Report on the doctoral thesis: **Flat Relative Mittag-Leffler Modules and Approximations.**

by RNDr. Asmae Ben Yassine

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Let R be a ring, for any right module M and any family of left modules $\mathcal{F} = \{Q_i\}_{i\in A}$ there is a canonical map $\rho_{\mathcal{F}}\colon M \otimes_R \prod_{i\in A} Q_i \to \prod_{i\in A} M \otimes_R Q_i$. The map $\rho_{\mathcal{F}}$ is onto for any $\mathcal{F}$ if and only if M is finitely generated, and it is bijective if and only if M is finitely presented. The class of Mittag-Leffler modules is the class of modules for which $\rho_{\mathcal{F}}$ is injective for any $\mathcal{F}$, if we restrict the possibilities for the choice of the family $\mathcal{F}$ to a class of modules $\mathcal{Q}$ we get the class of relative $\mathcal{Q}$-Mittag-Leffler modules.

Mittag-Leffler modules were introduced, and systematically studied, by Raynaud and Gruson in their fundamental paper *Crières the platitude et projectivité* (Invent. Math. **13** (1971), 1–89). In his 1994 habilitation P. Rothmaller introduced and studied the relative version of Mittag-Leffler modules.

Raynaud and Gruson's motivation was a question by Grothendieck coming from algebraic geometry, that needed a good theory for the descent of flatness and projectivity via a morphism between commutative rings. Drinfeld in 2006 recovered the theory of Mittag-Leffler modules and its geometric flavour suggesting replacing projective modules by the flat Mittag-Leffler ones in the definition of infinite dimensional vector bundles on a scheme X. This awoke the interest for Mittag-Leffler modules and, specially, for the class of flat Mittag-Leffler modules.

The proof that one Tilting modules are of finite type by Bazzoni and Herbera (2008), as well as the solution of the Baer Splitting problem by Angeleri-Hügel, Bazzoni and Herbera (2008) also revealed the unexpected importance of the relative Mittag-Leffler modules in homological algebra problems.

Since then, a systematic study of (relative) Mittag-Leffler modules and, in particular, of (relative) flat Mittag-Leffler modules has been undertaken. The results that have been proved show that these classes live in the boundary of many interesting conditions, for example flat Mittag-Leffler modules are a Kaplansky class (J. Šaroch and J. Trlifaj, 2012) but they are not deconstructible (D. Herbera and J. Trlifaj, 2012). Moreover, new and surprising applications of relative Mittag-Leffler conditions have been found, specially remarkable are the ones to Approximation Theory due to Angeleri-Hügel, Šaroch and Trlifaj

published in (2018).

The thesis under review makes a further step in the study of flat relative Mittag-Leffler modules and its applications, mainly, to Approximation Theory. The author is able to do interesting new contributions in a difficult topic that, in the thesis, are organized in three papers. The first one is already accepted for publication in Journal of Algebra and its Applications, and the other two are submitted for publication. I have no doubt that these two papers contain results of enough quality to deserve to be published in prestigious specialized journals. The three papers can be found in Arxiv in open access source.

A short summary of the three papers:

- In the first paper studies some structure properties of the class of relative Mittag-Leffler modules that are analogous to the ones satisfied in the non-relative case. For example, it is proved that such modules are determined by their countably presented submodules.

  One of the main results of the paper proves that the class of flat $\mathcal{Q}$-Mittag-Leffler modules is covering if and only if it is closed under direct limits. Confirming what is known as Enoch's conjecture for these classes of modules.

  The ideas and tools developed in this paper are the first step towards the further developments in the rest of the thesis.

- The second paper deals with the geometrical background of the topic: the classical result of ascent and descent of flat Mittag-Leffler modules is refined to the relative setting. This allows to prove the Zarisky locallity of a suitable notion of quasi-coherent sheaves associated with flat relative Mittag-Leffler modules.

- The third paper recovers de topic of approximation theory now for a general class $\mathcal{C}$ of modules. The topic studied are dualization methods to relate right and left approximations. Salce's Lemma gives a way to do that when the class $\mathcal{C}$ fits in a cotorsion pair, but the general case is completely open. Surprisingly enough, the authors show that some of such dualization process require the extra use of large cardinal principles.

  For example, it is shown that being pre-enveloping/pre-covering imply suitable closure properties of the class $\mathcal{C}$. The converse is proved under the assumption that weak/Vopěnka's Principle holds. As usual in this Ph.D. Thesis, the class of flat Mittag-Leffler modules is specially interesting in this context, providing examples and counterexamples in ZFC.

With no doubt, the thesis proves the author's ability for creative scientific work and also her ability to master very different concepts and to use them in a deep way.