Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Bc. Jana Burešová

Kapitálový požadavek ke kreditnímu riziku

Katedra pravděpodobnosti a matematické statistiky

Vedoucí diplomové práce: RNDr. Stanislav Keprta

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Děkuji panu RNDr. Stanislavu Keprtovi za poskytnuté konzultace a velmi cenné připomínky a podněty při zpracování mé diplomové práce.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a s jejím zveřejňováním.

V Praze dne 10.12.2008 Bc. Jana Burešová

Contents

Název práce: Kapitálový požadavek ke kreditnímu riziku
Autor: Bc. Jana Burešová
Katedra (ústav): Katedra pravděpodobnosti a matematické statistiky
Vedoucí diplomové práce: RNDr. Stanislav Keprta, Raiffeisenbank
e-mailová adresa vedoucího: Stanislav.Keprta@rb.cz

Abstrakt: V předložené práci studujeme způsob stanovení výše kapitálového požadavku ke kreditnímu riziku, který byl navržen Basilejským výborem pro bankovní dohled a který je po zaintegrování do evropské legislativy, t.j. od 1.1.2007 závazný pro banky a další finanční instituce v členských státech Evropské unie.

Diplomová práce nejprve uvádí základní prvky a principy celého dokumentu z roku 2004 známého pod názvem Basel II (The New Basel Capital Accord, Nová kapitálová dohoda). Poté následuje detailní rozbor části prvního pilíře, která se zabývá kreditním rizikem. V další kapitole jsou popsány tři matematické modely, ze kterých je výpočet kapitálového požadavku odvozen. Na základě teoretických znalostí těchto modelů a s pomocí výsledků studií provedených Basilejským výborem je postupně vysvětlena každá část výsledného vzorce, ktery je pak aplikován na hypotetické úvěrové portfolio banky. V závěru práce jsou porovnány výstupy basilejského vzorce pro dané portfolio se simulací reality na základě daných předpokladů.

Klíčová slova: kapitálový požadavek, default, úvěrové portfolio

Title: Capital requirement for credit risk
Author: Bc. Jana Burešová
Department: Department of probability and mathematical statistics
Supervisor: RNDr. Stanislav Keprta, Raiffeisenbank
Supervisor's e-mail address: Stanislav.Keprta@rb.cz

Abstract: In the present work we study the process of determination of capital requirement for credit risk that is recommended by the Basel Commitee for Banking Supervision to implement into national legislation and that is also obligatory for all European banks since it is a part of the Capital Requirement Directive.

At the beginning of this thesis, basic principles and three pillars of The New Basel Capital Accord (2004), better known as Basel II, are described. After focusing on the part of the first pillar dealing with credit risk, different approaches to credit risk measurement are introduced. The most important formula for the advanced internal-ratings based approach is then analyzed under the settings of mathematical models it is based on. In the last chapter, the output of the formula for capital requirement calculated for a given hypothetic portfolio is compared to the estimate of unexpected loss, that the requirement should correspond to.

Keywords: capital requirement, default, loan portfolio

Chapter 1

Úvod / Introduction

Kreditní riziko patří z důvodu obvykle velkého objemu úvěrů, hypoték a půjček na straně aktiv k nejpodstatnějším rizikům působícím na finanční instituce. Je to riziko, že dlužník banky nesplní své závazky podle dohodnutých podmínek. Banky si jsou tohota rizika vědomi a proto vytváří různé rezervy a opravné položky. Ale ty jsou obvykle ve výši očekávané ztráty.

Basilejský výbor pro bankovní dohled v tzv. dokumentu Basel II stanovil výpočet takové výše kapitálu, aby banky byly chráněny ve většine případech. Tj. tvrdí, že je třeba mimo již vytvořených rezerv ponechat ještě další kapitál, tak aby banka, potažmo vklady drobných střadatelů, byla chráněna na 99,9%.

Basel II byl vypracován vrámci harmonizace národních legislativ. Vedle kreditního rizika se zabývá i otázkami operačního a tržního rizika, dohledu a tržní dispilíny. Byl zaintergrován do evropské legislativy prostřednictvím směrnice Evropského parlamentu a Rady 2006/48/ES o přístupu k činnosti úvěrových institucí a o jejím výkonu a směrnicí 2006/49/ES o kapitálové přiměřenosti investičních podniků a úvěrových institucí (obě ze dne 14.června 2006) a je proto závazný pro všechny banky v členských státech Evropské unie. Do české legislativy byl implementován vyhláškou České národní banky č.123/2007 Sb., o obezřetném podnikání bank, spořitelních a úvěrních družstev a obchodníků s cennými papíry.

Tento dokument jako celek je popisován v druhé kapitole, kdežto ve třetí kapitole jsme se zaměřili pouze na oblast kreditního rizika. Basel II bankám umožňuje (pokud splní určité požadavky) vybrat si ze tří různých přístupů k výpočtu rizikových vah používaných při stanovení kapitálového požadavku. Tyto přístupy se liší především způsobem vyčíslení rizikových komponent vzorce. Těmi jsou pravděpodobnost selhání (defaultu), ztráta při selhání, expozice a doba splatnosti. Dále se tato práce zabývá matematickými modely, na nichž Basel II staví, a detaily vzorce stanoveného pro výpočet kapitálové přiměřenosti a jeho provázanost s uvedenými modely. Poslední kapitola porovnává doposud popisovaný vzorec s výsledky simulace pro hypotetické portfolio prováděných v softwaru Wolfram Mathematica 6.0. Zabýváme se poměrem mezi kapitálovým požadavkem vypočítaným podle Basel II pro různé vstupní hodnoty portfolia a odhadem neočekávané ztráty toho samého portfolia. Kapitálový požadavek by právě měl pokrýt tuto neočekávanou ztrátu, která je stanovena jako rozdíl mezi 99,9-procentním kvantilem rozdělení ztráty a střední hodnotou tohoto rozdělení, a proto nás zajímá, zda a kdy je tento podíl blízko jedné.

Bank´s assets consist from a large part of credits and mortgages and the bank is therefore influenced by credit risk, risk that a borrower does not meet his liability in full. Banks are aware of this risk but they only care about the expected loss.

The procedure of how to deal with unexpected loss and how much capital to hold is given by the New Basel Capital Accord (known as Basel II). This document issued by the Basel Commitee on Banking Supervision in June 2004 was later transformed into a directive of the European Union and is obligatory for all European banks.

We describe this document in the second chapter, the part dealing with credit risk and basic principles for assessing capital requirement for credit risk is discussed in the third chapter. Banks are allowed to choose between three distinct approaches, subject to some minimum requirements. This approaches differ in the ability provide own estimates of risk components. These are default probability, loss given default, exposure and effective maturity. The fourth chapter includes three mathematical models Basel II is based on and serves as an introduction to the capital requirement assessing in chapter five. The formula and also its characteristics are explained here in details. In the last chapter we would like to compare the capital requirement formula with the estimated unexpected loss for a hypothetic portfolio. For simulations we use the Wolfram Mathematica 6.0 software.

Chapter 2

The New Basel Capital Accord

In June 2004 the Basel Committee on Banking Supervision (BCBS)[1] issued a *Revised Framework on International Convergence of Capital Measurement and Capital Standards* (also known as *Basel II* or the *New Basel Capital Accord*) [4]. This Revised Framework will be applied on a consolidated basis to internationally active banks and should serve as the basis for further national implementation processes in order to protect depositors. It is also intended to encourage ongoing improvements in banks' risk management. A bank is supposed to know its loan portfolio better than the supervisor.

After extensive dialogue with industry participants and national supervisors, Committee replaced the *1988 Accord* (with several amendments) to improve the way regulatory capital requirements reflect underlying risks, the greatest disadvantage of the prior modification. Basel II is also designed to better reflect the developement of financial markets in the 90´s of 20^{th} century, for example, the creation of asset securitisation structures. Another aspect to the modification was the feedback received from banks participat-

[1]The Basel Committee on Banking Supervision is a committee of banking supervisory authorities which was established by the central-bank Governors of the Group of Ten countries at the end of 1974. It usually meets at the Bank for International Settlements in Basel, Switzerland, where its permanent Secretariat is located.

ing in several impact studies, altogether more than 350 banks of varying size and levels of complexity from more than 40 countries.

The New Basel Capital Accord consists of three pillars:

- Minimum capital requirements,
- Supervisory review of capital adequacy, and
- Market discipline.

The minimum capital requirement is determined by three different kinds of risk faced by banks: credit risk[2], market risk[3] and operational risk[4]. The procedure of calculations is described in paragraphs 40 and 44: *In calculating the capital ratio, the denominator or total risk-weighted assets will be determined by multiplying the capital requirements for market risk* ($\mathrm{CR}_{\mathrm{market}}$) *and operational risk* ($\mathrm{CR}_{\mathrm{operational}}$) *by 12.5 (i.e. the reciprocal of the minimum capital ratio of 8%) and adding the resulting figures to the sum of risk-weighted assets* (RWA) *compiled for credit risk. The ratio will be calculated using regulatory capital as the numerator. The total capital ratio must be no lower than 8%.*

$$\frac{\text{Regulatory Capital}}{1.06 \times \mathrm{RWA} + 12.5 \times (\mathrm{CR}_{\mathrm{market}} + \mathrm{CR}_{\mathrm{operational}})} \geq 8\% \qquad (2.1)$$

Banks are allowed to use distinct approaches to the measurement of capital requirements for credit and operational risk. As for market risk, certain basic requirements for positions eligible to receive trading book capital treatment are specified in Basel II, Part 2, Section VI. We will focus on credit risk in following chapters.

[2]Risk that the counterparty is no longer able to pay back the promised payment.

[3]Arising from trading activities of banks.

[4]Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems.

Paragraph 14 of the New Basel Capital Accord provides explanation for the factor 1.06: *The Committee believes it is important to reiterate its objectives regarding the overall level of minimum capital requirements. These are to broadly maintain the aggregate level of such requirements, while also providing incentives to adopt the more advanced risk-sensitive approaches of the revised Framework. To attain the objective, the Committee applies a scaling factor to the risk-weighted asset amounts for credit risk under the IRB approach. The current best estimate of the scaling factor using quantitative impact study data is 1.06.*

The pillar II *Supervisory review of capital adequacy* ensures the effectivity of both sides when pointing to the need for banks to assess their capital adequacy positions relative to their overall risks, and for supervisors to review and take appropriate actions in response to those assessments.

The purpose of the third pillar *Market discipline* is to complement pillar I and II by providing a set of disclosure requirements that allow market participants to assess key information about a bank's risk profile.

Chapter 3

Overview of Credit Risk Approaches in Basel II

3.1 Introduction to the Credit Loss

Under the New Basel Capital Accord, banks are allowed to choose among three approaches to calculate their capital requirement to cover credit risk: the standardised approach, the foundation internal ratings-based (IRB) approach and the advanced IRB approach. Before describing these three methods in details, we need to introduce the basis of how the credit loss is derived.

Let n denote the number of loans in the portfolio. The annual **credit loss of the portfolio**[1] is given by

$$L_n = \sum_{i=1}^{n} D_i \times LGD_i \times EAD_i,$$

where

- D_i is a Bernoulli random variable. $D_i = 1$ if the i-th obligor defaults within a one-year interval, $D_i = 0$ otherwise. The **default probabil-**

[1] Note that values of losses are all considered to be positive.

ity of the i-th obligor over a one-year time horizon PD_i is then the unconditional expected value of this variable,

- **loss given default** of the i-th instrument LGD_i represents random percentage of exposure the bank might lose in case of borrower´s default in the particular year. This percentage can depend on the type and amount of collateral. It can be also represented by the recovery rate. In this case $LGD_i = 1-$ *recovery rate*. For instance, if a default event leads to the recovery rate of 40%, LGD_i is then 60% of the exposure.

- EAD_i is the i-th instrument´s **exposure at default**, which is the economic value of the claim on the counterparty at the time of default. It is also a random variable because we don´t know when the default occurs (if it occurs) and what will the economic value developement be.

The total loss experienced in a particular year is a random variable. It is impossible for the bank to know the next year´s loss. There exist, nevertheless, some figures a bank is able to forecast. **Expected Losses (EL)** perform the average level of total credit losses a bank can reasonably expect to suffer. It is a cost component and is therefore covered by provisions and revenues. The expected loss for the i-th obligor can be written as

$$EL_i = PD_i \times LGD_i \times EAD_i. \tag{3.1}$$

Losses above the expected level are usually referred to as **Unexpected Losses (UL)** (as shown in figure 3.1, source [3]). Bank has some means that could absorb a part of unexpected losses (e.g. interest rates) but in the event of peak losses, additional capital is needed. On the other hand, holding too much capital for such a case means less profit for the bank. Thus, banks and their regulatory supervisors must carefully balance the risks of holding capital and the profit from that.

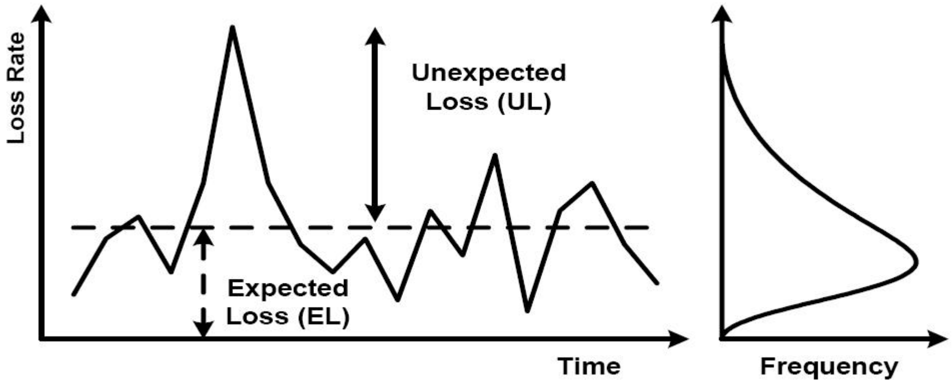

Figure 3.1: An example of loss rate experience

Capital is required so that it should protect the bank in most cases. We usually talk about the 99.9% confidence level. For this purpose we need to introduce the **Value-at-Risk (VaR)** variable[2]. Under our settings, it is defined as the q-th quantile of the distribution function of total credit loss at a given confidence level q. It holds

$$P[L_n > \mathrm{VaR}_q(L_n)] = 1 - q. \tag{3.2}$$

The sum of expected and unexpected losses is then set to equal the Value-at-Risk (VaR) at this confidence level (see figure 3.2 , source [3]).

$$\mathrm{UL} = \mathrm{VaR}_q(L_n) - \mathrm{EL}. \tag{3.3}$$

In addition, Basel II requires banks to undertake credit risk stress tests to underpin all these calculations. As described in paragraphs 434 and 435: *Stress testing must involve identifying possible events or future changes in economic conditions that could have unfavourable effects on a bank's credit exposures. The test to be employed must be meaningful and reasonably conservative. Individual banks may develop different approaches to undertaking*

[2]Value-at-Risk as a risk measure is easy to comprehend but it is difficult to calculate in case of insufficient data.

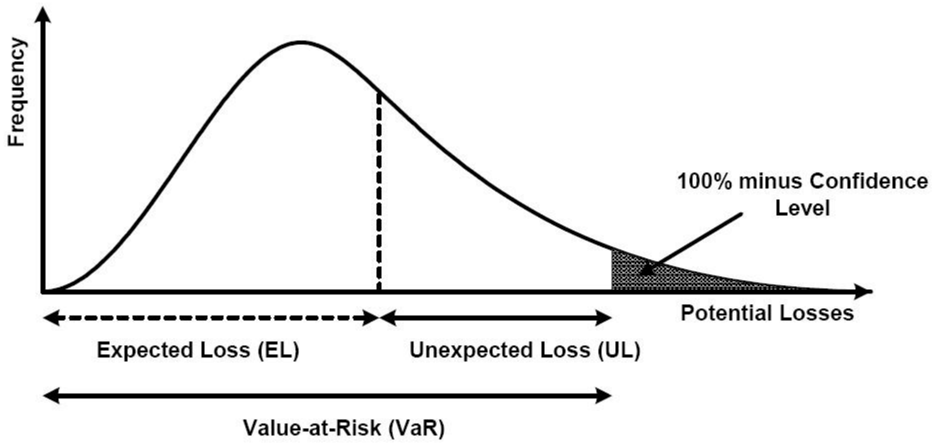

Figure 3.2: Typical loss rate density function

this stress test requirement, depending on their circumstances. For this purpose, the objective is not to require banks to consider worst-case scenarios. The bank's stress test in this context should, however, consider at least the effect of mild recession scenarios. In this case, one example might be to use two consecutive quarters of zero growth to assess the effect on the bank's PDs, LGDs and EADs, taking account – on a conservative basis – of the bank's international diversification.

3.2 Credit Risk - The Standardised Approach

The standardised approach is an extenstion of the 1988 Accord, with more sophisticated classification of categories for credit risk. To determine the risk weights in the standardised approach, banks may use assessments by external credit assessment institutions recognised as eligible for capital purposes by national supervisors, divide their portfolios into classes and then follow the numbers given in table 3.1. The risk category of 150% is newly introduced. Consideration of collaterals is another advantage of this extention. Claims secured by residential property (in the table labeled as *Mortgages*) will be weighted by 35%.

There are two possible options for claims on banks. Option 1 is based on the risk weight of the sovereign country where the bank is situated, but one category less favourable. Option 2 (subject to some floors) is based on the assessment of the bank itself. Preferential risk weights labeled with ST (short term) can be applied to claims with original maturity of 3 months or less. In general, these short term claims are evaluated one category more favourable.

After setting-up the risk weights and deriving the exposure value the way given in section 2.II. of the New Basel Capital Accord, capital for a given loan is derived as

$$Capital\ Requirement = Risk\ Weight \times Exposure \times 8\%.$$

	Assessment					
Claim on	AAA to AA-	A+ to A-	BBB+ to BBB-	BB+ to BB-	Below B-	Unrated
Sovereigns	0	20	50	100	150	100
Banks	20 20	50 50 LT 20 ST	100 50 LT 20 ST	100 100 LT 50 ST	150 150	100 50 LT 20 ST
Corporates	20	50	100	100	150	100
Mortgages						35
Retail						75

Table 3.1: An example of percentage risk weights in the standardised approach

3.3 Credit Risk - The Internal Ratings-Based Approach

Since 1988, when the first Capital Accord was published, the financial world has changed dramatically and there was a growing demand within the financial world for a new capital accord. Basel II should serve as an operative and more flexible mean for the banking system how to measure the potentially right level of capital and for the supervisors how to set their legal framework.

Under the IRB approach bank must categorize instruments into classes of assets with different underlying risk characteristics as well as under the standard approach. Asset classes are defined in paragraphs 215 - 243 and differ in three key elements:
(*i*) *risk components,*
(*ii*) *risk-weight functions* by which risk components are transformed into risk weighted assets, and
(*iii*) *minimum requirements* that must be met in order for a bank to use the IRB approach for a given asset class.

According to paragraph 180 of the New Basel Capital Accord the risk components include measures of the

- probability of default (PD),
- loss given default (LGD),
- the exposure at default (EAD), and
- effective maturity (M).

Paragraph 245 provides the difference between the foundation IRB approach and the advanced IRB approach: *Under the foundation approach, as a general rule, banks provide their own estimates of PD and rely on supervisory estimates for other risk components. Under the advanced approach,* *banks provide more of their own estimates of PD, LGD and EAD, and their own calculation of M, subject to meeting minimum standards.*

Under the foundation IRB approach, senior claims on corporates, sovereigns and banks not secured by recognised collateral will be assigned a 45% LGD and all subordinated claims on corporates, sovereigns and banks a 75% LGD. Effective maturity (M) will be 2.5 years except for repo-style transactions where the effective maturity will be 6 months. As for EAD, 100% of on-balance sheet items must be taken into account. But also off-balance sheet items are taken into consideration. Prescribed credit conversion factors are applied to them.

On the basis of the work of Merton [9] and Vasicek [10], BCBS decided to adopt the assumptions of a normal distribution for the systematic and idiosyncratic risk factors of a credit portfolio and the technique of computing PDs. For the model behind the capital requirements function BCBS assumes even more, the model should be *portfolio-invariant*, i.e. capital required for any given loan should only depend on the risk of that loan and must not depend on the portfolio it is added to. Basel II is therefore built on the Gordy´s thoughts [8] because it can be shown that only so called Asymptotic Single Risk Factor (ASRF) Models fulfil such a condition of portfolio-invariance. These three models will be described in next chapter.

Chapter 4

Mathematical Foundations of Risk Weight Formulas

4.1 Merton´s Model

The Merton´s model refers to a model proposed by Robert C. Merton in 1974 for assessing the credit risk of a company by characterizing the company's equity as a Europian call option on its assets. Many studies published in the first half of the 70´s were focused on the time structure of interest rates but Merton was the first economist dealing with probabilities of default. His paper called *On the Pricing of Corporate Debt: The Risk Structure of Interest Rates* [9] the basis for further extentions.

Merton assumes a *diffusion-type stochastic process* for the value of the firm at time t V_t. This process is given by a stochastic differential equation

$$\mathrm{d}V_t = rV_t\mathrm{d}t + \sigma V_t\mathrm{d}z_t, \tag{4.1}$$

where r is the (constant) instantaneous expected rate of return on the firm per unit of time, σ^2 is the (constant) instantaneous variance of the return on the firm per unit of time and z_t is a standard Wiener process with following properties.

- $z_0 = 0$,
- z_t is almost surely continuous,
- z_t has independent increments with distribution $z_t - z_s \sim \mathcal{N}(0, t-s)$, $t > s \geq 0$.

For the Wiener process it holds that the change $\mathrm{d}z_t$ during a small period of time $\mathrm{d}t$ could be rewritten as $\mathrm{d}z_t = \varepsilon_t \sqrt{\mathrm{d}t}$, where ε_t is a *random drawing from a standard normal distribution* $\mathcal{N}(0,1)$.

Consequently, since

$$\mathrm{E}\left(\frac{\mathrm{d}V_t}{V_t}\right) = \mathrm{E}(r\mathrm{d}t + \sigma \mathrm{d}z_t) = r\mathrm{d}t,$$

and

$$\mathrm{Var}\left(\frac{\mathrm{d}V_t}{V_t}\right) = \mathrm{Var}(r\mathrm{d}t + \sigma \mathrm{d}z_t) = \mathrm{Var}(\sigma\varepsilon_t\sqrt{\mathrm{d}t}) = \sigma^2\mathrm{d}t,$$

the relative change in the firm value is normally distributed with the expectation of $r\mathrm{d}t$ and the variance of $\sigma^2\mathrm{d}t$, denoted by

$$\frac{\mathrm{d}V_t}{V_t} \sim \mathcal{N}\Big(r\mathrm{d}t, \sigma^2\mathrm{d}t\Big).$$

For further calculations we need to introduce the *Itô´s Lemma*: Let y_t be governed by a diffusion process

$$\mathrm{d}y_t = \mu_t(y_t, t)\mathrm{d}t + \sigma_t(y_t, t)\mathrm{d}z_t.$$

Let $g = g(y_t, t)$ be twice differentiable. Then

$$\mathrm{d}g = \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial y_t}\mathrm{d}y_t + \frac{1}{2}\frac{\partial^2 g}{\partial y_t^2}(\mathrm{d}y_t)^2.$$

For our purposes, we substitute $y_t = V_t$ and define $g = \ln V_t$. Thus,

$$\mathrm{d}\ln V_t = 0\mathrm{d}t + \frac{1}{V_t}(rV_t\mathrm{d}t + \sigma V_t\mathrm{d}z_t) - \frac{1}{2}\frac{1}{V_t^2}(rV_t\mathrm{d}t + \sigma V_t\mathrm{d}z_t)^2 = \left(r - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}z_t.$$

When considering the time period from time 0 to time T, we can derive the distribution of the firm value at a future time T.

$$\ln V_T - \ln V_0 = \int_0^T \mathrm{d} \ln V_t = \Big(r - \frac{\sigma^2}{2}\Big)T + \sigma \mathrm{d}z_T$$

$$\mathrm{E}(\ln V_T - \ln V_0) = \left(r - \frac{\sigma^2}{2}\right)T,$$

$$\mathrm{Var}(\ln V_T - \ln V_0) = \sigma^2 T.$$

Following the notation of the standard Wiener process, where ε is a random drawing from the standard normal distribution, $\ln V_T$ and V_T can be expressed as

$$\ln V_T = \ln V_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon_T, \tag{4.2}$$

$$V_T = V_0 \exp\left(\Big(r - \frac{\sigma^2}{2}\Big)T + \sigma\sqrt{T}\varepsilon_T\right). \tag{4.3}$$

The distribution of $\ln V_t$ is therefore normal,

$$\ln V_T \sim \mathcal{N}\left(\ln V_0 + \Big(r - \frac{\sigma^2}{2}\Big)T, \sigma^2 T\right), \tag{4.4}$$

and V_T follows lognormal distribution.

In order to treat the firm equity as a call option we need to suppose following: there are only two classes of claims, debt and equity. The firm promised to pay an amount of B to the bondholders on the specified date T. In the event this payment is not met, the bondholders take over the company. From the stockholders´ point of view, they either get what is left over after the debt paying on the date T, or they lose whole investment. This characteristics could be represented by

$$S_T = \max(V_T - B, 0), \tag{4.5}$$

where S_T denotes the value of firm equity at time T and V_T the value of the firm at time T.

This expression shows that the equity is a call option on the value of the firm (its assets) with a strike price equal to the book value of the liabilities (promised payment). **The Black-Scholes formula**[1] for our option tells us

$$S_0(V_0, \tau) = V_0 N(x_1) - Be^{-r\tau} N(x_2)^2, \tag{4.6}$$

where S_0 is the current value of equity depending on current value of the firm V_0 and time to maturity τ ,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\Big(-\frac{1}{2} z^2 \Big) \mathrm{d}z$$

is the cumulative distribution function (CDF) of the standard normal distribution,

$$x_1 = \frac{1}{\sigma\sqrt{\tau}} \Big[\ln \frac{V_0}{B} + (r + \frac{\sigma^2}{2})\tau \Big],$$

and

$$x_2 = x_1 - \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}} \Big[\ln \frac{V_0}{B} + (r - \frac{\sigma^2}{2})\tau \Big].$$

The default condition $V_T < B$ (the firm cannot pay the debt because the value of assets is less than the value of liabilitites) specifies the unconditional probability of default as

$$p = P[V_T < B].$$

As given by 4.5, $S_T = 0$ in this case and the bondholders took over the whole

[1]First introduced by Fisher Black and Myron Scholes in 1973. Merton and Scholes received the 1997 Nobel Prize in Economics for this and related work. Black was mentioned as a contributor.

[2]Note that $Be^{-r\tau}$ is the discounted debt.

company. Subtituting V_T as the expression in 4.3 yields

$$\begin{aligned}
p &= P\left[V_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon_T\right) < B\right] \\
&= P\left[\ln V_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon_T < \ln B\right] \\
&= P\left[\sigma\sqrt{T}\varepsilon_T < \ln B - \ln V_0 - \left(r - \frac{\sigma^2}{2}\right)T\right] \\
&= P\left[\varepsilon_T < -\frac{\ln(V_0/B) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}\right] \\
&= P[\varepsilon_T < -x_2] \\
&= N(-x_2).
\end{aligned}$$

The advantage of this model is the advantage of the Black-Scholes formula, i.e. the ability to compute desired results only from a few values that are either known or can be easily computed. Disadvantages, however, are given by the represention of $\mathrm{d}V_t$ as a diffusion process with constant loadings. In reality, the expected rate of return as well as the variance of return depend on time.

4.2 Vasicek´s Model

The following thoughts have been published in Oldrich Alfons Vasicek´s papers *Loan Portfolio Value* [10]. The core settings of Vasicek´s model are originated from Merton´s model, however, specific assumptions about the loss portfolio are made.

Consider a portfolio consisting of n loans in equal dollar amounts. Let the probability of default on any single loan be p, and assume that the asset values of the borrowing companies are correlated with a coefficient ρ for any two companies. We will further assume that loans have the same term T. (Such a large number of assumptions is a great weakness of this model. Especially when the portfolios do not consist of loans from one industry in order to diversify the risk.)

For purposes of this section we can derive *the portfolio percentage gross loss* L as

$$L = \frac{1}{n} \sum_{i=1}^{n} L_i,$$

where L_i is the gross loss on the i-th loan and $L_i = 1$ if the i-th borrower defaults, $L_i = 0$ otherwise.

Given the central limit theorem, the portfolio loss distribution would converge to a normal distribution as the number of loans increases if the defaults on the loans in our portfolio were independent of each other. Because it is not such a case, the conditions of the central limit theorem are not satisfied and L is not asymtotically normal. However, there exists a limiting distribution as explained later.

Following the Merton´s model and the equation 4.2, the logarithm of the asset value of the i-th firm at time T could be expressed as

$$\ln V_{iT} = \ln V_{i0} + \left(r_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i \sqrt{T} \varepsilon_{iT}. \tag{4.7}$$

The variables ε_i are jointly standard normal distributed. Vasicek proposes to rewrite them as

$$\varepsilon_i = a_i Y + b_i Z_i,$$

where Y, Z_1, Z_2, $\ldots$, Z_n are mutually independent (and without loss of generality) standard normal variables. The random variable Y can be interpreted as a portfolio common factor (systematic risk) over the interval $(0, T)$, e.g. economic index, and the set of variables $Z_1, Z_2, ..., Z_n$ as idiosyncratic risk factors. Then the term $Y\sqrt{\rho}$ is the company´s exposure to the common risk factor and the term $Z_i\sqrt{1-\rho}$ represents the company specific risk.

In order to ensure next assumtion of equal pairwise correlation coefficient ρ and in order to get the standard normal ditribution of ε_i, constants a_i needs to equal $\pm\sqrt{\rho}$ as well as b_i needs to equal $\pm\sqrt{1-\rho}$ for every i. Then

$$\mathrm{E}(\varepsilon_i) = \mathrm{E}(a_i Y + b_i Z_i) = a_i \mathrm{E}(Y) + b_i \mathrm{E}(Z_i) = 0,$$

$$\begin{aligned} \mathrm{Var}(\varepsilon_i) &= \mathrm{E}(\varepsilon_i^2) - \mathrm{E}(\varepsilon_i)^2 = \mathrm{E}\Big((a_i Y + b_i Z_i)^2\Big) - 0 \\ &= a_i^2 \mathrm{E}(Y)^2 + 2a_i b_i \mathrm{E}(Y Z_i) + b_i^2 \mathrm{E}(Z_i)^2 \\ &= a_i^2 + b_i^2 = \rho + (1-\rho) = 1, \end{aligned}$$

$$\begin{aligned} \mathrm{Corr}(\varepsilon_i, \varepsilon_j) &= \mathrm{Cov}(\varepsilon_i, \varepsilon_j) = \mathrm{Cov}(a_i Y + b_i Z_i, a_j Y + b_j Z_j) \\ &= a_i a_j \mathrm{Cov}(Y, Y) + a_i b_j \mathrm{Cov}(Y, Z_j) + b_i a_j \mathrm{Cov}(Z_i, Y) + b_i b_j \mathrm{Cov}(Z_i, Z_j) \\ &= a_i a_j = \rho. \end{aligned}$$

Vasicek denotes

$$a_i = \sqrt{\rho}\ ,\ b_i = \sqrt{1-\rho}\ ,\ \forall i = 1, ..., n,$$

and therefore

$$\varepsilon_i = \sqrt{\rho} Y + \sqrt{1-\rho} Z_i.$$

Based on all the settings mentioned above, we will evaluate the conditional probability of default, conditional on the realization of Y. This can be interpreted as assuming various scenarios for the economy, determining the probability of a given portfolio loss under each scenario and then weighting each scenario by its likelihood. When the common factor is fixed, the probability of default of any single obligor is

$$\begin{aligned}
p(Y) &= P[L_i = 1 \mid Y] \\
&= P[\varepsilon_i < -x_2 \mid Y] \\
&= P[\sqrt{1-\rho}Z_i + \sqrt{\rho}Y < -x_2 \mid Y] \\
&= P\left[Z_i < \frac{-x_2 - \sqrt{\rho}Y}{\sqrt{1-\rho}} \mid Y\right] \\
&= P\left[Z_i < \frac{N^{-1}(\bar{p}) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right] \\
&= N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right).
\end{aligned}$$

Whereas the value of $p(Y)$ provides the loan default probability under given scenario, the value $\bar{p}$ is the average of the conditional probabilities over all scenarios.

We decomposed ε_i so that Y represents the common factor. Then if specific value of Y is assumed, the variables L_i are iid variables with a finite variance and the conditions of the law of large numbers hold. The portfolio loss conditional on Y converges to its expectation $p(Y)$ as $n \to \infty$. Then

$$
\begin{aligned}
P[L \leq x] &= P[p(Y) \leq x] \\
&= P\left[N\Big(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\Big) \leq x\right] \\
&= P\Big[N^{-1}(\bar{p}) - \sqrt{\rho}Y \leq \sqrt{1-\rho}N^{-1}(x)\Big] \\
&= P\left[Y \geq \frac{N^{-1}(\bar{p}) - \sqrt{1-\rho}N^{-1}(x)}{\sqrt{\rho}}\right] \\
&= N\left(\frac{\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p})}{\sqrt{\rho}}\right).
\end{aligned}
$$

4.3 Gordy´s Model

Michael B. Gordy, member of Board of Governors of the Federal Reserve System, published in 2002 paper *A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules* [8]. In this paper he shows that ratings-based capital rules can be reconciled by the class of credit VaR models because of the following proposition. Contributions to VaR are portfolio-invariant only if (i) the portfolio is *asymptotically fine-grained*, and (ii) there is only a single risk factor driving correlations across obligors.

What does it mean *portfolio-invariance* and *asymptotically fine-grained*? Capital charges are portfolio-invariant if capital charges on given instrument depend only on its own characteristics, but not on the characterteristics of the portfolio in which the instrument is held. The latter term could be explained as the property that no single exposure in the portfolio can account for more than an arbitrarily small share of total portfolio exposure. Idiosyncratic risk is totally diversified away.

Let X represent the one-dimensional risk factor, which is generally drawn from a known distribution. When assuming that all the correlations in default events are explained by this risk factor (e.g. industrial index), then conditional on X, the remaining risk must be idiosyncratic to the individual obligor.

We are interested in the borrower´s default which happens if and only if the asset return drops bellow some threshold value γ_i. Let R_i represent the return on the i-th obligor´s assets. R_i is given by

$$R_i = \psi_i \varepsilon_i - \omega_i X,$$

where the $\varepsilon_i,\ i = 1, 2, ..., n$, are independent identically distributed drawings from the standard normal distribution and represent the idiosyncratic risk factors. The weights ψ_i and ω_i[3] are scaled so that R_i has mean zero, variance

[3] Gordy used the minus sign before $\omega_i X$ only to get the $p_i(x)$ function increasing in x.

one, as in Vasicek´s model. The same as in Vasicek´s model is also the conditional default probability of the i-th obligor $p_i(X)$:

$$p_i(X) = P[R_i \leq \gamma_i \mid X] = N((\gamma_i + \omega_i x)/\psi_i),$$

where N is the standard normal CDF and γ_i the threshold value.

Gordy better introduces a new variable that should give better outcome than default status. Let the random variable U_i denote *loss per dollar exposure* of the i-th obligor and A_i the propriate exposure. The usual assumption of conditional independence of defaults (the ε_i´s are *iid*) is extended to conditional independence of the U_i´s. Formally, Gordy assumes that the $U_i,\ i = 1, 2, ..., n$, are bounded in the unit interval and, conditional on X, are mutually independent.

For the portfolio of n obligors, define *the portfolio loss ratio* L_n[4] as the ratio of total loss to total portfolio exposure, i.e.

$$L_n = \frac{\sum_{i=1}^n U_i A_i}{\sum_{i=1}^n A_i}.$$

The q-th quantile of the distribution of this loss ratio is denoted by $\alpha_q(L_n)$.

We should now better remind that the portfolio is asymptotically fine-grained, i.e. the share of the largest single exposure in total portfolio exposure vanishes to zero as the number of exposures in the portfolio increases.

Under these quite general assumtions, Gordy derives two important propositions[5] about the distribution of L_n. The former one says that the conditional ditribution of L_n degenerates to its conditional expectation as $n \to \infty$, i.e.

$$\text{conditional on } X = x,\ L_n - E[L_n \mid x] \to 0, \text{ almost surely.}$$

The latter one helps us if we wish to know the variance of the loss ratio. Then we can use the variance of the conditional expectation.

$$\mathrm{Var}[L_n] - \mathrm{Var}[\mathrm{E}[L_n \mid X]] \to 0.$$

[4] Instead of former $L = \frac{1}{n}\sum_{i=1}^n L_i$.

[5] For details and proofs see [8].

However, what really interests us is how to derive the q-th quantile of the portfolio loss distibution $\alpha_q(L_n)$. Let´s look at the Proposition 3 in the Gordy´s paper. In intuitive terms, it allows us to substitute the quantiles of $\mathrm{E}[L_n \mid X]$ (which typically are relatively easy to calculate) for the corresponding quantiles of the loss ratio L_n as the portfolio becomes large. Gordy emphasizes that we have obtained this results with very minimal restrictions on the portfolio and the nature of credit risk. Each asset may be of quite varied probability of default, expected loss given default, and exposure size. More importantly, these three characteristics of the distribution of L_n hold even if the systematic risk factor X is a vector of any finite length and with any distribution.

For following results it is required that the distribuiton of L_n be *nice*.[6]

$$\alpha_q(\mathrm{E}[L_n \mid X]) = \mathrm{E}[L_n \mid \alpha_q(X)],$$

$$P[L_n \leq \mathrm{E}[L_n \mid \alpha_q(X)]] \to q,$$

and

$$\mid \alpha_q(L_n) - \mathrm{E}[L_n \mid \alpha_q(X)] \mid \to 0.$$

After stating all these properties we can conclude, that the q-th quantile of the loss ratio can be asymptotically calculated as

$$\alpha_q(L_n) = \alpha_q(\mathrm{E}(L_n \mid X)) = \mathrm{E}(L_n \mid \alpha_q(X)) = \frac{\sum_{i=1}^{n} \mathrm{E}(U_i \mid \alpha_q(X))A_i}{\sum_{i=1}^{n} A_i} \tag{4.8}$$

and that the essence of calculating the asymptotic capital requirement for expected and unexpected loss on loan i is the evaluation of $\mathrm{E}(U_i \mid \alpha_q(X))$.

[6]Such requirements are quite complicated in order not to exclude some hedging instruments from the portfolio. For details see [8]

Chapter 5

IRB Risk Weighted Functions

5.1 General Principles

In this chapter we assume that risk components are already estimated by bank (subject to supervisory requirements in Basel II, Part2, Section III.H) or overtaken from the supervisor. We have to distinguish between *average PDs* and *conditional PDs*. Average PDs are in fact those computed by banks. These probabilities of default reflect expected default rates under normal business conditions and are estimated from observed values.

Whereas conditional PDs are derived from average PDs using a special mapping function (see section 4.2) to reflect default rates given an appropriately conservative value of the risk factor, the same for all loans.

Expected loss (given by average PD) should be covered by banks on an ongoing basis (by provisions, corrections and write-offs), because it represents a cost component of the lending business. The unexpected loss (given by conditional PD), on the other side, relates to potentially large losses that occur rather seldomly. Up to the Third Consultative Paper of BCBS from April 2003, EL was also included in the risk weight assets as well. That means capital was required in order to cover the total Value at Risk. Nowadays required capital relates to UL only. However, banks have to demostrate that they build adequiate provisions against EL.

Throughout this section, PD and LGD are measured as decimals, and EAD is measured as currency (e.g. euros). When computing the capital requirement for a single loan as a percentage of its exposure we have to derive the unexpected loss from equations 3.3, 3.1.

$$\begin{aligned} UL &= \alpha_q(L) - EL \\ &= \alpha_q(L) - PD \times LGD, \end{aligned}$$

where PD denotes the average default probability.

Let´s now remember formulas in the section Vasicek´s Model, especially those ones on page 29. $p(Y)$ denotes the conditional default probability and $p(Y) = P[L_i = 1|Y]$. Under the Basel II settings, D_i is the Bernoulli random variable and percentage loss of the i-th obligor is given by the product of D_i and LGD_i. In contrast to PDs, Basel II does not transform average LGDs into conditional LGDs. Instead, banks are asked to report LGDs that reflect economic-downturn conditions caused by expected recession. It follows that LGD does not depend on the risk factor and can be therefore treated as a constant. Returning to the Vasicek´s model under the Basel II settings and substituting $q = 0.999$, for a single loan we get

$$\begin{aligned} 0.999 &= P[L < \alpha_{0.999}(L)] \\ &= P[LGD \times p(Y) < \alpha_{0.999}(L)] \\ &= P[p(Y) < \alpha_{0.999}(L)/LGD] \\ &= N\left(\frac{\sqrt{1-\rho}N^{-1}(\alpha_{0.999}(L)/LGD) - N^{-1}(PD)}{\sqrt{\rho}} \right). \end{aligned}$$

We have to modify this equation in order to get the VaR value:

$$\sqrt{\rho} \times N_{-1}(0.999) + N_{-1}(0.999) = \sqrt{1-\rho} N^{-1}(\alpha_{0.999}(L)/LGD)$$

$$\alpha_{0.999}(L) = LGD \times N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right).$$

Substituting $\alpha_{0.999}(L)$ in the formula for unexpected loss we get

$$UL = LGD \times N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right) - LGD \times PD. \qquad (5.1)$$

This last expression is the basis for the Basel II capital requirement formula for a given loan. Let´s now introduce it and afterwards focus on some details.

5.2 Basel II Capital Requirement Formula

Capital requirement K for exposures not in default:

$$K = LGD\left[N\Big(\frac{N^{-1}(PD) + \sqrt{R}N^{-1}(0.999)}{\sqrt{1-R}}\Big) - PD\right]\frac{1 + (M - 2.5)b(PD)}{1 - 1.5b(PD)}1.06,$$

where $N(x)$ denotes the cumulative distribution function of the standard normal distribution, R is the correlation coefficient, M maturity and $b(x)$ a function of PDs given as

$$b(PD) = (0.08451 - 0.05898 \times \log(PD))^2.$$

For defaulted assets, i.e. $PD = 1$, $EL = LGD$ and consequently the conditional PD (the N term) equals one. In this case capital requirement is equal to zero. But we don´t yet know the realized recovery rate for this loan and thus LGD. Therefore, BCBS has underlined the need of EL and LGD being estimated separately. In particular, Basel II says: *The capital requirement K for a defaulted exposure is equal to the greater of zero and the difference between its LGD and the bank's best estimate of expected loss.*

The risk-weighted asset amount RWA for a given exposure is then the following product:

$$RWA = K \times 12.5 \times EAD.$$

5.2.1 Asset Correlation

In the Gordy´s ASRF model, ω_i denotes the sensitivity of the i-th obligor to the risk factor X that we would interprete as the state of global economy. ω_i is in other words the degree of the obligor's exposure to the systematic risk factor and may be expressed as the asset correlation. This variable, in short, shows how the asset value of one borrower depends on the asset value of another borrower. Likewise, correlation could be described as the dependence of the asset value of a borrower on the general state of the economy - all borrowers are linked to each other by this single risk factor. It should be noted that asset correlation does not equal the default correlation.

Because different asset classes perform different dependence on the economy, correlations should be treated separately for each asset class. This can be also motivated by example of two portfolios with identical expected loss but different asset correlations in figure 5.1, source [3]. The solid curve performs low variation and weak dependence on the systematic risk factor. This character is usually retained by a retail portfolio. Defaults of retail customers tend to be more idiosyncratic and less dependent on the economic cycle than corporate defaults.

When calibrating the model, analysis of data provided by G10 supervisors revealed two systematic dependencies described in [3]:

1. Asset correlations decrease with increasing PDs. (The higher the risk, the higher the individual risk component.)

2. Asset correlations increase with firm size. (The larger the firm, the higher its dependency on the business cycle.)

The asset correlation function exhibits both dependencies. It also includes two limits for correlations (results of the above mentioned analysis), that differ according to asset classes. Correlations between these limits are modelled by an exponential weighting function with coefficient k. This factor

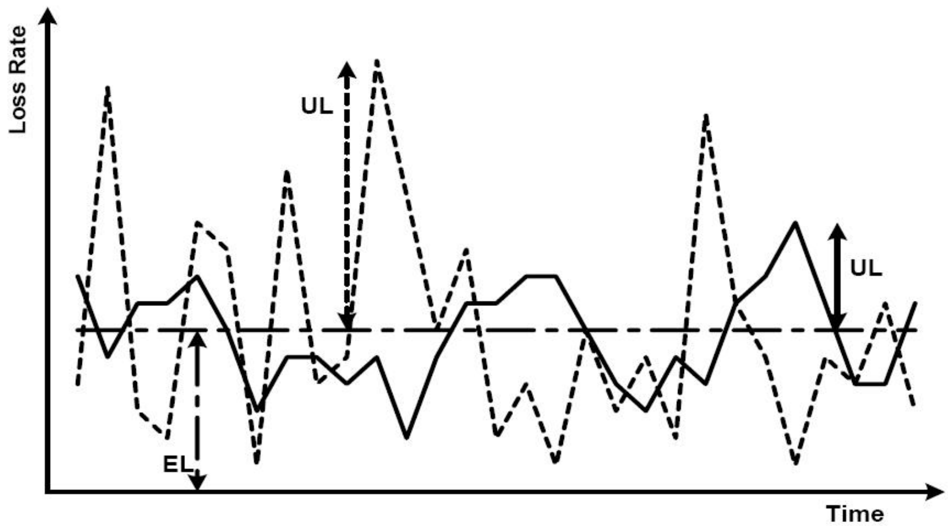

Figure 5.1: Two different loss experiences

influences the shape of our correlation function. The higher the k, the more concave the curve. We will now focus on formulas for selected asset classes separately.

For **corporate exposures**, the formula for R is the most complicated. The coefficient k is set at 50, the lower bound at 12% for high PDs and upper bound at 24% for low PDs (the first and second row of the formula, respectively) and a size adjustment is applied (the third row; S denotes the level of annual sales). The size adjustment affects only borrowers with annual sales of 50 milion EUR or less. For borrowers with 5 milion EUR or less annual sales, the size adjustment takes the value of 0.04 (maximum adjustment).

$$
\begin{aligned}
R &= 0.12 \times \frac{1 - \exp(-50 \times PD)}{1 - \exp(-50)} \\
&+ 0.24 \times \left[1 - \frac{1 - \exp(-50 \times PD)}{1 - \exp(-50)}\right] \\
&- 0.04 \times \left(1 - \frac{S - 5}{45}\right).
\end{aligned}
$$

The asset correlation function for **bank and sovereign exposures** looks very similar (only the size adjustment is ommited):

$$
R = 0.12 \times \frac{1 - \exp(-50 \times PD)}{1 - \exp(-50)} + 0.24 \times \left[1 - \frac{1 - \exp(-50 \times PD)}{1 - \exp(-50)}\right].
$$

Correlation R for **retail exposures** differs quite a lot. Firstly, the coefficient k is lower. It is set at 35. Secondly, limits for high and low PDs are modified to 3% and 16%, respectively. Finally, no size adjustment is applied for retail borrowers. Many analyses showed that there are also differences within the class of retail exposures. The correlation coefficient for residential mortgages seems to be constant and quite high at the level of 15%[1], the correlation coefficient for revolving retail exposures (loan without a fixed number of payments) was set at 4%. In other cases:

$$
R = 0.03 \times \frac{1 - \exp(-35 \times PD)}{1 - \exp(-35)} + 0.16 \times \left[1 - \frac{1 - \exp(-35 \times PD)}{1 - \exp(-35)}\right].
$$

The dependence of the correlation function on the k-factor and on limits is shown in figures 5.2, 5.3 and 5.4.

[1]Explained in the next section.

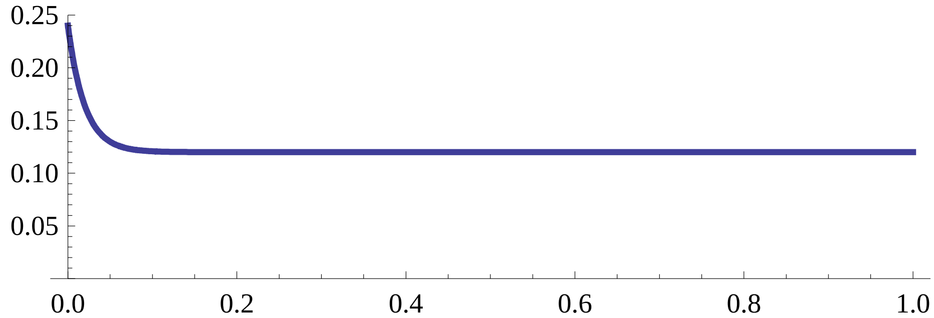

Figure 5.2: The asset correlation function for corporate exposures for all possible PDs (without size adjustment)

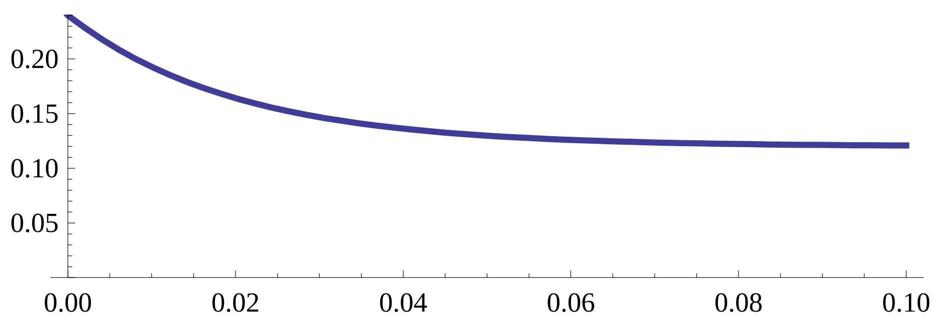

Figure 5.3: Detail of the asset correlation function for corporate exposures for PDs up to 10% (without size adjustment)

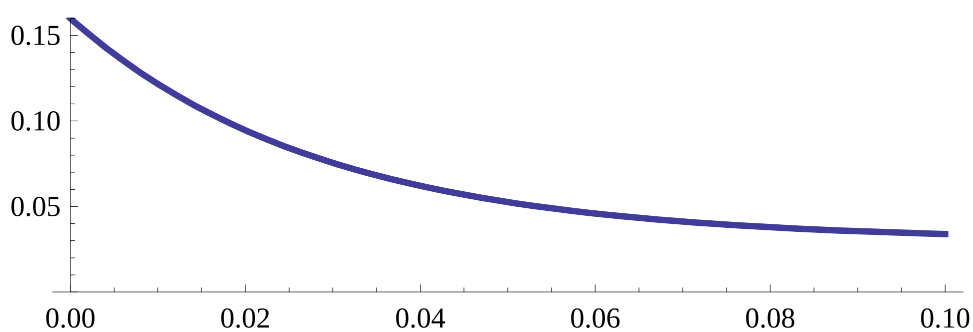

Figure 5.4: Detail of the asset correlation function for retail exposures for PDs up to 10%

5.2.2 Maturity Adjustment

We would now discuss the maturity adjustment (MA) part of the formula for calculating capital requirement for credit risk (the last fraction in the above mentioned formula):

$$MA = \frac{1 + (M - 2.5) \times b(PD)}{1 - 1.5 \times b(PD)}.$$

Credit portfolio usually consists of loans with different maturities. The long-term credits are naturally riskier than the short-term credits. Intuitively, our forecasts for next few months may be quite useful but not for the horizon of ten years. Consequently, the capital requirement should increase with maturity.

Moreover, maturity adjustment would also depend on the default probability as a consequence of mark-to-market valuation of loans. Potential down-grades in future would affect high PDs borrower in stronger way than borrowers with low PDs. A risk averse investor takes into account different risk-adjusted discount factors. The actual form of the Basel II maturity adjustment has been derived by applying a specific mark-to-market credit risk model. The output of this model is a matrix of VaR measures for a range of rating grades and maturities as scheduled in table 5.1.

	Maturity				
PD grade	1 year	2 years	3 years	4 years	5 years
1	VaR(1,1)	VaR(1,2)	VaR(1,3)	VaR(1,4)	VaR(1,5)
2	VaR(2,1)	VaR(2,2)	VaR(2,3)	VaR(2,4)	VaR(2,5)
3	VaR(3,1)	VaR(3,2)	VaR(3,3)	VaR(3,4)	VaR(3,5)
...	VaR(...,1)	VaR(...,2)	VaR(...,3)	VaR(...,4)	VaR(...,5)

Table 5.1: Matrix of VaR measures

Maturity adjustments are ratios of these VaR figures to the VaR of a *standard* maturity, which was set to be 2.5 years. This standard maturity was chosen with regard to the fixed maturity assumption of the Basel II foundation IRB approach. However, the Basel II maturity adjustment function includes smoothed VaR figures. As described in [3], the regression function was chosen in order to perform following properties.

1. The adjustments are linear and increasing in the maturity M.
$$\frac{\partial MA}{\partial M} = \frac{b(PD)}{1 - 1.5 \times b(PD)} = const. > 0$$
This inequality holds for all PDs. For the characteristics of $b(PD)$ and $\frac{\partial MA}{\partial M}$ see figures 5.5 and 5.6.

2. The slope of the adjustment function with respect to M decreases as the PD increases.
$$\frac{\partial MA}{\partial M \partial PD} = -\frac{0.16434 \times b(PD)^{3/2}}{x\left(1 - 1.5 \times b(PD)\right)^2} - \frac{0.10956 \times b(PD)^{1/2}}{x\left(1 - 1.5 \times b(PD)\right)} < 0$$
See figure 5.7.

3. For a maturity of one year the function yields the value of one and hence the resulting capital requirements coincide with the ones derived from Gordy´s ASRF model.

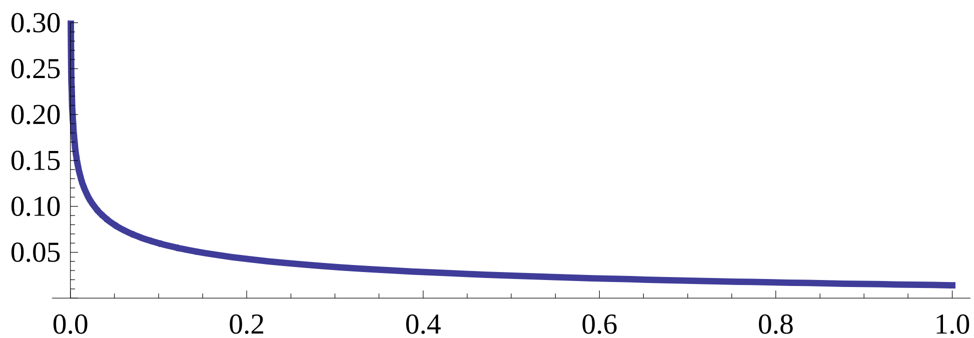

Figure 5.5: The regression function $b(PD)$

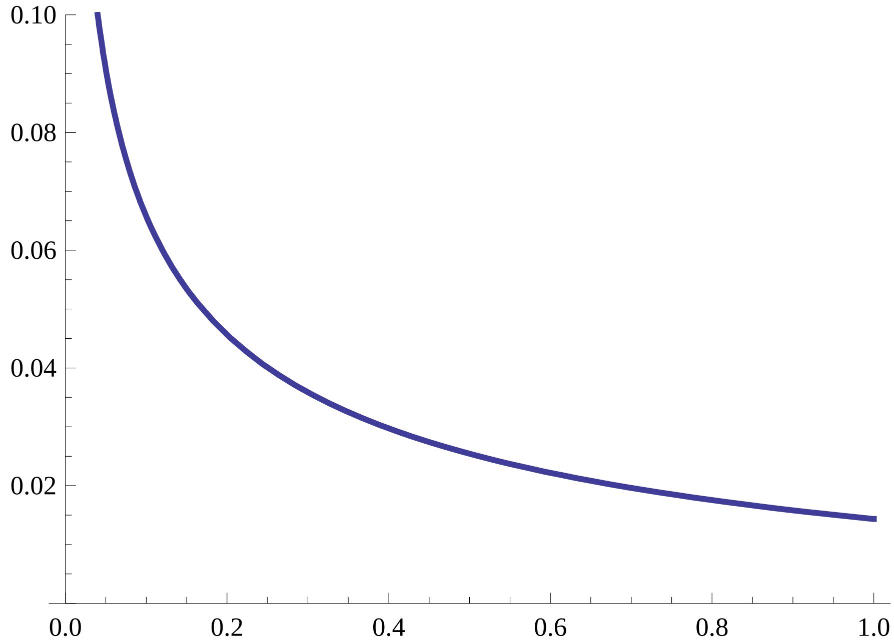

Figure 5.6: The first derivate of the adjustment function with respect to M

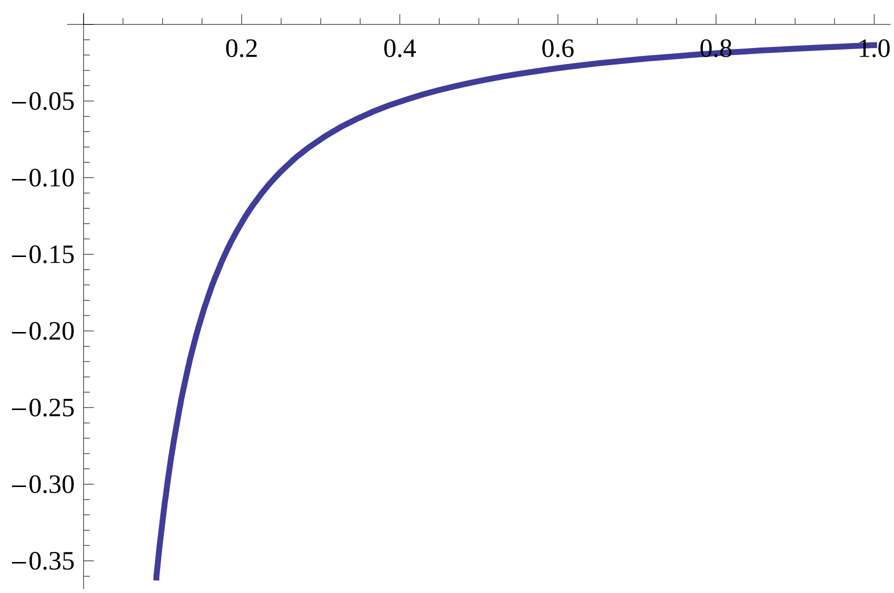

Figure 5.7: The second derivate of the adjustment function with respect to M and PD

It is important to mention that retail risk weight functions do not include maturity adjustments. In the absence of sufficient data for retail borrowers, maturity effects have been left as an implicit driver in the asset correlation and no separate maturity adjustment is required. In this context, [3] gives also explanation of the relatively high mortgage correlations: *not only are mortgage losses strongly linked to the mortgage collateral value and the effects of the overall economy on that collateral, but they have usually long maturities that drive the asset correlation upwards as well.*

Chapter 6

Application of the Basel II Capital Requirement Formula

Let´s now suppose we have a portfolio of 1000 identical loan instruments. All of them have the same PD, the same LGD and also the same correlation to the economy. Moreover, they have the same maturity and exposure (nominal value). Let the maturity and the exposure be equal to one year and 1 EUR, respectively. After one year, there are only two posibilities for the status of a borrower. He either pays the whole amount back or he defaults, i.e. he is not able to pay the whole debt. We may consider interest rate not to be an important factor for our purposes.

Under these settings we would like to compare the Basel II capital requirement for this portfolio with simulated unexpected loss. But what should be the values of risk components in our example? We will use some basic limits given in the New Basel Capital Accord, in particular in the foundation IRB approach part.

6.1 Input

- As stated in paragraph 285, under all circumstances, **probability of default** should be greater than 0.03%, i.e. 0.0003. In our case when the portfolio is not so large, it would be excessively small. The minimum level of PD=0.001 should be sufficient in this case. And what about the maximum level? In practice, PDs greater than 0.2 occur very seldomly. This is caused by the fact that entities with very high probability of default are rated as *bad* (and vice versa) and would not get any credit. An example of probabilities of default for different rating categories is given by figure 6.1. The whole transition matrix is used in more advanced credit risk models that consider also the change in rating and not only defaulted/not defaulted.

- **Loss given default** is recommended to equal 0.35 for secured senior claims on corporates (simplified number), 0.45 for senior claims on corporates not secured by recognised collateral, and 0.75 for all subordinated exposures. We will therefore emphasize output for these values.

- As already said, nominal value of each loan equals 1 EUR and interest rate does not influence our numbers. Therefore **exposure at default** equals 1 EUR.[1]

- The most complicated issue in our simulation assumptions is how to deal with **correlation**. Since we don´t want to be influenced by the Basel II ideas, we will not use the formula for R stated there. However, we will follow upper and lower limits for correlations and also the

[1]This could be also treated as providing the credit on the discounted basis, i.e. the borrower gets $1/(1+r)$ EUR at the beginning of the year (r is the appropriate annual interest rate) and should pay 1 EUR at the end of that year.

shape[2] of asset correlation function explained in section 5.2.1. As we do not insist on class of corporate exposures or class of retail exposures, we may fix only one lower limit and one upper limit. Let these limits equal 0.03 and 0.24, respectively.

Initial rating	**Rating at year-end (%)**							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

Source: Standard & Poor's CreditWeek (15 April 96)

Figure 6.1: One-year transition matrix with default propabilities in the last column.

6.2 Capital Requirement

As for computation of the capital requirement, we are allowed to simplify the formula because of M=1. To get value for the whole portfolio we have to multiply it by the number of instruments in the portfolio and also by the exposure at default.

$$K = LGD\left[N\Big(\frac{N^{-1}(PD)+\sqrt{R}N^{-1}(0.999)}{\sqrt{1-R}}\Big)-PD\right]\times 1000\times 1.06\times 1EUR.$$

[2] Correlation decreases with increasing PD.

6.3 Simulation of the Unexpected Loss

As described earlier, unexpected loss is the difference between some quantile and the expected loss, that are descriptive characteristics of some distribution. Since we are not able to get the real distribution function of credit loss, we have to compute the empirical cumulative ditribution function as its estimate.

Basic thoughts behind the single loss simulation procedure follow.

In order to get the default status and the number of defaults, we have to generate a random variable representing the risk of each loan as described in the section Vasicek´s Model.

$$\varepsilon_i = \sqrt{R}\, Y + \sqrt{1-R}\, Z_i,$$

where $Y, Z_i, i = 1, 2, ..., 1000$, are random drawings from the standard normal distribution. In this way we obtain pairwise correlated standard normal variables ε_i.

Cumulative distribution function of this risk variables is then compared to the given PD. We get the default status:

$$\begin{aligned} D_i &= 1, \text{ if} N(\varepsilon_i) < \text{PD}, \\ &= 0, \text{ if} N(\varepsilon_i) \geq \text{PD}, \end{aligned}$$

and the number of defaults in this portfolio (denoted by NDef) as sum of D_i over all i.

$$\text{NDef} = \sum_{i=1}^{1000} D_i$$

For each default we generate another random variable representing particular loss given default. LGD usually follows beta distribution that is a continuous distribution defined on the interval [0,1] and parametrized by two shape parametres, α and β. We compute these numbers from two equations that characterize the dependence of expected value and variance on α

and β.

$$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta},$$

$$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

where $X \sim \mathrm{Beta}(\alpha, \beta)$, i.e. random variable X follows beta ditribution with parametres α and β.

Expected value is given by our inputed value of LGD, variance is fixed to a small number (0.025) in order not to get a U-shaped beta distribution.[3] Fixed variance could not be suitable for low or high values of LGD such as 0.1 and 0.9. But as mentioned above, our inputed LGDs are bounded by 0.35 and 0.75. We should better give an example. Let LGD (inputed) equal 0.75 and the variance 0.025. Solving above mentioned equations, we get α=4.875 and β=1.625. However, substituting e.g. 0.1 for the variance, we get $\alpha = 0.65625$, $\beta = 0.21875$. Both density functions are shown in figure 6.1.

At this point we know what the distribution of simulated losses (given default) is, for further purposes denoted by lgd_i for the i-th defaulted obligor. But is it sufficient to generate iid random variables from this distribution? As many studies have shown, losses given default are usually also correlated. But we are not able to do that directly. Therefore we should generate correlated random variable using following proposition.

Let F_1 be the cumulative distribution function of X_1. Then $F_1(X_1)$ follows the uniform distribution over the interval [0,1] ($F_1(X_1) \sim U(0,1)$). Moreover, let $U \sim U(0,1)$. Then $q_{F_1}(U)$ follows a ditribution with CDF F_1. q_{F_1} is the quantile function corresponding to the CDF F_1. [4]

[3] We would like to built this model on a peak-shaped distribution of loss given default. However, it is still the objective of many discussions among practitioners how does the distribution of LGD in fact look like.

[4] F_1 does not necessarily have to be continuous.

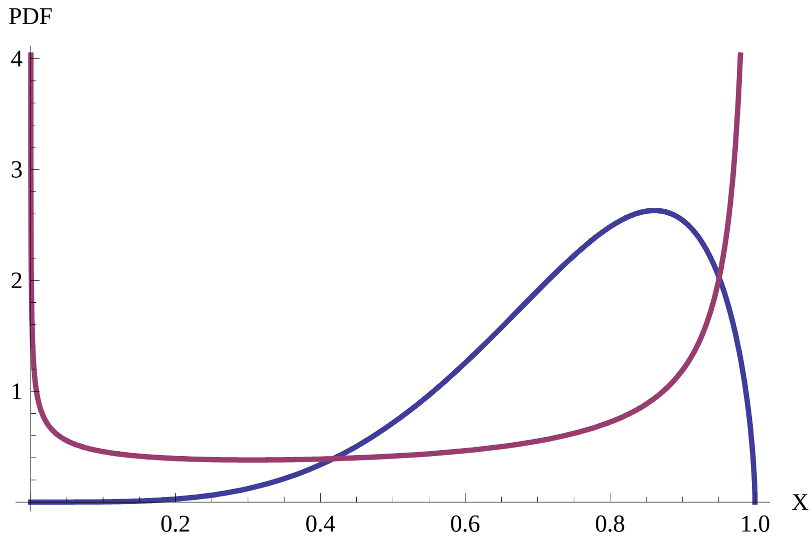

Figure 6.2: Two density functions with expected value of 0.75, one of Beta(4.875,1.625) - blue curve, and one of Beta(0.65625,0.21875) - purple curve.

Let´s follow this proposition by parts. We set X_1 equal to a standard normally distributed variable, F_1 from the first part is then the CDF of standard normal distribution denoted by N. However, F_1 from the second part could be another CDF and for our objective the CDF of beta distribution. So we generate new normal variables λ_i that are correlated by the same factor as ε_i´s (as we suppose). Using this proposition, we get correlated lgd_i´s as:

$$lgd_i = q_B(N(\lambda_i)),$$

where q_B is the quantile function of a given beta distribution and N is the CDF of standard normal distribution.

In the final step we should obtain the total credit loss of our hypothetic portfolio. This is done by multiplying all simulated losses from the last step by the exposure and adding them together[5], i.e.

$$loss = \sum_{i=1}^{NDef} lgd_i \times 1EUR.$$

In order to get the empirical distribution function of credit loss, we have to repeat this simulation many times. So we generate new ε_i´s and using this variables we compute new $NDef$ and new lgd_i´s. How many times? Number of simulations is influenced by the quantile. Our task is to provide a good estimate of the 0.999-th quantile. So we should talk about the number of simulations that is of the order of magnitude of 10,000. Let the start value be 10,000. Then we will see.

Running a code with large number of simulations will probably be quite time-consuming. The modification of computing single losses for more than one combination of PD, R and LGD should save some time. Particularly:

- We compute ε_i´s for all given correlation factors in one step using the same values of Y and Z_i´s.
- We derive the numbers of defaults for all given Rs and PDs in one step and the maximum of them MaxNDef.
- We generate new MaxNDef+1 drawings from standard normal distribution in order to get $lambda_i$´s for all Rs.
- For all possible combinations of PD, R and LGD given by (PD_k,R_l,LGD_m), the single loss is computed as

$$loss = \sum_{i=1}^{NDef_{kl}} lgd_{ilm} \times 1EUR.$$

[5]This corresponds to the definition of annual credit loss of the portfolio on page 13.

At this point, we are able to compute 10,000 simulations[6] of single loss for a given input and we would like to get estimate for quantile and expected value. For simplicity, we sort the losses from smallest to biggest and take the 9,990-th value in sequence as theestimate for the 0.999-th quantile. Average is obviously very good estimate for the expected value. Our estimate of the unexpected value is therefore the difference between the 9,990-th biggest loss and the average single loss.

[6] After a couple of simulations we found out that running the programm took quite a lot of time and so we did not increase the number of simulations.

6.4 Results

It is quite difficult to describe the dependence of some figure on a three-dimensional input. However, these three determinants are somehow connected. As already said, we will follow the Basel II idea of asset correlation function. As shown in figures 5.2, 5.3 and 5.4, the lower the PD, the higher the correlation. It is also good to see that asset correlation function is almost constant for PDs higher than 0.1.

We should also note that the connection between PDs and LGDs exists as well. A borrower with low probability of default would probably not be asked to provide a good collateral and vice versa. On the other side, bad creditors have to give some guarantee (and ensure lower LGD) in order to get a credit. The lower loss given default is a compensation for the higher default probability.

Let´s describe the difference between Basel II capital requirement and estimated unexpected loss as a relative number, i.e. the ratio of these variables. Ratio would fit our needs at most, as both numbers may be of quite varying size. In case, this ratio is greater one, capital requirement exceeds the simulated loss and we may consider it as quiet conservative. But one thing we cannot forget is the issue of reporting LGD. When computing the estimate of unexpected loss, the average level of LGD is used. But for the purpose of the Basel II formula *downturn* LGDs should be estimated. This figure will be a bit higher than the average and the capital requirement as well.

So what results do we expect? We suppose that the ratio is a bit bellow one for usual combinations of PD, R and LGD and for values mentioned in the foundation IRB approach (LGD=0.35, 0.45, 0.75).

Let´s now focus on the ratio for four categories of PDs and two categories of LGDs.

6.4.1 PDs lower than 0.01

The first category includes PDs smaller than 0.01 that are connected to very high correlations. We have to consider highest values for corporate exposures as well as for retail exposure that are in the level of 24% (20% with the maximum size adjustment) and 16%, respectively. Loans assigned by very low PDs should be considered as save and do not require a good collateral.

We did computations for typical values of LGD at the level of 65%, 75% and 85% and for one lower value only to see what the tendency is. Figure 6.4.1 shows the full range from zero to one as an example. All following figures show details and values around one. Basel II capital requirement for credit risk derived under these settings is quite sufficient. Especially when we expect in to be even higher for *downturn* LGDs. A bit higher variance for PD=0.001 is caused by the number of instruments in portfolio.

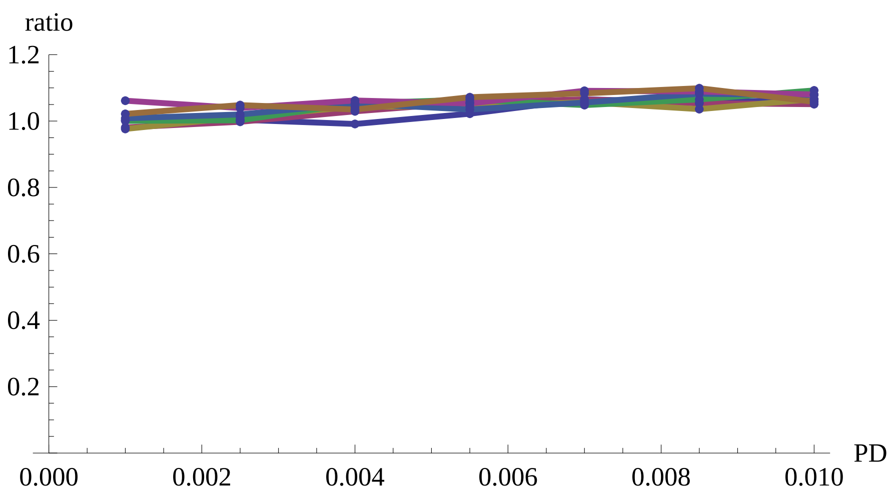

Figure 6.3: LGD=0.75 & R=0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24

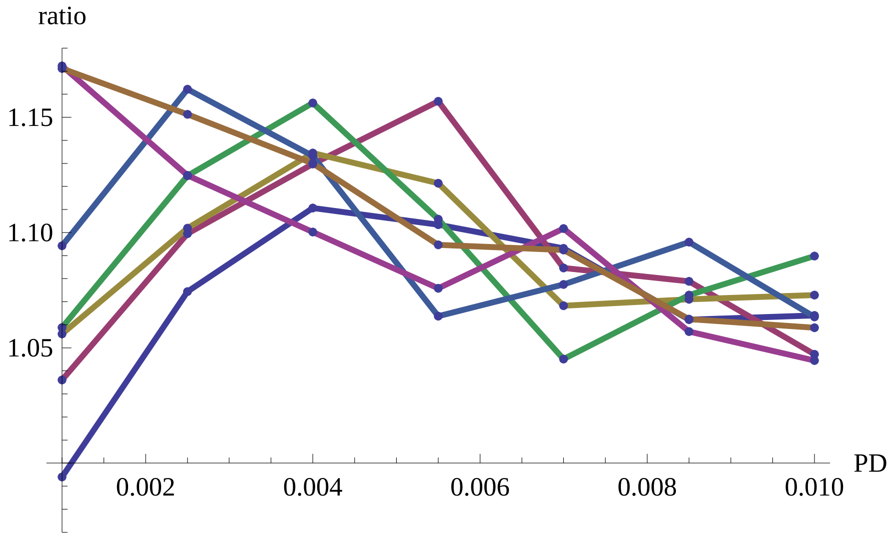

Figure 6.4: LGD=0.25 & R=0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24.

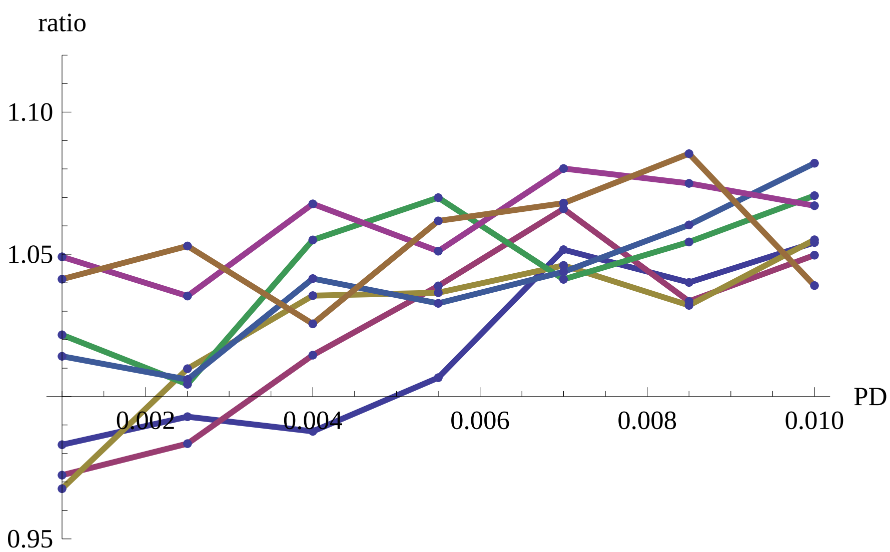

Figure 6.5: LGD=0.65 & R=0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24

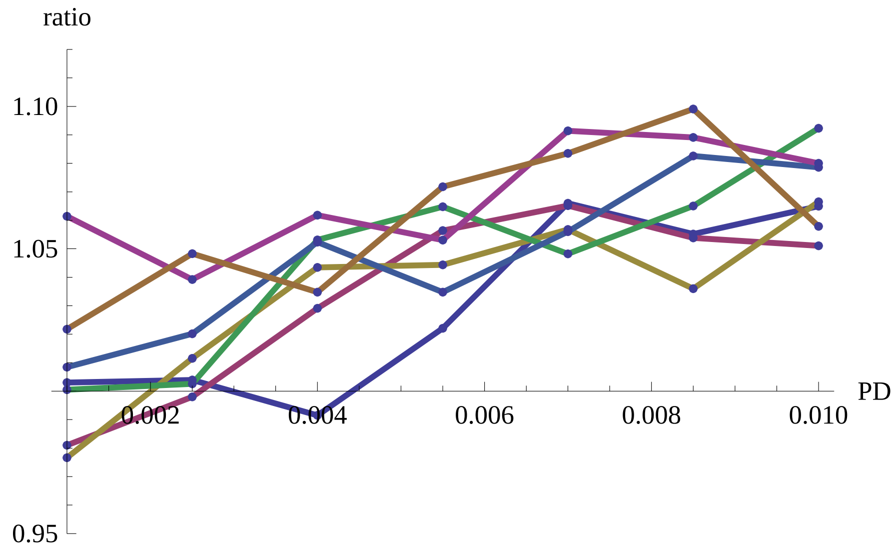

Figure 6.6: LGD=0.75 & R=0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24

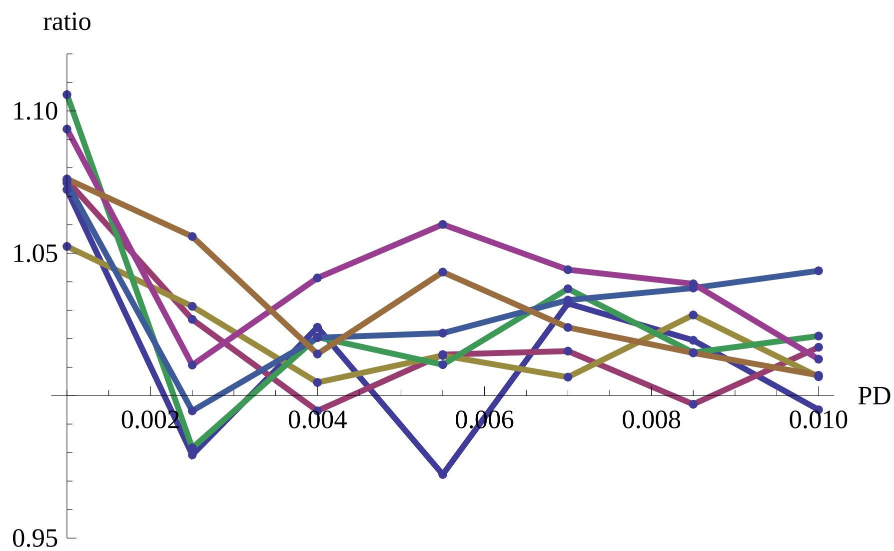

Figure 6.7: LGD=0.85 & R=0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24

6.4.2 PDs between 0.01 and 0.09

The second group is characterized by not so high, but still varying correlations. This corresponds to PDs from the interval <0.01,0.1>. Quite different values of loss given default may be settled for these PDs but these values are not likely to be high.

Following three figures show that the Basel II formula fits better for lower LGD. The ratio of the requirement and the simulated unexpected loss is below one (but very close to one) for LGD=20%. For the *downturn* LGD would the requirement perfectly fit the estimate. For higher LGDs is the bank forced to hold more money that it would probably need.

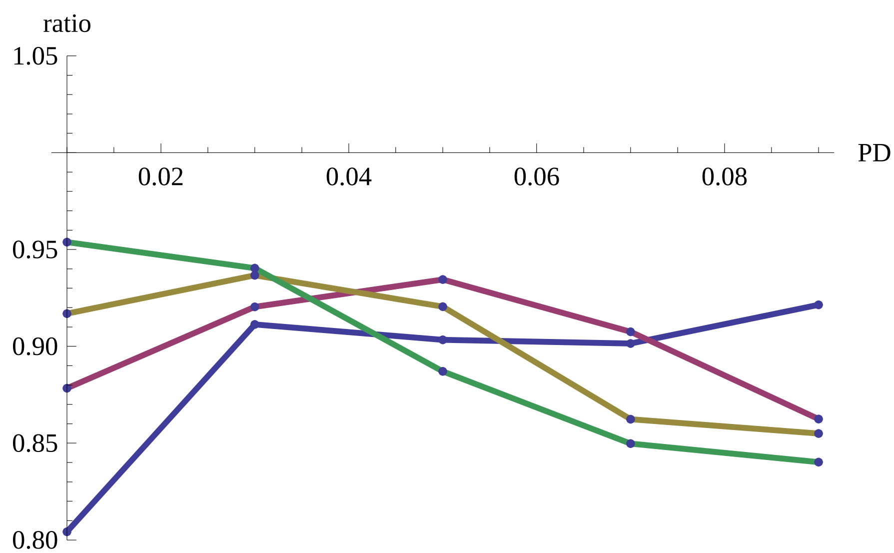

Figure 6.8: LGD=0.2 & R=0.04, 0.08, 0.12, 0.16

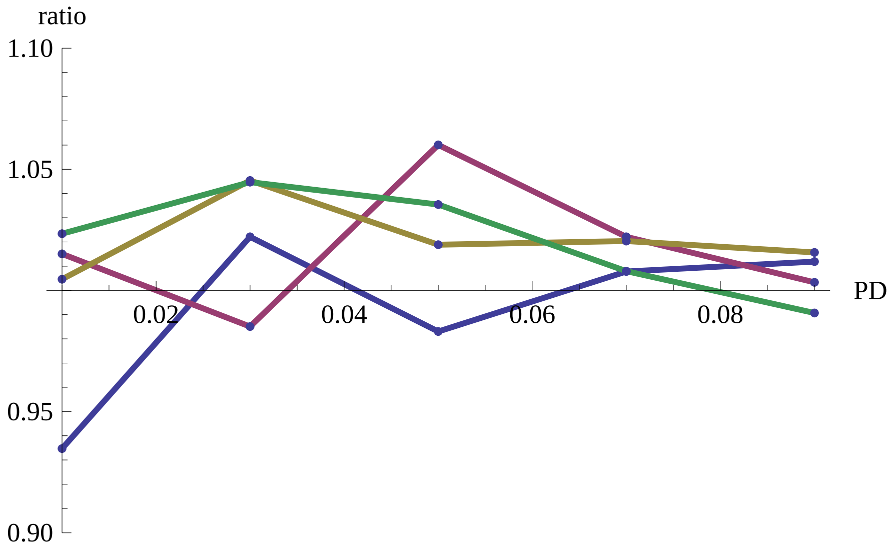

Figure 6.9: LGD=0.4 & R=0.04, 0.08, 0.12, 0.16

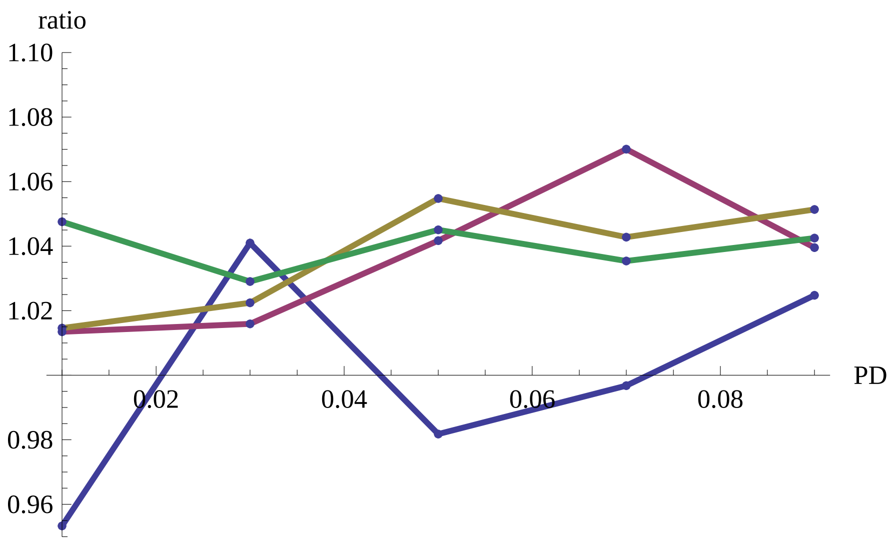

Figure 6.10: LGD=0.6 & R=0.04, 0.08, 0.12, 0.16

6.4.3 PDs between 0.09 and 0.20

Third group includes PDs between 0.1 and 0.2. This level of PD is still good enough to get a credit and sets the level of correlation to equal 0.03 for retail exposures, 0.08 for corporate exposures with maximum size adjustment and 0.12 for bank exposures or corporate exposures without any size adjustment. Because of the higher default probability we did simulations for low LGDs, but we wanted also to see the developement for a higher value.

Now we would try to explain results in figure 6.4.3. Why is the ratio so low? This is caused by the average LGD intead of the *downturn*. The ratio for average LGD of 15% corresponds to 0.7, then the *downturn* LGD that provides the ratio=1 should equal roughly 21%, ceteris paribus. This represents the 73% quantile of the beta distribution with expected value 0.15 and variance 0.025.

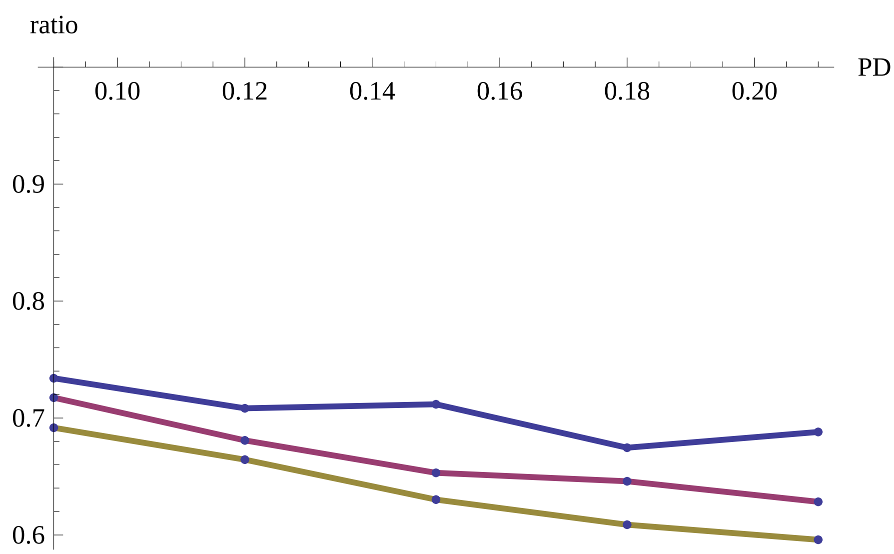

Figure 6.11: LGD=0.15 & R=0.03, 0.08, 0.12

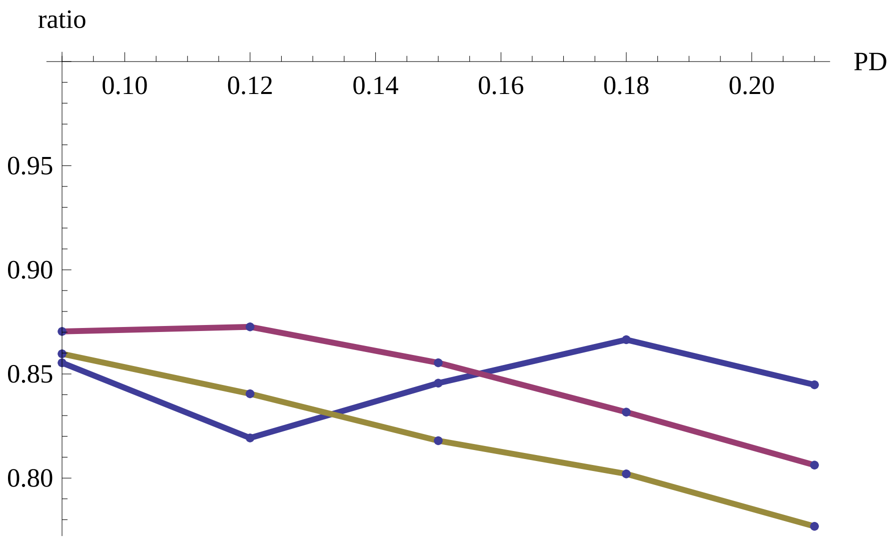

Figure 6.12: LGD=0.25 & R=0.03, 0.08, 0.12

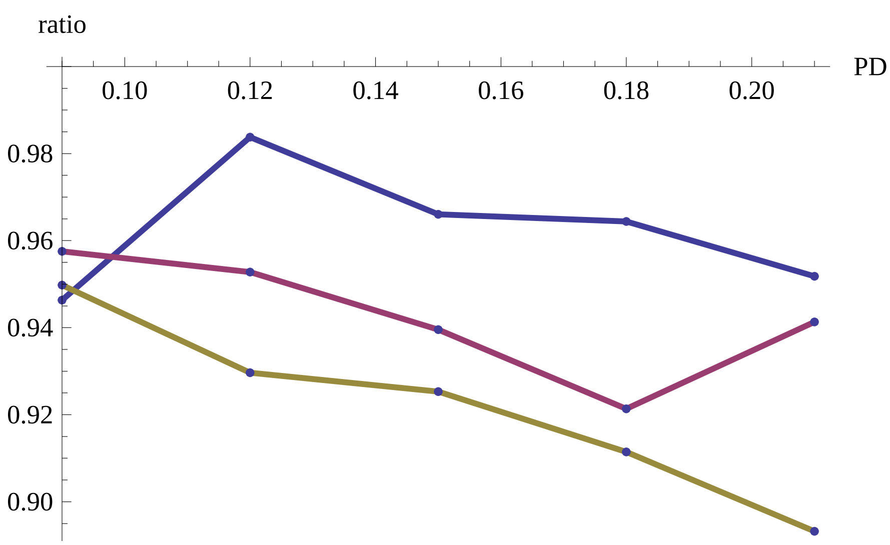

Figure 6.13: LGD=0.35 & R=0.03, 0.08, 0.12

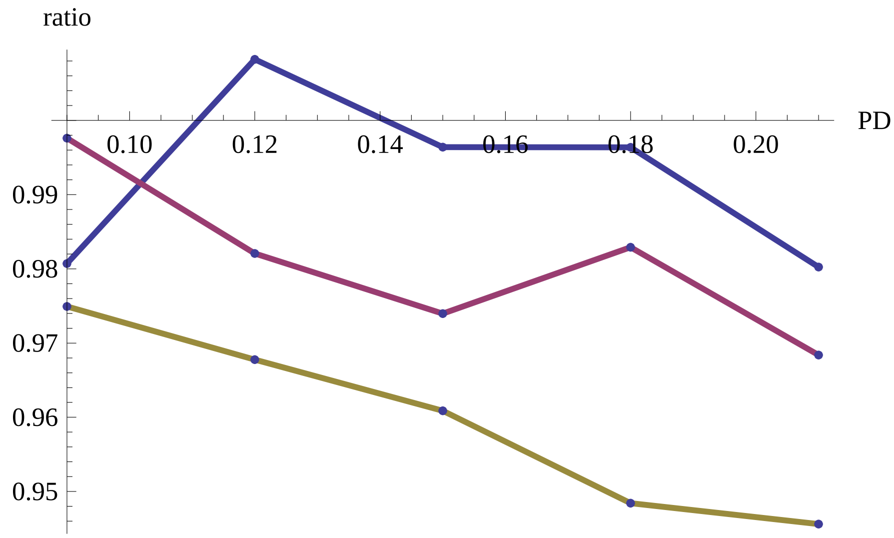

Figure 6.14: LGD=0.45 & R=0.03, 0.08, 0.12

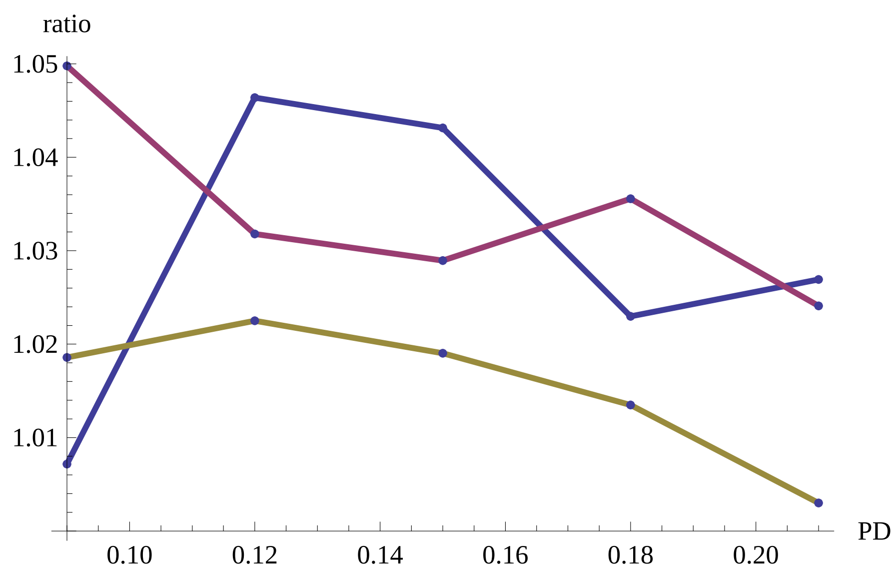

Figure 6.15: LGD=0.75 & R=0.03, 0.08, 0.12

6.4.4 PDs higher than 0.2

Finally, the last category of PDs (i.e. PDs higher than 0.2) should indicate bad loans. We do not want to say that it is impossible for a portfolio to include such a loan but it is less likely. Of course, it depends on the strategy of each bank, whether it is desirable to have a number of risky loans in the portfolio because of higher profit but this issue is not an objective of this thesis.

The ratio is for high PDs very low. Let´s follow the idea mentioned in the results for the last group of PDs. For example, given PD=0.9, R=0.03, LGD=0.15 (the lower blue line) the ratio equals 0.2, i.e. estimated unexpected loss is five times higher than the requirement. What should be the *correct* value of LGD in order to get the ratio=1? It should be 75%. This loan is too risky (PD=0.9) that the *very good* collateral (averge LGD=0.15) cannot be considered as *very good* for the purposes of the Basel II computations.

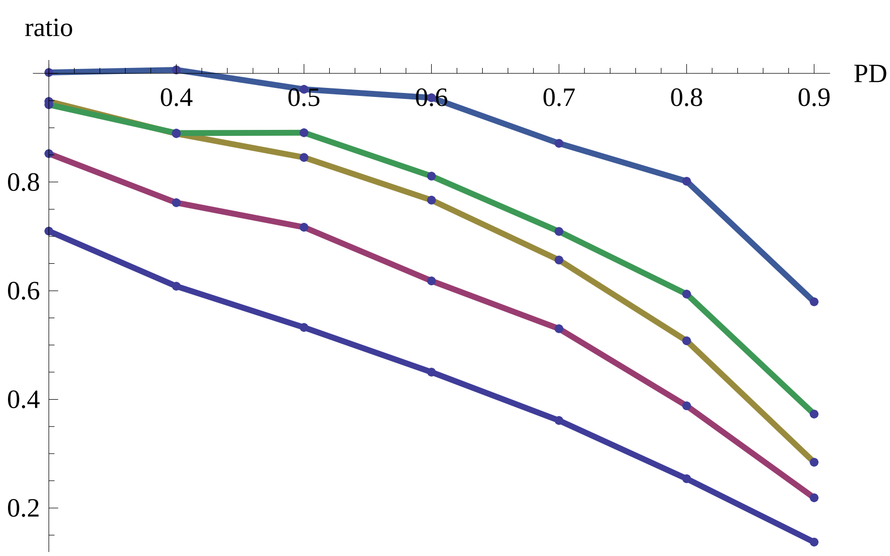

Figure 6.16: R=0.03 & LGD=0.15, 0.25, 0.35, 0.45, 0.75

6.4.5 Lower LGDs

We would like also to see, what is the dependency of the ratio on the loss given default. Together with lower LGDs are very often higher PDs. Using results from following figures, we could say that because the ratios for LGD=0.3, 0.4, 0.5 are very close to one, the capital requirement is sufficient and the *downturn* LGD does not differ much. Whereas the right value of LGD for the requirement should be higher in case the average LGD equals 0.1 or 0.2.

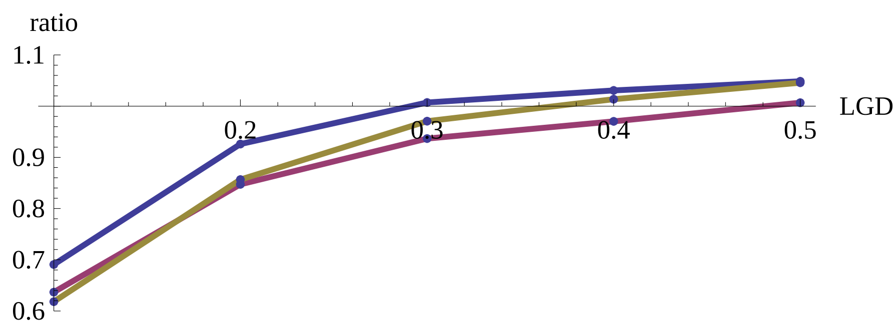

Figure 6.17: R=0.03 (retail exposures) & PD=0.1, 0.15, 0.2.

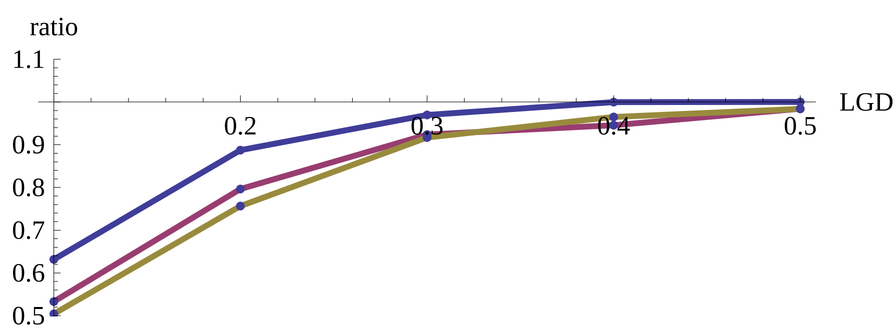

Figure 6.18: R=0.12 (corporate exposure) & PD=0.1, 0.15, 0.2.

6.4.6 Higher LGDs

In these figures, whenever the ratio is close to one, the *downturn* LGD would be certainly very close to the average value. For average LGD=0.4 should we make the downturn value higher, on the other side it could be a bit less for average LGD=0.9.

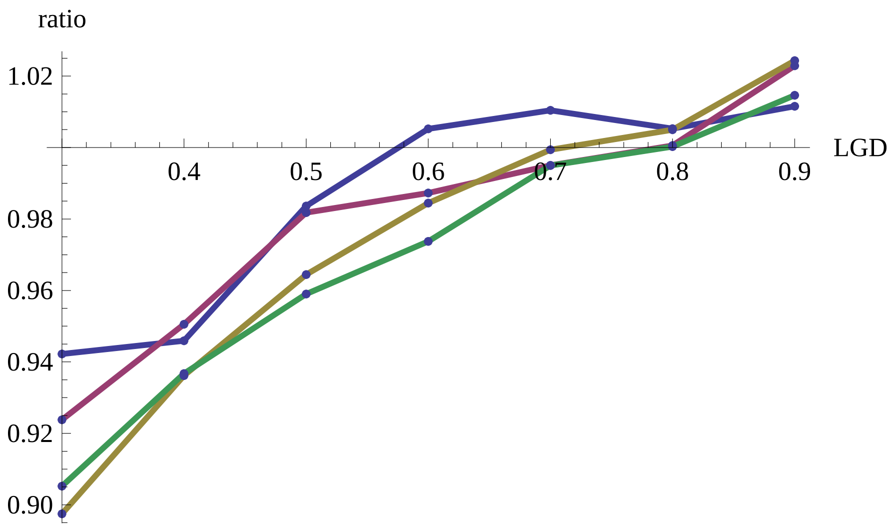

Figure 6.19: R=0.14 & PD= 0.005, 0.01, 0.02, 0.05

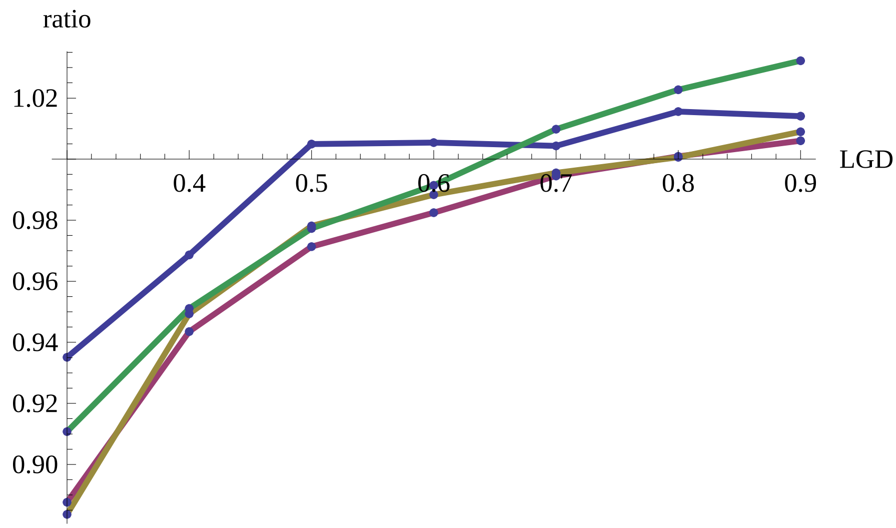

Figure 6.20: R=0.18 & PD=0.005, 0.01, 0.02, 0.05

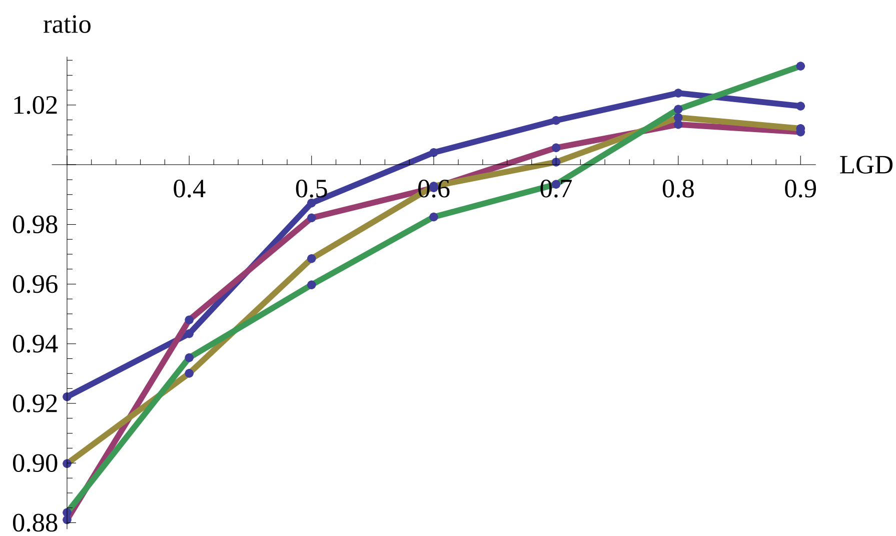

Figure 6.21: R=0.20 & PD=0.005, 0.01, 0.02, 0.05

Chapter 7

Závěr / Conclusions

V této diplomové práci jsme popisovali postup při výpočtu kapitálového požadavku ke kreditnímu riziku způsobem určeným předpisem Basel II. Uvedli jsme důvody a způsob vzniku vzorečku, předpoklady, na kterých Basel II staví a které často vychází z provedených studií, a zmínili jsme rovněž, že se doposud diskutují některé části. V kapitole zabývající se srovnáním výpočtů podle daného vzorce a odhadem jeho reálného ekvivalentu jsme došli k výsledkům, že pro dané portfolio s posuzovanými vstupními vlastnostmi basilejský vzorec přibližně vyhovuje. Připomínáme ale, že výsledky se musí interpretovat s ohledem na omezený počet jednotlivých simulací, počet instrumentů v úvěrovém portfoliu a předpoklady pro vzorové portfolio.

This thesis included the Basel II procedure of assessing capital requirement for the credit risk, its developement, assumptions it is based on and also a notice that still now exist discussions about the treatment of some figures. The conclusion of the last chapter where the Basel II formula is compared to estimates achieved by simulations should be following: Considering the number of simulations, the number of instuments in the portfolio and also the characteristics of the hypothetic portfolio, we may conclude that the formula provides quite good figures.

Appendix A

Mathematica notebook

```
ClearAll;

(* functions and procedures *)

betadist[lgd_]:=BetaDistribution[
α/.Solve[{α/(α+β) == lgd, αβ/((α+β)^2(α+β+1)) == 0.025}, {α,β}][[1,1]],
β/.Solve[{α/(α+β) == lgd, αβ/((α+β)^2(α+β+1)) == 0.025}, {α,β}][[1,2]]];

ndefaults[pd_, risk_]:=Length[Select[Boole[Thread[CDF[ndist, risk] < pd]], #==1&]];

LgdriskForOne[risk_, beta_]:=Quantile[beta, CDF[ndist, risk]];

Loss[lgdrisk_, ndef_]:=Total[Take[lgdrisk, ndef]];

OneSimulation[pd_, rho_, BetaDist_]:=
Module[{Idiosyncratic, Economy, Risk, Idiosyncratic2, Economy2, Risk2,
NumberOfDefaults, lgdrisk},
Idiosyncratic = RandomReal[ndist, nportfolio];
Economy = RandomReal[ndist];
Risk = Table[Sqrt[1−RHO]Idiosyncratic+Sqrt[RHO]Economy, {RHO, rho}];
NumberOfDefaults = Table[ndefaults[PD, RISK], {PD, pd}, {RISK, Risk}];
Idiosyncratic2 = RandomReal[ndist, Max[NumberOfDefaults]];
Economy2 = RandomReal[ndist];
Risk2 = Table[Sqrt[1−RHO]Idiosyncratic2+Sqrt[RHO]Economy2, {RHO, rho}];
```

```
lgdrisk = Table[LgdriskForOne[risk, beta], {risk, Risk2}, {beta, BetaDist}];
Table[Table[Loss[LGDRISK, NDEF],
{LGDRISK, lgdrisk[[i, All]]}, {NDEF, NumberOfDefaults[[All, i]]}], {i, 1, Length[rho], 1}]
];

UL[pd_, rho_, lgd_]:=
Module[{lossdistrib, beta, ULoss},
beta = Map[betadist, lgd];
lossdistrib = Table[OneSimulation[pd, rho, beta], {nsimul}];
Table[Sort[lossdistrib[[All, i, j, k]]][[Ceiling[nsimul q]]]−Mean[lossdistrib[[All, i, j, k]]],
{i, 1, Length[rho], 1}, {j, 1, Length[lgd], 1}, {k, 1, Length[pd], 1}]
];

Requirement[pd_, rho_, lgd_]:=
nportfolio×1.06×Table[(LGDCDF[ndist, Quantile[ndist, PD]/Sqrt[1−RHO]
+ Sqrt[RHO/(1 − RHO)]Quantile[ndist, 0.999]] − LGDPD),
{RHO, rho}, {LGD, lgd}, {PD, pd}];

(* input *)

ndist = NormalDistribution[];
nportfolio = 1000;
nsimul = 10000;
q = 0.999;

pd = Range[0.001, 0.01, 0.015];
rho = Range[0.12, 0.24, 0.02];
lgd = Range[0.15, 0.75, 0.2];

(* calculations itself *)

simul = UL[pd, rho, lgd];
basel = Requirement[pd, rho, lgd];
Export["simulation.xls", simul];
Export["requirement.xls", basel];

(* dimensions of the output :
 Length[rho] (sheets) × Length[lgd] (rows) × Length[pd] (columns) *)
```

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