(bz - \sin bz\Big)\,. \tag{1.175b}$$

Moreover, one can also choose to focus on the case when $bz \ll 1$. That is to say, we will assume that the length z is much smaller than the oscillation length $l_{\mathrm{osc}} = 2\pi/b$, so that the EM wave $A_\perp$ and the scalar wave ϕ remain coherent as they propagate through the magnetic medium. Notice that these are stronger conditions than the ones imposed by (1.171) and (1.172). Substituting again explicitly for our typical laboratory setup as above (recalling from section 1.1.5 that $n - 1 \approx 10^{-17}$ for air at pressure $10^{-9}\,\mathrm{Pa}$), we would get

$$\frac{\omega}{c}(n_\perp - 1) \simeq 10^{-10}\,\mathrm{m}^{-1}\,, \tag{1.176}$$

so that one can safely assume $\Delta_\perp z \ll 1$ for any values of z relevant for a laboratory. If, moreover, the mass of the scalar is small enough as well, so that we can put $bz \ll 1$, the expressions (1.175) then simplify as

$$\eta(z) \approx \frac{1}{2}\Theta^2 b^2 z^2\,, \tag{1.177a}$$

$$\varphi(z) \approx \omega(n_\perp - n_\parallel)z + \frac{1}{6}\Theta^2 b^3 z^3\,. \tag{1.177b}$$

On the other hand, if the mass m_0 of the scalar were large enough, so that the length z over which the beam propagates through the magnetic field is much larger than the oscillation length $l_{\rm osc}$, then harmonic contributions to the expressions (1.175) average out and we end up with

$$\langle \eta(z) \rangle = \Theta^2\,, \tag{1.178a}$$

$$\langle \varphi(z) \rangle = \omega(n_\perp - n_\parallel)z + \Theta^2 b z\,. \tag{1.178b}$$

Having non-zero η and ϕ results into two observable effects: 1. a *rotation* by an angle $\delta\theta$ of the direction in which the linearly polarized EM wave oscillates in the plane perpendicular to the line of propagation, and, 2. appearance of a small *ellipticity* $\delta\psi$ in the linearly polarized EM wave. See also figure 1.3.

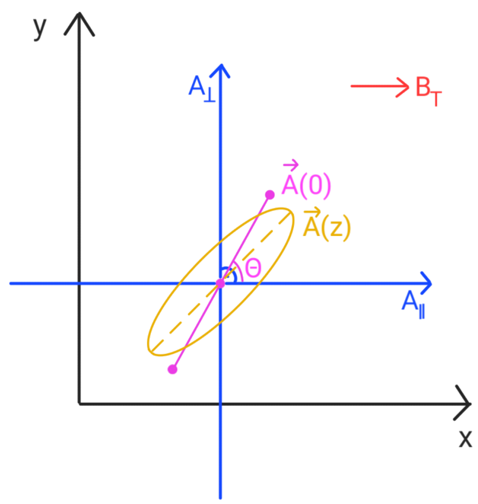

Figure 1.3: Rotation of the polarization plane and induced ellipticity of an EM wave due to mixing.

Rotation $\delta\theta(z)$ and induced ellipticity $\delta\psi(z)$

In order to quantify these effects, let θ denote the angle which the direction in which the EM field initially oscillates makes with the direction of the magnetic field, that is

$$\tan\theta = \frac{A_\perp(0)}{A_\parallel(0)}\,. \tag{1.179}$$

Then, due to the EM wave propagating a distance z in the magnetic field, we obtain a change

$$\delta(\tan\theta) = \frac{1}{\cos^2\theta}\delta\theta = -\eta(z)\frac{A_\perp(0)}{A_\parallel(0)} = -\eta(z)\tan\theta\,, \tag{1.180}$$

that is

$$\delta\theta(z) = -\eta(z)\tan\theta\cos^2\theta = -\frac{1}{2}\eta(z)\sin 2\theta\,. \tag{1.181}$$

Secondly, the induced phase delay $\varphi(z)$ between the two modes $A_\perp$ and $A_\parallel$ will cause EM polarization vector to trace out a very thin ellipse instead of being linearly polarized (as it was before entering the magnetic field). This minor semi-axis of this ellipse can be seen to satisfy

$$A_b \approx -\varphi A_\perp(0)\cos\theta \approx -\varphi A_\parallel(0)\sin\theta\,, \tag{1.182}$$

while the major semi-axis is simply

$$A_a \approx \sqrt{|A_\perp(0)|^2 + |A_\parallel(0)|^2}\,. \tag{1.183}$$

Defining the induced ellipticity $\delta\psi$ as the ratio

$$\delta\psi = \frac{A_b}{A_a} \tag{1.184}$$

of the shorter and the longer axis of the thin ellipse, we therefore obtain

$$\delta\psi(z) = -\varphi(z)\cos\theta\frac{A_\perp(0)}{\sqrt{|A_\perp(0)|^2 + |A_\parallel(0)|^2}} = -\frac{1}{2}\varphi(z)\sin 2\theta\,. \tag{1.185}$$

Summarizing, having propagated the EM wave a distance z through the region with the magnetic field, one should observe a rotation $\delta\theta(z)$ of the plane in which the linearly polarized EM wave oscillates, as well as a small induced ellipticity $\delta\psi(z)$, which, in terms of the variables of our setup and fundamental constants, satisfy (under the assumption that the EM waves and the scalar wave do not decohere, that is $z \ll \frac{1}{b} \sim l_{\mathrm{osc}}$)

$$\delta\theta(z) \approx -\frac{1}{16}g_0^2 B_{\mathrm{T}}^2 z^2 \sin 2\theta\,, \tag{1.186a}$$

$$\delta\psi(z) \approx -\frac{1}{2}(n_\perp - n_\parallel)\omega z \sin 2\theta + \\ -\frac{1}{96}\frac{g_0^2 B_{\mathrm{T}}^2 z^3}{\omega}\left[m_0^2 + 2(n_\perp - 1)\omega^2\right]\sin 2\theta\,. \tag{1.186b}$$

Notice that both of these measurable quantities are maximized when we set up our experiment so that $\theta = \frac{\pi}{4}$. Also note, that in the coherent limit $z \ll l_{\mathrm{osc}}$, the rotation $\delta\theta$ is not contaminated by the refractive indices and is independent of the mass m_0 of the scalar. It can therefore be used to directly measure the coupling constant g_0. Also note that $n_\perp$ and ω determine a mass scale (for $n_\perp > 1$)

$$m_{\mathrm{c}} = \sqrt{2(n_\perp - 1)}\omega\,, \tag{1.187}$$

which decides, whether m_0 effectively drops out from the ellipticity $\delta\psi(z)$: if the mass m_0 of the scalar is significantly less then m_{c}, the ellipticity cannot be used to find it out by measurement. Substituting again the values for our typical laboratory setup ($n_\perp - 1 = 10^{-17}$, $\omega = 2.4\,\mathrm{eV}$) one finds $m_{\mathrm{c}} = 10^{-8}\,\mathrm{eV}$.

Similarly one could write down the averaged values of $\delta\theta$ and $\delta\psi$ in the case $z \gg l_{\text{osc}}$, that is

$$\langle \delta\theta(z) \rangle = -\frac{1}{2} \frac{g_0^2 B_{\mathrm{T}}^2 \omega^2}{[m_0^2 + 2(n_\perp - 1)\omega^2]^2} \sin 2\theta \,, \tag{1.188a}$$

$$\langle \delta\psi(z) \rangle = -\frac{1}{2}(n_\perp - n_\parallel)\omega z \sin 2\theta - \frac{1}{4} \frac{g_0^2 B_{\mathrm{T}}^2}{m_0^2 + 2(n_\perp - 1)\omega^2} \omega z \sin 2\theta \,. \tag{1.188b}$$

Again, we note that if $m_0 \ll m_{\mathrm{c}}$, it drops out from the measurement of both $\langle \delta\theta(z) \rangle$ and $\langle \delta\psi(z) \rangle$. Also note, that the average induced ellipticity $\langle \delta\psi(z) \rangle$ becomes linear in z, where the presence of the mixing with the scalar field is felt through a small change in the slope of this dependence.

1.3.2 Light-shining-through-wall experiments

Another experimental setup which one could envisage is the following: a source of EM waves (laser beam) is directed towards a concrete wall which stops all photons. Before hitting the wall, the beam has to pass through a region with magnetic field with components B_{L} and B_{T}. As a consequence, part of the photons are converted into the scalar particles which only interact very weakly with the matter the wall is made of and thus pass through undisturbed to the other side. There the scalar particles pass through another region of constant magnetic field so that some of them are converted back to photons and may be detected. We will now quantify the probability of making a positive measurement of an EM wave on the other side of the wall.

Initially, let us prepare the system in a pure $A_\perp$ state

$$\Psi_{\mathrm{i}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv \Psi_{A_\perp} \,. \tag{1.189}$$

This is to maximize the fraction of photons converted to the massive scalar particles, as the $A_\parallel$ polarization does not participate in the oscillations. After the beam propagates over a distance z_1 through a region with constant magnetic field B_{T}, the beam will be in a state

$$\mathsf{U}(z_1, 0)\Psi_{\mathrm{i}} \,. \tag{1.190}$$

When the beam hits the wall, the EM component will be stopped while the scalar component passes through to the other side unimpeded. Hence, on the other side of the wall, the system should find itself in the state

$$\Pi_\phi \mathsf{U}(z_1, 0)\Psi_{\mathrm{i}} \,, \tag{1.191}$$

where by Π_ϕ, we have denoted the projector on the one-dimensional subspace spanned by the scalar fluctuations, namely

$$\Pi_\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{1.192}$$

After this pure-scalar beam enters the second region with the constant magnetic field $B_{\rm T}$, a non-zero $A_\perp$ component should regenerate. Hence, in the final state (assuming the thickness of this second region along the z axis to be z_2)

$$\Psi_{\rm f} = \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{\rm i} \tag{1.193}$$

one should be able to find an admixture of a pure $A_\perp$ state $\Psi_{\rm f} \equiv \Psi_{A_\perp} = (0,0,1)$, namely

$$(\Psi_{\rm f})^\dagger \Psi_{\rm i} = (\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp} \neq 0\,. \tag{1.194}$$

Since both the initial and the final states are normalized, the corresponding transition probability $P(A_\perp \to \phi \to A_\perp)$ (that is, the *photon regeneration probability* in a light-shining-through-wall experiment) is then computed as

$$\begin{align} P(A_\perp \to \phi \to A_\perp) &= |(\Psi_{\rm f})^\dagger \Psi_{\rm i}|^2 \tag{1.195a}\\ &= |(\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp}|^2\,. \tag{1.195b}\end{align}$$

Going through the intermediate steps of the beam evolution, we obtain

$$\begin{align} \mathsf{U}(z_1,0)\Psi_{A_\perp} &= \frac{1}{1+\Theta^2} e^{-i\omega n_\perp z_1} \begin{pmatrix} 0 \\ \Theta(e^{-ib\Theta^2 z_1} - e^{ib(1+\Theta^2)z_1}) \\ (\Theta^2 e^{ib(1+\Theta^2)z_1} + e^{-ib\Theta^2 z_1}) \end{pmatrix}, \tag{1.196a}\\ \Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp} &= \frac{1}{1+\Theta^2} e^{-i\omega n_\perp z_1} \Theta(e^{-ib\Theta^2 z_1} - e^{ib(1+\Theta^2)z_1}) \begin{pmatrix} 0\\1\\0\end{pmatrix}, \tag{1.196b}\end{align}$$

as well as

$$\begin{align} \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp} &= \frac{\Theta}{(1+\Theta^2)^2} \begin{pmatrix} 0 \\ e^{ib(1+\Theta^2)z_2} + \Theta^2 e^{-ib\Theta^2 z_2} \\ \Theta(e^{-ib\Theta^2 z_2} - e^{ib(1+\Theta^2)z_2}) \end{pmatrix} \times \nonumber\\ &\quad \times (e^{-ib\Theta^2 z_1} - e^{ib(1+\Theta^2)z_1}) e^{-i\omega n_\perp (z_1+z_2)}, \tag{1.197a}\\ (\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp} &= \frac{\Theta^2}{(1+\Theta^2)^2} (e^{-ib\Theta^2 z_2} - e^{ib(1+\Theta^2)z_2}) \times \nonumber\\ &\quad \times (e^{-ib\Theta^2 z_1} - e^{ib(1+\Theta^2)z_1}) e^{-i\omega n_\perp (z_1+z_2)}\,. \tag{1.197b}\end{align}$$

Incorporating the assumption (1.172) for both regions and keeping only terms quadratic in the mixing angle Θ (which needs to be very small in order for the small-mixing scenario to take place), this simplifies as

$$\begin{align} (\Psi_{\rm f})^\dagger \Psi_{\rm i} &= (\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_\phi \mathsf{U}(z_1,0)\Psi_{A_\perp} \tag{1.198}\\ &= \Theta^2 e^{-i\omega n_\perp (z_1+z_2)} \left(1 - e^{ibz_1}\right)\left(1 - e^{ibz_2}\right). \tag{1.199}\end{align}$$

Hence, we finally obtain the photon regeneration probability

$$\begin{align} P(A_\perp \to \phi \to A_\perp) &= |(\Psi_{\rm f})^\dagger \Psi_{\rm i}|^2 \tag{1.200a}\\ &= 4\Theta^4 (1-\cos bz_1)(1-\cos bz_2) \tag{1.200b}\end{align}$$

$$= 16\Theta^4 \sin^2\frac{bz_1}{2}\sin^2\frac{bz_2}{2}\,. \tag{1.200c}$$

In the case that $bz_1 \ll 1$ and $bz_2 \ll 1$, that is, the sizes of both regions with magnetic field are much less then the oscillation length l_{osc} (i.e. that the EM waves and the scalars do not decohere as they propagate through the magnetic regions) and substituting in terms of the physical parameters of the system and fundamental constants, we finally obtain the regeneration probability

$$P(A_\perp \to \phi \to A_\perp) \approx \frac{1}{16}g_0^4 B_{\mathrm{T}}^4 z_1^2 z_2^2\,. \tag{1.201}$$

We can see that this is insensitive to the mass m_0 of the scalar. On the other hand, if the opposite regime takes place, that is if we have both $z_1 \gg l_{\mathrm{osc}}$ and $z_2 \gg l_{\mathrm{osc}}$, the dependence on z_1 and z_2 gets averaged out and we obtain the mean regeneration probability

$$\langle P(A_\perp \to \phi \to A_\perp)\rangle = 4\Theta^4 = \frac{4g_0^4 B_{\mathrm{T}}^4\omega^4}{[m_0^2 + 2(n_\perp - 1)\omega^2]^4}\,. \tag{1.202}$$

For m_0 much greater or at least comparable to $m_{\mathrm{c}} = \sqrt{2(n_\perp - 1)}\omega$, this is sensitive to both g_0 and m_0.

1.3.3 Relative intensity decrease

We have already noted above that, as a consequence of the mixing, one should observe a relative decrease in the intensity of $\perp$-polarized photons in a laser beam passing through the magnetic field. In particular, assuming that initially there was no $A_\parallel$ component in the beam, we can write (based on the above result (1.147c))

$$\frac{|A_\perp(0)|^2 - |A_\perp(z)|^2}{|A_\perp(0)|^2} = P(\Psi_A \to \Psi_\phi; z) = 4\Theta^2\sin^2\frac{bz}{2} \tag{1.203}$$

in the small mixing scenario. Hence, the relative decrease of the $A_\perp$ intensity oscillates with length $l_{\mathrm{osc}} = \frac{2\pi}{b}$ and has amplitude $\alpha = 4\Theta^2$. Both l_{osc} and α are (in principle) measurable quantities, which we can express as

$$l_{\mathrm{osc}} = \frac{4\pi\omega}{m_0^2 + 2(n_\perp - 1)\omega^2}\,, \tag{1.204a}$$

$$\alpha = \frac{4g_0^2 B_{\mathrm{T}}^2\omega^2}{[m_0^2 + 2(n_\perp - 1)\omega^2]^2}\,. \tag{1.204b}$$

Thus, in the ideal case when one is able to measure both l_{osc} and α (and provided that that m_0 is greater, or at least comparable to m_{c}, so that it does not drop out), we could in principle extract both parameters g_0 and m_0 of the massive scalar lagrangian.

2. Mixing of photons with massive spin-2 field

Let us now turn to describing the mixing of a massive spin-2 particle with photons propagating in a constant magnetic background field. We will follow ideas of [10]. In contrast with the spin-0 case (which was discussed in the preceding chapter), we will see that the massive spin-2 has five physical degrees of freedom (polarizations). While a priori, one has to assume that all of these can mix with the EM field, we will find that only three polarizations enter non-trivially and the remaining two decouple from the dynamics. Also, this time *both* polarizations of the EM field will participate in the oscillations.

2.1 Non-interacting massive graviton

A free massive spin-two particle is represented by a rank-2 symmetric tensor which we will denote as $\chi_{\mu\nu}$. Its free dynamics is described by the lagrangian found by Fierz and Pauli [15]

$$\begin{aligned}\mathcal{L}_{\mathrm{FP}} = \frac{1}{2}(\partial_\rho\chi_{\mu\nu})(\partial^\rho\chi^{\mu\nu}) - (\partial_\mu\chi^{\mu\nu})(\partial^\rho\chi_{\rho\nu}) + (\partial_\rho\chi^{\rho\nu})(\partial_\nu{\chi^\mu}_\mu) +& \\ - \frac{1}{2}(\partial_\nu{\chi^\mu}_\mu)(\partial^\nu{\chi^\mu}_\mu) - \frac{m^2}{2}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{m^2}{2}({\chi^\mu}_\mu)^2\,,& \quad (2.1)\end{aligned}$$

where m denotes the mass of the spin-2 particle. To derive the equations of motion, let us consider directly varying the corresponding action and then setting variation to zero, that is

$$\delta S_{\mathrm{FP}} = \delta\int d^4x\,\mathcal{L}_{\mathrm{FP}} = 0\,. \quad (2.2)$$

In detail, we have

$$\begin{aligned}0 &= \delta\int d^4x\,\Big[\frac{1}{2}(\partial_\rho\chi_{\mu\nu})(\partial^\rho\chi^{\mu\nu}) - (\partial_\mu\chi^{\mu\nu})(\partial^\rho\chi_{\rho\nu}) + (\partial_\rho\chi^{\rho\nu})(\partial_\nu{\chi^\mu}_\mu) + \\ &\qquad - \frac{1}{2}(\partial_\nu{\chi^\mu}_\mu)(\partial^\nu{\chi^\mu}_\mu) - \frac{m^2}{2}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{m^2}{2}({\chi^\mu}_\mu)^2\Big] \quad (2.3a)\\ &= \int d^4x\,\Big[(\partial_\rho\chi_{\mu\nu})\delta(\partial^\rho\chi^{\mu\nu}) - 2(\partial_\mu\chi^{\mu\nu})\delta(\partial^\rho\chi_{\rho\nu}) + \delta(\partial_\rho\chi^{\rho\nu})(\partial_\nu{\chi^\mu}_\mu) + \\ &\qquad + (\partial_\rho\chi^{\rho\nu})\partial_\nu\delta{\chi^\mu}_\mu - (\partial_\nu{\chi^\mu}_\mu)\partial^\nu\delta{\chi^\mu}_\mu - m^2\chi_{\mu\nu}\delta\chi^{\mu\nu} + m^2({\chi^\mu}_\mu)\delta{\chi^\mu}_\mu\Big]. \quad (2.3b)\end{aligned}$$

Integrating by parts and throwing away the boundary terms (assuming suitable boundary conditions at infinity, as usual) we thus obtain

$$\begin{aligned}0 &= \int d^4x\,\Big[-\partial^\rho\partial_\rho\chi_{\mu\nu}\delta\chi^{\mu\nu} + 2\partial^\rho\partial_\mu\chi^{\mu\nu}\delta\chi_{\rho\nu} - \partial_\nu\partial_\rho\delta\chi^{\rho\nu}{\chi^\mu}_\mu + \\ &\qquad - \partial_\nu\partial_\rho\chi^{\rho\nu}\delta{\chi^\mu}_\mu + \partial^\nu\partial_\nu{\chi^\mu}_\mu\delta{\chi^\mu}_\mu - m^2\chi_{\mu\nu}\delta\chi^{\mu\nu} + m^2{\chi^\mu}_\mu\delta{\chi^\mu}_\mu\Big] \quad (2.4a)\\ &= \int d^4x\,\Big[-\partial^\rho\partial_\rho\chi^{\mu\nu} + 2\partial^\mu\partial_\rho\chi^{\rho\nu} - \partial^\nu\partial^\mu{\chi^\rho}_\rho - \partial_\rho\partial_\sigma\chi^{\rho\sigma}\eta^{\mu\nu} +\end{aligned}$$

$$+\,\partial^\rho\partial_\rho\chi^\sigma{}_\sigma\eta^{\mu\nu} - m^2\chi^{\mu\nu} + m^2\chi^\sigma{}_\sigma\eta^{\mu\nu}\Big]\delta\chi^{\mu\nu}\,. \tag{2.4b}$$

Since we are varying the action with respect to the tensor $\chi_{\mu\nu}$ which is symmetric, the variation $\delta\chi^{\mu\nu}$ only probes the symmetric part of the integrand. Hence, for an arbitrary but symmetric variation $\delta\chi^{\mu\nu}$, the least-action principle (2.2) implies that the *symmetric part* of the integrand in (2.4b) needs to vanish. This gives us the (free) equations of motion

$$\begin{aligned}0 = -\partial^\rho\partial_\rho\chi^{\mu\nu} + \partial^\mu\partial_\rho\chi^{\rho\nu} + \partial^\nu\partial_\rho\chi^{\rho\mu} - \partial^\nu\partial^\mu\chi^\rho{}_\rho - \partial_\rho\partial_\sigma\chi^{\rho\sigma}\eta^{\mu\nu} +\\ + \partial^\rho\partial_\rho\chi^\sigma{}_\sigma\eta^{\mu\nu} - m^2\chi^{\mu\nu} + m^2\chi^\sigma{}_\sigma\eta^{\mu\nu}\,.\end{aligned} \tag{2.5}$$

2.1.1 Dynamical constraints

We are now going to simplify the equation of motion (2.5) by noticing, that it actually implies the constraints

$$0 = \partial_\mu\chi^{\mu\nu}\,, \tag{2.6a}$$
$$0 = \chi_\mu{}^\mu\,. \tag{2.6b}$$

First, let us take the trace of the equation (2.5), that is, we multiply it by $\eta_{\mu\nu}$. This gives

$$\begin{aligned}0 = -\partial^\rho\partial_\rho\chi^\mu{}_\mu + \partial_\mu\partial_\rho\chi^{\rho\mu} + \partial_\nu\partial_\rho\chi^{\rho\nu} - \partial^\mu\partial_\mu\chi^\rho{}_\rho - 4\partial_\rho\partial_\sigma\chi^{\rho\sigma} +\\ + 4\partial^\rho\partial_\rho\chi^\sigma{}_\sigma - m^2\chi^\mu{}_\mu + 4m^2\chi^\sigma{}_\sigma\end{aligned} \tag{2.7}$$

so that after relabeling the summation indices, this reduces to

$$0 = -2\partial^\rho\partial_\rho\chi^\mu{}_\mu + 2\partial_\mu\partial_\rho\chi^{\rho\mu} - 3m^2\chi^\mu{}_\mu\,. \tag{2.8}$$

We therefore obtain that the trace of the massive graviton field is algebraically dependent on its derivatives as

$$\chi^\mu{}_\mu = \frac{2}{3m^2}\partial_\rho(-\partial^\rho\chi^\mu{}_\mu + \partial_\mu\chi^{\rho\mu})\,. \tag{2.9}$$

Furthermore, we can calculate the divergence of the equation of motion (2.5). Namely, let us apply on it the operator ∂_μ. We obtain

$$\begin{aligned}0 = -\partial^\rho\partial_\rho\partial_\mu\chi^{\mu\nu} + \partial_\mu\partial^\mu\partial_\rho\chi^{\rho\nu} + \partial_\mu\partial^\nu\partial_\rho\chi^{\rho\mu} - \partial^\nu\partial_\mu\partial^\mu\chi^\rho{}_\rho - \partial^\nu\partial_\rho\partial_\sigma\chi^{\rho\sigma} +\\ + \partial^\nu\partial^\rho\partial_\rho\chi^\sigma{}_\sigma - m^2\partial_\mu\chi^{\mu\nu} + m^2\partial^\nu\chi^\sigma{}_\sigma\,.\end{aligned} \tag{2.10}$$

After relabelling the indices, we can notice that the first and the second term cancel, the third and the fifth term cancel and finally that the fourth and the sixth term cancel as well. This means that only the mass-terms will survive to give us the constraint

$$0 = -\partial_\mu\chi^{\mu\nu} + \partial^\nu\chi^\sigma{}_\sigma\,. \tag{2.11}$$

Substituting this back into (2.9), we therefore obtain

$$\chi^\mu{}_\mu = 0\,. \tag{2.12}$$

Plugging this in turn back into (2.11), we finally have

$$\partial_\mu \chi^{\mu\nu} = 0\,. \tag{2.13}$$

Substituting the constraints (2.12) and (2.13) back into the original equations of motion (2.5), these simplify as

$$(\Box + m^2)\chi^{\mu\nu} = 0\,. \tag{2.14}$$

Note that the tensor field $\chi^{\mu\nu}$ is symmetric, meaning that it generally has 10 independent degrees of freedom. However, the constraints encoded by the equations (2.12) and (2.13) eliminate 5 of these, which leaves us with only 5 independent elements. This means that a massive spin-2 particle has 5 degrees of freedom, as appropriate for a spin-2 representation of the Poincare group with a non-zero value of the 4-momentum Casimir invariant [46]. It will therefore be useful to decompose the field $\chi^{\mu\nu}$ into a basis of the five spin-two polarizations with some coefficients (amplitudes).

2.1.2 Warm-up: polarization of a massive vector

Just for the sake of illustrating how polarizations work, let us first focus on the much simpler case of a massive vector (spin-1) boson A_μ. The free action for such a “massive photon” can be written as [47]

$$\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\nu A^\nu \right), \tag{2.15}$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By varying this action, one can derive the corresponding equations of motion as

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu + m^2 A^\nu = 0\,. \tag{2.16}$$

Calculating their divergence

$$\begin{aligned} 0 &= \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\nu \partial^\nu \partial_\mu A^\mu + m^2 \partial_\nu A^\nu && \text{(2.17a)} \\ &= \partial_\nu \Box A^\nu - \Box \partial_\mu A^\mu + m^2 \partial_\nu A^\nu && \text{(2.17b)} \end{aligned}$$

we can note that the first two terms are actually zero, so that the mass-term yields a dynamical (*transversality*) constraint

$$\partial_\nu A^\nu = 0\,. \tag{2.18}$$

Thanks to this, the equations of motion can be simplified as

$$(\Box + m^2) A^\nu = 0\,. \tag{2.19}$$

Consider now a plane wave of momentum p, where as before, p is chosen to be greater than zero so that the photon moves in the positive direction of the z-axis. The four-momentum k can then be written as $k^\mu = (\omega(p), 0, 0, p)$. That is, we will write

$$A^\rho(t, x; p) = A^\rho(p) e^{ik_\mu x^\mu}\,, \tag{2.20}$$

where

$$A^\mu(p) = (A^0(p), A^1(p), A^2(p), A^3(p)) \tag{2.21}$$

are the amplitudes of oscillation of A_μ in different directions. Now, let us substitute this plane-wave ansatz into both the equation of motion (2.19) and the constraint (2.18). First, we have

$$\begin{aligned} 0 &= (\Box + m^2)A^\rho & \text{(2.22a)} \\ &= (-k^2 + m^2)A^\rho(p) & \text{(2.22b)} \\ &= 0\,, & \text{(2.22c)} \end{aligned}$$

that is

$$k^2 = \omega(p)^2 - p^2 = m^2 \tag{2.23}$$

and so the dispersion relation for the massive photon reads (fixing $\omega > 0$)

$$\omega(p) = \sqrt{p^2 + m^2}\,, \tag{2.24}$$

as expected. After plugging into the constraint (2.18), we also have

$$\partial_\rho A^\rho = ik_\rho A^\rho(p) e^{ik_\mu x^\mu} = 0 \tag{2.25}$$

which implies that

$$k_\rho A^\rho(p) = 0\,. \tag{2.26}$$

In detail, this means that

$$0 = k^0 A^0 - pA^3 = \omega(p)A^0 - pA^3 \tag{2.27}$$

and so

$$\omega(p)A^0 = pA^3\,. \tag{2.28}$$

This gives a relation between two of the four polarizations. The polarizations A^1 and A^2 are independent. Hence, the total number of independent degrees of freedom is three. Introducing the usual pure right- and left-handed massless photon polarization vectors

$$\begin{aligned} \epsilon_+^\rho &= (0, -1, -i, 0)\,, & \text{(2.29a)} \\ \epsilon_-^\rho &= (0, +1, -i, 0)\,, & \text{(2.29b)} \end{aligned}$$

it is therefore convenient to decompose (using (2.28))

$$\begin{aligned} A^\rho(p) &= (0, A^1(p), A^2(p), 0) + (A^0(p), 0, 0, A^3(p)) & \text{(2.30a)} \\ &= \sum_{a=+,-} \gamma_a(p)\epsilon_a^\rho + \frac{m}{\omega(p)} A^3(p)(\tfrac{p}{m}, 0, 0, \tfrac{\omega(p)}{m})\,, & \text{(2.30b)} \end{aligned}$$

where

$$\gamma^+(p) = \frac{iA^2(p) - A^1(p)}{2}\,, \tag{2.31a}$$

$$\gamma^{-}(p) = \frac{iA^2(p) + A^1(p)}{2}\,, \tag{2.31b}$$

and, in the case of a more generic dispersion relation than (2.24), m^2 should be understood as $\omega(p)^2 - p^2$. Hence, denoting the amplitude of longitudinal polarization $\frac{m}{\omega(p)}A^3(p)$ as $\gamma^0(p)$ and introducing the corresponding longitudinal basis vector

$$\epsilon_0^\rho = (\tfrac{p}{m}, 0, 0, \tfrac{\omega(p)}{m})\,, \tag{2.32}$$

it is possible to rewrite

$$A^\rho(p) = \sum_{a=+,0,-} \gamma^a(p)\epsilon_a^\rho(p)\,. \tag{2.33}$$

Hence, in terms of the three polarization vectors (2.29) and (2.32), a generic plane wave can be expanded in the form

$$A^\rho(t,x;p) = \Big[\sum_{a\in\{+,0,-\}} \gamma_a(p)\epsilon_a^\rho(p)\Big] e^{ik_\mu x^\mu} \tag{2.34}$$

with corresponding amplitudes $\gamma^+(p)$,$\gamma^0(p)$ and $\gamma^-(p)$. Notice that three polarization vectors ϵ_a^ρ introduced above are orthonormal in the sense that

$$(\epsilon_a)_\mu(\epsilon_b)^\mu = \delta_{ab}\,. \tag{2.35}$$

They are also *transverse*, namely they satisfy

$$k_\mu\epsilon_a^\mu = 0\,, \tag{2.36}$$

which is just a reflection of the fact that they implement the constraint $\partial_\mu A^\mu = 0$.

2.1.3 Polarizations of a massive spin-2 field

The procedure of decomposing a generic massive spin-2 plane wave into basis polarizations will be similar to the case of a massive vector considered in the previous subsection, but a bit more complicated. Recall that from the Fierz-Pauli action S_{FP}, we have inferred the equations of motion

$$(\Box + m^2)\chi^{\mu\nu} = 0 \tag{2.37}$$

together with the transverse and traceless constraints

$$\partial_\mu\chi^{\mu\nu} = 0\,, \tag{2.38a}$$

$$\chi^\mu{}_\mu = 0\,. \tag{2.38b}$$

Let us again consider a plane-wave solution with momentum p, where p is chosen to be greater than zero so that the graviton moves in the positive direction of the z-axis. The four-momentum k can be written as $k^\mu = (\omega(p), 0, 0, p)$. Then we can put

$$\chi^{\rho\sigma}(t,x;p) = \chi^{\rho\sigma}(p)e^{ik_\mu x^\mu}\,. \tag{2.39}$$

The amplitudes of oscillation in different directions can be explicitly represented by a 4 by 4 matrix

$$\chi^{\mu\nu}(p) = \begin{pmatrix} \chi^{00}(p) & \chi^{0i}(p) \\ \chi^{i0}(p) & \chi^{ij}(p) \end{pmatrix}\,, \tag{2.40}$$

where $\chi^{0i} = \chi^{i0}$ and $\chi^{ij} = \chi^{ji}$. Plugging the plane wave ansatz (2.39) into equation of motion (2.37) would again give the condition

$$k^2 = m^2 \tag{2.41}$$

and hence the expected mass-shell relation

$$\omega(p) = \sqrt{p^2 + m^2}\,. \tag{2.42}$$

The constraints on the polarizations can then be obtained by substituting the plane wave (2.39) into the constraints (2.38). Focusing first on the transversality condition (2.38a), we obtain

$$0 = \partial_\rho \chi^{\rho\sigma} = ik_\rho \chi^{\rho\sigma}(p) e^{ik_\mu x^\mu}\,, \tag{2.43}$$

that is

$$k_\rho \chi^{\rho\sigma}(p) = 0\,. \tag{2.44}$$

Second, the tracelessness condition (2.38b) gives

$$0 = \chi^\rho{}_\rho = \chi^\rho{}_\rho(p) e^{ik_\mu x^\mu}\,, \tag{2.45}$$

meaning that we have to require

$$\chi^\rho{}_\rho(p) = 0\,. \tag{2.46}$$

Evaluating the transversality constraint (2.44) explicitly in coordinates, we have

$$\begin{aligned} 0 &= k_\rho \chi^{\rho\sigma}(p) && \text{(2.47a)} \\ &= k^0 \chi^{0\sigma}(p) - p\chi^{3\sigma}(p) && \text{(2.47b)} \\ &= \omega(p)\chi^{0\sigma}(p) - p\chi^{3\sigma}(p)\,, && \text{(2.47c)} \end{aligned}$$

that is

$$\omega(p)\chi^{0\sigma}(p) = p\chi^{3\sigma}(p)\,. \tag{2.48}$$

Hence, the individual elements of $\chi^{\rho\sigma}(p)$ are constrained to satisfy

$$\begin{aligned} \chi^{00} &= \frac{p}{\omega}\chi^{30} = \frac{p}{\omega}\chi^{03}\,, && \text{(2.49a)} \\ \chi^{01} = \chi^{10} &= \frac{p}{\omega}\chi^{31} = \frac{p}{\omega}\chi^{13}\,, && \text{(2.49b)} \\ \chi^{02} = \chi^{20} &= \frac{p}{\omega}\chi^{32} = \frac{p}{\omega}\chi^{23}\,, && \text{(2.49c)} \\ \chi^{03} = \chi^{30} &= \frac{p}{\omega}\chi^{33}\,. && \text{(2.49d)} \end{aligned}$$

In particular, this gives

$$\chi^{00} = \frac{p^2}{\omega^2}\chi^{33} = \frac{p^2}{p^2+m^2}\chi^{33} = \Big(1 - \frac{m^2}{\omega^2}\Big)\chi^{33} \,. \tag{2.50}$$

Expanding explicitly also the tracelessness condition (2.46), we obtain

$$\begin{aligned}
0 &= \chi^{00} - \chi^{11} - \chi^{22} - \chi^{33} && (2.51a)\\
&= (1 - \frac{m^2}{\omega^2})\chi^{33} - \chi^{11} - \chi^{22} - \chi^{33} && (2.51b)\\
&= \frac{m^2}{\omega^2}\chi^{33} - \chi^{11} - \chi^{22} \,, && (2.51c)
\end{aligned}$$

that is

$$\chi^{11} + \chi^{22} = -\frac{m^2}{\omega^2}\chi^{33} \,. \tag{2.52}$$

It is therefore convenient to reparametrize χ^{11} and χ^{22} in terms of a single degree of freedom as

$$\begin{aligned}
\chi^{11} &= +\tilde{\chi}^{11} - \frac{1}{2}\frac{m^2}{\omega^2}\chi^{33} \,, && (2.53a)\\
\chi^{22} &= -\tilde{\chi}^{11} - \frac{1}{2}\frac{m^2}{\omega^2}\chi^{33} \,. && (2.53b)
\end{aligned}$$

In total, we can therefore parametrize $\chi^{\rho\sigma}(p)$ as

$$\chi^{\rho\sigma}(p) = \begin{pmatrix} \frac{p^2}{\omega^2}\chi^{33} & \frac{p}{\omega}\chi^{13} & \frac{p}{\omega}\chi^{23} & \frac{p}{\omega}\chi^{33} \\ \frac{p}{\omega}\chi^{13} & \tilde{\chi}^{11} - \frac{1}{2}\frac{m^2}{\omega^2}\chi^{33} & \chi^{12} & \chi^{13} \\ \frac{p}{\omega}\chi^{23} & \chi^{12} & -\tilde{\chi}^{11} - \frac{1}{2}\frac{m^2}{\omega^2}\chi^{33} & \chi^{23} \\ \frac{p}{\omega}\chi^{33} & \chi^{13} & \chi^{23} & \chi^{33} \end{pmatrix} \tag{2.54}$$

From this form, we can see that there are only five independent degrees of freedom, specifically: $\tilde{\chi}^{11}$, χ^{12}, χ^{33}, χ^{13} and χ^{23}. Introducing the basis polarizations[1]

$$\epsilon^{\rho\sigma}_{+2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{2.55a}$$

$$\epsilon^{\rho\sigma}_{\times 2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{2.55b}$$

$$\epsilon^{\rho\sigma}_{+1} = \frac{-1}{m\sqrt{2}}\begin{pmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & \omega \\ 0 & 0 & \omega & 0 \end{pmatrix} \,, \tag{2.55c}$$

[1] Again, in the generic case where the spin-2 wave may not satisfy the exact dispersion relation (2.42), we should understand m^2 as being equal to $\omega(p)^2 - p^2$.

$$\epsilon^{\rho\sigma}_{\times 1} = \frac{1}{m\sqrt{2}} \begin{pmatrix} 0 & p & 0 & 0 \\ p & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \end{pmatrix} , \tag{2.55d}$$

$$\epsilon^{\rho\sigma}_{0} = \frac{1}{m^2}\sqrt{\frac{2}{3}} \begin{pmatrix} 2p^2 & 0 & 0 & 2p\omega \\ 0 & -m^2 & 0 & 0 \\ 0 & 0 & -m^2 & 0 \\ 2p\omega & 0 & 0 & 2\omega^2 \end{pmatrix} , \tag{2.55e}$$

the matrix $\chi^{\rho\sigma}(p)$ can therefore be decomposed as

$$\chi^{\rho\sigma} = \chi_{+2}(p)\epsilon^{\rho\sigma}_{+2} + \chi_{\times 2}(p)\epsilon^{\rho\sigma}_{\times 2} + \chi_{+1}(p)\epsilon^{\rho\sigma}_{+1} + \chi_{\times 1}(p)\epsilon^{\rho\sigma}_{\times 1} + \chi_{0}(p)\epsilon^{\rho\sigma}_{0} , \tag{2.56}$$

where the amplitudes $\chi_a(p)$ can be expressed in terms of the original independent parameters $\tilde{\chi}^{11}$, χ^{12}, χ^{33}, χ^{13} and χ^{23}. Overall, we can therefore rewrite the massive spin-2 plane wave as

$$\chi^{\rho\sigma}(t, x; p) = \Bigg[\sum_{a\in\{+2,\times 2,+1,\times 1,0\}} \chi_a(p)\epsilon^{\rho\sigma}_a \Bigg] e^{ik_\mu x_\mu} . \tag{2.57}$$

Similarly to the case of the massive vector, the polarizations $\epsilon^{\rho\sigma}_a$ were chosen to be orthonormal, that is

$$(\epsilon_a)^{\rho\sigma}(\epsilon_b)_{\rho\sigma} = \delta_{ab} . \tag{2.58}$$

They also transverse and traceless, namely

$$0 = k_\mu(\epsilon_a)^{\mu\nu} , \tag{2.59a}$$
$$0 = (\epsilon_a)^{\mu}{}_{\nu} \tag{2.59b}$$

as they are implementing the constraints (2.38).

2.2 Interaction with the EM field

In this section we would like to consider coupling the massive spin-2 particle to the electromagnetic fields (massless photons). That is, we generally want to consider a lagragian of the form

$$\mathcal{L} = \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}} \tag{2.60}$$

where $\mathcal{L}_{\text{EM}}$ is the (usual) free Maxwell lagrangian, $\mathcal{L}_{\text{FP}}$ is the Fierz-Pauli lagrangian (2.1) and $\mathcal{L}_{\text{int}}$ is some interaction term.

2.2.1 Coupling to the EM stress-energy tensor

In particular, we will couple the massive spin-2 field to the full electromagnetic stress-energy tensor

$$(T_{\text{EM}})^{\mu\nu} = F^{\mu}{}_{\alpha}F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} . \tag{2.61}$$

This can be motivated by considering the massless case, where this coupling arises by expanding the Einstein-Hilbert action coupled to the covariantized Maxwell action (see the following chapter for details). In chapter 4, we will also observe that the coupling of $\chi_{\mu\nu}$ to the EM stress energy tensor follows from the bimetric action. Let us therefore consider the interaction lagrangian of the form

$$\mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}}\chi_{\rho\sigma}(T_{\text{EM}})^{\rho\sigma} = \frac{g}{\sqrt{2}}\chi_{\rho\sigma}(F^{\rho}{}_{\nu}F^{\nu\sigma} + \frac{1}{4}\eta^{\rho\sigma}F_{\mu\nu}F^{\mu\nu})\,. \tag{2.62}$$

Varying the corresponding action as before, one obtains the equations of motion

$$\begin{aligned}0 = -\partial^{\rho}\partial_{\rho}\chi^{\mu\nu} + \partial^{\mu}\partial_{\rho}\chi^{\rho\nu} + \partial^{\nu}\partial_{\rho}\chi^{\rho\mu} - \partial^{\nu}\partial^{\mu}\chi^{\rho}{}_{\rho} - \partial_{\rho}\partial_{\sigma}\chi^{\rho\sigma}\eta^{\mu\nu} + \\ + \partial^{\rho}\partial_{\rho}\chi^{\sigma}{}_{\sigma}\eta^{\mu\nu} - m^2\chi^{\mu\nu} + m^2\chi^{\sigma}{}_{\sigma}\eta^{\mu\nu} + \frac{g}{\sqrt{2}}(T_{\text{EM}})^{\mu\nu}\,,\end{aligned} \tag{2.63a}$$

$$0 = \partial_{\mu}F^{\mu\alpha} - \sqrt{2}g\partial_{\nu}\Big(\chi^{\mu\alpha}F_{\mu}{}^{\nu} - \chi^{\mu\nu}F_{\mu}{}^{\alpha} + \frac{1}{2}\chi^{\rho}{}_{\rho}F^{\nu\alpha}\Big)\,. \tag{2.63b}$$

Let us now see what kind of constraints on $\chi^{\mu\nu}$ these equations imply by calculating the trace and the divergence of the graviton equation of motion. As a preparation, let us first calculate trace and divergence of the EM stress-energy tensor. For the trace we have

$$(T_{\text{EM}})^{\mu}{}_{\mu} = F_{\mu\alpha}F^{\alpha\mu} + \frac{1}{4}\eta^{\mu}{}_{\mu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.64a}$$

$$= -F_{\mu\alpha}F^{\mu\alpha} + F_{\alpha\beta}F^{\alpha\beta} \tag{2.64b}$$

$$= 0\,, \tag{2.64c}$$

while for the divergence, we obtain

$$\partial_{\mu}(T_{\text{EM}})^{\mu\nu} = \partial_{\mu}(F^{\mu}{}_{\alpha}F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \tag{2.65a}$$

$$= (\partial_{\mu}F^{\mu}{}_{\alpha})F^{\alpha\nu} + F^{\mu}{}_{\alpha}\partial_{\mu}F^{\alpha\nu} + \frac{1}{2}\eta^{\mu\nu}F^{\alpha\beta}\partial_{\mu}F_{\alpha\beta} \tag{2.65b}$$

We can substitute for the first term from (2.63b) so that it is $\mathcal{O}(gF\partial(\chi F))$. The remaining two terms can be manipulated as

$$F^{\mu}{}_{\alpha}\partial_{\mu}F^{\alpha\nu} + \frac{1}{2}\eta^{\mu\nu}F^{\alpha\beta}\partial_{\mu}F_{\alpha\beta} =$$

$$= \frac{1}{2}F_{\mu\alpha}\partial^{\mu}F^{\alpha\nu} + \frac{1}{2}F_{\mu\alpha}\partial^{\mu}F^{\alpha\nu} + \frac{1}{2}F^{\alpha\beta}\partial^{\nu}F_{\alpha\beta} \tag{2.66a}$$

$$= \frac{1}{2}F_{\mu\alpha}\partial^{\mu}F^{\alpha\nu} - \frac{1}{2}F_{\mu\alpha}\partial^{\alpha}F^{\mu\nu} + \frac{1}{2}F^{\mu\alpha}\partial^{\nu}F_{\mu\alpha} \tag{2.66b}$$

$$= -\frac{1}{2}F_{\mu\alpha}(\partial^{\mu}F^{\nu\alpha} + \partial^{\alpha}F^{\mu\nu} + \partial^{\nu}F^{\alpha\mu}) \tag{2.66c}$$

$$= 0\,, \tag{2.66d}$$

where the final step holds due to the Bianchi identity. Hence, in total this gives

$$\partial_{\mu}(T_{\text{EM}})^{\mu\nu} = \mathcal{O}(gF\partial(\chi F))\,. \tag{2.67}$$

Calculating now the trace of the whole graviton equation of motion, the result (2.64) implies that the entire interacting part drops out and we are therefore left with

$$\partial_\mu(\partial_\rho \chi^{\rho\mu} - \partial^\mu \chi^\sigma{}_\sigma) = \frac{3}{2} m^2 \chi^\mu{}_\mu \,, \tag{2.68}$$

as in the case of the free massive spin-2 particle. Secondly, for the divergence of graviton equation of motion, we get

$$0 = -m^2 \partial_\mu \chi^{\mu\nu} + m^2 \partial^\nu \chi^\sigma{}_\sigma + \frac{g}{\sqrt{2}} \partial_\mu (T_{\text{EM}})^{\mu\nu} \tag{2.69a}$$

$$0 = -m^2 \partial_\mu \chi^{\mu\nu} + m^2 \partial^\nu \chi^\sigma{}_\sigma + \mathcal{O}(g^2 F \partial(\chi F)) \tag{2.69b}$$

that is

$$\partial_\mu \chi^{\mu\nu} - \partial^\nu \chi^\sigma{}_\sigma = \mathcal{O}(\tfrac{g^2}{m^2} F \partial(\chi F)) \,. \tag{2.70}$$

This means that upto higher-order terms (which have to be dropped if our analysis is to be perturbatively consistent), we obtain

$$\frac{3}{2} m^2 \chi^\mu{}_\mu = \partial_\mu(\partial_\rho \chi^{\rho\mu} - \partial^\mu \chi^\sigma{}_\sigma) = 0 \,. \tag{2.71}$$

Plugging this back into (2.70), we therefore recover the transverse-traceless constraints

$$\chi^\mu{}_\mu = 0 \,, \tag{2.72a}$$

$$\partial_\mu \chi^{\mu\nu} = 0 \,. \tag{2.72b}$$

Hence, we can conclude that the coupling of $\chi_{\mu\nu}$ to the EM stress-energy tensor does not violate the transversality and tracelessness of $\chi_{\mu\nu}$, so that it can be expanded in the usual massive spin-2 polarizations as

$$\chi^{\mu\nu} = \sum_{a \in \{+2, \times 2, +1, \times 1, 0\}} \chi_a \epsilon_a^{\rho\sigma} \,. \tag{2.73}$$

Substituting the constraints (2.72) into the equations of motion (2.63a) and (2.63b), we get

$$0 = -(\Box + m^2) \chi^{\mu\nu} + \frac{g}{\sqrt{2}} (T_{\text{EM}})^{\mu\nu} \tag{2.74}$$

for the massive graviton, as well as

$$0 = \partial_\mu F^{\mu\alpha} - \sqrt{2} g \chi^{\mu\alpha} \partial_\nu F_\mu{}^\nu - \sqrt{2} g F_\mu{}^\nu \partial_\nu \chi^{\mu\alpha} + \sqrt{2} g \chi^{\mu\nu} \partial_\nu F_\mu{}^\alpha \tag{2.75}$$

for the photon, where we note that the second term contains the divergence of F and so is of higher order in g (by substituting from the photon equation of motion into itself). In total, we therefore obtain the equations of motion

$$(\Box + m^2) \chi^{\mu\nu} = \frac{g}{\sqrt{2}} (T_{\text{EM}})^{\mu\nu} \,, \tag{2.76a}$$

$$\partial_\mu F^{\mu\alpha} = \sqrt{2} g \Big(F_\mu{}^\nu \partial_\nu \chi^{\mu\alpha} - \chi^{\mu\nu} \partial_\nu F_\mu{}^\alpha \Big) \,. \tag{2.76b}$$

Expanding around the usual magnetic background (1.14), these can be straightforwardly linearized as

$$(\Box + m^2) \chi^{\mu\nu} = \frac{g}{\sqrt{2}} \Big[(F_{\text{ext}})^\mu{}_\alpha F^{\alpha\nu} + F^\mu{}_\alpha F_{\text{ext}}^{\alpha\nu} + \frac{1}{2} \eta^{\mu\nu} F_{\alpha\beta} F_{\text{ext}}^{\alpha\beta} \Big] \,, \tag{2.77a}$$

$$\Box A^\alpha - \partial^\alpha (\partial_\mu A^\mu) = \sqrt{2} g (F_{\text{ext}})_\mu{}^\nu \partial_\nu \chi^{\mu\alpha} \,. \tag{2.77b}$$

2.2.2 Gauge fixing

Recall that in the case of the scalar – photon mixing, we were able to fix for the EM field the Lorentz gauge (1.37) while also fixing the corresponding residual gauge symmetry by putting $A^0 = 0$. However, we can easily observe that repeating exactly this in the case of the massive spin-2 particle leads to an inconsistency. Indeed, considering the $\alpha = 0$ component of the linearized equation of motion (2.77b), we obtain

$$\Box A^0 - \partial^0(\partial_\mu A^\mu) = \sqrt{2}g(F_{\text{ext}})_\mu{}^\nu \partial_\nu \chi^{\mu 0}\,. \tag{2.78}$$

Assuming the usual ansatz of a plane wave propagating along the z-direction, we can evaluate the RHS to obtain

$$\begin{aligned}\Box A^0 - \partial^0(\partial_\mu A^\mu) &= \sqrt{2}g(F_{\text{ext}})_2{}^3 \partial_3 \chi^{20} &(2.79a)\\ &= \sqrt{2}gipB_{\text{T}}\epsilon^{20}_{+1}\chi_{+1} &(2.79b)\\ &= -igB_{\text{T}}\frac{p^2}{\sqrt{\omega^2-p^2}}\chi_{+1}\,, &(2.79c)\end{aligned}$$

where we have noticed that only for $a = +1$ do we obtain a non-zero 20 entry in the polarization tensor. Hence, we obtain that we cannot possibly fix Lorentz gauge *and* simultaneously fix $A^0 = 0$, because then the LHS of (2.79) would vanish while the RHS is clearly non-zero. Instead we can

1. either fix a generalization of the Lorentz gauge which would be found so as to enable setting $A^0 = 0$,
2. or, fix the usual Lorentz gauge but then allow for a non-zero A^0 (and, consequently, a non-zero A^3).

While we will focus only on the first method, we have checked that they both lead to the same result. First, we can see that fixing the deformed Lorentz gauge[2]

$$\partial_\mu A^\mu = -\sqrt{2}g\int^t d\tau\,(F_{\text{ext}})_\mu{}^\nu \partial_\nu \chi^{\mu 0} \tag{2.80}$$

and substituting back into the equation (2.78) cancels the RHS and leads to the equation

$$\Box A^0 = 0\,. \tag{2.81}$$

Since the residual gauge symmetry of the gauge condition (2.80) is the same as the residual gauge symmetry of the undeformed Lorentz gauge (the RHS of (2.80) does not transform under the EM gauge transformation) we can use the same analysis as in the previous chapter to check that in this case, A^0 *can* be gauged away. Evaluating the gauge-constraint (2.80) explicitly in coordinates for a plane wave solution, we obtain

$$i(\omega A^0 - pA^3) = gB_{\text{T}}\frac{p^2}{\omega\sqrt{\omega^2-p^2}}\chi_{+1}\,. \tag{2.82}$$

[2]Since we are assuming a plane-wave form for both both the EM and the massive spin-2 field, the integral $\int^t d\tau$ should be thought of as simply $1/(i\omega(p))$.

Substituting A^0, we observe that contrary to the case of the scalar – photon mixing, we are now getting

$$A^3 = -gB_{\mathrm{T}} \frac{p}{i\omega\sqrt{\omega^2 - p^2}} \chi_{+1} \neq 0\,. \tag{2.83}$$

As a quick consistency check, let us verify that A^3 given by (2.83) satisfies the equation of motion (2.77b) for $\alpha = 3$. Indeed, for the LHS we get

$$\begin{aligned} \Box A^3 - \partial^3(\partial_\mu A^\mu) &= -(-\omega^2 + p^2) g B_{\mathrm{T}} \frac{p}{i\omega\sqrt{\omega^2 - p^2}} \chi_{+1} + \\ &\qquad - igB_{\mathrm{T}} \frac{p^3}{\omega\sqrt{\omega^2 - p^2}} \chi_{+1} \qquad (2.84a) \\ &= -igB_{\mathrm{T}} \frac{\omega p}{\sqrt{\omega^2 - p^2}} \chi_{+1}\,, \qquad (2.84b) \end{aligned}$$

while the RHS evaluates to

$$\begin{aligned} \sqrt{2} g (F_{\mathrm{ext}})_\mu{}^\nu \partial_\nu \chi^{\mu 3} &= \sqrt{2} g (F_{\mathrm{ext}})_2{}^3 \partial_3 \chi^{23} \qquad (2.85a) \\ &= \sqrt{2} i g B_{\mathrm{T}} p (\epsilon_{+1})^{23} \chi_{+1} \qquad (2.85b) \\ &= -igB_{\mathrm{T}} \frac{\omega p}{\sqrt{\omega^2 - p^2}} \chi_{+1}\,, \qquad (2.85c) \end{aligned}$$

which verifies that the equation of motion holds. To summarize, in the explicit evaluation of the equations of motion for the remaining (independent) degrees of freedom $A^1, A^2, \chi_{\times 2}, \chi_{+2}, \chi_{\times 1}, \chi_{+1}, \chi_0$, we will use the gauge constraints

$$\begin{aligned} \partial_\mu A^\mu &= -\sqrt{2} g \int^t d\tau\, (F_{\mathrm{ext}})_\mu{}^\nu \partial_\nu \chi^{\mu 0}\,, \qquad (2.86a) \\ A^0 &= 0\,, \qquad (2.86b) \\ A^3 &= -gB_{\mathrm{T}} \frac{p}{i\omega\sqrt{\omega^2 - p^2}} \chi_{+1}\,. \qquad (2.86c) \end{aligned}$$

2.2.3 Mixing equations

Substituting the expansion (2.73) of the massive spin-2 field $\chi^{\mu\nu}$ in terms of a basis ε_a for the five polarizations and using the orthonormality relations (2.58), one can rewrite the massive graviton equation of motion (2.77a) as

$$\begin{aligned} &(\Box + m^2)\chi_i = \\ &\quad = \frac{g}{\sqrt{2}} (\epsilon_i)_{\mu\nu} \Big[(F_{\mathrm{ext}})^\mu{}_\alpha F^{\alpha\nu} + F^\mu_\alpha (F_{\mathrm{ext}})^{\alpha\nu} + \frac{1}{2} \eta^{\mu\nu} F_{\alpha\beta} (F_{\mathrm{ext}})^{\alpha\beta} \Big] \qquad (2.87a) \\ &\quad = \sqrt{2} g (\epsilon_i)_{\mu\nu} (F_{\mathrm{ext}})^\mu{}_\alpha F^{\alpha\nu}\,, \qquad (2.87b) \end{aligned}$$

where we have used the symmetry in the indices μ, ν and also the fact that the polarizations $(\epsilon_i)_{\mu\nu}$ are traceless. Furthermore, assuming that both the EM field and the massive spin-2 field are simple plane waves, we can use the transversality $(\epsilon_i)_{\mu\nu} k^\nu = 0$ of the massive spin-2 polarization to simplify

$$\begin{aligned} (\Box + m^2)\chi_i &= \sqrt{2} g (F_{\mathrm{ext}})^\mu{}_\alpha \Big[(\epsilon_i)_{\mu\nu} \partial^\alpha A^\nu - (\epsilon_i)_{\mu\nu} i k^\nu A^\alpha \Big] \qquad (2.88a) \\ &= \sqrt{2} g (F_{\mathrm{ext}})^\mu{}_\alpha (\epsilon_i)_{\mu\nu} \partial^\alpha A^\nu \qquad (2.88b) \end{aligned}$$

$$= \sqrt{2} g (F_{\mathrm{ext}})^{2}{}_{3} (\epsilon_i)_{2\nu} \partial^3 A^\nu \tag{2.88c}$$

$$= -\sqrt{2} g B_{\mathrm{T}} (\epsilon_i)_{2\nu} i p A^\nu \, . \tag{2.88d}$$

Employing the relativistic approximation (1.50) for the Klein-Gordon operator $\Box + m^2$, one can recast the massive spin-2 equation of motion into the form

$$0 = (\omega - p + \Delta)\chi_i + \frac{g B_{\mathrm{T}}}{\omega} \frac{1}{\sqrt{2}} (\epsilon_i)_{2\nu} i p A^\nu \, . \tag{2.89}$$

Furthermore, substituting for A^0 and A^3 from the gauge constraints (2.86) and recalling that $A^1 = -A_\parallel$, $A^2 = -A_\perp$, one obtains

$$0 = (\omega - p + \Delta)\chi_i - \frac{g B_{\mathrm{T}}}{\omega} \frac{1}{\sqrt{2}} i p \Big[(\epsilon_i)_{21} A_\parallel + (\epsilon_i)_{22} A_\perp \Big] + \\ + \frac{1}{2} \delta_{i,+1} \left(\frac{g B_{\mathrm{T}}}{\omega} \right)^2 \frac{p^2}{\omega^2 - p^2} \chi_{+1} \, . \tag{2.90}$$

Before proceeding further, let us discuss the last term in (2.90): this has originated from substituting for A^3 from the gauge constraints (2.86) and clearly only contributes to the equation of motion for χ_{+1}. However, this spin-2 polarization will be shown to decouple from the mixing problem and will therefore represent a well-defined eigenstate with mass m. We are therefore safe to substitute $\omega^2 - p^2 = m^2$, so that the last term in (2.90) becomes $\mathcal{O}(\frac{g^2 B^2}{m^2})$ and one can therefore neglect it by the token of perturbative consistency of the initial lagrangian. Substituting the different possible choices for the index i one by one, one obtains five equations of motion for the five massive spin-2 polarizations

$$0 = (\omega - p - \Delta)\chi_{+1} \, , \tag{2.91a}$$

$$0 = (\omega - p - \Delta)\chi_{\times 1} \, , \tag{2.91b}$$

$$0 = (\omega - p - \Delta)\chi_{\times 2} + a_2 p i A_\parallel \, , \tag{2.91c}$$

$$0 = (\omega - p - \Delta)\chi_{+2} + a_2 p i A_\perp \, , \tag{2.91d}$$

$$0 = (\omega - p - \Delta)\chi_{0} + a_0 p i A_\perp \, , \tag{2.91e}$$

where we have introduced the parameters

$$a_2 = + \frac{g B_{\mathrm{T}}}{2\omega} \, , \tag{2.92}$$

$$a_0 = - \frac{g B_{\mathrm{T}}}{\sqrt{3}\omega} \, . \tag{2.93}$$

On the other hand, focusing on the $\alpha = 1, 2$ components of the EM equation of motion (2.77b) (and realizing that, as usual, we have $\partial^1 = 0 = \partial^2$), we first obtain

$$0 = \Box A^1 - \sqrt{2} g (F_{\mathrm{ext}})_{2}{}^{3} \partial_3 \chi^{21} \, , \tag{2.94a}$$

$$0 = \Box A^2 - \sqrt{2} g (F_{\mathrm{ext}})_{2}{}^{3} \partial_3 \chi^{22} \, . \tag{2.94b}$$

Substituting in terms of $A_\parallel$, $A_\perp$ and decomposing the spin-2 field into the usual basis of polarizations, we eventually obtain

$$0 = (\omega - p) i A_\parallel + a_2 p \chi^{2\times} \, , \tag{2.95a}$$

$$0 = (\omega - p)iA_\perp + a_2 p\chi^{2+} + a_0 p\chi^0 \,. \tag{2.95b}$$

In total, expressing a_0 in terms of a_2 and, as in the case of the massive scalar, introducing effective photon mass terms $\Delta_\parallel$, $\Delta_\perp$ for the EM polarizations $A_\parallel$, $A_\perp$ (and relabelling $iA \to A$, as with the scalar), we can write all equations of motion in the matrix form as

$$0 = (\omega - p - \Delta)\chi_{+1} \,, \tag{2.96a}$$

$$0 = (\omega - p - \Delta)\chi_{\times 1} \,, \tag{2.96b}$$

$$0 = \begin{pmatrix} \omega - p - \Delta & a_2 p \\ a_2 p & \omega - p + \Delta_\parallel \end{pmatrix} \begin{pmatrix} \chi_{\times 2} \\ A_\parallel \end{pmatrix} \,, \tag{2.96c}$$

$$0 = \begin{pmatrix} \omega - p - \Delta & 0 & a_2 p \\ 0 & \omega - p - \Delta & a_0 p \\ a_2 p & a_0 p & \omega - p + \Delta_\perp \end{pmatrix} \begin{pmatrix} \chi_{+2} \\ \chi_0 \\ A_\perp \end{pmatrix} \,. \tag{2.96d}$$

This form clearly emphasizes the mixing between various EM and spin-2 polarizations: the parallel EM polarization $A_\parallel$ mixes with the $\chi_{\times 2}$ spin-2 polarization, while the EM polarization $A_\perp$ which is transverse to the external magnetic field, mixes with the χ_{+2} and χ_0 spin-2 polarizations.

2.3 Searching for mass eigenstates

Similarly as in the scalar case, we use the condition that the determinant of the matrices has to be zero in order for the equations to have solutions to determine the possible values of p in terms of the energy ω.

2.3.1 Decoupled spin-2 polarizations

Starting with the decoupled spin-2 polarizations χ_{+1} and $\chi_{\times 1}$, we simply get the (linearized) dispersion relations

$$p_{+1}(\omega) = \omega - \Delta \,, \tag{2.97a}$$

$$p_{\times 1}(\omega) = \omega - \Delta \,, \tag{2.97b}$$

which are telling us that both χ_{+1} and $\chi_{\times 1}$ propagate as eigenstates of definite mass which is directly given by the parameter m entering the lagrangian (2.1). The modes would therefore propagate as

$$\chi_{+1}(z) = \chi_{+1}(0)e^{-i(\omega-\Delta)z} \,, \tag{2.98a}$$

$$\chi_{\times 1}(z) = \chi_{\times 1}(0)e^{-i(\omega-\Delta)z} \,. \tag{2.98b}$$

2.3.2 $\chi_{\times 2}$–$A_\parallel$ mixing

Second, we note that the mixing problem of the EM polarization $A_\parallel$ with the spin-2 polarization $\chi_{\times 2}$ is precisely isomorphic to the 2-state mixing considered extensively in the previous chapter for the transverse EM polarization $A_\perp$ and the massive scalar ϕ, meaning that we can straightforwardly write down the results. In particular, one obtains the dispersion relations

$$2\left(1 - a_2^2\right)p^{(1)}_{\times 2,\parallel}(\omega) = 2\omega + \Delta_\parallel - \Delta - \sqrt{D_\parallel} \,, \tag{2.99a}$$

$$2\Big(1-a_2^2\Big)p^{(2)}_{\times 2,\parallel}(\omega) = 2\omega + \Delta_\parallel - \Delta + \sqrt{D_\parallel}\,, \tag{2.99b}$$

where the discriminant $D_\parallel$ can be expressed as

$$D_\parallel = (\Delta - \Delta_\parallel - 2\omega)^2 - 4\Big(1-a_2^2\Big)\Big[\omega(\Delta_\parallel - \Delta) + \omega^2 - \Delta\Delta_\parallel\Big]\,. \tag{2.100}$$

These dispersion relations describe the propagation of two mass eigenstates. The corresponding two directions $(\chi^{(1)}_{\times 2}, A^{(1)}_\parallel)$ and $(\chi^{(2)}_{\times 2}, A^{(2)}_\parallel)$ in the $\chi_{\times 2}$–$A_\parallel$ flavour space are specified by the relations

$$\frac{A^{(1)}_\parallel}{\chi^{(1)}_{\times 2}} = \frac{-a_2 p^{(1)}_{\times 2,\parallel}}{\omega - p^{(1)}_{\times 2,\parallel} + \Delta_\parallel} \equiv -\tan\Theta^{(1)}_{\times 2,\parallel}\,, \tag{2.101a}$$

$$\frac{\chi^{(2)}_{\times 2}}{A^{(2)}_\parallel} = \frac{-a_2 p^{(2)}_{\times 2,\parallel}}{\omega - p^{(2)}_{\times 2,\parallel} - \Delta} \equiv +\tan\Theta^{(2)}_{\times 2,\parallel}\,. \tag{2.101b}$$

Assuming the relativistic approximation $\Delta, \Delta_\parallel \ll \omega$, as well as the small-mixing scenario $\Theta^{(1)}_{\times 2,\parallel} \approx \Theta^{(2)}_{\times 2,\parallel} \equiv \Theta_{\times 2,\parallel} \ll 1$ (which arises when $gB_\mathrm{T} \ll \Delta + \Delta_\parallel$), we obtain the dispersion relations

$$p^{(1)}_{\times 2,\parallel}(\omega) = \omega - \Delta - \frac{1}{4}\frac{g^2B_\mathrm{T}^2}{\Delta + \Delta_\parallel}\,, \tag{2.102a}$$

$$p^{(2)}_{\times 2,\parallel}(\omega) = \omega + \Delta_\parallel + \frac{1}{4}\frac{g^2B_\mathrm{T}^2}{\Delta + \Delta_\parallel}\,, \tag{2.102b}$$

for the $\chi_{\times 2}$-like and the $A_\parallel$-like mode, respectively. The mixing angle $\Theta_{\times 2,\parallel}$ becomes simply

$$\Theta_{\times 2,\parallel} = \frac{1}{2}\frac{gB_\mathrm{T}}{\Delta + \Delta_\parallel}\,. \tag{2.103}$$

If we furthermore introduce the mass parameter $b_\times$ as

$$b_\times \equiv \Delta + \Delta_\parallel\,, \tag{2.104}$$

we can rewrite

$$p^{(1)}_{\times 2,\parallel}(\omega) = \omega n_\parallel - b_\times(1 + \Theta^2_{\times 2,\parallel})\,, \tag{2.105a}$$

$$p^{(2)}_{\times 2,\parallel}(\omega) = \omega n_\parallel + b_\times\Theta^2_{\times 2,\parallel}\,. \tag{2.105b}$$

This enables us to write down the general solution for the $\chi_{\times 2}$–$A_\parallel$ oscillations in the form

$$\begin{aligned} e^{i\omega n_\parallel z}\begin{pmatrix}\chi_{\times 2}(z)\\ A_\parallel(z)\end{pmatrix} &= \\ &= \frac{1}{1+\Theta^2_{\times 2,\parallel}}\begin{pmatrix}1\\ -\Theta_{\times 2,\parallel}\end{pmatrix}\Big[\chi_{\times 2}(0) - A_\parallel(0)\Theta_{\times 2,\parallel}\Big]e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + \\ &\quad + \frac{1}{1+\Theta^2_{\times 2,\parallel}}\begin{pmatrix}\Theta_{\times 2,\parallel}\\ 1\end{pmatrix}\Big[A_\parallel(0) + \chi_{\times 2}(0)\Theta_{\times 2,\parallel}\Big]e^{-ib_\times\Theta^2_{\times 2,\parallel}z}\,. \end{aligned} \tag{2.106}$$

As in the case of the massive spin-0 mixing with the photon (which was considered in the previous chapter), the evolution equation (2.106) can be recast in terms of a unitary trasfer matrix $\mathsf{U}_{\times 2,\parallel}(z,0)$, see (2.142) below.

2.3.3 χ_{+2}–χ_0–$A_\perp$ mixing

Finally, let us address the problem of finding the mass eigenstates in the coupled system of χ_{+2}, χ_0 and $A_\perp$ polarizations. Naively, it might seem as if we will have to force ourselves to go through the tedious work of diagonalizing the Hamiltonian in a coupled system of three particle flavours by sheer brute force. However, introducing the notation

$$a_+ = \sqrt{a_0^2 + a_2^2} = \sqrt{\frac{7}{3}}\frac{gB_{\mathrm{T}}}{2\omega} = \sqrt{\frac{7}{3}}a_2\,, \tag{2.107}$$

one can conveniently observe that upon rotating the polarizations as

$$\begin{pmatrix} \chi_+ \\ \chi'_+ \\ A_\perp \end{pmatrix} = R \begin{pmatrix} \chi_{+2} \\ \chi_0 \\ A_\perp \end{pmatrix}\,, \tag{2.108}$$

where the rotation matrix R is defined as

$$R = \begin{pmatrix} \frac{a_2}{a_+} & +\frac{a_0}{a_+} & 0 \\ -\frac{a_0}{a_+} & \frac{a_2}{a_+} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{7}} \begin{pmatrix} \sqrt{3} & -2 & 0 \\ 2 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{2.109}$$

one can rotate the (hamiltonian) matrix entering the equation of motion (2.96d) to obtain

$$\begin{aligned} R \begin{pmatrix} \omega - p - \Delta & 0 & a_2 p \\ 0 & \omega - p - \Delta & a_0 p \\ a_2 p & a_0 p & \omega - p + \Delta_\perp \end{pmatrix} R^{-1} = \\ = \begin{pmatrix} \omega - p - \Delta & 0 & a_+ p \\ 0 & \omega - p - \Delta & 0 \\ a_+ p & 0 & \omega - p + \Delta_\perp \end{pmatrix}\,. \end{aligned} \tag{2.110}$$

The equation of motion (2.96d) therefore becomes

$$0 = (\omega - p - \Delta)\chi'_+\,, \tag{2.111a}$$

$$0 = \begin{pmatrix} \omega - p - \Delta & a_+ p \\ a_+ p & \omega - p + \Delta_\perp \end{pmatrix} \begin{pmatrix} \chi_+ \\ A_\perp \end{pmatrix}\,. \tag{2.111b}$$

We conclude that the polarization χ'_+ decouples and is described by the dispersion relation

$$p'_+(\omega) = \omega - \Delta\,, \tag{2.112}$$

so that it represents an eigenstate of definite mass which is equal to m. On the other hand for the polarizations χ_+ and $A_\perp$, we again obtain a 2-flavour mixing problem. Imposing the condition that the determinant of the matrix in the equation of motion (2.111b), one obtains the dispersion relations

$$2\left(1 - a_+^2\right)p^{(1)}_{+,\perp}(\omega) = 2\omega + \Delta_\perp - \Delta - \sqrt{D_\perp}\,, \tag{2.113a}$$

$$2\left(1 - a_+^2\right)p^{(2)}_{+,\perp}(\omega) = 2\omega + \Delta_\perp - \Delta + \sqrt{D_\perp}\,, \tag{2.113b}$$

with the discriminant

$$D_{\perp} = (\Delta - \Delta_{\perp} - 2\omega)^2 - 4\Big(1 - a_+^2\Big)\Big[\omega(\Delta_{\perp} - \Delta) + \omega^2 - \Delta\Delta_{\perp}\Big]\,. \tag{2.114}$$

The two mass eigenstates described by these relations correspond to two directions $(\chi_+^{(1)}, A_\perp^{(1)})$ and $(\chi_+^{(2)}, A_\perp^{(2)})$ in the χ_+–$A_\perp$ flavour space, which define the mixing angles $\Theta_{+,\perp}^{(1)}$ and $\Theta_{+,\perp}^{(2)}$. These satisfy

$$\frac{A_\perp^{(1)}}{\chi_+^{(1)}} = \frac{-a_+ p_{+,\perp}^{(1)}}{\omega - p_{+,\perp}^{(1)} + \Delta_\perp} \equiv -\tan\Theta_{+,\perp}^{(1)}\,, \tag{2.115a}$$

$$\frac{\chi_+^{(2)}}{A_\perp^{(2)}} = \frac{-a_+ p_{+,\perp}^{(2)}}{\omega - p_{+,\perp}^{(2)} - \Delta} \equiv +\tan\Theta_{+,\perp}^{(2)}\,. \tag{2.115b}$$

Adopting the ultrarelativistic regime $\Delta, \Delta_\perp \ll \omega$ and focusing again on the small-mixing scenario $\Theta_{+,\perp}^{(1)} \approx \Theta_{+,\perp}^{(2)} \equiv \Theta_{+,\perp} \ll 1$ (which arises when $gB_{\mathrm{T}} \ll \Delta + \Delta_\perp$), we obtain the dispersion relations

$$p_{+,\perp}^{(1)}(\omega) = \omega - \Delta - \frac{7}{12}\frac{g^2 B_{\mathrm{T}}^2}{\Delta + \Delta_\perp}\,, \tag{2.116a}$$

$$p_{+,\perp}^{(2)}(\omega) = \omega + \Delta_\perp + \frac{7}{12}\frac{g^2 B_{\mathrm{T}}^2}{\Delta + \Delta_\perp}\,. \tag{2.116b}$$

The first one describes a χ_+-like mode, while the second one represents a $A_\perp$-like mode. The mixing angle $\Theta_{+,\perp}$ becomes simply

$$\Theta_{+,\perp} = \sqrt{\frac{7}{3}}\frac{1}{2}\frac{gB_{\mathrm{T}}}{\Delta + \Delta_\perp}\,. \tag{2.117}$$

Introducing the mass parameter b_+ as

$$b_+ \equiv \Delta + \Delta_\perp\,, \tag{2.118}$$

we can rewrite

$$p_{+,\perp}^{(1)}(\omega) = \omega n_\perp - b_+(1 + \Theta_{+,\perp}^2)\,, \tag{2.119a}$$

$$p_{+,\perp}^{(2)}(\omega) = \omega n_\perp + b_+\Theta_{+,\perp}^2\,. \tag{2.119b}$$

This enables us to write down the general solution for the χ_+–$A_\perp$ oscillations in the form

$$\begin{aligned} e^{i\omega n_\perp z}\begin{pmatrix}\chi_+(z)\\ A_\perp(z)\end{pmatrix} &= \\ &= \frac{1}{1+\Theta_{+,\perp}^2}\begin{pmatrix}1\\ -\Theta_{+,\perp}\end{pmatrix}\Big[\chi_+(0) - A_\perp(0)\Theta_{+,\perp}\Big]e^{ib_+(1+\Theta_{+,\perp}^2)z} + \\ &\quad + \frac{1}{1+\Theta_{+,\perp}^2}\begin{pmatrix}\Theta_{+,\perp}\\ 1\end{pmatrix}\Big[A_\perp(0) + \chi_+(0)\Theta_{+,\perp}\Big]e^{-ib_+\Theta_{+,\perp}^2 z}\,. \end{aligned} \tag{2.120}$$

At the same time, the decoupled mode χ_+' evolves simply as

$$\chi_+'(z) = \chi_+'(0)e^{-i(\omega-\Delta)z}\,. \tag{2.121}$$

See (2.142) for the unitary transfer matrix which compactly expresses the evolution given by (2.120).

2.4 Observable effects

We will now consider performing the same experiments and observations as in the previous chapter in the case of the massive scalar particle. We will again observe various quantities to oscillate upon varying length of the path z over which the beam travels through the magnetic field. In particular, we will find the oscillation lengths

$$l_{\mathrm{osc},\times} = \frac{2\pi}{b_\times}\,, \tag{2.122a}$$

$$l_{\mathrm{osc},+} = \frac{2\pi}{b_+}\,, \tag{2.122b}$$

due to $A_\parallel$–$\chi_\times$ mixing and the $A_\perp$–χ_+ mixing, respectively.

2.4.1 Effects on photon polarization

As in the scalar case, let prepare the system in a linearly polarized pure-EM initial state (where $\chi_{\mu\nu}(0) = 0$). Substituting into (2.120) and (2.106), we find that the propagation of the EM fields in the region with the magnetic field turned on is described by

$$e^{i\omega n_\parallel z} A_\parallel(z) = \frac{1}{1+\Theta^2_{\times 2,\parallel}}\left[\Theta^2_{\times 2,\parallel} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times \Theta^2_{\times 2,\parallel} z}\right] A_\parallel(0)\,, \tag{2.123a}$$

$$e^{i\omega n_\perp z} A_\perp(z) = \frac{1}{1+\Theta^2_{+,\perp}}\left[\Theta^2_{+,\perp} e^{ib_+(1+\Theta^2_{+,\perp})z} + e^{-ib_+ \Theta^2_{+,\perp} z}\right] A_\perp(0)\,. \tag{2.123b}$$

Contrary to the massive scalar case, we observe that for massive spin-2, both the $A_\parallel$ mode, as well as the $A_\perp$ mode evolve non-trivially, as they propagate through the magnetic environment. Assuming as in the scalar case that we are in the regime

$$\omega(n_\perp - n_\parallel) z \ll 1\,, \tag{2.124}$$

as well as

$$b_\times \Theta^2_{\times 2,\parallel} z \ll 1\,, \tag{2.125a}$$

$$b_+ \Theta^2_{+,\perp} z \ll 1\,, \tag{2.125b}$$

the ratio $A_\perp(z)/A_\parallel(z)$ changes only by a small amount, namely

$$\frac{A_\perp(z)}{A_\parallel(z)} = \frac{A_\perp(0)}{A_\parallel(0)} \frac{1+\Theta^2_{\times 2,\parallel}}{1+\Theta^2_{+,\perp}} \frac{\Theta^2_{+,\perp} e^{ib_+(1+\Theta^2_{+,\perp})z} + e^{-ib_+\Theta^2_{+,\perp} z}}{\Theta^2_{\times 2,\parallel} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times\Theta^2_{\times 2,\parallel} z}} e^{-i\omega(n_\perp - n_\parallel)z} \tag{2.126a}$$

$$\begin{aligned}
&\approx \frac{A_\perp(0)}{A_\parallel(0)}\Big(1 + \Theta^2_{+,\perp} e^{ib_+ z} - ib_+\Theta^2_{+,\perp} z - \Theta^2_{\times 2,\parallel} e^{ib_\times z} - i(\Delta_\perp - \Delta_\parallel)z + \\
&\qquad + ib_\times \Theta^2_{\times 2,\parallel} z + \Theta^2_{\times 2,\parallel} - \Theta^2_{+,\perp}\Big)
\end{aligned} \tag{2.126b}$$

$$\begin{aligned}
&= \frac{A_\perp(0)}{A_\parallel(0)}\Big[1 - 2\Theta^2_{+,\perp}\sin^2\frac{b_+ z}{2} + 2\Theta^2_{\times 2,\parallel}\sin^2\frac{b_\times z}{2} - i(\Delta_\perp - \Delta_\parallel)z + \\
&\qquad - i\Theta^2_{+,\perp}\Big(b_+ z - \sin b_+ z\Big) + i\Theta^2_{\times 2,\parallel}\Big(b_\times z - \sin b_\times z\Big)\Big]\,.
\end{aligned} \tag{2.126c}$$

Given this, we can easily identify the relative change in the amplitude $\eta(z)$, as well as the phase delay $\varphi(z)$ (see (1.169) for their definitions) as

$$\eta(z) = 2\Theta_{+,\perp}^2 \sin^2 \frac{b_+ z}{2} - 2\Theta_{\times 2,\parallel}^2 \sin^2 \frac{b_\times z}{2}\,, \tag{2.127a}$$

$$\varphi(z) = (b_+ - b_\times)z + \Theta_{+,\perp}^2\Big(b_+ z - \sin b_+ z\Big) - \Theta_{\times 2,\parallel}^2\Big(b_\times z - \sin b_\times z\Big)\,, \tag{2.127b}$$

where we have noted that $\Delta_\perp - \Delta_\parallel = b_+ - b_\times$. Furthermore, let us first assume that the EM and the spin-2 wave do not decohere as they pass through the region with the magnetic field. In other words, let us assume that $z \ll l_{\mathrm{osc},\times}$ and $z \ll l_{\mathrm{osc},+}$ (that is, $b_+ z \ll 1$ and $b_\times z \ll 1$). We can approximate the expressions for $\eta(z)$ and $\varphi(z)$ as

$$\eta(z) \approx \frac{1}{6} g^2 B_{\mathrm{T}}^2 z^2\,, \tag{2.128a}$$

$$\varphi(z) \approx (n_\perp - n_\parallel)\omega z + \frac{1}{36}\frac{g^2 B_{\mathrm{T}}^2 z^3}{\omega}\left[m^2 + 2\Big(n_\perp - 1 + \frac{3}{4}(n_\perp - n_\parallel)\Big)\omega^2\right]. \tag{2.128b}$$

After the particle has passed through the magnetic field, one should measure that the plane in which the electric field oscillates has rotated by an angle $\delta\theta(z)$ and that the linearly polarized wave has acquired a small ellipticity $\delta\psi(z)$. Under the assumption that $z \ll l_{\mathrm{osc},\times}$ and $z \ll l_{\mathrm{osc},+}$ (namely that coherence is retained), these are given by

$$\delta\theta(z) \approx -\frac{1}{12} g^2 B_{\mathrm{T}}^2 z^2 \sin 2\theta\,, \tag{2.129a}$$

$$\begin{aligned}\delta\psi(z) \approx& -\frac{1}{2}(n_\perp - n_\parallel)\omega z\Big(1 + \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2\Big)\sin 2\theta + \\ &- \frac{1}{72}\frac{g^2 B_{\mathrm{T}}^2 z^3}{\omega}\Big[m^2 + 2(n_\perp - 1)\omega^2\Big]\sin 2\theta\,.\end{aligned} \tag{2.129b}$$

It is now in order to comment on the differences with respect to the expressions (1.186) which were obtained for the massive scalar. In particular one should be interested in whether the massive spin-2 and massive spin-0 case could be told apart by measuring $\delta\theta(z)$ and $\delta\psi(z)$ in the limit $z \ll l_{\mathrm{osc},\times}$ and $z \ll l_{\mathrm{osc},+}$. First, we notice [10] that by performing a redefinition $g' = \frac{4}{3}g^2$ of the coupling constant, one finds that except for the factor of

$$1 + \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2\,, \tag{2.130}$$

appearing in the first line in (2.129b), one ends up with expressions for $\delta\theta(z)$ and $\delta\psi(z)$ which are identical to those for the massive scalar. So the problem reduces to the question, in which regime the factor (2.130) becomes important – if its effect is negligible, this experiment is unable to distinguish between massive spin-0 and massive spin-2. This will very much depend on the relative magnitude of the terms

$$m^2\,, \qquad (n_\perp - 1)\omega^2 \qquad \text{and} \qquad (n_\perp - n_\parallel)\omega^2\,. \tag{2.131}$$

In particular, in order for (2.130) to play any role, we need $(n_\perp - n_\parallel)\omega^2$ to be at least comparable with $(n_\perp - 1)\omega^2$ and much greater then m^2. [3] In practice, for the

[3]The price one would have to pay for being able to distinguish the two particle species in this regime would be the inability to measure the mass m, as it then completely drops out from both $\delta\theta(z)$ and $\delta\psi(z)$.

typical laboratory setup we have considered in the previous chapter ($\hbar\omega = 2.4\,\mathrm{eV}$, chamber with air pressure $10^{-9}\,\mathrm{Pa}$ and magnetic field $B_\mathrm{T} = 10\,\mathrm{T}$), one would need $m \ll m_\mathrm{c} \simeq 10^{-8}\,\mathrm{eV}$ (see (1.187) and below) in order for the m^2 to be suppressed compared to the other two terms listed in (2.131). However, even if this were satisfied, one would need to arrange for the difference $n_\parallel - n_\perp$ to be comparable with $n_\perp - 1$. In practice, substituting our typical laboratory values, one finds that $n_\parallel - n_\perp \simeq 10^{-22}$ (dominated by vacuum birefringence) is much smaller than $n_\perp - 1 \simeq 10^{-17}$. We can therefore conclude that it would be difficult to distinguish the massive spin-2 from massive spin-0 by measuring $\delta\theta(z)$ and $\delta\psi(z)$ in the regime where $z \ll l_{\mathrm{osc},\times}$ and $z \ll l_{\mathrm{osc},+}$.[10]

Let us also briefly discuss the regime $z \gg l_{\mathrm{osc},\times}$ and $z \gg l_{\mathrm{osc},+}$, where the oscillations are averaged out. One then obtains the average relative decrease and phase delay

$$\langle\eta(z)\rangle = g^2 B_\mathrm{T}^2 \omega^2 \left[\frac{7}{3}(m^2 + 2(n_\perp - 1)\omega^2)^{-2} - (m^2 + 2(n_\parallel - 1)\omega^2)^{-2}\right], \quad (2.132\mathrm{a})$$

$$\langle\varphi(z)\rangle = (n_\perp - n_\parallel)\omega z + \frac{1}{2} g^2 B_\mathrm{T}^2 \omega z \left[\frac{7}{3}(m^2 + 2(n_\perp - 1)\omega^2)^{-1} + \right.$$
$$\left. - (m^2 + 2(n_\parallel - 1)\omega^2)^{-1}\right], \quad (2.132\mathrm{b})$$

which in turn give

$$\langle\delta\theta(z)\rangle = -\frac{1}{2} g^2 B_\mathrm{T}^2 \omega^2 \left[\frac{7}{3}(m^2 + 2(n_\perp - 1)\omega^2)^{-2} + \right.$$
$$\left. - (m^2 + 2(n_\parallel - 1)\omega^2)^{-2}\right] \sin 2\theta\,, \quad (2.133\mathrm{a})$$

$$\langle\delta\psi(z)\rangle = -\frac{1}{2}(n_\perp - n_\parallel)\omega z \sin 2\theta +$$
$$-\frac{1}{4} g^2 B_\mathrm{T}^2 \omega z \left[\frac{7}{3}(m^2 + 2(n_\perp - 1)\omega^2)^{-1} + \right.$$
$$\left. - (m^2 + 2(n_\parallel - 1)\omega^2)^{-1}\right] \sin 2\theta\,. \quad (2.133\mathrm{b})$$

It is not difficult to see that analogous discussion applies as in the limit $z \ll l_{\mathrm{osc},\times}$ and $z \ll l_{\mathrm{osc},+}$: unless one can arrange for $n_\parallel - n_\perp$ to be at least comparable with $n_\perp - 1$ and, at the same time $(n_\perp - 1)\omega^2$ to be much greater than m^2, the measurement of $\langle\delta\theta(z)\rangle$ and $\langle\delta\psi(z)\rangle$ is unable to discern the massive spin-2 case from the massive spin-0 case.

2.4.2 LSW experiments

Let us now calculate the photon regeneration probability in the light shining through wall experiment. The experimental setup was described in detail in the first chapter. This time, photon can be converted into massive graviton instead of scalar particle, considered in previous chapter. However, the massive graviton, like the scalar, does not interact with the wall material and can thus pass through to the other side where it can be converted back into a photon in the magnetic field. Notice that since in the massive spin-2 case, the $A_\parallel$ polarization also participates in the oscillations, it may now regenerate on the other side of the wall alongside the polarization $A_\perp$. Recalling the formalism we introduced in the case of the scalar, the *total* probability of photon regeneration will therefore be equal to

$$P(A \to \chi \to A) = \frac{|(\Psi_{A_\perp})^\dagger \Psi_{\mathrm{i}}|^2}{|\Psi_{\mathrm{i}}|^2} + \frac{|(\Psi_{A_\parallel})^\dagger \Psi_{\mathrm{i}}|^2}{|\Psi_{\mathrm{i}}|^2} \,. \tag{2.134}$$

where Ψ_{i} denotes the initial state and Ψ_{f} denotes the final state and the individual terms correspond to the regeneration probability of $A_\parallel$ and $A_\perp$ respectively given the initial state Ψ_{i}.

Let us start with a general pure photon state of the form

$$\Psi_{\mathrm{i}} = \begin{pmatrix} 0 \\ A_\parallel(0) \\ 0 \\ A_\perp(0) \end{pmatrix} \,. \tag{2.135}$$

Similar to the scalar case, the final state can be computed as

$$\Psi_{\mathrm{f}} = \mathsf{U}(z_2, 0)\Pi_\chi \mathsf{U}(z_1, 0)\Psi_{\mathrm{i}} \,, \tag{2.136}$$

where Π_χ is the projector on the subspace spanned by the massive graviton fluctuations, namely

$$\Pi_\chi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{2.137}$$

It remains to find the form of the transfer matrix U, given that now there are two pairs of polarizations which separately undergo two-flavour mixing ($\chi_{\times 2}$–$A_\parallel$ and χ_+–$A_\perp$), as well as one decoupled polarization χ'_+. That means that it will be convenient to factorize (in an analogy with the first chapter)

$$\mathsf{U} = \mathsf{U}_{\times 2,\parallel}\mathsf{U}_{+,\perp}\mathsf{U}'_+ \,. \tag{2.138}$$

where the constituent matrices $\mathsf{U}_{\times 2,\parallel}$, $\mathsf{U}_{+,\perp}$ and U'_+ can be parametrized as

$$\mathsf{U}_{\times 2,\parallel}(z, 0) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$U_{\times 2,\parallel}$}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \,, \tag{2.139}$$

as well as

$$\mathsf{U}_{+,\perp}(z, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$U_{+,\perp}$}} & 0 \\ 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.140}$$

and

$$\mathsf{U}'_{+}(z,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & U'_{+} \end{pmatrix}. \tag{2.141}$$

As we can see that the mixing equations for each pair of polarizations take a form very similar to what was encountered in the ϕ-$A_\perp$ problem, it is not difficult to write down the corresponding submatrices $U_{\times 2,\parallel}$, $U_{+,\perp}$, U'_{+}, namely

$$\begin{aligned} U_{\times 2,\parallel}(z,0) = \frac{e^{-i\omega n_\parallel z}}{1+\Theta^2_{\times 2,\parallel}} \times \\ \times \begin{pmatrix} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + \Theta^2_{\times 2,\parallel} e^{-ib_\times \Theta^2_{\times 2,\parallel} z} & \Theta_{\times 2,\parallel}(e^{-ib_\times \Theta^2_{\times 2,\parallel} z} - e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z}) \\ \Theta_{\times 2,\parallel}(e^{-ib_\times \Theta^2_{\times 2,\parallel} z} - e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z}) & \Theta^2_{\times 2,\parallel} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times \Theta^2_{\times 2,\parallel} z} \end{pmatrix}, \end{aligned} \tag{2.142a}$$

$$\begin{aligned} U_{+,\perp}(z,0) = \frac{e^{-i\omega n_\perp z}}{1+\Theta^2_{+,\perp}} \times \\ \times \begin{pmatrix} e^{ib_+(1+\Theta^2_{+,\perp})z} + \Theta^2_{+,\perp} e^{-ib_+ \Theta^2_{+,\perp} z} & \Theta_{+,\perp}(e^{-ib_+ \Theta^2_{+,\perp} z} - e^{ib_+(1+\Theta^2_{+,\perp})z}) \\ \Theta_{+,\perp}(e^{-ib_+ \Theta^2_{+,\perp} z} - e^{ib_+(1+\Theta^2_{+,\perp})z}) & \Theta^2_{+,\perp} e^{ib_+(1+\Theta^2_{+,\perp})z} + e^{-ib_+ \Theta^2_{+,\perp} z} \end{pmatrix}, \end{aligned} \tag{2.142b}$$

and

$$U'_{+} = e^{-i(\omega-\Delta)z}. \tag{2.143}$$

Since χ'_{+} does not mix with any other mode, it will not affect our LSW setup. This allows us to ignore the last row and column in all matrices throughout the calculation. After some manipulations, we would have found

$$\begin{aligned} \frac{|(\Psi_{A_\parallel})^\dagger \mathsf{U}(z_2,0)\Pi_\chi \mathsf{U}(z_1,0)\Psi_{\mathrm{i}}|^2}{|\Psi_{\mathrm{i}}|^2} &= \\ &\approx \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\Theta^4_{\times 2,\parallel}\Big|(1-e^{ib_\times z_1})(1-e^{ib_\times z_2})\Big|^2 && (2.144\text{a}) \\ &= \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2} \times 16\Theta^4_{\times 2,\parallel}\sin^2\frac{b_\times z_1}{2}\sin^2\frac{b_\times z_2}{2}, && (2.144\text{b}) \end{aligned}$$

as well as

$$\begin{aligned} \frac{|(\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_\chi \mathsf{U}(z_1,0)\Psi_{\mathrm{i}}|^2}{|\Psi_{\mathrm{i}}|^2} &= \\ &\approx \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\Theta^4_{+,\perp}\Big|(1-e^{ib_+ z_1})(1-e^{ib_+ z_2})\Big|^2 && (2.145\text{a}) \\ &= \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2} \times 16\Theta^4_{+,\perp}\sin^2\frac{b_+ z_1}{2}\sin^2\frac{b_+ z_2}{2}. && (2.145\text{b}) \end{aligned}$$

Thus, the overall regeneration probability can be expressed as

$$P(A\to\chi\to A) = \frac{16}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\Big[|A_\parallel(0)|^2\Theta^4_{\times 2,\parallel}\sin^2\frac{b_\times z_1}{2}\sin^2\frac{b_\times z_2}{2} +$$

$$+ |A_\perp(0)|^2 \Theta^4_{+,\perp} \sin^2 \frac{b_+ z_1}{2} \sin^2 \frac{b_+ z_2}{2} \Big] \tag{2.146a}$$

$$= \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\parallel \to \chi_{\times 2} \to A_\parallel) +$$

$$+ \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\perp \to \chi_+ \to A_\perp) \,, \tag{2.146b}$$

where we define

$$P(A_\parallel \to \chi_{\times 2} \to A_\parallel) = 16 \Theta^4_{\times 2,\parallel} \sin^2 \frac{b_\times z_1}{2} \sin^2 \frac{b_\times z_2}{2} \,, \tag{2.147a}$$

$$P(A_\perp \to \chi_+ \to A_\perp) = 16 \Theta^4_{+,\perp} \sin^2 \frac{b_+ z_1}{2} \sin^2 \frac{b_+ z_2}{2} \,. \tag{2.147b}$$

These are the (elementary) probabilities of regenerating the polarization $A_\parallel$ (or $A_\perp$) given that we start with a pure $A_\parallel$ (or $A_\perp$) beam (noticing as well that the mixed transition probabilities $P(A_\parallel \to \chi_{\times 2} \to A_\perp)$ and $P(A_\perp \to \chi_{\times 2} \to A_\parallel)$ both vanish). As expected, since the two photon polarizations do not interact with each other through mixing with the massive spin-2 field, we observe that they could have been treated individually.

We can further simplify the results for the situation where the sizes of both regions with the magnetic field are much smaller than the oscillation lengths (2.122) for the two independent 2-flavour mixing problems, i.e.

$$b_\times z_1, b_+ z_1 \ll 1 \,, \tag{2.148a}$$

$$b_\times z_2, b_+ z_2 \ll 1 \,. \tag{2.148b}$$

In this case the massive spin-2 waves and EM waves will not decohere as they propagate through the magnetic field. After substituting for the mixing angles $\Theta_{\times 2,\parallel}$ and $\Theta_{+,\perp}$ and mass parameters $b_\times$ and b_+, we arrive at the result

$$P(A \to \chi \to A) \approx \frac{1}{16} g^4 B_\mathrm{T}^4 z_1^2 z_2^2 \frac{|A_\parallel(0)|^2 + \frac{49}{9}|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \,. \tag{2.149}$$

As in the case of the massive scalar, this does not depend on the mass parameter m. We can notice that in order to maximize the regeneration probability, one should take the initial beam to consist purely of the $A_\perp$ polarization. On the other hand, preparing the system in a pure $A_\parallel$ initial state would enable us to distinguish between the massive graviton and the scalar through this experiment, because, while in the massive spin-2 case, one should expect a positive measurement on the other side of the wall, in the scalar particle case, no signal would be obtained.

On the other hand, if the opposite regime takes place, that is if we have both $b_\times z_1, b_+ z_1 \gg 1$ and $b_\times z_2, b_+ z_2 \gg 1$ (z_1 and z_2 much greater than the two oscillation lengths (2.122)), we obtain the mean regeneration probability

$$\langle P(A \to \chi \to A) \rangle =$$

$$= \frac{4}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \Big[|A_\parallel(0)|^2 \Theta^4_{\times 2,\parallel} + |A_\perp(0)|^2 \Theta^4_{+,\perp} \Big] \tag{2.150a}$$

$$= \frac{4g^4 B_\mathrm{T}^4 \omega^4}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \left[\frac{|A_\parallel(0)|^2}{(m^2 + 2(n_\parallel - 1)\omega^2)^4} + \right.$$
$$\left. + \frac{49}{9} \frac{|A_\perp(0)|^2}{(m^2 + 2(n_\perp - 1)\omega^2)^4} \right] . \quad (2.150b)$$

We observe that the dependence on z_1 and z_2 has been traded for the dependence on the mass m and the refractive indices $n_\parallel$, $n_\perp$.

2.4.3 Relative intensity decrease

As a consequence of the mixing, one should again expect to observe a relative decrease in the intensity of the photon beam propagating in an external magnetic field. This time the problem becomes slightly more complicated, because both photon polarizations undergo mixing. Let us start with a pure-photon state $\Psi_i = (0, A_\parallel(0), 0, A_\perp(0))$ and consider the small mixing scenario. The total relative intensity decrease can then be computed as

$$\frac{|A(0)|^2 - |A(z)|^2}{|A(0)|^2} =$$
$$= \frac{1}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \left[|A_\parallel(0)|^2 \frac{|A_\parallel(0)|^2 - |A_\parallel(z)|^2}{|A_\parallel(0)|^2} + \right.$$
$$\left. + |A_\perp(0)|^2 \frac{|A_\perp(0)|^2 - |A_\perp(z)|^2}{|A_\perp(0)|^2} \right] \quad (2.151a)$$
$$= \frac{1}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \left[|A_\parallel(0)|^2 P(A_\parallel \to \chi_{\times 2}) + \right.$$
$$\left. + |A_\perp(0)|^2 P(A_\perp \to \chi_+) \right] \quad (2.151b)$$
$$= \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \times 4\Theta^2_{\times 2,\parallel} \sin^2\left(\frac{b_\times z}{2}\right) +$$
$$+ \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} \times 4\Theta^2_{+,\perp} \sin^2\left(\frac{b_+ z}{2}\right) , \quad (2.151c)$$

namely as a weighted sum of the two partial relative intensity decreases which one would measure for each of the two photon polarizations (we have realized that $|A(0)|^2 = |A_\parallel(0)|^2 + |A_\perp(0)|^2$ and then used the results for the two-flavour mixing derived already for the massive spin-0). We observe that depending on the initial polarization of the beam, the intensity will oscillate with two different amplitudes

$$\alpha_\times = \frac{4g^2 B_\mathrm{T}^2 \omega^2}{[m^2 + 2(n_\parallel - 1)\omega^2]^2} , \quad (2.152a)$$
$$\alpha_+ = \frac{7}{3} \frac{4g^2 B_\mathrm{T}^2 \omega^2}{[m^2 + 2(n_\perp - 1)\omega^2]^2} , \quad (2.152b)$$

and oscillation lengths

$$l_{\mathrm{osc},\times} = \frac{4\pi\omega}{m^2 + 2(n_\parallel - 1)\omega^2} , \quad (2.153a)$$

$$l_{\mathrm{osc},+} = \frac{4\pi\omega}{m^2 + 2(n_\perp - 1)\omega^2} \,. \tag{2.153b}$$

While one should generally expect the oscillation lengths to be roughly the same (unless $n_\perp - n_\parallel$ is at least comparable to $n_\perp - 1$ and m^2 is suppressed relative to $2(n_\perp - 1)\omega^2$), the two ampitudes, being proportional to the squares of the corresponding mixing angles, will differ by a factor of $7/3$.

3. Mixing of photons in GR

Let us now carefully discuss the case of mixing a massless graviton with a photon, which also goes by the name of the *Gertsenshtein-Zel'dovich effect.*[48, 49] Although one could naively think that the dynamics of the oscillations in the massless spin-2 case could be recovered by simply taking the $m \to 0$ limit of the massive spin-2 results derived in the previous chapter, this turns out to be incorrect due to the fact that in the massive case, there are additional degrees of freedom which do not decouple even for arbitrarily small value of m (namely the χ_0 mode). This is the famous *van Dam-Veltman-Zakharov (vDVZ) discontinuity.* [16, 17] The massless spin-2 case therefore calls for a separate treatment.

Note that one has to be cautious about the signs because, unlike most of the literature on Einstein gravity, in this thesis we are working with the mostly minus (West Coast) convention for the metric signature.

3.1 The Einstein-Maxwell theory

We will start with an action derived by combining the standard general relativity Lagrangian and the general-covariant electromagnetic Lagrangian $\mathcal{L}_{\text{EM}}$, namely [50]

$$S = \int d^4x \sqrt{-g}\Big(-\frac{1}{2\kappa}R + \mathcal{L}_{\text{EM}}\Big)\,, \tag{3.1}$$

where $g = \det g_{\mu\nu}$ denotes the determinant of the spacetime metric, R denotes Ricci scalar and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant. In the natural units (2), it is related to the Planck mass $m_{\text{Pl}} \simeq 2.4 \times 10^{18}$ GeV as

$$\frac{1}{2\kappa} = m_{\text{Pl}}^2\,. \tag{3.2}$$

In particular, for the EM-part of the action (3.1) we have

$$\begin{aligned} S_{\text{EM,int}} &= \int d^4x \sqrt{-g}\mathcal{L}_{\text{EM}} && (3.3\text{a}) \\ &= -\frac{1}{4}\int d^4x \sqrt{-g}F_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\,, && (3.3\text{b}) \end{aligned}$$

where by denoting it as $S_{\text{EM,int}}$, we have emphasized the fact that it contains not only the Maxwell kinetic term, but also the interactions of the EM field with the gravitational field. Rewriting $F_{\mu\nu}$ into covariant form and using the symmetry of the Christoffel symbols in the lower indices, we have

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu && (3.4\text{a}) \\ &= \partial_\mu A_\nu - \Gamma_{\nu\mu}{}^{\alpha}A_\alpha - (\partial_\nu A_\mu - \Gamma_{\mu\nu}{}^{\alpha}A_\alpha) && (3.4\text{b}) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, && (3.4\text{c}) \end{aligned}$$

so that the EM field-strength tensor has the same form as in the flat space. It is also straightforward to see that the Bianchi identity for the EM field-strength tensor continues to hold in curved space. In particular, one has

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} =$$

$$= \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} - \Gamma^\delta{}_{\alpha\beta} F_{\delta\gamma} - \Gamma^\delta{}_{\alpha\gamma} F_{\beta\delta} - \Gamma^\delta{}_{\beta\gamma} F_{\delta\alpha} + \\ - \Gamma^\delta{}_{\beta\alpha} F_{\gamma\delta} - \Gamma^\delta{}_{\gamma\alpha} F_{\delta\beta} - \Gamma^\delta{}_{\gamma\beta} F_{\alpha\delta} \,, \quad (3.5)$$

where, since $F_{\alpha\beta}$ can be expressed in terms of ordinary partial derivatives as in (3.4c), the first three terms vanish by the same argument as in the flat space, while the remaining six terms vanish by recalling the symmetry of the Christoffel symbols in its the lower indices. In total, one therefore has

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \,. \quad (3.6)$$

If we now vary the action (3.1) with respect to the metric, we get the well-known Einstein's field equations of general relativity, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (T_{\text{EM}})_{\mu\nu} \,, \quad (3.7)$$

where we define the Maxwell stress-energy tensor $(T_{\text{EM}})_{\mu\nu}$ on a generic curved background as

$$(T_{\text{EM}})_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} \mathcal{L}_{\text{EM}} \right) = 2 \frac{\partial \mathcal{L}_{\text{EM}}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}_{\text{EM}} \,. \quad (3.8)$$

Substituting the explicit form of $\mathcal{L}_{\text{EM}}$ from (3.3), we obtain

$$(T_{\text{EM}})_{\mu\nu} = F_{\mu\alpha} g^{\alpha\beta} F_{\beta\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} g^{\rho\alpha} g^{\sigma\beta} \,. \quad (3.9)$$

It is straightforward to see that in the flat background $g_{\mu\nu} = \eta_{\mu\nu}$, the expression (3.9) coincides with the expression (2.61) used in the previous chapter. On the other hand, by varying the action with respect to A_μ, we would obtain the equations of motion for the EM field, namely

$$0 = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) \,. \quad (3.10)$$

We can note that for any anti-symmetric tensor field $\Omega^{\alpha\beta}$, one can rewrite the covariant divergence as

$$\nabla_\alpha \Omega^{\alpha\beta} = \partial_\alpha \Omega^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\delta} \Omega^{\delta\beta} + \Gamma^\beta{}_{\alpha\delta} \Omega^{\alpha\delta} \quad (3.11a)$$

$$= \partial_\alpha \Omega^{\alpha\beta} + \frac{1}{\sqrt{-g}} \partial_\delta (\sqrt{-g}) \Omega^{\delta\beta} \quad (3.11b)$$

$$= \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} \Omega^{\alpha\beta}) \,, \quad (3.11c)$$

where in the second equality, we have used the symmetry of the Christoffel symbol in its lower indices and also noted that

$$\Gamma^\alpha{}_{\alpha\delta} = \frac{1}{\sqrt{-g}} \partial_\delta (\sqrt{-g}) \,. \quad (3.12)$$

It follows that one can rewrite

$$\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) = \nabla_\alpha (F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) \,, \quad (3.13)$$

so that the equation of motion (3.10) for the EM field can be equivalently restated as

$$0 = g^{\mu\alpha}\nabla_\alpha F_{\mu\nu} \,. \tag{3.14}$$

Using the equation of motion (3.14), as well as the Bianchi identity (3.6), one can then prove the covariant conservation of $(T_{\mathrm{EM}})_{\mu\nu}$, namely

$$\begin{aligned}
g^{\sigma\mu}\nabla_\sigma\Big((T_{\mathrm{EM}})_{\mu\nu}\Big) &= g^{\sigma\mu}\nabla_\sigma\Big(F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu}\Big) + \frac{1}{4}g^{\sigma\mu}\nabla_\sigma\Big(g_{\mu\nu}F_{\rho\tau}F_{\alpha\beta}g^{\rho\alpha}g^{\tau\beta}\Big) && (3.15\text{a})\\
&= g^{\sigma\mu}F_{\mu\alpha}g^{\alpha\beta}\nabla_\sigma F_{\beta\nu} + \frac{1}{4}\nabla_\nu\Big(F_{\rho\tau}F_{\alpha\beta}g^{\rho\alpha}g^{\tau\beta}\Big) && (3.15\text{b})\\
&= F_{\mu\sigma}g^{\alpha\mu}g^{\sigma\beta}\nabla_\alpha F_{\beta\nu} + \frac{1}{2}F_{\rho\tau}g^{\rho\alpha}g^{\tau\beta}\nabla_\nu F_{\alpha\beta} && (3.15\text{c})\\
&= \frac{1}{2}F_{\mu\sigma}g^{\alpha\mu}g^{\sigma\beta}(\nabla_\alpha F_{\beta\nu} + \nabla_\beta F_{\nu\alpha} + \nabla_\nu F_{\alpha\beta}) && (3.15\text{d})\\
&= 0\,. && (3.15\text{e})
\end{aligned}$$

Finally, computing the trace of the EM stress energy tensor, one gets

$$\begin{aligned}
g^{\mu\nu}(T_{\mathrm{EM}})_{\mu\nu} &= F_{\mu\alpha}g^{\mu\nu}g^{\alpha\beta}F_{\beta\nu} + \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}g^{\rho\alpha}g^{\sigma\beta} && (3.16\text{a})\\
&= -F_{\mu\alpha}F_{\nu\beta}g^{\mu\nu}g^{\alpha\beta} + F_{\rho\sigma}F_{\alpha\beta}g^{\rho\alpha}g^{\sigma\beta} && (3.16\text{b})\\
&= 0\,, && (3.16\text{c})
\end{aligned}$$

namely that the EM stress tensor is traceless. Hence, taking the trace of the field equation (3.7), one concludes that on any classical field configuration, the Ricci scalar needs to vanish. The equation of motion (3.7) can therefore be equivalently recast as

$$R_{\mu\nu} = \kappa(T_{\mathrm{EM}})_{\mu\nu} \,. \tag{3.17}$$

3.2 Linearization around a background

Let us start by considering the expansion of the Einstein-Hilbert-Maxwell action (3.1) around a classical background, that is around field configurations $\overline{g}_{\mu\nu}$ and $\overline{A}_\mu$, which satisfy the classical equations of motion (3.7) and (3.10) (the coupled system of Einstein-Maxwell field equations). To achieve this goal, we will expand the fields $g_{\mu\nu}$ and A_μ in fluctuations $\delta g_{\mu\nu}$, δA_μ around their background values $\overline{g}_{\mu\nu}$, $\overline{A}_\mu$ as

$$\begin{aligned}
g_{\mu\nu} &= \overline{g}_{\mu\nu} + \delta g_{\mu\nu}\,, && (3.18\text{a})\\
A_\mu &= \overline{A}_\mu + \delta A_\mu\,. && (3.18\text{b})
\end{aligned}$$

We would like to find the corresponding expansion of the action (3.1) in powers of $\delta g_{\mu\nu}$ and δA_μ. We therefore seek to write the action (3.1) in the form

$$S\Big[\overline{g}_{\mu\nu} + \delta g_{\mu\nu}, \overline{A}_\mu + \delta A_\mu\Big] =$$

$$= S\Big[\overline{g}_{\mu\nu}, \overline{A}_\mu\Big] + \delta^{(1)} S\Big[\overline{g}_{\mu\nu}, \overline{A}_\mu\Big] + \frac{1}{2!}\delta^{(2)} S\Big[\overline{g}_{\mu\nu}, \overline{A}_\mu\Big] + \dots\,, \tag{3.19}$$

where the first variation $\delta^{(1)}S$ automatically vanishes since the background fields satisfy the equations of motion, while the second variation $(1/2!)\delta^{(2)}S$ provides a quadratic action for the fluctuations $\delta g_{\mu\nu}$, δA_μ whose extremization yields the linearized equations of motion. Higher-order variations ($\dots$ in (3.19)) will provide higher-order interactions between the fluctuations $\delta g_{\mu\nu}$ and δA_μ. Finally, the on-shell action $S[\overline{g}_{\mu\nu}, \overline{A}_\mu]$ can be treated as a constant shift with no impact on the dynamics of the fluctuations.

3.2.1 First variation

Let us warm up by reviewing first the calculation of the first variation. One can first find

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}\,, \tag{3.20a}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\delta g_{\alpha\beta}\,. \tag{3.20b}$$

Using these results, one can derive the variation of the Christoffel symbols

$$\begin{aligned}
\delta\Gamma^{\alpha}{}_{\beta\gamma} &= \frac{1}{2}\delta g^{\alpha\rho}(\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\gamma\beta}) + \\
&\qquad + \frac{1}{2} g^{\alpha\rho}(\partial_\beta \delta g_{\rho\gamma} + \partial_\gamma \delta g_{\rho\beta} - \partial_\rho \delta g_{\gamma\beta}) && \text{(3.21a)}\\
&= -g^{\alpha\rho}\Gamma^{\nu}{}_{\beta\gamma}\delta g_{\rho\nu} + \frac{1}{2} g^{\alpha\rho}(\partial_\beta \delta g_{\rho\gamma} + \partial_\gamma \delta g_{\rho\beta} - \partial_\rho \delta g_{\gamma\beta}) && \text{(3.21b)}\\
&= \frac{1}{2} g^{\alpha\rho}\Big(\partial_\beta \delta g_{\rho\gamma} - \Gamma^{\nu}{}_{\beta\gamma}\delta g_{\rho\nu} - \Gamma^{\nu}{}_{\beta\rho}\delta g_{\gamma\nu} + \\
&\qquad + \partial_\gamma \delta g_{\rho\beta} - \Gamma^{\nu}{}_{\gamma\beta}\delta g_{\rho\nu} - \Gamma^{\nu}{}_{\gamma\rho}\delta g_{\beta\nu} + \\
&\qquad - \partial_\rho \delta g_{\gamma\beta} + \Gamma^{\nu}{}_{\beta\rho}\delta g_{\gamma\nu} + \Gamma^{\nu}{}_{\gamma\rho}\delta g_{\beta\nu}\Big) && \text{(3.21c)}\\
&= \frac{1}{2} g^{\alpha\rho}(\nabla_\beta \delta g_{\rho\gamma} + \nabla_\gamma \delta g_{\rho\beta} - \nabla_\rho \delta g_{\gamma\beta})\,, && \text{(3.21d)}
\end{aligned}$$

(where based on (3.21d), we note that as opposed to $\Gamma^{\alpha}{}_{\beta\gamma}$, the variation $\delta\Gamma^{\alpha}{}_{\beta\gamma}$ is a rank $(1,2)$ tensor field), the variation of the Riemann tensor

$$\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\sigma} + \\
&\qquad + \Gamma^{\rho}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\nu\sigma} + \Gamma^{\lambda}{}_{\nu\sigma}\delta\Gamma^{\rho}{}_{\mu\lambda} - \Gamma^{\rho}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\lambda}{}_{\mu\sigma}\delta\Gamma^{\rho}{}_{\nu\lambda} && \text{(3.22a)}\\
&= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\mu\sigma}\delta\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\lambda} + \\
&\qquad - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\lambda}{}_{\nu\sigma}\delta\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\nu\mu}\delta\Gamma^{\rho}{}_{\sigma\lambda} && \text{(3.22b)}\\
&= \nabla_\mu \delta\Gamma^{\rho}{}_{\nu\sigma} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\sigma}\,, && \text{(3.22c)}
\end{aligned}$$

the variation of the Ricci tensor

$$\begin{aligned}
\delta R_{\sigma\nu} &= \nabla_\rho \delta\Gamma^{\rho}{}_{\nu\sigma} - \nabla_\nu \delta\Gamma^{\rho}{}_{\rho\sigma} && \text{(3.23a)}\\
&= \frac{1}{2} g^{\rho\lambda}\nabla_\rho\nabla_\nu \delta g_{\lambda\sigma} + \frac{1}{2} g^{\rho\lambda}\nabla_\rho\nabla_\sigma \delta g_{\lambda\nu} - \frac{1}{2} g^{\rho\lambda}\nabla_\rho\nabla_\lambda \delta g_{\nu\sigma} +
\end{aligned}$$

$$- \frac{1}{2} g^{\rho\lambda} \nabla_\nu \nabla_\rho \delta g_{\lambda\sigma} - \frac{1}{2} g^{\rho\lambda} \nabla_\nu \nabla_\sigma \delta g_{\lambda\rho} + \frac{1}{2} g^{\rho\lambda} \nabla_\nu \nabla_\lambda \delta g_{\rho\sigma} \tag{3.23b}$$

$$= \frac{1}{2} g^{\rho\lambda} \nabla_\rho \nabla_\nu \delta g_{\lambda\sigma} + \frac{1}{2} g^{\rho\lambda} \nabla_\rho \nabla_\sigma \delta g_{\lambda\nu} - \frac{1}{2} \nabla^2 \delta g_{\nu\sigma} - \frac{1}{2} \nabla_\sigma \nabla_\nu \delta g_\lambda{}^\lambda \tag{3.23c}$$

and, finally, the variation of the Ricci scalar

$$\delta R = \delta\Big(g^{\nu\sigma} R_{\nu\sigma} \Big) \tag{3.24a}$$

$$= -R_{\nu\sigma} g^{\nu\alpha} g^{\sigma\beta} \delta g_{\alpha\beta} + g^{\nu\sigma} \delta R_{\nu\sigma} \tag{3.24b}$$

$$= -R^{\alpha\beta} \delta g_{\alpha\beta} + g^{\nu\sigma} g^{\rho\lambda} \nabla_\rho \nabla_\nu \delta g_{\lambda\sigma} - \nabla^2 \delta g_\lambda{}^\lambda \, . \tag{3.24c}$$

Hence, the lagrangian density of the pure-gravitational Einstein-Hilbert action is varied as

$$\delta(\sqrt{-g} R) = -\sqrt{-g} \Big(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \Big) \delta g_{\alpha\beta} + \\ + \nabla^\rho \nabla^\sigma \Big[\sqrt{-g} (\delta g_{\rho\sigma} - g_{\rho\sigma} \delta g_\lambda{}^\lambda) \Big] \, , \tag{3.25}$$

where the second term is a total derivative and will therefore give vanishing contribution upon assuming suitable boundary conditions. On the other hand, varying the EM part of the lagrangian density with respect to the metric gives

$$\delta_g(\sqrt{-g} \mathcal{L}_{\mathrm{EM}}) = \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\mathrm{EM}}) \delta g^{\mu\nu} \tag{3.26a}$$

$$= -\frac{1}{2} \Big[\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\mathrm{EM}}) \Big] \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \tag{3.26b}$$

$$= -\frac{1}{2} (T_{\mathrm{EM}})^{\alpha\beta} \sqrt{-g} \delta g_{\alpha\beta} \, , \tag{3.26c}$$

where we have recognized the definition (3.8) of the EM stress-energy tensor. Varying the EM lagrangian density with respect to A_μ gives (up to a total derivative which will result into a boundary term that can be made to vanish)

$$\delta_A(\sqrt{-g} \mathcal{L}_{\mathrm{EM}}) = \partial_\alpha (\sqrt{-g} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) \delta A_\beta \tag{3.27a}$$

$$= \sqrt{-g} \nabla_\alpha (F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) \delta A_\beta \, , \tag{3.27b}$$

where we have made use of the result (3.13). The first variation of the action (3.1) can therefore be computed as

$$\delta^{(1)} S = \frac{1}{2\kappa} \int d^4x \, \sqrt{-g} \Big(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} - \kappa (T_{\mathrm{EM}})^{\alpha\beta} \Big) \delta g_{\alpha\beta} + \\ + \int d^4x \, \sqrt{-g} \nabla_\alpha (F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}) \delta A_\beta \, , \tag{3.28}$$

which indeed manifestly vanishes upon substituting the background fields $\overline{g}_{\mu\nu}$, $\overline{A}_\mu$ satisfying the equations of motion (3.7) and (3.14).

3.2.2 Second variation

Let us continue with computing the second variation. When evaluated at some classical field configurations $\overline{g}_{\mu\nu}$, $\overline{A}_\mu$, it will give us the kinetic part of the action for the small fluctuations around such a background. It will be useful to decompose the second variation as

$$\delta^{(2)}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]=\delta^{(2)}_{gg}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]+\delta^{(2)}_{AA}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]+2\delta^{(2)}_{Ag}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]\,, \tag{3.29}$$

that is into a term quadratic in δg, quadratic in δA and a mixing term which is proportional to $\delta g\delta A$. In order to simplify the ensuing analysis, from now on we will assume that the background configuration $\overline{g}_{\mu\nu}$, $\overline{A}_\mu$ is just the flat-space vacuum, namely

$$\overline{g}_{\mu\nu}=\eta_{\mu\nu}\,,\qquad \overline{A}_\mu=0\,. \tag{3.30}$$

For this choice we simply have $\delta^{(2)}_{Ag}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]=0$: indeed, one should expect that in the flat-space vacuum, the metric and EM fluctuations should propagate as mass eigenstates with no mixing. At the same time the kinetic term $\delta^{(2)}_{AA}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]$ for the EM field A_μ in the linearized action around the flat-space configuration (3.30) clearly has to come out as the usual flat-space Maxwell action, that is

$$\delta^{(2)}_{AA}S\Big[\overline{g}_{\mu\nu}=\eta_{\mu\nu},\overline{A}_\mu=0\Big]\equiv S_{\mathrm{EM}}[A]=-\frac{1}{4}\int d^4x\,F_{\mu\nu}F^{\mu\nu} \tag{3.31}$$

(Indeed, this immediately follows from the fact, that we are taking second variation of a term which was already quadratic in the fluctuations.) We are therefore left with determining the metric-metric contribution $\delta^{(2)}_{gg}S\Big[\overline{g}_{\mu\nu},\overline{A}_\mu\Big]$ to the kinetic part of the linearized action.

Metric-metric part

In calculating $\delta^{(2)}_{g,g}S[\overline{g}_{\mu\nu},\overline{A}_\mu]$, it is useful to first note that we can write

$$\begin{aligned}\delta_g\Bigg[\sqrt{-g}\Big(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}-\kappa(T_{\mathrm{EM}})^{\alpha\beta}\Big)\Bigg]&=\\ &=(2\sqrt{-g}\delta g^{\beta\nu}g_{\nu\sigma}+\delta^\beta_\sigma\delta\sqrt{-g})\Big(R^{\alpha\sigma}-\frac{1}{2}Rg^{\alpha\sigma}-\kappa(T_{\mathrm{EM}})^{\alpha\sigma}\Big)+\\ &\quad+\sqrt{-g}g^{\alpha\mu}g^{\beta\nu}\Big(\delta R_{\mu\nu}-\frac{1}{2}\delta Rg_{\mu\nu}-\frac{1}{2}R\delta g_{\mu\nu}-\kappa\delta_g(T_{\mathrm{EM}})_{\mu\nu}\Big)\,.\end{aligned} \tag{3.32}$$

Substituting for classical field configuration $\overline{g}_{\mu\nu}=\eta_{\mu\nu}$, $\overline{A}_\mu=0$, the first term and the term proportional to R both vanish. We also have

$$\begin{aligned}\delta R_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\delta R&=\\ &=\frac{1}{2}\Big(\delta^\sigma_\mu\eta^{\rho\lambda}\partial_\rho\partial_\nu+\delta^\sigma_\nu\eta^{\rho\lambda}\partial_\rho\partial_\mu-\delta^\lambda_\mu\delta^\sigma_\nu\partial^2+\\ &\qquad-\eta^{\lambda\sigma}\partial_\mu\partial_\nu-\eta_{\mu\nu}\eta^{\tau\sigma}\eta^{\rho\lambda}\partial_\rho\partial_\tau+\eta_{\mu\nu}\eta^{\lambda\sigma}\partial^2\Big)\delta g_{\lambda\sigma}\end{aligned} \tag{3.33a}$$

$$=\frac{1}{2}\Box_{\mu\nu}{}^{\rho\sigma}\delta g_{\rho\sigma}\,, \tag{3.33b}$$

where we have used the flat-space equation of motion $R^{\rho\sigma}=0$ and we have introduced the background wave operator

$$\Box_{\mu\nu}{}^{\lambda\sigma}=\delta^\sigma_\mu\eta^{\rho\lambda}\partial_\rho\partial_\nu+\delta^\sigma_\nu\eta^{\rho\lambda}\partial_\rho\partial_\mu-\delta^\lambda_\mu\delta^\sigma_\nu\partial^2+$$

$$- \eta^{\lambda\sigma}\partial_\mu\partial_\nu - \eta_{\mu\nu}\eta^{\tau\sigma}\eta^{\rho\lambda}\partial_\rho\partial_\tau + \eta_{\mu\nu}\eta^{\lambda\sigma}\partial^2 \,. \tag{3.34}$$

Calculating $\delta_g(T_{\mathrm{EM}})_{\mu\nu}$ and evaluating on a classical background field configuration, we would have found that all terms contain at least two powers of the background EM field which we are setting to 0. Hence, we conclude that $\delta_g(T_{\mathrm{EM}})_{\mu\nu}$ when evaluated on a flat-space background. In total, varying the first term in (3.28) with respect to the metric and substituting for the flat-space background, one obtains

$$\delta_g\left[\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} - \kappa(T_{\mathrm{EM}})^{\alpha\beta}\right)\right]\Bigg|_{\overline{g}=\eta,\overline{A}=0} = \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}\Box_{\mu\nu}{}^{\rho\sigma}\delta g_{\rho\sigma} \,. \tag{3.35}$$

Denoting the metric fluctuations as $\delta g_{\mu\nu} = h_{\mu\nu}$ one therefore obtains the action

$$\begin{aligned} S_{\mathrm{grav}} = \frac{1}{8\kappa}\int d^4x\, h^{\mu\nu}(&\delta^\sigma_\mu\eta^{\rho\lambda}\partial_\rho\partial_\nu + \delta^\sigma_\nu\eta^{\rho\lambda}\partial_\rho\partial_\mu - \delta^\lambda_\mu\delta^\sigma_\nu\partial^2 + \\ &- \eta^{\lambda\sigma}\partial_\mu\partial_\nu - \eta_{\mu\nu}\eta^{\tau\sigma}\eta^{\rho\lambda}\partial_\rho\partial_\tau + \eta_{\mu\nu}\eta^{\lambda\sigma}\partial^2)h_{\rho\sigma} \qquad (3.36)\\ = \frac{1}{8\kappa}\int d^4x\,(&h^{\mu\nu}\partial^\alpha\partial_\mu h_{\alpha\nu} + h^{\mu\nu}\partial^\alpha\partial_\nu h_{\alpha\mu} - h^{\mu\nu}\partial_\mu\partial_\nu h_\alpha{}^\alpha + \\ &- h^{\mu\nu}\Box h_{\mu\nu} - h_\mu{}^\mu\partial^\alpha\partial^\beta h_{\alpha\beta} + h_\mu{}^\mu\Box h_\alpha{}^\alpha) \,. \qquad (3.37) \end{aligned}$$

Using the symmetry in indices μ and ν and integrating by parts assuming the boundary terms to vanish, we finally end up with the expression

$$\begin{aligned} S_{\mathrm{grav}} = \frac{1}{4\kappa}\int d^4x \Bigg[&\frac{1}{2}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\rho h_\mu{}^\mu\partial^\rho h_\alpha{}^\alpha + \\ &+ \partial_\mu h^{\mu\nu}\partial_\nu h_\alpha{}^\alpha - (\partial_\mu h^{\mu\nu})(\partial^\alpha h_{\alpha\nu})\Bigg] \,. \end{aligned} \tag{3.38}$$

Hence, rescaling the metric fluctuation as

$$H_{\mu\nu} = -\frac{1}{2\sqrt{\kappa}}h_{\mu\nu} \,, \tag{3.39}$$

we obtain

$$\begin{aligned} S_{\mathrm{grav}}[H] = \int d^4x \Bigg[&\frac{1}{2}\partial_\rho H^{\mu\nu}\partial^\rho H_{\mu\nu} - \frac{1}{2}\partial_\rho H_\mu{}^\mu\partial^\rho H_\alpha{}^\alpha + \\ &+ \partial_\mu H^{\mu\nu}\partial_\nu H_\alpha{}^\alpha - (\partial_\mu H^{\mu\nu})(\partial^\alpha H_{\alpha\nu})\Bigg] \,. \end{aligned} \tag{3.40}$$

The integrand then clearly agrees with the $m \to 0$ limit of the Fierz-Pauli lagrangian (2.1) for a massive spin-2 particle.

3.2.3 Interactions

Higher-order interactions between the gravity fluctuations $\delta g_{\alpha\beta}$ and the EM fluctuations δA_α can be worked out by computing higher-order variations of the action (3.1). This procedure of course becomes increasingly cumbersome with the increasing order. We will therefore only explicitly derive the interaction which we will need for later applications.

In the previous chapters, the mixing problem between the photon and other particles (scalar and massive spin 2) was considered from the perspective of an interacting action linearized around an empty flat spacetime. Starting with such an action, we expanded the fields around a classical solution corresponding to a constant magnetic field and observed how the cubic interaction vertices of the type photon–photon–(scalar/massive spin 2) gave rise to the mixing. In order to be able to perform an analogous derivation in the case of the massless graviton, let us therefore focus on deriving this interaction from the action (3.1) by varying it once with respect to the metric and twice with respect to the EM field (in arbitrary order). Calculating the second variation with respect to the EM field is straightforward, as the Einstein-Hilbert-Maxwell action is quadratic in the EM fields: one can simply replace the field-strength tensor $F_{\mu\nu}$ with its fluctuation $\delta F_{\mu\nu}$ to obtain

$$\frac{1}{2!}\delta^{(2)}_{AA}S = -\frac{1}{4}\int d^4x\,\sqrt{-g}\delta F_{\mu\nu}\delta F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\,. \tag{3.41}$$

Varying once more with respect to the metric, one can see (as a direct consequence of the definition (3.8)) that the metric fluctuation $\delta g_{\alpha\beta}$ will couple to the EM stress-energy tensor for the fluctuations δA_α. In an empty flat background $\overline{g}_{\alpha\beta} = \eta_{\alpha\beta}$, $\overline{A}_\alpha = 0$, this becomes (denoting $\delta g_{\alpha\beta} = h_{\alpha\beta}$ and relabelling $\delta A_\alpha \to A_\alpha$)

$$S_{\text{int}} = -\frac{1}{2}\int d^4x\left(F_{\mu\rho}F^{\rho}{}_{\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)h^{\mu\nu}\,. \tag{3.42}$$

After the rescaling (3.39) and a redefinition

$$g = \sqrt{2\kappa} = \frac{1}{m_{\text{Pl}}}\,, \tag{3.43}$$

of the coupling constant, we therefore obtain

$$S_{\text{int}}[H,A] = \frac{g}{\sqrt{2}}\int d^4x\left(F_{\mu\rho}F^{\rho}{}_{\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)H^{\mu\nu}\,. \tag{3.44}$$

This we recognize as the same form of interaction which we eventually considered for the massive spin-2 particle in the previous chapter.

3.2.4 Summary

In total, specializing to the case of a flat empty spacetime $\overline{g}_{\alpha\beta} = \eta_{\alpha\beta}$, $\overline{A}_\alpha = 0$, we can combine the results (3.40), (3.31) and (3.44) to obtain the action

$$S[H,A] = S_{\text{grav}} + S_{\text{EM}} + S_{\text{int}} \tag{3.45a}$$

$$\begin{aligned} &= \int d^4x\left[\frac{1}{2}\partial_\rho H^{\mu\nu}\partial^\rho H_{\mu\nu} - \frac{1}{2}\partial_\rho H_\mu{}^\mu\partial^\rho H_\alpha{}^\alpha + \right. \\ &\qquad\qquad \left. + \partial_\mu H^{\mu\nu}\partial_\nu H_\alpha{}^\alpha - (\partial_\mu H^{\mu\nu})(\partial^\alpha H_{\alpha\nu})\right] + \\ &\quad - \frac{1}{4}\int d^4x\,F_{\mu\nu}F^{\mu\nu} + \\ &\quad + \frac{g}{2}\int d^4x\left(F_{\mu\rho}F^{\rho}{}_{\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)H^{\mu\nu}\,. \end{aligned} \tag{3.45b}$$

This can be directly compared with the action for the massive spin-2 particle considered in the previous chapter (setting $m \to 0$).

3.3 Derivation of the mixing equations

We will now discuss in detail how the mixing between the EM modes and the metric modes on a constant magnetic background arises.

3.3.1 Linearized gauge symmetries

Action (3.45) is invariant under the gravity gauge transformation (order by order in the coupling constant g)

$$\delta_d H_{\mu\nu} = \partial_\mu d_\nu + \partial_\nu d_\mu \,, \tag{3.46a}$$

$$\delta_d A_\mu = -\frac{g}{\sqrt{2}}(A_\alpha \partial_\mu d^\alpha + d^\alpha \partial_\alpha A_\mu)\,, \tag{3.46b}$$

(where we have introduced a gauge parameter d_μ), which originates from infinitesimal diffeomorphisms. We could derive this transformation by considering the infintesimal change of coordinates $(x')^\mu = x^\mu + \frac{g}{\sqrt{2}}\epsilon d^\mu$ and investigating how the linearized metric and the vector field A_μ transform. While it can be proved, that the action (3.45) is invariant under the transformation (3.46) by directly substituting the transformation in the action, we can argue for the invariance by just noting that (3.45) comes from the original action (3.1), which is manifestly generally-covariant. The gauge symmetry (3.46) will later be used to fix two physical polarizations $\times 2$, $+2$ for graviton.

It is also easy to prove that the action (3.45) is invariant under the electromagnetic gauge transformation

$$\delta_\lambda H_{\mu\nu} = 0\,, \tag{3.47a}$$

$$\delta_\lambda A_\mu = \partial_\mu \lambda\,. \tag{3.47b}$$

We simply note that this transformation does not affect the graviton at all and that the rest of S depends only on $F_{\mu\nu}$, which is a gauge-invariant quantity by itself. As usual, the EM gauge symmetry (3.47) will allow us to fix the two physical polarizations for the photon.

3.3.2 Linearized equations of motion

The equations of motion corresponding to the action (3.45) can be obtained by the standard procedure, namely by varying the action we have found. We can already predict that they will come out the same as in the case of the massive graviton, only that the mass terms will vanish. The variation of S_{grav} with respect to $H_{\mu\nu}$, after integrating by parts, throwing away the boundary terms and symmetrizing the integrand (since we are varying with respect to a symmetric tensor) gives

$$\begin{aligned}\delta_H S_{\text{grav}} = \int d^4x\, \delta H_{\mu\nu}\Big(&- \partial^\rho\partial_\rho H^{\mu\nu} + \partial^\mu\partial_\rho H^{\rho\nu} + \partial^\nu\partial_\rho H^{\rho\mu} - \partial^\nu\partial^\mu H^\rho{}_\rho + \\ &- \partial_\rho\partial_\sigma H^{\rho\sigma}\eta^{\mu\nu} + \partial^\rho\partial_\rho H^\sigma{}_\sigma \eta^{\mu\nu}\Big)\,. \end{aligned} \tag{3.48}$$

which would indeed reproduce the free part of the equation of motion for a massless graviton. Variation of $S_{\text{EM}} + S_{\text{int}}$ with respect to $H_{\mu\nu}$ only receives contributions from the interacting part, namely

$$\delta_H(S_{\text{EM}} + S_{\text{int}}) = \frac{g}{\sqrt{2}}\delta \int d^4x\, H_{\mu\nu}\Big(F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\beta\alpha}F^{\beta\alpha}\Big) \tag{3.49a}$$

$$= \frac{g}{\sqrt{2}} \int d^4x \, \delta H_{\mu\nu} \Big(F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \,. \tag{3.49b}$$

Since the integrand of (3.49b) is already symmetric in the indices μ, ν, we can immediately write down the corresponding interacting equation of motion

$$\begin{aligned} 0 = -\partial^\rho \partial_\rho H^{\mu\nu} + \partial^\mu \partial_\rho H^{\rho\nu} + \partial^\nu \partial_\rho H^{\rho\mu} - \partial^\nu \partial^\mu H^\rho{}_\rho - \partial_\rho \partial_\sigma H^{\rho\sigma} \eta^{\mu\nu} + \\ + \partial^\rho \partial_\rho H^\sigma{}_\sigma \eta^{\mu\nu} + \frac{g}{\sqrt{2}} \Big(F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \,. \end{aligned} \tag{3.50}$$

To obtain the equation of motion for the EM field, let us vary the action with respect to A_μ. We already know what the variation of S_{EM} looks like from (1.13b). Furthermore, the variation of S_{grav} will be zero and finally, for the variation of S_{int} we can write

$$\delta_A S_{\mathrm{int}} = \frac{g}{\sqrt{2}} \delta_A \int d^4x \, H_{\mu\nu} \Big(F^{\mu\alpha} F_\alpha{}^\nu + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \tag{3.51a}$$

$$= \sqrt{2} g \int d^4x \, H_{\mu\nu} F^{\mu\alpha} \Big(\delta F_\alpha{}^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\beta\alpha} \delta F_{\beta\alpha} \Big) \tag{3.51b}$$

$$= \sqrt{2} g \int d^4x \, H^{\mu\nu} \Big[F_\mu{}^\alpha (\partial_\alpha \delta A_\nu - \partial_\nu \delta A_\alpha) + \frac{1}{2} \eta_{\mu\nu} F^{\beta\alpha} \partial_\beta \delta A_\alpha \Big] \tag{3.51c}$$

$$= \sqrt{2} g \int d^4x \, \Big[(H^{\mu\alpha} F_\mu{}^\nu - H^{\mu\nu} F_\mu{}^\alpha) + \frac{1}{2} H_\rho{}^\rho F^{\nu\alpha} \Big] \partial_\nu \delta A_\alpha \,. \tag{3.51d}$$

Integrating by parts, one can then read of the EM equation of motion as

$$0 = \partial_\mu F^{\mu\alpha} - \sqrt{2} g \partial_\nu \Big(H^{\mu\alpha} F_\mu{}^\nu - H^{\mu\nu} F_\mu{}^\alpha + \frac{1}{2} H_\rho{}^\rho F^{\nu\alpha} \Big) \,. \tag{3.52}$$

In order to make contact with the standard treatment of gravitational waves, let us rewrite the equations of motion (3.50) and (3.52) in terms of the trace-reversed metric perturbation

$$\tilde{H}^{\mu\nu} = H^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} H_\rho{}^\rho \,. \tag{3.53}$$

This clearly satisfies

$$\tilde{H}_\rho{}^\rho = -H_\rho{}^\rho \,. \tag{3.54}$$

First, we can rewrite the graviton equation of motion as

$$\begin{aligned} 0 = & -\Box \tilde{H}^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} \Box \tilde{H}_\rho{}^\rho + \partial^\mu \partial_\rho \tilde{H}^{\rho\nu} - \frac{1}{2} \partial^\mu \partial^\nu \tilde{H}_\rho{}^\rho + \partial^\nu \partial_\rho \tilde{H}^{\rho\mu} + \\ & - \frac{1}{2} \partial^\mu \partial^\nu \tilde{H}_\rho{}^\rho - \partial^\nu \partial^\mu \tilde{H}^\rho{}_\rho + \frac{1}{2} \partial^\mu \partial^\nu (4 \tilde{H}_\rho{}^\rho) - \partial_\rho \partial_\sigma \tilde{H}^{\rho\sigma} \eta^{\mu\nu} + \\ & + \frac{1}{2} \Box \tilde{H}_\alpha{}^\alpha \eta^{\mu\nu} - \eta^{\mu\nu} \Box \tilde{H}_\sigma{}^\sigma + \frac{g}{\sqrt{2}} \Big(F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \end{aligned} \tag{3.55a}$$

$$\begin{aligned} = & -\Box \tilde{H}^{\mu\nu} + \partial^\mu \partial_\rho \tilde{H}^{\rho\nu} + \partial^\nu \partial_\rho \tilde{H}^{\rho\mu} - \partial_\rho \partial_\sigma \tilde{H}^{\rho\sigma} \eta^{\mu\nu} + \\ & + \frac{g}{\sqrt{2}} \Big(F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \end{aligned} \tag{3.55b}$$

while the EM equation of motion can be recast as

$$0 = \partial_\mu F^{\mu\alpha} - \sqrt{2} g \partial_\nu \Big(\tilde{H}^{\mu\alpha} F_\mu{}^\nu - \frac{1}{2} \tilde{H}_\rho{}^\rho F^{\alpha\nu} + \\ - \tilde{H}^{\mu\nu} F_\mu{}^\alpha + \frac{1}{2} \tilde{H}_\rho{}^\rho F^{\nu\alpha} - \frac{1}{2} \tilde{H}_\rho{}^\rho F^{\nu\alpha} \Big) \tag{3.56a}$$

$$= \partial_\mu F^{\mu\alpha} - \sqrt{2} g \partial_\nu \Big(\tilde{H}^{\mu\alpha} F_\mu{}^\nu - \tilde{H}^{\mu\nu} F_\mu{}^\alpha + \frac{1}{2} \tilde{H}_\rho{}^\rho F^{\nu\alpha} \Big) . \tag{3.56b}$$

Altogether, we therefore end up with the pair of equations

$$0 = -\Box \tilde{H}^{\mu\nu} + \partial^\mu \partial_\rho \tilde{H}^{\rho\nu} + \partial^\nu \partial_\rho \tilde{H}^{\rho\mu} - \partial_\rho \partial_\sigma \tilde{H}^{\rho\sigma} \eta^{\mu\nu} + \\ + \frac{g}{\sqrt{2}} \Big(F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\beta\alpha} F^{\beta\alpha} \Big) \tag{3.57a}$$

$$0 = \partial_\mu F^{\mu\alpha} - \sqrt{2} g \partial_\nu \Big(\tilde{H}^{\mu\alpha} F_\mu{}^\nu - \tilde{H}^{\mu\nu} F_\mu{}^\alpha + \frac{1}{2} \tilde{H}_\rho{}^\rho F^{\nu\alpha} \Big) \tag{3.57b}$$

for the fields $\tilde{H}_{\mu\nu}$ and A_μ.

3.3.3 Propagation on a magnetic EM background

As in the previous chapters we would like to expand the fields in small fluctuations around a weak magnetic background to simplify the analysis of the equations of motion. In this case, let us expand

$$F^{\mu\nu} \to F^{\mu\nu}_{\text{ext}} + F^{\mu\nu} , \tag{3.58a}$$

$$\tilde{H}^{\mu\nu} \to \tilde{H}^{\mu\nu}_{\text{ext}} + \tilde{H}^{\mu\nu} , \tag{3.58b}$$

where the background fields denoted by the subscript "ext" satisfy the above derived equations of motion (3.57). Specifically, the field-strength tensor $F^{\mu\nu}_{\text{ext}}$ has the already well-known form (1.14), so it is assumed to be constant in a region of space $|r| \lesssim L$ and to vanish otherwise. It satisfies

$$\partial_\mu F^{\mu\nu}_{\text{ext}} = 0 \tag{3.59}$$

everywhere. As we have mentioned, we, strictly speaking, also have to turn on $\tilde{H}_{\text{ext}}$ which would satisfy

$$-\Box \tilde{H}^{\mu\nu}_{\text{ext}} + \partial^\mu \partial_\rho \tilde{H}^{\rho\nu}_{\text{ext}} + \partial^\nu \partial_\rho \tilde{H}^{\rho\mu}_{\text{ext}} - \partial_\rho \partial_\sigma \tilde{H}^{\rho\sigma}_{\text{ext}} \eta^{\mu\nu} = -\frac{g}{\sqrt{2}} T^{\mu\nu}_{\text{ext}} , \tag{3.60}$$

that is, schematically, $\partial^2 \tilde{H}_{\text{ext}} \sim g B^2$. This would imply that $\tilde{H}_{\text{ext}} \sim g B^2 r^2$ inside the region $|r| \lesssim L$ and $\tilde{H}_{\text{ext}} \sim \frac{g B^2 L^3}{r}$ outside. Hence $g H_{\text{ext}} \lesssim g^2 B^2 L^2$, so in order to ensure that $S_{\text{int}} \ll S_{\text{EM}}$ (so that it can be treated as a perturbation) and neglect H_{ext}, we need the external magnetic field to be weak in the sense that

$$g^2 B^2 L^2 \ll 1 . \tag{3.61}$$

Note that this is the same condition which we have found for the massless limit of the scalar particle. Also note that restoring SI units, we can express

$$g^2 B^2 L^2 = \frac{16 \pi G}{\mu_0 c^4} B^2 L^2 \simeq 3.3 \times 10^{-37} \left(\frac{B}{\text{T}} \right)^2 \left(\frac{L}{\text{m}} \right)^2 . \tag{3.62}$$

Hence, the assumption $g^2B^2L^2 \ll 1$ will certainly be valid in any conceivable laboratory setup ($L \sim 1\,\mathrm{m}$, $B \sim 10\,\mathrm{T}$), as well as in known astrophysical environments, such as around a pulsar ($L \sim 10\,\mathrm{km}$, $B \sim 10^9\,\mathrm{T}$).

From now on, we will therefore assume that we can neglect H_{ext} and write the equations of motion linearized around F_{ext} as

$$\begin{aligned}\Box\tilde{H}^{\mu\nu} - \partial^\mu\partial_\rho\tilde{H}^{\rho\nu} - \partial^\nu\partial_\rho\tilde{H}^{\rho\mu} + \partial_\rho\partial_\sigma\tilde{H}^{\rho\sigma}\eta^{\mu\nu} &= \\ = \frac{g}{\sqrt{2}}\Big[F^{\mu\alpha}_{\mathrm{ext}}F_\alpha{}^\nu + F^{\mu\alpha}(F_{\mathrm{ext}})_\alpha{}^\nu + \frac{1}{2}\eta^{\mu\nu}F^{\alpha\beta}_{\mathrm{ext}}F_{\alpha\beta}\Big]\end{aligned} \tag{3.63}$$

and

$$\partial_\mu F^{\mu\alpha} = \sqrt{2}g\Big[(F_{\mathrm{ext}})_\mu{}^\nu\partial_\nu\tilde{H}^{\mu\alpha} - (F_{\mathrm{ext}})_\mu{}^\alpha\partial_\nu\tilde{H}^{\mu\nu} + \frac{1}{2}F^{\nu\alpha}_{\mathrm{ext}}\partial_\nu\tilde{H}^\rho{}_\rho\Big]\,. \tag{3.64}$$

3.3.4 Gauge fixing

We will now use the gravity gauge transformation (3.46), as well as the EM gauge transformation (3.47) to isolate the physical propagating degrees of freedom. Taking the trace of the gravity gauge transformation, we obtain

$$\delta_d H_\rho{}^\rho = 2\partial_\mu d^\mu\,, \tag{3.65}$$

which gives the gauge transformation for the trace-reversed perturbation as

$$\delta_d\tilde{H}^{\mu\nu} = \delta_d H^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\delta_d H_\rho{}^\rho \tag{3.66a}$$

$$= \partial^\mu d^\nu + \partial^\nu d^\mu + \eta^{\mu\nu}\partial_\rho d^\rho\,. \tag{3.66b}$$

We now want to fix a gauge so that

$$\Box\tilde{H}^{01} = 0\,, \tag{3.67a}$$

$$\Box\tilde{H}^{02} = 0\,, \tag{3.67b}$$

$$\Box\tilde{H}^{03} = 0\,, \tag{3.67c}$$

$$\Box\tilde{H}_\rho{}^\rho = 0\,, \tag{3.67d}$$

$$\Box A^0 = 0\,. \tag{3.67e}$$

This will then allow us to consistently set

$$\tilde{H}^{01} = \tilde{H}^{02} = \tilde{H}^{03} = \tilde{H}_\rho{}^\rho = A^0 = 0\,, \tag{3.68}$$

as it is typically done in the case of freely propagating gravitational and EM waves. Introducing the shorthand notation

$$(\delta T_{\mathrm{EM}})^{\mu\nu} = F^{\mu\alpha}_{\mathrm{ext}}F_\alpha{}^\nu + F^{\mu\alpha}(F_{\mathrm{ext}})_\alpha{}^\nu + \frac{1}{2}\eta^{\mu\nu}F^{\alpha\beta}_{\mathrm{ext}}F_{\alpha\beta}\,, \tag{3.69}$$

the equations of motion (3.63) and (3.64) can be expanded in the respective components as

$$\Box\tilde{H}^{01} - \partial^0\partial_\rho\tilde{H}^{\rho1} = \frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{01}\,, \tag{3.70a}$$

$$\Box\tilde{H}^{02} - \partial^0\partial_\rho\tilde{H}^{\rho 2} = \frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{02}\,, \tag{3.70b}$$

$$\Box\tilde{H}^{03} - \partial^0\partial_\rho\tilde{H}^{\rho 3} - \partial^3\partial_\rho\tilde{H}^{\rho 0} = \frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{03}\,, \tag{3.70c}$$

$$\Box\tilde{H}_\rho{}^\rho + 2\partial_\rho\partial_\sigma\tilde{H}^{\rho\sigma} = \frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})_\rho{}^\rho\,, \tag{3.70d}$$

$$\Box A^0 - \partial^0\partial_\mu A^\mu = \frac{g}{\sqrt{2}}(F_{\mathrm{ext}})_\mu{}^\nu\partial_\nu\tilde{H}^{\mu 0}\,, \tag{3.70e}$$

where the second term on the left hand side of the equation (3.70d) can be written as $2\partial_\rho\partial_\sigma\tilde{H}^{\rho\sigma} = 2\partial_0\partial_\sigma\tilde{H}^{0\sigma} + 2\partial_3\partial_\sigma\tilde{H}^{3\sigma}$ and on the right hand side of the same equation, we obtain $(\delta T_{\mathrm{EM}})_\rho{}^\rho = 0$. We have also used the fact that $F^{\mu 0}_{\mathrm{ext}} = 0$, since there is no background electric field. Hence, we need to fix a gauge such that

$$\partial^0\partial_\rho\tilde{H}^{\rho 1} = -\frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{01}\,, \tag{3.71a}$$

$$\partial^0\partial_\rho\tilde{H}^{\rho 2} = -\frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{02}\,, \tag{3.71b}$$

$$\partial^0\partial_\rho\tilde{H}^{\rho 3} + \partial^3\partial_\rho\tilde{H}^{\rho 0} = -\frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{03}\,, \tag{3.71c}$$

$$\partial_0\partial_\sigma\tilde{H}^{0\sigma} + \partial_3\partial_\sigma\tilde{H}^{3\sigma} = 0\,, \tag{3.71d}$$

$$\partial^0\partial_\mu A^\mu = -\frac{g}{\sqrt{2}}(F_{\mathrm{ext}})_\mu{}^\nu\partial_\nu\tilde{H}^{\mu 0}\,. \tag{3.71e}$$

Having in mind a plane-wave solution for both the graviton and the EM field, the first two equations in (3.71) can be solved as (for a plane wave)

$$\partial_\rho\tilde{H}^{\rho 1} = -\frac{g}{\sqrt{2}}\frac{1}{i\omega}(\delta T_{\mathrm{EM}})^{01}\,, \tag{3.72a}$$

$$\partial_\rho\tilde{H}^{\rho 2} = -\frac{g}{\sqrt{2}}\frac{1}{i\omega}(\delta T_{\mathrm{EM}})^{02}\,. \tag{3.72b}$$

At the same time, denoting $\partial_\rho\tilde{H}^{\rho 0} = f^0$ and $\partial_\rho\tilde{H}^{\rho 3} = f^3$, we can rewrite the second pair of equations in (3.71) as

$$\partial_0 f^3 - \partial_3 f^0 = -\frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{03}\,, \tag{3.73a}$$

$$\partial_0 f^0 + \partial_3 f^3 = 0\,. \tag{3.73b}$$

These can therefore be solved for f^0 and f^3 by putting

$$f^0 = -\partial_3\psi \tag{3.74a}$$

$$f^3 = +\partial_0\psi\,, \tag{3.74b}$$

where the scalar function ψ solves the Poisson equation

$$(\partial_0^2 + \partial_3^2)\psi = \frac{g}{\sqrt{2}}(\delta T_{\mathrm{EM}})^{03}\,. \tag{3.75}$$

The solution then has the form

$$\psi = \frac{g}{\sqrt{2}}\int d\tau d\xi\, G(t,z;\tau,\xi)(\delta T_{\mathrm{EM}})^{03}\,, \tag{3.76}$$

where Poisson kernel $G(t,z;\tau,\xi)$ is given by the Fourier transform of $-\frac{1}{\omega^2+p^2}$. Thus, for a plane wave solution, we may simply write

$$\psi = -\frac{g}{\sqrt{2}}\frac{1}{\omega^2+p^2}(\delta T_{\mathrm{EM}})^{03}\,. \tag{3.77}$$

Hence, in total, we will be fixing de Donder gauge deformed by the interaction

$$\begin{aligned}
\partial_\rho \tilde{H}^{\rho 1} &= -\frac{g}{\sqrt{2}}\frac{1}{i\omega}(\delta T_{\mathrm{EM}})^{01}\,, && (3.78a)\\
\partial_\rho \tilde{H}^{\rho 2} &= -\frac{g}{\sqrt{2}}\frac{1}{i\omega}(\delta T_{\mathrm{EM}})^{02}\,, && (3.78b)\\
\partial_\sigma \tilde{H}^{0\sigma} &= -\frac{g}{\sqrt{2}}\frac{ip}{\omega^2+p^2}(\delta T_{\mathrm{EM}})^{03}\,, && (3.78c)\\
\partial_\sigma \tilde{H}^{3\sigma} &= -\frac{g}{\sqrt{2}}\frac{i\omega}{\omega^2+p^2}(\delta T_{\mathrm{EM}})^{03}\,, && (3.78d)
\end{aligned}$$

which we will accompany with the constraints

$$\tilde{H}^{01} = \tilde{H}^{02} = \tilde{H}^{03} = \tilde{H}_\rho{}^\rho = 0\,. \tag{3.79}$$

Those can be shown to fix all residual gauge symmetry. Combining (3.78) together with (3.79), we can realize that also the polarizations $\tilde{H}^{00}$, $\tilde{H}^{33}$, $\tilde{H}^{31}$, $\tilde{H}^{32}$ and A^3 are now fixed. In particular, for a plane wave solution it follows that

$$\begin{aligned}
\tilde{H}^{31} &= -\frac{g}{\sqrt{2}}\frac{1}{p\omega}(\delta T_{\mathrm{EM}})^{01}\,, && (3.80a)\\
\tilde{H}^{32} &= -\frac{g}{\sqrt{2}}\frac{1}{p\omega}(\delta T_{\mathrm{EM}})^{02}\,, && (3.80b)\\
\tilde{H}^{33} &= +\frac{g}{\sqrt{2}}\frac{\omega}{p}\frac{1}{\omega^2+p^2}(\delta T_{\mathrm{EM}})^{03}\,, && (3.80c)\\
\tilde{H}^{00} &= -\frac{g}{\sqrt{2}}\frac{p}{\omega}\frac{1}{\omega^2+p^2}(\delta T_{\mathrm{EM}})^{03}\,. && (3.80d)
\end{aligned}$$

From the trace constraint $\tilde{H}_\rho{}^\rho$, we can also fix the polarization $\tilde{H}^{22}$ in terms of $\tilde{H}^{11}$, namely

$$\begin{aligned}
\tilde{H}^{22} &= -\tilde{H}^{11} + \tilde{H}^{00} - \tilde{H}^{33} && (3.81a)\\
&= -\tilde{H}^{11} - \frac{g}{\sqrt{2}}\frac{1}{\omega p}(\delta T_{\mathrm{EM}})^{03}\,. && (3.81b)
\end{aligned}$$

Finally, for the photon, we note that we can actually fix

$$\partial_\mu A^\mu = -\sqrt{2}g\frac{1}{i\omega}(F_{\mathrm{ext}})_\mu{}^\nu\partial_\nu\tilde{H}^{\mu 0} = 0\,, \tag{3.82}$$

where in the second step, we have recalled that $(F_{\mathrm{ext}})_{0\nu} = 0$ and used the constraints (3.79). This means that we can consistently impose

$$A^0 = A^3 = 0\,. \tag{3.83}$$

We are now in a position to write down the equations of motion for the remaining four degrees of freedom, namely $A_1 = A_\parallel$, $A_2 = A_\perp$ as well as[1]

$$H^{11} = H_{+2}(\epsilon_{+2})^{11} = -\frac{1}{\sqrt{2}}H_{+2}\,, \tag{3.84a}$$

$$H^{12} = H_{\times 2}(\epsilon_{\times 2})^{12} = +\frac{1}{\sqrt{2}}H_{\times 2}\,. \tag{3.84b}$$

3.3.5 Mixing equations

Employing the gauge constraints derived in the previous section in the equation (3.63), it becomes straightforward to write down the equations of motion for the two graviton polarizations $H_{\times 2}$ and H_{+2}, namely

$$\Box H_{\times 2} = +g(\delta T_{\mathrm{EM}})^{12}\,, \tag{3.85a}$$

$$\Box H_{+2} = -g(\delta T_{\mathrm{EM}})^{11}\,, \tag{3.85b}$$

Recalling the expression (3.69) for $(\delta T_{\mathrm{EM}})^{\mu\nu}$, it is straightforward to evaluate

$$(\delta T_{\mathrm{EM}})^{11} = +B_{\mathrm{T}}\partial_3 A_\perp\,, \tag{3.86a}$$

$$(\delta T_{\mathrm{EM}})^{12} = -B_{\mathrm{T}}\partial_3 A_\parallel\,, \tag{3.86b}$$

which, in turn, yields the equations

$$\Box H_{\times 2} = -gB_{\mathrm{T}}\partial_3 A_\parallel\,, \tag{3.87a}$$

$$\Box H_{+2} = -gB_{\mathrm{T}}\partial_3 A_\perp\,. \tag{3.87b}$$

As usual, assuming plane wave solutions in the ultrarelativistic regime, where $\omega \approx p$ and $\Box = -\omega^2 + p^2 \approx -2\omega(\omega - p)$ and substituting $\partial_3 = -ip$, we therefore obtain

$$(\omega - p)H_{\times 2} + a_2 p(iA_\parallel) = 0\,, \tag{3.88a}$$

$$(\omega - p)H_{+2} + a_2 p(iA_\perp) = 0\,, \tag{3.88b}$$

where we have introduced the notation

$$a_2 = \frac{gB_{\mathrm{T}}}{2\omega}\,. \tag{3.89}$$

Similarly, substituting the gauge constraints into the EM equation of motion (3.64) for $A_1 = A_\parallel$ and $A_2 = A_\perp$ and neglecting $\mathrm{O}(g^2B^2L^2)$ terms, one obtains equations of motion

$$\Box A_\parallel = +gB_{\mathrm{T}}\partial_3 H_{\times 2}\,, \tag{3.90a}$$

$$\Box A_\perp = +gB_{\mathrm{T}}\partial_3 H_{+2}\,, \tag{3.90b}$$

that is

$$(\omega - p)iA_\parallel + a_2 p H_{\times 2} = 0\,, \tag{3.91a}$$

[1] From now on, since we are imposing the constraint that the trace $\tilde{H}_\rho{}^\rho$ vanishes, we will lose the distinction between $\tilde{H}_{\mu\nu}$ and $H_{\mu\nu}$.

$$(\omega - p)iA_\perp + a_2 p H_{+2} = 0\,. \tag{3.91b}$$

In total, including also refractive indices for the propagating EM modes and relabelling as usual $iA \to A$, we get mixing equations

$$0 = \begin{pmatrix} \omega - p & a_2 p \\ a_2 p & \omega - p + \Delta_\parallel \end{pmatrix} \begin{pmatrix} H_{\times 2} \\ A_\parallel \end{pmatrix}, \tag{3.92a}$$

$$0 = \begin{pmatrix} \omega - p & a_2 p \\ a_2 p & \omega - p + \Delta_\perp \end{pmatrix} \begin{pmatrix} H_{+2} \\ A_\perp \end{pmatrix}. \tag{3.92b}$$

On a magnetic background B_T, one should therefore observe oscillations between the $H_{\times 2}$ polarization of the graviton and the $A_\parallel$ polarization of the photon, as well as oscillations between the H_{+2} polarization of the graviton and the $A_\perp$ polarization of the photon. This is the Gertsenshtein-Zeldovich effect.

3.4 Identifying mass eigenstates

Since we are dealing with two separate systems exhibiting 2-flavour mixing (the $H_{\times 2}$–$A_\parallel$ pair, as well as the H_{+2}–$A_\perp$ pair) we can directly apply the machinery of 2-flavour mixing, which was discussed in great detail in chapter 1. As usual, the dispersion relations will greatly simplify by assuming the relativistic approximation $\Delta_\parallel, \Delta_\perp \ll \omega$. On top of that, we will also assume that the parameters

$$y_\parallel = \frac{gB_\mathrm{T}}{\Delta_\parallel}, \tag{3.93a}$$

$$y_\perp = \frac{gB_\mathrm{T}}{\Delta_\perp}. \tag{3.93b}$$

are kept small, namely $y_\parallel, y_\perp \ll 1$. This in turn will guarantee that we stay in the small-mixing regime.

3.4.1 $H_{\times 2}$–$A_\parallel$ mixing

Adapting the results obtained for the mixing of a massive scalar with the $A_\perp$ polarization of the EM field for the case of a *massless* spin-2 polarization $H_{\times 2}$ (thus setting the mass terms Δ to zero), we end up with the dispersion relations

$$2\left(1 - a_2^2\right) p^{(1)}_{\times 2,\parallel}(\omega) = 2\omega + \Delta_\parallel - \sqrt{D_\parallel}\,, \tag{3.94a}$$

$$2\left(1 - a_2^2\right) p^{(2)}_{\times 2,\parallel}(\omega) = 2\omega + \Delta_\parallel + \sqrt{D_\parallel}\,, \tag{3.94b}$$

as well as the discriminant

$$D_\parallel = (\Delta_\parallel + 2\omega)^2 - 4\omega\left(1 - a_2^2\right)(\Delta_\parallel + \omega)\,. \tag{3.95}$$

The two directions $(H^{(1)}_{\times 2}, A^{(1)}_\parallel)$ and $(H^{(2)}_{\times 2}, A^{(2)}_\parallel)$ in the $H_{\times 2}$–$A_\parallel$ flavour space corresponding to the respective mass eigenstates are specified by the mixing angles $\Theta^{(1)}_{\times 2,\parallel}$, $\Theta^{(2)}_{\times 2,\parallel}$, which are given by the relations

$$\frac{A^{(1)}_\parallel}{H^{(1)}_{\times 2}} = \frac{-a_2 p^{(1)}_{\times 2,\parallel}}{\omega - p^{(1)}_{\times 2,\parallel} + \Delta_\parallel} \equiv -\tan\Theta^{(1)}_{\times 2,\parallel}\,, \tag{3.96a}$$

$$\frac{H^{(2)}_{\times 2}}{A^{(2)}_{\parallel}} = \frac{-a_2 p^{(2)}_{\times 2,\parallel}}{\omega - p^{(2)}_{\times 2,\parallel}} \equiv + \tan \Theta^{(2)}_{\times 2,\parallel} \,. \tag{3.96b}$$

Assuming the ultrarelativistic and small-mixing approximations then yields simplified dispersion relations

$$p^{(1)}_{\times 2,\parallel}(\omega) = \omega - \frac{1}{4}\frac{g^2 B_{\mathrm{T}}^2}{\Delta_{\parallel}} \,, \tag{3.97a}$$

$$p^{(2)}_{\times 2,\parallel}(\omega) = \omega + \Delta_{\parallel} + \frac{1}{4}\frac{g^2 B_{\mathrm{T}}^2}{\Delta_{\parallel}} \,, \tag{3.97b}$$

for the $H_{\times 2}$-like and the $A_{\parallel}$-like mode, respectively. The mixing angles then satisfy

$$\Theta^{(1)}_{\times 2,\parallel} \approx \Theta^{(2)}_{\times 2,\parallel} \equiv \Theta_{\times 2,\parallel} \ll 1 \,. \tag{3.98}$$

where $\Theta_{\times 2,\parallel}$ is given simply by

$$\Theta_{\times 2,\parallel} = \frac{1}{2}\frac{g B_{\mathrm{T}}}{\Delta_{\parallel}} \,. \tag{3.99}$$

Finally, introducing the parameter $b_{\times}$ as

$$b_{\times} \equiv \Delta_{\parallel} \,, \tag{3.100}$$

we can rewrite

$$p^{(1)}_{\times 2,\parallel}(\omega) = \omega n_{\parallel} - b_{\times}(1 + \Theta^2_{\times 2,\parallel}) \,, \tag{3.101a}$$

$$p^{(2)}_{\times 2,\parallel}(\omega) = \omega n_{\parallel} + b_{\times}\Theta^2_{\times 2,\parallel} \,. \tag{3.101b}$$

This enables us to write down the general solution for the $H_{\times 2}$–$A_{\parallel}$ oscillations in the form

$$\begin{aligned} e^{i\omega n_{\parallel} z} \begin{pmatrix} H_{\times 2}(z) \\ A_{\parallel}(z) \end{pmatrix} &= \\ &= \frac{1}{1+\Theta^2_{\times 2,\parallel}} \begin{pmatrix} 1 \\ -\Theta_{\times 2,\parallel} \end{pmatrix} \Big[H_{\times 2}(0) - A_{\parallel}(0)\Theta_{\times 2,\parallel} \Big] e^{i b_{\times}(1+\Theta^2_{\times 2,\parallel})z} + \\ &\quad + \frac{1}{1+\Theta^2_{\times 2,\parallel}} \begin{pmatrix} \Theta_{\times 2,\parallel} \\ 1 \end{pmatrix} \Big[A_{\parallel}(0) + H_{\times 2}(0)\Theta_{\times 2,\parallel} \Big] e^{-i b_{\times}\Theta^2_{\times 2,\parallel} z} \,. \end{aligned} \tag{3.102}$$

Altogether we observe that the $H_{\times 2}$–$A_{\parallel}$ oscillations can be obtained by simply taking the $m \to 0$ limit of the massive spin-2 case.

Note that as in the massive spin-0 case, the evolution equation (3.102) may be expressed in terms a unitary transfer matrix $\mathsf{U}_{\times 2,\parallel}(z,0)$ acting on an initial state $(A_{\parallel}(0), H_{\times 2}(0))$. See (3.147a) below. One would then find a transition probability

$$P(A_{\parallel} \to H_{\times 2}) = 4\Theta^2_{\times 2,\parallel} \sin^2 \frac{b_{\times} z}{2} \,, \tag{3.103}$$

which can be seen to oscillate with length $l_{\mathrm{osc},\times} = \frac{2\pi}{b_{\times}}$ and amplitude $\alpha_{\times} = 4\Theta^2_{\times 2,\parallel}$.

3.4.2 H_{+2}–$A_\perp$ mixing

In the case of the interacting system of the graviton polarization H_{+2} and the photon polarization $A_\perp$, we analogously obtain dispersion relations

$$2\Big(1-a_2^2\Big)p^{(1)}_{+2,\perp}(\omega) = 2\omega + \Delta_\perp - \sqrt{D_\perp}\,, \tag{3.104a}$$

$$2\Big(1-a_2^2\Big)p^{(2)}_{+2,\perp}(\omega) = 2\omega + \Delta_\perp + \sqrt{D_\perp}\,, \tag{3.104b}$$

as well as the discriminant

$$D_\perp = (\Delta_\perp + 2\omega)^2 - 4\omega\Big(1-a_2^2\Big)(\Delta_\perp + \omega)\,. \tag{3.105}$$

The mixing angles $\Theta^{(1)}_{+2,\perp}$, $\Theta^{(2)}_{+2,\perp}$, which specify the departure of the corresponding two mass eigenstates from the graviton-like and the photon-like state in the H_{+2}–$A_\perp$ flavour space, are given by the formulae

$$\frac{A^{(1)}_\perp}{H^{(1)}_{+2}} = \frac{-a_2 p^{(1)}_{+2,\perp}}{\omega - p^{(1)}_{+2,\perp} + \Delta_\perp} \equiv -\tan\Theta^{(1)}_{+2,\perp}\,, \tag{3.106a}$$

$$\frac{H^{(2)}_{+2}}{A^{(2)}_\perp} = \frac{-a_2 p^{(2)}_{+2,\perp}}{\omega - p^{(2)}_{+2,\perp}} \equiv +\tan\Theta^{(2)}_{+2,\perp}\,. \tag{3.106b}$$

In the ultrarelativistic regime $\Delta_\perp \ll \omega$ and small-mixing regime $y_\perp \ll 1$, the dispersion relations simplify as

$$p^{(1)}_{+2,\perp}(\omega) = \omega - \frac{1}{4}\frac{g^2 B_\mathrm{T}^2}{\Delta_\perp}\,, \tag{3.107a}$$

$$p^{(2)}_{+2,\perp}(\omega) = \omega + \Delta_\perp + \frac{1}{4}\frac{g^2 B_\mathrm{T}^2}{\Delta_\perp}\,, \tag{3.107b}$$

while the mixing angle becomes

$$\Theta^{(1)}_{+2,\perp} \approx \Theta^{(2)}_{+2,\perp} \equiv \Theta_{+2,\perp} = \frac{1}{2}\frac{gB_\mathrm{T}}{\Delta_\perp} \ll 1\,. \tag{3.108}$$

Introducing the mass parameter b_+ as

$$b_+ \equiv \Delta_\perp\,, \tag{3.109}$$

we can rewrite

$$p^{(1)}_{+2,\perp}(\omega) = \omega n_\perp - b_+(1+\Theta^2_{+2,\perp})\,, \tag{3.110a}$$

$$p^{(2)}_{+2,\perp}(\omega) = \omega n_\perp + b_+\Theta^2_{+2,\perp}\,. \tag{3.110b}$$

This enables us to write down the general solution for the H_{+2}–$A_\perp$ oscillations in the form

$$e^{i\omega n_\perp z}\begin{pmatrix} H_{+2}(z) \\ A_\perp(z)\end{pmatrix} =$$

$$= \frac{1}{1+\Theta^2_{+2,\perp}}\begin{pmatrix} 1 \\ -\Theta_{+2,\perp}\end{pmatrix}\Big[H_{+2}(0) - A_\perp(0)\Theta_{+2,\perp}\Big]e^{ib_+(1+\Theta^2_{+2,\perp})z} +$$

$$+\frac{1}{1+\Theta_{+2,\perp}^2}\begin{pmatrix}\Theta_{+2,\perp}\\ 1\end{pmatrix}\Big[A_\perp(0)+H_{+2}(0)\Theta_{+2,\perp}\Big]e^{-ib_+\Theta_{+2,\perp}^2 z}\,. \quad (3.111)$$

Again, one can re-express the evolution equation (3.111) in terms a unitary transfer matrix $\mathsf{U}_{+2,\perp}(z,0)$, as in (3.147b) below. It is then possible to calculate the transition probability

$$P(A_\parallel \to H_{+2}) = 4\Theta_{+2,\perp}^2 \sin^2\frac{b_+ z}{2}\,, \quad (3.112)$$

which can be seen to oscillate with length $l_{\mathrm{osc},\times} = \frac{2\pi}{b_\times}$ and amplitude $\alpha_\times = 4\Theta_{+2,\perp}^2$.

We point out that in contrast with the case of the $H_{\times 2}$–$A_\parallel$ oscillations, the H_{+2}–$A_\perp$ oscillations *cannot* be obtained as a simple $m \to 0$ limit of the massive spin-2 story: taking the massless limit, we would have predicted a mixing angle which is larger by a factor of $\sqrt{7/3}$! This can be seen as a direct (and, as we will see below, hypothetically observable) manifestation of the vDVZ discontinuity [16, 17].

3.5 Observable effects

Let us again consider performing the measurements and observations of the kind considered in previous chapters. We will compare our results with the predictions which were made for the massive spin-2 and spin-0 particles. Moreover, this time we will have an advantage of (in principle) knowing all parameters of the problem numerically so we will be to make explicit predictions.

3.5.1 Mixing angles, oscillation lengths and conversion probabilities

Let us first try and explicitly evaluate the mixing angles and oscillation lengths in a number of experimentally / observationally relevant setups. Since both the mixing angles and oscillation lengths differ between the $\times 2$ case and the $+2$ case by a factor of $\Delta_\parallel/\Delta_\perp = (n_\parallel - 1)/(n_\perp - 1) = \mathcal{O}(1)$ and we will be interested in orders of magnitude only, we can focus on just one of the two angles / lengths, without loosing generality. Reinstating units and substituting for the Δ parameters in terms of the refractive index n, the corresponding mixing angle can be expressed as

$$\Theta = \frac{1}{2}\sqrt{\frac{16\pi G}{\mu_0 c^4}}\frac{cB_{\mathrm{T}}}{\omega}(n-1)^{-1} \simeq 5.7\times 10^{-26}\,(n-1)^{-1}\,\frac{\mathrm{eV}}{\hbar\omega}\frac{B}{\mathrm{T}}\,, \quad (3.113)$$

while for the associated small-mixing oscillation length, we can write

$$l_{\mathrm{osc}} = 1.24\times 10^{-6}\,(n-1)^{-1}\,\frac{\mathrm{eV}}{\hbar\omega}\,\mathrm{m}\,. \quad (3.114)$$

Laboratory

Considering first the laboratory setup of the previous chapters ($\hbar\omega = 2.4\,\mathrm{eV}$, $n_\parallel - 1 \approx n_\perp - 1 \simeq 10^{-17}$, $B_{\mathrm{T}} \simeq 10\,\mathrm{T}$), we would have obtained

$$\Theta \simeq 10^{-8} \ll 1\,, \quad (3.115a)$$

$$l_{\rm osc} \simeq 10^{11}\,{\rm m}\,. \tag{3.115b}$$

While the mixing angle clearly comes out to be quite small a number, the oscillation length is comparable to one astronomical unit (Sun-to-Earth distance). In usual laboratory conditions, one should be therefore safe to assume that $z \ll l_{\rm osc}$.

Neutron stars

Second, let us consider the environment around a neutron star (see also [51, 52, 53] for recent discussions of the Gertsenshtein-Zeldovich effect and its realisation near pulsars as a possible explanation of the fast radio bursts). For this kind of a system, the actual magnetic field is of course far from constant (both in space and time), but at the very least, we should be able to get to correct ballpark estimates. We will take the magnetic fields around a neutron star to be as large as $B_{\rm T} \simeq 10^9\,{\rm T}$ (which we fix). The nature and magnitude of the dominant contribution to the refractive index will then strongly depend on ω. While for the vacuum contribution, we obtain

$$n^{(\rm vac)} - 1 \simeq 10^{-5} \tag{3.116}$$

independently of ω, the free electrons will contribute (taking the typical electron density as $N_{\rm e} \simeq 10^{18}\,{\rm m}^{-3}$ [5] and recalling the formula (1.63) which gives the plasma frequency $\hbar\omega_{\rm p} \cong 4 \times 10^{-5}\,{\rm eV}$)

$$n^{(\rm gas)} - 1 \simeq -\frac{1}{2}\left(\frac{4 \times 10^{-5}\,{\rm eV}}{\omega}\right)^2 . \tag{3.117}$$

Hence, we can conclude that for photons with energies higher than $\simeq 10^{-2}\,{\rm eV}$, the vacuum contribution will dominate and one obtains

$$\Theta \simeq 10^{-11}\,\frac{\rm eV}{\hbar\omega}\,, \tag{3.118a}$$

$$l_{\rm osc} \simeq 10^{-1}\,\frac{\rm eV}{\hbar\omega}\,{\rm m}\,. \tag{3.118b}$$

Since the typical size of a neutron star is at most $10^4\,{\rm m}$, we will be well within the regime $z \gg l_{\rm osc}$. One would therefore obtain photon-graviton transition probability $P(A \to H) \approx 2\Theta^2 \simeq 10^{-22}\,(\frac{\rm eV}{\hbar\omega})^2$. We can see that since the vacuum refractive indices are proportional to the square of the magnetic field, one can achieve larger values for the mixing angle by taking smaller $B_{\rm T}$. For instance, for magnetic white dwarfs with $B_{\rm T} = 10^4\,{\rm T}$ and taking $\hbar\omega = 10\,{\rm eV}$ then, assuming that the refractive indices are still dominated by the vacuum contribution, we would have obtained $\Theta \simeq 10^{-8}$ and $l_{\rm osc} \simeq 1\,{\rm km}$ [5], giving the transition probability $P(A \to H) \simeq 10^{-16}$.

On the other hand, we notice that for the value of ω around $10^{-2}\,{\rm eV}$ to $10^{-3}\,{\rm eV}$ (microwaves), the plasma contribution to the refractive index (for the above-given electron number density around neutron stars) may actually balance the vacuum contribution so that a MSW-like resonance is achieved [5]. In such a case, one would observe *maximum mixing* between the photons and gravitons ($\Theta \approx \frac{\pi}{4}$) so that the transition probability $P(A \to H)$ would oscillate with amplitude equal to 1 with large-mixing oscillation length (recalling the result (1.165))

$$l_{\rm osc} = \frac{\pi}{gB_{\rm T}} \simeq 10^9\,{\rm m}\,. \tag{3.119}$$

Given the typical dimensions of neutron stars, we will clearly have $z \ll l_{\mathrm{osc}}$ and so the full conversion of photons into gravitons will not be achieved (for that we would need to maintain stronger magnetic fields over larger distances). Recalling (1.166) and setting $z = 10^4\,\mathrm{m}$, the corresponding conversion probability of photons into gravitons (or vice versa) will roughly go as

$$P(A \to H; z) \approx P(H \to A; z) \approx g^2 B_{\mathrm{T}}^2 z^2 \simeq 10^{-11}\,. \tag{3.120}$$

Hence, given an energetic enough emission of gravitational waves (of the appropriate frequency) passing around a neutron star, substantial power may be converted into photons.

Galactic fields

Finally, let us consider conversion in galactic magnetic fields. For ordinary spiral galaxies, we can take $B_{\mathrm{T}} \simeq 2 \times 10^{-9}\,\mathrm{T}$ as well as $N_{\mathrm{e}} \simeq 10^6\,\mathrm{m}^{-3}$ and $z \simeq 20\,\mathrm{kpc}$, while for starburst galaxies, we will assume the values $B_{\mathrm{T}} \simeq 7{\times}10^{-8}\,\mathrm{T}$ (dominated by the large scale anisotropic component) as well as $N_{\mathrm{e}} \simeq 10^9\,\mathrm{m}^{-3}$ and $z \simeq 1\,\mathrm{kpc}$ [3]. This yields the plasma frequencies (orders of magnitude)

$$\hbar\omega_{\mathrm{p}} \simeq \begin{cases} 10^{-11}\,\mathrm{eV} & \text{(spiral)} \\ 10^{-9}\,\mathrm{eV} & \text{(starburst)} \end{cases}, \tag{3.121}$$

while the vacuum refractive indices are given by the formulae (1.75) as

$$n^{(\mathrm{vac})} - 1 \simeq \begin{cases} 10^{-41} & \text{(spiral)} \\ 10^{-38} & \text{(starburst)} \end{cases}. \tag{3.122}$$

Hence, we see that in both spiral and starburst galaxies, the plasma refractive index would dominate over the vacuum one for $\omega < 10\,\mathrm{GeV}$. One would then obtain mixing angle

$$\Theta_{\omega<10\,\mathrm{GeV}} \simeq \begin{cases} 10^{-11}\,\frac{\hbar\omega}{\mathrm{eV}} & \text{(spiral)} \\ 10^{-13}\,\frac{\hbar\omega}{\mathrm{eV}} & \text{(starburst)} \end{cases} \tag{3.123}$$

The corresponding small-mixing oscillation length would be

$$l_{\mathrm{osc},\omega<10\,\mathrm{GeV}} \simeq \begin{cases} 10^{-3}\,\frac{\hbar\omega}{\mathrm{eV}}\,\mathrm{kpc} & \text{(spiral)} \\ 10^{-7}\,\frac{\hbar\omega}{\mathrm{eV}}\,\mathrm{kpc} & \text{(starburst)} \end{cases}, \tag{3.124}$$

so given the above values of z, one can achieve both the coherent regime $z \ll l_{\mathrm{osc}}$ (for high enough ω) with conversion probability

$$P(A \to H) \approx \frac{1}{4} g^2 B_{\mathrm{T}}^2 z^2 \simeq 10^{-12} \tag{3.125}$$

(valid for both spiral and starburst), as well as the averaged-oscillation regime $z \gg l_{\mathrm{osc}}$ (for small-enough ω) with conversion probabilities

$$\langle P(A \to H)\rangle \approx 2\Theta^2 \simeq \begin{cases} 10^{-21}\,(\frac{\hbar\omega}{\mathrm{eV}})^2 & \text{(spiral)} \\ 10^{-26}\,(\frac{\hbar\omega}{\mathrm{eV}})^2 & \text{(starburst)} \end{cases}, \tag{3.126}$$

which will always be less than the coherent probability (3.125). Around $\omega \simeq 10\,\mathrm{GeV}$, one could obtain a resonant regime with strong mixing $\Theta_{\omega\simeq 10\,\mathrm{GeV}} \simeq \frac{\pi}{4}$ and large-mixing oscillation length

$$l_{\mathrm{osc},\omega\simeq 10\,\mathrm{GeV}} \simeq \begin{cases} 10^9\,\mathrm{kpc} & \text{(spiral)} \\ 10^7\,\mathrm{kpc} & \text{(starburst)} \end{cases}, \tag{3.127}$$

so that clearly $z \ll l_{\mathrm{osc}}$. Total conversion is therefore not achieved and one ends up again with the probability (3.125). Finally, for $\omega > 10\,\mathrm{GeV}$, the vacuum refractive indices dominate over the plasma ones, so that one obtains mixing a small angle

$$\Theta_{\omega > 10\,\mathrm{GeV}} \simeq \begin{cases} 10^7\,\frac{\mathrm{eV}}{\hbar\omega} & \text{(spiral)} \\ 10^6\,\frac{\mathrm{eV}}{\hbar\omega} & \text{(starburst)} \end{cases} \tag{3.128}$$

with small-mixing oscillation length

$$l_{\mathrm{osc}} \simeq \begin{cases} 10^{16}\,\frac{\mathrm{eV}}{\hbar\omega}\,\mathrm{kpc} & \text{(spiral)} \\ 10^{13}\,\frac{\mathrm{eV}}{\hbar\omega}\,\mathrm{kpc} & \text{(starburst)} \end{cases}, \tag{3.129}$$

meaning that even for the most energetic photons $\omega \simeq 10 - 100\,\mathrm{TeV}$, one has $z \simeq l_{\mathrm{osc}}$ and otherwise $z \ll l_{\mathrm{osc}}$ (namely the coherent regime). The transition probability is therefore again given by (3.125).

Remarkably, we therefore conclude that the result $P(A \to H) \simeq 10^{-12}$ holds over quite a universal range of values of ω for both spiral and starburst galaxies. This is a similar value as the one obtained in (3.119) for the resonant regime in the case of conversion near a neutron star. However, for the galactic conversion, we did not have to fine-tune ω so as to achieve such a (relatively) high conversion probability.

3.5.2 Effects on photon polarization

We will consider a linearly polarized beam of photons and let it propagate through a region of constant magnetic field B_{T} of thickness z along the line of propagation. For the sake of brevity, we will focus on the small mixing scenario. See also [54] for a similar analysis of the imprints on the photon polarization due to photon-graviton mixing.

The two polarizations $A_\parallel$ and $A_\perp$ then evolve as

$$A_\parallel(z) = \frac{1}{1+\Theta^2_{\times 2,\parallel}}\left[\Theta^2_{\times 2,\parallel} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times \Theta^2_{\times 2,\parallel} z}\right] e^{-i\omega n_\parallel z} A_\parallel(0)\,, \tag{3.130a}$$

$$A_\perp(z) = \frac{1}{1+\Theta^2_{+2,\perp}}\left[\Theta^2_{+2,\perp} e^{ib_+(1+\Theta^2_{+2,\perp})z} + e^{-ib_+ \Theta^2_{+2,\perp} z}\right] e^{-i\omega n_\perp z} A_\perp(0)\,. \tag{3.130b}$$

Notice that now the relation

$$b_\times \Theta^2_{\times 2,\parallel} z \ll 1 \tag{3.131}$$

already follows (in the small mixing case) from the previous assumptions: we can write

$$b_\times \Theta^2_{\times 2,\parallel} z = \Theta_{\times 2,\parallel} g B_{\mathrm{T}} z\,, \tag{3.132}$$

where $\Theta_{\times 2,\parallel} \ll 1$ as a consequence of the small-mixing assumption, while, as it was argued above, we need to take $gB_{\mathrm{T}}z \ll 1$ so as to guarantee perturbative consistency of the action (3.45) on the constant magnetic background. Similarly, we can obtain $b_+\Theta^2_{+2,\perp}z \ll 1$. In particular, as we have numerically checked above, these should be valid assumptions both in laboratory experiments, as well as for environments around neutron stars.

Let us also briefly discuss validity of the second assumption needed to show that the ratio $A_\perp/A_\parallel$ only changes by a very small amount, namely $\omega(n_\perp - n_\parallel)z \ll 1$. We have already checked that this is true in our model laboratory setup. For a neutron star, we have seen above that for photons $\hbar\omega \geq 10^{-2}\,\mathrm{eV}$ and for $B = 10^9\,\mathrm{T}$, the difference $n_\perp - n_\parallel$ will be independent of ω as it is dominated by the vacuum birefringence. Using the formula (1.78), it can be evaluated as $n_\perp - n_\parallel \simeq 10^{-6}$. We then obtain

$$\frac{\omega}{c}(n_\perp - n_\parallel) \simeq 20\,\frac{\hbar\omega}{\mathrm{eV}}\,\mathrm{m}^{-1}\,. \tag{3.133}$$

that is, considering typical dimensions of neutron stars $z = 10^4\,\mathrm{m}$, one always ends up having $\omega(n_\perp - n_\parallel)z \gg 1$, so that it is no longer true that the change in the ratio $A_\perp/A_\parallel$ will be small.

Assuming both $b\Theta^2 z \ll 1$, $\omega(n_\perp - n_\parallel)z \ll 1$ (such as in a laboratory), it follows that the evolution of the ratio $A_\perp/A_\parallel$ can be approximated as

$$\frac{A_\perp(z)}{A_\parallel(z)} = \frac{A_\perp(0)}{A_\parallel(0)}\,\frac{1+\Theta^2_{\times 2,\parallel}}{1+\Theta^2_{+2,\perp}}\,\frac{\Theta^2_{+2,\perp}e^{ib_+(1+\Theta^2_{+2,\perp})z} + e^{-ib_+\Theta^2_{+2,\perp}z}}{\Theta^2_{\times 2,\parallel}e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times\Theta^2_{\times 2,\parallel}z}}\,e^{-i\omega(n_\perp - n_\parallel)z} \tag{3.134a}$$

$$\approx \frac{A_\perp(0)}{A_\parallel(0)}\Big[1 - \Theta^2_{+2,\perp} + \Theta^2_{\times 2,\parallel} - i\omega(n_\perp - n_\parallel)z + \\ - ib_+\Theta^2_{+2,\perp}z + ib_\times\Theta^2_{\times 2,\parallel}z + \Theta^2_{+2,\perp}e^{ib_+z} - \Theta^2_{\times 2,\parallel}e^{ib_\times z}\Big] \tag{3.134b}$$

$$= \frac{A_\perp(0)}{A_\parallel(0)}\Big[1 - 2\Theta^2_{+2,\perp}\sin^2\frac{b_+z}{2} + 2\Theta^2_{\times 2,\parallel}\sin^2\frac{b_\times z}{2} - i\omega(n_\perp - n_\parallel)z + \\ - i\Theta^2_{+2,\perp}(b_+z - \sin b_+z) + i\Theta^2_{\times 2,\parallel}(b_\times z - \sin b_\times z)\Big]\,. \tag{3.134c}$$

This gives the relative change $\eta(z)$ in the $A_\perp$ amplitude, as well as the phase delay $\varphi(z)$ as

$$\eta(z) = 2\Theta^2_{+2,\perp}\sin^2\frac{b_+z}{2} - 2\Theta^2_{\times 2,\parallel}\sin^2\frac{b_\times z}{2}\,, \tag{3.135a}$$

$$\varphi(z) = \omega(n_\perp - n_\parallel)z + \Theta^2_{+2,\perp}(b_+z - \sin b_+z) - \Theta^2_{\times 2,\parallel}(b_\times z - \sin b_\times z)\,. \tag{3.135b}$$

Furthermore, let us again consider working in the regime when $b_+z \ll 1$, as well as $b_\times z \ll 1$ so that the EM wave and the massless spin-2 wave remain coherent. Recall that for our model experimental setup, we found that $\frac{\omega}{c}(n_\parallel - 1)z \approx \frac{\omega}{c}(n_\perp - 1)z \simeq 10^{-10}\,\mathrm{m}^{-1}$ so assuming coherence is justified in this case. We can then first approximate $\eta(z)$ as

$$\eta(z) \approx 2\Theta^2_{+2,\perp}\frac{b_+^2z^2}{4} - 2\Theta^2_{\times 2,\parallel}\frac{b_\times^2 z^2}{4} \tag{3.136a}$$

$$= \frac{1}{8} g^2 B_{\mathrm{T}}^2 z^2 - \frac{1}{8} g^2 B_{\mathrm{T}}^2 z^2 \tag{3.136b}$$

$$= 0 \,. \tag{3.136c}$$

Hence, in the coherent regime, contrary to all the previously considered cases, at order $\mathcal{O}(g^2 B^2 z^2)$ there is no relative change in the $A_\perp$ amplitude (and therefore no rotation $\delta\theta(z)$ of the polarization plane) due to the mixing of the photon with the graviton in an external magnetic field. However, note that $\eta(z)$ and consequently $\delta\theta(z)$ receive non-zero (higher order) contributions upon considering loop effects in QED [55]. See [54] for a detailed calculation.

Second, the phase delay can be approximated as

$$\varphi(z) \approx \omega(n_\perp - n_\parallel) z + \frac{1}{6} \Theta_{+2,\perp}^2 b_+^3 z^3 - \frac{1}{6} \Theta_{\times 2,\parallel}^2 b_\times^3 z^3 \tag{3.137a}$$

$$= \omega(n_\perp - n_\parallel) z + \frac{1}{24} g^2 B_{\mathrm{T}}^2 (b_+ - b_\times) z^3 \tag{3.137b}$$

$$= \omega(n_\perp - n_\parallel) z \Big(1 + \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2 \Big) \tag{3.137c}$$

where we have noted that

$$b_+ - b_\times = \Delta_\perp - \Delta_\parallel = (n_\perp - n_\parallel)\omega \,. \tag{3.138}$$

This yields the corresponding induced ellipticity

$$\delta\psi(z) \approx -\frac{1}{2} \omega (n_\perp - n_\parallel) z \Big(1 + \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2 \Big) \sin 2\theta \,. \tag{3.139}$$

We can therefore conclude, that as a result of the photon-graviton mixing, one can find an increase of the induced ellipticity by a factor of

$$1 + \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2 \tag{3.140}$$

compared to a situation when there were no mixing, that is $\delta\psi_0(z) = -\frac{1}{2}\omega(n_\perp - n_\parallel) z$. That is, the corresponding relative increase in ellipticity can be expressed as

$$\frac{\delta\psi(z) - \delta\psi_0(z)}{\delta\psi_0(z)} = \frac{1}{24} g^2 B_{\mathrm{T}}^2 z^2 \simeq 1.4 \times 10^{-38} \left(\frac{B_{\mathrm{T}}}{\mathrm{T}} \right)^2 \left(\frac{z}{\mathrm{m}} \right)^2 , \tag{3.141}$$

which, in our laboratory setup, comes out as an extremely small number. We can therefore conclude that although it was found in section 3.5.1 that the mixing angles in our laboratory setup is not so small, the fact that the two polarizations $A_\parallel$ and $A_\perp$ are influenced by the mixing comparably means that this effect becomes (most probably) unobservable through the measurement of $\delta\theta(z)$ and $\delta\psi(z)$.

Finally, note that as it is typically the case, due to the vDVZ discontinuity, the results for the massless spin-2 particle cannot be derived by simply taking the $m \to 0$ of the massive spin-2 computation.

3.5.3 LSW experiments

As in the previous chapters, we will now be interested in the probability of photon regeneration when passing through a wall, where on each side we turn on a background transverse magnetic field. In contrast to the considerations in Chapters 1 and 2, note that now we are dealing with a channel in which an LSW experiment could yield a signal even with the Standard Model framework. Interestingly the only other such channel is mediated by the conversion of photons into neutrino-antineutrino pairs [54, 56]. At leading order, this proceeds via an electron triangle and a Z-boson exchange, or a mixed loop of an electron and a W-boson. Nevertheless, this second channel appears to be strongly suppressed with respect to the graviton channel, so the calculation which we are about to embark on, would provide for the only significant LSW mechanism within the boundaries of the Standard Model.

The experimental setup and the idea is the same as before. Photons can only be measured on the other side of the wall if they are converted before they reach the wall, this time into gravitons (which practically do not interact with the medium). Once these are transformed back into photons in the magnetic field on the other side of the wall, the beam becomes detectable again. Initially, let us generally prepare the system in a pure photon state with general mixture of parallel and perpendicular polarization

$$\Psi_{\mathrm{i}} = \begin{pmatrix} 0 \\ A_{\parallel}(0) \\ 0 \\ A_{\perp}(0) \end{pmatrix} . \tag{3.142}$$

Following the ideas from the first chapter, we could then write for the final state measured on the other side of the wall

$$\Psi_{\mathrm{f}} = \mathsf{U}(z_2, 0)\Pi_H \mathsf{U}(z_1, 0)\Psi_{\mathrm{i}} \,, \tag{3.143}$$

where by Π_H, we have denoted the projector on the subspace spanned by both graviton fluctuations, namely

$$\Pi_H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.144}$$

The transfer matrix U may be expressed as a product of two matrices $\mathsf{U}_{\times 2,\parallel}$ and $\mathsf{U}_{+2,\perp}$ acting on the subspaces spanned by the $H_{\times 2}$, $A_{\parallel}$ flavours and the H_{+2}, $A_{\perp}$ flavours, respectively. In particular, we can write

$$\mathsf{U} = \mathsf{U}_{\times 2,\parallel}\mathsf{U}_{+2,\perp} \,, \tag{3.145}$$

where

$$\mathsf{U}_{\times 2,\parallel}(z,0) = \begin{pmatrix} \begin{matrix} U_{\times 2,\parallel} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} , \tag{3.146a}$$

$$\mathsf{U}_{+2,\perp}(z,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$U_{+2,\perp}$}} \\ 0 & 0 & & \end{pmatrix} . \tag{3.146b}$$

The submatrices $U_{\times 2,\parallel}(z,0)$ and then read $U_{+2,\perp}(z,0)$

$$U_{\times 2,\parallel}(z,0) = \frac{e^{-i\omega n_\parallel z}}{1+\Theta^2_{\times 2,\parallel}} \times \\ \times \begin{pmatrix} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + \Theta^2_{\times 2,\parallel} e^{-ib_\times \Theta^2_{\times 2,\parallel} z} & \Theta_{\times 2,\parallel}(e^{-ib_\times \Theta^2_{\times 2,\parallel} z} - e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z}) \\ \Theta_{\times 2,\parallel}(e^{-ib_\times \Theta^2_{\times 2,\parallel} z} - e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z}) & \Theta^2_{\times 2,\parallel} e^{ib_\times(1+\Theta^2_{\times 2,\parallel})z} + e^{-ib_\times \Theta^2_{\times 2,\parallel} z} \end{pmatrix} , \tag{3.147a}$$

$$U_{+2,\perp}(z,0) = \frac{e^{-i\omega n_\perp z}}{1+\Theta^2_{+2,\perp}} \times \\ \times \begin{pmatrix} e^{ib_+(1+\Theta^2_{+2,\perp})z} + \Theta^2_{+2,\perp} e^{-ib_+ \Theta^2_{+2,\perp} z} & \Theta_{+2,\perp}(e^{-ib_+ \Theta^2_{+2,\perp} z} - e^{ib_+(1+\Theta^2_{+2,\perp})z}) \\ \Theta_{+2,\perp}(e^{-ib_+ \Theta^2_{+2,\perp} z} - e^{ib_+(1+\Theta^2_{+2,\perp})z}) & \Theta^2_{+2,\perp} e^{ib_+(1+\Theta^2_{+2,\perp})z} + e^{-ib_+ \Theta^2_{+2,\perp} z} \end{pmatrix} . \tag{3.147b}$$

We can notice that the LSW experiment for photon-graviton mixing does not mix the two photon polarizations: the $A_\parallel$ mode is converted by the magnetic field into the $H_{\times 2}$ mode of the graviton, which passes through the wall and converts back into the $A_\parallel$ mode on the other side. Analogously for the $A_\perp$ mode. Hence, the total regeneration probability $P(A \to H \to A)$ can be expressed in terms of elementary regeneration probabilities for the two photon polarizations $P(A_\parallel \to H \to A_\parallel)$ and $P(A_\perp \to H \to A_\perp)$ as

$$P(A \to H \to A) = \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\parallel \to H \to A_\parallel) + \\ + \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\perp \to H \to A_\perp) , \tag{3.148}$$

while we have

$$P(A_\parallel \to H \to A_\perp) = 0 , \tag{3.149a}$$
$$P(A_\perp \to H \to A_\parallel) = 0 . \tag{3.149b}$$

We can therefore simplify our analysis by computing $P(A_\parallel \to H \to A_\parallel)$ and $P(A_\perp \to H \to A_\perp)$ first. To this end, let us first consider performing two separate measurements: in the first, the system is initially prepared in the (normalized) pure-$A_\parallel$ state

$$\Psi_{A_\parallel} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \tag{3.150}$$

while in the second, we prepared the system to be in the pure-$A_\perp$ state

$$\Psi_{A_\perp} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{3.151}$$

We can then simply write (recalling that both $\Theta_{\times 2,\parallel}$ and $\Theta_{+2,\perp}$ are small angles)

$$\begin{aligned}
P(A_\parallel \to H \to A_\parallel) &= |(\Psi_{A_\parallel})^\dagger \mathsf{U}(z_2,0)\Pi_H \mathsf{U}(z_1,0)\Psi_{A_\parallel}|^2 && (3.152a)\\
&\approx \Theta^4_{\times 2,\parallel}\left|(1-e^{ib_\times z_1})(1-e^{ib_\times z_2})\right|^2 && (3.152b)\\
&= 16\Theta^4_{\times 2,\parallel}\sin^2\frac{b_\times z_1}{2}\sin^2\frac{b_\times z_2}{2}\,, && (3.152c)
\end{aligned}$$

as well as

$$\begin{aligned}
P(A_\perp \to H \to A_\perp) &= |(\Psi_{A_\perp})^\dagger \mathsf{U}(z_2,0)\Pi_H \mathsf{U}(z_1,0)\Psi_{A_\perp}|^2 && (3.153a)\\
&\approx \Theta^4_{+2,\perp}\left|(1-e^{ib_+ z_1})(1-e^{ib_+ z_2})\right|^2 && (3.153b)\\
&= 16\Theta^4_{+2,\perp}\sin^2\frac{b_+ z_1}{2}\sin^2\frac{b_+ z_2}{2}\,. && (3.153c)
\end{aligned}$$

In total, substituting (3.152c) and (3.153c) into (3.148), we obtain the overall photon regeneration probability

$$\begin{aligned}
P(A \to H \to A) = \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2} &\times 16\Theta^4_{\times 2,\parallel}\sin^2\frac{b_\times z_1}{2}\sin^2\frac{b_\times z_2}{2}+\\
+\frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2} &\times 16\Theta^4_{+2,\perp}\sin^2\frac{b_+ z_1}{2}\sin^2\frac{b_+ z_2}{2}\,. \quad (3.154)
\end{aligned}$$

We notice that this can be maximized by a pure-$A_\parallel$ beam if $P(A_\parallel \to H \to A_\parallel) > P(A_\perp \to H \to A_\perp)$ and vice versa if $P(A_\parallel \to H \to A_\parallel) < P(A_\perp \to H \to A_\perp)$.

Again, if $b_\times z_1, b_+ z_1 \ll 1$ and $b_\times z_2, b_+ z_2 \ll 1$, that is, the sizes of both regions with magnetic field are very small compared to the oscillation length, we can simplify the above derived relation (using the definitions of the mixing angles Θ and parameters b) as

$$P(A \to H \to A) \approx \frac{1}{16} g^4 B_\mathrm{T}^4 z_1^2 z_2^2\,. \quad (3.155)$$

Notice that unlike for the previously considered particles, all dependence on the initial photon polarizations $A_\parallel$ and $A_\perp$ has dropped out in this limit. On the other hand, in the opposite regime, where $b_\times z_1, b_+ z_1 \gg 1$ and $b_\times z_2, b_+ z_2 \gg 1$, we could average over the oscillations and get the probability

$$\begin{aligned}
&\langle P(A \to H \to A)\rangle =\\
&\quad= \frac{4}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\left[|A_\parallel(0)|^2\Theta^4_{\times 2,\parallel}+|A_\perp(0)|^2\Theta^4_{+2,\perp}\right] && (3.156a)\\
&\quad= \frac{g^4 B_\mathrm{T}^4}{4\omega^4}\frac{1}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\left[\frac{|A_\parallel(0)|^2}{(n_\parallel-1)^4}+\frac{|A_\perp(0)|^2}{(n_\perp-1)^4}\right], && (3.156b)
\end{aligned}$$

where the dependence on z_1 and z_2 has completely dropped out.

Laser experiments

Let us evaluate the photon regeneration probability $P(A \to H \to A)$ for our typical laboratory setup with $\hbar\omega = 2.4\,\mathrm{eV}$ and $n_\parallel - 1 \approx n_\perp - 1 \simeq 10^{-17}$ (see

Chapter 1 for introduction). For these values, note that since we have already found the mixing length $l_{\rm osc} \simeq 10^{11}\,$m (see above), in a laboratory we can safely put $z_1, z_2 \ll l_{\rm osc}$. Substituting into (3.155), we obtain

$$P(A \to H \to A) \simeq 6.8 \times 10^{-75} \left(\frac{B_{\rm T}}{\rm T}\right)^4 \left(\frac{z_1}{\rm m}\right)^2 \left(\frac{z_2}{\rm m}\right)^2 , \tag{3.157}$$

so that considering a (quite optimistic) laboratory experiment with $B_{\rm T} \simeq 10\,$T and $z_1 \simeq 10\,{\rm m} \simeq z_2$, we would obtain $P(A \to H \to A) \simeq 10^{-66}$. Assuming our laser has a power of 100 W (the actual power of the laser used in the ALPS experiment at DESY was 35 W [57]), so that every second, it is emitting 2.6×10^{20} photons with energy $\hbar\omega = 2.4\,$eV, we can see that it would take about 10^{46} years (on average) for a single photon to be measured on the other side of the wall. This is more than the current age of the universe (by many orders of magnitude).

Cosmic double-conversion

Let us perform one final piece of calculation. As we have discussed in the introduction, one possible explanation for the recent observations of the ultra-high energy photons coming from a GRB would be an LSW-type scenario but on cosmic scales: in the GRB host galaxy, the photons would be converted into weakly interacting particles (such as gravitons) and then regenerated back in a magnetic field near to the observer at Earth, thus avoiding the energy loss through the interaction with CMB and EBL. We can therefore estimate the significance of such a phenomenon happening in the graviton channel by calculating the regeneration probability $P(A \to H \to A)$. As we have discussed above, graviton would be the only viable channel for this double conversion within the framework of the Standard Model (the neutrino-antineutrino channel being suppressed). Since we will be interested into an order-of-magnitude estimate only, we will not be considering effects due to expansion of the universe.

Let us take $\hbar\omega = 100\,$TeV and assume that the two conversions were mediated by the magnetic fields in the GRB host galaxy and the Milky Way. Since we are making just an estimate, we will simply take these two magnetic fields to have the same magnitude and extent. We will also assume same free-electron densities in both galaxies. Recalling the result (3.125) which, in particular, was valid for $\omega \simeq 100\,$TeV, we can then approximately write

$$P(A \to H \to A) \simeq \frac{1}{16} g^4 B_{\rm T}^4 z^4 \simeq 10^{-25} . \tag{3.158}$$

Also note, that the total energy emitted by GRB221009A was estimated as $10^{55}\,{\rm erg} \simeq 10^{67}\,$eV [58]. Assuming for the sake of simplicity, that all of this energy was radiated through the 100 TeV photons (very optimistic to say the least), that would translate to 10^{53} high-energy photons emitted during the event. Were it not for the interaction with CMB and EBL, the number of photons hitting the cross-section of the Earth would then have been

$$10^{53} \times \left(\frac{6370\,{\rm km}}{0.6\,{\rm Gpc}}\right)^2 \simeq 10^{16} , \tag{3.159}$$

where for the distance to the progenitor of GRB221009A, we have substituted 0.6 Gpc (based on its redshift $z \simeq 0.15$). Combining the results (3.158) and

(3.159) (and assuming that no photon would have made it to the Earth without undergoing the double conversion), we can therefore conclude that on average, one photon would make it once in every 10^9 GRB221009A-like events. Hence, the oscillations between photons and massless gravitons alone do not seem to be strong enough an effect to explain the observations of 100 TeV photons, meaning that one should consider physics beyond the standard model (see [3] for the study which is based on an axion-mediated conversion).

4. Mixing of photons in bigravity

Finally, let us discuss the mixing of both massless and massive spin-2 particles with the EM fluctuations. We will do so in the consistent framework of the bimetric theory of gravity.[20, 21, 59]

4.1 Linearized bigravity

We will first see how the linearized action for spin-2 (massive and massless) and EM fluctuations arises from the full bimetric theory.

4.1.1 The bimetric action

Let us consider an action for two symmetric tensor fields $g_{\mu\nu}$, $f_{\mu\nu}$ of the form

$$S[g, f, A] = S_0[g, f] + \int d^4x \sqrt{-g}\mathcal{L}_{\rm EM} \,, \tag{4.1}$$

where the pure spin-2 part is given as [21, 22]

$$S_0[g, f] = -\frac{1}{2\kappa_g} \int d^4x \left[\sqrt{-g}R(g) + \alpha^2\sqrt{-f}R(f) + \frac{\alpha^2}{\kappa_g}\sqrt{-g}V(S; \beta_n)\right] . \tag{4.2}$$

Here κ_g is a coupling constant which will later be related to the Einstein gravitational constant, while α is a dimensionless parameter which measures the relative interaction strength of the two spin-2 fields. Also, $\mathcal{L}_{\rm EM}$ is the usual Maxwell lagrangian, which is covariantized with respect to the metric $g_{\mu\nu}$, namely

$$\mathcal{L}_{\rm EM} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} \,, \tag{4.3}$$

where $F_{\mu\nu} = (\nabla_g)_\mu A_\nu - (\nabla_g)_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$. Finally, the interaction term between the spin-2 fields g and f is given in terms of the potential $V(S; \beta_n)$ which is defined so as to ensure the absence of the Boulware-Deser ghost. First, let us introduce the square-root matrix S through the relation

$$S^\rho{}_\sigma S^\sigma{}_\nu = g^{\rho\mu}f_{\mu\nu} \,. \tag{4.4}$$

Then, in terms of S, the potential V can be expressed as

$$V(S; \beta_n) = \sum_{n=0}^{4} \beta_n e_n(S) \,, \tag{4.5}$$

where β_n are five dimensionless parameters and the symmetric polynomials $e_n(S)$ are defined as

$$e_n(S) = S^{\mu_1}{}_{[\mu_1} \dots S^{\mu_n}{}_{\mu_n]} \,. \tag{4.6}$$

More explicitly, these read

$$e_0(S) = 1 \,, \tag{4.7a}$$

$$e_1(S) = \mathrm{tr}(S)\,, \tag{4.7b}$$

$$e_2(S) = \frac{1}{2}\Big[\mathrm{tr}(S)^2 - \mathrm{tr}(S^2)\Big]\,, \tag{4.7c}$$

$$e_3(S) = \frac{1}{6}\Big[\mathrm{tr}(S)^3 - 3\mathrm{tr}(S)\mathrm{tr}(S^2) + 2\mathrm{tr}(S^3)\Big]\,, \tag{4.7d}$$

$$e_4(S) = \frac{1}{24}\Big[\mathrm{tr}(S)^4 - 6\mathrm{tr}(S^2)\mathrm{tr}(S)^2 + 3\mathrm{tr}(S^2)^2 + 8\mathrm{tr}(S)\mathrm{tr}(S^3) - 6\mathrm{tr}(S^4)\Big]\,, \tag{4.7e}$$

where, since we work in four dimensions, we can note that $e_4(S)$ can be identified with the determinant of S, namely

$$e_4(S) = \det S\,. \tag{4.8}$$

Also, attempting to continue the expansion of V in terms of symmetric polynomials $e_n(S)$ beyond $n = 4$ using the definition (4.6), we would have identically found 0 at each $n > 4$ because we would have been antisymmetrizing over more indices than there are dimensions.

Notice that β_0 and β_4 give cosmological constant terms for the tensors g and f respectively: indeed, noting that

$$\det(S)^2 = \det(S^2) = \frac{\det f}{\det g}\,, \tag{4.9}$$

one can see that β_0 and β_4 contribute into the action with the following terms

$$\begin{aligned} -\frac{\alpha^2}{2\kappa_g^2}\int d^4x\,\Big[\sqrt{-g}\Big(\beta_0 + \beta_4 \det S\Big)\Big] &= \\ = -\frac{\alpha^2}{2\kappa_g^2}\int d^4x\,\sqrt{-g}\beta_0 &- \frac{\alpha^2}{2\kappa_g^2}\int d^4x\,\sqrt{-f}\beta_4\,. \end{aligned} \tag{4.10}$$

On the other hand, the couplings β_1, β_2, β_3 give non-linear interactions between $g_{\mu\nu}$ and $f_{\mu\nu}$.

Furthermore, observe that the pure spin-2 action (4.2) is symmetric under the simultaneous replacements

$$g_{\mu\nu} \to \alpha^2 f_{\mu\nu}\,, \tag{4.11a}$$

$$f_{\mu\nu} \to \alpha^{-2} g_{\mu\nu}\,, \tag{4.11b}$$

$$\beta_n \to \alpha^{2n-4}\beta_{4-n}\,. \tag{4.11c}$$

Finally, note that we have minimally coupled the Maxwell action to the spin-2 field $g_{\mu\nu}$ only: again, in order to ensure the absence of ghost instabilities, one can only couple matter to either $g_{\mu\nu}$ or $f_{\mu\nu}$. Since the action otherwise treats g and f symmetrically, we have chosen to couple the EM field to $g_{\mu\nu}$ without loss of generality.

4.1.2 Equations of motion

Varying the action (4.1) with respect to the spin-2 fields $g_{\mu\nu}$, one obtains the equations of motion

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \frac{\alpha^2}{2\kappa_g}V_{\mu\nu}(g,f) = \kappa_g (T_{\mathrm{EM}})_{\mu\nu}\,, \tag{4.12a}$$

$$R_{\mu\nu}(f) - \frac{1}{2}R(f)f_{\mu\nu} + \frac{1}{2\kappa_g}\tilde{V}_{\mu\nu}(g,f) = 0\,, \tag{4.12b}$$

where we have defined

$$V_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}V)\,, \tag{4.13a}$$

$$\tilde{V}_{\mu\nu} = -\frac{2}{\sqrt{-f}}\frac{\partial}{\partial f^{\mu\nu}}(\sqrt{-g}V)\,, \tag{4.13b}$$

and

$$(T_{\mathrm{EM}})_{\mu\nu} = +\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_{\mathrm{EM}})\,. \tag{4.14}$$

It is possible to explicitly evaluate

$$V_{\mu\nu} = g_{\mu\rho}\sum_{n=0}^{3}(-1)^n\beta_n(Y_{(n)})^\rho{}_\nu(S)\,, \tag{4.15a}$$

$$\tilde{V}_{\mu\nu} = f_{\mu\rho}\sum_{n=0}^{3}(-1)^n\beta_{4-n}(Y_{(n)})^\rho{}_\nu(S^{-1})\,. \tag{4.15b}$$

where, for $n = 0, \ldots, 3$, we have defined tensors

$$(Y_{(n)})^\rho{}_\nu(S) = \sum_{k=0}^{n}(-1)^k e_k(S)[S^{n-k}]^\rho{}_\nu\,. \tag{4.16}$$

Finally, for the EM field A_μ, we get the (curved) Maxwell equations

$$0 = g^{\mu\alpha}\nabla_\alpha F_{\mu\nu}\,. \tag{4.17}$$

4.1.3 Proportional solutions

Studying the classical solutions of the coupled set of equations of motion (4.12) and (4.17) for $g_{\mu\nu}$, $f_{\mu\nu}$ and A_μ in full generality seems like a formidable task. Let us therefore restrict ourselves on considering the so-called *proportional solutions*, where the classical field configurations take the form

$$\bar{f}_{\mu\nu} = c^2\bar{g}_{\mu\nu}\,, \tag{4.18}$$

with $\bar{A}_\mu = 0$. Note that without losing any generality, we can always rescale the spin-2 field $f_{\mu\nu}$ and suitably redefine the couplings α and β_n so as to be able to set $c^2 = 1$. This is facilitated by the fact that $f_{\mu\nu}$ is not coupled to any matter. Given this choice, we have $\bar{g}^{-1}\bar{f} = 1$ and so we can put $\bar{S} = 1$. This gives

$$e_0(\bar{S}) = 1\,,\quad e_1(\bar{S}) = 4\,,\quad e_2(\bar{S}) = 6\,,\quad e_3(\bar{S}) = 4\,,\quad e_4(\bar{S}) = 1\,, \tag{4.19}$$

which in turn yields

$$(Y_{(0)})^\rho{}_\nu(\bar{S}) = +\delta^\rho{}_\nu\,, \tag{4.20a}$$

$$(Y_{(1)})^\rho{}_\nu(\bar{S}) = -3\delta^\rho{}_\nu\,, \tag{4.20b}$$

$$(Y_{(2)})^\rho{}_\nu(\bar{S}) = +3\delta^\rho{}_\nu\,, \tag{4.20c}$$

$$(Y_{(3)})^{\rho}{}_{\nu}(\bar{S}) = -\delta^{\rho}{}_{\nu} \,. \tag{4.20d}$$

Substituting into (4.15), the equations of motion (4.12) therefore become (recall that we are restricting $\bar{A}_\mu = 0$)

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}_{\mu\nu} + \Lambda_g \bar{g}_{\mu\nu} = 0\,, \tag{4.21a}$$

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}_{\mu\nu} + \Lambda_f \bar{g}_{\mu\nu} = 0\,, \tag{4.21b}$$

where we have introduced the cosmological constants

$$\Lambda_g = \frac{\alpha^2}{2\kappa_g}(\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3)\,, \tag{4.22a}$$

$$\Lambda_f = \frac{1}{2\kappa_g}(\beta_4 + 3\beta_3 + 3\beta_2 + \beta_1)\,, \tag{4.22b}$$

for the two spin-2 fields $g_{\mu\nu}$ and $f_{\mu\nu}$. Hence, in order for the theory to actually admit such proportional solutions, we have to ensure that $\Lambda_f = \Lambda_g$, which corresponds to fixing one of the parameters β_n. We will actually make an even stronger restriction of the β-parameter space by requiring that

$$\Lambda_f = \Lambda_g = 0\,, \tag{4.23}$$

so that the theory admits flat spacetimes

$$\bar{g}_{\mu\nu} = \bar{f}_{\mu\nu} = \eta_{\mu\nu}\,. \tag{4.24}$$

This means that we will have to require that the linear constraints

$$0 = \beta_0 + 3\beta_1 + 3\beta_2 + \beta_3\,, \tag{4.25a}$$

$$0 = \beta_4 + 3\beta_3 + 3\beta_2 + \beta_1\,, \tag{4.25b}$$

on the β parameters are satisfied.

4.1.4 Expanding in fluctuations

Assuming that appropriate restrictions in the β parameter space have been made, we would now like to expand the bimetric theory in fluctuations around the flat proportional background, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{4.26a}$$

$$f_{\mu\nu} = \eta_{\mu\nu} + \ell_{\mu\nu}\,, \tag{4.26b}$$

$$A_\mu = A_\mu\,. \tag{4.26c}$$

Expanding the action (4.1) in powers of the fluctuations, one would have found an interacting action for $h_{\mu\nu}$, $\ell_{\mu\nu}$ and A_μ, which couples $h_{\mu\nu}$ and $\ell_{\mu\nu}$ already at quadratic order. As we have already gone through a similar derivation in great detail for the massless spin-2 field in section 3.2, let us just summarize the results. See for instance [22] for more details.

Spin-2 part

In particular, the quadratic part of the expanded pure spin-2 part (4.2) of the full action (4.1) reads

$$
\begin{aligned}
S_0^{(2)}[h,\ell] = \frac{1}{2\kappa_g}\int d^4x \Big[&\frac{1}{4}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} - \frac{1}{4}\partial_\rho h_\mu{}^\mu \partial^\rho h_\alpha{}^\alpha + \\
&+\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h_\alpha{}^\alpha - \frac{1}{2}(\partial_\mu h^{\mu\nu})(\partial^\alpha h_{\alpha\nu})\Big] + \\
+\frac{\alpha^2}{2\kappa_g}\int d^4x \Big[&\frac{1}{4}\partial_\rho \ell^{\mu\nu}\partial^\rho \ell_{\mu\nu} - \frac{1}{4}\partial_\rho \ell_\mu{}^\mu \partial^\rho \ell_\alpha{}^\alpha + \\
&+\frac{1}{2}\partial_\mu \ell^{\mu\nu}\partial_\nu \ell_\alpha{}^\alpha - \frac{1}{2}(\partial_\mu \ell^{\mu\nu})(\partial^\alpha \ell_{\alpha\nu})\Big] + \\
+\frac{1}{2\kappa_g}\int d^4x\, \frac{\tilde{M}^2}{4}\Big[&(h^\mu{}_\mu)^2 - h_{\mu\nu}h^{\mu\nu} + (\ell^\mu{}_\mu)^2 - \ell_{\mu\nu}\ell^{\mu\nu} + \\
&+2(h_{\mu\nu}\ell^{\mu\nu} - h^\mu{}_\mu \ell^\nu{}_\nu)\Big]\,, \qquad (4.27)
\end{aligned}
$$

where we have introduced a mass scale

$$\tilde{M}^2 = \frac{\alpha^2}{2\kappa_g}(\beta_1 + 2\beta_2 + \beta_3)\,. \qquad (4.28)$$

In other words, due to the presence of the term

$$2(h_{\mu\nu}\ell^{\mu\nu} - h^\mu{}_\mu \ell^\nu{}_\nu) \qquad (4.29)$$

in the last line of (4.27), the fluctuations $h_{\mu\nu}$ and $\ell_{\mu\nu}$ are not true mass eigenstates of the theory, as they do not diagonalize the kinetic term of the action. Hence, one has to perform a rotation in the space of the fluctuations in order to obtain true mass eigenstates. Denoting these by $H_{\mu\nu}$ and $\chi_{\mu\nu}$, it turns out that one can write

$$
\begin{aligned}
H_{\mu\nu} &= \frac{1}{2}\frac{1}{\sqrt{\kappa_g(1+\alpha^2)}}(h_{\mu\nu} + \alpha^2\ell_{\mu\nu})\,, && (4.30a)\\
\chi_{\mu\nu} &= \frac{1}{2}\frac{\alpha}{\sqrt{\kappa_g(1+\alpha^2)}}(\ell_{\mu\nu} - h_{\mu\nu})\,, && (4.30b)
\end{aligned}
$$

or, vice versa,

$$
\begin{aligned}
h_{\mu\nu} &= 2\sqrt{\frac{\kappa_g}{1+\alpha^2}}(H_{\mu\nu} - \alpha\chi_{\mu\nu})\,, && (4.31a)\\
\ell_{\mu\nu} &= 2\sqrt{\frac{\kappa_g}{1+\alpha^2}}(H_{\mu\nu} + \alpha^{-1}\chi_{\mu\nu})\,. && (4.31b)
\end{aligned}
$$

In terms of the new variables $H_{\mu\nu}$ and $\chi_{\mu\nu}$, the quadratic part of the action then becomes diagonal and reads

$$S_0^{(2)}[H,\chi] = \int d^4x \Big[\frac{1}{2}\partial_\rho H^{\mu\nu}\partial^\rho H_{\mu\nu} - \frac{1}{2}\partial_\rho H_\mu{}^\mu \partial^\rho H_\alpha{}^\alpha +$$

$$
\left. + \partial_\mu H^{\mu\nu}\partial_\nu H_\alpha{}^\alpha - (\partial_\mu H^{\mu\nu})(\partial^\alpha H_{\alpha\nu}) \right] +
$$

$$
+ \int d^4x \left[\frac{1}{2}\partial_\rho \chi^{\mu\nu}\partial^\rho \chi_{\mu\nu} + \partial_\mu \chi^{\mu\nu}\partial_\nu \chi_\alpha{}^\alpha - (\partial_\mu \chi^{\mu\nu})(\partial^\alpha \chi_{\alpha\nu}) + \right.
$$

$$
\left. - \frac{1}{2}\partial_\rho \chi_\mu{}^\mu \partial^\rho \chi_\alpha{}^\alpha - \frac{(m^{(\chi)})^2}{2}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{(m^{(\chi)})^2}{2}(\chi^\mu{}_\mu)^2 \right], \quad (4.32)
$$

where we notice that for the fluctuation $\chi_{\mu\nu}$, one has obtained precisely the Fierz-Pauli lagrangian (2.1) with spin-2 mass

$$
(m^{(\chi)})^2 = \frac{(\beta_1 + 2\beta_2 + \beta_3)(1 + \alpha^2)}{2\kappa_g} . \quad (4.33)
$$

We can therefore conclude that the bimetric theory with action (4.2), expanded around a flat proportional background, can be understood in terms of two mass eigenstates: a massless spin-2 field $H^{\mu\nu}$, as well as a massive spin-2.

It is interesting to note that since only the metric $g_{\mu\nu}$ couples to ordinary matter, one should expect astrophysical objects to source waves which initially excite only the $g_{\mu\nu}$ field. However, since this is not a mass eigenstate, then, as the wave propagates, a non-zero $f_{\mu\nu}$ amplitude will regenerate. This again leads to an oscillation phenomenon. Combined with the observational data from the LIGO experiment, this can be used to derive interesting constraints on the bimetric parameter space [60]. These oscillations will not be important for our considerations as we will directly work with the mass eigenstates $H_{\mu\nu}$ and $\chi_{\mu\nu}$.

Coupling to the EM field

Let us now analyze, how the new degrees of freedom $H_{\mu\nu}$ and $\chi_{\mu\nu}$ couple to the EM field A_μ. This we will do by linearizing the EM term in (4.1) (which only depends on $g_{\mu\nu}$) by substituting the expansion (4.26). Keeping the kinetic term for A_μ, as well as the cubic coupling AAh, we first obtain

$$
\int d^4x \sqrt{-g}\mathcal{L}_{\text{EM}} = -\frac{1}{4}\int d^4x \, F_{\mu\nu}F^{\mu\nu} +
$$

$$
- \frac{1}{2}\int d^4x \left(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) h_{\mu\nu} \quad (4.34a)
$$

$$
\equiv S_{\text{EM}} + S_{\text{int}} , \quad (4.34b)
$$

where the kinetic part S_{EM} of the action for A_μ is just the flat-space Maxwell action. In the interaction term S_{int}, we can substitute for the field redefinition (4.31) to obtain[1]

$$
S_{\text{int}} = \frac{g^{(H)}}{\sqrt{2}}\int d^4x \left(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) H_{\mu\nu} +
$$

$$
+ \frac{g^{(\chi)}}{\sqrt{2}}\int d^4x \left(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) \chi_{\mu\nu} , \quad (4.35)
$$

[1]For the sake of convenience, we also rescale the fluctuation $H_{\mu\nu} \to -H_{\mu\nu}$ so that the two couplings $g^{(H)}$ and $g^{(\chi)}$ have same sign. This has no effect on the kinetic part of the action.

where we have identified

$$g^{(H)} = \sqrt{\frac{2\kappa_g}{1+\alpha^2}}\,, \tag{4.36a}$$

$$g^{(\chi)} = \sqrt{\frac{2\kappa_g}{1+\alpha^2}}\alpha\,. \tag{4.36b}$$

Hence, comparing with the HAA coupling obtained for the massless spin-2 field in the preceding chapter (in particular the interaction term (3.44) and the rescaling (3.39)), we conclude what we can identify

$$\kappa = \frac{\kappa_g}{1+\alpha^2}\,, \tag{4.37}$$

where $\kappa = 8\pi G/c^4$ is the Einstein gravitational constant.

Total linearized action

We can summarize by saying that the bimetric theory yields the action

$$\begin{aligned}
S[H,\chi,A] = \int d^4x \Bigg[&\frac{1}{2}\partial_\rho H^{\mu\nu}\partial^\rho H_{\mu\nu} - \frac{1}{2}\partial_\rho H_\mu{}^\mu \partial^\rho H_\alpha{}^\alpha + \\
&+ \partial_\mu H^{\mu\nu}\partial_\nu H_\alpha{}^\alpha - (\partial_\mu H^{\mu\nu})(\partial^\alpha H_{\alpha\nu})\Bigg] + \\
+ \int d^4x \Bigg[&\frac{1}{2}\partial_\rho \chi^{\mu\nu}\partial^\rho \chi_{\mu\nu} + \partial_\mu \chi^{\mu\nu}\partial_\nu \chi_\alpha{}^\alpha - (\partial_\mu \chi^{\mu\nu})(\partial^\alpha \chi_{\alpha\nu}) + \\
&- \frac{1}{2}\partial_\rho \chi_\mu{}^\mu \partial^\rho \chi_\alpha{}^\alpha - \frac{(m^{(\chi)})^2}{2}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{(m^{(\chi)})^2}{2}(\chi^\mu{}_\mu)^2\Bigg] + \\
- \frac{1}{4}\int d^4x\, &F_{\mu\nu}F^{\mu\nu} + \\
+ \frac{g^{(H)}}{\sqrt{2}}\int d^4x &\left(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)H_{\mu\nu} + \\
+ \frac{g^{(\chi)}}{\sqrt{2}}\int d^4x &\left(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)\chi_{\mu\nu}
\end{aligned} \tag{4.38}$$

for the massless spin-2 fluctuation $H_{\mu\nu}$, massive spin-2 fluctuation $\chi_{\mu\nu}$ and the EM fluctuation A_μ. Moreover, we have learned that the coupling constants $g^{(H)}$, $g^{(\chi)}$ and the mass $m^{(\chi)}$ of the spin-2 field $\chi_{\mu\nu}$ are given in terms of the parameters of the bimetric theory and the Einstein gravitational constant as

$$g^{(H)} = \sqrt{2\kappa} = \frac{1}{m_{\text{Pl}}}\,, \tag{4.39a}$$

$$g^{(\chi)} = \sqrt{2\kappa}\alpha = \frac{\alpha}{m_{\text{Pl}}} = \alpha g^{(H)}\,, \tag{4.39b}$$

and

$$m^{(\chi)} = \sqrt{\frac{\beta_1 + 2\beta_2 + \beta_3}{2\kappa}} = \sqrt{\beta_1 + 2\beta_2 + \beta_3}\,m_{\text{Pl}}\,, \tag{4.40}$$

where $m_{\text{Pl}} \simeq 2.4 \times 10^{18}\,\text{GeV}$ is the Planck mass.

4.1.5 Mixing equations in a magnetic background

We observe that the action (4.38) is just a combination of the actions considered in chapters 2 and 3 with precisely the same forms of the couplings of the massive and massless spin-2 fields to the EM field. We can therefore spare ourselves a detailed derivation of the mixing equations for the bimetric fluctuations in a background magnetic field and just simply write down the result based on what we obtained in previous chapters. In particular, including also vacuum birefringence for the propagating EM modes, we get mixing equations

$$0 = (\omega - p - \Delta^{(\chi)})\chi_{+1}\,, \tag{4.41a}$$

$$0 = (\omega - p - \Delta^{(\chi)})\chi_{\times 1}\,, \tag{4.41b}$$

$$0 = \begin{pmatrix} \omega - p & 0 & a_2^{(H)} p \\ 0 & \omega - p - \Delta^{(\chi)} & a_2^{(\chi)} p \\ a_2^{(H)} p & a_2^{(\chi)} p & \omega - p + \Delta_{\parallel} \end{pmatrix} \begin{pmatrix} H_{\times 2} \\ \chi_{\times 2} \\ A_{\parallel} \end{pmatrix}, \tag{4.41c}$$

$$0 = \begin{pmatrix} \omega - p & 0 & 0 & a_2^{(H)} p \\ 0 & \omega - p - \Delta^{(\chi)} & 0 & a_2^{(\chi)} p \\ 0 & 0 & \omega - p - \Delta^{(\chi)} & a_0^{(\chi)} p \\ a_2^{(H)} p & a_2^{(\chi)} p & a_0^{(\chi)} p & \omega - p + \Delta_{\perp} \end{pmatrix} \begin{pmatrix} H_{+2} \\ \chi_{+2} \\ \chi_0 \\ A_{\perp} \end{pmatrix}, \tag{4.41d}$$

where we have defined

$$a_2^{(H)} = +\frac{g^{(H)} B_{\mathrm{T}}}{2\omega}\,, \tag{4.42a}$$

$$a_2^{(\chi)} = +\frac{g^{(\chi)} B_{\mathrm{T}}}{2\omega}\,, \tag{4.42b}$$

$$a_0^{(\chi)} = -\frac{g^{(\chi)} B_{\mathrm{T}}}{\sqrt{3}\omega}\,, \tag{4.42c}$$

as well as

$$\Delta^{(\chi)} = \frac{(m^{(\chi)})^2}{2\omega}\,. \tag{4.43}$$

Using the above identifications (4.39) and (4.40), the quantities $a_2^{(H)}$, $a_2^{(\chi)}$, $a_0^{(\chi)}$ and $\Delta^{(\chi)}$ can, in turn, be expressed in terms of the parameters α, β_i of the bimetric theory and the Einstein gravitational constant κ.

4.2 Searching for mass eigenstates

As in the previous chapters, we will now identify the mass eigenstates of the system propagating in a background magnetic field by setting the determinants of the matrices appearing in (4.41) to zero and solving for p in terms of ω to find the corresponding dispersion relations.

4.2.1 Decoupled massive spin-2 polarizations

In as the case of the isolated system of an EM field interacting with a massive spin-2 field, we find that the spin-2 polarizations $+1$ and $\times 1$ decouple from the rest and satisfy the massive dispersion relations

$$p_{+1}(\omega) = \omega - \Delta^{(\chi)}\,, \tag{4.44a}$$

$$p_{\times 1}(\omega) = \omega - \Delta^{(\chi)}\,, \tag{4.44b}$$

thus representing two freely propagating particles, each of them with mass m_χ. Their propagation is described simply as

$$\chi_{+1}(z) = \chi_{+1}(0)e^{-i(\omega-\Delta^{(\chi)})z}\,, \tag{4.45a}$$

$$\chi_{\times 1}(z) = \chi_{\times 1}(0)e^{-i(\omega-\Delta^{(\chi)})z}\,. \tag{4.45b}$$

4.2.2 $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ mixing

We will now consider the problem of identifying the mass eigenstates in the system of coupled polarizations $H_{\times 2}$, $\chi_{\times 2}$ and $A_\parallel$. Although the corresponding equation of motion (4.41c) looks very similar to the one considered in the case of 3-flavour mixing for the photon-massive spin 2 system (equation (2.96d)), there is a crucial difference which prevents us from using the same method to deal with the system: the two flavours ($H_{\times 2}$, $\chi_{\times 2}$) coupled to the EM polarization $A_\parallel$ have *different masses*. As a result, there is no apparent way of performing a rotation in the flavour space to decouple one of the polarizations and convert the problem to a simple 2-flavour mixing. Instead, we will make use of the fact as a consequence of perturbative consistency of the lagrangian, as well as due to the ultrarelativistic approximation, the numbers $a_2^{(H)}$ and $a_2^{(\chi)}$ need to be very small, namely

$$a_2^{(H)}, a_2^{(\chi)} \ll 1\,. \tag{4.46}$$

Hence, writing

$$a_2^{(H)} = a\alpha_2^{(H)}\,, \tag{4.47}$$

$$a_2^{(\chi)} = a\alpha_2^{(\chi)}\,, \tag{4.48}$$

where $a \ll 1$ is some characteristic scale of the couplings $a_2^{(H)}$, $a_2^{(\chi)}$ we can search for the mass eigenstates perturbatively in powers a. This should definitely be an admissible treatment in the case when *small mixing* occurs between all flavours, namely when the directions in the $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ flavour space corresponding to the new mass eigenstates depart not too much from the three axes. Since, at the same time, the small-mixing assumption is justified experimentally, we will focus on this from now on.

To this end, let us start by expressing the matrix appearing in (4.41c) as

$$M_\times(p) \equiv \begin{pmatrix} \omega - p & 0 & a_2^{(H)}p \\ 0 & \omega - p - \Delta^{(\chi)} & a_2^{(\chi)}p \\ a_2^{(H)}p & a_2^{(\chi)}p & \omega - p + \Delta_\parallel \end{pmatrix} = M_\times^{[0]}(p) + aM_\times^{[1]}(p)\,, \tag{4.49}$$

where we have introduced matrices

$$M_\times^{[0]}(p) = \begin{pmatrix} \omega - p & 0 & 0 \\ 0 & \omega - p - \Delta^{(\chi)} & 0 \\ 0 & 0 & \omega - p + \Delta_\parallel \end{pmatrix}\,, \tag{4.50a}$$

$$M_\times^{[1]}(p) = \begin{pmatrix} 0 & 0 & \alpha_2^{(H)} \\ 0 & 0 & \alpha_2^{(\chi)} \\ \alpha_2^{(H)} & \alpha_2^{(\chi)} & 0 \end{pmatrix} p\,. \tag{4.50b}$$

In this way, one can set up perturbation theory in a around the system of decoupled polarizations $H_{\times 2}$, $A_\parallel$, $\chi_{\times 2}$ and observe how the mixing arises through $\mathcal{O}(a)$ perturbations. Considering the smallness of a, we will be content with keeping only leading terms in the expansion in the powers of a.

First we note, that expanding the determinant of $M_\times(p)$ in powers of a, we can write (still exactly in a at this point)

$$\begin{aligned} &\det M_\times(p) = \\ &\qquad = \det M_\times^{[0]}(p) + a^2 p^2 \Big[(\alpha_2^{(\chi)})^2 (p-\omega) + (\alpha_2^{(H)})^2 (\Delta^{(\chi)} - \omega + p) \Big]\,, \end{aligned} \tag{4.51}$$

where we simply have

$$\det M_\times^{[0]}(p) = (\omega - p)(\omega - p - \Delta^{(\chi)})(\omega - p + \Delta_\parallel)\,. \tag{4.52}$$

Clearly, the $\mathcal{O}(a^0)$ condition

$$\det M_\times^{[0]}(p) = 0 \tag{4.53}$$

yields the $\mathcal{O}(a^0)$ dispersion relations

$$p_{\times 2}^{(H),[0]}(\omega) = \omega\,, \tag{4.54a}$$
$$p_{\times 2}^{(\chi),[0]}(\omega) = \omega - \Delta^{(\chi)}\,, \tag{4.54b}$$
$$p_\parallel^{(A),[0]}(\omega) = \omega + \Delta_\parallel\,, \tag{4.54c}$$

which correspond to three propagating mass eigenstates, which, in the $H_{\times 2}$–$A_\parallel$–$\chi_{\times 2}$ flavour space, are associated with the directions

$$e_{\times 2}^{(H),[0]} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_{\times 2}^{(\chi),[0]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_\parallel^{(A),[0]} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.55}$$

We will now treat the full dispersion relations, which are the solution of the condition

$$\det M_\times(p) = 0\,, \tag{4.56}$$

as a-deformations of the $\mathcal{O}(a^0)$ dispersion relations (4.54). In particular, let us parametrize these full dispersion relations as

$$p_{\times 2}^{(H)}(\omega) = p_{\times 2}^{(H),[0]}(\omega) + a^2 p_{\times 2}^{(H),[2]}(\omega) + \mathcal{O}(a^4)\,, \tag{4.57a}$$
$$p_{\times 2}^{(\chi)}(\omega) = p_{\times 2}^{(\chi),[0]}(\omega) + a^2 p_{\times 2}^{(\chi),[2]}(\omega) + \mathcal{O}(a^4)\,, \tag{4.57b}$$
$$p_\parallel^{(A)}(\omega) = p_\parallel^{(A),[0]}(\omega) + a^2 p_\parallel^{(A),[2]}(\omega) + \mathcal{O}(a^4) \tag{4.57c}$$

and the full normalized mass eigenstates as

$$e_{\times 2}^{(H)} = e_{\times 2}^{(H),[0]} + a e_{\times 2}^{(H),[1]} + \mathcal{O}(a^2)\,, \tag{4.58a}$$

$$e_{\times 2}^{(\chi)} = e_{\times 2}^{(\chi),[0]} + a e_{\times 2}^{(\chi),[1]} + \mathcal{O}(a^2)\,, \tag{4.58b}$$

$$e_{\parallel}^{(A)} = e_{\parallel}^{(A),[0]} + a e_{\parallel}^{(A),[1]} + \mathcal{O}(a^2)\,. \tag{4.58c}$$

We can then substitute the a-expansions (4.57) into the full equations (4.56) and then (using the fact that at $\mathcal{O}(a^0)$, these are solved as a consequence of the fact that the undeformed dispersion relations (4.54) solve the $\mathcal{O}(a^0)$ condition (4.53)) obtain at $\mathcal{O}(a^2)$ *linear* conditions for the subleading contributions

$$p_{\times 2}^{(H),[2]}(\omega)\,, \qquad p_{\times 2}^{(\chi),[2]}(\omega)\,, \qquad p_{\parallel}^{(A),[2]}(\omega)$$

into the full dispersion relations. This procedure yields $\mathcal{O}(a^2)$ linear conditions

$$0 = p_{\times 2}^{(H),[2]}(\omega)\Delta^{(\chi)}\Delta_{\parallel} + \omega^2(\alpha_2^{(H)})^2\Delta^{(\chi)}\,, \tag{4.59a}$$

$$0 = p_{\times 2}^{(\chi),[2]}(\omega)\Delta^{(\chi)}(\Delta^{(\chi)} + \Delta_{\parallel}) + (\omega - \Delta^{(\chi)})^2(\alpha_2^{(\chi)})^2\Delta^{(\chi)}\,, \tag{4.59b}$$

$$\begin{aligned} 0 = {} & p_{\parallel}^{(A),[2]}(\omega)\Delta_{\parallel}(\Delta_{\parallel} + \Delta^{(\chi)}) + \\ & - (\omega - \Delta_{\parallel})^2\Big[(\alpha_2^{(\chi)})^2\Delta_{\parallel} + (\alpha_2^{(H)})^2(\Delta^{(\chi)} + \Delta_{\parallel})\Big]\,. \end{aligned} \tag{4.59c}$$

In the ultrarelativistic limit $\Delta^{(\chi)}, \Delta_{\parallel} \ll \omega$, these can be solved to give

$$a^2 p_{\times 2}^{(H),[2]}(\omega) = -\frac{1}{4}\frac{(g^{(H)})^2 B_{\mathrm{T}}^2}{\Delta_{\parallel}}\,, \tag{4.60a}$$

$$a^2 p_{\times 2}^{(\chi),[2]}(\omega) = -\frac{1}{4}\frac{(g^{(\chi)})^2 B_{\mathrm{T}}^2}{\Delta_{\parallel} + \Delta^{(\chi)}}\,, \tag{4.60b}$$

$$a^2 p_{\parallel}^{(A),[2]}(\omega) = +\frac{1}{4}\frac{(g^{(H)})^2 B_{\mathrm{T}}^2}{\Delta_{\parallel}} + \frac{1}{4}\frac{(g^{(\chi)})^2 B_{\mathrm{T}}^2}{\Delta_{\parallel} + \Delta^{(\chi)}}\,. \tag{4.60c}$$

Similarly we can also compute the corrections

$$e_{\times 2}^{(H),[1]}\,, \qquad e_{\times 2}^{(\chi),[1]}\,, \qquad e_{\parallel}^{(A),[1]} \tag{4.61}$$

to the corresponding directions $e_{\times 2}^{(H)}$, $e_{\times 2}^{(\chi)}$, $e_{\parallel}^{(A)}$ in the $H_{\times 2}$–$\chi_{\times 2}$–$A_{\parallel}$ flavour space. We will do so by solving the equations

$$M_{\times}(p_{\times 2}^{(H)})e_{\times 2}^{(H)} = 0\,, \tag{4.62a}$$

$$M_{\times}(p_{\times 2}^{(\chi)})e_{\times 2}^{(\chi)} = 0\,, \tag{4.62b}$$

$$M_{\times}(p_{\parallel}^{(A)})e_{\parallel}^{(A)} = 0\,, \tag{4.62c}$$

order by order in a. At order $\mathcal{O}(a^0)$ these conditions read

$$0 = M_{\times}^{[0]}(p_{\times 2}^{(H),[0]})e_{\times 2}^{(H),[0]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta^{(\chi)} & 0 \\ 0 & 0 & +\Delta_{\parallel} \end{pmatrix} e_{\times 2}^{(H),[0]}\,, \tag{4.63a}$$

$$0 = M_{\times}^{[0]}(p_{\times 2}^{(\chi),[0]})e_{\times 2}^{(\chi),[0]} = \begin{pmatrix} \Delta^{(\chi)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta^{(\chi)} + \Delta_{\parallel} \end{pmatrix} e_{\times 2}^{(\chi),[0]}\,, \tag{4.63b}$$

$$0 = M_\times^{[0]}(p_\parallel^{(A),[0]})e_\parallel^{(A),[0]} = \begin{pmatrix} -\Delta_\parallel & 0 & 0 \\ 0 & -\Delta_\parallel - \Delta^{(\chi)} & 0 \\ 0 & 0 & 0 \end{pmatrix} e_\parallel^{(A),[0]} \,, \tag{4.63c}$$

which are clearly solved by the eigenvectors (4.55). At order $\mathcal{O}(a^1)$, we obtain conditions[2]

$$0 = M_\times^{[0]}(p_{\times 2}^{(H),[0]})e_{\times 2}^{(H),[1]} + M_\times^{[1]}(p_{\times 2}^{(H),[0]})e_{\times 2}^{(H),[0]} \,, \tag{4.64a}$$

$$0 = M_\times^{[0]}(p_{\times 2}^{(\chi),[0]})e_{\times 2}^{(\chi),[1]} + M_\times^{[1]}(p_{\times 2}^{(\chi),[0]})e_{\times 2}^{(\chi),[0]} \,, \tag{4.64b}$$

$$0 = M_\times^{[0]}(p_\parallel^{(A),[0]})e_\parallel^{(A),[1]} + M_\times^{[1]}(p_\parallel^{(A),[0]})e_\parallel^{(A),[0]} \,, \tag{4.64c}$$

or, after substituting,

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta^{(\chi)} & 0 \\ 0 & 0 & +\Delta_\parallel \end{pmatrix} e_{\times 2}^{(H),[1]} + \omega \begin{pmatrix} 0 \\ 0 \\ \alpha_2^{(H)} \end{pmatrix} \,, \tag{4.65a}$$

$$0 = \begin{pmatrix} \Delta^{(\chi)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta^{(\chi)} + \Delta_\parallel \end{pmatrix} e_{\times 2}^{(\chi),[1]} + (\omega - \Delta^{(\chi)}) \begin{pmatrix} 0 \\ 0 \\ \alpha_2^{(\chi)} \end{pmatrix} \,, \tag{4.65b}$$

$$0 = \begin{pmatrix} -\Delta_\parallel & 0 & 0 \\ 0 & -\Delta_\parallel - \Delta^{(\chi)} & 0 \\ 0 & 0 & 0 \end{pmatrix} e_\parallel^{(A),[1]} + (\omega + \Delta_\parallel) \begin{pmatrix} \alpha_2^{(H)} \\ \alpha_2^{(\chi)} \\ 0 \end{pmatrix} \,. \tag{4.65c}$$

This leads to the solutions (assuming, as usual, that $\Delta_\parallel, \Delta^{(\chi)} \ll \omega$)

$$a e_{\times 2}^{(H),[1]} = -\Theta_{\times 2}^{(H)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a\beta^{(H)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \tag{4.66a}$$

$$a e_{\times 2}^{(\chi),[1]} = -\Theta_{\times 2}^{(\chi)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a\beta^{(H)} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,, \tag{4.66b}$$

$$a e_\parallel^{(A),[1]} = \begin{pmatrix} \Theta_{\times 2}^{(H)} \\ \Theta_{\times 2}^{(\chi)} \\ 0 \end{pmatrix} + a\beta^{(A)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,, \tag{4.66c}$$

where we have introduced the mixing angles

$$\Theta_{\times 2}^{(H)} = \frac{g^{(H)} B_{\mathrm{T}}}{2\Delta_\parallel} \,, \tag{4.67a}$$

$$\Theta_{\times 2}^{(\chi)} = \frac{g^{(\chi)} B_{\mathrm{T}}}{2(\Delta_\parallel + \Delta^{(\chi)})} \,, \tag{4.67b}$$

which, consistent with the weak-mixing scenario, we assume to be small, that is

$$\Theta_{\times 2}^{(H)}, \Theta_{\times 2}^{(\chi)} \ll 1 \,. \tag{4.68}$$

[2]Note that here we can continue substituting the $\mathcal{O}(a^0)$ results for the momenta, because momenta receive first corrections only at second order in a.

Observe that we did not forget to add elements from the kernel of the $M_\times^{[0]}$ matrices in (4.66). However, the apparently arbitrary coefficients $\beta^{(H)}$, $\beta^{(\chi)}$, $\beta^{(A)}$ can be all set to zero by demanding that the perturbed eigenstates remain normalized.

Altogether, we can summarize that after diagonalization, the 3-flavour system $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ gives rise to mass eigenstates whose propagation is, in the ultrarelativistic and weak-mixing limit, described by the dispersion relations

$$p_{\times 2}^{(H)}(\omega) = \omega - \frac{1}{4}\frac{(g^{(H)})^2 B_\mathrm{T}^2}{\Delta_\parallel} + \mathcal{O}(a^4)\,, \tag{4.69a}$$

$$p_{\times 2}^{(\chi)}(\omega) = \omega - \Delta^{(\chi)} - \frac{1}{4}\frac{(g^{(\chi)})^2 B_\mathrm{T}^2}{\Delta_\parallel + \Delta^{(\chi)}} + \mathcal{O}(a^4)\,, \tag{4.69b}$$

$$p_\parallel^{(A)}(\omega) = \omega + \Delta_\parallel + \frac{1}{4}\frac{(g^{(H)})^2 B_\mathrm{T}^2}{\Delta_\parallel} + \frac{1}{4}\frac{(g^{(\chi)})^2 B_\mathrm{T}^2}{\Delta^{(\chi)} + \Delta_\parallel} + \mathcal{O}(a^4)\,, \tag{4.69c}$$

giving, in turn, a graviton-like state, a massive spin-2 like state and a photon-like state. In the $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ flavour space, these are represented by the directions

$$e_{\times 2}^{(H)} = \frac{1}{\sqrt{1 + (\Theta_{\times 2}^{(H)})^2}} \begin{pmatrix} 1 \\ 0 \\ -\Theta_{\times 2}^{(H)} \end{pmatrix}\,, \tag{4.70a}$$

$$e_{\times 2}^{(\chi)} = \frac{1}{\sqrt{1 + (\Theta_{\times 2}^{(\chi)})^2}} \begin{pmatrix} 0 \\ 1 \\ -\Theta_{\times 2}^{(\chi)} \end{pmatrix}\,, \tag{4.70b}$$

$$e_\parallel^{(A)} = \frac{1}{\sqrt{1 + (\Theta_{\times 2}^{(H)})^2 + (\Theta_{\times 2}^{(\chi)})^2}} \begin{pmatrix} \Theta_{\times 2}^{(H)} \\ \Theta_{\times 2}^{(\chi)} \\ 1 \end{pmatrix}\,, \tag{4.70c}$$

where the mixing angles are given by (4.67). If we furthermore introduce the parameters $b_{\times 2}^{(H)}$ and $b_{\times 2}^{(\chi)}$ as

$$b_{\times 2}^{(H)} \equiv \Delta_\parallel\,, \tag{4.71a}$$

$$b_{\times 2}^{(\chi)} \equiv \Delta_\parallel + \Delta^{(\chi)}\,, \tag{4.71b}$$

we can rewrite the dispersion relations (4.69) as

$$p_{\times 2}^{(H)}(\omega) = \omega n_\parallel - b_{\times 2}^{(H)}\Big[1 + (\Theta_{\times 2}^{(H)})^2\Big] + \mathcal{O}(a^4)\,, \tag{4.72a}$$

$$p_{\times 2}^{(\chi)}(\omega) = \omega n_\parallel - b_{\times 2}^{(\chi)}\Big[1 + (\Theta_{\times 2}^{(\chi)})^2\Big] + \mathcal{O}(a^4)\,, \tag{4.72b}$$

$$p_\parallel^{(A)}(\omega) = \omega n_\parallel + b_{\times 2}^{(H)}(\Theta_{\times 2}^{(H)})^2 + b_{\times 2}^{(\chi)}(\Theta_{\times 2}^{(\chi)})^2 + \mathcal{O}(a^4)\,. \tag{4.72c}$$

This enables us to write down the general solution for the $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ oscillations in the form

$$e^{i\omega n_\parallel z} \begin{pmatrix} H_{\times 2}(z) \\ \chi_{\times 2}(z) \\ A_\parallel(z) \end{pmatrix} =$$

$$= \frac{1}{1+(\Theta_{\times2}^{(H)})^2}\begin{pmatrix}1\\0\\-\Theta_{\times2}^{(H)}\end{pmatrix}\left[H_{\times2}(0)-A_\parallel(0)\Theta_{\times2}^{(H)}\right]e^{ib_{\times2}^{(H)}\left[1+(\Theta_{\times2}^{(H)})^2\right]z}+$$

$$+\frac{1}{1+(\Theta_{\times2}^{(\chi)})^2}\begin{pmatrix}0\\1\\-\Theta_{\times2}^{(\chi)}\end{pmatrix}\left[\chi_{\times2}(0)-A_\parallel(0)\Theta_{\times2}^{(\chi)}\right]e^{ib_{\times2}^{(\chi)}\left[1+(\Theta_{\times2}^{(\chi)})^2\right]z}+$$

$$+\frac{1}{1+(\Theta_{\times2}^{(H)})^2+(\Theta_{\times2}^{(\chi)})^2}\begin{pmatrix}\Theta_{\times2}^{(H)}\\\Theta_{\times2}^{(\chi)}\\1\end{pmatrix}\times$$

$$\times\left[A_\parallel(0)+H_{\times2}(0)\Theta_{\times2}^{(H)}+\chi_{\times2}(0)\Theta_{\times2}^{(\chi)}\right]e^{-i\left[b_{\times2}^{(H)}(\Theta_{\times2}^{(H)})^2+b_{\times2}^{(\chi)}(\Theta_{\times2}^{(\chi)})^2\right]z}. \quad (4.73)$$

4.2.3 H_{+2}–χ_{+2}–χ_0–$A_\perp$ mixing

Finally, we are faced with the problem of finding mass eigenstates in a system of 4-flavour mixing. We will now demonstrate that this can be reduced to the simple case of 2-flavour mixing in two steps. First, defining a new parameter

$$a_+^{(\chi)}=\sqrt{(a_0^{(\chi)})^2+(a_2^{(\chi)})^2}\,, \quad (4.74)$$

let us perform a rotation

$$\begin{pmatrix}H_{+2}\\\chi_+\\\chi_+'\\iA_\perp\end{pmatrix}=R_{\chi\chi}\begin{pmatrix}H_{+2}\\\chi_{+2}\\\chi_0\\iA_\perp\end{pmatrix}, \quad (4.75)$$

using a matrix

$$R_{\chi\chi}=\begin{pmatrix}1&0&0&0\\0&\frac{a_2^{(\chi)}}{a_+^{(\chi)}}&+\frac{a_0^{(\chi)}}{a_+^{(\chi)}}&0\\0&-\frac{a_0^{(\chi)}}{a_+^{(\chi)}}&\frac{a_2^{(\chi)}}{a_+^{(\chi)}}&0\\0&0&0&1\end{pmatrix}=\frac{1}{\sqrt7}\begin{pmatrix}1&0&0&0\\0&\sqrt3&-2&0\\0&2&\sqrt3&0\\0&0&0&1\end{pmatrix}. \quad (4.76)$$

The four by four matrix which enters the equation of motion (4.41d) then becomes

$$R_{\chi\chi}\begin{pmatrix}\omega-p&0&0&a_2^{(H)}p\\0&\omega-p-\Delta^{(\chi)}&0&a_2^{(\chi)}p\\0&0&\omega-p-\Delta^{(\chi)}&a_0^{(\chi)}p\\a_2^{(H)}p&a_2^{(\chi)}p&a_0^{(\chi)}p&\omega-p+\Delta_\perp\end{pmatrix}R_{\chi\chi}^{-1}=$$

$$=\begin{pmatrix}\omega-p&0&0&a_2^{(H)}p\\0&\omega-p-\Delta^{(\chi)}&0&a_+^{(\chi)}p\\0&0&\omega-p-\Delta^{(\chi)}&0\\a_2^{(H)}p&a_+^{(\chi)}p&0&\omega-p+\Delta_\perp\end{pmatrix}, \quad (4.77)$$

so that the equation of motion can be recast as

$$0=(\omega-p-\Delta^{(\chi)})\chi_+'\,, \quad (4.78a)$$

$$0 = \begin{pmatrix} \omega - p & 0 & a_2^{(H)} p \\ 0 & \omega - p - \Delta^{(\chi)} & a_+^{(\chi)} p \\ a_2^{(H)} p & a_+^{(\chi)} p & \omega - p + \Delta_\perp \end{pmatrix} \begin{pmatrix} H_{+2} \\ \chi_+ \\ iA_\perp \end{pmatrix} . \tag{4.78b}$$

The polarization χ'_+ therefore decouples from the rest and satisfies the dispersion relation

$$p'_+(\omega) = \omega - \Delta^{(\chi)} , \tag{4.79}$$

so that it propagates as a mode with definite mass m_χ. We are then still left with the coupled system of the three polarizations H_{+2}, χ_+ and $A_\perp$, which will be dealt with using perturbation theory, as in the case of the $H_{\times 2}$–$\chi_{\times 2}$–$A_\parallel$ oscillations. Directly applying the results of the previous subsection, we obtain that in the ultrarelativistic and weak-mixing limit, the propagation of the three mass eigenstates is described by the dispersion relations

$$p_{+2}^{(H)}(\omega) = \omega - \frac{1}{4}\frac{(g^{(H)})^2 B_\mathrm{T}^2}{\Delta_\perp} + \mathcal{O}(a^4) , \tag{4.80a}$$

$$p_{+}^{(\chi)}(\omega) = \omega - \Delta^{(\chi)} - \frac{7}{12}\frac{(g^{(\chi)})^2 B_\mathrm{T}^2}{\Delta_\perp + \Delta^{(\chi)}} + \mathcal{O}(a^4) , \tag{4.80b}$$

$$p_{\perp}^{(A)}(\omega) = \omega + \Delta_\perp + \frac{1}{4}\frac{(g^{(H)})^2 B_\mathrm{T}^2}{\Delta_\perp} + \frac{7}{12}\frac{(g^{(\chi)})^2 B_\mathrm{T}^2}{\Delta^{(\chi)} + \Delta_\perp} + \mathcal{O}(a^4) . \tag{4.80c}$$

These give, in turn, a graviton-like state, a massive spin-2 like state and a photon-like state. In the H_{+2}–χ_+–$A_\perp$ flavour space, these are represented by the directions

$$e_{+2}^{(H)} = \frac{1}{\sqrt{1 + (\Theta_{+2}^{(H)})^2}} \begin{pmatrix} 1 \\ 0 \\ -\Theta_{+2}^{(H)} \end{pmatrix} , \tag{4.81a}$$

$$e_{+}^{(\chi)} = \frac{1}{\sqrt{1 + (\Theta_{+}^{(\chi)})^2}} \begin{pmatrix} 0 \\ 1 \\ -\Theta_{+}^{(\chi)} \end{pmatrix} , \tag{4.81b}$$

$$e_{\perp}^{(A)} = \frac{1}{\sqrt{1 + (\Theta_{+2}^{(H)})^2 + (\Theta_{+}^{(\chi)})^2}} \begin{pmatrix} \Theta_{+2}^{(H)} \\ \Theta_{+}^{(\chi)} \\ 1 \end{pmatrix} , \tag{4.81c}$$

where the mixing angles $\Theta_{+2}^{(H)}$, $\Theta_{+}^{(\chi)}$ are given by the expressions

$$\Theta_{+2}^{(H)} = \frac{g^{(H)} B_\mathrm{T}}{2\Delta_\perp} , \tag{4.82a}$$

$$\Theta_{+}^{(\chi)} = \sqrt{\frac{7}{3}} \frac{g^{(\chi)} B_\mathrm{T}}{2(\Delta_\perp + \Delta^{(\chi)})} ., \tag{4.82b}$$

In line with the small-mixing assumption, one needs to assume that

$$\Theta_{+2}^{(H)}, \Theta_{+}^{(\chi)} \ll 1 . \tag{4.83}$$

If we furthermore introduce the parameters $b_{+2}^{(H)}$ and $b_{+}^{(\chi)}$ as

$$b_{+2}^{(H)} \equiv \Delta_\perp \,, \tag{4.84a}$$

$$b_{+}^{(\chi)} \equiv \Delta_\perp + \Delta^{(\chi)} \,, \tag{4.84b}$$

we can rewrite the dispersion relations (4.80) as

$$p_{+2}^{(H)}(\omega) = \omega n_\perp - b_{+2}^{(H)}\Big[1 + (\Theta_{+2}^{(H)})^2\Big] + \mathcal{O}(a^4)\,, \tag{4.85a}$$

$$p_{+}^{(\chi)}(\omega) = \omega n_\perp - b_{+}^{(\chi)}\Big[1 + (\Theta_{+}^{(\chi)})^2\Big] + \mathcal{O}(a^4)\,, \tag{4.85b}$$

$$p_{\perp}^{(A)}(\omega) = \omega n_\perp + b_{+2}^{(H)}(\Theta_{+2}^{(H)})^2 + b_{+}^{(\chi)}(\Theta_{+}^{(\chi)})^2 + \mathcal{O}(a^4)\,. \tag{4.85c}$$

This enables us to write down the general solution for the H_{+2}–χ_+–$A_\perp$ oscillations in the form

$$\begin{aligned}
e^{i\omega n_\perp z}\begin{pmatrix} H_{+2}(z) \\ \chi_+(z) \\ A_\perp(z)\end{pmatrix} = \\
= \frac{1}{1+(\Theta_{+2}^{(H)})^2}\begin{pmatrix} 1 \\ 0 \\ -\Theta_{+2}^{(H)}\end{pmatrix}\Big[H_{+2}(0) - A_\perp(0)\Theta_{+2}^{(H)}\Big] e^{ib_{+2}^{(H)}\left[1+(\Theta_{+2}^{(H)})^2\right]z} + \\
+ \frac{1}{1+(\Theta_{+}^{(\chi)})^2}\begin{pmatrix} 0 \\ 1 \\ -\Theta_{+}^{(\chi)}\end{pmatrix}\Big[\chi_{+}(0) - A_\perp(0)\Theta_{+}^{(\chi)}\Big] e^{ib_{+}^{(\chi)}\left[1+(\Theta_{+}^{(\chi)})^2\right]z} + \\
+ \frac{1}{1+(\Theta_{+2}^{(H)})^2+(\Theta_{+}^{(\chi)})^2}\begin{pmatrix} \Theta_{+2}^{(H)} \\ \Theta_{+}^{(\chi)} \\ 1\end{pmatrix} \times \\
\times \Big[A_\perp(0) + H_{+2}(0)\Theta_{+2}^{(H)} + \chi_+(0)\Theta_{+}^{(\chi)}\Big] e^{-i\left[b_{+2}^{(H)}(\Theta_{+2}^{(H)})^2 + b_{+}^{(\chi)}(\Theta_{+}^{(\chi)})^2\right]z}\,.
\end{aligned} \tag{4.86}$$

At the same time, the decoupled mode χ'_+ evolves simply as

$$\chi'_+(z) = \chi'_+(0)e^{-i(\omega-\Delta^{(\chi)})z}\,. \tag{4.87}$$

4.3 Observable effects

Let us now discuss implications of the mixing between an EM wave, a massive spin-2 wave and a massless spin-2 wave in various experimental and observational setups.

Of course, since the realization of the massive bigravity in nature remains hypothetical, we do not have the luxury of being able to substitute precise numerical values for the parameters α and $m^{(\chi)}$. However, it turns out [22] that one can severely constrain the ranges of α and $m^{(\chi)}$ by demanding that, at the same time, the massive spin-2 particle explains the observed abundance of Dark Matter in the present-day universe. We will therefore evaluate some of the observables which we consider below for the most optimistic values of α and $m^{(\chi)}$ (that is, smallest allowed $m^{(\chi)}$ and largest allowed Planck mass ratio α) which are still permitted by the requirement that the massive graviton is the elusive DM particle. We will consider two different scenarios following two different paradigms for modelling dark matter in cosmology: the heavy spin-2 DM and the ultra-light spin-2 DM.

For the first case, the corresponding bounds on the bimetric parameters are displayed in figure 4.1 which was adapted from [22][3,4] In particular, the DM hypothesis restricts the bigravity parameter space by three requirements: 1. ability to treat the bigravity theory perturbatively, 2. stability of the massive spin-2 particle, so that it is a viable DM candidate, 3. high-enough production rate in the early universe, so that the abundance of the massive spin-2 particles can match the observed DM content in our universe. Thus, for the purposes of explicit evaluation of some of the observables considered below, we shall adopt the most optimistic values

$$m^{(\chi)} \simeq 1\,\mathrm{TeV}\,, \qquad \alpha \simeq 10^{-11}\,. \tag{4.88}$$

At the same time, such a small value of α ensures compatibility of the predictions of the bimetric theory with standard GR and explains why massive spin-2 particles have not been observed in colliders. The coupling constant $g^{(\chi)}$ can be correspondingly evaluated as

$$g^{(\chi)} = \alpha g^{(H)} \simeq 4.2 \times 10^{-30}\,\mathrm{GeV}^{-1}\,. \tag{4.89}$$

As a consequence, if the massive spin-2 particle with such a high value of $m^{(\chi)}$ is to participate in the mixing with photons, we will have to take roughly $\omega > 10\,\mathrm{TeV}$, such as in the case of the ultra-high energetic photons observed by the Carpet-2 experiment [2]. In fact, mixing of such photons with the bimetric massive spin-2 particles could in principle explain, why we were able to observe these photons in the first place. Indeed, assuming that such photons were, in their host galaxy, converted into massive spin-2 particles and then regenerated back in our Galaxy, they would in fact avoid losing their energy through the interaction with the cosmic microwave background and extragalactic background light (with the subsequent production of e^+e^- pairs) and thus allow themselves to be detected upon arriving to Earth through what is effectively a light-shining-through-wall mechanism. In particular, in the numerical evaluations of various observable quantities below, we will be often taking

$$\omega \simeq 100\,\mathrm{TeV}\,, \tag{4.90}$$

which ensures that the ultrarelativistic condition $m^{(\chi)} \ll \omega$ holds.

On the other hand, one can obtain different bounds on the bimetric parameters by assuming that the massive spin-2 field is to provide the ultra-light dark matter. Such bounds were indeed obtained [24] from constraints given by pulsar timing. For the mass range

$$10^{-23}\,\mathrm{eV} < m^{(\chi)} < 10^{-17}\,\mathrm{eV}\,, \tag{4.91}$$

[3] In that paper, the Fierz-Pauli mass of the massive spin-2 field χ is denoted by m_{FP}, whereas here we denote it by $m^{(\chi)}$.

[4] The lower bound on the spin-2 particle mass can be reduced somewhat below the TeV scale by considering self-interactions of χ [61].

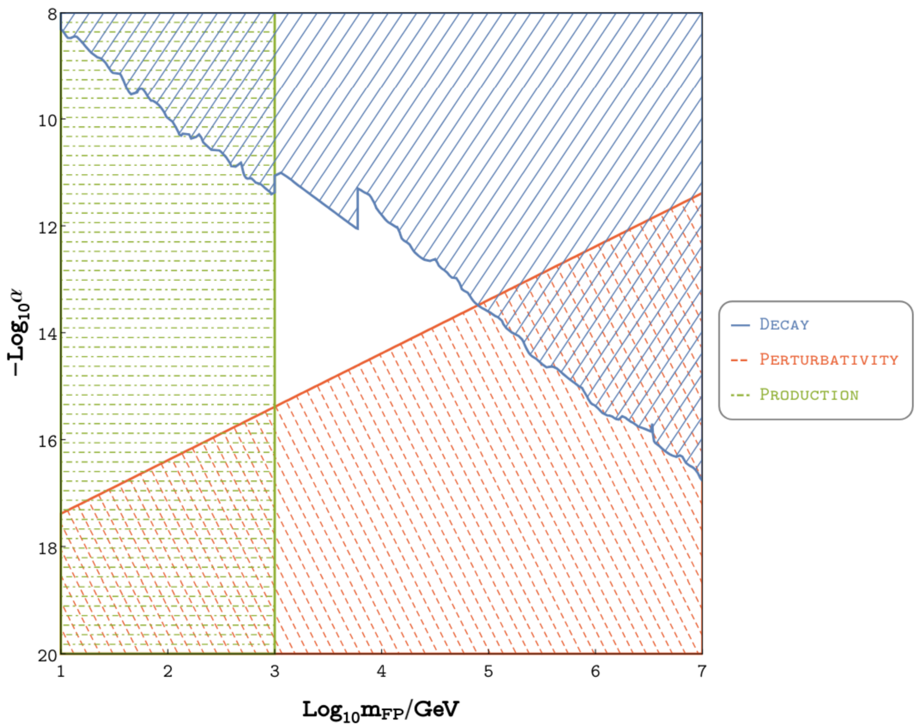

Figure 4.1: Bounds on the massive bigravity parameters α and $m_{\mathrm{FP}} \equiv m^{(\chi)}$. Figure taken from [22].

it turns out that one can constrain

$$\alpha \simeq 10^{-5}\,. \tag{4.92}$$

Thus, one gets a significantly stronger relative coupling of the two spin two fields than in the heavy spin-2 case. Moreover, the massive spin-2 field will be ultra-relativistic for all practically conceivable values of ω. In fact, in this scenario, the mass $m^{(\chi)}$ is so small that in most experimental / observational setups we have up to now considered, the term $\Delta^{(\chi)}$ will be completely negligible compared to $\Delta_\perp$ and $\Delta_\parallel$. Hence, for the massive spin-2 contribution to the mixing in this regime, one should mostly obtain qualitatively similar results as for the massless, except for having the coupling rescaled by α and, of course a relative factor of $\sqrt{7/3}$ between the two mixing angles $\Theta^{(\chi)}_{\times 2}$ and $\Theta^{(\chi)}_{+}$.

Finally, while it would be very elegant if DM and high-energy photons shared the same explanation, one should bear in mind that nothing prevents the bimetric massive spin-2 from providing a good explanation for the observations of 100 TeV photons while, at the same time, failing to account for the observed DM abundance (by, for instance, violating the bounds coming from production in the early universe and stability). Thus, even if one finds that the bimetric massive spin-2 field with the DM-consistent parameters (4.88) does not yield large enough LSW regeneration probabilities, there could still be allowed regions in the bimetric parameter space where it does so. It would only come at a price of having to search for another explanation for DM.

4.3.1 Mixing angles, oscillation lengths and conversion probabilities

To get some idea about evolution of the coupled system of the photon with the massive and massless spin-2 in an external magnetic field, let us first briefly discuss evaluating the various mixing angles and oscillation lengths appearing in the problem. Recall from the previous chapter that for the massless spin-2 mixing angles, one obtains

$$\Theta^{(H)} \simeq \frac{1}{2}\sqrt{\frac{16\pi G}{\mu_0 c^4}}\frac{cB_{\mathrm{T}}}{\omega}(n-1)^{-1} \simeq 5.7\times 10^{-26}\,(n-1)^{-1}\,\frac{\mathrm{eV}}{\hbar\omega}\frac{B}{\mathrm{T}}\,. \tag{4.93}$$

On the other hand, for the massive spin-2 mixing angles one obtains (forgetting about the relative factor of $\sqrt{7/3}$ between the two massive spin-2 mixing angles as we are interested in order of magnitude only)

$$\Theta^{(\chi)} \simeq \alpha\,\frac{2(n-1)\omega^2}{(m^{(\chi)})^2+2(n-1)\omega^2}\Theta^{(H)}\,. \tag{4.94}$$

Similarly, for the small-mixing oscillation length of the massive spin-2, one obtains

$$l_{\mathrm{osc}}^{(\chi)} = \frac{2(n-1)\omega^2}{(m^{(\chi)})^2+2(n-1)\omega^2}l_{\mathrm{osc}}^{(H)}\,, \tag{4.95}$$

while the large-mixing scenario oscillation length $\frac{\pi}{g^{(\chi)}B_{\mathrm{T}}}$ would have been enhanced by a factor of $1/\alpha$.

ULDM case

First, we note that the massive spin-2 mixing angles will be suppressed relative to the massless ones by a factor of α which, for both of the above mentioned scenaria (heavy spin-2 DM and ULDM), is very small. Furthermore, in the ULDM regime, the mass term will be mostly negligible with respect to the refractive indices so we may approximately write

$$\Theta^{(\chi)} \simeq \alpha\Theta^{(H)}\,, \qquad l_{\mathrm{osc}}^{(\chi)} \simeq l_{\mathrm{osc}}^{(H)} \qquad (\mathrm{ULDM})\,. \tag{4.96}$$

This means that the conversion probabilities $P(A\to\chi)$ and $P(\chi\to A)$ will be suppressed with respect to $P(A\to H)$ and $P(H\to A)$ by a factor of α^2, while the double-conversion probability $P(A\to\chi\to A)$ will be suppressed by α^4.

Heavy spin-2 DM case

On the other hand, in the heavy spin-2 scenario with $100\,\mathrm{TeV} > \omega > 10\,\mathrm{TeV}$, for small-enough $n-1$ one will typically have $(m^{(\chi)})^2$ comparable with or dominating over $2(n-1)\omega^2$, thus causing further suppression.[5] In that case we can write

$$\Theta^{(\chi)} \simeq \alpha\,\frac{2(n-1)\omega^2}{(m^{(\chi)})^2}\Theta^{(H)}\,, \quad l_{\mathrm{osc}}^{(\chi)} \simeq \frac{2(n-1)\omega^2}{(m^{(\chi)})^2}l_{\mathrm{osc}}^{(H)} \quad (\text{heavy spin-2})\,. \tag{4.97}$$

[5] There could be a small window where $2(n-1)\omega^2$ dominates over $(m^{(\chi)})^2$. This would give relations analogous to the ULDM case (4.96) and therefore similar suppression of transition probabilities.

In the incoherent case $z \gg l_{\rm osc}^{(H)}$, the massive spin-2 will also oscillate incoherently (since (4.97) gives $l_{\rm osc}^{(\chi)} < l_{\rm osc}^{(H)}$) and the transition probabilities $P(A \to \chi)$ will be suppressed relative to $P(A \to H)$ by a factor of $4\alpha^2\,(n-1)^2\omega^4/(m^{(\chi)})^4$. In the coherent case $z \ll l_{\rm osc}^{(H)}$, we have to distinguish between two cases: 1. the suppression of $l_{\rm osc}$ is mild so that we also have $z \ll l_{\rm osc}^{(\chi)}$ (the massive as well as massless spin-2 oscillate coherently), or, 2. the suppression of $l_{\rm osc}$ is strong so that the massive spin-2 oscillation becomes incoherent, that is $z \gg l_{\rm osc}^{(\chi)}$ (while the massless spin-2 continues to oscillate coherently). For scenario no. 1, the transition probabilities $P(A \to \chi)$ will be suppressed by a factor of α^2 while for scenario number 2, they will be suppressed even more strongly by a factor of $\alpha^2(l_{\rm osc}^{(\chi)}/z)^2$.

Resonant enhancement?

Notice that in principle, one could think about achieving a relative enhancement of the massive spin-2 mixing angle and oscillation length with respect to the massless case by resonantly tuning

$$\frac{m^{(\chi)}}{\omega} \approx \sqrt{2(1-n)}\,. \tag{4.98}$$

This can only be possible when the refractive index is dominated by the free electrons so that $\Delta_\parallel, \Delta_\perp < 0$. In such cases, the resonance condition (4.98) could be solved for quite a wide range of ω by tuning the plasma frequency

$$\omega_{\rm p} \approx m^{(\chi)} \tag{4.99}$$

in media with $n < 1$. In the ULDM case, this would require the free electron number density from the range $N_{\rm e} \simeq 10^{-6} - 10^{-18}\,{\rm m}^{-3}$. Such low values seem to be impossible to achieve in our universe (even in intergalactic space, one has $N_{\rm e} \simeq 10^2\,{\rm m}^{-3}$, see for instance [62]). Similarly, in the heavy spin-2 regime, one would have to arrange for a plasma frequency $\omega_{\rm p} \simeq 1\,{\rm TeV}$. This would correspond to $N_{\rm e} \simeq 10^{51}\,{\rm m}^{-3}$, which in turn gives a mass density (due to electrons and the same number of protons so as to balance the total charge) of about $10^{24}\,{\rm kg\,m}^{-3}$. Since this is significantly larger than density of a typical neutron star (by a factor of 10^7), we conclude that such $\omega_{\rm p}$ is unphysical. Hence, we have to conclude that for the two massive spin-2 paradigms (ULDM and heavy spin-2), the resonant enhancement of $\Theta^{(\chi)}$ relative to $\Theta^{(H)}$ seems not likely to be achievable.

We have found that within the two regions of the bimetric parameter space specified by the ULDM and heavy spin-2 DM paradigms, there does not seem to be an opportunity for the effects due to the massive spin-2 to be comparable with the effects due to massless spin-2 (which were extensively discussed in the preceding chapter), let alone be enhanced relative to them. However, as we have remarked above, it may be possible for the theory of bigravity to be realized in nature in a setting which is different from the above two candidate descriptions of Dark Matter (possibly with some intermediate value of the Fierz-Pauli mass $m^{(\chi)}$ or higher values of α). With this in mind, as well as for the sake of completeness, let us now derive a number of general results for various observable quantities.

4.3.2 Effects on photon polarization

Letting a linearly-polarized laser beam propagate over a distance z through a region with constant magnetic field B_{T} (that is, taking the solutions (4.73) and (4.86) with the initial conditions $\chi_{\mu\nu}(0) = H_{\mu\nu}(0) = 0$), one obtains that the EM polarizations $A_\parallel$ and $A_\perp$ evolve as

$$
\begin{aligned}
e^{i\omega n_\parallel z}\frac{A_\parallel(z)}{A_\parallel(0)} = {} & \frac{1}{1+(\Theta^{(H)}_{\times 2})^2}(\Theta^{(H)}_{\times 2})^2 e^{ib^{(H)}_{\times 2}\left[1+(\Theta^{(H)}_{\times 2})^2\right]z} + \\
& + \frac{1}{1+(\Theta^{(\chi)}_{\times 2})^2}(\Theta^{(\chi)}_{\times 2})^2 e^{ib^{(\chi)}_{\times 2}\left[1+(\Theta^{(\chi)}_{\times 2})^2\right]z} + \\
& + \frac{1}{1+(\Theta^{(H)}_{\times 2})^2+(\Theta^{(\chi)}_{\times 2})^2} e^{-i\left[b^{(H)}_{\times 2}(\Theta^{(H)}_{\times 2})^2 + b^{(\chi)}_{\times 2}(\Theta^{(\chi)}_{\times 2})^2\right]z}\,, \quad (4.100a)\\
e^{i\omega n_\perp z}\frac{A_\perp(z)}{A_\perp(0)} = {} & \frac{1}{1+(\Theta^{(H)}_{+2})^2}(\Theta^{(H)}_{+2})^2 e^{ib^{(H)}_{+2}\left[1+(\Theta^{(H)}_{+2})^2\right]z} + \\
& + \frac{1}{1+(\Theta^{(\chi)}_{+})^2}(\Theta^{(\chi)}_{+})^2 e^{ib^{(\chi)}_{+}\left[1+(\Theta^{(\chi)}_{+})^2\right]z} + \\
& + \frac{1}{1+(\Theta^{(H)}_{+2})^2+(\Theta^{(\chi)}_{+})^2} e^{-i\left[b^{(H)}_{+2}(\Theta^{(H)}_{+2})^2 + b^{(\chi)}_{+}(\Theta^{(\chi)}_{+})^2\right]z}\,. \quad (4.100b)
\end{aligned}
$$

Making the usual set of assumptions

$$
\begin{aligned}
b^{(H)}_{\times 2}(\Theta^{(H)}_{\times 2})^2 z &\ll 1\,, && (4.101a)\\
b^{(\chi)}_{\times 2}(\Theta^{(\chi)}_{\times 2})^2 z &\ll 1\,, && (4.101b)\\
b^{(H)}_{+2}(\Theta^{(H)}_{+2})^2 z &\ll 1\,, && (4.101c)\\
b^{(\chi)}_{+}(\Theta^{(\chi)}_{+})^2 z &\ll 1\,, && (4.101d)
\end{aligned}
$$

as well as

$$
\omega(n_\perp - n_\parallel)z \ll 1\,, \quad (4.102)
$$

we can obtain the relative decrease $\eta(z)$ in the $A_\perp$ amplitude, as well as the phase delay $\varphi(z)$ as

$$
\begin{aligned}
\eta(z) = {} & 2(\Theta^{(H)}_{+2})^2 \sin^2\frac{b^{(H)}_{+2}z}{2} - 2(\Theta^{(H)}_{\times 2})^2 \sin^2\frac{b^{(H)}_{\times 2}z}{2} + \\
& + 2(\Theta^{(\chi)}_{+2})^2 \sin^2\frac{b^{(\chi)}_{+2}z}{2} - 2(\Theta^{(\chi)}_{\times 2})^2 \sin^2\frac{b^{(\chi)}_{\times 2}z}{2}\,, \quad (4.103a)\\
\varphi(z) = {} & \omega(n_\perp - n_\parallel)z + \\
& + (\Theta^{(H)}_{+2})^2(b^{(H)}_{+2}z - \sin b^{(H)}_{+2}z) - (\Theta^{(H)}_{\times 2})^2(b^{(H)}_{\times 2}z - \sin b^{(H)}_{\times 2}z) + \\
& + (\Theta^{(\chi)}_{+})^2(b^{(\chi)}_{+}z - \sin b^{(\chi)}_{+}z) - (\Theta^{(\chi)}_{\times 2})^2(b^{(\chi)}_{\times 2}z - \sin b^{(\chi)}_{\times 2}z)\,. \quad (4.103b)
\end{aligned}
$$

Further approximating these expressions by assuming that coherence between the EM wave and the massive and massless spin-2 waves is retained, namely

$$
b^{(H)}_{\times 2}z \ll 1\,, \qquad b^{(\chi)}_{\times 2}z \ll 1\,, \qquad b^{(H)}_{+2}z \ll 1\,, \qquad b^{(\chi)}_{+}z \ll 1\,, \quad (4.104)
$$

one obtains

$$\eta(z) \approx \frac{1}{6}(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^2\,, \tag{4.105a}$$

$$\begin{aligned}\varphi(z) \approx \omega(n_\perp - n_\parallel) z\Big(1 + \frac{1}{24}(g^{(H)})^2 B_{\mathrm{T}}^2 z^2 + \frac{1}{24}(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^2\Big) + \\ + \frac{1}{36}\frac{(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^3}{\omega}\Big[(m^{(\chi)})^2 + 2(n_\perp - 1)\omega^2\Big]\,.\end{aligned} \tag{4.105b}$$

These then give rise to the rotation $\delta\theta(z)$ of the photon polarization plane, as well as to the induced ellipticity $\delta\psi(z)$ of the beam, which can be expressed as

$$\delta\theta(z) \approx -\frac{1}{12}(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^2 \sin 2\theta\,, \tag{4.106a}$$

$$\begin{aligned}\delta\psi(z) \approx -\frac{1}{2}\omega(n_\perp - n_\parallel) z\Big(1 + \frac{1}{24}(g^{(H)})^2 B_{\mathrm{T}}^2 z^2 + \frac{1}{24}(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^2\Big)\sin 2\theta + \\ - \frac{1}{72}\frac{(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^3}{\omega}\Big[(m^{(\chi)})^2 + 2(n_\perp - 1)\omega^2\Big]\sin 2\theta\,.\end{aligned} \tag{4.106b}$$

Note that regarding the distinguishability of the massive spin-2 particle from a massive spin-0 particle through this measurement, the same discussion applies as we have made at the end of section 2.4.1. In addition to this, we can see that in order for the measurement to tell the difference between spin-2 and spin-2, the $(g^{(\chi)})^2 B_{\mathrm{T}}^2 z^2$ term in the first line of (4.106b) must not be dominated by the $(g^{(H)})^2 B_{\mathrm{T}}^2 z^2$ term. That is to say, the massive spin-2 coupling $g^{(\chi)}$ has to be at least comparable to the gravitational coupling $g^{(H)}$ (which, for instance, does not happen neither for the heavy spin-2 DM regime, nor the ULDM regime). Otherwise, as we have pointed out in section 2.4.1, one can perform a rescalling of $g^{(\chi)}$ and the expressions for $\delta\theta(z)$ and $\delta\psi(z)$ will become identical to those one would have obtained for a massive spin-0 field.

4.3.3 LSW experiments

Following the same ideas as in the previous three chapters, we can calculate the regeneration probability of photon when the beam is passing through a wall with magnetic fields on both sides. The total regeneration probability $P(A \to \chi + H \to A)$ can be again expressed in terms of elementary regeneration probabilities for the two photon polarizations $P(A_\parallel \to \chi_{\times 2} + H_{\times 2} \to A_\parallel)$ and $P(A_\perp \to \chi_+ + H_{+2} \to A_\perp)$ as

$$\begin{aligned}P(A \to \chi + H \to A) = \\ = \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\parallel \to \chi_{\times 2} + H_{\times 2} \to A_\parallel) + \\ + \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2} P(A_\perp \to \chi_+ + H_{+2} \to A_\perp)\,.\end{aligned} \tag{4.107}$$

Thus let us first consider doing two measurements, in the first one the system will be initially prepared in pure $A_\parallel$ state $\Psi_{A_\parallel} = (0,0,0,0,1,0,0,0,0)$, while in the second one, the system will be initially in pure $A_\perp$ state $\Psi_{A_\perp} = (0,0,0,0,0,0,0,0,1)$

in the full 9-flavour space. The projector $\Pi_{\chi H}$ on the spin-2 flavours will hence be represented by a 9 by 9 matrix

$$\Pi_{\chi H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.108}$$

However, since the polarizations χ_{+1}, $\chi_{\times 1}$ and χ'_{+} do not mix with the rest, we can just ignore the rows and columns 1, 2 and 8 in the matrix, as well as corresponding components in the initial state vector. While, formally speaking, the transfer matrix U will also be 9 by 9, due to the three decoupled polarizations, we can limit ourselves to a 6 by 6 matrix $\tilde{\mathsf{U}}$ which can be factorized as

$$\tilde{\mathsf{U}} = \tilde{\mathsf{U}}_{\times,\parallel}\tilde{\mathsf{U}}_{+,\perp} , \tag{4.109}$$

where the matrices $\tilde{\mathsf{U}}_{\times,\parallel}$ and $\tilde{\mathsf{U}}_{+,\perp}$ act non-trivially only on the $H_{\times 2}$-$\chi_{\times 2}$-$A_{\parallel}$ flavour space and the H_{+2}-χ_{+}-$A_{\perp}$ flavour space, respectively. They take on a familiar form

$$\tilde{\mathsf{U}}_{\times,\parallel} = \begin{pmatrix} & & & 0 & 0 & 0 \\ & \tilde{U}_{\times,\parallel} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.110}$$

and

$$\tilde{\mathsf{U}}_{+,\perp} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \tilde{U}_{+,\perp} & \\ 0 & 0 & 0 & & & \end{pmatrix} , \tag{4.111}$$

while the exact form of the submatrices $\tilde{U}_{\times,\parallel}$ and $\tilde{U}_{+,\perp}$ (this time quite complicated) can be deduced from the evolution equations (4.73) and (4.86) derived above. Performing all the steps of a (by now surely familiar) calculation, we can arrive at the probabilities (considering all Θs to be small)

$$P(A_{\parallel} \to \chi_{\times 2} + H_{\times 2} \to A_{\parallel}) =$$

$$= \left|(\Psi_{A_{\parallel}})^{\dagger}\tilde{\mathsf{U}}(z_2, 0)\Pi_{\chi H}\tilde{\mathsf{U}}(z_1, 0)\Psi_{A_{\parallel}}\right|^2 \tag{4.112a}$$

$$\approx \Big|(\Theta^{(H)}_{\times 2})^2(-e^{ib^{(H)}_{\times 2} z_2} + 1)(-e^{ib^{(H)}_{\times 2} z_1} + 1)+$$

$$+ (\Theta^{(\chi)}_{\times 2})^2(-e^{ib^{(\chi)}_{\times 2} z_2} + 1)(-e^{ib^{(\chi)}_{\times 2} z_1} + 1)\Big|^2 \tag{4.112b}$$

$$= 16(\Theta^{(H)}_{\times 2})^4 \sin^2\left(\frac{b^{(H)}_{\times 2} z_1}{2}\right)\sin^2\left(\frac{b^{(H)}_{\times 2} z_2}{2}\right)+$$

$$+ 16(\Theta^{(\chi)}_{\times 2})^4 \sin^2\left(\frac{b^{(\chi)}_{\times 2} z_1}{2}\right)\sin^2\left(\frac{b^{(\chi)}_{\times 2} z_2}{2}\right)+$$

$$+ 32(\Theta^{(H)}_{\times 2})^2(\Theta^{(\chi)}_{\times 2})^2 \cos\left(\frac{(b^{(\chi)}_{\times 2} - b^{(H)}_{\times 2})(z_1+z_2)}{2}\right)\times$$

$$\times \sin\left(\frac{b^{(\chi)}_{\times 2} z_1}{2}\right)\sin\left(\frac{b^{(\chi)}_{\times 2} z_2}{2}\right)\sin\left(\frac{b^{(H)}_{\times 2} z_1}{2}\right)\sin\left(\frac{b^{(H)}_{\times 2} z_2}{2}\right) \tag{4.112c}$$

and

$$P(A_\perp \to \chi_+ + H_{+2} \to A_\perp) =$$

$$= \left|(\Psi_{A_\perp})^\dagger \tilde{\mathsf{U}}(z_2,0)\Pi_{\chi+H}\tilde{\mathsf{U}}(z_1,0)\Psi_{A_\perp}\right|^2 \tag{4.113a}$$

$$\approx \Big|(\Theta^{(H)}_{+2})^2(-e^{ib^{(H)}_{+2} z_2} + 1)(-e^{ib^{(H)}_{+2} z_1} + 1)+$$

$$+ (\Theta^{(\chi)}_{+})^2(-e^{ib^{(\chi)}_{+} z_2} + 1)(-e^{ib^{(\chi)}_{+} z_1} + 1)\Big|^2 \tag{4.113b}$$

$$= 16(\Theta^{(H)}_{+2})^4 \sin^2\left(\frac{b^{(H)}_{+2} z_1}{2}\right)\sin^2\left(\frac{b^{(H)}_{+2} z_2}{2}\right)+$$

$$+ 16(\Theta^{(\chi)}_{+})^4 \sin^2\left(\frac{b^{(\chi)}_{+} z_1}{2}\right)\sin^2\left(\frac{b^{(\chi)}_{+} z_2}{2}\right)+$$

$$+ 32(\Theta^{(H)}_{+2})^2(\Theta^{(\chi)}_{+})^2 \cos\left(\frac{(b^{(\chi)}_{+} - b^{(H)}_{+2})(z_1+z_2)}{2}\right)\times$$

$$\times \sin\left(\frac{b^{(\chi)}_{+} z_1}{2}\right)\sin\left(\frac{b^{(\chi)}_{+} z_2}{2}\right)\sin\left(\frac{b^{(H)}_{+2} z_1}{2}\right)\sin\left(\frac{b^{(H)}_{+2} z_2}{2}\right). \tag{4.113c}$$

We can notice that in the case of both elementary regeneration probabilities $P(A_\parallel \to \chi_{\times 2} + H_{\times 2} \to A_\parallel)$ and $P(A_\perp \to \chi_+ + H_{+2} \to A_\perp)$, the result is given by not just a plain sum of the regeneration probabilities derived earlier for the massive and massless spin-2 field individualy. Instead, it also contains a new interference term. This is caused by having two independent particle species (H and χ) propagating through the wall which are then separately converted back into EM waves upon passing through the second region with magnetic field. These two regenerated EM waves are then superposed (summed) at the level of wavefunctions but not at the level of probabilities. There is indeed some level of analogy with the famous double-slit experiment: the two EM waves which have regenerated from the massive spin-2 and the massless spin-2 wave, respectively, can be identified with the two wavefronts propagating from each slit towards the screen.

Finally, substituting from (4.112c) and (4.113c) into (4.107), the overall regeneration probability will therefore read

$$P(A \to \chi + H \to A) =$$

$$= \frac{|A_\parallel(0)|^2}{|A_\parallel(0)|^2 + |A_\perp(0)|^2}\times$$

$$
\begin{aligned}
&\times \Bigg[16(\Theta^{(H)}_{\times 2})^4 \sin^2\left(\frac{b^{(H)}_{\times 2} z_1}{2}\right)\sin^2\left(\frac{b^{(H)}_{\times 2} z_2}{2}\right)+ \\
&\quad + 16(\Theta^{(\chi)}_{\times 2})^4 \sin^2\left(\frac{b^{(\chi)}_{\times 2} z_1}{2}\right)\sin^2\left(\frac{b^{(\chi)}_{\times 2} z_2}{2}\right)+ \\
&\quad + 32(\Theta^{(H)}_{\times 2})^2(\Theta^{(\chi)}_{\times 2})^2 \cos\left(\frac{(b^{(\chi)}_{\times 2} - b^{(H)}_{\times 2})(z_1+z_2)}{2}\right)\times \\
&\quad\quad \times \sin\left(\frac{b^{(\chi)}_{\times 2} z_1}{2}\right)\sin\left(\frac{b^{(\chi)}_{\times 2} z_2}{2}\right)\sin\left(\frac{b^{(H)}_{\times 2} z_1}{2}\right)\sin\left(\frac{b^{(H)}_{\times 2} z_2}{2}\right)\Bigg]+ \\
&+ \frac{|A_\perp(0)|^2}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\times \\
&\times \Bigg[16(\Theta^{(H)}_{+2})^4 \sin^2\left(\frac{b^{(H)}_{+2} z_1}{2}\right)\sin^2\left(\frac{b^{(H)}_{+2} z_2}{2}\right)+ \\
&\quad + 16(\Theta^{(\chi)}_{+})^4 \sin^2\left(\frac{b^{(\chi)}_{+} z_1}{2}\right)\sin^2\left(\frac{b^{(\chi)}_{+} z_2}{2}\right)+ \\
&\quad + 32(\Theta^{(H)}_{+2})^2(\Theta^{(\chi)}_{+})^2 \cos\left(\frac{(b^{(\chi)}_{+} - b^{(H)}_{+2})(z_1+z_2)}{2}\right)\times \\
&\quad\quad \times \sin\left(\frac{b^{(\chi)}_{+} z_1}{2}\right)\sin\left(\frac{b^{(\chi)}_{+} z_2}{2}\right)\sin\left(\frac{b^{(H)}_{+2} z_1}{2}\right)\sin\left(\frac{b^{(H)}_{+2} z_2}{2}\right)\Bigg]. \qquad (4.114)
\end{aligned}
$$

In the case that $b^{(\chi)}_{+} z_1, b^{(\chi)}_{\times 2} z_1, b^{(H)}_{+2} z_1, b^{(H)}_{\times 2} z_1 \ll 1$ as well as $b^{(\chi)}_{+} z_2, b^{(\chi)}_{\times 2} z_2, b^{(H)}_{+2} z_2,$ $b^{(H)}_{\times 2} z_2 \ll 1$, that is, the sizes of both regions with magnetic field are much less then the oscillation lengths (i.e. the coherent regime) and after substituting for Θs and zs, we finally get

$$
\begin{aligned}
&P(A_\perp \to \chi + H \to A_\perp) \approx \\
&\quad \approx \frac{1}{16} B_{\mathrm{T}}^4 z_1^2 z_2^2 \frac{1}{|A_\parallel(0)|^2+|A_\perp(0)|^2}\times \qquad (4.115) \\
&\quad\quad \times \left[|A_\parallel(0)|^2\Big(g^{(H)^2}+g^{(\chi)^2}\Big)^2 + |A_\perp(0)|^2\Big(g^{(H)^2}+\frac{7}{3}g^{(\chi)^2}\Big)^2\right]. \qquad (4.116)
\end{aligned}
$$

Again, this is not sensitive to the masses of the particles.

Analysis in the opposite mode, when the sizes of the regions with the magnetic field are larger than the oscillation lengths, would be complicated in this case. This is due to the appearance of the interference term, whose averaging would depend on the hierarchy of magnitudes of the b parameters.

Conclusion

In this thesis we have studied the mixing of EM fluctuations with nearly massless particles in an external magnetic field, focusing mainly on the spin-2 case. The first three chapters were devoted to a discussion of the mixing phenomena for the photon-scalar system, photon-massive spin-2 system and photon-massless spin-2 system, respectively. In each particular case, we started from the corresponding lagrangian and linearized around a constant magnetic background in order to obtain the corresponding mixing equations in the ultrarelativistic regime. In the process, we have also made a detailed analysis of the possible refractive indices which one may have to deal with in various environments and which may in general differ for different polarizations, thus giving rise to birefringence. Diagonalizing the mixing equations, we have identified the corresponding mass eigenstates and wrote down general expressions for the evolution of the given states in time and space. For this purpose, we introduced mixing angles that measure the distance of the mass eigenstates to the pure-flavour states. As it is conventional in the literature, we also introduced the formalism of transfer matrices that enabled us to continue working in terms of more compact expressions. While discussing the observational aspects, we have mostly focused on the small mixing scenario as it generally appears to be more relevant for real setups, although we have made a number of comments on the occurrence of the resonant large-mixing regime for certain fine-tuned conditions. The possibly observable effects we have analyzed include the effects the mixing may have on photon polarization (rotation and induced ellipticity), transition probabilities and the light-shining-through-wall experiments.

The simplest case we started with in chapter 1, the scalar, was mainly intended for illustrative purposes. We used it to demonstrate the entire procedure in detail, which then later helped us to somewhat streamline the discussion in the following chapters. In particular, in chapter 1 we have discussed in detail the effects due to the environment in which our particle beam propagates. We have seen that these manifest themselves mainly through effective refractive indices and include the Cotton-Mouton effect, plasma birefringence, as well as the vacuum birefringence which we have derived in detail starting from the Euler-Heisenberg lagrangian. In addition to small mixing, we have also analyzed in quite some detail the opposite case, namely the strong (Mikheyev-Smirnov-Wolfenstein-like) mixing, which occurs whenever the refractive index exactly balances the scalar mass term (for this to be possible, we saw that we need $n < 1$). In general, the mixing equations we have derived taught us that the field ϕ can only oscillate into the perpendicular component of the photon polarization, while the parallel polarization decouples and evolves independently of the rest. In the final section on observational aspects, we have first discussed the relative amplitude decrease and phase delay of the two EM polarizations in a linearly polarized wave, which were induced by the oscillations with the scalar. These were seen to result in the polarization plane rotation, as well as into an induced ellipticity. In writing down our results for these observables, we have distinguished between two possible cases: 1. when the scalar and EM wave remain coherent, namely when the dimensions of the magnetic region are much smaller than the oscillation length, and 2. when, on the contrary, the oscillation length is much smaller than the magnetic region, which causes the oscillations to average out. We derived general expressions for the probability of photon regeneration in the "light-shining-through-wall" experiments, as well as for the relative decrease in the total intensity of a laser beam propagating through the magnetic field in response to the oscillations.

We subsequently repeated the analysis for the case of the massive graviton in chapter 2. Motivated by the following chapters discussing the Einstein-Maxwell theory and the bimetric theory coupled to electromagnetism, we wrote down a cubic interaction lagrangian coupling the EM and massive spin-2 fields in a way that preserves the transversality and tracelessness of the graviton. Having arrived at the mixing equations for the 7 polarizations (five for the massive spin-2 field χ and two for the EM field A) in a background magnetic field, we have observed that not all of these can interact through mixing. In particular, the two polarizations $\chi_{\times 1}$ and χ_{+1} of the massive spin-2 have decoupled, while the remaining three polarizations $\chi_{\times 2}$, χ_{+2} and χ_0 participate in the oscillations with the photon polarizations in such a way so that $A_{\parallel}$ can oscillate into $\chi_{\times 2}$ only, while $A_{\perp}$ can oscillate into both χ_{+2} and χ_0. Moreover, performing a suitable rotation in the χ_{+2}-χ_0 flavour subspace (following the idea of [10]), we have defined new flavours χ_+ and its orthogonal complement χ'_+, such that only χ_+ remains coupled to the $A_{\perp}$ photon, thus ending up with two pairs of flavours, each separately undergoing 2-flavour mixing with coupling related by a factor of $\sqrt{7/3}$. When discussing the observational aspects, we have focused on describing the differences and similarities with the scalar case with the goal of determining whether the photon oscillations mediated by massive spin-0 and massive spin-2 can be told apart by experiment. In particular, we have found that from the point of view of measuring the polarization rotation and induced ellipticity, the spin-0 and spin-2 (after performing a suitable coupling redefinition) can only differ by contributions of negligible magnitude under typical laboratory conditions. On the other hand, in the case of the light shining through wall experiments, we have seen that one is able to fully exploit the fact that massless spin-2 mixes with both EM polarizations to arrange for experimental conditions which can unambiguously tell the difference between spin-2 and spin-0. Finally, we have explicitly observed that no matter how small the mass of the spin-2 particle is taken to be, the expressions we have derived for various observables continue to be affected by the presence of the scalar polarization χ_0 through the aforementioned relative factor of $\sqrt{7/3}$ between the $A_{\parallel}$ and $A_{\perp}$ mixing angles, in line with the spirit of the vDVZ discontinuity.

The treatment of the *exactly* massless spin-2 particle (a.k.a. massless graviton) mixing with the photon in a background magnetic field (the Gertsenshtein-Zeldovich effect) therefore necessitated a separate chapter. Before repeating the by-now-standard procedure leading up to the derivation of the mixing equations, we have first focused on rigorously deriving the correct form of the cubic interaction between two photons and one graviton starting from the Einstein-Hilbert-Maxwell action for GR coupled to electromagnetism. Seeing that due to the presence of both the EM gauge symmetry and the GR gauge (diffeomorphism) symmetry the system will now propagate only four degrees of freedom, we have went find that these pairwise oscillate into one another: $A_{\parallel}$ into $H_{\times 2}$ and $A_{\perp}$ into H_{+2}. Inspecting the obtained formulae for various quantities (such as the mixing angles) we were able to convince ourselves that these do not agree with the $m \to 0$ limit of the results from chapter 2. In discussing the observational aspects of photon oscillations mediated by massless gravitons, we have taken advantage of the (in-principle) numerical knowledge of all parameters entering the problem, as the coupling constant g relates very simply to the Planck mass and the refractive indices may be evaluated given that we specify a particular experimental / observational setup. This enabled us to estimate the magnitude of the mixing angles and oscillation lengths: first in laboratory environment and then also for photons passing in the vicinity of neutron star, as well as for photons propagating through galactic magnetic fields. Interestingly, in both of these two astrophysical configurations, we were able to identify regimes which allowed for not-so-negligible photon-graviton transition probabilities (of the order 10^{-11} for the neutron star and 10^{-12} for the galactic fields). What is more, while we found that the photon energy in the neutron star case had to be finely tuned to a MSW-like resonance, the conversion probability for the galactic fields we were able to maintain for a large range of ω. Going on to study the corresponding double conversion in an LSW experiment, we confirmed that the photon-graviton oscillations would yield signals so weak, that they could not be measured in a present-day laboratory. Applying the double-conversion results to the problem of having observed high-energy photons emitted by GRBs, we have found that the photon-graviton mixing does not appear to provide a conversion channel strong enough to explain these observations. Since this already was the most significant channel within the framework of Standard Model, we have concluded that the situation possibly calls for a beyond-standard-model explanation. Finally, we have also seen that the photon-graviton oscillations would not contribute at the leading order to the rotation of the polarization plane and would only have a very weak effect on the induced ellipticity.

Finally, in the last chapter we have, for the first time, considered the combined mixing of and EM wave, a massive spin-2 wave and a massless spin-2 wave. Working within the self-consistent framework of the bimetric theory of gravity, we have verified that the linearized action for the fluctuations of the aforementioned three fields consists of the exact same interactions as the ones considered in chapters 2 and 3. In the process, we were also able to express all coupling constants in terms of the Einstein's constant (which, in turn, can be expressed using Planck mass) and the parameters of bimetric theory. Identifying the structure of the relevant cubic interactions to be already familiar from the preceding chapters, we were able to immediately write down the corresponding mixing equations for the 9 polarizations involved. Seeing three of these polarizations to decouple straight away, we have recognized that the remaining modes can be organized into two oscillating triplets (each containing one EM polarization, one massive spin-2 and one massless spin-2 polarization). Diagonalizing each of these perturbatively order by order in the couplings, we have identified the corresponding mass eigenstates together with their disperion relations. This allowed us to write down the evolution equations describing oscillations in each triplet. We then went on to describe the parameter space of the bimetric theory, focusing on two regions which have been discussed in the literature from the perspective of providing Dark Matter candidates: the heavy spin-2 DM region and the ULDM region. Comparing the massive spin-2 mixing angles, oscillation lengths an conversion probabilities with the massless ones within each of these two parameter regions, we have found there there appears to be no setting in which the massive conversion probabilities $P(A \to \chi)$ would *not* be suppressed relative to the massless conversion probabilities $P(A \to H)$. In other words, tuning the parameters of the bimetric theory so as to explain use it to explain DM appears to be disqualifying it from simultaneously explaining the high-energy GRB photon observations (recall the massless conversion probabilities were already claimed not to be high enough so as to provide explanation to the GRB photon observations). Finally, we have provided general results for a number of observables with no reference to particular values of the couplng α and FP mass $m^{(\chi)}$, in particular finding new interference terms for the LSW regeneration probabilities, which were not present in the 2-flavour mixing dynamics.

Having seen that the bimetric theory does not seem to able to provide simultaneous explanation to both Dark Matter and high-energy photon observations (at least for the two considered DM paradigms), one has to consider the possibility that either one of these phenomena (or both) are in need for a different explanation. As we have already mentioned, an alternative candidate has been proposed as the mediating particle in the kind of double-conversion process which saves the high-energy photons from having been lost along the way from the GRB to the Earth, namely the axion. In particular, promising results on the photon survival rate have been reported in [3] for the range of axion masses $m_a \simeq (10^{-11} - 10^{-7})\,\mathrm{eV}$ and pseudoscalar coupling $g_a \simeq (3-5) \times 10^{-12}\,\mathrm{GeV}^{-1}$. Notice that this coupling is greater than the gravitational coupling $g^{(H)} = \frac{1}{m_{\mathrm{Pl}}}$ by about seven orders of magnitude. Trying to adopt such a coupling strength for the massive spin-2 field χ in the bimetric theory at these values of the FP mass would not only make it an invalid DM candidate (a possibility which we are in principle happy to allow for) but one would also immediately run into a clash with local gravity tests. In the future, it would be interesting to investigate whether there exists a healthy regime within the bimetric parameter space which would be consistent with the massive and massless spin-2 particles mediating the processes which could explain the high-energy photon observations from GRBs.

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List of Abbreviations

LHAASO	Large High Altitude Air-shower Observatory
CMB	cosmic microwave background
EBL	extragalactic background light
EM	electromagnetic
CP	charge conjugation and parity
vDVZ	van Dam – Veltman – Zakharov
dRGT	de Rham – Gabadadze – Tolley
WIMP	weakly interacting massive particle
Λ CDM	Λ cold dark matter
ULDM	ultra light dark matter
CDM	cold dark matter
QED	quantum electrodynamics
LSW	light-shining through-wall
LHS	left hand side
RHS	right hand side
MSW	Mikheyev–Smirnov–Wolfenstein
ALPS	Any Light Particle Search
DESY	Deutsches Elektronen-Synchrotron
GRB	gamma ray burst
LIGO	Laser Interferometer Gravitational-Wave Observatory
DM	dark matter
GR	general relativity
FP	Fierz – Pauli

List of Figures