REFEREE REPORT

THESIS: *Flat Relative Mittag-Leffler and Approximations*
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The thesis under review is in the area of abstract algebra, more precisely in Module Theory. The results obtained are related to the class of relatively flat Mittag-Leffler modules and to approximation theory.

After introducing the notation and general results needed in the sequel, the second chapter is devoted to the study of the class $\mathcal{D}_{\mathcal{Q}}$ consisting of *flat relative Mittag-Leffler modules*, where $\mathcal{Q}$ is a fixed class of modules. Recall that a module M is $\mathcal{Q}$-*Mittag Leffler* if the natural homomorphism from $M \otimes_R \prod_{i\in I} N_i \to \prod_{i\in I} M \otimes_R N_i$ is a monomorphism for every set of modules $\{N_i \mid i \in I\}$ belonging to $\mathcal{Q}$. This notion generalizes the classical one of *Mittag-Leffler*, introduced by Raynaud and Gruson [2], which is recovered when $\mathcal{Q} = \text{Mod-}R$.

The main objective of this chapter is to extend the structure and properties of Mittag-Leffler modules to $\mathcal{Q}$-Mittag-Leffler. In this way, the well-known result of Herbera and Trlifaj [1] stating that the flat Mittag-Leffler modules are the $\aleph_1$-projective modules (in the sense that they have an $\aleph_1$-dense system $\mathcal{S}$ consisting of countably generated projective modules, i. e., a direct system $\mathcal{S}$ of submodules of the module which (1) is closed under unions of countable well-ordered chains and (2) satisfies that any countable subset of the module is contained in a submodule in $\mathcal{S}$). This result is extended in Proposition 2.3.4 in two directions:

1. A module M is flat $\mathcal{Q}$-Mittag-Leffler if and only if M has a κ-dense system consisting of $\leq \kappa$-presented flat $\mathcal{Q}$-Mittag-Leffler **pure** submodules of M, where $\kappa = \text{card}(R) + \aleph_0$.

2. If $R \in \mathcal{Q}$, a module M is flat $\mathcal{Q}$-Mittag-Leffler if and only if it has a $\aleph_1$-dense system $\mathcal{S}$ of countably presented flat $\mathcal{Q}$-Mittag-Leffler modules, which are not necessarily pure, but are $\mathcal{Q}$-*pure* in the sense that the inclusion $K \leq M$ remains monic when tensoring by a module in $\mathcal{Q}$ for any $K \in \mathcal{S}$.

As a consequence of this result, it is proven the useful criterion to compare two classes of flat relative Mittag-Leffler modules: $\mathcal{D}_{\mathcal{Q}}$ and $\mathcal{D}_{\mathcal{Q}'}$ are equal if and only if they contain the same countably presented modules.

The other interesting result about relative Mittag-Leffler modules is the extension to this setting of the well-known result by J. Săroch [3]: the class of all flat Mittag-Leffler modules $\mathcal{D}_{\text{Mod-}R}$ is precovering if and only if the ring is right perfect, i. e., the class $\mathcal{D}_{\text{Mod-}R}$ coincides with the class of all flat modules. Then, in Theorem 2.3.6., it is proven that $\mathcal{D}_{\mathcal{Q}}$ is precovering if and only if $\mathcal{D}_{\mathcal{Q}}$ coincides with the class of flat modules. Even more is derived: $\mathcal{D}_{\mathcal{Q}}$ is covering if and only if $\mathcal{D}_{\mathcal{Q}}$ is closed under direct limit, so that the class $\mathcal{D}_{\mathcal{Q}}$ satisfies *Enoch's conjecture*.

In the last section of Chapter 2, the results of the previous sections are applied to the *f-projective* modules, i. e., the flat $\mathcal{Q}$-Mittag-Leffler modules where $\mathcal{Q} = \{R\}$. Among other results, the description of these modules over right semihereditary rings is obtained (Proposition 2.4.8): M is f-projective if and only if every finitely generated submodule of M is projective. Moreover, it is proved that if f-projective modules are characterized in this way, the ring has to be right semihereditary.

In Chapter 3 it is studied whether the notion *flat relative Mittag-Leffler* is geometric, so that it can be useful in the category of quasi-coherent sheaves over a scheme. If A is any commutative ring, we can use the equivalence between the category of A-modules and the category of quasi-coherent sheaves on Spec(A) to define any property $\mathbf{P}$ of modules in the category of quasi-coherent sheaves. If X is a non-affine scheme with structure sheaf $\mathcal{O}_X$, then the property $\mathbf{P}$ is usually considered locally on sheaves: a quasi-coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is *locally* ***P****-quasi-coherent* if for every affine open subset U of X, $\mathcal{M}(U)$ satisfies the property $\mathbf{P}$ as an $\mathcal{O}_X(U)$-module. Then, this local notion is considered geometrical if it does not depend on the open affine covering of X, that is, the quasi-coherent sheaf $\mathcal{M}$ is locally $\mathbf{P}$-quasi-coherent if and only if there exists an open affine covering $\mathbf{V}$ of X such that $\mathcal{M}(U)$ satisfies $\mathbf{P}$ as a $\mathcal{O}_X(U)$-module for any U in the covering $\mathbf{V}$. In this case, we say that *the notion locally* ***P****-quasi-coherent is Zariski local.*

The main results regarding Zariski locality obtained in Chapter 3 are the following:

1. The locally f-projective quasi-coherent property is Zariski local.
2. The locally flat n-Mittag-Leffler quasi-coherent property is Zariski local. The locally flat n-Mittag-Leffler quasi-coherent sheaves are the so-called *n-Drinfeld vector bundles.*

These results are a direct consequence of Theorem 3.4.5, where the following general assertion is proven: given a class $\mathcal{R}$ of commutative rings and a class of definable categories $\{\mathcal{Q}_R \mid R \in \mathcal{R}\}$, where $\mathcal{Q}_R \subseteq \text{Mod-}R$ for each $R \in \mathcal{R}$, the property locally flat $\mathcal{Q}$-Mittag-Leffler is Zariski local on the class of locally $\mathcal{R}$-schemes provided that there are some compatibility conditions between the classes $\mathcal{Q}_R$ and the rings in $\mathcal{R}$.

In the last chapter of the thesis the relationship between the closure under certain operations of a class of modules $\mathcal{F}$, and the existence of $\mathcal{F}$-approximations ($\mathcal{F}$-precovers and $\mathcal{F}$-preenvelopes) is studied. In this study, some set theoretical principles are assumed: the Vopěnka and Weak Vopěnka principles.

For example, it is known that if $\mathcal{F}$ is closed under direct summands and is preenveloping, then $\mathcal{F}$ is closed under direct products. In Lemma 4.3.1 it is proved that, assuming the Weak Vopěnka principle, the converse is true.

For precovering classes, the situation is different, because the class of flat Mittag-Leffler modules is closed under direct summands and direct sums, but it is precovering if and only if the ring is right perfect. Therefore, the dual result of Lemma 4.3.1 is not true. However, a partial dual can be obtained for classes $\mathcal{F}$ which are closed under homomorphic images assuming the Vopěnka principle: $\mathcal{F}$ is precovering if and only if it is closed under direct sums if and only if it is of the form $\text{Gen}(M)$ for a single module M (the first equivalence is always true; the last one uses Vopěnka's principle).

The most interesting result of Chapter 4 is Theorem 4.3.11, where a partial converse of Lemma 4.3.1 is proved. If any class $\mathcal{F}$ of abelian grups satisfying that

1. is covering and closed under homomorphic images, and
2. is generated by a class of $\aleph_1$-free modules,

is of the form $\text{Gen}(M)$ for some abelian group M, then the Weak Vopěnka's principle holds.

In my opinion, the thesis contains new results in two active research areas of Module Theory, Mittag-Leffler modules, and approximation theory. Flat Mittag-Leffler modules have been intensely investigated in the last years, since they have been used in the solution of certain problems of module theory, and they have been proposed in the setting of quasi-coherent sheaves over a scheme. Chapters 2 and 3 give new results related to the flat relative Mittag-Leffler modules in these two directions.

Regarding approximation theory, there are some open problems in the area which, maybe, will require the use of set theoretical principles to solve them. The results in Chapter 4 can be considered as a piece of research in this direction.

For all these reasons, I think that the thesis "Flat Relative Mittag-Leffler Modules and Approximations" contains relevant results in modern research areas and evidence of the author's ability to create original scientific work.

References

[1] Dolors Herbera and Jan Trlifaj. Almost free modules and Mittag-Leffler conditions. *Adv. Math.*, 229(6):3436–3467, 2012.

[2] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de “platification” d’un module. *Invent. Math.*, 13:1–89, 1971.

[3] Jan Šaroch. Approximations and Mittag-Leffler conditions the tools. *Israel J. Math.*, 226(2):737–756, 2018.