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Stability of stationary flows of non-Newtonian heat conducting fluid in 2D

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Abstract: This thesis aims to study the Navier-Stokes-Fourier problem with the entropy equation. In particular, we want to define the notion of a solution and prove its existence. We approach this problem by modifying techniques used in several papers studying the generalized NSF system and the entropy equality and we want to conclude similar results. We are treating the two-dimensional case as opposed to the more frequent 3D case, hence we were able to relax conditions on the initial data. Firstly, we formulate the definition of a weak solution and impose sufficient conditions to prove its existence. In particular, we will require a bound $p \geq 2$ for the power-law index of the Cauchy stress tensor. Next, we show that there exists a solution to Navier-Stokes-Fourier system $(\boldsymbol{u}, \vartheta)$ fulfilling our definition. Lastly, we show that this solution additionally fulfills the entropy equality for $\eta = \log \vartheta$.

Keywords: Navier, Stokes, Fourier, entropy equality, renormalized solutions

Contents

Introduction

We want to present a mathematical analysis of a model describing mechanical and thermal changes shown by unsteady flows of incompressible non-Newtonian fluids in fixed bounded two-dimensional domains. The incompressibility of the fluid is exhibited by equation (2), the mechanical and thermal changes respect the balance of linear momentum (1) and the balance of energy (3). We treat the case where the viscosity of the fluid (i.e., its resistance to deformation) and the heat conductivity (i.e., the ability of a fluid to conduct heat) depend on the temperature. As we consider that the fluid is non-Newtonian, we expect the viscosity to change disproportionally to the velocity of the fluid. Thus, in our model, the viscous part of the Cauchy stress tensor $\boldsymbol{\mathcal{S}}$ depends non-linearly on the velocity gradient.

Additionally, we introduce the entropy equality (4), which on a formal level corresponds to the balance of energy divided by temperature. It is a desirable equality for proving the stability of a solution since its terms possess better regularity properties.

As regards the boundary conditions, we consider the homogeneous Dirichlet boundary condition for the velocity, so we expect no fluid exchange with the exterior and also the velocity of the fluid to slow down to zero near the boundary. We expect a thermal interaction with the exterior, which is described by a non-homogeneous Dirichlet boundary condition. This condition is time-independent, so we suppose that the heating (or cooling) of the system is constant in time.

Let us now formulate the problem more rigorously. We study the generalized Navier–Stokes–Fourier system with the entropy equation

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{\mathcal{S}}) + \nabla p = \boldsymbol{f} \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{2}$$

$$\partial_t \vartheta + \operatorname{div}(\vartheta \boldsymbol{u}) - \operatorname{div}(\boldsymbol{q}) = \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \tag{3}$$

$$\partial_t \eta + \operatorname{div}(\eta \boldsymbol{u}) - \operatorname{div}(\frac{\boldsymbol{q}}{\vartheta}) = \frac{\boldsymbol{\mathcal{S}} : D\boldsymbol{u}}{\vartheta} - \frac{\boldsymbol{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{4}$$

in $Q := (0,T) \times \Omega$ with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $T > 0$. Here $\boldsymbol{u} : Q \to \mathbb{R}^2$ denotes the velocity field, $D\boldsymbol{u} := (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ is the symmetric part of the velocity gradient $\nabla \boldsymbol{u}$, $p : Q \to \mathbb{R}$ is the pressure, $\vartheta : Q \to \mathbb{R}$ is the temperature, $\eta = \log \vartheta$ is the entropy, $\boldsymbol{f} : Q \to \mathbb{R}^2$ denotes the external body forces, $\boldsymbol{\mathcal{S}} : Q \to \mathbb{R}^{2\times 2}$ denotes the viscous part of the Cauchy stress tensor, and $\boldsymbol{q} : Q \to \mathbb{R}^2$ is the heat flux. The system (1)–(4) is completed by the initial and boundary conditions

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega \times (0,T) \\ \vartheta &= \vartheta_b && \text{on } \partial\Omega \times (0,T) \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega \\ \vartheta(0) &= \vartheta_0 && \text{in } \Omega. \end{aligned} \tag{5}$$

For given $\boldsymbol{f} : Q \to \mathbb{R}$, $\vartheta_b : \partial\Omega \to \mathbb{R}$, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^2$ and $\vartheta_0 : \Omega \to \mathbb{R}$ we search for $\boldsymbol{u}$, ϑ and p solving the system (1)–(5).

We will assume, that the heat flux is represented by the Fourier law. That is

$\boldsymbol{q} = \boldsymbol{q}^*(\vartheta)$ and for all $\vartheta \in \mathbb{R}$ it holds

$$\boldsymbol{q}^*(\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{6}$$

where the heat conductivity $\kappa : \mathbb{R} \to (0, \infty)$ is a continuous function, satisfying for some constants $0 < \underline{\kappa}, \overline{\kappa} < \infty$,

$$0 < \underline{\kappa} \le \kappa \le \overline{\kappa} < \infty. \tag{7}$$

Furthermore, we assume that $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$, where $\boldsymbol{\mathcal{S}}^* : (0,\infty) \times \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$ is a continuous mapping. Additionally, $\boldsymbol{\mathcal{S}}$ fulfills the following set of conditions

$$\begin{aligned}
(\boldsymbol{\mathcal{S}}^*(\vartheta, D_1) - \boldsymbol{\mathcal{S}}^*(\vartheta, D_2)) : (D_1 - D_2) \ge 0 \\
\boldsymbol{\mathcal{S}}^*(\vartheta, D_1) : D_1 \ge \underline{\nu}|D_1|^p - \overline{\nu} \\
|\boldsymbol{\mathcal{S}}^*(\vartheta, D_1)| \le \overline{\nu}(1 + |D_1|)^{p-1} \\
\boldsymbol{\mathcal{S}}^*(\vartheta, 0) = 0
\end{aligned} \tag{8}$$

for some $0 < \underline{\nu}, \overline{\nu} < \infty$, for all $\vartheta \in \mathbb{R}_+, D_1, D_2 \in \mathbb{R}^{2\times 2}_{sym}$ and for $p \ge 2$, where the parameter p is called the power law index of $\boldsymbol{\mathcal{S}}$.

For simplicity, let us introduce $\hat{\vartheta}$ solving

$$-\operatorname{div} \boldsymbol{q}^*(\hat{\vartheta}) = 0 \quad \text{in } \Omega, \qquad \hat{\vartheta} = \vartheta_b \quad \text{on } \partial\Omega, \qquad \hat{\vartheta} \in L^\infty(\Omega) \cap W^{1,2}(\Omega). \tag{9}$$

We will assume that such $\hat{\vartheta}$ exists and is uniquely defined.

1. Notation

Ω	Bounded Lipschitz subdomain of $\mathbb{R}^2$.
$L^\gamma(\Omega)$	Standard Lebesgue space with a norm $\|\cdot\|_{L^\gamma(\Omega)}$, $\gamma \in [1,\infty]$.
$W^{1,\gamma}(\Omega)$	Standard Sobolev space with a norm $\|\cdot\|_{W^{1,\gamma}(\Omega)}$, $\gamma \in [1,\infty]$.
$\|\cdot\|_\gamma$	Shorter notation for $\|\cdot\|_{L^\gamma(\Omega)}$, $\gamma \in [1,\infty]$.
$\|\cdot\|_{1,\gamma}$	Shorter notation for $\|\cdot\|_{W^{1,\gamma}(\Omega)}$, $\gamma \in [1,\infty]$.
$C_0^\infty(\Omega)$	Smooth functions with compact support in Ω.
$\mathcal{D}(0,T)$	Smooth functions compactly supported on $(0,T)$.
$W^{1,p}_{0,\mathrm{div}}(\Omega)$	$\{\boldsymbol{u} \in (W^{1,p}(\Omega))^2; \boldsymbol{u} = 0 \text{ on } \partial\Omega; \operatorname{div}\boldsymbol{u} = 0\}$.
$L^2_{0,\mathrm{div}}$	$\overline{\{\boldsymbol{u} \in (C_0^\infty(\Omega))^2; \operatorname{div}\boldsymbol{u} = 0\}}^{\|\cdot\|_2}$.
V, $V(\Omega)$	Shorter notation for $W^{1,p}_{0,\mathrm{div}}(\Omega)$.
$\langle\cdot,\cdot\rangle_X$	Duality pairing between functional from X^* and function from X, sometimes we omit the space X if it is clear from the context.
$\mathbb{R}^{2\times 2}_{sym}$	Space of symmetric 2×2 matrices.
$\omega_\varepsilon(x)$	Regularization kernel $\varepsilon^{-2}\omega(x/\varepsilon)$, where $\omega : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative radially symmetric function, $C_0^\infty(\mathbb{R}^2)$, such that it support lies in a unit ball and $\int_{\mathbb{R}^2} \omega \, dx = 1$.
C	Constant dependent only on initial data, unless denoted otherwise (e.g. $C(M)$). It can vary from one inequality to another.
$L^\gamma(0,T,X)$	Standard Bochner space with a norm $\|\cdot\|_{L^\gamma(0,T;X)}$, $\gamma \in [1,\infty]$, X Banach space on Ω.
$C([0,T],X)$	Space of vector-valued continuous functions with a norm $\|\cdot\|_{C([0,T];X)}$, X Banach space on Ω.
$\|\cdot\|_{L^\gamma(X)}$	Shorter notation for $\|\cdot\|_{L^\gamma(0,T;X)}$.
$\boldsymbol{\mathcal{S}}^{N,M}$	Abbreviation for $\boldsymbol{\mathcal{S}}^*\left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right)$.
$\boldsymbol{\mathcal{S}}^{N}$	Abbreviation for $\boldsymbol{\mathcal{S}}^*\left(\vartheta^{N}, D\boldsymbol{u}^{N}\right)$.
$⨍_A^B \, dt$	Normed integral $\frac{1}{\lvert B-A\rvert}\int_A^B \, dt$.
IBP	Integration by parts.
GT	Use of the Gelfand triple to express the duality pairing as a scalar product.

Note that for vector-valued functions, we will simplify the notation by writing e.g. $L^\gamma(0,T,L^2_{0,\mathrm{div}})$ instead of $L^\gamma(0,T,L^2_{0,\mathrm{div}}(\Omega))$.

2. Preliminaries

Let us recall some known results that will be used in the upcoming chapters.

Lemma 1 (Carathéodory's existence theorem)**.** *Let $n \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $\boldsymbol{x}_0 \in \mathbb{R}^n$, and $\varepsilon > 0$, $\delta > 0$. Consider equation*

$$\partial_t \boldsymbol{x}(t) = F(t, \boldsymbol{x}(t)), \tag{2.1}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{2.2}$$

where F is defined on $R = I \times G := \{(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n : |t - t_0| \leq \varepsilon,\ |\boldsymbol{x} - \boldsymbol{x}_0| \leq \delta\}$. We say that $\boldsymbol{x}$ is a solution *to* (2.1)-(2.2) *on interval $J \subseteq I$, $t_0 \in J$, if $\boldsymbol{x} \in C^1(J)$, the equality* (2.1) *holds for all $t \in J$, and $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. We say that $\boldsymbol{x}$ is a* maximal solution *to* (2.1)-(2.2) *if it is a solution and there is no solution $\tilde{\boldsymbol{x}}$ on $\tilde{J} \supset J$ such that $\tilde{\boldsymbol{x}} = \boldsymbol{x}$ on J.*

Let F satisfy the following conditions:

- *$F(t, x)$ is continuous in x for each fixed $t \in I$,*
- *$F(t, x)$ is measurable in t for each fixed $x \in G$,*
- *there is a Lebesgue-integrable function $M : [t_0 - \varepsilon, t_0 + \varepsilon] \to [0, \infty)$ such that $|F(t, x)| \leq M(t)$ for all $(t, x) \in R$.*

Then there exists a maximal solution to (2.1)*.*

Proof. For proof of the lemma see [1], Theorem 2.1.1. □

Definition 1 (Gelfand triple)**.** *Let X be a separable reflexive Banach space such that there exists a Hilbert space H, where $X \hookrightarrow H$ densely. Then we call the triple X, $H \cong H^*$, and X^* the* Gelfand triple.

For Gelfand triple, it holds that also $H^* \hookrightarrow X^*$ densely. Let $I : X \to H$ represent the dense embedding, and let $\Phi : H^* \to H$ represent the identification of H and H^* through the Riesz representation theorem. We then may define $i : X \to X^*$ as

$$\langle ix, y \rangle_X = (Ix, Iy)_H = \left\langle \Phi^{-1} Ix, Iy \right\rangle_H,$$

where $x, y \in X$. More details on the Gelfand triple and its properties can be found in Section 4.2 in [2].

Lemma 2. *Let $g : (0, \infty)^2 \to [0, \infty)$ be a continuous function such that there exists a constant $D > 0$ for which it holds*

$$|\partial_1 g(\sigma, \tau)| + |\partial_1^2 g(\sigma, \tau)| + |\partial_2 \partial_1 g(\sigma, \tau)| \leq D$$

for all $\sigma \geq \mu$ and $0 < \underline{\tau} \leq \tau \leq \overline{\tau}$. Furthermore, let $\hat{\vartheta}$ be defined as in (9) *and $s \in L^2(0, T; W^{1,2})$ such that $\partial_t s \in L^2(0, T; (W_0^{1,2})^*)$ and $s \geq \mu$ a.e. in Q. Then $\|g(s, \hat{\vartheta})\|_1$ is an absolutely continuous function on the interval $(0, T)$ and*

$$\partial_t \|g(s, \hat{\vartheta})\|_1 = \left\langle \partial_t s, \partial_1 g(s, \hat{\vartheta}) \right\rangle_{W_0^{1,2}}. \tag{2.3}$$

Proof. We know, by Lemma 5.3.19 in [3], that there exist functions $\{s_n\}_{n=1}^{\infty} \in \left(C^{\infty}([0,T];W^{1,2})\right)^{\mathbb{N}}$ such that

$$s_n \to s \text{ in } L^2(0,T;W^{1,2}), \qquad \partial_t s_n \to \partial_t s \text{ in } L^2(0,T;(W_0^{1,2})^*). \tag{2.4}$$

Without the loss of generality we can assume $s_n \geq \mu$ for all $n \in \mathbb{N}$. For all s_n it holds

$$\partial_t s_n \left(\partial_1 g(s_n, \hat{\vartheta})\right) = \partial_t[g(s_n, \hat{\vartheta})], \quad \text{a.e. in } Q$$

since g is Lipschitz continuous in the first component (by the boundedness of $\partial_1 g$) and s_n has a weak time derivative. Also, $g = |g|$ by the assumption on non-negativity of g. We thus have $\|g(s_n, \hat{\vartheta})\|_1 = \int_\Omega g(s_n, \hat{\vartheta})\, dx$. For $\varphi \in \mathcal{D}(0,T)$ we can compute

$$\begin{aligned}
&\int_0^T \left\langle \partial_t s_n, \partial_1 g(s_n, \hat{\vartheta}) \right\rangle \varphi \, dt \overset{\text{GT}}{=} \int_Q \partial_t s_n \left(\partial_1 g(s_n, \hat{\vartheta})\right) \varphi \, d(t,x) = \\
&= \int_Q \partial_t[g(s_n, \hat{\vartheta})] \varphi \, d(t,x) = -\int_0^T \int_\Omega g(s_n, \hat{\vartheta})\, dx \partial_t \varphi \, dt = -\int_0^T \|g(s_n, \hat{\vartheta})\|_1 \partial_t \varphi \, dt,
\end{aligned} \tag{2.5}$$

where we could use the property of the Gelfand triple since for all $t \in (0,T)$ it holds $\partial_t s_n(t) \in W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$.

We know that up to a subsequence $s_n \to s$ and $\nabla s_n \to \nabla s$ almost everywhere in Q by (2.4), thus also $\partial_1 g(s_n, \hat{\vartheta})$ and $\nabla \partial_1 g(s_n, \hat{\vartheta})$ converge almost everywhere. Additionally, $s_n \to s$ and $\nabla s_n \to \nabla s$ in $L^2(Q)$, hence there exist functions $m \in L^2(Q)$ and $m_g \in L^2(Q)$ such that for all $n \in \mathbb{N}$ it holds $|s| + |s_n| \leq m$ and $|\nabla s| + |\nabla s_n| \leq m_g$. We can use the Lebesgue Dominated Convergence Theorem to show

$$\begin{aligned}
&\partial_1 g(s_n, \hat{\vartheta}) \to \partial_1 g(s, \hat{\vartheta}) && \text{in } L^2(0,T;L^2), \text{ and} \\
&\nabla \partial_1 g(s_n, \hat{\vartheta}) \to \nabla \partial_1 g(s, \hat{\vartheta}) && \text{in } L^2(0,T;L^2).
\end{aligned}$$

since

$$|\partial_1 g(s_n, \hat{\vartheta}) - \partial_1 g(s, \hat{\vartheta})|^2 \leq 4D^2 \in L^1(Q),$$

and

$$\begin{aligned}
&|\nabla \partial_1 g(s_n, \hat{\vartheta}) - \nabla \partial_1 g(s, \hat{\vartheta})|^2 \leq 4D^2 \left(|\nabla s_n|^2 + 2|\nabla \hat{\vartheta}|^2 + |s|^2\right) \leq \\
&\leq 8D^2 \left(|m_g|^2 + |\nabla \hat{\vartheta}|^2\right) \in L^1(Q).
\end{aligned}$$

Furthermore, $|g(s_n, \hat{\vartheta})| \leq D|s_n| + C$ by the estimate on $\partial_1 g$. Thus Lebesgue Dominated Convergence Theorem again implies

$$\int_0^T \int_\Omega g(s_n, \hat{\vartheta})\, dx \partial_t \varphi \, dt \to \int_0^T \int_\Omega g(s, \hat{\vartheta})\, dx \partial_t \varphi \, dt,$$

since

$$|g(s_n, \hat{\vartheta}) - g(s, \hat{\vartheta})| \leq 2D|m| + 2C \in L^1(Q).$$

We can now take the limit $n \to \infty$ in (2.5) to obtain

$$\int_0^T \partial_t \|g(s,\hat{\vartheta})\|_1 \varphi \, dt = -\int_0^T \|g(s,\hat{\vartheta})\|_1 \partial_t \varphi \, dt =$$
$$= \int_0^T \left\langle \partial_t s, \partial_1 g(s,\hat{\vartheta}) \right\rangle \varphi \, dt.$$

Since

$$\left\langle \partial_t s, \partial_1 g(s,\hat{\vartheta}) \right\rangle \in L^1(0,T),$$

we can use Theorem 2.17 in [4] to conclude

$$\|g(s,\hat{\vartheta})\|_1 \in AC(0,T).$$

□

Lemma 3 (Aubin-Lions)**.** *Let X_0, X_1 and X be Banach spaces such that X_0 is compactly embedded in X and X is embedded in X_1. Let X_0, X_1 be reflexive, $1 \leq \alpha_0, \alpha_1 < \infty$ and $T < \infty$. Then $\{u \in L^{\alpha_0}(0,T,X_0); \partial_t u \in L^{\alpha_1}(0,T,X_1)\}$ is compactly embedded in $L^{\alpha_0}(0,T,X)$.*

Proof. The lemma is a consequence of Corollary 9 in [5]. □

Lemma 4 (Korn's inequality)**.** *Let $\Omega \in \mathcal{C}^{0,1}$ be a bounded domain in $\mathbb{R}^2$ and $q \in (1,\infty)$. There exists a positive constant C depending only on q and Ω such that for all $\boldsymbol{u} \in W^{1,q}(\Omega)^2$ with $Tr\,\boldsymbol{u} \in L^2(\partial\Omega)$ it holds*

$$C\|\boldsymbol{u}\|_{1,q} \leq \|D\boldsymbol{u}\|_q + \|Tr\,\boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

Proof. The lemma with a proof can be found in [6] as Lemma 1.11. □

Lemma 5. *Let $\Omega \in \mathcal{C}^{0,1}$ be a bounded domain in $\mathbb{R}^2$, $z \in L^p(\Omega) \cap L^q(\Omega)$, $1 \leq p < q \leq \infty$. Then $z \in L^r$ for all $r \in [p,q]$ and*

$$\|z\|_r \leq \|z\|_p^{\frac{p(q-r)}{r(q-p)}} \|z\|_q^{\frac{q(r-p)}{r(q-p)}} \qquad \textit{for } q < \infty,$$
$$\|z\|_r \leq \|z\|_p^{\frac{p}{r}} \|z\|_\infty^{\frac{r-p}{r}} \qquad \textit{for } q = \infty.$$

Proof. Firstly, let us show the case for q finite. Let us take

$$\|z\|_r^r = \int_\Omega |z|^r \, dx = \int_\Omega |z|^{\frac{p(q-r)}{q-p}} |z|^{\frac{q(r-p)}{q-p}} \, dx \overset{\text{Hölder}}{\leq}$$
$$\overset{\text{Hölder}}{\leq} \||z|^{\frac{p(q-r)}{q-p}}\|_{\frac{q-p}{q-r}} \||z|^{\frac{q(r-p)}{q-p}}\|_{\frac{q-p}{r-p}} = \|z\|_p^{\frac{p(q-r)}{(q-p)}} \|z\|_q^{\frac{q(r-p)}{(q-p)}}.$$

If $q = \infty$, we have

$$\|z\|_r^r = \int_\Omega |z|^r \, dx = \int_\Omega |z|^p |z|^{r-p} \, dx \leq$$
$$\leq \||z|^{r-p}\|_\infty \int_\Omega |z|^p \, dx = \|z\|_\infty^{r-p} \|z\|_p^p.$$

□

Lemma 6 (Gagliardo-Nirenberg interpolation inequality)**.** *Let $\Omega \in \mathcal{C}^{0,1}$ be a bounded domain in $\mathbb{R}^2$, $z \in W^{1,s}(\Omega) \cap L^q(\Omega)$, and $1 \leq q < \infty$. Then*

1. *If $s < 2$, then $z \in L^r(\Omega)$ for $r \leq \frac{2s}{2-s}$ and for $q \leq r \leq \frac{2s}{2-s}$ there exists a constant C such that:*
$$\|z\|_r \leq C\|z\|_{1,s}^{\frac{2s(q-r)}{r(2q-sq-2s)}}\|z\|_q^{\frac{q(2r-sr-2s)}{r(2q-sq-2s)}}.$$
2. *If $s = 2$, then $z \in L^r(\Omega)$ for $r < \infty$ and for $q \leq r < \infty$ there exists a constant C such that:*
$$\|z\|_r \leq C\|z\|_{1,2}^{\frac{r-q}{r}}\|z\|_q^{\frac{q}{r}}.$$
3. *If $s > 2$, then $z \in L^r(\Omega)$ for $r \leq \infty$ and for $q \leq r \leq \infty$ there exists a constant C such that:*
$$\|z\|_r \leq C\|z\|_{1,s}^{\frac{2s(q-r)}{r(2q-sq-2s)}}\|z\|_q^{\frac{q(2r-sr-2s)}{r(2q-sq-2s)}}.$$

Proof. For a proof of the lemma see Theorem 2.2 in [7]. □

Lemma 7. *Let X, H, X^* form a Gelfand triple on Ω, and $r \in (1,\infty)$. Let $z \in L^r(0,T;X)$ and $\partial_t z \in L^{r'}(0,T;X^*)$. Then $z = \tilde{z}$ almost everywhere on $(0,T)$, where $\tilde{z} \in C([0,T];H)$. Moreover, the mapping $t \to \|z(t)\|_H^2$ is weakly differentiable and*
$$\partial_t\|z(t)\|_H^2 = 2\left\langle \partial_t z(t), z(t)\right\rangle_X$$
for a.e. $t \in [0,T]$.

Proof. The statement and the proof can be found in Section 4.1 in [8]. □

Lemma 8. *Let $W^{2,2}_{0,\mathrm{div}} := \overline{\{\boldsymbol{u} \in (C_0^\infty(\Omega))^2; \operatorname{div}\boldsymbol{u} = 0\}}^{\|\cdot\|_{2,2}}$. Then*
$$W^{2,2}_{0,\mathrm{div}} = \{\boldsymbol{u} \in (W^{2,2}(\Omega))^2; \boldsymbol{u} = 0 = \nabla\boldsymbol{u} \text{ on } \partial\Omega; \operatorname{div}\boldsymbol{u} = 0\}$$
and there exists a countable set $\{\lambda_r\}_{r=1}^\infty$ and a corresponding family of functions $\{\boldsymbol{w}_r\}_{r=1}^\infty$ such that

- *for all $r, s \in \mathbb{N}$ it holds $\int_\Omega \boldsymbol{w}_r\boldsymbol{w}_s\,dx = \delta_{rs}$,*
- *$1 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, and $\lambda_r \to \infty$ for $r \to \infty$,*
- *$\{\boldsymbol{w}_r\}_{r=1}^\infty$ forms a basis of $W^{2,2}_{0,\mathrm{div}}(\Omega)$,*
- *for all $\boldsymbol{\varphi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)$ and all $r \in \mathbb{N}$ it holds $\int_\Omega \nabla^2\boldsymbol{w}_r\nabla^2\boldsymbol{\varphi}\,dx = \lambda_r\int_\Omega \boldsymbol{w}_r\boldsymbol{\varphi}\,dx$,*
- *for all $r, s \in \mathbb{N}$ it holds $\int_\Omega \frac{\nabla^2\boldsymbol{w}_r}{\sqrt{\lambda_r}}\frac{\nabla^2\boldsymbol{w}_s}{\sqrt{\lambda_s}}\,dx = \delta_{rs}$.*

Moreover, defining $H^N := span\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N\}$ (a linear hull) and
$$P^N(\boldsymbol{\varphi}) = \sum_{r=1}^N \int_\Omega \boldsymbol{w}_r\boldsymbol{\varphi}\,dx\boldsymbol{w}_r : W^{2,2}_{0,\mathrm{div}} \to H^N,$$
we get
$$\|P^N\|_{\mathcal{L}(W^{2,2}_{0,\mathrm{div}}, W^{2,2}_{0,\mathrm{div}})} \leq 1.$$

Proof. See Theorem 4.11 in the appendix of [9]. □

3. Existence and Uniqueness of a Weak Solution

Let $C_0^\infty(\Omega)$ denote a set of smooth and compactly supported functions in Ω. We then define

$$V(\Omega) = W^{1,p}_{0,\mathrm{div}}(\Omega) := \overline{\{\boldsymbol{u} \in (C_0^\infty(\Omega))^2; \mathrm{div}\,\boldsymbol{u}0\}}^{\|\cdot\|_{1,p}},$$

and

$$L^2_{0,\mathrm{div}}(\Omega) := \overline{\{\boldsymbol{u} \in (C_0^\infty(\Omega))^2; \mathrm{div}\,\boldsymbol{u} = 0\}}^{\|\cdot\|_2}.$$

Furthermore, we know that

$$\begin{aligned} W^{1,p}_{0,\mathrm{div}}(\Omega) &= \{\boldsymbol{u} \in (W^{1,p}(\Omega))^2; \boldsymbol{u} = 0 \text{ on } \partial\Omega; \mathrm{div}\,\boldsymbol{u} = 0\}, \\ L^2_{0,\mathrm{div}}(\Omega) &= \overline{W^{1,2}_{0,\mathrm{div}}(\Omega)}^{\|\cdot\|_2}, \end{aligned}$$

which can be found in sections III.2 and III.4 of [10]. We know that

$$\begin{aligned} &V(\Omega) \hookrightarrow L^2_{0,\mathrm{div}}(\Omega) \text{ densely, and} \\ &W_0^{1,2}(\Omega) \hookrightarrow L^2_0(\Omega) \text{ densely.} \end{aligned}$$

In addition, $V(\Omega)$, $W_0^{1,2}(\Omega)$ are separable reflexive Banach spaces, and $L^2_{0,\mathrm{div}}(\Omega)$, $L^2_0(\Omega)$ are Hilbert spaces. Consequently,

$$(V(\Omega), L^2_{0,\mathrm{div}}(\Omega), (V(\Omega))^*) \text{ and } (W_0^{1,2}(\Omega), L^2_0(\Omega), (W_0^{1,2}(\Omega))^*)$$

form Gelfand triplets and we can identify dualities $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_{W_0^{1,2}}$ with the scalar products using $L^2_{0,\mathrm{div}}(\Omega)$, $L^2_0(\Omega)$ for functionals that are regular enough.

Let us now define the weak solution to the problem (1)–(5).

Definition 2 (Weak solution)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $T > 0$. Assume that $\boldsymbol{\mathcal{S}}^*$ and κ satisfy* (7)–(8) *with $p \geq 2$. Additionally, assume that $\boldsymbol{f}$, $\boldsymbol{u}_0$, ϑ_0, $\hat{\vartheta}$ fulfill*

$$\boldsymbol{f} \in L^{p'}(0,T;V^*)\,,\ \boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega), \tag{3.1}$$

$$\vartheta_0 \in L^1(\Omega),\ \hat{\vartheta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \tag{3.2}$$

$$\mu := \min\{\operatorname*{ess\,inf}_{x\in\Omega} \hat{\vartheta}(x), \operatorname*{ess\,inf}_{x\in\Omega} \vartheta_0(x)\} > 0. \tag{3.3}$$

We define a weak solution *to* (1)–(5) *as a quadruplet $(\boldsymbol{u}, \boldsymbol{\mathcal{S}}, \vartheta, \eta)$ fulfilling*

$$\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}}) \cap L^p(0,T;V), \tag{3.4}$$

$$\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*),\ \boldsymbol{\mathcal{S}} \in L^{p'}(Q, \mathbb{R}^{2\times 2}_{sym}), \tag{3.5}$$

$$\vartheta \in L^\infty(0,T;L^1),\ (\vartheta)^\alpha \in L^2(0,T;W^{1,2}) \qquad \alpha \in (0,1/2)\,, \tag{3.6}$$

$$\vartheta \in L^r(Q) \qquad r \in [1,2)\,, \tag{3.7}$$

$$\vartheta - \hat{\vartheta} \in L^s(0,T;W_0^{1,s}) \qquad s \in [1,4/3)\,, \tag{3.8}$$

$$\eta \in L^2(0,T;W^{1,2}) \cap L^q(Q) \qquad q \in [1,\infty), \tag{3.9}$$

and satisfying (1)–(4) *in the following sense:*
Momentum equation: *The Cauchy stress is of the form* $\boldsymbol{S} = \boldsymbol{S}^*(\vartheta, D\boldsymbol{u})$ *a.e. in* Q*, it holds* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *in* $L^2_{0,\mathrm{div}}(\Omega)$*, and for all* $\boldsymbol{w} \in L^p(0,T;V)$

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{w}\rangle_V \, dt + \int_0^T \int_\Omega \boldsymbol{S} : D\boldsymbol{w} \, dx \, dt \\ = \int_0^T \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : D\boldsymbol{w} \, dx \, dt + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}\rangle_V \, dt;\end{aligned} \tag{3.10}$$

Internal energy balance: *Temperature satisfies the minimum principle* $\vartheta \geq \mu$ *a.e. in* Q*. There exists a set of full measure* $S \subseteq [0,T]$ *such that* $\vartheta(t) \to \vartheta_0$ *in* $L^1(\Omega)$ *as* $S \ni t \to 0+$*. Additionally, for all* $\varphi \in C_0^\infty((-\infty,T) \times \Omega)$

$$\begin{aligned}-\int_0^T \int_\Omega \vartheta \partial_t \varphi \, dx \, dt - \int_0^T \int_\Omega \vartheta \boldsymbol{u} \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_\Omega \kappa(\vartheta) \nabla \vartheta \cdot \nabla \varphi \, dx \, dt \\ = \int_0^T \int_\Omega \boldsymbol{S} : D\boldsymbol{u} \varphi \, dx \, dt + \int_\Omega \vartheta_0 \varphi(0) \, dx;\end{aligned} \tag{3.11}$$

Entropy equation: *Entropy is given as* $\eta = \ln \vartheta$ *a.e. in* Q*,* $\eta_0 := \ln \vartheta_0$ *and for all* $\varphi \in C_0^\infty((-\infty,T) \times \Omega)$

$$\begin{aligned}-\int_0^T \int_\Omega \eta \partial_t \varphi \, dx \, dt - \int_0^T \int_\Omega \eta \boldsymbol{u} \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_\Omega \kappa(\vartheta) \nabla \eta \cdot \nabla \varphi \, dx \, dt \\ = \int_0^T \int_\Omega \frac{1}{\vartheta} \boldsymbol{S} : D\boldsymbol{u} \varphi \, dx \, dt + \int_0^T \int_\Omega \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \varphi \, dx \, dt + \int_\Omega \eta_0 \varphi(0) \, dx.\end{aligned} \tag{3.12}$$

We can now formulate the main theorem of the thesis.

Theorem 9 (Existence of a solution fulfilling entropy equality)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary and* $T > 0$*. Assume that* $\boldsymbol{S}^*$ *and* κ *satisfy* (7)–(8) *with* $p \geq 2$*. Then for any data* $\boldsymbol{f}$*,* $\boldsymbol{u}_0$*,* ϑ_0*,* $\hat{\vartheta}$ *fulfilling* (3.1)–(3.3)*, there exists a weak solution* $(\boldsymbol{u}, \boldsymbol{S}, \vartheta, \eta)$ *to* (1)–(5)*.*

4. Proof of Theorem 9

Firstly, following the methods used in the article [11], we will prove the existence of a weak solution satisfying (3.10) and (3.11). Such an approach needs to be modified since the article treats the case with Navier's slip boundary condition for temperature whereas we consider a nonhomogeneous Dirichlet condition.

We then show that the found solution fulfills (3.12) adopting the methods from [12].

4.1 Galerkin approximations

Let us consider the basis $\{\boldsymbol{w}_j\}_{j=0}^{\infty}$ of $W^{2,2}_{0,\mathrm{div}}(\Omega)$ from Lemma 8. Note that for every $j \in \mathbb{N}$, $\|\boldsymbol{w}_j\|_{1,q}$ is bounded for all $q \in [1,\infty)$ by the Sobolev Emmbeding Theorem. Furthermore, we define $\{w_j\}_{j=0}^{\infty}$ as an orthonormal basis of $L^2(\Omega)$ that is orthogonal in $W_0^{1,2}(\Omega)$, such that for any $\psi \in W_0^{1,2}(\Omega)$ it holds

$$\int_\Omega \nabla w_k \nabla \psi \, dx = \lambda_k \int_\Omega w_k \psi \, dx. \tag{4.1}$$

Such basis can be constructed by eigenfunctions of the Laplace operator in Ω subject to the homogeneous Dirichlet boundary condition, see Theorem 6.5.1. in [13] for more details.

For fixed $N, M \in \mathbb{N}$, let us define

$$\begin{aligned} \boldsymbol{u}^{N,M}(x,t) &= \sum_{i=1}^{N} c_i^{N,M}(t) \boldsymbol{w}_i(x), \\ \vartheta^{N,M}(x,t) &= \left(\sum_{i=1}^{M} d_i^{N,M}(t) w_i(x) \right) + \hat{\vartheta}, \end{aligned}$$

such that the vector $(\boldsymbol{c}^{N,M}, \boldsymbol{d}^{N,M}) = (c_1^{N,M}, \dots, c_N^{N,M}, d_1^{N,M}, \dots, d_M^{N,M}) : (0,T^*) \to \mathbb{R}^{N+M}$, for $T^* \in (0,T)$, is the maximal solution to the following system of ordinary differential equations

$$\begin{aligned} &\int_\Omega \partial_t \boldsymbol{u}^{N,M} \boldsymbol{w}_j \, dx + \int_\Omega \boldsymbol{\mathcal{S}}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right) : D\boldsymbol{w}_j \, dx = \\ &\qquad\qquad = \int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla(\boldsymbol{w}_j) \, dx + \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle, \\ &\int_\Omega \partial_t \vartheta^{N,M} w_k \, dx + \int_\Omega \kappa(\vartheta^{N,M}) \nabla \vartheta^{N,M} \cdot (\nabla w_k) \, dx = \\ &= \int_\Omega \boldsymbol{\mathcal{S}}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right) : D\boldsymbol{u}^{N,M} w_k \, dx + \int_\Omega \vartheta^{N,M} \boldsymbol{u}^{N,M} \cdot (\nabla w_k) \, dx, \end{aligned} \tag{4.2}$$

where $j \in \{1, \dots, N\}$ and $k \in \{1, \dots, M\}$. System of equations (4.2) can be rewritten into

$$\partial_t c_j^{N,M}(t) = -\int_\Omega \boldsymbol{\mathcal{S}}^*\left(\left(\sum_{i=1}^M d_i^{N,M}(t)w_i\right)+\hat{\vartheta}, \sum_{i=1}^N c_i^{N,M}(t)D\boldsymbol{w}_i\right) : D\boldsymbol{w}_j\,dx$$
$$+\int_\Omega \sum_{i=1}^N\sum_{l=1}^N c_i^{N,M}(t)c_l^{N,M}(t)(\boldsymbol{w}_i\otimes\boldsymbol{w}_l) : \nabla(\boldsymbol{w}_j)\,dx + \langle \boldsymbol{f},\boldsymbol{w}_j\rangle\,,$$
$$\partial_t d_k^{N,M}(t) = -\int_\Omega \kappa(\sum_{i=1}^M d_i^{N,M}(t)w_i+\hat{\vartheta})(\sum_{l=1}^M d_l^{N,M}(t)\nabla w_i + \nabla\hat{\vartheta})\cdot(\nabla w_k)\,dx$$
$$+\int_\Omega \sum_{i=1}^N c_i^{N,M}(t)\boldsymbol{\mathcal{S}}^*\left(\sum_{i=1}^M d_i^{N,M}(t)w_i+\hat{\vartheta}, \sum_{i=1}^N c_i^{N,M}(t)D\boldsymbol{w}_i\right) : D\boldsymbol{w}_i w_k\,dx$$
$$+\int_\Omega \sum_{i=1}^N c_i^{N,M}(t)\left(\sum_{l=1}^M d_l^{N,M}(t)w_l+\hat{\vartheta}\right)\boldsymbol{w}_i\cdot(\nabla w_k)\,dx.$$

We hence obtain a system of $N+M$ equations

$$\begin{aligned}\partial_t c_j^{N,M}(t) &= F_j(t,\boldsymbol{c}^{N,M}(t),\boldsymbol{d}^{N,M}(t)), \quad \forall j\in\{1,\dots,N\},\\ \partial_t d_k^{N,M}(t) &= G_k(\boldsymbol{c}^{N,M}(t),\boldsymbol{d}^{N,M}(t)), \quad \forall k\in\{1,\dots,M\}.\end{aligned} \tag{4.3}$$

Note, that functions $\boldsymbol{F}$ and $\boldsymbol{G}$ are measurable in t and continuous in $(\boldsymbol{c}^{N,M},\boldsymbol{d}^{N,M})$. We equip (4.3) with the following initial conditions:

$$\begin{aligned}\boldsymbol{u}^{N,M}(x,0) &= \boldsymbol{u}_0^N(x) \quad &&\text{on } \Omega,\\ \vartheta^{N,M}(x,0) &= \vartheta_0^{N,M}(x) &&\text{on } \Omega,\end{aligned} \tag{4.4}$$

where $\boldsymbol{u}_0^N(x) = \sum_{i=1}^N c_{i,0}^N\boldsymbol{w}_i(x)$ is the projection of $\boldsymbol{u}_0$ onto the linear hull of $\{\boldsymbol{w}_j\}_{j=0}^N$ in $L^2(\Omega)$. We define $\vartheta_0^{N,M}$ in the following manner. Set

$$\tilde{\vartheta}_0 = \begin{cases}\vartheta_0, & \text{if } x\in\Omega,\\ \mu, & \text{if } x\in\mathbb{R}^2\setminus\Omega,\end{cases}$$

and for regularization kernel $\omega_{\frac{1}{N}}$ set $\vartheta_0^N = \tilde{\vartheta}_0 * \omega_{\frac{1}{N}}$. Since $\tilde{\vartheta}_0\geq\mu$ almost everywhere in $\mathbb{R}^2$, the same holds for ϑ_0^N. We then define $\vartheta_0^{N,M}$ as a projection of ϑ_0^N onto the linear hull of $\{w_j\}_{j=0}^M$ in $L^2(\Omega)$, hence $\vartheta_0^{N,M} = \sum_{j=1}^M d_{0,j}^{N,M}w_j$.

Note, that

$$\begin{aligned}\vartheta_0^{N,M} &\xrightarrow{M\to\infty} \vartheta_0^N \quad &&\text{in } L^2(\Omega),\\ \boldsymbol{u}_0^N &\xrightarrow{N\to\infty} \boldsymbol{u}_0 &&\text{in } \left(L^2(\Omega)\right)^2,\\ \vartheta_0^N &\xrightarrow{N\to\infty} \vartheta_0 &&\text{in } L^1(\Omega),\end{aligned} \tag{4.5}$$

where the first two convergences hold due to the completeness of an orthonormal basis in Hilbert spaces, and the last convergence is given by properties of a regularization kernel.

Moreover, for $k<M$ and $j<N$, G_k depends only on $(\boldsymbol{c}^{N,M},\boldsymbol{d}^{N,M})$, and

$$F_j(t,\boldsymbol{c}^{N,M},\boldsymbol{d}^{N,M}) = R_j(\boldsymbol{c}^{N,M},\boldsymbol{d}^{N,M}) + \langle \boldsymbol{f}(t),\boldsymbol{w}_j\rangle\,,$$

where $\langle \boldsymbol{f},\boldsymbol{w}_j\rangle \in L^1(0,T)$. Thus $|\boldsymbol{F}|+|\boldsymbol{G}|$ has an integrable majorant $M(t)$ on a certain right neighbourhood of $t=0$. By Lemma 1, we obtain the existence of a maximal solution to the system (4.3)–(4.4) until $T^*\leq T$. Showing that $\boldsymbol{c}^{N,M}$ and $\boldsymbol{d}^{N,M}$ are bounded will imply the existence of the solution until time T by the Theorem of Maximal Interval of Existence.

4.2 Estimates independent of M

We wish to show estimates on $\boldsymbol{c}^{N,M}$ and $\boldsymbol{d}^{N,M}$ uniform with respect to M. That will imply the existence of a solution to (4.3) for all $t \in (0,T)$, and additionally it will allow us to find converging subsequences of $\vartheta^{N,M}$ and $\boldsymbol{u}^{N,M}$. Hence, we multiply the j-th equation in (4.2) by $c_j^{N,M}$, sum over $j = 1, \dots, N$ and integrate until time $t \le T^*$. This gives us

$$\int_0^t \left\langle \partial_t \boldsymbol{u}^{N,M}, \boldsymbol{u}^{N,M} \right\rangle d\tau + \int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right) : D\boldsymbol{u}^{N,M} \, dx \, d\tau =$$

$$= \int_0^t \int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla(\boldsymbol{u}^{N,M}) \, dx \, d\tau + \int_0^t \left\langle \boldsymbol{f}, \boldsymbol{u}^{N,M} \right\rangle d\tau.$$

Using estimate (8) on $\boldsymbol{\mathcal{S}}^*$, Hölder's inequality, and identity

$$\int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla(\boldsymbol{u}^{N,M}) \, dx = \int_\Omega \boldsymbol{u}^{N,M} \cdot \nabla \left(\frac{|\boldsymbol{u}^{N,M}|^2}{2} \right) dx =$$

$$= -\int_\Omega (\operatorname{div} \boldsymbol{u}^{N,M}) \frac{|\boldsymbol{u}^{N,M}|^2}{2} \, dx = 0$$

we will obtain

$$\int_0^t \frac{1}{2} \partial_t \|\boldsymbol{u}^{N,M}\|_2^2 \, d\tau + \int_0^t \int_\Omega \underline{\nu} |D\boldsymbol{u}^{N,M}|^p - \overline{\nu} \, dx \, d\tau \le \int_0^t \|\boldsymbol{u}^{N,M}\|_V \|\boldsymbol{f}\|_{V^*} \, d\tau.$$

By Lemma 4, Young's inequality, and the fact that $\boldsymbol{f} \in L^{p'}(0,T;V^*)$ we have

$$\frac{1}{2} \|\boldsymbol{u}^{N,M}(t)\|_2^2 + \underline{\nu} \int_0^t \|\boldsymbol{u}^{N,M}\|_V^p \, d\tau \le C + \frac{\underline{\nu}}{2} \int_0^t \|\boldsymbol{u}^{N,M}\|_V^p \, d\tau,$$

hence

$$\|\boldsymbol{u}^{N,M}(t)\|_2^2 + \int_0^t \|\boldsymbol{u}^{N,M}\|_V^p \, d\tau \le C.$$

Since this estimate works for any $t \le T^*$, we have the estimate independent of M and N on the whole interval $(0, T^*)$, hence

$$\sup_{t \in (0,T^*)} \|\boldsymbol{u}^{N,M}(t)\|_2^2 + \int_0^{T^*} \|\boldsymbol{u}^{N,M}\|_V^p \, d\tau \le C. \tag{4.6}$$

The estimate of the first term on the left-hand side furthermore implies that

$$\sup_{j \le N} \sup_{t \in (0,T^*)} |c_j^{N,M}(t)|^2 \le C. \tag{4.7}$$

We now multiply the $(M+k)$-th equation in (4.2) by $d_k^{N,M}$, sum over $k = 1, \dots, M$ and integrate until the time $t \le T^*$. We obtain

$$\int_0^t \left\langle \partial_t \vartheta^{N,M}, \vartheta^{N,M} - \hat{\vartheta} \right\rangle d\tau + \int_0^t \int_\Omega \kappa(\vartheta^{N,M}) \nabla \vartheta^{N,M} \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta}) \, dx \, d\tau =$$

$$= \int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^* : D\boldsymbol{u}^{N,M} (\vartheta^{N,M} - \hat{\vartheta}) + \vartheta^{N,M} \boldsymbol{u}^{N,M} \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta}) \, dx \, d\tau.$$

By the identity

$$\int_\Omega (\vartheta^{N,M} - \hat{\vartheta})\boldsymbol{u}^{N,M} \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta})\, dx = \int_\Omega \boldsymbol{u}^{N,M} \cdot \nabla \left(\frac{(\vartheta^{N,M} - \hat{\vartheta})^2}{2} \right) dx = \\ = - \int_\Omega (\operatorname{div} \boldsymbol{u}^{N,M}) \frac{(\vartheta^{N,M} - \hat{\vartheta})^2}{2}\, dx = 0, \tag{4.8}$$

the estimates on κ and $\boldsymbol{S}^*$, and the manipulation with $\hat{\vartheta}$, we can rewrite the equation as

$$\int_0^t \frac{1}{2}\partial_t \|\vartheta^{N,M} - \hat{\vartheta}\|_2^2 + \underline{\kappa}\|\nabla(\vartheta^{N,M} - \hat{\vartheta})\|_2^2 + \int_\Omega \kappa(\vartheta^{N,M})\nabla\hat{\vartheta} \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta})\, dx\, d\tau \leq \\ \leq \int_0^t \int_\Omega \overline{\nu} \left(1 + |D\boldsymbol{u}^{N,M}|\right)^{p-1} |D\boldsymbol{u}^{N,M}| \left|\vartheta^{N,M} - \hat{\vartheta}\right| + \hat{\vartheta}\boldsymbol{u}^{N,M} \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta})\, dx\, d\tau.$$

Using

$$\int_0^t \int_\Omega \overline{\nu} \left(1 + |D\boldsymbol{u}^{N,M}|\right)^{p-1} |D\boldsymbol{u}^{N,M}| \left|\vartheta^{N,M} - \hat{\vartheta}\right| dx\, d\tau \leq \\ \leq \int_0^t \int_\Omega C \left(1 + |D\boldsymbol{u}^{N,M}|\right)^{p} \left|\vartheta^{N,M} - \hat{\vartheta}\right| dx\, d\tau \leq \\ \leq C \int_0^t \int_\Omega |D\boldsymbol{u}^{N,M}|^p \left|\vartheta^{N,M} - \hat{\vartheta}\right| + \left|\vartheta^{N,M} - \hat{\vartheta}\right| dx\, d\tau \leq \\ \leq \int_0^t \|D\boldsymbol{u}^{N,M}\|_{2p}^p \|\vartheta^{N,M} - \hat{\vartheta}\|_2 + C\|\vartheta^{N,M} - \hat{\vartheta}\|_2\, d\tau,$$

and

$$\int_0^t \int_\Omega (\hat{\vartheta}\boldsymbol{u}^{N,M} - \kappa(\vartheta^{N,M})\nabla\hat{\vartheta}) \cdot \nabla(\vartheta^{N,M} - \hat{\vartheta})\, dx\, d\tau \leq \\ \leq \int_0^t \int_\Omega (\hat{\vartheta}|\boldsymbol{u}^{N,M}| + \overline{\kappa}|\nabla\hat{\vartheta}|)|\nabla(\vartheta^{N,M} - \hat{\vartheta})|\, dx\, d\tau \leq \\ \leq \int_0^t (\|\hat{\vartheta}\|_\infty \|\boldsymbol{u}^{N,M}\|_2 + \overline{\kappa}\|\nabla\hat{\vartheta}\|_2)\|\nabla(\vartheta^{N,M} - \hat{\vartheta})\|_2\, d\tau$$

together with Poincaré's inequality yields into

$$\frac{1}{2}\|(\vartheta^{N,M} - \hat{\vartheta})(t)\|_2^2 - \frac{1}{2}\|(\vartheta^{N,M} - \hat{\vartheta})(0)\|_2^2 + C\int_0^t \|\vartheta^{N,M} - \hat{\vartheta}\|_{1,2}^2\, d\tau \leq \\ \leq \int_0^t C\left(1 + \|D\boldsymbol{u}^{N,M}\|_{2p}^p + \|\hat{\vartheta}\|_\infty \|\boldsymbol{u}^{N,M}\|_2 + \overline{\kappa}\|\nabla\hat{\vartheta}\|_2\right) \|\vartheta^{N,M} - \hat{\vartheta}\|_{1,2}\, d\tau \leq \\ \overset{\text{Young}}{\leq} \int_0^t C\left(1 + \|D\boldsymbol{u}^{N,M}\|_{2p}^{2p} + \|\hat{\vartheta}\|_\infty^2 \|\boldsymbol{u}^{N,M}\|_2^2 + \overline{\kappa}\|\nabla\hat{\vartheta}\|_2^2\right) + \delta\|\vartheta^{N,M} - \hat{\vartheta}\|_{1,2}^2\, d\tau,$$

where Young's inequality is used in such a way that δ is small enough. We observe that $\|(\vartheta^{N,M} - \hat{\vartheta})(0)\|_2^2$ and $\int_0^t \|\nabla\hat{\vartheta}\|_2^2\, d\tau$ are finite by (4.5) and (3.2). Furthermore, $\|\hat{\vartheta}\|_\infty \int_0^t \|\boldsymbol{u}^{N,M}\|_2^2\, d\tau$ is bounded, since $\|\hat{\vartheta}\|_\infty$ is finite by (3.2), and

$$\int_0^t \|\boldsymbol{u}^{N,M}\|_2^2\, d\tau \leq \int_0^{T^*} \|\boldsymbol{u}^{N,M}\|_2^2\, d\tau \leq C$$

by (4.6). Finally, we can estimate

$$\int_0^t \|D\boldsymbol{u}^{N,M}\|_{2p}^{2p}\,d\tau \le \sup_{j\le N}\|D\boldsymbol{w}_j\|_{2p}^{2p}\int_0^t\left(\sum_{j=1}^N c_j^{N,M}\right)^{2p} d\tau \le C(N). \tag{4.9}$$

This holds since $\|\boldsymbol{w}_j\|_{1,2p}\le C$ for every j by the choice of the basis $\{\boldsymbol{w}_j\}_{j=1}^N$, and since

$$\int_0^t\left(\sum_{j=1}^N\left|c_j^{N,M}\right|\right)^{2p} d\tau \le N^{2p}T^*\sup_{j\le N}\|c_j^{N,M}\|_{L^\infty(0,T^*)}^{2p},$$

which is bounded by (4.7).

Thus

$$\|(\vartheta^{N,M}-\hat{\vartheta})(t)\|_2^2+\int_0^t\|\vartheta^{N,M}-\hat{\vartheta}\|_{1,2}^2\,d\tau\le C(N). \tag{4.10}$$

The estimate (4.10) holds for any $t\le T^*$, so in particular we have

$$\sup_{t\in(0,T^*)}\|(\vartheta^{N,M}-\hat{\vartheta})(t)\|_2^2+\int_0^{T^*}\|\vartheta^{N,M}-\hat{\vartheta}\|_{1,2}^2\,dt\le C(N), \tag{4.11}$$

hence also

$$\sup_{k\le M}\sup_{t\in(0,T^*)}\|d_k^{N,M}(t)\|_2^2\le C(N). \tag{4.12}$$

Since $(\boldsymbol{c}^{N,M},\boldsymbol{d}^{N,M})=(c_1^{N,M},\dots,c_N^{N,M},d_1^{N,M},\dots d_M^{N,M})$ is a maximal solution of (4.3)–(4.4) uniformly bounded with respect to time, it is defined until time T by the Theorem of Maximal Interval of Existence. We thus obtain the estimates

$$\begin{aligned}&\sup_{t\in(0,T)}\|\boldsymbol{u}^{N,M}(t)\|_2^2+\int_0^T\|\boldsymbol{u}^{N,M}\|_V^p\le C\\ &\sup_{t\in(0,T)}\|(\vartheta^{N,M}-\hat{\vartheta})(t)\|_2^2+\int_0^T\|\vartheta^{N,M}-\hat{\vartheta}\|_{1,2}^2\,dt\le C(N),\end{aligned} \tag{4.13}$$

We now need to estimate the norm of the time derivatives of $\vartheta^{N,M}$ and $\boldsymbol{u}^{N,M}$ uniformly in M. Firstly, we show that $\partial_t\vartheta^{N,M}\in L^2(0,T;(W_0^{1,2})^*)$. Due to the orthonormality of $\{w_j\}$ in L^2, we can write for any $t\in(0,T)$ and for any $\varphi\in L^2(0,T;W_0^{1,2}(\Omega))$:

$$\begin{aligned}&\left\langle\partial_t\vartheta^{N,M},\varphi\right\rangle_{W_0^{1,2}(\Omega)}(t)=\left\langle\partial_t(\vartheta^{N,M}-\hat{\vartheta}),\varphi\right\rangle(t)=\int_\Omega\partial_t(\vartheta^{N,M}-\hat{\vartheta})(t)\varphi(t)\,dx=\\ &=\int_\Omega\sum_{i=1}^M\partial_t d_i^{N,M}(t)w_i\varphi(t)\,dx=\int_\Omega\partial_t(\vartheta^{N,M}-\hat{\vartheta})(t)\varphi^M(t)\,dx=\\ &=\int_\Omega\partial_t\vartheta^{N,M}(t)\varphi^M(t)\,dx,\end{aligned}$$

where φ^M denotes a projection of φ onto the linear hull of $\{w_j\}_{j=1}^M$ in $L^2(\Omega)$. Let

$\boldsymbol{\mathcal{S}}^{N,M} := \boldsymbol{\mathcal{S}}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right)$ for the simplicity of notation. Then we have

$$\begin{aligned}
&\|\partial_t\vartheta^{N,M}\|_{(L^2(W_0^{1,2})^*)} = \sup_{\|\varphi\|_{L^2(W_0^{1,2})}\leq 1} \left|\int_Q \partial_t\vartheta^{N,M}\varphi^M \, d(t,x)\right| \overset{(4.2)}{=} \\
&= \sup_\varphi \left|\int_Q (\vartheta^{N,M}\boldsymbol{u}^{N,M} - \kappa(\vartheta^{N,M})\nabla\vartheta^{N,M})\cdot(\nabla\varphi^M) + \boldsymbol{\mathcal{S}}^{N,M} : D\boldsymbol{u}^{N,M}\varphi^M \, d(t,x)\right| \leq \\
&\leq \sup_\varphi \int_Q (|\vartheta^{N,M}||\boldsymbol{u}^{N,M}| + \overline{\kappa}|\nabla\vartheta^{N,M}|)|\nabla\varphi^M| + |\boldsymbol{\mathcal{S}}^{N,M} : D\boldsymbol{u}^{N,M}||\varphi^M| \, d(t,x) \leq \\
&\leq \sup_\varphi \int_0^T C(1 + \|D\boldsymbol{u}^{N,M}\|_{2p}^p + \|\vartheta^{N,M}\|_4\|\boldsymbol{u}^{N,M}\|_4 + \|\nabla\vartheta^{N,M}\|_2)\|\varphi^M\|_{1,2} \, dt \leq \\
&\leq \sup_\varphi \int_0^T C(\|D\boldsymbol{u}^{N,M}\|_{2p}^{2p} + \|\vartheta^{N,M}\|_4^4 + \|\boldsymbol{u}^{N,M}\|_4^4 + \|\nabla\vartheta^{N,M}\|_2^2) \, dt \|\varphi^M\|_{L^2(W_0^{1,2})}.
\end{aligned}$$

We know that $\|D\boldsymbol{u}^{N,M}\|_{L^{2p}(Q)} \leq C(N)$ by (4.9) and $\|\nabla\vartheta^{N,M}\|_{L^2(Q)} \leq C(N)$ by (4.13). Furthermore, we use that $\int_0^T \|\vartheta^{N,M}\|_4^4 \, dt$ and $\int_0^T \|\boldsymbol{u}^{N,M}\|_4^4 \, dt$ are bounded by (4.13) using Lemma 6. Additionally, it holds that

$$\|\varphi^M\|_{L^2(W_0^{1,2})} \leq \|\varphi\|_{L^2(W_0^{1,2})}$$

in $W_0^{1,2}$ by the orthogonality of $\{w_j\}_{j=1}^\infty$ together with (4.1). Hence

$$\|\partial_t\vartheta^{N,M}\|_{(L^2(W_0^{1,2})^*)} \leq \sup_{\|\varphi\|_{L^2(W_0^{1,2})}\leq 1} C(N)\|\varphi\|_{L^2(W_0^{1,2})} \leq C(N). \tag{4.14}$$

Analogously, we show that $\partial_t\boldsymbol{u}^{N,M} \in L^{p'}(0,T;(W_{0,\text{div}}^{2,2})^*)$ and the estimate is uniform with respect to both M and N. Take any $\boldsymbol{\varphi} \in L^p(0,T;W_{0,\text{div}}^{2,2})$ and any $t \in (0,T)$, using properties of the basis from Theorem 8 we have

$$\begin{aligned}
&\left\langle \partial_t\boldsymbol{u}^{N,M}, \boldsymbol{\varphi}\right\rangle_{W_{0,\text{div}}^{2,2}}(t) = \sum_{i=1}^N \partial_t c_i^{N,M}(t) \left\langle \boldsymbol{w}_i, \sum_{j=1}^N \int_\Omega \boldsymbol{w}_j\boldsymbol{\varphi}(t) \, dy \boldsymbol{w}_j\right\rangle_{W_{0,\text{div}}^{2,2}} \overset{\text{GT}}{=} \\
&= \sum_{i=1}^N \partial_t c_i^{N,M}(t) \int_\Omega \boldsymbol{w}_i \sum_{j=1}^N \int_\Omega \boldsymbol{w}_j\boldsymbol{\varphi}(t) \, dy \boldsymbol{w}_j \, dx = \\
&= \int_\Omega \left(\sum_{i=1}^N \partial_t c_i^{N,M}(t)\boldsymbol{w}_i\right)\left(\sum_{j=1}^N \int_\Omega \boldsymbol{w}_j\boldsymbol{\varphi}(t) \, dy \boldsymbol{w}_j\right) dx = \int_\Omega \partial_t\boldsymbol{u}^{N,M}(t)\boldsymbol{\varphi}^N(t) \, dx.
\end{aligned}$$

Function $\boldsymbol{\varphi}^N(t)$ is defined as follows

$$\boldsymbol{\varphi}^N(t) := P^N(\boldsymbol{\varphi}(t)) = \sum_{j=1}^N \int_\Omega \boldsymbol{w}_j\boldsymbol{\varphi}(t) \, dy \boldsymbol{w}_j \in H^N,$$

where P^N and H^N were introduced in Lemma 8. We could use the property of the Gelfand triple since for all $j \in \mathbb{N}$ it holds $\boldsymbol{w}_j \in W_{0,\text{div}}^{2,2}(\Omega) \hookrightarrow L_{0,\text{div}}^2(\Omega)$. Due to the previous equality we can estimate

$$
\begin{aligned}
\|\partial_t \boldsymbol{u}^{N,M}\|_{L^{p'}((W^{2,2}_{0,\mathrm{div}})^*)} &= \sup_{\|\boldsymbol{\varphi}\|_{L^p(W^{2,2}_{0,\mathrm{div}})}\le 1} \left| \int_Q \partial_t \boldsymbol{u}^{N,M} \boldsymbol{\varphi}^N \, d(t,x) \right| \overset{(4.2)}{=} \\
&= \sup_{\boldsymbol{\varphi}} \left| \int_Q (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M} - \boldsymbol{\mathcal{S}}^{N,M}) \nabla \boldsymbol{\varphi}^N \, d(t,x) + \int_0^T \left\langle \boldsymbol{f}, \boldsymbol{\varphi}^N \right\rangle \, dt \right| \le \\
&\le \sup_{\boldsymbol{\varphi}} \int_Q \left(|\boldsymbol{u}^{N,M}|^2 + |\boldsymbol{\mathcal{S}}^{N,M}| \right) |\nabla \boldsymbol{\varphi}^N| \, d(t,x) + \int_0^T \left\langle \boldsymbol{f}, \boldsymbol{\varphi}^N \right\rangle \, dt \le \\
&\le \sup_{\boldsymbol{\varphi}} \int_0^T \left(\|\boldsymbol{u}^{N,M}\|_{2p'}^2 + C\left(1 + \|D\boldsymbol{u}^{N,M}\|_p^{p-1}\right) + \|\boldsymbol{f}\|_{V^*} \right) \|\boldsymbol{\varphi}^N\|_V \, dt \le \\
&\le \sup_{\boldsymbol{\varphi}} \int_0^T \left(\|\boldsymbol{u}^{N,M}\|_{2p'}^2 + C\left(1 + \|D\boldsymbol{u}^{N,M}\|_p^{\frac{p}{p'}}\right) + \|\boldsymbol{f}\|_{V^*} \right) \|\boldsymbol{\varphi}^N\|_{W^{2,2}_{0,\mathrm{div}}} \, dt \le \\
&\le \sup_{\boldsymbol{\varphi}} \left(\|\boldsymbol{u}^{N,M}\|_{L^{2p'}(Q)} + C\left(1 + \|D\boldsymbol{u}^{N,M}\|_{L^p(Q)}\right) + \|\boldsymbol{f}\|_{L^p(V^*)} \right) \|\boldsymbol{\varphi}^N\|_{L^{p'}(W^{2,2}_{0,\mathrm{div}})}.
\end{aligned}
$$

We know, that $\|\boldsymbol{u}^{N,M}\|_{L^{2p'}(Q)} \le C\|\boldsymbol{u}^{N,M}\|_{L^4(Q)}$, which is bounded by Lemma 6 and estimate (4.13). It remains to show that for all $t \in (0,T)$ the following implication holds

$$
\|\boldsymbol{\varphi}(t)\|_{W^{2,2}_{0,\mathrm{div}}} \le 1 \implies \|\boldsymbol{\varphi}^N(t)\|_{W^{2,2}_{0,\mathrm{div}}} \le 1.
$$

This is true since

$$
\|\boldsymbol{\varphi}^N(t)\|_{W^{2,2}_{0,\mathrm{div}}} = \|P^N(\boldsymbol{\varphi}(t))\|_{W^{2,2}_{0,\mathrm{div}}} \le \|P^N\|_{\mathcal{L}(W^{2,2}_{0,\mathrm{div}}, W^{2,2}_{0,\mathrm{div}})} \|\boldsymbol{\varphi}(t)\|_{W^{2,2}_{0,\mathrm{div}}} \le 1.
$$

We can thus conclude

$$
\|\partial_t \boldsymbol{u}^{N,M}\|_{L^{p'}((W^{2,2}_{0,\mathrm{div}})^*)} \le C, \tag{4.15}
$$

where the constant is independent of both N and M.

4.3 Limit $M \to \infty$

Due to estimates (4.6)–(4.15) that are independent of M, for any fixed $N \in \mathbb{N}$, we can find a not relabeled subsequence $\{\boldsymbol{u}^{N,M}, \vartheta^{N,M}\}_{M=1}^\infty$ such that for $M \to \infty$ we will obtain the following convergences

$$
\begin{aligned}
\partial_t \boldsymbol{u}^{N,M} &\rightharpoonup \partial_t \boldsymbol{u}^N && \text{in } L^{p'}(0,T;(W^{2,2}_{0,\mathrm{div}})^*), && (4.16)\\
\boldsymbol{u}^{N,M} &\rightharpoonup^* \boldsymbol{u}^N && \text{in } L^\infty(0,T;L^2_{0,\mathrm{div}}), && (4.17)\\
\boldsymbol{u}^{N,M} &\rightharpoonup \boldsymbol{u}^N && \text{in } L^p(0,T;V), && (4.18)\\
\partial_t \vartheta^{N,M} &\rightharpoonup \partial_t \vartheta^N && \text{in } L^2(0,T;(W^{1,2}_0)^*), && (4.19)\\
\vartheta^{N,M} &\rightharpoonup^* \vartheta^N && \text{in } L^\infty(0,T;L^2), && (4.20)\\
\vartheta^{N,M} &\rightharpoonup \vartheta^N && \text{in } L^2(0,T;W^{1,2}). && (4.21)
\end{aligned}
$$

In this section the convergence results are considered up to a subsequence.

We use Lemma 3 to conclude, that

$$\{\vartheta^{N,M} \in L^2(0,T;W^{1,2}); \partial_t \vartheta^{N,M} \in L^2(0,T;(W_0^{1,2})^*)\} \hookrightarrow\hookrightarrow L^2(0,T;L^2),$$

hence we get $\vartheta^{N,M} \to \vartheta^N$ in $L^2(0,T;L^2)$. Fixing $0 < \varepsilon << 1$ and using Lemma 6 we get the following estimate for $r < 4$ we can estimate

$$\begin{aligned}
&\int_0^T \|\vartheta^{N,M} - \vartheta^N\|_r^r \, dt \le C \int_0^T \|\vartheta^{N,M} - \vartheta^N\|_{1,2}^{r-2} \|\vartheta^{N,M} - \vartheta^N\|_2^2 \, dt \le \\
&\le \sup_t \|\vartheta^{N,M} - \vartheta^N\|_2^{2-\varepsilon} \int_0^T \|\vartheta^{N,M} - \vartheta^N\|_{1,2}^{r-2} \|\vartheta^{N,M} - \vartheta^N\|_2^{\varepsilon} \, dt \le \\
&\le \tilde{C} \left(\int_0^T \|\vartheta^{N,M} - \vartheta^N\|_{1,2}^{\frac{2}{2-\varepsilon}(r-2)} \, dt \right)^{\frac{2-\varepsilon}{2}} \left(\int_0^T \|\vartheta^{N,M} - \vartheta^N\|_2^2 \, dt \right)^{\frac{\varepsilon}{2}}.
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen in such a way that $\frac{r-2}{2-\varepsilon} \le 1$ if $r < 4$, we obtain

$$\vartheta^{N,M} \to \vartheta^N \quad \text{in } L^r(0,T;L^r) \text{ for } r \in [1,4). \tag{4.22}$$

Similarly, we have

$$\{\boldsymbol{u}^{N,M} \in L^p(0,T;V); \partial_t \boldsymbol{u}^{N,M} \in L^{p'}(0,T;(W_{0,\mathrm{div}}^{2,2})^*)\} \hookrightarrow\hookrightarrow L^p(0,T;L^p),$$

hence we get $\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N$ in $L^p(0,T;L^p)$, which also implies $\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N$ in $L^2(0,T;L^2)$. Fixing $0 < \varepsilon << 1$ and using Lemma 6 for $r \in (2,2p)$ we get

$$\begin{aligned}
&\int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_r^r \, dt \le C \int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_{1,p}^{\frac{p(2-r)}{2-2p}} \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^{\frac{2r-2p-pr}{2-2p}} \, dt \le \\
&\le C \sup_t \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^{\frac{2r-2p-pr}{2-2p}-\varepsilon} \int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_{1,p}^{\frac{p(2-r)}{2-2p}} \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^{\varepsilon} \, dt \le \\
&\le \tilde{C} \left(\int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_{1,p}^{\frac{p(2-r)}{2-2p}\frac{2}{2-\varepsilon}} \, dt \right)^{\frac{2-\varepsilon}{2}} \left(\int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^2 \, dt \right)^{\frac{\varepsilon}{2}}.
\end{aligned}$$

Since for $r < 2p$, $\varepsilon > 0$ can be chosen such that $\frac{2-r}{(2-\varepsilon)(1-p)} \le 1$, we obtain even better strong convergence result

$$\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N \qquad \text{in } L^r(0,T;L^r) \text{ for } r \in [1,2p). \tag{4.23}$$

For fixed $j \le N$ and fixed $k \le M$ we multiply the j-th and $(N+k)$-th equation in (4.2) by $\varphi \in \mathcal{D}(0,T)$ and integrate over time. That gives us

$$\begin{aligned}
&\int_Q \partial_t \boldsymbol{u}^{N,M} \varphi \boldsymbol{w}_j \, d(t,x) + \int_Q \boldsymbol{S}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right) : D\boldsymbol{w}_j \varphi \, d(t,x) = \\
&\qquad = \int_Q \varphi(\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla \boldsymbol{w}_j \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \varphi \boldsymbol{w}_j \rangle \, dt, \\
&\int_Q \partial_t \vartheta^{N,M} w_k \varphi \, d(t,x) + \int_Q \kappa(\vartheta^{N,M}) \varphi \nabla \vartheta^{N,M} \cdot \nabla w_k \, d(t,x) = \\
&= \int_Q \boldsymbol{S}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right) : D\boldsymbol{u}^{N,M} w_k \varphi \, d(t,x) + \int_Q \varphi \vartheta^{N,M} \boldsymbol{u}^{N,M} \cdot \nabla w_k \, d(t,x).
\end{aligned} \tag{4.24}$$

We now wish to pass the limit $M \to \infty$ in these equations. First, let us show that

$$\int_Q \varphi(\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla \boldsymbol{w}_j \, d(t,x) \to \int_Q \varphi(\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{w}_j \, d(t,x). \qquad (4.25)$$

This holds since

$$\begin{aligned}
&\left| \int_Q \varphi(\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M} - \boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{w}_j \, d(t,x) \right| \leq \\
&\leq C \int_0^T |\varphi| \|(\boldsymbol{u}^{N,M} - \boldsymbol{u}^N)\|_3 \left(\|\boldsymbol{u}^{N,M}\|_3 + \|\boldsymbol{u}^{N,M}\|_3 \right) \|\nabla \boldsymbol{w}_j\|_3 \, dt \leq \\
&\leq C \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_{L^2(L^3)} \left(\|\boldsymbol{u}^{N,M}\|_{L^2(L^3)} + \|\boldsymbol{u}^N\|_{L^2(L^3)} \right) \stackrel{M \to \infty}{\to} 0.
\end{aligned}$$

Now let us show, that

$$\int_Q \boldsymbol{S}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right) : D\boldsymbol{w}_j \varphi \, d(t,x) \to \int_Q \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N \right) : D\boldsymbol{w}_j \varphi \, d(t,x).$$

By (4.22), we obtain that

$$\vartheta^{N,M} \to \vartheta^N \qquad \text{almost everywhere in } Q. \qquad (4.26)$$

Let us now show that $D\boldsymbol{u}^{N,M}$ also converges almost everywhere. We have

$$\begin{aligned}
&\|D\boldsymbol{u}^{N,M} - D\boldsymbol{u}^N\|_{L^p(Q)}^p = \int_0^T \|D\boldsymbol{u}^{N,M} - D\boldsymbol{u}^N\|_p^p \, dt \leq \\
&\leq \int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_{1,p}^p \, dt \leq C(N) \int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^p \, dt \leq \\
&\leq C(N) \sup_{t \in (0,T)} \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^{p-2} \int_0^T \|\boldsymbol{u}^{N,M} - \boldsymbol{u}^N\|_2^2 \, dt \stackrel{M \to \infty}{\to} 0,
\end{aligned}$$

where the second inequality holds since all norms are equivalent on a finite dimensional space H^N. This gives us

$$D\boldsymbol{u}^{N,M} \to D\boldsymbol{u}^N \qquad \text{in } L^p(0,T;L^p), \qquad (4.27)$$

and consequently

$$D\boldsymbol{u}^{N,M} \to D\boldsymbol{u}^N \qquad \text{almost everywhere in } Q.$$

By the continuity of $\boldsymbol{S}^*$ we have

$$\boldsymbol{S}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right) \to \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N \right) \qquad \text{almost everywhere in } Q. \qquad (4.28)$$

Furthermore, $|\boldsymbol{S}^* \left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M} \right)| \leq C \left(1 + |D\boldsymbol{u}^{N,M}|^{p-1} \right)$ by the third inequality in (8). Additionally, by (4.27), for fixed N there exist a function $m^N : Q \to (0,\infty)$ such that $|D\boldsymbol{u}^{N,M}| + |D\boldsymbol{u}^N| \leq m$ and $m \in L^p(Q)$. Thus, by the Lebesgue Dominated Convergence Theorem

$$\boldsymbol{S}^{N,M} \to \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N \right) \qquad \text{in } L^{p'}(0,T;L^{p'}), \qquad (4.29)$$

since

$$|\boldsymbol{S}^{N,M} - \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N\right)|^{p'} \leq C(1 + |D\boldsymbol{u}^{N,M}|^p + |D\boldsymbol{u}^N|^p) \leq C(1 + m^p) \in L^1(Q).$$

Hence,

$$\int_Q \boldsymbol{S}^{N,M} : D\boldsymbol{u}^{N,M} w_j \varphi \, d(t,x) \to \int_Q \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N w_j \varphi \, d(t,x),$$

since $\boldsymbol{S}^{N,M}$ converges strongly in $L^{p'}(Q)$ by (4.29) and $D\boldsymbol{u}^{N,M}$ converges strongly in $L^p(Q)$ by (4.27). Lastly, we want to show

$$\int_Q \kappa(\vartheta^{N,M}) \varphi \nabla \vartheta^{N,M} \cdot \nabla w_k \, d(t,x) \to \int_Q \kappa(\vartheta^N) \varphi \nabla \vartheta^N \cdot \nabla w_k \, d(t,x). \tag{4.30}$$

We have $\nabla \vartheta^{N,M} \rightharpoonup \nabla \vartheta^N$ in $L^2(Q)$ by (4.21). Also

$$\kappa(\vartheta^{N,M}) \varphi \nabla w_k \to \kappa(\vartheta^N) \varphi \nabla w_k \qquad \text{in } L^2(Q)$$

due to (4.26), the integrable majorant $\left|\kappa(\vartheta^{N,M}) \varphi \nabla w_k\right|^2 \leq (\overline{\kappa} \|\varphi\|_\infty |\nabla \boldsymbol{w}_k|)^2 \in L^2$, and the Lebesgue Dominated Convergence Theorem. Hence, (4.30) holds true.

Since the convergences of the other integrals in (4.24) are straightforward, we can take the limit $M \to \infty$. The convergences hold for any $\varphi \in \mathcal{D}(0,T)$, so we can use the Fundamental Lemma of the Calculus of Variations and obtain a system of equations independent of M, that holds almost everywhere in $(0,T)$:

$$\begin{aligned} &\left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{w}_j \right\rangle_{W^{2,2}_{0,\mathrm{div}}} + \int_\Omega \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{w}_j \, dx = \\ &\qquad = \int_\Omega (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{w}_j \, dx + \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle, \qquad \text{for all } j \in \{1, 2, \ldots, N\} \end{aligned} \tag{4.31}$$

$$\begin{aligned} &\left\langle \partial_t \vartheta^N, w_k \right\rangle_{W^{1,2}_0} + \int_\Omega \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla w_k \, dx = \\ &= \int_\Omega \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N w_k \, dx + \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla w_k \, dx, \quad \text{for all } k \in \mathbb{N}. \end{aligned} \tag{4.32}$$

Additionally, linear hull of $\{w_k\}$ is dense in $W^{1,2}_0(\Omega)$. Hence for any $\psi \in W^{1,2}_0(\Omega)$ we have a sequence $\{\psi^M\}_{M=1}^\infty$ such that $\psi^M \in \mathit{span}\{w_1, \ldots, w_M\}$, and $\psi^M \to \psi$ in $W^{1,2}_0(\Omega)$. We can thus consider (4.32), where w_k is replaced by ψ^M, pass to the limit $M \to \infty$, and conclude

$$\begin{aligned} &\left\langle \partial_t \vartheta^N, \psi \right\rangle_{W^{1,2}_0} + \int_\Omega \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \psi \, dx = \\ &= \int_\Omega \boldsymbol{S}^* \left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N \psi \, dx + \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \psi \, dx, \quad \forall \psi \in W^{1,2}_0(\Omega), \end{aligned} \tag{4.33}$$

almost everywhere on $(0,T)$.

It remains to show that the initial conditions are attained. For all M, we have the equality $\boldsymbol{c}^{N,M}(0) = \boldsymbol{c}^N_0$. To conclude $\boldsymbol{u}^N(x,0) = \sum_{j=1}^N c^N_j(0) \boldsymbol{w}_j = \sum_{j=1}^N c^N_{j,0} \boldsymbol{w}_j = \boldsymbol{u}^N_0(x)$, for all $x \in \Omega$, it is thus enough to prove that for all $j \in \{1, \ldots, N\}$ it holds

$$c^{N,M}_j \to c^N_j \qquad \text{in } C([0,T]). \tag{4.34}$$

For each $j \in \{1, \dots, N\}$ it holds

$$\left\langle \partial_t \boldsymbol{u}^{N,M}, \boldsymbol{w}_j \right\rangle_{W^{2,2}_{0,\mathrm{div}}} = \sum_{i=1}^{N} \partial_t c_i^{N,M} \left\langle \boldsymbol{w}_i, \boldsymbol{w}_j \right\rangle_{W^{2,2}_{0,\mathrm{div}}} = \partial_t c_j^{N,M}.$$

Thus $\|\partial_t c_j^{N,M}\|_{L^{p'}(0,T)} \le C$ by (4.15). Furthermore, $\|c_j^{N,M}\|_{L^\infty(0,T)} \le C$ by (4.7). We thus have $\|c_j^{N,M}\|_{W^{1,p'}(0,T)} \le C$, where the estimates are again uniform with respect to M. We can hence find a weakly converging not relabeled subsequence such that

$$c_j^{N,M} \rightharpoonup c_j^N \qquad \text{in } W^{1,p'}(0,T).$$

Convergence (4.34) then holds by the Sobolev embedding

$$W^{1,p'}(0,T) \hookrightarrow\hookrightarrow C([0,T]).$$

Furthermore, using Lemma 7 for

$$\vartheta^N - \hat{\vartheta} \in L^2(0,T; W_0^{1,2})$$

and

$$\partial_t(\vartheta^N - \hat{\vartheta}) = \partial_t \vartheta^N \in L^2(0,T; (W_0^{1,2})^*),$$

we conclude $\vartheta^N - \hat{\vartheta} \in C([0,T]; L^2)$, and thus

$$\vartheta^N \in C([0,T]; L^2), \tag{4.35}$$

since $\hat{\vartheta}$ is independent of t. Let us now show, that $\vartheta^N(x,0) = \vartheta_0^N(x)$. Take $\tau \in \mathcal{D}(-\infty, T)$, $\tau(0) \neq 0$, then

$$\int_Q \partial_t \vartheta^{N,M} w_k \tau \, d(t,x) \overset{\text{IBP}}{=} -\int_Q \vartheta^{N,M} w_k \partial_t \tau \, d(t,x) - \int_\Omega \vartheta_0^{N,M} w_k \tau(0) \, dx.$$

Taking the limit in $M \to \infty$ and using (4.20), (4.21) and (4.5), we have

$$\int_Q \partial_t \vartheta^N w_k \tau \, d(t,x) = -\int_Q \vartheta^N w_k \partial_t \tau \, d(t,x) - \int_\Omega \vartheta_0^N w_k \tau(0) \, dx.$$

Also

$$\int_0^T \left\langle \partial_t \vartheta^N, w_k \right\rangle_{W_0^{1,2}} \tau \, dt \overset{\text{IBP}}{=} -\int_Q \vartheta^N w_k \partial_t \tau \, d(t,x) - \int_\Omega \vartheta^N(0) w_k \tau(0) \, dx.$$

Hence $\vartheta^N(0) = \vartheta_0^N$ in $L^2(\Omega)$.

4.4 Minimum principle

We wish to show that $\vartheta^N \ge \mu$ for almost all $(t,x) \in (0,T) \times \Omega$, where μ is defined in (3.3). Hence, we set

$$\varphi^N(\tau, x) := \max\{-1, \min\{0, \vartheta^N(\tau,x) - \mu\}\} \le 0$$

as a test function in (4.33) and integrate until $t \in (0,T)$. We need to check that $\varphi^N(\tau) \in W_0^{1,2}(\Omega)$ for almost all $\tau \in (0,T)$. By (3.3) and $\vartheta^N = \hat{\vartheta}$ on the boundary, we know that $\varphi^N(\tau,\cdot) = 0$ for a.a. $\tau \in (0,T)$ in the sense of traces. Additionally,

$$|\nabla \varphi^N| = |\nabla \vartheta^N \chi_{\{\vartheta^N \in (\mu-1,\mu)\}}| \leq |\nabla \vartheta^N|, \tag{4.36}$$

thus $\varphi^N(\tau) \in W_0^{1,2}(\Omega)$ for a.a. $\tau \in (0,T)$. Let use Lemma 2 to identify the duality $\left\langle \partial_t \vartheta^N, \varphi^N \right\rangle$. If we consider

$$g(s) := \frac{1}{2}(s-\mu)^2 \chi_{\{s\in(\mu-1,\mu)\}} + (\mu - \frac{1}{2} - s)\chi_{\{s\leq \mu-1\}},$$

then $g'(\vartheta^N) = \varphi^N$. Also $|g'(s)| + |g''(s)| \leq D$ and $g(s) \geq 0$. Lastly, note that

$$g > 0 \text{ on } (-\infty,\mu), \text{ and} \qquad\qquad g = 0 \text{ on } [\mu,\infty) \tag{4.37}$$

Function g fulfills the assumptions of Lemma 2, so we can conclude

$$\left\langle \partial_t \vartheta^N, \varphi^N \right\rangle_{W_0^{1,2}} = \partial_t \|g(\vartheta^N\|_1.$$

Using φ^N as a test function in (4.33) thus gives us

$$\int_0^t \partial_t \int_\Omega g(\vartheta^N)\,dx\,d\tau + \int_0^t \int_\Omega \left(\kappa(\vartheta^N)\nabla\vartheta^N - \vartheta^N \boldsymbol{u}^N\right)\cdot \nabla\varphi^N))\,dx\,d\tau =$$
$$= \int_0^t \int_\Omega \boldsymbol{S}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N \varphi^N\,dx\,d\tau \leq 0,$$

where the inequality holds by the first and the last property in (8). By a simple computation, we have

$$\int_0^t \partial_t \int_\Omega g(\vartheta^N)\,dx\,d\tau + \int_0^t \int_\Omega \kappa(\vartheta^N)|\nabla\vartheta^N|^2 \chi_{\{\vartheta^N\in(\mu-1,\mu)\}}\,dx\,d\tau \leq$$
$$\leq \int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla g'(\vartheta^N)\,dx\,d\tau \overset{\text{IBP}}{=} -\int_0^t \int_\Omega \nabla g(\vartheta^N)\cdot \boldsymbol{u}^N\,dx\,d\tau \overset{\text{IBP}}{=} 0.$$

Since the second and the third term on the left-hand side are positive, we have

$$0 \geq \int_0^t \partial_t \int_\Omega g(\vartheta^N)\,dx\,d\tau = \|g(\vartheta^N(t))\|_2^2 - \|g(\vartheta_0^N)\|_2^2.$$

This yields into

$$\|g(\vartheta^N(t))\|_2^2 \leq 0,$$

since $\vartheta_0^N \geq \mu$ a. e. by definition and so $g(\vartheta_0^N) = 0$ a.e. on Ω by (4.37). Hence, $g(\vartheta^N(t)) = 0$ almost everywhere on Ω and by (4.37) we can conclude

$$\vartheta^N \geq \mu \qquad\qquad \text{for a. a. } (t,x) \in (0,T)\times\Omega. \tag{4.38}$$

To summarize the last four sections, we have functions

$$\boldsymbol{u}^N : Q \to \mathbb{R}^2 \qquad\qquad \vartheta^N : Q \to \mathbb{R}$$

such that

$$\begin{aligned}
&\boldsymbol{u}^N \in L^\infty(0,T;L^2_{0,\text{div}}) \cap L^p(0,T;V), \quad \partial_t \boldsymbol{u}^N \in L^{p'}(0,T;(W^{2,2}_{0,\text{div}})^*),\\
&\vartheta^N \in C([0,T];L^2) \cap L^2(0,T;W^{1,2}), \quad \partial_t \vartheta^N \in L^2(0,T;(W^{1,2}_0)^*),\\
&\vartheta^N \geq \mu \text{ a.a. on } Q.
\end{aligned} \tag{4.39}$$

Furthermore $\boldsymbol{u}^N$ and ϑ^N fulfill the system (4.31)-(4.33), and

$$\begin{aligned}
&\boldsymbol{u}^N(x,0) = \boldsymbol{u}_0^N(x) \text{ for all } x \in \Omega,\\
&\vartheta^N(0) = \vartheta_0^N \text{ in } L^2(\Omega).
\end{aligned}$$

4.5 Estimates independent of N

We now wish to provide estimates of $\boldsymbol{u}^N$, ϑ^N, $\partial_t \boldsymbol{u}^N$, and $\partial_t \vartheta^N$ that are uniform with respect to N. This will allow us to derive weakly convergent subsequences in the next section. Since the estimates of $\boldsymbol{u}^{N,M}$ and $\partial_t \boldsymbol{u}^{N,M}$ in Section 4.2 were uniform with respect to N, we can apply the same procedure as at the beginning of Section 4.2 and observe, that $\boldsymbol{u}^N$ satisfies for all $t \in [0,T]$

$$\int_0^t \frac{1}{2}\partial_t \|\boldsymbol{u}^N\|_2^2 \, d\tau + \int_0^t \int_\Omega \underline{\nu}|D\boldsymbol{u}^N|^p - \overline{\nu} \, dx \, d\tau \leq \int_0^t \|\boldsymbol{u}^N\|_V \|\boldsymbol{f}\|_{V^*} \, d\tau,$$

which implies that $\boldsymbol{u}^N$ also satisfies

$$\sup_{t \in (0,T)} \|\boldsymbol{u}^N(t)\|_2^2 + \|\boldsymbol{u}^N\|^p_{L^p(V)} \leq C. \tag{4.40}$$

Additionally, by the same steps as in (4.15), we have

$$\|\boldsymbol{u}^N\|_{L^{p'}((W^{2,2}_{0,\text{div}})^*)} \leq C. \tag{4.41}$$

We now wish to provide estimates for ϑ^N uniform with respect to N. For $\alpha \in (0,1)$, we consider

$$\psi = 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha,$$

where K is a constant such that $K > \|\hat{\vartheta}\|_\infty$. It is zero at the boundary because $\hat{\vartheta} = \vartheta^N$ at the boundary. By the minimum principle (4.39) and the positivity of α, we have $\psi \in L^\infty(Q)$. Also

$$\nabla \psi = \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \in L^2(Q), \tag{4.42}$$

so $\psi(t) \in W^{1,2}_0(\Omega)$ for a.e. $t \in (0,T)$. Thus we can take $\psi(t)$ as a test function in (4.33) and integrate until time $t \in (0,T)$. We hence have

$$\begin{aligned}
&\int_0^t \left\langle \partial_t \vartheta^N, 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha \right\rangle_{W^{1,2}_0} d\tau +\\
&\int_0^t \int_\Omega \left(\kappa(\vartheta^N)\nabla \vartheta^N - \vartheta^N \boldsymbol{u}^N\right) \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \, dx \, d\tau =\\
&= \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \left(1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha\right) dx \, d\tau.
\end{aligned} \tag{4.43}$$

Since $\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N$ is positive, we can estimate

$$\int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \left(1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha\right) dx\, d\tau \le \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \, d(t,x) \le C,$$

which holds by Hölder's inequality, (4.40), and the third part of (8).

We define $H^\alpha : (0,\infty)^2 \to \mathbb{R}$ as

$$H^\alpha(s,\sigma) = \int_0^s \left(\frac{K}{K + \tau - \sigma}\right)^\alpha d\tau$$

and we observe that there exist c_1, $c_2 > 0$ such that for all $s \ge \mu$ and $K \le \underline{\sigma} \le \sigma \le \overline{\sigma}$ it holds

$$c_1(\alpha) s^{1-\alpha} \le H^\alpha(s,\sigma) \le c_2(\alpha) s^{1-\alpha}. \tag{4.44}$$

Let us use Lemma 2 for $g(\vartheta^N, \hat{\vartheta}) := \vartheta^N - H^\alpha(\vartheta^N, \hat{\vartheta})$. The function g is non-negative since $g(\sigma,\sigma) = 0$ and

$$\partial_1 g(s,\sigma) = 1 - \left(\frac{K}{K + s - \sigma}\right)^\alpha$$

is positive for $s > \sigma$ and negative for $s < \sigma$. Furthermore,

$$|\partial_1 g(s,\hat{\vartheta})| + |\partial_1^2 g(s,\hat{\vartheta})| + |\partial_2\partial_1 g(s,\hat{\vartheta})| < D.$$

We can thus use Lemma 2 to conclude that $\|g(\vartheta^N,\hat{\vartheta})\|_1$ is an absolutely continuous function on the interval $(0,T)$ and

$$\partial_t \|g(\vartheta^N,\hat{\vartheta})\|_1 = \left\langle \partial_t \vartheta^N, 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha \right\rangle_{W_0^{1,2}}. \tag{4.45}$$

Let us now rewrite

$$\kappa(\vartheta^N)\nabla\vartheta^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^{\alpha+1} \frac{\nabla\vartheta^N - \nabla\hat{\vartheta}}{K} =$$
$$\frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} |\nabla\vartheta^N|^2 - \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \nabla\vartheta^N \cdot \nabla\hat{\vartheta}$$

and treat each part separately. Firstly, by (7), we know that

$$\int_0^t \int_\Omega \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} |\nabla\vartheta^N|^2 \, dx\, d\tau$$

is non-negative. For the second part, we have

$$\int_0^t \int_\Omega \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \nabla\vartheta^N \cdot \nabla\hat{\vartheta} \, dx\, d\tau \le$$
$$\le \int_0^t \int_\Omega \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} |\nabla\vartheta^N||\nabla\hat{\vartheta}| \, dx\, d\tau \overset{\text{Young}}{\le}$$
$$\le \int_0^t \int_\Omega C|\nabla\hat{\vartheta}|^2 + \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \frac{|\nabla\vartheta^N|^2}{4} \, dx\, d\tau.$$

Lastly, we need to estimate

$$\int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}} \right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \, dx \, d\tau =$$
$$= \int_0^t \int_\Omega (\vartheta^N - \hat{\vartheta}) \boldsymbol{u}^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}} \right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \, dx \, d\tau +$$
$$+ \int_0^t \int_\Omega \hat{\vartheta} \boldsymbol{u}^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}} \right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \, dx \, d\tau.$$

The first part is equal to zero since

$$\int_0^t \int_\Omega (\vartheta^N - \hat{\vartheta}) \boldsymbol{u}^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}} \right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \, dx \, d\tau =$$
$$= \frac{\alpha}{K} \int_0^t \int_\Omega \boldsymbol{u}^N \cdot \nabla \xi(\vartheta^N - \hat{\vartheta}) \, dx \, d\tau \overset{\text{IBP}}{=} \frac{\alpha}{K} \int_0^t \int_\Omega \operatorname{div} \boldsymbol{u}^N \xi(\vartheta^N - \hat{\vartheta}) \, dx \, d\tau = 0,$$

where ξ is defined as follows

$$\xi(s) := \int_0^s \tau \left(\frac{K}{K + \tau} \right)^{\alpha+1} d\tau.$$

To estimate the second part we consider

$$\int_0^t \int_\Omega \left| \hat{\vartheta} \boldsymbol{u}^N \cdot \alpha \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}} \right)^{\alpha+1} \frac{\nabla \vartheta^N - \nabla \hat{\vartheta}}{K} \right| dx \, d\tau \leq$$
$$C(\alpha) \int_0^t \int_\Omega \left|\hat{\vartheta}\right| \left|\boldsymbol{u}^N\right| \left|\nabla \hat{\vartheta}\right| \, dx \, d\tau + C(\alpha) \int_0^t \int_\Omega \frac{\left|\hat{\vartheta}\right| \left|\boldsymbol{u}^N\right| \left|\nabla \vartheta^N\right|}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \, dx \, d\tau \overset{\text{Hölder}}{\leq}$$
$$\leq C(\alpha) \|\hat{\vartheta}\|_\infty \left(\|\boldsymbol{u}^N\|_{L^1(L^2)} \|\nabla \hat{\vartheta}\|_2 + \int_0^t \int_\Omega \frac{\left|\boldsymbol{u}^N\right| \left|\nabla \vartheta^N\right|}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \, dx \, d\tau \right) \overset{\text{Young}}{\leq}$$
$$\leq C(\alpha) + C(\alpha) \|\boldsymbol{u}^N\|_{L^2(L^2)} + \int_0^t \int_\Omega \frac{\alpha \kappa(\vartheta^N) K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \frac{|\nabla \vartheta^N|^2}{4} \, dx \, d\tau.$$

Putting all the estimates together we can rewrite (4.43) as

$$\int_0^t \partial_t \|g(\vartheta^N)\|_1 \, d\tau + \int_0^t \int_\Omega \frac{\alpha \kappa(\vartheta^N) K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \frac{|\nabla \vartheta^N|^2}{2} \, dx \, d\tau \leq C(\alpha).$$

We will now write C instead of $C(\alpha)$ for clearer notation. Note, however, that the constants still depend on alpha. The estimates hold for any $t \in (0, T)$, so we can write

$$\sup_{t \in (0,T)} \|g(\vartheta^N(t))\|_1 + \int_Q \frac{\alpha \kappa(\vartheta^N) K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}} \frac{|\nabla \vartheta^N|^2}{2} \, d(t, x) \leq$$
$$\leq C + \|g(\vartheta^N(0))\|_1 \leq C.$$

We thus conclude

$$\|g(\vartheta^N)\|_{L^\infty(L^1)} \leq C.$$

This implies

$$\|\vartheta^N\|_{L^\infty(L^1)} \leq C, \tag{4.46}$$

since

$$\int_\Omega |\vartheta^N|\,dx \leq \int_\Omega |g(\vartheta^N)| + |H^\alpha(\vartheta^N)|\,dx \leq \|g(\vartheta^N)\|_1 + \int_\Omega c_2(\vartheta^N)^{1-\alpha}\,dx \overset{\text{Young}}{\leq}$$
$$\leq \|g(\vartheta^N)\|_1 + C + \frac{1}{2}\int_\Omega |\vartheta^N|\,dx.$$

Additionally,

$$C\int_Q \frac{|\nabla\vartheta^N|^2}{\left(K+\vartheta^N-\hat{\vartheta}\right)^{\alpha+1}}\,d(t,x) \leq \int_Q \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K+\vartheta^N-\hat{\vartheta}\right)^{\alpha+1}}\frac{|\nabla\vartheta^N|^2}{2}\,d(t,x) \leq C, \tag{4.47}$$

which gives us that for all $\alpha \in (0,1)$ it holds

$$\int_Q \left(\frac{1}{\vartheta^N}\right)^{\alpha+1}\left|\nabla\vartheta^N\right|^2\,d(t,x) \leq C. \tag{4.48}$$

It is important to notice that we cannot let $\alpha \to 0+$, since from (4.47) we would have

$$\int_Q \frac{|\nabla\vartheta^N|^2}{\left(K+\vartheta^N-\hat{\vartheta}\right)^{\alpha+1}}\,d(t,x) \leq C\frac{2}{\underline{\kappa}\alpha K^\alpha} \overset{\alpha\to 0+}{\to} \infty$$

and we would thus lose estimate (4.48). We can now use (4.48) to estimate

$$\int_Q \left|\nabla\left((\vartheta^N)^{\frac{1-\alpha}{2}} - \hat{\vartheta}^{\frac{1-\alpha}{2}}\right)\right|^2\,d(t,x) \leq C\int_Q \left(\frac{1}{\vartheta^N}\right)^{\alpha+1}\left|\nabla\vartheta^N\right|^2\,d(t,x)$$
$$+\,C\int_Q \left(\frac{1}{\hat{\vartheta}}\right)^{\alpha+1}\left|\nabla\hat{\vartheta}\right|^2 + 2\left(\frac{1}{\vartheta^N}\right)^{\frac{\alpha+1}{2}}\left(\frac{1}{\hat{\vartheta}}\right)^{\frac{\alpha+1}{2}}\nabla\vartheta^N\cdot\nabla\hat{\vartheta}\,d(t,x) \overset{(4.48)}{\leq} C,$$

and we conclude

$$\forall s<\infty : (\vartheta^N)^{\frac{1-\alpha}{2}} - (\hat{\vartheta})^{\frac{1-\alpha}{2}} \in L^2(0,T;W_0^{1,2}) \hookrightarrow L^2(0,T;L^s(\Omega)).$$

From the upper bound on $\hat{\vartheta}$, we have that

$$\forall s<\infty : (\vartheta^N)^{\frac{1-\alpha}{2}} \in L^2(0,T;L^s(\Omega)). \tag{4.49}$$

For fixed $s<\infty$, denote $q := \frac{s(1-\alpha)}{2}$ and note that

$$\int_0^T \|\vartheta^N\|_q^{1-\alpha}\,dt = \int_0^T \left(\int_\Omega (\vartheta^N)^{s\frac{1-\alpha}{2}}\,dx\right)^{\frac{2}{s}}\,dt \overset{(4.49)}{\leq} C. \tag{4.50}$$

We now use Lemma 5 together with (4.46) to estimate

$$\int_0^T \|\vartheta^N\|_r^r \, dt \le \int_0^T \|\vartheta^N\|_1^{\frac{q-r}{q-1}} \|\vartheta^N\|_q^{\frac{q(r-1)}{q-1}} \, dt \le C \int_0^T \|\vartheta^N\|_q^{\frac{q(r-1)}{q-1}} \, dt,$$

which is finite when

$$1-\alpha \ge \frac{q(r-1)}{q-1} = \frac{s(1-\alpha)(r-1)}{s(1-\alpha)-2} \overset{s\to\infty}{\to} (r-1).$$

Hence $\vartheta^N \in L^r(Q)$ for $r < 2-\alpha$. Taking $\alpha \to 0+$ we have

$$\|\vartheta^N\|_{L^r(Q)} \le C \qquad \text{for } r \in [1,2) . \tag{4.51}$$

Let us now estimate the gradient of ϑ^N using estimate (4.48):

$$\int_Q |\nabla\vartheta^N|^t \, d(t,x) = \int_Q \left(|\nabla\vartheta^N| \left(\frac{1}{\vartheta^N}\right)^{\frac{\alpha+1}{2}} \right)^{\frac{2t}{2}} \left(\vartheta^N\right)^{\frac{t(\alpha+1)}{2}} d(t,x) \overset{\text{Hölder}}{\le}$$

$$\overset{\text{Hölder}}{\le} \left(\int_Q |\nabla\vartheta^N|^2 \left(\frac{1}{\vartheta^N}\right)^{(\alpha+1)} d(t,x) \right)^{\frac{t}{2}} \left(\int_Q \left(\vartheta^N\right)^{\frac{t(\alpha+1)}{2-t}} d(t,x) \right)^{\frac{2-t}{2}} \overset{(4.48)}{\le}$$

$$\le C \left(\int_Q \left(\vartheta^N\right)^{\frac{t(\alpha+1)}{2-t}} d(t,x) \right)^{\frac{2-t}{2}} .$$

That is finite if

$$\frac{t(\alpha+1)}{2-t} < 2 \iff t < \frac{4}{3+\alpha}.$$

Hence, taking $\alpha \to 0+$ we obtain

$$\|\nabla\vartheta^N\|_{L^t(Q)} \le C \qquad \text{for } t \in \left[1, \frac{4}{3}\right) . \tag{4.52}$$

Finally, we consider $\psi \in L^\infty(0,T;W_0^{1,5})$ as a test function in (4.33) and integrate over time. We obtain

$$\int_0^T | \left\langle \partial_t\vartheta^N, \psi \right\rangle | \, dt \le \int_Q \left(\overline{\kappa} |\nabla\vartheta^N| + |\vartheta^N \boldsymbol{u}^N| \right) |\nabla\psi| + \overline{\nu}(1 + |D\boldsymbol{u}^N|^p)|\psi| \, d(t,x) \le$$

$$\le C \int_0^T \left(\|\nabla\vartheta^N\|_{\frac{5}{4}} + \|\vartheta^N\|_{\frac{10}{3}} \|\boldsymbol{u}^N\|_2 \right) \|\nabla\psi\|_5 + \|\psi\|_1 + \|D\boldsymbol{u}^N\|_p^p \|\psi\|_\infty \, dt \le$$

$$\le C\|\psi\|_{L^\infty(W^{1,5})} \left(\|\nabla\vartheta^N\|_{L^1(L^{\frac{5}{4}})} + \|\vartheta^N\|_{L^1(L^{\frac{10}{3}})} \|\boldsymbol{u}^N\|_{L^\infty(L^2)} + \|D\boldsymbol{u}^N\|_{L^p(Q)}^p \right) .$$

Note, that $\|\vartheta^N\|_{L^1(L^{\frac{10}{3}})} \le C$ uniformly with respect to N since

$$\|\vartheta^N\|_{\frac{10}{3}} \le C \|\vartheta^N\|_{1,\frac{5}{4}}$$

by Sobolev's embedding, and $\|\vartheta^N\|_{L^1(W^{1,\frac{5}{4}})} \le C$ by (4.51) and (4.52). The remaining terms are estimated by (4.52) and (4.40) Thus

$$\|\partial_t\vartheta^N\|_{L^1\left((W_0^{1,5})^*\right)} \le C. \tag{4.53}$$

Since all the estimates are uniform with respect to N, we can proceed to limit passage $N \to \infty$.

4.6 Limit $N \to \infty$

From the uniform estimates in the previous section, we obtain the following convergences for not-relabeled subsequences:

$$\begin{aligned}
&\partial_t \boldsymbol{u}^N \rightharpoonup \partial_t \boldsymbol{u} && \text{in } L^{p'}(0,T;(W^{2,2}_{0,\mathrm{div}})^*), && (4.54)\\
&\boldsymbol{u}^N \rightharpoonup^* \boldsymbol{u} && \text{in } L^{\infty}(0,T;L^2_{0,\mathrm{div}}), && (4.55)\\
&\boldsymbol{u}^N \rightharpoonup \boldsymbol{u} && \text{in } L^{p}(0,T;V), && (4.56)\\
&\boldsymbol{u}^N \to \boldsymbol{u} && \text{in } L^{r}(0,T;L^r) \text{ for } r \in [1,2p), && (4.57)\\
&\boldsymbol{\mathcal{S}}^N \rightharpoonup \boldsymbol{\mathcal{S}} && \text{in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^{2\times 2}_{sym})). && (4.58)\\
&\vartheta^N \rightharpoonup \vartheta && \text{in } L^{s}(0,T;L^s) \text{ for } s \in [1,2), && (4.59)\\
&\nabla\vartheta^N \rightharpoonup \nabla\vartheta && \text{in } L^{t}(0,T;L^t) \text{ for } t \in [1,4/3), && (4.60)\\
&(\vartheta^N)^\alpha \rightharpoonup (\vartheta)^\alpha && \text{in } L^{2}(0,T;W^{1,2}) \text{ for } \alpha \in (0,1/2), && (4.61)
\end{aligned}$$

where convergence (4.57) is obtained by the same procedure as in (4.23).

We use Lemma 3 to show stronger convergence results for ϑ^N. We have

$$\{\vartheta^N \in L^r(0,T,W^{1,r}(\Omega)); \partial_t\vartheta^N \in L^1\left(0,T;\left(W^{1,5}_0(\Omega)\right)^*\right)\} \hookrightarrow\hookrightarrow L^r(0,T,L^{\frac{2r}{2-r}}),$$

for $r \in \left[1,\frac{4}{3}\right)$. Hence for non-relabeled subsequences, we have

$$\begin{aligned}
&\vartheta^N \to \vartheta && \text{in } L^r(0,T,L^{\frac{2r}{2-r}}) \text{ for } r\in[1,4/3), && (4.62)\\
&\vartheta^N \to \vartheta && \text{almost everywhere in } Q. && (4.63)
\end{aligned}$$

If we combine (4.59) with (4.63) and use Vitali convergence theorem, we have

$$\vartheta^N \to \vartheta \qquad \text{in } L^s(0,T,L^s) \text{ for } s \in [1,2). \tag{4.64}$$

Additionally, we have

$$\vartheta \in L^\infty(0,T;L^1) \tag{4.65}$$

by (4.46) and (4.64).

We now test equations (4.31) by $\varphi \in \mathcal{D}(0,T)$ and equation (4.33) by $\psi \in \mathcal{D}((-\infty,T)\times\Omega)$, and we obtain the following equalities

$$\begin{aligned}
&\int_0^T \left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{w}_j \right\rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \, dt + \int_Q \boldsymbol{\mathcal{S}}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{w}_j \varphi \, d(t,x) = \\
&= \int_Q (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla(\boldsymbol{w}_j)\varphi \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \varphi \boldsymbol{w}_j\rangle_V \, dt, \forall j \in \{0,\dots,N\}.
\end{aligned} \tag{4.66}$$

$$\begin{aligned}
&\int_0^T \left\langle \partial_t \vartheta^N, \psi \right\rangle_{W^{1,2}_0} dt + \int_Q \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi \, d(t,x) = \\
&= \int_Q \boldsymbol{\mathcal{S}}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N \psi \, d(t,x) + \int_Q \vartheta^N \boldsymbol{u}^N \cdot \nabla\psi \, d(t,x).
\end{aligned} \tag{4.67}$$

Taking the limit $N \to \infty$ in (4.66), using convergences (4.54)-(4.58) we have

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{w}_j \rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \, dt + \int_0^T \int_\Omega \boldsymbol{\mathcal{S}} : D\boldsymbol{w}_j \varphi \, dx \, dt = \\ = \int_0^T \int_\Omega \varphi(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{w}_j \, dx \, dt + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle_V \varphi \, dt,\end{aligned} \tag{4.68}$$

for all $j \in \mathbb{N}$, and all $\varphi \in \mathcal{D}(0,T)$. The convergence of the convective term was managed similarly as in (4.25). Additionally, linear hull of $\{\boldsymbol{w}_k\}$ is dense in $W^{2,2}_{0,\mathrm{div}}(\Omega)$. Hence for any $\boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)$ we have a sequence

$$\boldsymbol{\psi}^M \in span\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_M\}$$

such that $\boldsymbol{\psi}^M \to \boldsymbol{\psi}$ in $W^{2,2}_{0,\mathrm{div}}(\Omega)$. We can thus take a limit passage in (4.68) and conclude

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{\psi} \rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \, dt + \int_Q \varphi \boldsymbol{\mathcal{S}} : D\boldsymbol{\psi} \, d(t,x) = \\ = \int_Q \varphi(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_V \varphi \, dt, \forall \boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega).\end{aligned} \tag{4.69}$$

Finally, we want equation (4.69) to hold for all $\boldsymbol{w} \in L^p(0,T;V)$ instead of the products $\varphi\boldsymbol{\psi}$, where $\varphi \in \mathcal{D}(0,T)$ and $\boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)$. For that, we need to formulate an auxiliary lemma

Lemma 10. *For any function $\boldsymbol{w} \in L^p(0,T,V)$, there exist sequences $\{\varphi^n\}_{n=1}^\infty$ and $\{\boldsymbol{\psi}^n\}_{n=1}^\infty$ such that $\varphi^n \in \mathcal{D}(0,T)$, $\boldsymbol{\psi}^n \in W^{2,2}_{0,\mathrm{div}}(\Omega)$, and $\sum_{i=1}^n \boldsymbol{\psi}^i \varphi^i$ converges to $\boldsymbol{w}$ in $L^p(0,T;V)$-norm as $n \to \infty$.*

Proof. Pick $\varepsilon > 0$, by part (iii) of Theorem 5.2.8. in [3], there exists $\tilde{\boldsymbol{w}} \in C([0,T];V)$ such that

$$\|\boldsymbol{w} - \tilde{\boldsymbol{w}}\|_{L^p(V)} < \frac{\varepsilon}{3}.$$

By continuity of $\tilde{\boldsymbol{w}}$, for any $t \in [0,T]$ there exists $\delta_t > 0$ such that for all $\tau \in I_t := (t - \delta_t, t + \delta_t) \cap [0,T]$ it holds

$$\|\tilde{\boldsymbol{w}}(\tau) - \tilde{\boldsymbol{w}}(t)\|_V < \frac{\varepsilon}{3T^{\frac{1}{p}}}.$$

Since $[0,T]$ is a compact, we can find a finite covering $\{\tilde{I}_{t^i}\}_{i=1}^m$ of $[0,T]$. By taking $I_{t^i} := \tilde{I}_{t^i} \setminus \overline{\tilde{I}_{t^{i+1}}}$ for $i \in \{1, \dots, m-1\}$ and $I_{t^m} := \tilde{I}_{t^m}$ we obtain pairwise disjoint covering of $[0,T]$ up to a null set. Then for $\tau \in \bigcup_{i=1}^m I_{t^i}$ we have

$$\|\tilde{\boldsymbol{w}}(\tau) - \sum_{i=1}^m \chi_{I_{t^i}}(\tau)\tilde{\boldsymbol{w}}(t^i)\|_V < \frac{\varepsilon}{3T^{\frac{1}{p}}},$$

where for all $i \in \{1, \dots, m\}$ it holds $\chi_{I_{t^i}} \in L^p(0,T)$ and $\tilde{\boldsymbol{w}}(t^i) \in V(\Omega)$. For every $i \in \{1, \dots, m\}$ there exists $\boldsymbol{\psi}^i \in C_0^\infty(\Omega) \subset W^{2,2}_{0,\mathrm{div}}(\Omega)$ and $\varphi^i \in \mathcal{D}(0,T)$ such that

$$\|\chi_{I_{t^i}} - \varphi^i\|_{L^p(0,T)} < \frac{\varepsilon}{6m \max_{i \in \{1,\dots,m\}} \|\tilde{\boldsymbol{w}}(t^i)\|_{V(\Omega)}},$$

$$\|\tilde{\boldsymbol{w}}(t^i) - \boldsymbol{\psi}^i\|_{V(\Omega)} < \frac{\varepsilon}{6m \max_{i \in \{1,\dots,m\}} \|\varphi^i\|_{L^p(0,T)}}$$

by the density of the smooth functions. Thus

$$\|\sum_{i=1}^{m} \chi_{I_{ti}} \tilde{\boldsymbol{w}}(t^i) - \sum_{i=1}^{m} \varphi^i \boldsymbol{\psi}^i\|_{L^p(V)} \leq$$
$$\leq m \max_{i \in \{1,\dots,m\}} \left(\|\chi_{I_{ti}} - \varphi^i\|_{L^p(0,T)} \|\tilde{\boldsymbol{w}}(t^i)\|_{V(\Omega)} + \|\varphi^i\|_{L^p(0,T)} \|\tilde{\boldsymbol{w}}(t^i) - \boldsymbol{\psi}^i\|_{V(\Omega)} \right) < \frac{\varepsilon}{3},$$

which implies

$$\|\boldsymbol{w} - \sum_{i=1}^{m} \varphi^i \boldsymbol{\psi}^i\|_{L^p(V)} < \frac{\varepsilon}{3} + \|\tilde{\boldsymbol{w}} - \sum_{i=1}^{m} \chi_{I_{ti}} \tilde{\boldsymbol{w}}(t^i)\|_{L^p(V)} + \frac{\varepsilon}{3} < \varepsilon.$$

□

We will now show that the time derivative of $\boldsymbol{u}$ has better regularity. More precisely, we will show $\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*)$. We know that by the definition of the weak time derivative

$$\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{\psi} \rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \, dt = -\int_Q \boldsymbol{u} \partial_t(\boldsymbol{\psi}\varphi) \, d(t,x).$$

We thus have

$$-\int_Q \boldsymbol{u} \partial_t(\boldsymbol{\psi}\varphi) \, d(t,x) = \mathcal{F}(\boldsymbol{\psi}\varphi), \qquad \varphi \in \mathcal{D}(0,T), \boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega) \tag{4.70}$$

where

$$\mathcal{F}(\boldsymbol{\psi}\varphi) := \int_Q \varphi(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} - \varphi \boldsymbol{\mathcal{S}} : D\boldsymbol{\psi} \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \varphi \boldsymbol{\psi} \rangle_V \, dt.$$

Furthermore, $\mathcal{F}$ is bounded for all $\boldsymbol{w} \in L^p(0,T;V)$, since

$$|\mathcal{F}(\boldsymbol{w})| \leq \int_Q |\boldsymbol{u}|^2 |\nabla \boldsymbol{w}| + |\boldsymbol{\mathcal{S}}||D\boldsymbol{w}| \, d(t,x) + \int_0^T |\langle \boldsymbol{f}, \boldsymbol{w} \rangle_V| \, dt \leq$$
$$\leq \|\boldsymbol{u}\|^2_{L^{2p'}(Q)} \|\nabla \boldsymbol{w}\|_{L^p(Q)} + \|\boldsymbol{\mathcal{S}}\|_{L^{p'}(Q)} \|D\boldsymbol{w}\|_{L^p(Q)} + \|\boldsymbol{f}\|_{L^{p'}(V^*)} \|\boldsymbol{w}\|_{L^p(V)} \leq C.$$

The last estimate is true for $p > 2$ by (4.57), (4.58), and (3.1). For $p = 2 = p'$ we need to proceed more carefully with the term $\|\boldsymbol{u}\|_{L^{2p'}(Q)}$. From Lemma 6 we have

$$\int_0^T \|\boldsymbol{u}\|_4^4 \, dt \leq \int_0^T \|\boldsymbol{u}\|_{1,2}^2 \|\boldsymbol{u}\|_2^2 \, dt \leq \|\boldsymbol{u}\|_{L^2(V)} \|\boldsymbol{u}\|^2_{L^\infty(L^2)},$$

which is bounded by (4.55) and (4.56). We hence have $\mathcal{F} \in L^{p'}(0,T;V^*)$. Since the set *span*$\{\varphi\boldsymbol{\psi}; \varphi \in \mathcal{D}(0,T), \boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)\}$ is dense in $L^p(0,T;V)$ by Lemma 10, there exists a uniquely defined extension of $\partial_t \boldsymbol{u}$ (denoted again $\partial_t \boldsymbol{u}$) such that $\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*)$. This extension is defined by

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{w} \rangle_{L^p(V)} = \mathcal{F}(\boldsymbol{w}), \qquad \boldsymbol{w} \in L^p(0,T;V).$$

Hence, by Lemma 10 we can conclude that for every $\boldsymbol{w} \in L^p(0,T,V)$ it holds

$$\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{w} \rangle_V \, dt + \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{w} \, d(t,x) = = \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{w} \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w} \rangle_V \, dt. \tag{4.71}$$

Additionally, by Lemma 7, we have $\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}})$.

Now, we only need to identify $\boldsymbol{\mathcal{S}}$ with $\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$ by using the Minty trick. Recall $\boldsymbol{\mathcal{S}}^N := \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}^N)$. From the first property in (8) we have that for all $\tilde{D} \in L^p(Q; \mathbb{R}^{2\times 2}_{sym})$ it holds

$$
\begin{aligned}
0 \le \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D})) : (D\boldsymbol{u}^N - \tilde{D})\, d(t,x) = \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\, d(t,x) \\
- \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D}) : (D\boldsymbol{u}^N - \tilde{D})\, d(t,x) - \int_Q \boldsymbol{\mathcal{S}}^N : \tilde{D}\, d(t,x) = \\
= - \int_0^T \left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{u}^N \right\rangle_V dt + \int_Q (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{u}^N\, d(t,x) \\
+ \int_0^T \left\langle \boldsymbol{f}, \boldsymbol{u}^N \right\rangle_V dt - \int_Q \boldsymbol{\mathcal{S}}^N : \tilde{D} + \boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D}) : (D\boldsymbol{u}^N - \tilde{D})\, d(t,x),
\end{aligned}
\tag{4.72}
$$

where the last equality is obtained from (4.31) by the method shown at the beginning of Section 4.2. We recall that

$$
\int_Q (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{u}^N\, d(t,x) = 0 = \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{u}\, d(t,x).
$$

We want to take the limit of (4.72) as $N \to \infty$. The term $\int_0^T \left\langle \boldsymbol{f}, \boldsymbol{u}^N \right\rangle_V dt$ converges by (4.56), and $\int_Q \boldsymbol{\mathcal{S}}^N : \tilde{D}\, d(t,x)$ by (4.58). Furthermore,

$$
\int_Q \boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D}) : (D\boldsymbol{u}^N - \tilde{D})\, d(t,x) \to \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta, \tilde{D}) : (D\boldsymbol{u} - \tilde{D})\, d(t,x)
$$

since $D\boldsymbol{u}^N \rightharpoonup D\boldsymbol{u}$ in $L^p(Q)$ by (4.56), and $\boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D}) \to \boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D})$ in $L^{p'}(Q)$ by (4.63), continuity of $\boldsymbol{\mathcal{S}}^*$, the fact that $|\boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D})| + |\boldsymbol{\mathcal{S}}^*(\vartheta^N, \tilde{D})| < C(1 + \tilde{D}^{p-1}) \in L^{p'}(Q)$, and the Lebesgue Dominated Convergence Theorem. Lastly,

$$
\begin{aligned}
\liminf_{N\to\infty} \int_0^T \left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{u}^N \right\rangle_V dt = \liminf_{N\to\infty} \frac{1}{2} \left(\|\boldsymbol{u}^N(T)\|_2^2 - \|\boldsymbol{u}_0^N\|_2^2 \right) = \\
= \liminf_{N\to\infty} \frac{1}{2} \|\boldsymbol{u}^N(T)\|_2^2 - \frac{1}{2} \|\boldsymbol{u}_0\|_2^2
\end{aligned}
\tag{4.73}
$$

by (4.5). We need to show that $\boldsymbol{u}^N(T) \rightharpoonup \boldsymbol{u}(T)$ in $L^2(\Omega)$. Let us take $\boldsymbol{\varphi} \in \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N\}$ and a sequence $\{\eta_k\}_{k=1}^\infty$

$$
\eta_k = \begin{cases} kt & \text{on } [0, \frac{1}{k}], \\ 1 & \text{on } [\frac{1}{k}, T - \frac{1}{k}], \\ -k(t-T) & \text{on } [T - \frac{1}{k}, T]. \end{cases}
$$

Plugging $\boldsymbol{\varphi}\eta_k$ in (4.71) and integrating by parts, we have

$$
\begin{aligned}
- \fint_0^{\frac{1}{k}} \int_\Omega \boldsymbol{u}\boldsymbol{\varphi}\, dx\, dt + \fint_{T-\frac{1}{k}}^T \int_\Omega \boldsymbol{u}\boldsymbol{\varphi}\, dx\, dt + \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{\varphi}\eta_k\, d(t,x) = \\
= \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi}\eta_k\, d(t,x) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_V \eta_k\, dt.
\end{aligned}
$$

Letting $k \to \infty$ yields into

$$\begin{aligned} -\int_\Omega \boldsymbol{u}(0)\boldsymbol{\varphi}\, dx + \int_\Omega \boldsymbol{u}(T)\boldsymbol{\varphi}\, dx + \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{\varphi}\, d(t,x) = \\ = \int_Q (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{\varphi}\, d(t,x) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_V \, dt, \end{aligned} \tag{4.74}$$

since $\boldsymbol{u} \in C([0,T], L^2_{0,\mathrm{div}})$. Plugging $\boldsymbol{\varphi}\eta_k$ in (4.31), integrating by parts, and letting $k \to \infty$ results in

$$\begin{aligned} -\int_\Omega \boldsymbol{u}^N(0)\boldsymbol{\varphi}\, dx + \int_\Omega \boldsymbol{u}^N(T)\boldsymbol{\varphi}\, dx + \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{\varphi}\, d(t,x) = \\ = \int_Q (\boldsymbol{u}^N\otimes\boldsymbol{u}^N) : \nabla\boldsymbol{\varphi}^N\, d(t,x) + \int_0^T \left\langle \boldsymbol{f}, \boldsymbol{\varphi}^N\right\rangle_V \, dt, \end{aligned} \tag{4.75}$$

because $\boldsymbol{u}^N \in C([0,T], L^2_{0,\mathrm{div}})$. Passing to the limit $N \to \infty$ in (4.75) and comparing the result with (4.74) yields into $\lim_{N\to\infty} \int_\Omega \boldsymbol{u}^N(T)\boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{u}(T)\boldsymbol{\varphi}\, dx$ for any $\boldsymbol{\varphi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)$. However, since $\boldsymbol{u}(t) \in L^2_{0,\mathrm{div}}(\Omega)$ for all $t \in [0,T]$, we can conclude

$$\lim_{N\to\infty} \int_\Omega \boldsymbol{u}^N(T)\boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{u}(T)\boldsymbol{\varphi}\, dx \qquad \text{for all } \boldsymbol{\varphi} \in L^2(\Omega). \tag{4.76}$$

We can now use (4.76) together with the weak lower semicontinuity of the L^2-norm in (4.73) to estimate

$$\liminf_{N\to\infty} \int_0^T \left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{u}^N\right\rangle_V \, dt \geq \frac{1}{2}\left(\|\boldsymbol{u}(T)\|_2^2 - \|\boldsymbol{u}_0\|_2^2\right) = \int_0^T \langle \partial_t\boldsymbol{u}, \boldsymbol{u}\rangle_V \, dt.$$

We can hence take $N \to \infty$ in (4.72) to obtain

$$\begin{aligned} 0 \leq &-\int_0^T \langle \partial_t\boldsymbol{u}, \boldsymbol{u}\rangle_V \, dt + \int_Q (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{u}\, d(t,x) \\ &+ \int_0^T \langle \boldsymbol{f}, \boldsymbol{u}\rangle_V \, dt - \int_Q \boldsymbol{\mathcal{S}} : \tilde{D}\, d(t,x) - \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta, \tilde{D}) : (D\boldsymbol{u} - \tilde{D})\, d(t,x) = \\ &\overset{(4.71)}{=} \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, \tilde{D})) : (D\boldsymbol{u} - \tilde{D})\, d(t,x). \end{aligned} \tag{4.77}$$

In the last equality, we used that $\boldsymbol{u} \in L^p(0,T;V)$ and it is thus an admissible test function in (4.71).

We now take $\tilde{D} := D\boldsymbol{u} - \varepsilon W$ for some $W \in L^p(Q; \mathbb{R}^{2\times 2}_{sym})$ and $\varepsilon \in (0,1)$. Hence, we have

$$\begin{aligned} &0 \leq \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u} - \varepsilon W)) : \varepsilon W\, d(t,x), \text{ implying} \\ &0 \leq \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u} - \varepsilon W)) : W\, d(t,x) \overset{\varepsilon\to 0+}{\to} \\ &\overset{\varepsilon\to 0+}{\to} \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : W\, d(t,x), \text{ since } \boldsymbol{\mathcal{S}}^* \text{ is continuous.} \end{aligned}$$

Taking the same estimates for $-W$, we obtain

$$0 = \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : W \, d(t,x).$$

This holds for any $W \in L^p(Q; \mathbb{R}^{2\times 2}_{sym})$, in particular for all $W \in C_0^\infty(Q; \mathbb{R}^{2\times 2}_{sym})$, and so by the Fundamental Lemma of Calculus of Variations we have $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$ almost everywhere in Q.

We thus have $\boldsymbol{u} \in L^p(0,T;V)$ such that $\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*)$

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{w} \rangle_V \, dt + \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) : D\boldsymbol{w} \, d(t,x) = \\ = \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{w} \, d(t,x) + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w} \rangle_V \, dt, \forall \boldsymbol{w} \in L^p(0,T;V).\end{aligned} \tag{4.78}$$

Furthermore, it holds $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in $L^2(\Omega)$. This can be shown in the same way as the equality $\vartheta^N(0) = \vartheta_0^N$ by the end of Section 4.3. We have thus found $\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}}) \cap L^p(0,T,V)$ such that $\partial_t \boldsymbol{u} \in L^{p'}(0,T,V^*)$ and $\boldsymbol{u}$ fulfills the *Momentum equation* (3.10).

Let us now focus on the convergence of (4.67). By the definition of the weak time-derivative term, we have

$$\begin{aligned}\int_0^T \left\langle \partial_t \vartheta^N, \psi \right\rangle_{W_0^{1,2}} dt = -\int_Q \vartheta^N \partial_t \psi \, d(t,x) - \int_\Omega \vartheta_0^N \psi(0) \, dx \\ \overset{(4.59),(4.5)}{\to} -\int_Q \vartheta \partial_t \psi \, d(t,x) - \int_\Omega \vartheta_0 \psi(0) \, dx.\end{aligned} \tag{4.79}$$

For the convective term, it holds

$$\begin{aligned}&\int_Q \left| \vartheta^N \boldsymbol{u}^N \cdot \nabla \psi - \vartheta \boldsymbol{u} \cdot \nabla \psi \right| d(t,x) \leq \\ &\leq \|\nabla \psi\|_{L^\infty(Q)} \left(\int_Q \left| \vartheta^N - \vartheta \right| \left| \boldsymbol{u}^N \right| d(t,x) + \int_Q |\vartheta| \left| \boldsymbol{u}^N - \boldsymbol{u} \right| d(t,x) \right) \leq \\ &\leq C \left(\|\vartheta^N - \vartheta\|_{L^{\frac{9}{5}}(Q)} \|\boldsymbol{u}^N\|_{L^{\frac{9}{4}}(Q)} + \|\vartheta\|_{L^{\frac{9}{5}}(Q)} \|\boldsymbol{u}^N - \boldsymbol{u}\|_{L^{\frac{9}{4}}(Q)} \right) \to 0.\end{aligned} \tag{4.80}$$

Moreover, we have the convergence

$$\int_Q \nabla \vartheta^N \cdot \kappa(\vartheta^N) \nabla \psi \, d(t,x) \to \int_Q \nabla \vartheta \cdot \kappa(\vartheta) \nabla \psi \, d(t,x), \tag{4.81}$$

since $\nabla \vartheta^N \rightharpoonup \nabla \vartheta$ in $L^{\frac{5}{4}}(Q)$, and $\kappa(\vartheta^N) \nabla \psi \to \kappa(\vartheta) \nabla \psi$ in $L^5(Q)$ due to the convergence $\kappa(\vartheta^N) \to \kappa(\vartheta)$ almost everywhere, the fact that $\kappa \leq \overline{\kappa}$ on $(0,\infty)$, and the Lebesgue Dominated Convergence Theorem.

We will now show the convergence of

$$\int_Q \boldsymbol{\mathcal{S}}^* \left(\vartheta^N, D\boldsymbol{u}^N \right) : D\boldsymbol{u}^N \psi \, d(t,x).$$

Let us substitute $\tilde{D} := D\boldsymbol{u}$ in (4.72) and (4.77). This same procedure as in the Minty trick above yields to

$$0 \overset{(4.72)}{\leq} \lim_{N\to\infty} \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u})\, d(t,x) \leq$$
$$\leq \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : (D\boldsymbol{u} - D\boldsymbol{u}) = 0.$$

Hence

$$(\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) \to 0 \qquad \text{in } L^1(Q). \tag{4.82}$$

Additionally,

$$\boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) \rightharpoonup 0 \qquad \text{in } L^1(Q), \tag{4.83}$$

since for $w \in L^\infty(Q)$ we have

$$\lim_{N\to\infty} \left| \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x) \right| \leq$$
$$\leq \lim_{N\to\infty} \left| \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x) \right|$$
$$+ \lim_{N\to\infty} \left| \int_Q (\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x) \right|.$$

The first integral converges to zero, since $D\boldsymbol{u}^N \rightharpoonup D\boldsymbol{u}$ in $L^p(Q)$ and

$$\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) w \in L^{p'}(Q).$$

For the second integral, we estimate

$$\int_Q (\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x) \leq$$
$$\leq \|\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})\|_{L^{p'}(Q)} \left(\|D\boldsymbol{u}^N\|_{L^p(Q)} + \|D\boldsymbol{u}^N\|_{L^p(Q)} \right) \|w\|_{L^\infty(Q)} \to 0$$

due to the convergence (4.63), the Lebesgue Dominated Convergence Theorem, and the $L^{p'}$-growth and continuity of $\boldsymbol{\mathcal{S}}^*$.

For $w \in L^\infty(Q)$ we thus have

$$\int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N w\, d(t,x) = \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x)$$
$$+ \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u} w\, d(t,x) + \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w\, d(t,x) \overset{N\to\infty}{\to}$$
$$\to \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{u} w\, d(t,x),$$

where we have used convergences (4.82), (4.58), and (4.83). Hence

$$\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \rightharpoonup \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \qquad \text{in } L^1(Q). \tag{4.84}$$

We can now take the limit in (4.67) and obtain

$$
\begin{aligned}
&- \int_Q \vartheta \partial_t \psi \, d(t,x) + \int_Q \kappa(\vartheta) \nabla \vartheta \cdot \nabla \psi \, d(t,x) = \\
&= \int_Q \boldsymbol{S}^* (\vartheta, D\boldsymbol{u}) : D\boldsymbol{u} \psi \, d(t,x) + \int_Q \vartheta \boldsymbol{u} \cdot \nabla \psi \, d(t,x) + \int_\Omega \vartheta_0 \psi(0) \, dx
\end{aligned}
\tag{4.85}
$$

for all $\psi \in \mathcal{D}((-\infty, T) \times \Omega)$.

We have thus found $\vartheta \in L^\infty(0,T;L^1) \cap L^r(Q)$, for $r \in [1,2)$, such that

$$
\begin{aligned}
&\vartheta - \hat{\vartheta} \in L^s(0,T;W_0^{1,s}) \text{ for } s \in [1, 4/3)\,, \\
&(\vartheta)^\alpha \in L^2(0,T;W^{1,2}) \text{ for } \alpha \in (0, 1/2),
\end{aligned}
$$

and ϑ solves the *Internal energy balance* (3.11). Furthermore, minimal principle $\vartheta^N \geq \mu$, for $N \in \mathbb{N}$, together with (4.63) implies $\vartheta \geq \mu$ almost everywhere.

We want to show that the initial condition is attained, more specifically that there exists $S \subset (0,T)$ such that $(0,T) \setminus S$ is of measure zero and

$$
\vartheta(t) \to \vartheta_0 \qquad \text{in } L^1(\Omega) \text{ as } S \ni t \to 0+. \tag{4.86}
$$

We note that by (4.65) we have $\sqrt{\theta(t)} \in L^2(\Omega)$ for all $t \in (0,T)$. From (4.64) we know that there exists a set $S \subset (0,T)$, such that $|(0,T) \setminus S| = 0$ and for all $t \in S$ it holds

$$
\vartheta^N(t) \to \vartheta(t) \qquad \text{in } L^s(\Omega), s < 2, \text{ as } N \to \infty. \tag{4.87}
$$

Let us now show that

$$
\liminf_{S \ni t \to 0+} \int_\Omega \sqrt{\vartheta(t)} \varphi \, dx \geq \int_\Omega \sqrt{\vartheta_0} \varphi \, dx \qquad \text{for all } \varphi \in L^2(\Omega),\ \varphi \geq 0. \tag{4.88}
$$

We test the equation (4.33) by $\frac{\varphi}{\sqrt{\vartheta^N}}$, where $0 \leq \varphi \in C_0^\infty(\Omega)$, and integrate until the time $t \in S$. Note, that it is an admissible test function for (4.33) since $\frac{\varphi}{\sqrt{\vartheta^N(t)}} \in W_0^{1,2}(\Omega)$ for a.a. $t \in (0,T)$. Additionally, $\frac{\varphi}{\sqrt{\vartheta^N}} \in L^\infty(Q)$. Since $g(\vartheta^N) = 2\sqrt{\vartheta^N}$ fulfills the assumptions if Lemma 2, we can identify the duality $\left\langle \partial_t \vartheta^N, \frac{1}{\sqrt{\vartheta^N}} \right\rangle$. We obtain

$$
\begin{aligned}
&\int_0^t \partial_t \int_\Omega 2\sqrt{\vartheta^N} \varphi \, dx \, d\tau + \int_0^t \int_\Omega \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \frac{\varphi}{\sqrt{\vartheta^N}} \, dx \, d\tau = \\
&= \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \frac{\varphi}{\sqrt{\vartheta^N}} \, dx \, d\tau + \int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \frac{\varphi}{\sqrt{\vartheta^N}} \, dx \, d\tau,
\end{aligned}
$$

which can be rewritten into

$$
\begin{aligned}
&2 \int_\Omega \sqrt{\vartheta^N(t)} \varphi - \sqrt{\vartheta_0^N} \varphi \, dx + \int_0^t \int_\Omega \left(\kappa(\vartheta^N) \frac{\nabla \vartheta^N}{\sqrt{\vartheta^N}} - 2\sqrt{\vartheta^N} \boldsymbol{u}^N \right) \cdot \nabla \varphi \, dx \, d\tau = \\
&= \int_0^t \int_\Omega \kappa(\vartheta^N) \varphi \frac{|\nabla \vartheta^N|^2}{2(\vartheta^N)^{3/2}} \, dx \, d\tau + \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \frac{\varphi}{\sqrt{\vartheta^N}} \, dx \, d\tau \geq 0,
\end{aligned}
\tag{4.89}
$$

since by the zero divergence of $\boldsymbol{u}^N$ it holds

$$\int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \frac{\varphi}{\sqrt{\vartheta^N}}\, dx \overset{\text{IBP}}{=} -\int_\Omega \varphi \boldsymbol{u}^N \cdot \frac{\nabla \vartheta^N}{\sqrt{\vartheta^N}}\, dx \overset{\text{IBP}}{=} 2\int_\Omega \nabla \varphi \cdot \boldsymbol{u}^N \sqrt{\vartheta^N}\, dx.$$

We can now let $N \to \infty$ in the inequality (4.89). Using convergences (4.87), (4.5), (4.63), (4.61) and (4.57) yields to

$$2\int_\Omega \sqrt{\vartheta(t)}\varphi - \sqrt{\vartheta_0}\varphi\, dx + \int_0^t \int_\Omega \left(\kappa(\vartheta)\frac{\nabla \vartheta}{\sqrt{\vartheta}} - 2\sqrt{\vartheta}\boldsymbol{u}\right) \cdot \nabla \varphi\, dx\, d\tau \geq 0. \tag{4.90}$$

We can pass to the limit inferior $S \ni t \to 0+$ in (4.90). The term

$$\int_\Omega \left(\kappa(\vartheta)\frac{\nabla \vartheta}{\sqrt{\vartheta}} - 2\sqrt{\vartheta}\boldsymbol{u}\right) \cdot \nabla \varphi\, dx \in L^{5/4}(0,T),$$

hence

$$\left|\int_0^t \int_\Omega \left(\kappa(\vartheta)\frac{\nabla \vartheta}{\sqrt{\vartheta}} - 2\sqrt{\vartheta}\boldsymbol{u}\right) \cdot \nabla \varphi\, dx\, d\tau\right| \leq \left\|\left(\kappa(\vartheta)\frac{\nabla \vartheta}{\sqrt{\vartheta}} - 2\sqrt{\vartheta}\boldsymbol{u}\right) \cdot \nabla \varphi\right\|_{L^{5/4}(L^1)} |t|^{1/5}$$

converges to zero. We thus obtain

$$\liminf_{S \in t \to 0+} \int_\Omega \sqrt{\vartheta(t)}\varphi\, dx \geq \int_\Omega \sqrt{\vartheta_0}\varphi\, dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

Since $\sqrt{\vartheta} \in L^\infty(0,T;L^2)$ we can extend the result for all $\varphi \in L^2(\Omega)$ by the density of smooth functions. Hence (4.88) holds.

Next, we would like to obtain the following result

$$\lim_{S \ni t \to 0+} \int_\Omega \vartheta(t)\varphi\, dx = \int_\Omega \vartheta_0 \varphi \qquad \text{for } \varphi \in C_0^\infty(\Omega),\ 0 \leq \varphi \leq 1. \tag{4.91}$$

We test the equation (4.33) by $\varphi \in C_0^\infty(\Omega)$, $0 \leq \varphi \leq 1$ and integrate until the time $t \in S$. By the definition of a weak derivative

$$\int_0^t \left\langle \partial_t \vartheta^N, \varphi \right\rangle_{W_0^{1,2}} d\tau = \int_\Omega \vartheta^N(t)\varphi\, dx - \int_\Omega \vartheta_0^N \varphi\, dx$$

we have

$$\begin{aligned} \int_\Omega \vartheta^N(t)\varphi\, dx &= -\int_0^t \int_\Omega \kappa(\vartheta^N)\nabla \vartheta^N \cdot \nabla \varphi\, dx\, d\tau + \int_\Omega \vartheta_0^N \varphi\, dx \\ &+ \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \varphi\, dx\, d\tau + \int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \varphi\, dx\, d\tau. \end{aligned} \tag{4.92}$$

We now want to pass to the limit $N \to \infty$ in (4.92). The right-hand side converges by (4.81), (4.5), (4.84), and (4.80). The left-hand side converges by (4.87). We thus have

$$\begin{aligned} \int_\Omega \vartheta(t)\varphi\, dx &= -\int_0^t \int_\Omega \kappa(\vartheta)\nabla \vartheta \cdot \nabla \varphi\, dx\, d\tau + \int_\Omega \vartheta_0 \varphi\, dx \\ &+ \int_0^t \int_\Omega \boldsymbol{S} : D\boldsymbol{u}\varphi\, dx\, d\tau + \int_0^t \int_\Omega \vartheta \boldsymbol{u} \cdot \nabla \varphi\, dx\, d\tau. \end{aligned} \tag{4.93}$$

We can now let $S \ni t \to 0+$ in (4.93). The term

$$\int_0^t \int_\Omega \boldsymbol{S} : D\boldsymbol{u}\varphi + \vartheta \boldsymbol{u} \cdot \nabla\varphi - \kappa(\vartheta)\nabla\vartheta \cdot \nabla\varphi \, dx \, d\tau$$

converges to 0 by the Lebesgue Dominated Convergence Theorem as

$$\int_\Omega \boldsymbol{S} : D\boldsymbol{u}\varphi + \vartheta \boldsymbol{u} \cdot \nabla\varphi - \kappa(\vartheta)\nabla\vartheta \cdot \nabla\varphi \, dx \in L^1(0,T).$$

We thus obtain the desired estimate (4.91).

For $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$ we can estimate

$$\begin{aligned} 0 \le \lim_{S \ni t \to 0+} \int_\Omega ([\sqrt{\vartheta(t)} - \sqrt{\vartheta_0}]\sqrt{\varphi})^2 \, dx \le \lim_{S \ni t \to 0+} \int_\Omega \vartheta(t)\varphi + \vartheta_0 \varphi \, dx \\ - \liminf_{S \ni t \to 0+} 2 \int_\Omega \sqrt{\vartheta(t)}\sqrt{\vartheta_0}\varphi \, dx \overset{(4.91),(4.88)}{\le} 0. \end{aligned}$$

For any compact $K \subset \Omega$ we can find $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$ such that $\varphi = 1$ on K. The inequality above hence implies

$$\sqrt{\vartheta(t)} \to \sqrt{\vartheta_0} \qquad \text{in } L^2(K) \text{ for any compact } K \subset \Omega.$$

This yields into

$$\int_\Omega \sqrt{\vartheta(t)}\varphi \, dx \to \int_\Omega \sqrt{\vartheta_0}\varphi \, dx \qquad \text{for } \varphi \in C_0^\infty(\Omega) \text{ as } S \ni t \to)+ \tag{4.94}$$

as such φ are compactly supported. For any $\tilde{\varphi} \in L^2(\Omega)$ there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ such that

$$\varphi_n \to \tilde{\varphi} \qquad \text{in } L^2(\Omega) \tag{4.95}$$

and $\varphi_n \in C_0^\infty(\Omega)$ by the density of smooth functions. For given $\varepsilon > 0$ we can find $n \in \mathbb{N}$ such that it holds

$$\left(\|\vartheta\|_{L^\infty(L^1)}^2 + \|\vartheta_0\|_1^2\right) \|\tilde{\varphi} - \varphi_n\|_2 < \varepsilon/2$$

by (4.95). For such $n \in \mathbb{N}$ we can find $\delta > 0$ such that for all $t \in S \cap (0,\delta)$ we have

$$\left| \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\vartheta_0}\right) \varphi_n \, dx \right| < \varepsilon/2$$

by (4.94). We can thus estimate

$$\begin{aligned} \left| \int_\Omega \sqrt{\vartheta(t)}\tilde{\varphi} \, dx - \int_\Omega \sqrt{\vartheta_0}\tilde{\varphi} \, dx \right| \le \left| \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\vartheta_0}\right) \varphi_n \, dx \right| \\ + \int_\Omega \sqrt{\vartheta(t)} \, |\tilde{\varphi} - \varphi_n| \, dx + \int_\Omega \sqrt{\vartheta_0} \, |\tilde{\varphi} - \varphi_n| \, dx \le \\ \le \left| \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\vartheta_0}\right) \varphi_n \, dx \right| + \left(\|\vartheta\|_{L^\infty(L^1)}^2 + \|\vartheta_0\|_1^2\right) \|\tilde{\varphi} - \varphi_n\|_2 < \varepsilon. \end{aligned}$$

Hence

$$\sqrt{\vartheta(t)} \rightharpoonup \sqrt{\vartheta_0} \qquad\qquad \text{in } L^2(\Omega) \text{ as } S \ni t \to 0+. \tag{4.96}$$

Let us now show that

$$\limsup_{S\ni t\to 0+} \int_\Omega \vartheta(t) - 2\sqrt{\vartheta(t)}\sqrt{\hat\vartheta}\,dx \le \int_\Omega \int_\Omega \vartheta_0 - 2\sqrt{\vartheta_0}\sqrt{\hat\vartheta}\,dx. \tag{4.97}$$

We test equation (4.33) by $\psi := 1 - \sqrt{\frac{\hat\vartheta}{\vartheta^N}}$ and integrate until the time $t \in S$. Note, that ψ is an admissible test function since it is zero at the boundary, $\psi(t) \in L^\infty(\Omega)$ for a.a. $t \in S$ by (3.2) and the minimum principle $\vartheta^N \ge \mu$, and

$$\nabla\psi(t) = \left(-\frac{\nabla\hat\vartheta}{2\sqrt{\hat\vartheta}\sqrt{\vartheta^N}} + \frac{\sqrt{\hat\vartheta}\nabla\vartheta^N}{2(\vartheta^N)^{3/2}} \right)(t) \in L^2(\Omega), \quad \text{for a.a. } t \in S.$$

Let us use Lemma 2 for $g(\vartheta^N,\hat\vartheta) = \vartheta^N - 2\sqrt{\vartheta^N}\sqrt{\hat\vartheta} + \hat\vartheta$. Function g is positive and $|\partial_1 g(s,\sigma)| + |\partial_1^2 g(s,\sigma)| + |\partial_2\partial_1 g(s,\sigma)| \le D$. We can thus use Lemma 2 to obtain

$$\begin{aligned}&\int_\Omega \left(\vartheta^N - 2\sqrt{\vartheta^N}\sqrt{\hat\vartheta}\right)(t) + \hat\vartheta\,dx + \int_0^t \int_\Omega \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi\,dx\,d\tau = \\ &= \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N\psi + \vartheta^N\boldsymbol{u}^N \cdot \nabla\psi\,dx\,d\tau + \int_\Omega \vartheta_0^N - 2\sqrt{\vartheta_0^N}\sqrt{\hat\vartheta} + \hat\vartheta\,dx.\end{aligned}$$

By the non-negativity of the term

$$\int_0^t \int_\Omega \frac{\kappa(\vartheta^N)\sqrt{\hat\vartheta}}{2(\vartheta^N)^{3/2}} |\nabla\vartheta^N|^2\,dx\,d\tau$$

we have

$$\begin{aligned}&\int_\Omega \left(\vartheta^N - 2\sqrt{\vartheta^N}\sqrt{\hat\vartheta}\right)(t) + \hat\vartheta\,dx \le \int_\Omega \vartheta_0^N - 2\sqrt{\vartheta_0^N}\sqrt{\hat\vartheta} + \hat\vartheta\,dx \\ &+ \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \left(1 - \sqrt{\frac{\hat\vartheta}{\vartheta^N}}\right) dx\,d\tau + \int_0^t \int_\Omega \frac{\kappa(\vartheta^N)\nabla\hat\vartheta}{2\sqrt{\hat\vartheta}\sqrt{\vartheta^N}} \cdot \nabla\vartheta^N\,dx\,d\tau \\ &- \int_0^t \int_\Omega \sqrt{\vartheta^N}\boldsymbol{u}^N \cdot \frac{\nabla\hat\vartheta}{2\sqrt{\hat\vartheta}}\,dx + \int_0^t \int_\Omega \boldsymbol{u}^N(\vartheta^N)^{1/4} \cdot \frac{\sqrt{\hat\vartheta}\nabla\vartheta^N}{2(\vartheta^N)^{3/4}}\,dx\,d\tau.\end{aligned} \tag{4.98}$$

We want to pass to the limit $N \to \infty$ in (4.98). The term on the left-hand side of (4.98) converges by (4.87). The first term on the right-hand side converges by (4.5). Convergence of the second term on the right-hand side holds due to (4.63), (4.84), Egorov's Theorem, and Dunford-Pettis Theorem. A more detailed explanation is provided at the beginning of Section 4.7 for convergence (4.107). Convergence of the third term can be shown analogously to (4.81). The fourth term converges by (4.64), (4.57), (3.2), and (3.3), which can be shown similarly as in (4.80). Finally, the last term converges by (4.57), (4.64), and (4.61). We thus obtain

$$\begin{aligned}&\int_\Omega \left(\vartheta - 2\sqrt{\vartheta}\sqrt{\hat\vartheta}\right)(t) + \hat\vartheta \, dx \le \int_\Omega \vartheta_0 - 2\sqrt{\vartheta_0}\sqrt{\hat\vartheta} + \hat\vartheta \, dx \\ &+ \int_0^t \int_\Omega \boldsymbol{S} : D\boldsymbol{u}\left(1 - \sqrt{\frac{\hat\vartheta}{\vartheta}}\right) dx\, d\tau + \int_0^t \int_\Omega \frac{\kappa(\vartheta)\nabla\hat\vartheta}{2\sqrt{\hat\vartheta}\sqrt{\vartheta}} \cdot \nabla\vartheta \, dx\, d\tau \\ &- \int_0^t \int_\Omega \sqrt{\vartheta}\boldsymbol{u} \cdot \frac{\nabla\hat\vartheta}{2\sqrt{\hat\vartheta}} \, dx + \int_0^t \int_\Omega \boldsymbol{u} \cdot \frac{\sqrt{\hat\vartheta}\nabla\vartheta}{2\sqrt{\vartheta}} \, dx\, d\tau.\end{aligned} \tag{4.99}$$

Let us take the limit $S \ni t \to 0+$ in (4.99). Since all the terms on the right-hand side of (4.99) except for the first one converge to zero by the Lebesgue Dominated Convergence Theorem we will obtain

$$\limsup_{S\ni t\to 0+} \int_\Omega \left(\vartheta - 2\sqrt{\vartheta}\sqrt{\hat\vartheta}\right)(t) + \hat\vartheta \, dx \le \int_\Omega \vartheta_0 - 2\sqrt{\vartheta_0}\sqrt{\hat\vartheta} + \hat\vartheta \, dx,$$

which implies (4.97), since $\hat\vartheta$ is independent of t.

The convergence (4.97) yields into

$$\begin{aligned}&\limsup_{S\ni t\to 0+} \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta}\right)^2 dx = \limsup_{S\ni t\to 0+} \int_\Omega \vartheta(t) - 2\sqrt{\vartheta(t)}\sqrt{\hat\vartheta} + \hat\vartheta \, dx \overset{(4.97)}{\le} \\ &\le \int_\Omega \int_\Omega \vartheta_0 - 2\sqrt{\vartheta_0}\sqrt{\hat\vartheta} + \hat\vartheta \, dx = \int_\Omega \left(\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\right)^2 dx.\end{aligned} \tag{4.100}$$

Using (4.96) we have

$$\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta} \rightharpoonup \sqrt{\vartheta_0} - \sqrt{\hat\vartheta} \qquad \text{in } L^2(\Omega) \text{ as } S \ni t \to 0+,$$

which by the weak lower semicontinuity of the L^2-norm implies

$$\liminf_{S\ni t\to 0+} \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta}\right)^2 dx \ge \int_\Omega \left(\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\right)^2 dx. \tag{4.101}$$

If we combine (4.100) and (4.101) we have

$$\|\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta}\|_2 \to \|\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\|_2. \tag{4.102}$$

From (4.102) we can deduce

$$\sqrt{\vartheta(t)} \to \sqrt{\vartheta_0} \qquad \text{in } L^2(\Omega) \text{ as } S \ni t \to 0+, \tag{4.103}$$

since we have

$$\begin{aligned}&\int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\vartheta_0}\right)^2 dx = \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta} - (\sqrt{\vartheta_0} - \sqrt{\hat\vartheta})\right)^2 dx = \\ &= \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\hat\vartheta}\right)^2 + \left(\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\right)^2 dx \\ &- 2\int_\Omega \sqrt{\vartheta(t)}\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\sqrt{\vartheta_0} - \sqrt{\vartheta(t)}\sqrt{\hat\vartheta} + \hat\vartheta \, dx \overset{(4.102),(4.96)}{\to} \\ &\to \int_\Omega 2\left(\sqrt{\vartheta_0} - \sqrt{\hat\vartheta}\right)^2 - 2\left(\vartheta_0 - 2\sqrt{\hat\vartheta}\sqrt{\vartheta_0} + \hat\vartheta\right) = 0.\end{aligned}$$

Convergence (4.86) is indeed true by

$$\int_\Omega |\vartheta(t)-\vartheta_0|\,dx \le \int_\Omega \left(\sqrt{\vartheta(t)}-\sqrt{\vartheta_0}\right)^2 + |2\sqrt{\vartheta(t)}\sqrt{\vartheta_0}-2\vartheta_0|\,dx \overset{(4.103),(4.96)}{\to} 0.$$

Hence, we have shown that the initial condition ϑ_0 is indeed attained.

It only remains to prove that $\eta := \log\vartheta$ fulfills the *Entropy equation.*

4.7 Entropy equation

We want to show equality (3.12) following closely the method used in the article [12] since the difference between the 2D and 3D case is negligible. However, we provide more structured and detailed steps than the ones in the article.

We test (4.33) over Q by $\psi := \frac{\varphi}{\vartheta^N}$ where $\varphi \in C_0^\infty((-\infty,T)\times\Omega)$ and denote $\eta^N := \log\vartheta^N$. We note that

$$\nabla\vartheta^N \cdot \nabla\frac{\varphi}{\vartheta^N} = \nabla\eta^N\cdot\nabla\varphi - \frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2}\varphi,$$

and

$$\int_\Omega \vartheta^N \boldsymbol{u}^N\cdot\nabla\frac{\varphi}{\vartheta^N}\,dx = \int_\Omega \boldsymbol{u}^N\cdot\nabla\varphi - \varphi\boldsymbol{u}^N\cdot\nabla\eta^N\,dx \overset{\text{IBP}}{=} \int_\Omega \eta\boldsymbol{u}^N\cdot\nabla\varphi\,dx.$$

Additionally, we have

$$\begin{aligned}\int_0^T \left\langle \partial_t\vartheta^N, \frac{\varphi}{\vartheta^N}\right\rangle_{W_0^{1,2}} dt &= \int_0^T \left\langle \partial_t\eta^N,\varphi\right\rangle_{W_0^{1,2}} dt = \\ &= -\int_Q \eta^N\partial_t\varphi\,d(t,x) - \int_\Omega \eta_0^N\varphi(0)\,dx,\end{aligned}$$

where $\eta_0^N := \log\vartheta_0^N$. We thus obtain

$$\begin{aligned}&-\int_Q \eta^N\partial_t\varphi\,d(t,x) - \int_Q \eta^N\boldsymbol{u}^N\cdot\nabla\varphi\,d(t,x) + \int_Q \kappa(\vartheta^N)\nabla\eta^N\cdot\nabla\varphi\,d(t,x)\\ &= \int_Q \frac{1}{\vartheta^N}\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\varphi\,d(t,x) + \int_Q \kappa(\vartheta^N)\frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2}\varphi\,d(t,x) + \int_\Omega \eta_0^N\varphi(0)\,dx.\end{aligned} \tag{4.104}$$

We want to let $N\to\infty$. By (4.64) we have

$$\eta^N \to \eta \qquad \text{in } L^q(0,T,L^q) \text{ for } q\in[1,\infty), \tag{4.105}$$

where $\eta := \log\vartheta$. Moreover, $\nabla\eta^N = \frac{\nabla\vartheta^N}{\vartheta^N} \in L^2(Q)$ by (4.48). We can thus find a converging subsequence

$$\nabla\eta^N \rightharpoonup \nabla\eta \qquad \text{in } L^2(0,T,L^2), \tag{4.106}$$

and so $\eta \in L^2(0,T;W^{1,2})\cap L^q(Q)$ for $q\in[1,\infty)$.

The left-hand side of (4.104) converges due to (4.105), (4.57), (4.106), (4.63), the boundedness of κ and the Lebesgue Dominated Convergence Theorem. Also, the limit passage works in the last term of (4.104) due to (4.5). We want to show the convergence

$$\int_Q \frac{1}{\vartheta^N} \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \varphi \, d(t,x) \to \int_Q \frac{1}{\vartheta} \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \varphi \, d(t,x). \tag{4.107}$$

Using Egorov's Theorem we obtain from (4.63) that for every $\varepsilon > 0$ there exists $Q_\varepsilon \subset Q$ such that $|Q \setminus Q_\varepsilon| \leq \varepsilon$ and

$$\frac{\varphi}{\vartheta^N} \to \frac{\varphi}{\vartheta} \qquad \text{uniformly on } Q_\varepsilon. \tag{4.108}$$

We have

$$\int_Q \frac{\varphi}{\vartheta^N} \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N - \frac{\varphi}{\vartheta} \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \, d(t,x) = \int_Q \frac{\varphi}{\vartheta} \left(\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N - \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \right) d(t,x)$$
$$+ \int_{Q \setminus Q_\varepsilon} \left(\frac{\varphi}{\vartheta^N} - \frac{\varphi}{\vartheta} \right) \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \, d(t,x) + \int_{Q_\varepsilon} \left(\frac{\varphi}{\vartheta^N} - \frac{\varphi}{\vartheta} \right) \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \, d(t,x).$$

The first term converges to zero by (4.84), the last term convergence is given by

$$\left| \int_{Q_\varepsilon} \left(\frac{\varphi}{\vartheta^N} - \frac{\varphi}{\vartheta} \right) \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \, d(t,x) \right| \leq \|\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\|_{L^1(Q)} \sup_{Q_\varepsilon} \left| \frac{\varphi}{\vartheta^N} - \frac{\varphi}{\vartheta} \right| \overset{(4.108)}{\to} 0.$$

Lastly, for the second term we have

$$\left| \int_{Q \setminus Q_\varepsilon} \left(\frac{\varphi}{\vartheta^N} - \frac{\varphi}{\vartheta} \right) \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \, d(t,x) \right| \leq 2 \frac{\|\varphi\|_{L^\infty(Q)}}{\mu} \int_{Q \setminus Q_\varepsilon} \left| \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \right| d(t,x).$$

Since $\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N$ converges weakly in L^1, we have by the Dunford-Pettis Theorem that it is a uniformly integrable sequence, thus for all N it holds

$$\int_{Q \setminus Q_\varepsilon} \left| \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \right| d(t,x) \leq C(\varepsilon),$$

where $C(\varepsilon) \to 0+$ as $|Q \setminus Q_\varepsilon| = \varepsilon \to 0+$. Thus (4.107) holds.

However, the second term on the right-hand side is problematic, since

$$\kappa(\vartheta^N) \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} = \kappa(\vartheta^N) |\nabla \eta^N|^2$$

is uniformly bounded only in $L^1(Q)$, hence we cannot apriori extract a weakly converging subsequence. We will thus need to find a different way to show the strong convergence in $L^1(Q)$.

We first need to show the convergence almost everywhere of $\nabla \vartheta^N$. To do so, we will introduce two functions ψ_1 and ψ_2 that are regular enough to be used as test functions in (4.33). We will find such ψ_1 and ψ_2 using auxiliary cut-off functions. For any $k > 0$ we define

$$\mathcal{T}_k : \mathbb{R} \to [-k, k], \qquad \mathcal{T}_k(z) := \operatorname{sign}(z) \min\{|z|, k\}. \tag{4.109}$$

Let $\mathcal{G}_k$ denote such function that

$$\mathcal{G}_k' = \mathcal{T}_k \text{ on } \mathbb{R}, \qquad \mathcal{G}_k(0) = 0.$$

Note that for all $z \in \mathbb{R}$ it holds $|\mathcal{G}_k(z)| \le k|z|$. For $\varepsilon \in (0,k)$, let $\mathcal{T}_{k,\varepsilon} \in C^2(\mathbb{R})$ denote the mollification of $\mathcal{T}_k$, which is given by a convolution with ω_ϵ. For $\mathcal{T}_{k,\varepsilon}$ it holds

$$\begin{aligned} &\mathcal{T}_{k,\varepsilon}(z) = \mathcal{T}_k(z) && \text{if } |z| \le k-\varepsilon \text{ or } |z| \ge k+\varepsilon, \\ &|\mathcal{T}''_{k,\varepsilon}| \le \frac{C}{\varepsilon} && \text{on } \mathbb{R}, \\ &0 \le \mathcal{T}'_{k,\varepsilon} \le 1, \quad \mathcal{T}''_{k,\varepsilon} \le 0, \quad \mathcal{T}_{k,\varepsilon} \le \mathcal{T}_k && \text{on } (0,\infty). \end{aligned} \tag{4.110}$$

Let us fix $N, M, k \in \mathbb{N}$, where $k \ge 2\|\hat{\vartheta}\|_\infty$. For fixed $\varepsilon, \delta \in (0,k)$ we define

$$w_\varepsilon^{N,M} := \mathcal{T}_{k+\delta,\varepsilon}(\vartheta^N) - \mathcal{T}_{k,\varepsilon}(\vartheta^M).$$

Let us then consider a test function

$$\psi_1 := \mathcal{T}'_{k+\delta,\varepsilon}(\vartheta^N)\mathcal{T}_\delta(w_\varepsilon^{N,M})$$

in (4.33) for ϑ^N, and a test function

$$\psi_2 := \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\mathcal{T}_\delta(w_\varepsilon^{N,M})$$

in (4.33) for ϑ^M. We can consider ψ_1 and ψ_2 as test functions in (4.33) since for almost all $t \in (0,T)$ we have $\psi_1(t), \psi_2(t) \in W_0^{1,2}(\Omega)$. Especially, the functions have a zero trace since $\vartheta^N = \hat{\vartheta} = \vartheta^M$ on $\partial\Omega$ and $\mathcal{T}_{k+\delta,\varepsilon}(\hat{\vartheta}) = \hat{\vartheta} = \mathcal{T}_{k,\varepsilon}(\hat{\vartheta})$ by the choice of k and ε. Thus $w_\varepsilon^{N,M} = 0$ on the boundary. We observe that

$$\begin{aligned} \nabla\psi_1 &= \mathcal{T}''_{k+\delta,\varepsilon}(\vartheta^N)\nabla\vartheta^N\mathcal{T}_\delta(w_\varepsilon^{N,M}) + \mathcal{T}'_{k+\delta,\varepsilon}(\vartheta^N)\mathcal{T}'_\delta(w_\varepsilon^{N,M})\nabla w_\varepsilon^{N,M}, \\ \nabla\psi_2 &= \mathcal{T}''_{k,\varepsilon}(\vartheta^M)\nabla\vartheta^M\mathcal{T}_\delta(w_\varepsilon^{N,M}) + \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\mathcal{T}'_\delta(w_\varepsilon^{N,M})\nabla w_\varepsilon^{N,M}, \end{aligned}$$

where

$$\nabla w_\varepsilon^{N,M} = \mathcal{T}'_{k+\delta,\varepsilon}(\vartheta^N)\nabla\vartheta^N - \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\nabla\vartheta^M.$$

We hence have two equations

$$\begin{aligned} &\int_Q \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi_1 \, d(t,x) = -\int_0^T \left\langle \partial_t\vartheta^N, \psi_1 \right\rangle dt \\ &+ \int_Q \vartheta^N \boldsymbol{u}^N \cdot \nabla\psi_1 + \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \psi_1 \, d(t,x), \end{aligned} \tag{4.111}$$

$$\begin{aligned} &\int_Q \kappa(\vartheta^M)\nabla\vartheta^M \cdot \nabla\psi_2 \, d(t,x) = -\int_0^T \left\langle \partial_t\vartheta^M, \psi_2 \right\rangle dt \\ &+ \int_0^T \int_\Omega \vartheta^M \boldsymbol{u}^M \cdot \nabla\psi_2 + \boldsymbol{\mathcal{S}}^M : D\boldsymbol{u}^M \psi_2 \, d(t,x). \end{aligned} \tag{4.112}$$

We want to subtract (4.112) from (4.111) and pass to the limit in all the parameters $\varepsilon, \delta \to 0+$, and $k, N, M \to \infty$. This will give us a strong convergence $\nabla\vartheta^N \to \nabla\vartheta$ in $L^1(Q)$. Before we proceed, we should notice some auxiliary equalities

$$\begin{aligned} &\kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi_1 - \kappa(\vartheta^M)\nabla\vartheta^M \cdot \nabla\psi_2 = \\ &\Big(\kappa(\vartheta^N)|\nabla\vartheta^N|^2\mathcal{T}''_{k+\delta,\varepsilon}(\vartheta^N) - \kappa(\vartheta^M)|\nabla\vartheta^M|^2\mathcal{T}''_{k,\varepsilon}(\vartheta^M)\Big)\,\mathcal{T}_\delta(w_\varepsilon^{N,M}) + \\ &+ \kappa(\vartheta^N)\mathcal{T}'_\delta(w_\varepsilon^{N,M})|\nabla w_\varepsilon^{N,M}|^2 + \Big(\kappa(\vartheta^N) - \kappa(\vartheta^M)\Big)\,\nabla\mathcal{T}_{k,\varepsilon}(\vartheta^M) \cdot \nabla\mathcal{T}_\delta(w_\varepsilon^{N,M}), \end{aligned}$$

where

$$\nabla \mathcal{T}_{k,\varepsilon}(\vartheta^M) := \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\nabla\vartheta^M$$
$$\nabla \mathcal{T}_{\delta}(w_\varepsilon^{N,M}) := \mathcal{T}'_{\delta}(w_\varepsilon^{N,M})\nabla w_\varepsilon^{N,M}.$$

Moreover,

$$\left\langle \partial_t\vartheta^N, \psi_1\right\rangle - \left\langle \partial_t\vartheta^M, \psi_2\right\rangle = \left\langle \mathcal{T}'{}_{k+\delta,\varepsilon}(\vartheta^N)\partial_t\vartheta^N - \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\partial_t\vartheta^M, \mathcal{T}_\delta(w_\varepsilon^{N,M})\right\rangle = \\ = \left\langle \partial_t w_\varepsilon^{N,M}), \mathcal{T}_\delta(w_\varepsilon^{N,M})\right\rangle.$$

Finally, using the zero divergence of $\boldsymbol{u}^N$,

$$\begin{aligned}
&\int_Q \vartheta^N\boldsymbol{u}^N\cdot\nabla\psi_1 - \vartheta^M\boldsymbol{u}^M\cdot\nabla\psi_2\, d(t,x) \overset{\text{IBP}}{=} \\
&= -\int_Q \psi_1\nabla\vartheta^N\cdot\boldsymbol{u}^N - \psi_2\nabla\vartheta^M\cdot\boldsymbol{u}^M\, d(t,x) = \\
&= -\int_Q \left(\mathcal{T}'_{k+\delta,\varepsilon}(\vartheta^N)\nabla\vartheta^N\cdot\boldsymbol{u}^N - \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\nabla\vartheta^M\cdot\boldsymbol{u}^M\right)\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x) = \\
&= -\int_Q \left(\nabla\mathcal{T}_{k+\delta,\varepsilon}(\vartheta^N)\cdot\boldsymbol{u}^N - \nabla\mathcal{T}_{k,\varepsilon}(\vartheta^M)\cdot\boldsymbol{u}^M\right)\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x) = \\
&= -\int_Q \operatorname{div}\left(\mathcal{T}_{k+\delta,\varepsilon}(\vartheta^N)\boldsymbol{u}^N - \mathcal{T}_{k,\varepsilon}(\vartheta^M)\boldsymbol{u}^M\right)\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x) \overset{\text{IBP}}{=} \\
&= \int_Q \left(\mathcal{T}_{k+\delta,\varepsilon}(\vartheta^N)\boldsymbol{u}^N - \mathcal{T}_{k,\varepsilon}(\vartheta^M)\boldsymbol{u}^M\right)\cdot\nabla\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x).
\end{aligned}$$

We can thus subtract (4.112) from (4.111) and using the auxiliary equalities we obtain

$$\begin{aligned}
&\int_Q \kappa(\vartheta^N)\mathcal{T}'_\delta(w_\varepsilon^{N,M})|\nabla w_\varepsilon^{N,M}|^2\, d(t,x) = -\int_0^T \left\langle \partial_t w_\varepsilon^{N,M}, \mathcal{T}_\delta(w_\varepsilon^{N,M})\right\rangle dt \\
&+ \int_Q \left[\mathcal{T}_{k+\delta,\varepsilon}(\vartheta^N)\boldsymbol{u}^N - \mathcal{T}_{k,\varepsilon}(\vartheta^M)\boldsymbol{u}^M\right]\cdot\nabla\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x) \\
&+ \int_Q \left(\kappa(\vartheta^M) - \kappa(\vartheta^N)\right)\nabla\mathcal{T}_{k,\varepsilon}(\vartheta^M)\cdot\nabla\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x) \\
&+ \int_Q G^{N,M}\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x),
\end{aligned} \tag{4.113}$$

where we denoted

$$G^{N,M} := \left[\mathcal{T}'_{k+\delta,\varepsilon}(\vartheta^N)\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N - \mathcal{T}'_{k,\varepsilon}(\vartheta^M)\boldsymbol{\mathcal{S}}^M : D\boldsymbol{u}^M\right] \\ - \left[\kappa(\vartheta^N)|\nabla\vartheta^N|^2\mathcal{T}''_{k+\delta,\varepsilon}(\vartheta^N) - \kappa(\vartheta^M)|\nabla\vartheta^M|^2\mathcal{T}''_{k,\varepsilon}(\vartheta^M)\right].$$

Our first goal is to pass to the limit $\varepsilon \to 0+$ in equation (4.113) for fixed M, N, k, and δ. We notice that

$$\mathcal{T}_{k,\varepsilon} \to \mathcal{T}_k \qquad \text{uniformly on } \mathbb{R}, \text{ and} \tag{4.114}$$
$$\mathcal{T}'_{k,\varepsilon} \to \mathcal{T}'_k \qquad \text{on } \mathbb{R}\setminus\{-k,k\} \tag{4.115}$$

as $\varepsilon \to 0+$. Thus also

$$w_\varepsilon^{N,M} \to w^{N,M} := \mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta^M) \qquad \text{uniformly on } Q. \tag{4.116}$$

For the term on the left-hand side of (4.113) we have

$$\kappa(\vartheta^N)\mathcal{T}_\delta'(w_\varepsilon^{N,M})|\nabla w_\varepsilon^{N,M}|^2 \le 2\overline{\kappa}\left(|\nabla\vartheta^N|^2 + |\nabla\vartheta^M|^2\right) \in L^1(Q).$$

Additionally, for $\varepsilon \to 0+$ it holds

$$\kappa(\vartheta^N)\mathcal{T}_\delta'(w_\varepsilon^{N,M})|\nabla w_\varepsilon^{N,M}|^2 \to \kappa(\vartheta^N)\mathcal{T}_\delta'(w^{N,M})|\nabla w^{N,M}|^2 \qquad \text{a.e. in } Q.$$

by convergences (4.115) and (4.116). Thus, using the Lebesgue Dominated Convergence Theorem we have

$$\kappa(\vartheta^N)\mathcal{T}_\delta'(w_\varepsilon^{N,M})|\nabla w_\varepsilon^{N,M}|^2 \to \kappa(\vartheta^N)\mathcal{T}_\delta'(w^{N,M})|\nabla w^{N,M}|^2 \qquad \text{in } L^1(Q).$$

The same approach can be used for the second and the third term on the right-hand side of (4.113).

Let us now estimate the remaining terms uniformly with respect to ε. For the term containing the time derivative we have

$$\begin{aligned}
&-\int_0^T \left\langle \partial_t w_\varepsilon^{N,M}, \mathcal{T}_\delta(w_\varepsilon^{N,M})\right\rangle dt = -\int_Q \partial_t \mathcal{G}_\delta(w_\varepsilon^{N,M})\, d(t,x) \le \\
&-\int_\Omega \mathcal{G}_\delta(w_\varepsilon^{N,M}(T))\, dx + \int_\Omega \mathcal{G}_\delta(w_\varepsilon^{N,M}(0))\, dx \le \int_\Omega \mathcal{G}_\delta(w_\varepsilon^{N,M}(0))\, dx \le C\delta.
\end{aligned}$$

Lastly, we want to estimate the term $\int_Q G^{N,M}\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x)$. For the first part, we have

$$\begin{aligned}
&\int_Q \left|\mathcal{T}_{k+\delta,\varepsilon}'(\vartheta^N)\boldsymbol{S}^N : D\boldsymbol{u}^N - \mathcal{T}_{k,\varepsilon}'(\vartheta^M)\boldsymbol{S}^M : D\boldsymbol{u}^M\right| \left|\mathcal{T}_\delta(w_\varepsilon^{N,M})\right|\, d(t,x) \le \\
&\le \delta \int_Q \left|\boldsymbol{S}^N : D\boldsymbol{u}^N\right| + \left|\boldsymbol{S}^M : D\boldsymbol{u}^M\right|\, d(t,x) \le C\delta.
\end{aligned}$$

To estimate the second part of $G^{N,M}$ we introduce a function $\psi_m := 1 - \mathcal{T}_{m,\varepsilon}'(\vartheta^N)$ for $m \ge 2\|\hat{\vartheta}\|_\infty$ and $\varepsilon \le \frac{m}{2}$. Note that

$$\nabla\psi_m := -\mathcal{T}_{m,\varepsilon}''(\vartheta^N)\nabla\vartheta^N \in L^2(Q),$$

and $\mathcal{T}_{m,\varepsilon}'(\vartheta^N) = \mathcal{T}_{m,\varepsilon}'(\hat{\vartheta}) = \mathcal{T}_m'(\hat{\vartheta}) = 1$ on $\partial\Omega$. Hence, $\psi_m(t) \in W_0^{1,2}(\Omega)$ for almost all $t \in (0,T)$ and we can use it as a test function in (4.33). Such a choice leads to

$$\begin{aligned}
&\int_0^T \left\langle \partial_t\vartheta^N, 1 - \mathcal{T}_{m,\varepsilon}'(\vartheta^N)\right\rangle dt - \int_Q \kappa(\vartheta^N)\mathcal{T}_{m,\varepsilon}''(\vartheta^N)|\nabla\vartheta^N|^2\, d(t,x) = \\
&= \int_Q \left(1 - \mathcal{T}_{m,\varepsilon}'(\vartheta^N)\right)\boldsymbol{S}^N : D\boldsymbol{u}^N - \vartheta^N\boldsymbol{u}^N \cdot \mathcal{T}_{m,\varepsilon}''(\vartheta^N)\nabla\vartheta^N\, d(t,x).
\end{aligned} \tag{4.117}$$

We use Lemma 2 for $g(\vartheta^N) = \vartheta^N - \mathcal{T}_{m,\varepsilon}(\vartheta^N)$ to identify

$$\int_0^T \left\langle \partial_t\vartheta^N, 1 - \mathcal{T}_{m,\varepsilon}'(\vartheta^N)\right\rangle dt = \int_0^T \partial_t \int_\Omega \vartheta^N - \mathcal{T}_{m,\varepsilon}(\vartheta^N)\, dx\, dt.$$

Moreover, it holds

$$\int_Q \vartheta^N \boldsymbol{u}^N \cdot \mathcal{T}''_{m,\varepsilon}(\vartheta^N)\nabla\vartheta^N \, d(t,x) = \int_Q \boldsymbol{u}^N \cdot \nabla\xi(\vartheta^N)\, d(t,x) \overset{\text{IBP}}{=} 0,$$

where

$$\xi(s) := \int_0^s \tau\mathcal{T}''_{m,\varepsilon}(\tau)\, d\tau.$$

Thus, by the non-positivity of $\mathcal{T}''_{m,\varepsilon}(\vartheta^N)$, we can rewrite (4.117) as

$$\begin{aligned}
&\int_Q \kappa(\vartheta^N)\left|\mathcal{T}''_{m,\varepsilon}(\vartheta^N)\right| |\nabla\vartheta^N|^2\, d(t,x) = -\int_\Omega \left(\vartheta^N - \mathcal{T}_{m,\varepsilon}(\vartheta^N)\right)(T)\, dx\\
&+\int_Q \left(1-\mathcal{T}'_{m,\varepsilon}(\vartheta^N)\right)\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\, d(t,x) + \int_\Omega \vartheta_0^N - \mathcal{T}_{m,\varepsilon}(\vartheta_0^N)\, dx \leq \\
&\leq \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \chi_{\{\vartheta^N > \frac{m}{2}\}}\, d(t,x) + \int_\Omega \vartheta_0^N \chi_{\{\vartheta_0^N>\frac{m}{2}\}}\, dx \leq C,
\end{aligned} \tag{4.118}$$

where we have used that $\left(\vartheta^N - \mathcal{T}_{m,\varepsilon}(\vartheta^N)\right)(T) \geq 0$ by properties (4.110). Hence, for $k \geq 2\|\hat{\vartheta}\|_\infty$, we estimate the second part of $G^{N,M}$ as

$$\left|\int_Q \left[\kappa(\vartheta^N)|\nabla\vartheta^N|^2\mathcal{T}''_{k+\delta,\varepsilon}(\vartheta^N) - \kappa(\vartheta^M)|\nabla\vartheta^M|^2\mathcal{T}''_{k,\varepsilon}(\vartheta^M)\right]\mathcal{T}_\delta(w_\varepsilon^{N,M})\, d(t,x)\right| \leq C\delta,$$

which is uniform with respect to ε by (4.118).

All the remaining terms in (4.113) are thus estimated uniformly in ε. We can let $\varepsilon \to 0+$ to deduce

$$\begin{aligned}
&\underline{\kappa}\int_Q \mathcal{T}'_\delta(w^{N,M})|\nabla w^{N,M}|^2\, d(t,x) \leq C\delta\\
&+\int_Q \left(\kappa(\vartheta^M) - \kappa(\vartheta^N)\right)\nabla\mathcal{T}_k(\vartheta^M)\cdot\nabla\mathcal{T}_\delta(w^{N,M})\, d(t,x)\\
&+\int_Q \left[\mathcal{T}_{k+\delta}(\vartheta^N)\boldsymbol{u}^N - \mathcal{T}_k(\vartheta^M)\boldsymbol{u}^M\right]\cdot\nabla\mathcal{T}_\delta(w^{N,M})\, d(t,x).
\end{aligned} \tag{4.119}$$

We will additionally use that

$$\int_Q |\nabla\mathcal{T}_\delta(w^{N,M})|^2\, d(t,x) \leq \int_Q \mathcal{T}'_\delta(w^{N,M})|\nabla w^{N,M}|^2\, d(t,x) \tag{4.120}$$

for $\delta \in (0,1)$.

We now want to let $M, N \to \infty$ and lastly $\delta \to 0+$ in (4.119). We start with the third term on the right-hand side of (4.119). We define $w_\delta := \mathcal{T}_{k+\delta}(\vartheta) - \mathcal{T}_k(\vartheta)$ and using convergences (4.57), (4.61), and (4.63) we deduce

$$\begin{aligned}
&\lim_{\delta\to 0+}\lim_{N\to\infty}\lim_{M\to\infty}\int_Q \left[\mathcal{T}_{k+\delta}(\vartheta^N)\boldsymbol{u}^N - \mathcal{T}_k(\vartheta^M)\boldsymbol{u}^M\right]\cdot\nabla\mathcal{T}_\delta(w^{N,M})\, d(t,x) =\\
&= \lim_{\delta\to 0+}\int_Q [\mathcal{T}_{k+\delta}(\vartheta)\boldsymbol{u} - \mathcal{T}_k(\vartheta)\boldsymbol{u}]\cdot\mathcal{T}'_\delta(w_\delta)\nabla w_\delta\, d(t,x) = 0.
\end{aligned} \tag{4.121}$$

Note, that we can use convergence (4.61) while working with $\nabla \mathcal{T}_\delta(w^{N,M})$ since we have e.g.

$$\left| \int_Q \mathcal{T}_k(\vartheta^M) \boldsymbol{u}^M \cdot \mathcal{T}'_\delta(w^{N,M}) \left(\mathcal{T}'_{k+\delta}(\vartheta^N) \nabla \vartheta^N - \mathcal{T}'_k(\vartheta^M) \nabla \vartheta^M \right) d(t,x) \right| \leq$$
$$\leq k \int_Q (k+\delta) \left| \boldsymbol{u}^M \cdot \frac{\nabla \vartheta^N}{\vartheta^N} \right| + k \left| \boldsymbol{u}^M \cdot \frac{\nabla \vartheta^M}{\vartheta^M} \right| d(t,x).$$

To estimate the second term on the right-hand side of (4.119) note that

$$\{|w^{N,M}| < \delta\} \cap \{\vartheta^M < k\} = \{|\vartheta^N - \vartheta^M| < \delta\} \cap \{\vartheta^M < k\}.$$

Additionally, we will use that for all $N \in \mathbb{N}$ it holds $(\vartheta^N)^\alpha \in L^2(0,T;W^{1,2})$ for $\alpha \in (0,1/2)$ by (4.48), and $\vartheta^N \geq \mu$ by (4.39). Considering $\delta < k$, we can derive the following estimate:

$$\left| \int_Q \left(\kappa(\vartheta^M) - \kappa(\vartheta^N) \right) \mathcal{T}'_k(\vartheta^M) \nabla \vartheta^M \cdot \mathcal{T}'_\delta(w^{N,M}) \nabla w^{N,M} \, d(t,x) \right| \overset{\text{Young}}{\leq}$$
$$\leq C \int_Q \left| \kappa(\vartheta^M) - \kappa(\vartheta^N) \right| \left(|\nabla \vartheta^M|^2 + |\nabla \vartheta^N|^2 \right) \chi_{\{|w^{N,M}|<\delta\} \cap \{\vartheta^M < k\}} \, d(t,x) \leq$$
$$\leq C \int_Q \left| \kappa(\vartheta^M) - \kappa(\vartheta^N) \right| (2k)^2 \left(\frac{|\nabla \vartheta^M|^2}{(\vartheta^M)^2} + \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} \right) \chi_{\{|w^{N,M}|<\delta\} \cap \{\vartheta^M < k\}} \, d(t,x) \leq$$
$$\leq C(k) \sup_{(t,x) \in Q} \left(\left| \kappa(\vartheta^M) - \kappa(\vartheta^N) \right| \chi_{\{|w^{N,M}|<\delta\} \cap \{\vartheta^M < k\}} \right) \leq$$
$$\leq C(k) \sup_{(t,x) \in Q} \left(\left| \kappa(\vartheta^M) - \kappa(\vartheta^N) \right| \chi_{\{|\vartheta^N - \vartheta^M|<\delta\} \cap \{\vartheta^M < k\}} \right) \leq$$
$$\leq C(k) \sup_{l,s \in [\mu, 2k] : |s-l| < \delta} |\kappa(l) - \kappa(s)| \to 0 \tag{4.122}$$

as $\delta \to 0+$ by the uniform continuity of κ on $[\mu, 2k]$. Hence, taking the limits $N, M \to \infty$, $\delta \to 0+$ of the inequality (4.119) and using (4.121) and (4.122), we can conclude

$$\lim_{\delta \to 0+} \limsup_{N \to \infty} \limsup_{M \to \infty} \underline{\kappa} \int_Q |\nabla \mathcal{T}_\delta(w^{N,M})|^2 \, d(t,x) \overset{(4.120)}{\leq}$$
$$\leq \lim_{\delta \to 0+} \limsup_{N \to \infty} \limsup_{M \to \infty} \underline{\kappa} \int_Q \mathcal{T}'_\delta(w^{N,M}) |\nabla w^{N,M}|^2 \, d(t,x) = 0.$$

Furthermore,

$$\left| \nabla \mathcal{T}_\delta(w^{N,M}) \right| = \left| \mathcal{T}'_\delta(w^{N,M}) \left(\mathcal{T}'_{k+\delta}(\vartheta^N) \nabla \vartheta^N - \mathcal{T}'_k(\vartheta^M) \nabla \vartheta^M \right) \right| \leq \left| \nabla \vartheta^N \right| + \left| k^2 \frac{\nabla \vartheta^M}{(\vartheta^M)^2} \right|$$

and the right-hand side is bounded in $L^2(Q)$ uniformly with respect to M. We can hence extract a weakly converging subsequence. Also,

$$\mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta^M)) \to \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta)) \quad \text{ in } L^q(Q) \text{ for all } q > 1.$$

We can thus use the Lebesgue Dominated Convergence Theorem to identify the weak limit

$$\nabla \mathcal{T}_\delta(w^{N,M}) = \nabla[\mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta^M))] \rightharpoonup \nabla[\mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))]$$

in $L^2(Q)$ as $M \to \infty$. It thus holds

$$\lim_{\delta\to 0+} \limsup_{N\to\infty} \int_Q |\nabla \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))|^2 \, d(t,x) = 0, \tag{4.123}$$

by the weak lower semicontinuity of the L^2-norm.

Finally, we use Hölder's inequality, Chebyschev's inequality, and the fact that by (4.64) it holds

$$\int_Q |\vartheta^N| + |\vartheta| \, d(t,x) \le C \tag{4.124}$$

to obtain that

$$\begin{aligned}
&\int_Q |\nabla\vartheta^N - \nabla\vartheta| \, d(t,x) \le \int_Q |\nabla\vartheta^N - \nabla\vartheta| \chi_{\{|\vartheta^N - \vartheta| > \delta\} \cup \{|\vartheta^N| + |\vartheta| > \frac{k}{2}\}} \, d(t,x) \\
&+ \int_Q |\nabla \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))| \, d(t,x) \overset{\text{Hölder}}{\le} \\
&\le C \|\nabla\vartheta^N - \nabla\vartheta\|_{L^{\frac{9}{8}}(Q)} \left(\left| \{|\vartheta^N - \vartheta| > \delta\} \right|^{\frac{1}{9}} + \left| \{|\vartheta^N| + |\vartheta| > k/2\} \right|^{\frac{1}{9}} \right) \\
&+ C \left(\int_Q |\nabla \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))|^2 \, d(t,x) \right)^{\frac{1}{2}} \overset{\text{Chebyschev,(4.124)}}{\le} \\
&\le C \left(\int_Q |\nabla \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))|^2 \, d(t,x) \right)^{\frac{1}{2}} + \frac{C \|\vartheta^N - \vartheta\|_{L^1(Q)}^{\frac{1}{9}}}{\delta^{\frac{1}{9}}} + \frac{C}{k^{\frac{1}{9}}}.
\end{aligned}$$

We can therefore conclude, using convergence (4.64), that

$$\begin{aligned}
&\lim_{N\to\infty} \int_Q |\nabla\vartheta^N - \nabla\vartheta| \, d(t,x) \le \\
&\le C \left(\limsup_{N\to\infty} \int_Q |\nabla \mathcal{T}_\delta(\mathcal{T}_{k+\delta}(\vartheta^N) - \mathcal{T}_k(\vartheta))|^2 \, d(t,x) \right)^{\frac{1}{2}} + \frac{C}{k^{\frac{1}{9}}}.
\end{aligned}$$

The left-hand side is independent of δ and k, hence we may first let $\delta \to 0+$, which by (4.123) implies

$$\lim_{N\to\infty} \int_Q |\nabla\vartheta^N - \nabla\vartheta| \, d(t,x) \le \frac{C}{k^{\frac{1}{9}}}.$$

Lastly, we let $k \to \infty$ and since the left-hand side doesn't depend on k we obtain

$$\lim_{N\to\infty} \int_Q |\nabla\vartheta^N - \nabla\vartheta| \, d(t,x) \le 0.$$

This gives us the desired convergence

$$\nabla\vartheta^N \to \nabla\vartheta \qquad \text{in } L^1(Q), \tag{4.125}$$

and consequently (for a non-relabeled subsequence)

$$\nabla\vartheta^N \to \nabla\vartheta \qquad \text{almost everywhere in } Q. \tag{4.126}$$

We now wish to use (4.126) to prove

$$\kappa(\vartheta^N)\frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2} \to \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} \qquad \text{in } L^1(Q). \tag{4.127}$$

We start the proof by showing the strong convergence of $\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})$ in $L^2(Q)$ for any k. To do so, we would like to test (4.33) by $\psi := \mathcal{T}_k(\vartheta^N - \hat{\vartheta})$, let $N \to \infty$, and compare the limit with (3.11) for $\varphi := \mathcal{T}_k(\vartheta - \hat{\vartheta})$. However, such test function isn't admissible for (3.11), so we must proceed differently, without the use of (3.11). We fix an arbitrary Lebesgue point $T^* \in (0,T)$ of $\vartheta(\cdot,x) \in L^1(Q)$. Let Q^* denote the set $(0,T^*) \times \Omega$. We want to show

$$\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \to \sqrt{\kappa(\vartheta)}\nabla\mathcal{T}_k(\vartheta - \hat{\vartheta}) \qquad \text{in } L^2(Q^*), \tag{4.128}$$

using the following lemma for $f^N := \sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})$.

Lemma 11. *Let $A \subseteq Q$ and let $f^N, f : A \to \mathbb{R}$. Furthermore, let*

1. *$f^N \rightharpoonup f$ in $L^2(A)$, and*
2. *$\limsup_{N\to\infty} \|f^N\|^2_{L^2(A)} \le \|f\|^2_{L^2(A)}$.*

It then holds

$$f^N \to f \qquad \text{in } L^2(A).$$

Proof. We have

$$\begin{aligned}
&\lim_{N\to\infty} \|f^N - f\|^2_{L^2(A)} = \lim_{N\to\infty} \int_A (f^N)^2 - 2f^N f + f^2 \, d(t,x) \le \\
&\le \int_A f^2 \, d(t,x) - 2\lim_{N\to\infty}\int_A f^N f \, d(t,x) + \limsup_{N\to\infty}\int_A (f^N)^2 \, d(t,x) \overset{\text{assumptions 1,2}}{\le} 0.
\end{aligned}$$

□

We know that $\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})$ is bounded in $L^2(Q^*)$ uniformly with respect to N so we can find a weakly converging subsequence

$$\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \rightharpoonup K \qquad \text{in } L^2(Q^*).$$

Additionally, by (4.63) and (4.126) it holds

$$\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \to \sqrt{\kappa(\vartheta)}\nabla\mathcal{T}_k(\vartheta - \hat{\vartheta}) \qquad \text{almost everywhere in } Q^*.$$

We hence obtain

$$\sqrt{\kappa(\vartheta^N)}\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \rightharpoonup \sqrt{\kappa(\vartheta)}\nabla\mathcal{T}_k(\vartheta - \hat{\vartheta}) \qquad \text{in } L^2(Q^*). \tag{4.129}$$

It remains to show that

$$\limsup_{N\to\infty}\int_{Q^*} \kappa(\vartheta^N)|\nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})|^2 \, d(t,x) \le \int_{Q^*} \kappa(\vartheta)|\nabla\mathcal{T}_k(\vartheta - \hat{\vartheta})|^2 \, d(t,x).$$

To prove this, let us set $\psi := \mathcal{T}_k(\vartheta^N - \hat{\vartheta})$ as a test function in (4.33) and integrate until time t for some $t \in (T^*, T)$. Note, that $\vartheta^N \in C([0,T]; L^2)$ by (4.39) and $\hat{\vartheta}$ does not depend on time. Using integration by parts and the fact $\operatorname{div} \boldsymbol{u}^N = 0$ we have

$$\int_\Omega \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \nabla(\vartheta^N - \hat{\vartheta}) \cdot \boldsymbol{u}^N \, dx = \int_\Omega \nabla \mathcal{G}_k(\vartheta^N - \hat{\vartheta}) \cdot \boldsymbol{u}^N = 0,$$

thus

$$\int_\Omega \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \nabla \vartheta^N \cdot \boldsymbol{u}^N \, dx = \int_\Omega \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \nabla \hat{\vartheta} \cdot \boldsymbol{u}^N \, dx. \tag{4.130}$$

We can then use equality (4.130) and the identity

$$\begin{aligned}&\int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, dx \overset{\text{IBP}}{=} - \int_\Omega \operatorname{div}(\vartheta^N \boldsymbol{u}^N) \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, dx = \\ &= - \int_\Omega \nabla \vartheta^N \cdot \boldsymbol{u}^N \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, dx\end{aligned}$$

together with Lemma 2 for $g(\vartheta^N, \hat{\vartheta}) = \mathcal{G}_k(\vartheta^N - \hat{\vartheta})$ to derive

$$\begin{aligned}&\int_0^t \int_\Omega \kappa(\vartheta^N) \nabla(\vartheta^N - \hat{\vartheta}) \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, dx \, d\tau = \\ &= \int_0^t \int_\Omega \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \left(\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N - \nabla \hat{\vartheta} \cdot \boldsymbol{u}^N \right) - \kappa(\vartheta^N) \nabla \hat{\vartheta} \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, dx \, d\tau \\ &- \int_\Omega \mathcal{G}_k(\vartheta^N(t) - \hat{\vartheta}) - \mathcal{G}_k(\vartheta_0^N - \hat{\vartheta}) \, dx.\end{aligned} \tag{4.131}$$

Let us now estimate $\limsup_{N\to\infty} \int_{Q^*} \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, d(t,x)$ using equality (4.131). We have

$$\begin{aligned}&\int_{Q^*} \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, d(t,x) = \int_{Q^*} \kappa(\vartheta^N) \nabla \hat{\vartheta} \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, d(t,x) \\ &+ \int_{Q^*} \kappa(\vartheta^N) \nabla(\vartheta^N - \hat{\vartheta}) \cdot \nabla \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \, d(t,x).\end{aligned} \tag{4.132}$$

The second term of (4.132) is nonnegative by the nonnegativity of $\mathcal{T}_k'$. For $\delta \in (0, T - T^*)$ we can hence estimate

$$
\begin{aligned}
&\limsup_{N\to\infty} \int_{Q^*} \kappa(\vartheta^N)\nabla(\vartheta^N - \hat{\vartheta}) \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, d(t,x) \le \\
&\le \limsup_{N\to\infty} \fint_{T^*}^{T^*+\delta} \int_0^t \int_\Omega \kappa(\vartheta^N)\nabla(\vartheta^N - \hat{\vartheta}) \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, dx\, d\tau\, dt \overset{(4.131)}{=} \\
&= \limsup_{N\to\infty} \fint_{T^*}^{T^*+\delta} \int_0^t \int_\Omega \mathcal{T}_k(\vartheta^N - \hat{\vartheta}) \Big(\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N - \nabla\hat{\vartheta} \cdot \boldsymbol{u}^N \Big)\, dx\, d\tau\, dt \\
&- \liminf_{N\to\infty} \fint_{T^*}^{T^*+\delta} \int_0^t \int_\Omega \kappa(\vartheta^N)\nabla\hat{\vartheta} \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, dx\, d\tau\, dt \\
&- \liminf_{N\to\infty} \fint_{T^*}^{T^*+\delta} \int_\Omega \mathcal{G}_k(\vartheta^N(t) - \hat{\vartheta}) - \mathcal{G}_k(\vartheta_0^N - \hat{\vartheta})\, dx\, dt = \\
&= \fint_{T^*}^{T^*+\delta} \int_0^t \int_\Omega \mathcal{T}_k(\vartheta - \hat{\vartheta}) \Big(\boldsymbol{\mathcal{S}} : D\boldsymbol{u} - \nabla\hat{\vartheta} \cdot \boldsymbol{u} \Big) - \kappa(\vartheta)\nabla\hat{\vartheta} \cdot \nabla\mathcal{T}_k(\vartheta - \hat{\vartheta})\, dx\, d\tau\, dt \\
&- \fint_{T^*}^{T^*+\delta} \int_\Omega \mathcal{G}_k(\vartheta(t) - \hat{\vartheta}) - \mathcal{G}_k(\vartheta_0 - \hat{\vartheta})\, dx\, dt,
\end{aligned}
$$

where the last equality holds by convergences (4.63), (4.84), (4.57), (4.61), and (4.5). Since the left-hand side doesn't depend on δ, by letting $\delta \to 0+$ we have

$$
\begin{aligned}
&\limsup_{N\to\infty} \int_{Q^*} \kappa(\vartheta^N)\nabla(\vartheta^N - \hat{\vartheta}) \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, d(t,x) \le \\
&\le \int_{Q^*} \mathcal{T}_k(\vartheta - \hat{\vartheta}) \Big(\boldsymbol{\mathcal{S}} : D\boldsymbol{u} - \nabla\hat{\vartheta} \cdot \boldsymbol{u} \Big) - \kappa(\vartheta)\nabla\hat{\vartheta} \cdot \nabla\mathcal{T}_k(\vartheta - \hat{\vartheta})\, d(t,x) \\
&- \int_\Omega \mathcal{G}_k(\vartheta(T^*) - \hat{\vartheta}) - \mathcal{G}_k(\vartheta_0 - \hat{\vartheta})\, dx.
\end{aligned}
\tag{4.133}
$$

We used that T^* is a Lebesgue point of ϑ to obtain $\int_\Omega \mathcal{G}_k(\vartheta(T^*) - \hat{\vartheta})\, dx$. For the first term of (4.132) we have

$$
\int_{Q^*} \kappa(\vartheta^N)\nabla\hat{\vartheta} \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, d(t,x) \to \int_{Q^*} \kappa(\vartheta)\nabla\hat{\vartheta} \cdot \nabla\mathcal{T}_k(\vartheta - \hat{\vartheta})\, d(t,x) \tag{4.134}
$$

by (4.61) and (4.63). Taking the limit superior of (4.132) and using (4.133) and (4.134) we have

$$
\begin{aligned}
&\limsup_{N\to\infty} \int_{Q^*} \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\mathcal{T}_k(\vartheta^N - \hat{\vartheta})\, d(t,x) \le \\
&\le \int_{Q^*} \mathcal{T}_k(\vartheta - \hat{\vartheta}) \Big(\boldsymbol{\mathcal{S}} : D\boldsymbol{u} - \nabla\hat{\vartheta} \cdot \boldsymbol{u} \Big)\, d(t,x) \\
&- \int_\Omega \mathcal{G}_k(\vartheta(T^*) - \hat{\vartheta}) - \mathcal{G}_k(\vartheta_0 - \hat{\vartheta})\, dx.
\end{aligned}
\tag{4.135}
$$

We now want to estimate the expression on the right-hand side of (4.135) by $\int_{Q^*} \kappa(\vartheta)\nabla\vartheta \cdot \nabla\mathcal{T}_k(\vartheta - \hat{\vartheta})\, d(t,x)$. As mentioned before, setting $\varphi := \mathcal{T}_k(\vartheta - \hat{\vartheta})$ in (3.11) is not possible since $\mathcal{T}_k(\vartheta - \hat{\vartheta})$ is not regular enough. We hence need to find a suitable function to test (4.33) again. Let us consider $m, k > 0$ such that $m > k+1+\|\hat{\vartheta}\|_\infty$ and $\delta \in (0,1)$. Take any $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $t \in (0, T^*)$ and $\varepsilon \in (0, T - T^*)$. Let us now set $\psi := \frac{1}{\varepsilon}\mathcal{T}'_{m,\delta}(\vartheta^N)\varphi$ as a test function in (4.33) and integrate over the interval $(t, t+\varepsilon)$. Using Lemma 2 for $g(\vartheta^N, \hat{\vartheta}) := \mathcal{T}_{m,\delta}(\vartheta^N)$, we obtain

$$
\begin{aligned}
&\int_\Omega \partial_t \fint_t^{t+\varepsilon} \mathcal{T}_{m,\delta}(\vartheta^N)\, d\tau \varphi\, dx + \int_\Omega \fint_t^{t+\varepsilon} \kappa(\vartheta^N)\nabla \mathcal{T}_{m,\delta}(\vartheta^N)\, d\tau \cdot \nabla\varphi\, dx \\
&+ \int_\Omega \fint_t^{t+\varepsilon} \nabla \mathcal{T}_{m,\delta}(\vartheta^N)\cdot \boldsymbol{u}^N\, d\tau \varphi\, dx = \int_\Omega \fint_t^{t+\varepsilon} \mathcal{T}'_{m,\delta}(\vartheta^N)\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\, d\tau\varphi\, dx \\
&+ \int_\Omega \fint_t^{t+\varepsilon} \kappa(\vartheta^N)\mathcal{T}''_{m,\delta}(\vartheta^N)|\nabla\vartheta^N|^2\, d\tau\varphi\, dx,
\end{aligned}
\tag{4.136}
$$

where we have again used the zero divergence of $\boldsymbol{u}^N$ for

$$
\int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla\left(\mathcal{T}_{m,\delta}(\vartheta^N)\varphi\right)\, dx \overset{\text{IBP}}{=} \int_\Omega \mathcal{T}_{m,\delta}(\vartheta^N)\nabla\vartheta^N\varphi \cdot \boldsymbol{u}^N\, dx.
$$

We now note that any function $\tilde{\varphi} \in L^2(0,T^*; W_0^{1,2}) \cap L^\infty(Q^*)$ can be approximated by a sequence of step functions $\tilde{\varphi}_n := \sum_{i=1}^n \chi_{J_n}\varphi_n$, where J_n is a measurable subset of $(0, T^*)$ and $\varphi_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. More precisely, it holds

$$
\tilde{\varphi}_n \to \tilde{\varphi} \qquad\qquad \text{in } L^2(0,T^*; W_0^{1,2}) \cap L^\infty(Q^*).
$$

Using this while integrating (4.136) until T^* yields into

$$
\begin{aligned}
&\int_{Q^*} \partial_t \fint_t^{t+\varepsilon} \mathcal{T}_{m,\delta}(\vartheta^N)\, d\tau\tilde{\varphi}\, d(t,x) + \int_{Q^*} \fint_t^{t+\varepsilon} \kappa(\vartheta^N)\nabla\mathcal{T}_{m,\delta}(\vartheta^N)\, d\tau \cdot \nabla\tilde{\varphi}\, d(t,x) \\
&+ \int_{Q^*} \fint_t^{t+\varepsilon} \nabla\mathcal{T}_{m,\delta}(\vartheta^N)\cdot\boldsymbol{u}^N - \mathcal{T}'_{m,\delta}(\vartheta^N)\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N\, d\tau\tilde{\varphi}\, d(t,x) = \\
&= \int_{Q^*} \fint_t^{t+\varepsilon} \kappa(\vartheta^N)\mathcal{T}''_{m,\delta}(\vartheta^N)|\nabla\vartheta^N|^2\, d\tau\tilde{\varphi}\, d(t,x), \forall\tilde{\varphi} \in L^2(0,T^*; W_0^{1,2}) \cap L^\infty(Q^*).
\end{aligned}
$$

Passing to the limit inferior $N \to \infty$ and using convergences (4.63), (4.61), (4.57), and (4.84), we have for all $\tilde{\varphi} \in L^2(0,T^*; W_0^{1,2}) \cap L^\infty(Q^*)$:

$$
\begin{aligned}
&\int_{Q^*} \partial_t\vartheta_\varepsilon^{m,\delta}\tilde{\varphi}\, d(t,x) + \int_{Q^*} \fint_t^{t+\varepsilon} \kappa(\vartheta)\nabla\mathcal{T}_{m,\delta}(\vartheta)\, d\tau\cdot\nabla\tilde{\varphi}\, d(t,x) \\
&+ \int_{Q^*} \fint_t^{t+\varepsilon} \nabla\mathcal{T}_{m,\delta}(\vartheta)\cdot\boldsymbol{u} - \mathcal{T}'_{m,\delta}(\vartheta)\boldsymbol{\mathcal{S}} : D\boldsymbol{u}\, d\tau\tilde{\varphi}\, d(t,x) \geq \\
&\geq \liminf_{N\to\infty} \int_{Q^*} \fint_t^{t+\varepsilon} \kappa(\vartheta^N)\mathcal{T}''_{m,\delta}(\vartheta^N)|\nabla\vartheta^N|^2\, d\tau|\tilde{\varphi}|\, d(t,x) = \\
&= -\limsup_{N\to\infty} \int_{Q^*} \fint_t^{t+\varepsilon} \kappa(\vartheta^N)|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\, d\tau|\tilde{\varphi}|\, d(t,x),
\end{aligned}
\tag{4.137}
$$

where $\vartheta_\varepsilon^{m,\delta}(t,x) := \fint_t^{t+\varepsilon} \mathcal{T}_{m,\delta}(\vartheta(\tau,x))\, d\tau$. Since $\vartheta_\varepsilon^{m,\delta}(t,x) \in W^{1,\infty}(0,T; L^\infty)$ for any $\varepsilon > 0$ the integral with the time derivative is well defined. Let us estimate the last term of (4.137) from bellow. By taking the limit superior of (4.118) we derive the inequality

$$\begin{aligned}&\limsup_{N\to\infty}\int_Q \kappa(\vartheta^N)\left|\mathcal{T}''_{m,\varepsilon}(\vartheta^N)\right||\nabla\vartheta^N|^2\,d(t,x)\overset{(4.118)}{\leq}\\&\leq\int_Q \boldsymbol{S}:D\boldsymbol{u}\chi_{\{\vartheta>\frac{m}{2}\}}\,d(t,x)+\int_\Omega \vartheta_0\chi_{\{\vartheta_0>\frac{m}{2}\}}\,dx,\end{aligned}\tag{4.138}$$

where we used convergences (4.5), (4.63), (4.84), Egorov's Theorem and Dunford-Pettis Theorem. Additionally, using Fubini's Theorem we have

$$\begin{aligned}&\limsup_{N\to\infty}\int_{Q^*}\fint_t^{t+\varepsilon}\kappa(\vartheta^N)|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\,d\tau|\tilde{\varphi}|\,d(t,x)\leq\\&\leq\|\tilde{\varphi}\|_{L^\infty(Q^*)}\limsup_{N\to\infty}\int_{Q^*}\int_t^{t+\varepsilon}\frac{\kappa(\vartheta^N)}{\varepsilon}|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\,d\tau\,d(t,x)\overset{\text{Fubini}}{=}\\&=\|\tilde{\varphi}\|_{L^\infty(Q^*)}\limsup_{N\to\infty}\int_0^{T^*+\varepsilon}\int_\Omega\kappa(\vartheta^N)|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\int_{\max\{0,\tau-\varepsilon\}}^{\min\{\tau,T^*\}}\frac{1}{\varepsilon}\,dt\,dx\,d\tau\leq\\&\leq\|\tilde{\varphi}\|_{L^\infty(Q^*)}\limsup_{N\to\infty}\int_0^T\int_\Omega\kappa(\vartheta^N)|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\,dx\,d\tau\overset{(4.138)}{\leq}\\&\leq\|\tilde{\varphi}\|_{L^\infty(Q^*)}\left(\int_Q\boldsymbol{S}:D\boldsymbol{u}\chi_{\{\vartheta>\frac{m}{2}\}}\,d(t,x)+\int_\Omega\vartheta_0\chi_{\{\vartheta_0>\frac{m}{2}\}}\,dx\right),\end{aligned}$$

hence

$$\begin{aligned}&-\limsup_{N\to\infty}\int_{Q^*}\fint_t^{t+\varepsilon}\kappa(\vartheta^N)|\mathcal{T}''_{m,\delta}(\vartheta^N)||\nabla\vartheta^N|^2\,d\tau|\tilde{\varphi}|\,d(t,x)\geq\\&\geq-\|\tilde{\varphi}\|_{L^\infty(Q^*)}\left(\int_Q\boldsymbol{S}:D\boldsymbol{u}\chi_{\{\vartheta>\frac{m}{2}\}}\,d(t,x)+\int_\Omega\vartheta_0\chi_{\{\vartheta_0>\frac{m}{2}\}}\,dx\right).\end{aligned}\tag{4.139}$$

Let us now set $\tilde{\varphi}:=\mathcal{T}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})$ in (4.137). We are allowed to do that, since $\mathcal{T}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})\in L^2(0,T^*;W_0^{1,2})\cap L^\infty(Q^*)$. By (4.139) we obtain

$$\begin{aligned}&\int_{Q^*}\fint_t^{t+\varepsilon}\kappa(\vartheta)\nabla\mathcal{T}_{m,\delta}(\vartheta)\cdot\nabla\mathcal{T}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})\,d\tau\,d(t,x)\geq\\&\geq-\int_\Omega\mathcal{G}_k(\vartheta_\varepsilon^{m,\delta}(T^*)-\hat{\vartheta})-\mathcal{G}_k(\vartheta_\varepsilon^{m,\delta}(0)-\hat{\vartheta})\,dx\\&+\int_{Q^*}\left(\fint_t^{t+\varepsilon}\mathcal{T}'_{m,\delta}(\vartheta)\boldsymbol{S}:D\boldsymbol{u}-\nabla\mathcal{T}_{m,\delta}(\vartheta)\cdot\boldsymbol{u}\,d\tau\right)\mathcal{T}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})\,d(t,x)\\&-k\left(\int_Q\boldsymbol{S}:D\boldsymbol{u}\chi_{\{\vartheta>\frac{m}{2}\}}\,d(t,x)+\int_\Omega\vartheta_0\chi_{\{\vartheta_0>\frac{m}{2}\}}\,dx\right),\end{aligned}\tag{4.140}$$

where we have used that

$$\int_{Q^*}\partial_t\vartheta_\varepsilon^{m,\delta}\mathcal{T}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})\,d(t,x)=\int_{Q^*}\partial_t\left[\mathcal{G}_k(\vartheta_\varepsilon^{m,\delta}-\hat{\vartheta})\right]\,d(t,x).$$

We can now let $\varepsilon\to 0+$ in (4.140). We exploit that since T^* is a Lebesgue point it holds

$$\vartheta_\varepsilon^{m,\delta}(T^*)=\fint_{T^*}^{T^*+\varepsilon}\mathcal{T}_{m,\delta}(\vartheta(\tau))\,d\tau\to\mathcal{T}_{m,\delta}(\vartheta(T^*)),\quad\text{a.e. in }\Omega,$$

and that

$$\vartheta_\varepsilon^{m,\delta}(0) = \fint_0^\varepsilon \mathcal{T}_{m,\delta}(\vartheta(\tau))\,d\tau \to \mathcal{T}_{m,\delta}(\vartheta_0), \quad \text{a.e. in } \Omega$$

by (4.86). Limit passage yields to

$$\begin{aligned}
&\int_{Q^*} \kappa(\vartheta)\nabla\mathcal{T}_{m,\delta}(\vartheta)\cdot\nabla\mathcal{T}_k(\mathcal{T}_{m,\delta}(\vartheta)-\hat{\vartheta})\,d(t,x) \geq \\
&\geq -\int_\Omega \mathcal{G}_k(\mathcal{T}_{m,\delta}(\vartheta(T^*))-\hat{\vartheta}) - \mathcal{G}_k(\mathcal{T}_{m,\delta}(\vartheta_0)-\hat{\vartheta})\,dx \\
&+\int_{Q^*} (\mathcal{T}'_{m,\delta}(\vartheta)\boldsymbol{S}:D\boldsymbol{u} - \nabla\mathcal{T}_{m,\delta}(\vartheta)\cdot\boldsymbol{u})\,\mathcal{T}_k(\mathcal{T}_{m,\delta}(\vartheta)-\hat{\vartheta})\,d(t,x) \\
&- k\left(\int_Q \boldsymbol{S}:D\boldsymbol{u}\chi_{\{\vartheta>\frac{m}{2}\}}\,d(t,x) + \int_\Omega \vartheta_0\chi_{\{\vartheta_0>\frac{m}{2}\}}\,dx\right).
\end{aligned} \tag{4.141}$$

We note that

$$\mathcal{T}_k(\mathcal{T}_{m,\delta}(\vartheta)-\hat{\vartheta}) = \mathcal{T}_k(\vartheta-\hat{\vartheta})$$

for $m > k+1+\|\hat{\vartheta}\|_\infty$. Hence, letting $m\to\infty$ in (4.141) we get

$$\begin{aligned}
&\int_{Q^*} \kappa(\vartheta)\nabla\vartheta\cdot\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})\,d(t,x) \geq \int_{Q^*} (\boldsymbol{S}:D\boldsymbol{u} - \nabla\vartheta\cdot\boldsymbol{u})\,\mathcal{T}_k(\vartheta-\hat{\vartheta})\,d(t,x) \\
&-\int_\Omega \mathcal{G}_k(\vartheta(T^*)-\hat{\vartheta}) - \mathcal{G}_k(\vartheta_0-\hat{\vartheta})\,dx,
\end{aligned} \tag{4.142}$$

since $\mathcal{T}'_{m,\delta} \to 1$ almost everywhere. Using that

$$\int_\Omega \mathcal{T}_k(\vartheta-\hat{\vartheta})\nabla\vartheta\cdot\boldsymbol{u}\,dx = \int_\Omega \mathcal{T}_k(\vartheta-\hat{\vartheta})\nabla\hat{\vartheta}\cdot\boldsymbol{u}\,dx$$

by the same argument as in (4.130), we can rewrite (4.142) as

$$\begin{aligned}
&\int_{Q^*} \kappa(\vartheta)\nabla\vartheta\cdot\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})\,d(t,x) \geq \\
&\geq \int_{Q^*} \mathcal{T}_k(\vartheta-\hat{\vartheta})\left(\boldsymbol{S}:D\boldsymbol{u} - \nabla\hat{\vartheta}\cdot\boldsymbol{u}\right)\,d(t,x) \\
&-\int_\Omega \mathcal{G}_k(\vartheta(T^*)-\hat{\vartheta}) - \mathcal{G}_k(\vartheta_0-\hat{\vartheta})\,dx.
\end{aligned} \tag{4.143}$$

Inequality (4.143) gives us the estimate needed in (4.135) and we can conclude

$$\begin{aligned}
&\limsup_{N\to\infty} \int_{Q^*} \kappa(\vartheta^N)\nabla\vartheta^N\cdot\nabla\mathcal{T}_k(\vartheta^N-\hat{\vartheta})\,d(t,x) \leq \\
&\leq \int_{Q^*} \kappa(\vartheta)\nabla\vartheta\cdot\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})\,d(t,x).
\end{aligned} \tag{4.144}$$

By (4.144), (4.63), (4.61) and the positivity of κ we have

$$\begin{aligned}
&\limsup_{N\to\infty}\int_{Q^*}\kappa(\vartheta^N)|\nabla\mathcal{T}_k(\vartheta^N-\hat{\vartheta})|^2\,d(t,x)=\\
&=\limsup_{N\to\infty}\int_{Q^*}\kappa(\vartheta^N)\nabla(\vartheta^N-\hat{\vartheta})\cdot\nabla\mathcal{T}_k(\vartheta^N-\hat{\vartheta})\,d(t,x)\leq\\
&\leq\int_{Q^*}\kappa(\vartheta)\nabla(\vartheta-\hat{\vartheta})\cdot\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})\,d(t,x)=\\
&=\int_{Q^*}\kappa(\vartheta)|\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})|^2\,d(t,x).
\end{aligned}\tag{4.145}$$

Hence, by Lemma 11 we have the desired convergence (4.128) since the assumptions hold by (4.129) and (4.145).

Finally, we want to show that

$$\sqrt{\kappa(\vartheta^N)}\frac{\nabla\vartheta^N}{\vartheta^N}\to\sqrt{\kappa(\vartheta)}\frac{\nabla\vartheta}{\vartheta}\qquad\text{in } L^2(Q^*)\tag{4.146}$$

using Lemma 11 for $f^N:=\sqrt{\kappa(\vartheta^N)}\frac{\nabla\vartheta^N}{\vartheta^N}$. Since $\sqrt{\kappa(\vartheta^N)}\frac{\nabla\vartheta^N}{\vartheta^N}$ is uniformly bounded in $L^2(Q^*)$, we can extract a weakly converging subsequence. By (4.63), (4.126) it holds that

$$\sqrt{\kappa(\vartheta^N)}\frac{\nabla\vartheta^N}{\vartheta^N}\rightharpoonup\sqrt{\kappa(\vartheta)}\frac{\nabla\vartheta}{\vartheta}\qquad\text{in } L^2(Q^*).\tag{4.147}$$

To show that the second assumption of Lemma 11 holds, we rewrite

$$\begin{aligned}
&\int_{Q^*}\kappa(\vartheta^N)\frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2}\,d(t,x)=\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla\hat{\vartheta}}{\vartheta^N}\,d(t,x)\\
&+\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla(\vartheta^N-\hat{\vartheta})}{\vartheta^N}\,d(t,x)=\\
&=\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla(\vartheta^N-\hat{\vartheta})}{\vartheta^N}\chi_{\{|\vartheta^N-\hat{\vartheta}|>k\}}\,d(t,x)\\
&+\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla\mathcal{T}_k(\vartheta^N-\hat{\vartheta})}{\vartheta^N}+\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla\hat{\vartheta}}{\vartheta^N}\,d(t,x)
\end{aligned}\tag{4.148}$$

and let $N\to\infty$ on the right-hand side. Firstly, by convergences (4.128), (4.147), (4.63), and the minimum principle $\vartheta^N\geq\mu$, we have

$$\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla\mathcal{T}_k(\vartheta^N-\hat{\vartheta})}{\vartheta^N}\,d(t,x)\to\int_{Q^*}\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\frac{\nabla\mathcal{T}_k(\vartheta-\hat{\vartheta})}{\vartheta}\,d(t,x).\tag{4.149}$$

Similarly, by convergences (4.147), (4.63), the minimum principle $\vartheta^N\geq\mu$, and the fact $\nabla\hat{\vartheta}\in L^2(Q^*)$, it holds

$$\int_{Q^*}\kappa(\vartheta^N)\frac{\nabla\vartheta^N}{\vartheta^N}\cdot\frac{\nabla\hat{\vartheta}}{\vartheta^N}\,d(t,x)\to\int_{Q^*}\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\frac{\nabla\hat{\vartheta}}{\vartheta}\,d(t,x).$$

Lastly, for $k > \|\hat{\vartheta}\|_\infty$ the inequality $|\vartheta^N - \hat{\vartheta}| > k$ implies $\vartheta^N \geq k$, since $\vartheta^N, \hat{\vartheta} > 0$. Consequently, we can use Young's inequality to estimate

$$\left| \int_{Q^*} \kappa(\vartheta^N) \frac{\nabla \vartheta^N}{\vartheta^N} \cdot \frac{\nabla(\vartheta^N - \hat{\vartheta})}{\vartheta^N} \chi_{\{|\vartheta^N - \hat{\vartheta}| > k\}} \, d(t,x) \right| \leq$$

$$\leq 2\overline{\kappa} \int_{Q^*} \frac{|\nabla \vartheta^N|^2 + |\nabla \hat{\vartheta}|^2}{(\vartheta^N)^2} \chi_{\{|\vartheta^N - \hat{\vartheta}| > k\}} \, d(t,x) \leq$$

$$\leq 2\overline{\kappa} \int_{Q^*} \frac{|\nabla \vartheta^N|^2 + |\nabla \hat{\vartheta}|^2}{(\vartheta^N)^{1+\lambda} k^{1-\lambda}} \, d(t,x) \leq \frac{C(\lambda)}{k^{1-\lambda}},$$

where $\lambda \in (0,1)$. Note, that C is independent of N and k by the uniformity of estimates (4.48) and the minimum principle $\vartheta^N \geq \mu$. Furthermore, it holds

$$\int_{Q^*} \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \, d(t,x) = \int_{Q^*} \kappa(\vartheta) \frac{\nabla \vartheta}{\vartheta} \cdot \frac{\nabla(\vartheta - \hat{\vartheta})}{\vartheta} \chi_{\{|\vartheta - \hat{\vartheta}| > k\}} \, d(t,x)$$

$$+ \int_{Q^*} \kappa(\vartheta) \frac{\nabla \vartheta}{\vartheta} \cdot \frac{\nabla \mathcal{T}_k(\vartheta - \hat{\vartheta})}{\vartheta} \, d(t,x) + \int_{Q^*} \kappa(\vartheta) \frac{\nabla \vartheta}{\vartheta} \cdot \frac{\nabla \hat{\vartheta}}{\vartheta} \, d(t,x),$$

and also

$$\left| \int_{Q^*} \kappa(\vartheta) \frac{\nabla \vartheta}{\vartheta} \cdot \frac{\nabla(\vartheta - \hat{\vartheta})}{\vartheta} \chi_{\{|\vartheta - \hat{\vartheta}| > k\}} \, d(t,x) \right| \leq \frac{C(\lambda)}{k^{1-\lambda}} \tag{4.150}$$

by (4.61). We can thus take the limit in (4.148) and using (4.149)–(4.150) we conclude

$$\limsup_{N \to \infty} \int_{Q^*} \kappa(\vartheta^N) \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} \, d(t,x) \leq \int_{Q^*} \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \, d(t,x) + \frac{C(\lambda)}{k^{1-\lambda}}. \tag{4.151}$$

Since the left-hand side of (4.151) doesn't depend on k, we can take the limit $k \to \infty$ on both sides to obtain

$$\limsup_{N \to \infty} \int_{Q^*} \kappa(\vartheta^N) \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} \, d(t,x) \leq \int_{Q^*} \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \, d(t,x). \tag{4.152}$$

Lemma 11 together with (4.152) and (4.147) imply strong convergence (4.146). We can a priori construct the solution on the extended time interval $(0, 2T)$. Thus, T^* can be chosen bigger than T and so we obtain (4.146) in $L^2(Q)$.

By (4.146) and (4.147) we have

$$\kappa(\vartheta^N) \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} \to \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \qquad \text{in } L^1(Q),$$

because

$$\int_Q \left| \kappa(\vartheta^N) \frac{|\nabla \vartheta^N|^2}{(\vartheta^N)^2} - \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \right| d(t,x) \leq$$

$$\leq \int_Q \left(\sqrt{\kappa(\vartheta^N)} \frac{\nabla \vartheta^N}{\vartheta^N} - \sqrt{\kappa(\vartheta)} \frac{\nabla \vartheta}{\vartheta} \right)^2 d(t,x)$$

$$+ 2 \int_Q \left| \sqrt{\kappa(\vartheta)} \frac{\nabla \vartheta}{\vartheta} \right| \left| \sqrt{\kappa(\vartheta^N)} \frac{\nabla \vartheta^N}{\vartheta^N} - \sqrt{\kappa(\vartheta)} \frac{\nabla \vartheta}{\vartheta} \right| d(t,x) \to 0.$$

Since we have shown all the necessary convergences, we can now pass to the limit $N \to \infty$ in (4.104). Consequently, $\eta \in L^2(0,T; W^{1,2}) \cap L^q(Q), q \in [1, \infty)$ satisfies the *Entropy equation* (3.12). All the parts of the theorem are proven.

Conclusion

In the thesis, we provided a qualitative analysis of the system (1)–(5). More precisely, we determined the sufficient bound on the power-law index p of Cauchy-stress tensor $\boldsymbol{\mathcal{S}}$, which ensures the existence of a solution in two dimensions. The bound $p \geq 2$ differs from the three-dimensional case studied in [12] where $p \geq 11/5$. This difference stems from the need to use the Sobolev Embedding Theorem to estimate the convective term while obtaining the apriori estimates on the time derivative of the velocity. Additionally, we have shown that the constructed weak solution to Navier-Stokes-Fourier system (1)–(3) is regular enough to satisfy the entropy equality (4).

This qualitative analysis can serve as a cornerstone of nonlinear stability research. In fact, following the methods from [14], we should be able to show the existence of a steady weak solution to (1)–(5) for $\boldsymbol{f} = \mathbf{0}$ that is non-linearly stable and attracts all suitable weak solutions.

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