FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Anna Kneselová

Maximal non-compactness of operators and embeddings

Department of Mathematical Analysis

Supervisor of the bachelor thesis: prof. RNDr. Luboš Pick, CSc., DSc.
Study programme: General Mathematics
Study branch: Mathematical Analysis

Prague 2024

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank prof. Luboš Pick, the supervisor of this thesis, for providing me with any kind of help and advice I required.

Title: Maximal non-compactness of operators and embeddings

Author: Anna Kneselová

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: We will focus on studying the ball measure of non-compactness $\alpha(T)$ for various particular instances of embedding operators in sequence spaces. Our first main goal is to find necessary and sufficient conditions for an identity operator to be maximally non-compact. Next, we will focus on studying Lorentz sequence spaces $\ell^{p,q}$ and their basic properties. We will characterize the inclusions between Lorentz sequence spaces depending on the values of p and q. Then we will try to determine exact values of the norms of the identity operators between these embedded spaces. Lastly, we will determine whether these identity operators are maximally non-compact by using our general theorems.

Keywords: operator, embedding, maximal noncompactness, sequence spaces, normed linear spaces, rearrangements, Lorentz spaces

Contents

Introduction **2**

1 Notation and preliminaries **3**

2 Maximal non-compactness of embedding operators in sequence spaces **7**

3 Lorentz sequence spaces **13**

3.1 Inclusions of Lorentz sequence spaces 13

3.2 Maximal non-compactness of embedding operators in Lorentz sequence spaces 20

3.3 Examples of embedding operators that are not maximally non-compact 21

Bibliography **23**

Introduction

The compactness of mappings is one of the most important questions that have been studied in contemporary functional analysis. This field of mathematics is interesting not only from the theoretical point of view but, at the same time, has a wide array of applications in various parts of mathematics. However, especially in infinite-dimensional spaces, the mapping in question is often not compact.

The difference between compactness and non-compactness of an operator is however often too rough, and, therefore, it is worthwhile to study its various refinements.

One particular example of such a refinement is offered by studying the so-called measure of non-compactness.
The measure of non-compactness $\alpha(T)$ can attain any value between zero and the operator norm $||T||$ of the mapping $T\colon X \to Y$ in question, where X and Y are (alpha/quasi)-normed linear spaces. In the case when $\alpha(T)$ coincides with the operator norm, the mapping is called maximally non-compact. The concept of measure of non-compactness is a good device for quantifying *how bad* the non-compactness of a mapping is.

We will study this property for various particular instances of embedding (identity) operators in sequence spaces, focusing on Lorentz sequence spaces.

The thesis is divided into three parts. In the first part we introduce the definition of measure of non-compactness $\alpha(T)$, which we get from the article [1]. We also state and prove some of its basic properties, which follow from those of metric spaces and the operator norm, and are mentioned in the article without the proof.

In the second part we focus on generalisation of the example in the introduction of the article [1]. We formulate the prerequisites and state some general theorems which are addressing the question whether an identity operator between two embedded sequence spaces is maximally non-compact.

In the final third part we study the Lorentz sequence spaces $\ell^{p,q}$ and some of their basic properties. We prove that for an arbitrary combination of $p, q \in [1, \infty]$ such that $\min\{p, q\} < \infty$, the space $\ell^{p,q}$ is a subset of c_0 by using some elementary knowledge of series convergence from introductory courses of mathematical analysis. After that, we concentrate on finding the inclusions $\ell_{p_1,q_1} \hookrightarrow \ell_{p_2,q_2}$ between the Lorentz sequence spaces depending on the values of p_1, p_2 and q_1, q_2. We draw some of the inequalities and inclusions from the article [2]. Either we try to detail and complete their proofs, or we try to generalize them more. Then in some of the cases we also determine the exact values of the norms of the identity operators between these spaces.

Lastly we determine whether these identity operators are maximally non-compact by using our general theorems from the second part of the thesis.

1. Notation and preliminaries

We will throughout the text denote by $\mathbb{K}$ the field of scalars, that is, either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$.

Definition 1.1 (Norm, quasinorm and α-norm)**.** *Let X be a linear space over $\mathbb{K}$ and $\| \, . \, \|\colon X \to [0,\infty)$ satisfying:*

1. For all $x \in X\colon \|x\| = 0 \Leftrightarrow x = 0$,

2. For all $x \in X,\ t \in \mathbb{K}\colon \|tx\| = |t|\|x\|$,

3. For all $x, y \in X\colon \|x+y\| \leq \|x\| + \|y\|$.

Then $\|\cdot\|$ is called a norm*. If it satisfies 1., 2. and instead of 3.*

3.' There exists a constant $\alpha \in (0,1]$ such that for all $x, y \in X$:

$$\|x+y\|^\alpha \leq \|x\|^\alpha + \|y\|^\alpha,$$

or

3." There exists a constant $C \in \mathbb{R}$ such that for all $x, y \in X$:

$$\|x+y\| \leq C(\|x\| + \||y\|),$$

*then it is called an α-*norm *or a* quasinorm*, respectively.*

Remark 1.2. Below are some basic relations between norms, quasinorms and α-norms:

1. Every norm is a quasinorm with constant $C = 1$ and also an α-norm with $\alpha = 1$.
2. If $\alpha \leq \beta$ and $\|\cdot\|$ is a β-norm then $\|\cdot\|$ is also a α-norm.
3. Every α-norm is a quasinorm with constant $C = 2^{\frac{1}{\alpha}-1}$ but not every quasinorm is a α-norm.

The proofs can be find in the first chapter of [3, Proposition 1.5; 1.7; 1.8].

Definition 1.3. *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$. For $x \in X$, $A \subset X$ a $\alpha \in \mathbb{K}$ we define following sets:*

$$x + A = \{x + y;\ y \in A\},$$

$$\alpha A = \{\alpha y;\ y \in A\}.$$

Notation 1.4. *Let X be an (alpha/quasi)-normed linear space over $\mathbb{K}$. We will denote the closed unit ball by*

$$B_X := \{x \in X\colon \|x\| \leq 1\}$$

and the closed unit ball centered in $x \in X$ with radius $r > 0$ by

$$B_X(x,r) := \{y \in X\colon \|x-y\| \leq r\}.$$

Remark 1.5. For every $x \in X$ and $r > 0$ the following equality holds:

$$x + rB_X = \{x + ry; y \in B_X\} = B_X(x, r).$$

Definition 1.6 (Compact operator)**.** *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded mapping defined on X and taking values in Y. We say that the operator T is* compact*, if for every bounded sequence $\{x_n\}$ in X the image $\{T(x_n)\}$ has a convergent subsequence in Y.*

Definition 1.7 (Ball measure of non-compactness)**.** *Let X, Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded mapping defined on X and taking values in Y. The* ball measure of non-compactness $\alpha(T)$ *of T is defined as the infimum of radii $r > 0$ for which there exists a finite set of balls in Y of radii r that covers $T(B_X)$. In other words,*

$$\alpha(T) = \inf\{r > 0 : T(B_X) \subset \bigcup_{i=1}^{m} (y_i + rB_Y) \;\; y_i \in Y \;\&\; m \in \mathbb{N}\}.$$

Definition 1.8 (Operator norm)**.** *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded mapping defined on X and taking values in Y. We define the* operator norm $||T||$ *of T as follows:*

$$\|T\| = \sup_{x \in B_X} \|T(x)\|.$$

We shall now present a simple proof of the two-sided estimate for the measure of non-compactness.

Proposition 1.9. *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded linear mapping defined on X and taking values in Y. Then:*

$$0 \leq \alpha(T) \leq \|T\|.$$

Proof. We will prove each of the inequalities separately:

We can see that $\alpha(T) \geq 0$ easily from the definition of the ball measure of non-compactness, because infimum of positive radii r is always greater or equal to zero.

Take $\varrho > \|T\|$ arbitrarily. Then for every $x \in B_X$ one has $\|Tx\| \leq \|T\| < \varrho$, which in turn implies that $Tx \in B_Y(0, \varrho)$. Since x was arbitrary, this implies that

$$T(B_X) \subset \bigcup_{i=1}^{m=1} (0 + \varrho B_Y).$$

The last inclusion is true for all $\varrho > \|T\|$. Consequently, $\alpha(T) \leq \|T\|$. □

Definition 1.10. *Let X be a metric space and $A \subset X$. We say that A is:*

1. relatively compact *in X if the closure of A is compact.*
2. totally bounded *in X if for every real number $\varepsilon > 0$, there exists a finite collection of open balls of radius ε whose centers lie in X and whose union contains A.*

Lemma 1.11. *Let (X, ϱ) be a metric space and $A \subset X$. Then A is relatively compact in X if and only if every sequence $\{x_n\} \subset A$ has a subsequence that is convergent in X.*

Proof. "$\Rightarrow$": $\overline{A}$ is compact from the definition of relative compactness, therefore $\{x_n\} \subset A \subset \overline{A}$ has a convergent subsequence $\{x_{n_k}\}$ with a limit in $\overline{A}$.
"$\Leftarrow$": Let $\{x_n\} \subset \overline{A}$ be a sequence. Then for every $n \in \mathbb{N}$ exists $y_n \in A$ such that $x_n \in B_X(y_n, \frac{1}{n})$. According to our assumption $\{y_n\} \subset A$ has a convergent subsequence $\{y_{n_k}\}$ with a limit $y \in \overline{A}$. Then $\varrho(x_{n_k}, y) \leq \varrho(x_{n_k}, y_{n_k}) + \varrho(y_{n_k}, y) \leq \frac{1}{n_k} + \varrho(y_{n_k}, y) \to 0$ as $k \to \infty$. Hence $x_{n_k} \to y$ and we get that $\overline{A}$ is compact. $\square$

Remark 1.12. Let X be a metric space and $A \subset X$. We recall some basic and known properties of totally bounded and relatively compact sets.

1. A is totally bounded if and only if every sequence in A has a Cauchy subsequence.

2. Relative compactness of A always implies that A is totally bounded.
   This is because a sequence in A has a convergent subsequence with limit in $\overline{A}$ from relative compactness, and a convergent subsequence is a Cauchy subsequence, so A is totally bounded.

3. If X is complete, then A is relatively compact if and only if A is totally bounded.

There is an intimate relation between the situation when the measure of non-compactness of a given operator is zero and its compactness. We shall collect relations between these two notions in the following proposition.

Proposition 1.13. *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded linear mapping defined on X and taking values in Y. Then:*

1. *T is compact if and only if for every bounded $A \subset X$ the set $T(A)$ is relatively compact in Y.*

2. *If T is compact, then $\alpha(T) = 0$.*

3. *If Y is a complete space and $\alpha(T) = 0$, then T is compact.*

Proof. We will gradually prove each statement from the proposition.

1. "$\Leftarrow$": Let $A \subset X$ be bounded and let $\{y_n\}$ be a sequence in $T(A)$. Then for every $n \in \mathbb{N}$ there is $x_n \in A$ such that $y_n = T(x_n)$. Then $\{x_n\}$ is a bounded sequence in A and $\{y_n\} = \{T(y_n)\}$ has a convergent subsequence owing to the compactness of T. It follows that $T(A)$ is relatively compact from Lemma 1.11.

   "$\Rightarrow$": Let $A \subset X$ be bounded and and let $\{x_n\}$ be a sequence in A. Because $\{T(x_n)\} \subset T(A)$, there exists an increasing sequence of indices $\{n_k\}$ such that $\{T(x_{n_k})\}$ converges thanks to the relative compactness of $T(A)$. So T satisfies the definition of a compact operator.

2. Let T be compact. From (i) we have that for every bounded $A \subset X$ the set $T(A)$ is relatively compact in Y therefore $T(B_X)$ is relatively compact. Hence, $T(B_X)$ is totally bounded (as follows from Remark 1.12). Consequently, for every $\varepsilon > 0$, one has $T(B_X) \subset \bigcup_{i=1}^{m}(y_i + \varepsilon B_Y)$, in which $y_i \in Y$ for $i \in \{1, \dots, m\}$, and $m \in \mathbb{N}$. Therefore, $\alpha(T) = 0$.

3. Since $\alpha(T) = 0$, for every $\varepsilon > 0$ we have $T(B_X) \subset \bigcup_{i=1}^{m}(y_i + \varepsilon B_Y)$, where $y_i \in Y$ and $m \in \mathbb{N}$, which implies that $T(B_X)$ is totally bounded. Because Y is complete, this implies that $T(B_X)$ is relatively compact in Y.

   Now let $A \subset X$ be a bounded set and let $\{x_n\} \subset A$ be a sequence from A. Then there exists $r > 0$ such that $\{x_n\} \subset A \subset B_X(0, r)$. Then $\frac{1}{r}x_n \subset B_X$. Now, similarly to the proof of (i), $\{T(\frac{1}{r}x_n)\} \subset T(B_X)$, and there exists an increasing sequence of indices $\{n_k\}$ such that $\{T(\frac{1}{r}x_{n_k})\}$ converges owing to the relative compactness of $T(B_X)$. But then also, by the linearity of T and linearity of limits, we get that $\{T(x_n)\} = \{rT(\frac{1}{r}x_{n_k})\}$ converges. Therefore, for every bounded $A \subset X$ the set $T(A)$ is relatively compact in Y. So, from 1., T is compact.

□

Definition 1.14 (Maximal non-compactness)**.** *Let X and Y be (alpha/quasi)-normed linear spaces over $\mathbb{K}$ and let $T\colon X \to Y$ be a bounded mapping defined on X and taking values in Y. We say that T is* maximally non-compact *if*

$$\alpha(T) = \|T\|.$$

2. Maximal non-compactness of embedding operators in sequence spaces

Definition 2.1 (Sequence space ℓ_p)**.** *Let $1 \leq p < \infty$. Then we define the sequence space:*

$$\ell_p = \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{K}; \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \text{ with norm } \|\{x_n\}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

We also define

$$\ell_\infty = \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{K}; \sup_{n\in\mathbb{N}} |x_n| < +\infty\} \text{ with norm } \|\{x_n\}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Definition 2.2 (Sequence space c_0)**.** *We define*

$$c_0 = \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{K}; \lim_{n\to\infty} x_n = 0\} \text{ with norm } \|\{x_n\}\|_\infty.$$

Definition 2.3. *Let $(\ell, \|\cdot\|_\ell)$ and $(w, \|\cdot\|_w)$ be sequence (alpha/quasi)-normed linear spaces over $\mathbb{K}$. We say that ℓ is a* embedded *into w (we shall denote this as $\ell \hookrightarrow w$) if ℓ is a linear subspace of w and the identity operator is bounded in the sense that there exists a constant $C \geq 0$ such that for every $a \in \ell$ one has $a \in w$ and $\|a\|_w \leq C\|a\|_\ell$.*

Remark 2.4. For $1 \leq p \leq q < \infty$, the following inclusions clearly hold:

$$\ell_p \subset \ell_q \subset c_0 \subset \ell_\infty.$$

Definition 2.5 (Canonical sequence e^j)**.** *For $j \in \mathbb{N}$, we denote*

$$e^j = (e^j_n)_{n=1}^{\infty} = (0, \dots, 0, 1, 0, \dots),$$

where on the j-th position is 1 and 0 on all the others. We call every such sequence a canonical sequence*.*

We shall now present a proposition which nicely illustrates the principle of showing that a given identity operator between two embedded subspaces of c_0 is maximally non-compact. A particular case of this assertion can be found in [1].

Proposition 2.6. *Let ℓ and w be sequence (alpha/quasi)-normed linear spaces over $\mathbb{K}$ satisfying:*

1. *$\ell,\ w \subset c_0$,*
2. *$\ell \hookrightarrow w$,*
3. *the embedding $I\colon \ell \to w$ satisfies $\|I\| \leq 1$, and $e^j \in B_\ell$ for every $j \in \mathbb{N}$.*

Then I is maximally non-compact.

Proof. We will prove the proposition by contradiction. Let us assume that I is not maximally non-compact. Then $\alpha(I) < \|I\|$ and we take $\varrho \in (\alpha(I), \|I\|)$. Then from the definition of the ball measure of non-compactness we find $y^1, \dots, y^m \in w$ and $m \in \mathbb{N}$ such that:

$$I(B_\ell) \subset \bigcup_{i=1}^{m} (y^i + \varrho B_w).$$

Note that $y^i \in w \subset c_0$ for each $i \in \{1, \dots, m\}$. So from the definition of limit there exists $j_0 \in \mathbb{N}$ satisfying: $|(y^i)_j| < \|I\| - \varrho$ for all $j \geq j_0$ and for all $i \in \{1, \dots, m\}$. Now, the sequence $\|I\|e^j = (0, \dots,\ 0,\ \|I\|,\ 0, \dots)$, $j \geq j_0$, satisfies $\|I\|e^j \in I(B_\ell)$, but $\|I\|e^j \notin (y^i + \varrho B_w)$ for any $i \in \{1, \dots, m\}$. Because for every $z \in (y^i + \varrho B_w)$, we have $(z)_j < (\|I\| - \varrho) + \varrho = \|I\|$, so we get our contradiction, hence $\alpha(I) = \|I\|$, and I is maximally non-compact. □

We shall now present an example which illustrates Proposition 2.6.

Example 2.7. Let $1 \leq p < \infty$ and $I \colon \ell_p \to c_0$ be the identity operator. Then I is maximally non-compact.

1. From Remark 2.4 we know that $\ell_p \subset c_0$.

2. We will compute the operator norm of I. For every $x = (x_n)_{n=1}^{\infty} \in \ell_p$ it holds that:

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} = \|x\|_p.$$

   Moreover, $\|I(x)\|_\infty = \|x\|_\infty$, and so we get $\|I\| \leq 1$. We will show that the norm of I achieves the value 1 on the closed unit ball. For arbitrary $j \in \mathbb{N}$, we get $\|e^j\|_p = 1$ (i.e. $e^j \in B_{\ell_p}$), and also $\|I(e^j)\|_\infty = \|e^j\|_\infty = 1$. So we get that $\|I\| = 1$.

3. From computations in the previous step, $\ell_p \hookrightarrow c_0$ and the embedding operator I satisfies the prerequisites of Proposition 2.6, namely $\|I\| \leq 1$ and $e^j \in B_{\ell_p}$ for all $j \in \mathbb{N}$.

So I is maximally non-compact and $\alpha(I) = \|I\| = 1$.

We can notice that Proposition 2.6 limits us to the cases where the norm of the embedding operator is less or equal to 1. Later in this chapter we present a more general theorem addressing the maximal non-compactness in c_0 subspaces. But first we need to introduce some new terminology and definitions.

Notation 2.8. *We denote by* $\mathcal{P}(\mathbb{N})$ *the power set of the set* $\mathbb{N}$ *of all natural numbers. We endow the set* $\mathbb{N}$ *with the usual* σ*-finite counting measure* $m \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$.

Below we define a decreasing rearrangement for sequences. We use the definition from the book [4, Definition 1.1 and Definition 1.5; page 36–39] and reformulate it only for the special case where we set the space $(X, \mathcal{A}, \mu)$ with a σ-finite measure μ equal to $(\mathbb{N}, \mathcal{P}(\mathbb{N}), m)$.

Definition 2.9. *[Distribution function and decreasing rearrangement] Let us consider the measure space* $(\mathbb{N}, \mathcal{P}(\mathbb{N}), m)$*. For a sequence* $\{a\}_{n=1}^{\infty} \subset \mathbb{K}$ *we define its* distribution function $m_a \colon [0,\infty) \to [0,\infty]$ *in the following way:*

$$m_a(\omega) = m\{n \in \mathbb{N} \colon |a_n| > \omega\}$$

and the decreasing rearrangement $a^* \colon \mathbb{N} \to [0,\infty]$ *of the sequence* a *as:*

$$a_n^* = \inf\{\omega > 0 \colon m_a(\omega) \le n\}.$$

Definition 2.10 (Rearrangement-invariant lattice)**.** *Let* $(\ell, \|\cdot\|_\ell)$ *be a (alpha/quasi)-normed sequence space over* $\mathbb{K}$*. We say that* ℓ *is a* rearrangement-invariant lattice *if it satisfies the following axioms:*

1. *Let* $b \in \ell$ *and let* a *be a sequence such that for all* $n \in \mathbb{N} \colon |a_n| \le |b_n|$*, then also* $a \in \ell$ *and* $\|a\|_\ell \le \|b\|_\ell$*.*
2. *For every* $a \in \ell$ *it holds that* $\|a^*\|_\ell = ||a||_\ell$*.*

We can now state our first principal result, namely, a general theorem about the maximal non-compactness in c_0 subspaces.

Theorem 2.11. *Let* ℓ *and* w *be sequence (alpha/quasi)-normed linear spaces over* $\mathbb{K}$ *satisfying:*

1. $\ell \hookrightarrow w \subset c_0$*,*
2. ℓ*,* w *are rearrangement-invariant lattices,*
3. *the embedding operator* $I \colon \ell \to w$ *satisfies* $0 < \|I\| < \infty$*, where* $\|I\| = \sup\limits_{x \in B_\ell} \|x\|_w$*.*

Then the embedding operator I *is maximally non-compact.*

Proof. We will prove the proposition by contradiction. Let us assume that I is not maximally non-compact, then $\alpha(I) < \|I\|$. We take $\varrho \in (\alpha(I), \|I\|)$ and find $\lambda \in (0,1)$ such that:

$$\frac{\varrho}{1-\lambda} < \|I\|. \tag{2.1}$$

From the definition of $\|I\|$ we find $x \in B_\ell$ satisfying:

$$\|x\|_w > \frac{\varrho}{1-\lambda}. \tag{2.2}$$

Then from the definition of the ball measure of non-compactness we find $m \in \mathbb{N}$ and $y^1, \dots, y^m \in w$ such that

$$I(B_\ell) \subset \bigcup_{i=1}^{m} (y^i + \varrho B_w). \tag{2.3}$$

Let us now define a new sequence $\varepsilon = \{\varepsilon_k\}_{k=1}^{\infty}$. For all $k \in \mathbb{N}$ we set

$$\varepsilon_k = \lambda x_k^*. \tag{2.4}$$

Owing to (2.4), $\varepsilon \in c_0$. Then from the positive homogeneity of (alpha/quasi)-norm, our prerequisite that w is a rearrangement-invariant lattice, and inequality (2.2), we get

$$\|x^* - \varepsilon\|_w = \|\{x^*_k - \lambda x^*_k\}\|_w = \|(1-\lambda)x^*\|_w = (1-\lambda)\|x\|_w > \varrho. \tag{2.5}$$

Now $y^i \in w \subset c_0$ for each $i \in \{1, \dots, m\}$. So for every $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$, such that for every $k \geq 2$, one has $j_{k+1} > j_k$, and for all $i \in \{1, \dots, m\}$

$$|(y^i)_{j_k}| \leq \varepsilon_k. \tag{2.6}$$

Let us now define a new sequence:

$$a := \sum_{k=1}^{\infty} x^*_k e^{j_k}. \tag{2.7}$$

Then $a^* = x^*$. Consequently, $a \in B_\ell$, $\|a\|_\ell = \|x\|_\ell$ and $\|a\|_w = \|x\|_w$, since ℓ and w are rearrangement-invariant lattices and $x \in B_\ell$. Now for every $i \in \{1, \dots, m\}$ and every $k \in \mathbb{N}$ from inequality (2.6) and definition of a:

$$a_{j_k} - y^i_{j_k} = x^*_k - y^i_{j_k} \geq x^*_k - \varepsilon_k \geq 0. \tag{2.8}$$

This together with inequality (2.5) yields

$$\begin{aligned}
\|a - y^i\|_w = \|\sum_{k=1}^{\infty} e^{j_k} x^*_k - y^i\|_w &\geq \|\sum_{k=1}^{\infty} e^{j_k}(x^*_k - y^i_{j_k})\|_w \\
&\geq \|\sum_{k=1}^{\infty} e^{j_k}(x^*_k - \varepsilon_k)\|_w \\
&= \|x^* - \varepsilon\|_w \\
&> \varrho.
\end{aligned}$$

We showed that $a \notin (y^i + \varrho B_w)$ for all $i \in \{1, \dots, m\}$. Which gives us a contradiction with our claim (2.3) for any choice of $\varrho \in (\alpha(I), \|I\|)$. Therefore I is maximally non-compact, and $\alpha(I) = \|I\|$. □

We will now state an example of a maximally non-compact embedding, where the target space is not a subspace of c_0, and therefore Theorem 2.11 cannot be used.

Example 2.12. Let I be the embedding operator $I\colon c_0 \to \ell_\infty$. Then I is maximally non-compact.

Proof. First, we compute the operator norm of I. Let $x = \{x_n\}_{n=1}^{\infty} \in c_0$, then $\|I(x)\| = \|x\|_\infty$. So I is a linear isometry from c_0 to ℓ_∞, which implies that $\|I\| = 1$ and it achieves its norm on an arbitrary element $x \in B_{c_0}$ such that $\|x\|_\infty = 1$.

We will proceed with the proof by contradiction. Let us assume that I is not maximally non-compact. Then $\alpha(I) < \|I\| = 1$ and we take $\varrho \in (\alpha(I), 1)$. Then from the definition of the ball measure of non-compactness we find $y^1, \dots, y^m \in \ell_\infty$, where $m \in \mathbb{N}$, such that:

$$B_{c_0} \subset \bigcup_{i=1}^{m} (y^i + \varrho B_{\ell_\infty}).$$

We will define a sequence $a = (a_j)_{j=1}^{\infty}$ in the following way:

$$a_j = \begin{cases} 1, & \text{if } (y^j)_j < 0 \text{ and } j \in \{1, \dots, m\}, \\ -1, & \text{if } (y^j)_j \geq 0 \text{ and } j \in \{1, \dots, m\}, \\ 0, & \text{for } j > m. \end{cases}$$

Trivially $a \in c_0$, because it has only a finite number of non-zero elements. Also $\|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n| = 1$. So $a \in B_{c_0}$.

We will show that $a \notin y^i + \varrho B_{\ell_\infty}$ for any $i \in \{1, \dots, m\}$. Fix $i \in \{1, \dots, m\}$ and $j \in \mathbb{N}$. Then for all $z \in y^i + \varrho B_{\ell_\infty}$ it holds that $z_j \in [(y^i)_j - \varrho,\ (y^i)_j + \varrho]$. We have $\varrho \in (0, 1)$, thus for any $j \in \mathbb{N}$ it cannot happen that $\{1, -1\} \subset [(y^i)_j - \varrho,\ (y^i)_j + \varrho]$, because this interval can contain at most one of the elements of the set $\{1,\ -1\}$. Let us take $j \in \{1, \dots, m\}$ arbitrarily. From the definition of a it follows that the element $a_j \notin [(y^j)_j - \varrho,\ (y^j)_j + \varrho]$. So the whole sequence a cannot belong to the ball $(y^j + \varrho B_{\ell_\infty})$. This holds for every $j \in \{1, \dots, m\}$. And thus we have shown that $a \notin \bigcup_{i=1}^m (y^i + \varrho B_{\ell_\infty})$ and we get a contradiction. □

Our next aim is to find conditions under which an operator is not maximally non-compact. We will need the following new notion.

Definition 2.13 (Sequence space span)**.** *Let ℓ be a (alpha/quasi)-normed sequence space over $\mathbb{K}$. We define the* span σ_ℓ *of ℓ as follows:*

$$\sigma_\ell := \sup_{y \in B_\ell} \left(\sup_{n \in \mathbb{N}} y_n - \inf_{n \in \mathbb{N}} y_n \right). \tag{2.9}$$

We are now in a position to state and prove our next main result. It gives a general sufficient condition for an embedding operator into ℓ^∞ to avoid maximal non-compactness.

Theorem 2.14. *Let ℓ be a sequence (alpha/quasi)-normed linear space over $\mathbb{K}$ satisfying $\ell \hookrightarrow \ell^\infty$ and*

$$\|I\| \leq \sigma_\ell < 2\|I\|. \tag{2.10}$$

Then the embedding operator $I \colon \ell \to \ell^\infty$ is not maximally non-compact. Moreover, $\alpha(I) \leq \sigma_\ell / 2$.

Proof. Denote $\varrho \in (\sigma_\ell/2, \|I\|)$ and consider $m \in \mathbb{N}$ such that

$$\left(1 + \frac{1}{m}\right) \frac{\sigma_\ell}{2} < \varrho. \tag{2.11}$$

Define $\lambda_k = \frac{\sigma_\ell k}{2m}$ for $k = -m, \dots, m$ and let y^k be a constant sequence defined by $(y^k)_j = \lambda_k$ for every $j \in \mathbb{N}$ and $k = -m, \dots, m$. We will show that

$$B_\ell \subset \bigcup_{i=-m}^{m} (y^i + \varrho B_{\ell^\infty}), \tag{2.12}$$

proving $\alpha(I) \leq \varrho < \|I\|$. Assume $y \in B_\ell$. Then $y \in B_{\ell^\infty}(0, \|I\|)$ and $|y_j| \leq \|I\| \leq \sigma_\ell$ for every $j \in \mathbb{N}$. Now from (2.9) it follows that $\sup y - \inf y \leq \sigma_\ell$. We shall now distinguish three cases.

a) If $\inf y = -\sigma_\ell$, then $\sup y \in [-\sigma_\ell, 0]$. Thus, $y_j \in [-\sigma_\ell, 0]$ for each $j \in \mathbb{N}$, and we claim that $y \in y^{-m} + \varrho B_{\ell^\infty}$. Indeed, since $\varrho > \frac{\sigma_\ell}{2}$ and for every $j \in \mathbb{N}$ we have $(y^{-m})_j = \lambda_{-m} = \frac{\sigma_\ell(-m)}{2m} = -\frac{\sigma_\ell}{2}$, the claim follows.

b) If $\inf y \in (-\sigma_\ell, 0]$, then there is a unique $k \in \{-m+1, \dots, m\}$ such that $\inf y + \sigma_\ell/2 \in (\lambda_{k-1}, \lambda_k] = (\frac{\sigma_\ell(k-1)}{2m}, \frac{\sigma_\ell k}{2m}] \subset (-\sigma_\ell/2, \sigma_\ell/2]$. Then by the choice of $\varrho > \sigma_\ell/2$ and inequality (2.10),

$$\lambda_k + \varrho > \lambda_k + \sigma_\ell/2 \geq \inf y + \sigma_\ell \geq \sup y.$$

Here, we get the second inequality from

$$\inf y + \sigma_\ell \in (\lambda_{k-1} + \sigma_\ell/2, \lambda_k + \sigma_\ell/2].$$

On the other hand, using the definition of λ_k and inequality (2.11), we arrive at

$$\inf y > \lambda_{k-1} - \sigma_\ell/2 = \frac{\sigma_\ell k - \sigma_\ell - \sigma_\ell m}{2m} = \lambda_k - \left(1 + \frac{1}{m}\right)\frac{\sigma_\ell}{2} > \lambda_k - \varrho.$$

c) If $\inf y \in (0, \sigma_\ell]$, then $y_j \in [0, \sigma_\ell]$ for each $j \in \mathbb{N}$, hence, evidently, $y \in y^m + \varrho B_{\ell^\infty}$.

Altogether, we showed that $y \in y^k + \varrho B_{\ell^\infty}$, and (2.12) follows. Therefore $\alpha(I) \leq \sigma_\ell/2 < \|I\|$ and I is not maximally non-compact. $\square$

Remark 2.15. We could not use Theorem 2.14 for the embedding $I\colon c_0 \to \ell_\infty$ in Example 2.12, because $\sigma_{c_0} = 2$. It is achieved for example on the sequence $y = (1, -1, 0, 0, \dots) \in B_{c_0}$.

3. Lorentz sequence spaces

Definition 3.1 (Lorentz sequence spaces $\ell^{p,q}$)**.** *Let $p, q \in [1, \infty]$. We define* Lorentz sequence space $\ell^{p,q}$ *as a space of all sequences $a = \{a_n\}_{n=1}^{\infty} \subset \mathbb{K}$, for which the value $\|a\|_{p,q}$ defined below is finite:*

$$\|a\|_{p,q} = \begin{cases} \left(\sum_{n=1}^{\infty} (a_n^*)^q n^{\frac{q}{p}-1}\right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty \text{ and } 1 \leq p \leq \infty, \\ \sup_{n\in\mathbb{N}}\{n^{1/p} a_n^*\}, & \text{if } q = \infty \text{ and } 1 \leq p \leq \infty, \end{cases}$$

where $a^* = \{a_n^*\}_{n=1}^{\infty}$ is the decreasing rearrangement of a.

Remark 3.2. Throughout, we adopt the convention $1/\infty = 0$.

Observation 3.3. *Below are some of the basic properties of Lorentz sequence spaces which follow directly from the definition:*

1. $\|a\|_{p,q} = \left(\sum_{n=1}^{\infty} (a_n^*)^q n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} = \|n^{\frac{1}{p}-\frac{1}{q}} a^*\|_{\ell^q}$,
2. $\ell^{p,p} = \ell^p$.

Remark 3.4. If $1 \leq q \leq p \leq \infty$, then $\|\cdot\|_{p,q}$ is a norm. If $1 \leq p < q \leq \infty$, $\|\cdot\|_{p,q}$ is not a norm but it is a quasi-norm satisfying, for any $a, b \in \ell^{p,q}$,

$$\|a + b\|_{p,q} \leq 2^{1/p}(\|a\|_{p,q} + \|b\|_{p,q}).$$

The proof can be found in [2, p. 76, Proposition 1].

Observation 3.5. *Any Lorentz sequence space $\ell^{p,q}$, $p, q \in [1, \infty]$, trivially satisfies the axioms in Definition 2.10. Therefore, it is a rearrangement-invariant lattice.*

3.1 Inclusions of Lorentz sequence spaces

Lemma 3.6. *Let $p, q \in [1, \infty]$ and $\min\{p, q\} < \infty$, then $\ell^{p,q} \subset c_0$.*

Proof. Let us assume that $\ell^{p,q} \not\subset c_0$ and take $a \in \ell^{p,q} \setminus c_0$. Then there exists $\varepsilon > 0$, such that for all $n_0 \in \mathbb{N}$ there exists $n \geq n_0$, $n \in \mathbb{N}$: $|a_n| \geq \varepsilon$, which implies that $a_n^* \geq \varepsilon$ for $\forall n \in \mathbb{N}$. First, let us look at the case when $q < \infty$ and $p \in [1, \infty]$. Then:

$$\infty > \|a\|_{p,q} \geq \varepsilon \left(\sum_{n=1}^{\infty} (n^{\frac{1}{p}-\frac{1}{q}})^q\right)^{\frac{1}{q}} = \varepsilon \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} = \infty,$$

because for all $p \in [1, \infty]$ one has $(\frac{q}{p} - 1) \geq -1$, and we know that $\sum_{n=1}^{\infty} n^{\alpha}$ diverges for $\alpha \geq -1$. This gives us a contradiction with our assumption that $a \in \ell^{p,q}$. Similarly, for $q = \infty$ and $p \in [1, \infty)$ we get $\infty > \|a\|_{p,\infty} \geq \sup_{n\in\mathbb{N}}\{\varepsilon n^{1/p}\} = \infty$, which is a contradiction once again. $\square$

The following two lemmas occur in the article [2, p. 77-78]. However, the proofs are rather incomplete there. Here we present the complete proofs including full details.

Lemma 3.7. *Let $p \in [1,\infty]$, $q \in [1,\infty)$ and $a = \{a_n\}_{n=1}^{\infty} \in \ell_{p,q}$, then for all $n \in \mathbb{N}$ it holds that*

1. $a_n^* \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} n^{-\frac{1}{p}} \|a\|_{p,q}$, *if* $1 \leq p \leq q < \infty$,

2. $a_n^* \leq n^{-\frac{1}{p}} \|a\|_{p,q}$, *if* $1 \leq q < p \leq \infty$.

Proof. We will prove the individual cases:

1. Let $1 \leq p \leq q < \infty$ and fix $n \in \mathbb{N}$, Then:

$$\begin{aligned}
\|a\|_{p,q}^q &= \sum_{i=1}^{\infty} i^{\frac{q}{p}-1} (a_i^*)^q \\
&\geq \sum_{i=1}^{n} i^{\frac{q}{p}-1} (a_i^*)^q \\
&\geq (a_n^*)^q \sum_{i=1}^{n} i^{\frac{q}{p}-1} \\
&\geq (a_n^*)^q \frac{p}{q} \sum_{i=1}^{n} \left(i^{\frac{q}{p}} - (i-1)^{\frac{q}{p}} \right) \\
&= (a_n^*)^q \left(\frac{p}{q}\right) n^{\frac{q}{p}}.
\end{aligned}$$

Here, the second inequality follows from the fact that (a_i^*) is the decreasing rearrangement. The third inequality results from the Lagrange mean value theorem. Indeed, for all $i \in \mathbb{N}$, there exists $\xi \in (i-1, i)$, that satisfies for the function $f(t) = \frac{p}{q} t^{\frac{q}{p}}$, where $t \geq 0$ following equality:

$$f'(\xi) = \xi^{\frac{q}{p}-1} = \frac{p}{q} \left[i^{\frac{q}{p}} - (i-1)^{\frac{q}{p}} \right].$$

Also $i^{\frac{q}{p}-1} \geq \xi^{\frac{q}{p}-1}$, because $\frac{q}{p} \geq 1$ and $i > \xi > 1$. From the estimates above we get the third inequality and 1. is proven.

2. Analogously to the computations in 1. we get the following inequality:

$$\begin{aligned}
\|a\|_{p,q}^q &\geq (a_n^*)^q \sum_{i=1}^{n} i^{\frac{q}{p}-1} \\
&\geq (a_n^*)^q n n^{\frac{q}{p}-1} \\
&= (a_n^*)^q n^{\frac{q}{p}}.
\end{aligned}$$

Here the second inequality follows from the fact that for all $i \leq n$ it holds that $\frac{q}{p} \in [0,1)$. Therefore $\left(\frac{q}{p} - 1\right) \in [-1, 0)$, so we get $n^{\left(\frac{q}{p}-1\right)} \leq i^{\left(\frac{q}{p}-1\right)}$, and 2. is proven.

□

Lemma 3.8. *Let* $1 \le p \le \infty$, $1 \le q_1 < q_2 \le \infty$, *Then:*

$$\ell_{p,q_1} \hookrightarrow \ell_{p,q_2},$$

and for all $a \in \ell_{p,q_1}$ *it holds that*

1. $\|a\|_{p,q_2} \le \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|a\|_{p,q_1}$*; if* $p < q_1$,

2. $\|a\|_{p,q_2} \le \|a\|_{p,q_1}$*; if* $p \ge q_1$.

Proof. We will treat each case separately

1. (a) Let $q_2 < \infty$. From the first part of Lemma 3.7 we get:

$$\begin{aligned}
\|a\|_{p,q_2}^{q_2} &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p}-1} (a_n^*)^{q_2-q_1} (a_n^*)^{q_1} \\
&\le \sum_{n=1}^{\infty} n^{\frac{q_2}{p}-1} \left(\frac{q_1}{p}\right)^{\frac{q_2-q_1}{q_1}} n^{-\frac{q_2-q_1}{p}} \|a\|_{p,q_1}^{q_2-q_1} (a_n^*)^{q_1} \\
&= \left(\frac{q_1}{p}\right)^{\frac{q_2}{q_1}-1} \|a\|_{p,q_1}^{q_2-q_1} \sum_{n=1}^{\infty} n^{\frac{q_1}{p}-1} (a_n^*)^{q_1} \\
&= \left(\frac{q_1}{p}\right)^{\frac{q_2}{q_1}-1} \|a\|_{p,q_1}^{q_2-q_1} \|a\|_{p,q_1}^{q_1} \\
&= \left(\frac{q_1}{p}\right)^{\frac{q_2}{q_1}-1} \|a\|_{p,q_1}^{q_2}.
\end{aligned}$$

(b) Now, let $q_2 = \infty$:

$$\begin{aligned}
\|a\|_{p,q_2} &= \sup_{n\in\mathbb{N}} n^{\frac{1}{p}} a_n^* \\
&\le \sup_{n\in\mathbb{N}} n^{\frac{1}{p}} \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}} n^{-\frac{1}{p}} \|a\|_{p,q_1} \\
&= \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}} \|a\|_{p,q_1}.
\end{aligned}$$

Here the first inequality again follows from Lemma 3.7.

So 1. is proven.

2. Now let $p \ge q_1$. Then, the following inequality follows from the second part of Lemma 3.7:

$$\begin{aligned}
\|a\|_{p,q_2}^{q_2} &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p}-1} (a_n^*)^{q_2-q_1} (a_n^*)^{q_1} \\
&\le \sum_{n=1}^{\infty} n^{\frac{q_2}{p}-1} n^{-\frac{q_2-q_1}{p}} \|a\|_{p,q_1}^{q_2-q_1} (a_n^*)^{q_1} \\
&= \|a\|_{p,q_1}^{q_2-q_1} \sum_{n=1}^{\infty} n^{\frac{q_1}{p}-1} (a_n^*)^{q_1} \\
&= \|a\|_{p,q_1}^{q_2-q_1} \|a\|_{p,q_1}^{q_1} \\
&= \|a\|_{p,q_1}^{q_2}.
\end{aligned}$$

□

We will now concentrate on establishing the embeddings $\ell_{p_1,q_1} \hookrightarrow \ell_{p_2,q_2}$ between the Lorentz sequence spaces in dependence on the values of p_1, p_2 and q_1, q_2. Then we will try to calculate the exact value of the operator norms of these embeddings.

Lemma 3.9. *Let $1 \le p_1 < p_2 \le \infty$ and $q_1, q_2 \in [1, \infty]$. Then*

$$\ell_{p_1,q_1} \hookrightarrow \ell_{p_2,q_2},$$

and for all $a \in \ell_{p_1,q_1}$ holds:

1. $\|a\|_{p_2,q_2} \le \|a\|_{p_1,q_1}$, *if $q_1 \le p_1$,*
2. $\|a\|_{p_2,q_2} \le \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1}\right)^{1/q_2} \|a\|_{p_1,\infty}$, *if $p_1 < q_1 = \infty$ and $q_2 < \infty$,*
3. $\|a\|_{p_2,q_2} \le \left(\frac{q_1}{p_1}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|a\|_{p_1,q_1}$, *if $p_1 < q_1 < q_2 < \infty$,*
4. $\|a\|_{p_2,q_2} \le \|a\|_{p_1,q_1}$, *if $p_1 < q_1 < \infty$ and $q_1 \ge q_2$.*
5. $\|a\|_{p_2,q_2} \le \|a\|_{p_1,q_1}$, *if $q_1 = q_2 = \infty$.*

Proof. Let $1 \le p_1 < p_2 \le \infty$. We will prove the individual cases.

1. Let $q_1 \le p_1 < \infty$.

   a) Let $q_2 < \infty$. Then

$$\begin{aligned}
\|a\|_{p_2,q_2}^{q_2} &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1}(a_n^*)^{q_2} \\
&= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1}(a_n^*)^{q_2-q_1}(a_n^*)^{q_1} \\
&\le \|a\|_{p_1,q_1}^{q_2-q_1} \sum_{n=1}^{\infty} n^{-\frac{q_2-q_1}{p_1}}(a_n^*)^{q_1} n^{\frac{q_2}{p_2}-1} \\
&= \|a\|_{p_1,q_1}^{q_2-q_1} \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}}(a_n^*)^{q_1} n^{\frac{q_1}{p_1}-1} \\
&\le \|a\|_{p_1,q_1}^{q_2-q_1} \sum_{n=1}^{\infty} (a_n^*)^{q_1} n^{\frac{q_1}{p_1}-1} \\
&\le \|a\|_{p_1,q_1}^{q_2}.
\end{aligned}$$

   Here the first inequality follows from the second part of Lemma 3.7

$$a_n^* \le n^{-\frac{1}{p_1}} \|a\|_{p_1,q_1},$$

   and we get the second inequality from the fact that

$$n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}} < 1.$$

b) Now let $q_2 = \infty$ then:

$$\begin{aligned}\|a\|_{p_2,\infty} &= \sup_{n\in\mathbb{N}} a_n^* n^{\frac{1}{p_2}} \\ &\leq \sup_{n\in\mathbb{N}} n^{\frac{1}{p_2}} n^{-\frac{1}{p_1}} \|a\|_{p_1,q_1} \\ &= \|a\|_{p_1,q_1} \sup_{n\in\mathbb{N}} n^{\frac{1}{p_2}} n^{-\frac{1}{p_1}} \\ &\leq \|a\|_{p_1,q_1}.\end{aligned}$$

The first inequality follows from the second part of Lemma 3.7

$$a_n^* \leq n^{-\frac{1}{p_1}} \|a\|_{p_1,q_1}.$$

And we get the second inequality from the fact that

$$n^{\frac{1}{p_2}-\frac{1}{p_1}} \leq 1.$$

2. Let $p_1 < q_1 = \infty$ and $q_2 < \infty$, then:

$$\begin{aligned}\|a\|_{p_2,q_2}^{q_2} &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1} (a_n^*)^{q_2} \\ &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1} (a_n^*)^{q_2} n^{\frac{q_2}{p_1}} \\ &\leq \left(\sup_{n\in\mathbb{N}} a_n^* n^{\frac{1}{p_1}}\right)^{q_2} \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1} \\ &= \|a\|_{p_1,\infty}^{q_2} \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1},\end{aligned}$$

in which the sum $\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1}$ converges, because $\frac{q_2}{p_2} - \frac{q_2}{p_1} - 1 < -1$.

3. Let $p_1 < q_1 < \infty$ and $q_2 < \infty$, then:

$$\begin{aligned}\|a\|_{p_2,q_2}^{q_2} &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1} (a_n^*)^{q_2} \\ &= \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1} (a_n^*)^{q_2-q_1} (a_n^*)^{q_1} \\ &\leq \|a\|_{p_1,q_1}^{q_2-q_1} \left(\frac{q_1}{p_1}\right)^{\frac{q_2-q_1}{q_1}} \sum_{n=1}^{\infty} n^{-\frac{q_2-q_1}{p_1}} (a_n^*)^{q_1} n^{\frac{q_2}{p_2}-1} \\ &= \|a\|_{p_1,q_1}^{q_2-q_1} \left(\frac{q_1}{p_1}\right)^{\frac{q_2}{q_1}-1} \sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}} (a_n^*)^{q_1} n^{\frac{q_1}{p_1}-1} \\ &\leq \left(\frac{q_1}{p_1}\right)^{\frac{q_2}{q_1}-1} \|a\|_{p_1,q_1}^{q_2}.\end{aligned}$$

Here the first inequality follows from the first part of Lemma 3.7

$$a_n^* \leq \left(\frac{q_1}{p_1}\right)^{\frac{1}{q_1}} n^{-\frac{1}{p_1}} \|a\|_{p_1,q_1}$$

and we get the second inequality from the fact that

$$n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}} \leq 1.$$

4. Let $p_1 < q_1 < \infty$ and $q_1 \geq q_2$, then from the computations in the 3. part of the proof we have:

$$\|a\|_{p_2,q_2} \leq \left(\frac{q_1}{p_1}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \|a\|_{p_1,q_1},$$

where $\frac{q_1}{p_1} > 1$ because $p_1 < q_1 < \infty$. But $\frac{1}{q_1} - \frac{1}{q_2} \leq 0$ because $q_1 \geq q_2$. So $\left(\frac{q_1}{p_1}\right)^{\frac{1}{q_1}-\frac{1}{q_2}} \leq 1$ and $\|a\|_{p_2,q_2} \leq \|a\|_{p_1,q_1}$.

5. Let $q_1 = q_2 = \infty$, then:

$$\begin{aligned}\|a\|_{p_2,\infty} &= \sup_{n\in\mathbb{N}} a_n^* n^{\frac{1}{p_2}} \\ &\leq \sup_{n\in\mathbb{N}} n^{\frac{1}{p_1}} a_n^* \\ &= \|a\|_{p_1,q_1},\end{aligned}$$

where the inequality follows from $p_1 < p_2$, hence $n^{\frac{1}{p_1}} \geq n^{\frac{1}{p_2}}$.

□

Lemma 3.10. *Let $1 \leq p_1 < p_2 \leq \infty$ and $q_1 \in [1,\infty]$ and $q_2 \in [1,\infty]$, then the norm of the embedding operator $I\colon \ell_{p_1,q_1} \to \ell_{p_2,q_2}$ is equal to:*

1. $\|I\| = 1$, if $q_1 \leq p_1$,

2. $\|I\| = 1$, if $p_1 < q_1 < \infty$ and $q_1 \geq q_2$,

3. $\|I\| = 1$, if $q_1 = q_2 = \infty$,

4. $\|I\| = \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1}\right)^{1/q_2}$, if $p_1 < q_1 = \infty$ and $q_2 < \infty$.

Proof. We will prove the individual cases separately:

1. Let $q_1 \leq p_1$. We get the upper estimate for the operator norm from the first part of Lemma 3.9. So $\|I\| \leq 1$. Now we will show that I attains 1 on the closed unit ball. For arbitrary e^j we have that $(e^j)^* = e^1$ and:

$$\|e^j\|_{p_1,q_1} = \left(\sum_{n=1}^{\infty} n^{\frac{q_1}{p_1}-1}((e_n^j)^*)^{q_1}\right)^{\frac{1}{q_1}} = \left(1^{\frac{q_1}{p_1}-1} 1^{q_1}\right)^{\frac{1}{q_1}} = 1.$$

So $e^j \in B_{\ell_{p_1,q_1}}$. Now we need to show that also $\|I(e^j)\|_{p_2,q_2} = \|e^j\|_{p_2,q_2} = 1$.

a) Let $q_2 < \infty$. Then $\|e^j\|_{p_2,q_2} = 1$ follows from the computations above.

b) Let $q_2 = \infty$. Then $\|e^j\|_{p_2,q_2} = \sup_{n\in\mathbb{N}}((e_n^j)^*)n^{\frac{1}{p_2}} = 1$

It already follows that $\|I\| = 1$.

2. Let $p_1 < q_1 < \infty$ and $q_1 \geq q_2$. We get the upper estimate for the operator norm from the fourth part of Lemma 3.9. So $\|I\| \leq 1$. We can show that I attains 1 on the closed unit ball at an arbitrary e^j. The computations are same as in the proof of 1. So again $\|I\| = 1$.

3. Let $q_1 = q_2 = \infty$. We get the upper estimate for the operator norm from the fifth part of Lemma 3.9. So $\|I\| \leq 1$. We can show that I attains 1 on the closed unit ball at an arbitrary e^j. The computations are similar as in the proof of 1.
$$\|e^j\|_{p_1,q_1} = \sup_{n\in\mathbb{N}}((e^j_n)^*)n^{\frac{1}{p_1}} = 1 = \|e^j\|_{p_2,q_2}$$
So again $\|I\| = 1$.

4. Let $p_1 < q_1 = \infty$ and $q_2 < \infty$. Once again we get the upper estimate for the operator norm from the second part of Lemma 3.9.
Therefore $\|I\| \leq \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1}\right)^{1/q_2}$. Now we will again show that I achieves this upper estimate on the closed unit ball. Let us consider the sequence $a = a^* = (1, 2^{-\frac{1}{p_1}}, 3^{-\frac{1}{p_1}}, \ldots, n^{-\frac{1}{p_1}}, \ldots)$, then
$$\|a\|_{p_1,\infty} = \sup_{n\in\mathbb{N}} a^*_n n^{\frac{1}{p_1}} = \sup_{n\in\mathbb{N}} n^{-\frac{1}{p_1}} n^{\frac{1}{p_1}} = 1.$$
Hence $a \in B_{\ell_{p_1,\infty}}$ and we will now compute the norm of $\|I(a)\|_{p_2,q_2} = ||a||_{p_2,q_2}$.
$$\begin{aligned}\|a\|_{p_2,q_2} &= \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1}(a^*_n)^{q_2}\right)^{1/q_2}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-1}(n^{-\frac{1}{p_1}})^{q_2}\right)^{1/q_2}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q_2}{p_2}-\frac{q_2}{p_1}-1}\right)^{1/q_2}.\end{aligned}$$
So indeed I achieves it's upper estimate on the closed unit ball and 4. is proven.

□

Lemma 3.11. *Let $1 \leq p < q < \infty$, then the norm of the embedding operator $I\colon \ell_{p,q} \to \ell_{p,\infty}$ is:*
$$\|I\| = \left(\frac{q}{p}\right)^{\frac{1}{q}}.$$

Proof. $\|I\| \leq \left(\frac{q}{p}\right)^{\frac{1}{q}}$ from the first part of Lemma 3.8.
We want to show that there exists a sequence $\{a^n\}_{n=1}^{\infty}$ of nonzero elements, where $a^n \in \ell_{p,q}$ for all $n \in \mathbb{N}$, such that:
$$\lim_{n\to\infty} \frac{\|a^n\|_{p,\infty}}{\|a^n\|_{p,q}} = \left(\frac{q}{p}\right)^{\frac{1}{q_1}}. \tag{3.1}$$

This would already imply that $\|I\| = \left(\frac{q}{p}\right)^{\frac{1}{q_1}}$, since we would be able to get arbitrarily close to our upper estimate $\left(\frac{q}{p}\right)^{\frac{1}{q}}$ of the operator norm $\|I\|$.

For every $n \in \mathbb{N}$ set $a^n = (1, 1, 1, \dots, 1, 0, 0 \dots)$, where $a_i^n = 1$ for $i \in \{1, 2, \dots, n\}$ and $a_i^n = 0$ for $i > n$. Then for every $n \in \mathbb{N}$:

$$\|a^n\|_{p,\infty} = \sup_{i \le n} i^{\frac{1}{p}} = n^{\frac{1}{p}},$$

while

$$\|a^n\|_{p,q} = \left(\sum_{i=1}^{n} i^{\frac{q}{p}-1}\right)^{\frac{1}{q}}.$$

Now, from the theory of the Riemann integral, we know

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|a^n\|_{p,\infty}}{\|a^n\|_{p,q}} &= \lim_{n\to\infty} \frac{n^{\frac{1}{p}}}{\left(\sum_{i=1}^{n} i^{\frac{q}{p}-1}\right)^{\frac{1}{q}}} \\
&= \frac{1}{\left(\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{\frac{q}{p}-1} \frac{1}{n}\right)^{\frac{1}{q}}} \\
&= \left(\int_0^1 x^{\frac{q}{p}-1} dx\right)^{-\frac{1}{q}} \\
&= \left(\frac{q}{p}\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence $\{a^n\}_{n=1}^{\infty}$ satisfies (3.1) and we proved $\|I\| = \left(\frac{q}{p}\right)^{\frac{1}{q}}$. □

3.2 Maximal non-compactness of embedding operators in Lorentz sequence spaces

We shall now present examples of embeddings between Lorentz sequence spaces which illustrate Theorem 2.11.

Example 3.12. Let $p, q \in [1, \infty]$ and $\min\{p, q\} < \infty$. Then the embedding operator $I\colon \ell^{p,q} \to c_0$ is maximally non-compact.

1. $\ell^{p,q} \subset c_0$ from Lemma 3.6.

2. We find the norm of operator I. Fix $a = \{a_n\}_{n=1}^{\infty} \in \ell_{p,q}$.

   a) Let $q = \infty$. Then $\|I(a)\| = \|a\|_\infty = \sup_{n\in\mathbb{N}} |a_n| \le \sup_{n\in\mathbb{N}}\{n^{1/p} a_n^*\} = \|a\|_{p,q}$, where the inequality results from the fact that $n^{1/p} \ge 1$ for all $n \in \mathbb{N}$ and the Definition 2.9 of the decreasing rearrangement of a.

   b) Let $q < \infty$. Then $\|I(a)\| = \|a\|_\infty = \sup_{n\in\mathbb{N}} |a_n| \le \left(\sum_{n=1}^{\infty} (a_n^*)^q n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} = \|a\|_{p,q}$, where the inequality results from the Definition 2.9 of the decreasing rearrangement of a and because $\sup_{n\in\mathbb{N}} |a_n| = a_1^* = 1^{\frac{q}{p}-1}(a_1^*)^q$.

Thus $\|I\| \leq 1$ for any $p \in [1, \infty)$, $q \in [1, \infty]$. Now we will show that I achieves 1 on the closed unit ball. For an arbitrary e^j we have that $(e^j)^* = e^1$, therefore $\|e^j\|_{p,q} = \|e^1\|_{p,q} = 1$, thus $e^j \in B_{\ell_{p,q}}$ and also $\|I(e^j)\|_\infty = \|e^j\|_\infty = 1$, hence $\|I\|| = 1$.

So from the Theorem 2.11 the embedding operator I is maximally non-compact and $\alpha(I) = \|I\| = 1$.

Example 3.13. Let $1 \leq p_1 < p_2 \leq \infty$, $q_1, q_2 \in [1, \infty]$, $\min\{p_2, q_2\} < \infty$ and ℓ^{p_1,q_1}, ℓ^{p_2,q_2} Lorentz sequence spaces. Then the embedding operator $I \colon \ell^{p_1,q_1} \to \ell^{p_2,q_2}$ is maximally non-compact. We have

1. $\ell^{p_1,q_1} \hookrightarrow \ell^{p_2,q_2}$ and $0 < \|I\| < \infty$ from Lemma 3.9,
2. ℓ^{p_1,q_1}, $\ell^{p_2,q_2} \subset c_0$ from Lemma 3.6.

So the prerequisites from Theorem 2.11 are satisfied, the embedding operator I is maximally non-compact and $\alpha(I) = \|I\|$, where the values of $\|I\|$ for some individual relations between p_1, q_1 and q_2 are computed in Lemma 3.10 or Lemma 3.11.

3.3 Examples of embedding operators that are not maximally non-compact

We will now present examples of embeddings of Lorentz sequence spaces into ℓ^∞ which are not maximally non-compact. This will be shown by finding their span $\sigma_{\ell^{p,q}}$ and applying Theorem 2.14.

Example 3.14. Let $p \in [1, \infty)$, $q \in [1, \infty]$ and $p \leq q$. Then the embedding operator $I \colon \ell^{p,q} \to \ell^\infty$, $\|I\| = 1$ is not maximally non-compact. More precisely:

1. $\alpha(I) \leq 2^{-1/q} < 1$ when $q < \infty$,
2. $\alpha(I) \leq 2^{-1}(1 + 2^{-1/p}) < 1$ when $q = \infty$.

Proof. We have $\ell^{p,q} \subset c_0 \subset \ell^\infty$ from Lemma 3.6 and $||I|| = 1$ follows from the computations in Example 3.12

1. First we will have a look at the case when $q < \infty$. Denote $\sigma = 2^{1-1/q}$. Then $\|I\| \leq \sigma < 2\|I\|$. Assume $y \in B_{\ell^{p,q}}$, then $y \in B_{\ell^\infty}$ since $\|I\| = 1$, hence $|y_j| \leq 1 \leq \sigma$. We claim that

$$\sup y - \inf y \leq \sigma. \tag{3.2}$$

Indeed, given $\varepsilon > 0$, we find from the definition of supremum and infimum

$s, i \in \mathbb{N}$ such that $y_s > \sup y - \varepsilon$ and $y_i < \inf y + \varepsilon$. We get

$$\begin{aligned}
1 &\geq \|y\|_{p,q} \\
&= \left(\sum_{n=1}^{\infty} (y_n^*)^q n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} \\
&\geq \left(\sum_{n=1}^{\infty} (y_n^*)^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} (y_n)^q\right)^{\frac{1}{q}} \\
&\geq (|y_s|^q + |y_i|^q)^{1/q} \\
&\geq 2^{1/q-1}(|y_s| + |y_i|) \\
&> \frac{1}{\sigma}(\sup y - \inf y - 2\varepsilon),
\end{aligned}$$

where the second inequality follows from our prerequisite $p \leq q$ thus $n^{\frac{q}{p}-1} \geq 1$ and the fourth follows from the binomial expansion and the fact that

$$0 \leq 2^{1/q-1} \leq 1.$$

Now when sending $\varepsilon \to 0_+$ we get our claim. Therefore

$$\sigma_{\ell_{p,q}} \leq \sigma < 2\|I\|$$

and all the prerequisites of Theorem 2.14 are satisfied, so I is not maximally non-compact and $\alpha(I) \leq \sigma/2 < 1 = \|I\|$.

2. Now let $q = \infty$. We will proceed with the proof similarly to the way we did in the first part. The only difference is that we set $\sigma = 1 + 2^{-1/p}$. Assume $y \in B_{\ell^{p,\infty}}$. We again need to verify that the inequality

$$\sup y - \inf y \leq \sigma. \tag{3.3}$$

Let $y \in B_{\ell^{p,\infty}}$, then

$$||y||_{p,\infty} = \sup_{n\in\mathbb{N}}\{n^{1/p} y_n^*\} \leq 1, \text{ hence } \sup_{n\in\mathbb{N}}\{y_n\} \leq 1. \tag{3.4}$$

Inequality (3.4) implies that the decreasing rearrangement y^* of y must satisfy $y_n^* \leq n^{-1/p}$ for all $n \in \mathbb{N}$. So every sequence from $B_{\ell^{p,\infty}}$ has its decreasing rearrangement bounded from above by the decreasing sequence $\{n^{-1/p}\}_{n=1}^{\infty}$ and from below by $\{-n^{-1/p}\}_{n=1}^{\infty}$. We shall now distinguish two cases.

a) If $\sup y \leq 2^{-1/p}$, then from the observation above $\inf y \geq -1$ and inequality (3.3) holds.

b) If $\sup y > 2^{-1/p}$, then from the observations above y_n^* is bounded by the decreasing sequence $n^{-1/p}$, so there can be only one index $j \in \mathbb{N}$ such that $y_j > 2^{-1/p}$, therefore $y_1^* = y_j$ and we get that for all $i \neq j$: $|y_i| \leq 2^{-1/p}$, which implies that $\inf y \geq -2^{-1/p}$. Thus $\sup y - \inf y \leq 1 - (-2^{-1/p}) = \sigma$ holds for all $y \in B_{\ell^{p,q}}$, and inequality (3.3) follows.

We computed that $\sigma_{\ell_{p,q}} = \sigma = 1+2^{-1/p} < 2\|I\|$. Now again from Theorem 2.14 we get that I is not maximally non-compact and $\alpha(I) \leq \varrho = 2^{-1}(1+2^{-1/p}) < 1$.

□

Bibliography

[1] Jan Lang, Vít Musil, Miroslav Olšák, and Luboš Pick. Maximal non-compactness of Sobolev embeddings. *J. Geom. Anal.*, 31(9):9406–9431, 2021.

[2] Mikio Kato. On Lorentz spaces $l_{p,q}\{E\}$. *Hiroshima Math. J.*, 6(1):73–93, 1976.

[3] Kristýna Kuncová. Lorentz spaces, 2009. Available at http://hdl.handle.net/20.500.11956/27813.

[4] C. Bennett and R. Sharpley. *Interpolation of operators*. Academic Press Professional, Inc., USA, 1987.