Stanovisko k dizertační práci a vyjádření ke studentovi

Asmae Ben Yassine was accepted to the Ph.D. program at our Faculty in July 2019. Though she had applied for her student visa at the Czech embassy in Rabat already in August 2019, her visa application was granted only in January 2020, so that she started her studies under my supervison with an almost four months delay.

Her Ph.D. thesis titled **Flat Relative Mittag-Leffler Modules and Approximations** presents main results of our joint collaboration over the past several years. It consists of an introductory chapter, a paper published in J. Algebra and Its Appl., https://doi.org/10.1142/s0219498824502190, and of two preprints, https://arxiv.org/pdf/2208.00869 (submitted to J. Pure Appl. Algebra) and https://arxiv.org/abs/2401.11979 (submitted to Appl. Cat. Structures).

The original motivation for the published paper was to test a major open problem in the area, the Enochs' Conjecture, on a particular set of classes of modules, the flat relative Mittag-Leffler modules. We have realized quickly that Šaroch's proof that the class of all (absolute) flat Mittag-Leffler modules is precovering, iff it is covering, iff it is closed under direct limits, can be modified to include also the relative setting. Moreover, it turned out that classes of flat relative Mittag-Leffler modules, though not deconstructible in general, are completely determined by their countably presented modules. Asmae also suggested to consider in more detail the particular case of f-projective modules (that coincide with flat $\{R\}$-Mittag-Leffler modules by an old result of Ken Goodearl). This was done in the final section of the paper.

Flat Mittag-Leffler modules were instrumental in proving Zariski locality of the notion of a vector bundle (conjectured by Grothendieck, and proved by Raynaud and Gruson in 1971). A natural challenge was to extend the Zariski locality also to the relative setting. In the first preprint, we were originally able to prove such an extension only under additional finiteness assumptions, in the setting of Tor-orthogonal classes of finite type. But a recent result of Angermüller made it possible to extend our proof to all locally f-projective quasi-coherent sheaves over all schemes, and also, for each $n \geq 0$, to the class of all n-Drinfeld vector bundles over all noetherian schemes.

The second preprint concerns characterizations of approximation classes of modules which satisfy additional closure properties. There is a formal duality between the notions of a preenvelope (= left approximation) and a precover (= right approximation). In the particular case of special approximations, there is even an explicit duality provided by Salce's Lemma. In general, it has turned out that some results concerning approximations and closure properties can be dualized including their proofs, while other dualizations require large cardinal principles, and some even fail in ZFC, with counterexamples provided by classes of flat relative Mittag-Leffler modules. For example, a large cardinal principle called Vopěnka's Principle implies that $(*)$ *each covering class of modules closed under homomorphic images is of the form Gen(M) for a module M.* Moreover, property $(*)$, restricted to classes generated by flat Mittag-Leffler abelian groups, implies Weak Vopěnka's Principle. However, the dual of $(*)$ fails in ZFC: the class of all flat Mittag-Leffler abelian groups is closed under submodules, but it is not of the form Cog(M) for any abelian group M.

During her studies, Asmae presented the results of our joint research at two European conferences: *Homological Methods in Representation Theory* at Chiemsee in October 2021, and at the *Functor Categories, Model Theory, and Constructive Category Theory* conference at Univ. Málaga in July 2023. In my opinion, the research she has done during her doctoral studies in Prague provides a good start for a successful academic carrier, whether pursued in Europe or elsewhere.

Prague March 2, 2024 Prof. Jan Trlifaj (advisor)