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Flat Relative Mittag-Leffler Modules and Approximations

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I want to express my gratitude to my supervisor, Jan Trlifaj, for his patient guidance, encouragement, and valuable advice throughout my time as his student. I consider myself extremely fortunate to have had a supervisor who cared deeply about my work and responded promptly to my questions and queries.

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Abstract: The thesis presents the main results of our joint work with my supervisor on approximations of modules, with a primary emphasis on the class of flat relative Mittag-Leffler modules, Zariski locality of quasi-coherent sheaves associated to this class, and dualizations of approximations. We start by giving a characterization of the class $\mathcal{D}_{\mathcal{Q}}$ consisting of all flat relative Mittag-Leffler modules in terms of their local structure. Additionaly, we show that Enochs' Conjecture holds for all the classes $\mathcal{D}_{\mathcal{Q}}$. These results are then applied to the specific setting of f-projective modules. Our study then extends to the ascent and descent for relative versions of the Mittag-Leffler property along flat and faithfully flat homomorphisms of commutative rings. This investigation leads to results such as the Zariski locality of locally f-projective quasi-coherent sheaves for all schemes and, for each $n \geq 1$, the Zariski locality of n-Drinfeld vector bundles for all locally noetherian schemes. Shifting focus to general approximation classes of modules, we investigate possibilities of dualization in dependence on closure properties of these classes. While some proofs easily dualize, other require large cardinal principles: we show that Vopěnka's Principle implies that each covering class of modules closed under homomorphic images is of the form $\mathrm{Gen}\,(M)$ for a module M, and the latter, when restricted to classes generated by $\aleph_1$-free abelian groups, implies Weak Vopěnka's Principle. Flat Mittag-Leffler modules appear also in this context, since they in several cases yield limits for dualization.

Keywords: Approximations of modules, Enochs' Conjecture, flat relative Mittag-Leffler module, Zariski locality, Vopěnka's Principle.

Contents

Introduction

Approximation theory traces its origins back to Reinhold Baer's 1940 discovery of the injective hull of a “group with operators”. Since the late 1950s, injective envelopes, projective covers as well as pure-injective envelopes, have successfully been used in module theory of arbitrary rings. The general theory of preenvelopes and precovers (or left and right approximations) of modules was developed through independent research by Auslander, Reiten, and Smalø in the finite-dimensional case, and by Enochs and Xu for arbitrary modules.

It's worth noting that decomposability of classes of modules in homological algebra is typically rare. Essentially, decomposability is available only for projective modules, and for injective modules over right noetherian rings. On the other hand, *deconstructible* classes [41, 2.4] are abundant, and the deconstructibility leads to approximations of modules. With these approximations, one can perform relative homological algebra, where projective and injective modules are replaced by other classes of modules that better fit the specific settings under consideration.

A well-known result of Bass states that non-right perfect rings R are characterized by the existence of countably presented flat (right R-) modules that are not projective [4, 28.4]. Although projective modules can be decomposed into direct sums of countably generated submodules [4, 26.2], the situation is less straightforward for flat modules, as only a deconstruction theorem is available: if $\kappa = \operatorname{card} R + \aleph_0$, then each flat module M can be deconstructed into a transfinite extension of $\leq \kappa$-presented flat modules [27, 6.17]. That is, M possesses a continuous increasing chain of submodules, $(M_\alpha \mid \alpha \leq \sigma)$, such that $M_0 = 0$, $M_\sigma = M$, and for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is a $\leq \kappa$-presented flat module.

When answering Grothendieck's question concerning the Zariski locality of the notion of a vector bundle, Raynaud and Gruson introduced the intermediate class of (absolute) flat Mittag-Leffler modules, denoted by $\mathcal{FM}$, in their work [38]. Recall that a module M is *Mittag-Leffler*, if for each family $\mathcal{I} = (Q_i \mid i \in I)$ of left R-modules, the canonical group homomorphism $\varphi_{M,\mathcal{I}} : M \otimes_R \prod_{i \in I} Q_i \to \prod_{i\in I} M \otimes_R Q_i$ is monic (see the next chapter for unexplained terminology).

Let $\mathcal{P}$ and $\mathcal{F}$ denote the classes of all projective and flat modules, respectively. If R is not right perfect, then $\mathcal{P} \subsetneq \mathcal{FM} \subsetneq \mathcal{F}$. These classes are closed under transfinite extensions, however, unlike $\mathcal{P}$ and $\mathcal{F}$, the class $\mathcal{FM}$ is not deconstructible [27, 10.13]. Nonetheless, between $\mathcal{P}$ and $\mathcal{F}$, there exists an abundant supply of deconstructible classes closed under transfinite extensions: as κ varies over all infinite cardinals, the classes $\mathcal{FM}_\kappa$, which consist of *κ-restricted flat Mittag-Leffler modules* (= transfinite extensions of $\leq \kappa$-presented flat Mittag-Leffler modules), form a strictly increasing chain $(\mathcal{FM}_\kappa \mid \aleph_0 \leq \kappa)$ between $\mathcal{P}$ and $\mathcal{FM}$ [45]:

$$\mathcal{P} = \mathcal{FM}_{\aleph_0} \subsetneq \mathcal{FM}_{\aleph_1} \subsetneq \cdots \subsetneq \mathcal{FM}_\kappa \subsetneq \mathcal{FM}_{\kappa^+} \subsetneq \cdots \subsetneq \bigcup_{\aleph_0 \leq \kappa} \mathcal{FM}_\kappa = \mathcal{FM}.$$

For each $\kappa \geq \aleph_0$, the class $\mathcal{FM}_\kappa$ is obviously deconstructible, and hence precovering [27, 7.21], but the class $\mathcal{FM}$ fails these properties [41, 3.3].

If R is not right perfect, the classes of flat *relative* Mittag-Leffler modules provide a rich intermediate structure between the classes $\mathcal{FM}$ and $\mathcal{F}$. These are obtained by restricting the choice of the families $\mathcal{I}$ in the definition above: if $\mathcal{Q}$ is any class of left R-modules, then a module M is *$\mathcal{Q}$-Mittag-Leffler*, if the canonical group homomorphism $\varphi_{M,\mathcal{I}}$ is monic for each family $\mathcal{I} = (Q_i \mid i \in I)$ which consists of modules from $\mathcal{Q}$. Following [31], we will use $\mathcal{D}_\mathcal{Q}$ to denote the class of all flat $\mathcal{Q}$-Mittag-Leffler modules. Thus, if $\mathcal{Q}' \subseteq \mathcal{Q}$, we get the following inclusions

$$\mathcal{FM} = \mathcal{D}_{R\text{–Mod}} \subseteq \mathcal{D}_\mathcal{Q} \subseteq \mathcal{D}_{\mathcal{Q}'} \subseteq \mathcal{F}.$$

While there exists a proper class of classes $\mathcal{Q} \subseteq R$–Mod, there is only a set of different classes $\mathcal{D}_\mathcal{Q}$. As proved by Rothmaler [39, 2.2], $\mathcal{D}_\mathcal{Q} = \mathcal{D}_{\mathrm{Def}\,\mathcal{Q}}$ where $\mathrm{Def}\,\mathcal{Q}$ is the *definable closure* of $\mathcal{Q}$, that is, the least class of left R-modules containing $\mathcal{Q}$ and closed under direct products, direct limits, and pure submodules. Moreover, if $R \in \mathcal{Q}$, then the structure of the class $\mathcal{D}_\mathcal{Q}$ is completely determined by the countably presented modules in $\mathcal{D}_\mathcal{Q}$. Therefore, if $R \in \mathcal{Q}'$, then $\mathcal{D}_\mathcal{Q} = \mathcal{D}_{\mathcal{Q}'}$, iff $\mathcal{D}_\mathcal{Q}$ and $\mathcal{D}_{\mathcal{Q}'}$ contain the same countably presented modules, [12, 2.5]. Regarding the approximation properties of flat relative Mittag-Leffler modules, the situation is similar to the absolute case: the class $\mathcal{D}_\mathcal{Q}$ is precovering only if it coincides with the class of all flat modules [12, 2.6].

Another key aspect to highlight is the application of large cardinal principles in approximation theory, notably the Vopěnka's Principle. A recent application of this principle to approximation theory has been discussed in [14]: If Vopěnka's Principle is consistent, then it is also consistent that each cotorsion pair over any right hereditary ring is complete. However, by [17], it is consistent with ZFC that the Whitehead cotorsion pair $({}^\perp\mathbb{Z}, ({}^\perp\mathbb{Z})^\perp)$ is not complete.

The core of this thesis consists of three papers, the first of which has already been published online:

(i) A. Ben Yassine, J. Trlifaj, *Flat relative Mittag-Leffler modules and approximations*, J. Algebra and Its Appl., DOI: 10.1142/S0219498824502190, arXiv:2110.06792v2.

(ii) A. Ben Yassine, J. Trlifaj, *Flat relative Mittag-Leffler modules and Zariski locality*, submitted to J. of Pure and Applied Algebra, arXiv:2208.00869v2.

(iii) A. Ben Yassine, J. Trlifaj, *Dualizations of approximations, $\aleph_1$-projectivity, and Vopěnka's Principles*, submitted to Applied Categorical Structures, arXiv:2401.11979v1.

Before introducing the contents of these three papers, the first chapter will gather relevant aspects of approximation theory. Additionally, it will cover basic properties of (flat) Mittag-Leffler modules, quasi-coherent sheaves, Zariski locality and the Vopěnka's principles. This is done to ensure completeness and improve the readability of the following chapters.

Paper (i) investigates the structure and approximation properties of the class $\mathcal{D}_\mathcal{Q}$ in dependence on $\mathcal{Q}$. We prove that the classes $\mathcal{D}_\mathcal{Q}$ are determined by their countably presented modules. Additionally, we show that approximation properties of $\mathcal{D}_{\mathcal{Q}}$ depend completely on whether there exists a Bass module $N \notin \mathcal{D}_{\mathcal{Q}}$. In the final section, we apply these results to the particular setting of $\mathcal{Q} = \{R\}$, i.e., to the f-projective modules.

Concerning the second paper (ii), the aim is to refine the classic result on the ascent and descent of flat Mittag-Leffler modules to the relative setting and prove Zariski locality of the corresponding notions of flat quasi-coherent sheaves. In particular, we prove the Zariski locality of the notion of a locally f-projective quasi-coherent sheaf for all schemes, and for each $n \geq 1$, of the notion of an n-Drinfeld vector bundle for all locally noetherian schemes.

Finally, in (iii), we consider general approximation classes of modules and investigate if, and how, dualizations are possible assuming additional closure properties for these classes. While certain results can be straightforwardly dualized by employing dual arguments, other require the use of large cardinal principles, namely, Vopěnka's Principles. Flat Mittag-Leffler modules appear also in this context, as they yield limits for dualizations.

A brief note about the structure of the thesis: each of the four chapters has its own bibliography at the end. Therefore, any citation in square brackets refers to the list of references at the end of the chapter in which it appears. Despite the fact that the thesis is a collection of individual papers, it has a consistent numbering of chapters, sections, and theorems.

Chapter 1

Preliminaries and Main Theorems

The reader is supposed to be familiar with basic notions of module theory (see for instance [4]). The aim of this chapter is to present some basic and classical results concerning approximation theory, Mittag-Leffler modules and quasi-coherent sheaves which will be needed in the next chapters.

1.1 Introduction to approximation theory

In this section, we recall the basics of approximation theory for module categories over any associative ring with unit. We introduce preenvelopes and precovers and their minimal versions, envelopes and covers.

Many of the results presented in this section have been known for decades; they have contributed significantly to progress of the general theory and proved essential for the recent applications discussed in the upcoming chapters.

Let R be an associative and unital ring. By Mod–R we denote the category of all unitary right R-modules. When referring to a module without specifying the side and/or the ring, we consistently mean a right R-module. At times, duality will also take us to the category of all left R-modules, which is denoted by R–Mod.

Throughout this chapter, we denote the class of all projective modules by $\mathcal{P}$, the class of all injective modules by $\mathcal{I}$, and the class of all flat modules by $\mathcal{F}$.

1.1.1 Basic tools: direct and inverse limits

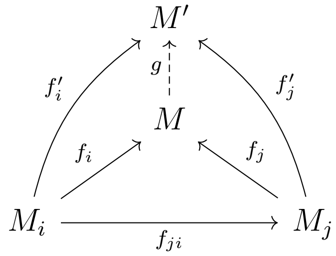
Direct and inverse limits are among the main constructions that yield new modules from classes of modules whose structure is already known.

For example, flat modules are obtained as direct limits of direct systems of finitely generated free modules. Let us now recall the general construction:

A partially ordered set $(I, \leq)$ is *upper directed*, provided that for all $i, j \in I$, there exists $k \in I$, such that $i \leq k$ and $j \leq k$.

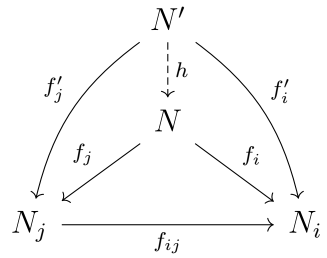
Definition 1.1.1. Given an upper directed set $(I, \leq)$, a system $\mathcal{D} = (M_i, f_{ji} \mid i \leq j \in I)$ is a *direct system* of modules, provided that $M_i (i \in I)$ are (right R-) modules and $f_{ji} (i \leq j \in I)$ are R-homomorphisms, such that $f_{ji} : M_i \to M_j$ for $i \leq j \in I$, $f_{ii} = id_{M_i}$, for all $i \in I$ and $f_{ki} = f_{kj} f_{ji}$, whenever $i \leq j \leq k \in I$.

Viewing $\mathcal{D}$ as a diagram in the category of all right R-modules, we can form its colimit $(M, f_i \mid i \in I)$. In particular, M is a module, and $f_i \in \mathrm{Hom}_R(M_i, M)$ satisfies $f_i = f_j f_{ji}$ for all $i \leq j \in I$. This colimit (or sometimes just the module M itself) is called the *direct limit* of the direct system $\mathcal{D}$. It is denoted by $\varinjlim_{i\in I} M_i$ (or just $\varinjlim \mathcal{D}$).
More precisely, the colimit is a *cocone* admitting the following universal property: for each cocone $(M', f'_i \mid i \in I)$ there is a unique homomorphism $g : M \to M'$ such that $gf_i = f'_i$ for each $i \in I$.



Dually, we define inverse limits of inverse systems of (right R-) modules. In this thesis, we will employ them mainly to define and investigate properties of Mittag-Leffler systems and Mittag-Leffler modules.

Definition 1.1.2. For an upper directed set $(I, \leq)$, a system $\mathcal{V} = (N_i, f_{ij} \mid i \leq j \in I)$ is an *inverse system* of modules, provided that N_i is a (right R-) module, $f_{ij} \in \mathrm{Hom}_R(N_j, N_i)$, $f_{ii} = id_{N_i}$, and $f_{ik} = f_{ij} f_{jk}$ for all $i \leq j \leq k \in I$.
Let $(N, f_i(i \in I))$ be the limit of $\mathcal{V}$ viewed as a diagram in the category Mod–R. So M is a module, and $f_i \in \mathrm{Hom}_R(N, N_i)$ satisfies $f_i = f_{ij} f_j$ for all $i \leq j \in I$. This limit (or just the module M itself) is called the *inverse limit* of the inverse system $\mathcal{V}$ and denoted by $\varprojlim_{i\in I} N_i$.
Specifically, the limit is a *cone* with the following universal property: for each cone $(N', f'_i(i \in I))$ there is a unique homomorphism $h : N' \to N$ such that $f_i h = f'_i$ for each $i \in I$.



Remark 1. In category theoretic language, direct limits are usually called filtered colimits, and inverse limits are the filtered limits. Since we deal with modules here, we stick to the terminology common in algebra.

1.1.2 Envelopes and covers

Let $\mathscr{C}$ be a class of R-modules. Throughout this section, we assume that $\mathscr{C}$ is closed under direct summands and isomorphic images.

Definition 1.1.3. For an R-module M, a module $C \in \mathscr{C}$ is called a *$\mathscr{C}$-envelope* of M if there is a homomorphism $\varphi : M \to C$ such that the following hold:

(1) For any homomorphism $\varphi' : M \to C'$ with $C' \in \mathscr{C}$, there is a homomorphism $f : C \to C'$ with $\varphi' = f\varphi$:

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & C \\ & \searrow_{\varphi'} & \Big\downarrow{\scriptstyle f} \\ & & C'. \end{array}$$

In other words, $\mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C') \to 0$ is exact for any $C' \in \mathscr{C}$;

(2) If an endomorphism $f : C \to C$ is such that $\varphi = f\varphi$, then f must be an automorphism.

If condition (1) is satisfied, and possibly not (2), we call $\varphi : M \to C$ a $\mathscr{C}$-*preenvelope.*
For simplicity, we sometimes refer to C or the map φ as a $\mathscr{C}$-envelope (preenvelope) of M. We say that $\mathscr{C}$ is a *preenveloping class*, (*enveloping class*) provided that each module has a $\mathscr{C}$-preenvelope ($\mathscr{C}$-envelope).
Note that if $M \to C$ is a $\mathscr{C}$-preenvelope and if $S \subseteq M$ is a direct summand of M, then $S \to M \to C$ is a $\mathscr{C}$-preenvelope of S. One can also prove easily that if $\varphi_1 : M \to C_1$ and $\varphi_2 : M \to C_2$ are two different $\mathscr{C}$-envelopes of M, then $C_1 \cong C_2$.

Example 1.1.4. (i) The embedding $M \hookrightarrow E(M)$ where $E(M)$ is the injective hull of M is evidently the $\mathcal{I}$-envelope of a module M.

(ii) A module M is *pure-injective*, provided that M is injective with respect to pure embeddings, that is, $\mathrm{Hom}_R(f, M) : \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M)$ is surjective for each pure embedding $f : A \subseteq B$. Let $\mathcal{PI}$ be the class of all pure-injective modules and let $PE(M)$ be the pure-injective hull of M (cf. [33, Chapter 7]). Then $M \to PE(M)$ is the $\mathcal{PI}$-envelope of M.

Hence, the classes $\mathcal{I}$ and $\mathcal{PI}$ are enveloping classes of modules.

Proposition 1.1.5. *[48, Proposition 1.2.2] Let M be an R-module. Assume that M has a $\mathscr{C}$-envelope and let $\varphi : M \to C$ be a $\mathscr{C}$-preenvelope. Then, there exist submodules D and K such that $C = D \oplus K$, and the composition $M \to C \to D$ is a $\mathscr{C}$-envelope.*

Proposition 1.1.6. *[48, Corollary 1.2.3] Let M be an R-module. Assume that M has a $\mathscr{C}$-envelope, and let $\varphi : M \to C$ be a $\mathscr{C}$-preenvelope. Then, φ is an envelope if and only if there is no direct sum decomposition $C = D \oplus K$ with $K \neq 0$ and $\mathrm{im}(\varphi) \subseteq D$.*

Dually, we have the following definition and properties for $\mathscr{C}$-covers.

Definition 1.1.7. For an R-module M, a module $C \in \mathscr{C}$ is called a $\mathscr{C}$-*cover* of M if there is a homomorphism $\varphi : C \to M$ such that the following hold:

(1) For any homomorphism $\varphi' : C' \to M$ with $C' \in \mathscr{C}$, there exists a homomorphism $f : C' \to C$ with $\varphi' = \varphi f$:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & M \\ & \nwarrow^{f} & \uparrow{\scriptstyle \varphi'} \\ & & C' \end{array}$$

or equivalently, $\mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M) \to 0$ is exact for any $C' \in \mathscr{C}$;

(2) If f is an endomorphism of C with $\varphi = \varphi f$, then f must be an automorphism.

If condition (1) is satisfied, and possibly not (2), we call $\varphi : C \to M$ a $\mathscr{C}$-*precover*.

A $\mathscr{C}$-cover (precover) is not necessarily surjective. Moreover, if $C \to M$ is a $\mathscr{C}$-precover of M, and $M \to S$ is the projection of M onto a direct summand S of M, then $C \to M \to S$ is a $\mathscr{C}$-precover of S. Additionally, one can show that if $\varphi_i : C_i \to M$, $i = 1, 2$, are two $\mathscr{C}$-covers, then $C_1 \cong C_2$.

We say that $\mathscr{C}$ is a *precovering class*, (*covering class*) provided that each module has a $\mathscr{C}$-precover ($\mathscr{C}$-cover).

Remark 2. These definitions were given close in time by Auslander and Smalø in [7] in the setting of finitely generated modules over artin algebras and by Enochs in [21] for arbitrary modules. However Auslander and Smalø used the terminology of *minimal right* and *minimal left approximations* instead of covers and envelopes, respectively.

Example 1.1.8. Since every module is a homomorphic image of a projective module, each module M admits a $\mathcal{P}$-precover. Moreover, M has a $\mathcal{P}$-cover if and only if it has a projective cover in the sense of Bass. Therefore, $\mathcal{P}$ is always a precovering class, and it is a covering class, if and only if R is a right perfect ring. See Theorem 1.1.18 below for more details.

One of our main interests is to determine for which classes $\mathscr{C}$, $\mathscr{C}$-covers exist.

Proposition 1.1.9. *[48, Theorem 1.2.7] Let M be an R-module and assume that M has a $\mathscr{C}$-cover. Let $\varphi : C \to M$ be a $\mathscr{C}$-precover. Then $C = D \oplus K$ for some submodules D and K such that the restriction $\varphi_{|D} : D \to M$ gives rise to a $\mathscr{C}$-cover of M and $K \subseteq \ker(\varphi)$.*

Proposition 1.1.10. *[48, Corollary 1.2.8] Let M be an R-module and assume that M has a $\mathscr{C}$-cover. Then a $\mathscr{C}$-precover $\varphi : C \to M$ is a cover if and only if there is no nonzero direct summand K of C contained in* $\ker(\varphi)$.

We list some essential closure properties related to covering and enveloping classes:

Lemma 1.1.11. *[44, Lemma 9.13] Let R be a ring and $\mathscr{C}$ be a class of modules closed under isomorphisms. Let $C \in \mathscr{C}$ and $D \oplus E = C$.*

(i) Assume that D has a $\mathscr{C}$-cover. Then $D \in \mathscr{C}$.
So if $\mathscr{C}$ is covering, then $\mathscr{C}$ is closed under direct summands.

(ii) Assume that D has a $\mathscr{C}$-envelope. Then $D \in \mathscr{C}$.
So if $\mathscr{C}$ is enveloping, then $\mathscr{C}$ is closed under direct summands.

Lemma 1.1.12. *[44, Lemma 9.14] Let R be a ring and $\mathscr{C}$ be a class of modules closed under isomorphisms and direct summands.*

(i) Assume that $\mathscr{C}$ is precovering. Then $\mathscr{C}$ is closed under direct sums.

(ii) Assume that $\mathscr{C}$ is preenveloping. Then $\mathscr{C}$ is closed under direct products.

Corollary 1.1.13. *Let R be a ring and $\mathscr{C}$ be a class of modules closed under isomorphisms.*

(i) Assume that $\mathscr{C}$ is covering. Then $\mathscr{C}$ is closed under direct summands and direct sums.

(ii) Assume that $\mathscr{C}$ is enveloping. Then $\mathscr{C}$ is closed under direct summands and direct products.

Up to this point, our discussion has evolved around envelopes and covers in a general sense. Now, we shift our focus to examine important specific examples.

Definition 1.1.14. An injective module I is called an *injective envelope* of M if M can be *essentially embedded* into I, i.e., there is an injection $\varphi : M \to I$ such that $\mathrm{Im}\,(\varphi) \cap K = 0$ for any submodule K of I only if $K = 0$.

Over any ring, Eckmann and Schopf [15] showed that every module M possesses an injective envelope, denoted by $E(M)$. This result, along with Matlis' structure theorem for injective modules [35], has played a crucial role in homological algebra and its application in commutative algebra (see [24, 43, 36]). The following illustrates the consistency between the notion of an injective envelope and the notion of an $\mathcal{I}$-envelope.

Proposition 1.1.15. *[48, Theorem 1.2.11] Let M be an R-module, and let $I \in \mathcal{I}$. Then the following assertions are equivalent:*

(i) $\varphi : M \to I$ is an $\mathcal{I}$-envelope;

(ii) $\varphi : M \to I$ is an injective envelope in the Eckmann-Schopf's sense.

Bass [8] introduced projective covers as the duals of injective envelopes. Surprisingly, the existence of projective covers is rather uncommon and requires the ring to be right perfect.

Definition 1.1.16. (1) For an R-module M, a submodule $S \subseteq M$ is said to be *small* or *superfluous* if for any submodule $D \subseteq M$, $S + D = M$ implies $D = M$. This is denoted by $S << M$.

(2) Let P be a projective R-module. A surjective homomorphism $\varphi : P \to M$ is called a *projective cover* if $\ker(\varphi) << P$.

The following result shows the consistency between projective covers and $\mathcal{P}$-covers.

Proposition 1.1.17. *[48, Theorem 1.2.12] Let M be an R-module, and let $P \in \mathcal{P}$. Then the following assertions are equivalent:*

(i) $\varphi : P \to M$ *is a* $\mathcal{P}$*-cover;*

(ii) $\varphi : P \to M$ *is a projective cover.*

Next, we present Bass' Theorem P [8], which addresses the existence of projective covers. A ring R is said to be *right perfect* if every R-module admits a projective cover. For additional terminology used in Bass' Theorem, we refer to [8] or [34].

Theorem 1.1.18. *[8, Theorem P] Let R be an associative ring and let J be its Jacobson radical. Then, the following statements are equivalent:*

(1) R is right perfect;

(2) R/J is semisimple and J is right T-nilpotent;

(3) R/J is semisimple and every nonzero R-module has a maximal submodule;

(4) Every flat R-module is projective;

(5) R satisfies the descending chain condition (DCC) for principal left ideals;

(6) Any direct limit of projective R-modules is projective.

1.1.3 Existence of left and right approximations

Assume the class $\mathscr{C}$ is as previously defined. We say that $\mathscr{C}$ is closed *under extensions* provided that for every short exact sequence of R-modules :

$$0 \to A \to B \to C \to 0$$

if both A and C are in $\mathscr{C}$, then $B \in \mathscr{C}$. For instance, $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{F}$ are closed under extensions.

Definition 1.1.19. Let $\mathscr{C}$ be a class of R-modules. We have the two associated classes:

$$\mathscr{C}^{\perp} = \{N \in R\text{–Mod} \mid \operatorname{Ext}^1_R(C, N) = 0, C \in \mathscr{C}\}$$
$$^{\perp}\mathscr{C} = \{N \in R\text{–Mod} \mid \operatorname{Ext}^1_R(N, C) = 0, C \in \mathscr{C}\}$$

$\mathscr{C}^{\perp}$ is called the *right orthogonal class* of $\mathscr{C}$, while $^{\perp}\mathscr{C}$ is called the *left orthogonal class* of $\mathscr{C}$.

Regarding the orthogonal operations mentioned above, one interesting question is when the following holds true:

$$\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp}) \text{ or } \mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}.$$

i.e., $(\mathscr{C}, \mathscr{C}^{\perp})$ or $({}^{\perp}\mathscr{C}, \mathscr{C})$ form a cotorsion pair in the sense of the next section.

It is clear that $\mathcal{P}^{\perp} = R$–Mod, and ${}^{\perp}(\mathcal{P}^{\perp}) = \mathcal{P}$. Similarly, we have ${}^{\perp}\mathcal{I} = R$–Mod and $({}^{\perp}\mathcal{I})^{\perp} = \mathcal{I}$.

Remark 3. The orthogonal operations are very handy for describing modules. For instance, let $\mathcal{FI}$ denotes the class of all *fp-injective* (or *absolutely pure*) modules, i.e., modules M that are pure submodules of every module containing them as submodules. It is well known that M is fp-injective if and only if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented modules N. Hence, $\mathcal{FI} = \mathcal{F}_P^{\perp}$, where $\mathcal{F}_P$ is the class of all finitely presented R-modules. On the other hand a finitely generated R-module is finitely presented if and only $\mathrm{Ext}^1_R(N, M) = 0$ for all fp-injective modules M (see Enochs [19] or Glaz [26, Theorem 2.1.10]).

Let us go back to $\mathscr{C}$-covers and envelopes.

Definition 1.1.20. Let M be an R-module.

(i) A $\mathscr{C}$-preenvelope $\varphi : M \to C$ of M is called *special*, provided that φ is injective and $\mathrm{Coker}(\varphi) \in {}^{\perp}\mathscr{C}$. In other words, a special $\mathscr{C}$-preenvelope φ of M is a morphism that fits into a short exact sequence

$$0 \to M \xrightarrow{\varphi} C \to D \to 0$$

with $C \in \mathscr{C}$ and $D \in {}^{\perp}\mathscr{C}$. Indeed, such φ is always a $\mathscr{C}$-preenvelope, since for each $C' \in \mathscr{C}$, $\mathrm{Ext}^1_R(D, C') = 0$ implies that the abelian group homomorphism $\mathrm{Hom}_R(\varphi, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective.

(ii) A $\mathscr{C}$-precover $\psi : C \to M$ of M is called *special*, provided that ψ is surjective and $\mathrm{Ker}(\psi) \in \mathscr{C}^{\perp}$. Again, a special $\mathscr{C}$-precover ψ of M is just a map that fits into a short exact sequence

$$0 \to B \to C \xrightarrow{\psi} M \to 0$$

where $C \in \mathscr{C}$ and $B \in \mathscr{C}^{\perp}$.

If $\mathscr{C}$ is a class of modules such that each module M has a special $\mathscr{C}$-preenvelope (special $\mathscr{C}$-precover), then $\mathscr{C}$ is called *special preenveloping* (*special precovering*).

The next two results are referred to as Wakamatsu's Lemmas.

Lemma 1.1.21. *[48, Lemma 2.1.1] Let M be an R-module, and let $\varphi : C \to M$ be a $\mathscr{C}$-cover of M. Assume that $\mathscr{C}$ is closed under extensions. Let $K = \ker(\varphi)$. Then $\mathrm{Ext}^1_R(C', K) = 0$ for any $C' \in \mathscr{C}$. In particular, each injective $\mathscr{C}$-envelope of M is special.*

Lemma 1.1.22. *[48, Lemma 2.1.2] Let M be an R-module, and let $\varphi : M \to C$ be a $\mathscr{C}$-envelope of M. Assume that $\mathscr{C}$ is closed under extensions. Let $D = \mathrm{Coker}(\varphi)$. Then $\mathrm{Ext}^1_R(D, C') = 0$ for all $C' \in \mathscr{C}$. In particular, each surjective $\mathscr{C}$-cover of M is special.*

Definition 1.1.23. Let $\mathscr{C}$ be a class of R-modules, and let M be an R-module. Let $\mathcal{E}xt(\mathscr{C}, M)$ denote the class of all extensions of M by $\mathscr{C}$. An extension $0 \to M \to E \to C \to 0$ with $C \in \mathscr{C}$ is called a *generator* for $\mathcal{E}xt(\mathscr{C}, M)$ if for any extension $0 \to M \to E' \to C' \to 0$ with $C' \in \mathscr{C}$, there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow g & & \downarrow f & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Furthermore, such a generator is said to be *minimal* provided that any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

always implies that f is an automorphism (so that g is too).

Note that if $0 \to M \to E \to C \to 0$ is a generator for $\mathcal{E}xt(\mathscr{C}, M)$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & E'' & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows and $C'' \in \mathscr{C}$, then $0 \to M \to E'' \to C'' \to 0$ is also a generator.

Example 1.1.24. For a given R-module M, any exact sequence $0 \to M \to I \to L \to 0$ with I injective gives a generator for $\mathcal{E}xt(\text{Mod-}R, M)$. Moreover if I is the injective envelope of M, then this generator is minimal.

In the following, we will see that the existence of a generator (resp. minimal generator) is merely related to the existence of a $\mathscr{C}^{\perp}$-preenvelope (resp. $\mathscr{C}^{\perp}$-envelope), for a given class $\mathscr{C}$.

Proposition 1.1.25. *[48, Proposition 2.2.1] Assume that the class $\mathscr{C}$ is closed under extensions, and assume that $0 \to M \to K \to C \to 0$ is a minimal generator for $\mathcal{E}xt(\mathscr{C}, M)$, then $K \in \mathscr{C}^{\perp}$.*

When having a generator, we aim to find the minimal one. The following theorem is one important result of this section:

Theorem 1.1.26. *[48, Theorem 2.2.2] Assume that the class $\mathscr{C}$ is closed under direct limits. If for an R-module M, $\mathcal{E}xt(\mathscr{C}, M)$ has a generator, then there must be a minimal generator.*

The proof can be given through the following three lemmas, which are listed here for the sake of completeness (the main ingredients were first presented by Enochs, see [21]).

Lemma 1.1.27. *[48, Lemma 2.2.3] Assume the class $\mathscr{C}$ is closed under direct limits. For an R-module M, if $0 \to M \to N \to C \to 0$ is a generator for $\mathcal{E}xt(\mathscr{C}, M)$, then there is a generator $0 \to M \to N' \to C' \to 0$ and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

such that for any generator $0 \to M \to N'' \to C'' \to 0$ and for any commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

we have that $\ker(g) = \ker(hg)$.

Lemma 1.1.28. *[48, Lemma 2.2.4] Assume the class $\mathscr{C}$ is closed under direct limits. If there exists a generator $0 \to M \to N \to C \to 0$ for $\mathcal{E}xt(\mathscr{C}, M)$, then there is a generator $0 \to M \to N' \to C' \to 0$ such that for any generator $0 \to M \to N'' \to C'' \to 0$ and any commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & M & \longrightarrow & N'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

g must be injective.

Lemma 1.1.29. *[48, Lemma 2.2.5] Assume the class $\mathscr{C}$ is closed under direct limits. If $0 \to M \to N \to C \to 0$ is a generator having the property stated in Lemma 1.1.28, then it is a minimal generator.*

Using Proposition 1.1.25 and Theorem 1.1.26, one can notice an explicit link between minimal genrators and envelopes, serving as a bridge between these different subjects:

Theorem 1.1.30. *[48, Theorem 2.2.6] Assume the class $\mathscr{C}$ is closed under extensions and under direct limits. For a given R-module M, if $\mathcal{E}xt(\mathscr{C}, M)$ has a generator, then M admits a $\mathscr{C}^{\perp}$-envelope.*

Equivalently,

Theorem 1.1.31. *[27, Theorem 5.27] Let R be a ring and M be an R-module. Let $\mathscr{C}$ be a class of modules closed under direct limits. Assume moreover that $\mathscr{C}$ is closed under extensions, and that M has a monic $\mathscr{C}^{\perp}$-preenvelope φ with* $\operatorname{Coker}(\varphi) \in \mathscr{C}$. *Then M has a $\mathscr{C}^{\perp}$-envelope.*

The preceding results have dual versions:

Proposition 1.1.32. *[48, Proposition 2.2.7] Assume $\mathscr{C}$ is closed under extensions and let M be an R-module. If $\varphi : C \to M$ is a $\mathscr{C}$-cover of M, then* $\ker(\varphi) \in \mathscr{C}^{\perp}$.

Theorem 1.1.33. *[48, Theorem 2.2.8] Assume $\mathscr{C}$ is closed under direct limits and let M be an R-module. If M has a $\mathscr{C}$-precover, then M has a $\mathscr{C}$-cover.*

The proof is similar to that of Theorem 1.1.26. The three required steps are analogous to Lemmas 1.1.27–1.1.29.

Lemma 1.1.34. *[48, Lemma 2.2.9] Assume $\mathscr{C}$ is closed under direct limits. If $C \to M$ is a $\mathscr{C}$-precover of M, then there is a precover $C' \to M$ and a commutative diagram*

$$\begin{array}{ccc} C & \longrightarrow & M \\ \big\downarrow{\scriptstyle g} & & \Big\| \\ C' & \longrightarrow & M \end{array}$$

such that for any precover $C'' \to M$ and any commutative diagram

$$\begin{array}{ccc} C' & \longrightarrow & M \\ \big\downarrow{\scriptstyle h} & & \Big\| \\ C'' & \longrightarrow & M \end{array}$$

we have $\ker(hg) = \ker(g)$.

Lemma 1.1.35. *[48, Lemma 2.2.10] Assume $\mathscr{C}$ is closed under direct limits. If M has a $\mathscr{C}$-precover, then there is a precover $C' \to M$ such that for any precover $C'' \to M$ and any commutative diagram*

$$\begin{array}{ccc} C' & \longrightarrow & M \\ \big\downarrow{\scriptstyle h} & & \Big\| \\ C'' & \longrightarrow & M \end{array}$$

h must be an injection.

Lemma 1.1.36. *[48, Lemma 2.2.11] If $\mathscr{C}$ is closed under direct limits and $C \to M$ is a $\mathscr{C}$-precover of M satisfying the condition of Lemma 1.1.35, then $C \to M$ is a $\mathscr{C}$-cover of M.*

1.1.4 Cotorsion pairs and approximations

The notion of a cotorsion pair, as presented below, yields a duality between special preenvelopes and precovers.

Definition 1.1.37. (1) Let $\mathcal{A}, \mathcal{B} \subseteq \text{Mod–}R$. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* (or a *cotorsion theory*), if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

(2) Let $\mathscr{L}$ be a class of modules. Then $\mathscr{L} \subseteq {}^{\perp}(\mathscr{L}^{\perp})$ as well as $\mathscr{L} \subseteq ({}^{\perp}\mathscr{L})^{\perp}$. Moreover, $\mathfrak{G}_{\mathscr{L}} = ({}^{\perp}(\mathscr{L}^{\perp}), \mathscr{L}^{\perp})$ and $\mathfrak{C}_{\mathscr{L}} = ({}^{\perp}\mathscr{L}, ({}^{\perp}\mathscr{L})^{\perp})$ are easily seen to be cotorsion pairs, called the cotorsion pairs *generated* and *cogenerated*, respectively, by the class $\mathscr{L}$. When $\mathscr{L}$ consists of a single module L, we will use ${}^{\perp}L$ and $L^{\perp}$ instead of ${}^{\perp}\{L\}$ and $\{L\}^{\perp}$, respectively.

(3) If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then the class $\mathcal{K}_{\mathfrak{C}} = \mathcal{A} \cap \mathcal{B}$ is called the *kernel* of $\mathfrak{C}$. Note that each element K of the kernel is a *splitter* (or an *exceptional module*), that is, K satisfies $\text{Ext}^1_R(K, K) = 0$.

Replacing $\text{Hom}(= \text{Ext}^0)$ with Ext^1, cotorsion pairs can be considered analogous to the classical (non-hereditary) torsion pairs.

Definition 1.1.38. For a class of (right resp. left) R-modules, $\mathscr{L}$, we define

$$\mathscr{L}^{\top} = \operatorname{Ker}\operatorname{Tor}_R^1(L, -) = \{N \in R\text{–Mod} \mid \operatorname{Tor}_R^1(L, N) = 0 \text{ for all } L \in \mathscr{L}\},$$

resp.

$$^{\top}\mathscr{L} = \operatorname{Ker}\operatorname{Tor}_R^1(-, L) = \{N \in \text{Mod–}R \mid \operatorname{Tor}_R^1(N, L) = 0 \text{ for all } L \in \mathscr{L}\},$$

$(\mathcal{A}, \mathcal{B})$ is called a *Tor-pair*, if $\mathcal{A} = {}^{\top}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\top}$. As Tor commutes with direct limits, both $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits. If necessary, for simplification, we will use $A^{\top}$ and ${}^{\top}B$ instead of $\{A\}^{\top}$ and ${}^{\top}\{B\}$, respectively, where $A \in \text{Mod–}R$ and $B \in R\text{–Mod}$.

Let B^c be the character module of B, defined as $\operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$. The following result holds:

Lemma 1.1.39. *[27, Lemma 5.17] Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a Tor-pair. Then $\mathcal{D} = (\mathcal{A}, \mathcal{A}^{\perp})$ is a cotorsion pair. Moreover, $\mathcal{D} = \mathfrak{C}_{\mathscr{L}}$, where $\mathscr{L} = \{B^c \mid B \in \mathcal{B}\} \subseteq \mathcal{PI}$.*

Before presenting examples of cotorsion pairs, we start by defining some key notions:

Definition 1.1.40. Let R be a ring, Q be its maximal left quotient ring and M be a left R-module.

(1) An element $r \in R$ is called a *non-zero-divisor* if, for every $r' \in R$, the conditions $r \cdot r' = 0$ or $r' \cdot r = 0$ imply that $r' = 0$.
We say that M is *torsion-free* if $\operatorname{Tor}_R^1(R/rR, M) = 0$ for each non-zero-divisor $r \in R$.

(2) M is called *Matlis cotorsion* if $\operatorname{Ext}_R^1(Q, M) = 0$.

(3) M is called *Enochs cotorsion* if $\operatorname{Ext}_R^1(F, M) = 0$ for all flat modules F.

(4) M is called *Warfield cotorsion* if $\operatorname{Ext}_R^1(T, M) = 0$ for all torsion-free modules T.

We will denote by $\mathcal{TF}$, $\mathcal{MC}$, $\mathcal{EC}$, and $\mathcal{RC}$ the classes of all torsion-free, Matlis cotorsion, Enochs cotorsion, and Warfield cotorsion modules, respectively.

Example 1.1.41. (i) Consider the case presented in Lemma 1.1.39, where $\mathcal{A} = \mathcal{F}$ and $\mathcal{B} = R\text{–Mod}$. Then $(\mathcal{F}, \mathcal{EC})$ forms a cotorsion pair, known as the *Enochs cotorsion pair*. In this case, $\mathcal{EC} = \mathcal{F}^{\perp}$ represents the class of all Enochs cotorsion modules, as defined in Definition 1.1.40(3).
Lemma 1.1.39 implies that the Enochs cotorsion pair is cogenerated by the class of all dual modules, and hence by the class $\mathcal{PI}$ as well. In particular, we have $\mathcal{PI} \subseteq \mathcal{EC}$ too.

(ii) An additional interesting case arises when $\mathcal{A} = \mathcal{TF}$, where $\mathcal{TF} = {}^{\top}\mathscr{L}$ and $\mathscr{L}$ represents the set of all cyclically presented left R-modules. We recall that a left R-module M is called *cyclically presented* if it can be expressed as R/Rr, where either $r = 0$ or $r \in R$ is a non-zero-divisor.
The elements of $\mathcal{TF}$ are the torsion-free modules in the sense of Definition

1.1.40(1). By Lemma 1.1.39, the pair $(\mathcal{TF}, \mathcal{RC})$ forms a cotorsion pair, known as the *Warfield cotorsion pair*. In this case, $\mathcal{RC} = \mathcal{TF}^{\perp}$ consists of all Warfield cotorsion modules, as defined in Definition 1.1.40(4).

Obviously, $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{TF}$, so $\mathcal{I} \subseteq \mathcal{RC} \subseteq \mathcal{EC}$ for any ring R.

We now shift our focus to approximations arising from cotorsion pairs.

Lemma 1.1.42. *[27, Corollary 5.19] Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. If $\mathcal{A}$ is a covering class, then $\mathcal{A}$ is special precovering. Similarly, if $\mathcal{B}$ is an enveloping class, then $\mathcal{B}$ is special preenveloping.*

Thanks to the notion of a cotorsion pair, Salce [40] discovered that special precovers and special preenvelopes go hand in hand, offering a remedy for the problem of not having duality involving all modules over a ring:

Lemma 1.1.43. *[27, Lemma 5.20] Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules. The following statements are equivalent:*

(1) Every module has a special $\mathcal{A}$-precover.

(2) Every module has a special $\mathcal{B}$-preenvelope.

Cotorsion pairs satisfying the equivalent conditions of the previous lemma, also known as Salce Lemma, are called *complete*.

Definition 1.1.44. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair.

(i) $\mathfrak{C}$ is called *perfect*, provided that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class.

(ii) $\mathfrak{C}$ is called *closed*, provided that $\mathcal{A} = \varinjlim \mathcal{A}$, that is, the class $\mathcal{A}$ is closed under forming direct limits in Mod–R.

Using Theorems 1.1.31 and 1.1.33, we obtain:

Corollary 1.1.45. *[27, Corollary 5.32] Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete and closed cotorsion pair. Then $\mathfrak{C}$ is perfect.*

The following theorem, presented in [18], proves the abundance of complete cotorsion pairs. Similar reasoning has been applied in homotopy theory since Quillen's fundamental work [37] known as the small object argument.

Theorem 1.1.46. *[27, Theorem 6.11] Let $\mathcal{S}$ be a set of modules.*

(1) Let M be a module. Then there is a short exact sequence

$$0 \to M \to A \to B \to 0,$$

where $A \in \mathcal{S}^{\perp}$ and B is $\mathcal{S}$-filtered. In particular, $M \to A$ is a special $\mathcal{S}^{\perp}$-preenvelope of M.

(2) The cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is complete.

Definition 1.1.47. Let R be a domain, Q be the quotient field of R and let M be an R-module. We say that M is *strongly flat*, provided that $\mathrm{Ext}^1_R(M, N) = 0$ for each Matlis cotorsion module N.

Denote by $\mathcal{SF}$ the class of all strongly flat modules. Then, we have $\mathcal{SF} = {}^{\perp}\mathcal{MC}$. Clearly, any projective module and any divisible torsion-free module is strongly flat. Since Q is a flat module (namely the localisation of R at 0), we have

$$\mathcal{I} \subseteq \mathcal{RC} \subseteq \mathcal{EC} \subseteq \mathcal{MC}$$

and hence

$$\mathcal{P} \subseteq \mathcal{SF} \subseteq \mathcal{F} \subseteq \mathcal{TF}$$

for any domain R.

By Theorem 1.1.46, $(\mathcal{SF}, \mathcal{MC})$ is a complete cotorsion pair (generated by Q). This cotorsion pair is called the *Matlis cotorsion pair*. Note that Mod–Q (= the class of all divisible torsion-free R-modules) is a subclass of Mod–R closed under extensions and direct limits and $(\text{Mod–}Q)^{\perp} = \mathcal{MC}$. Therefore, Theorems 1.1.46 and 1.1.31 directly imply:

Lemma 1.1.48. *[27, Corollary 7.42] Assume R is a domain. Then each module admits an $\mathcal{MC}$-envelope and a special $\mathcal{SF}$-precover.*

1.1.5 Enochs' Conjecture and some well-known examples

The converse of Theorem 1.1.31 fails in general. A counterexample illustrating this point is provided.

Example 1.1.49. Let R be a domain that is not almost perfect, meaning $\mathcal{SF} \neq \mathcal{F}$ (cf. [27, Theorem 7.56]). Consider $\mathscr{C} = \mathcal{SF}$. As shown earlier, $\mathscr{C}^{\perp} = \mathcal{MC}$. According to Lemma 1.1.48, $\mathscr{C}^{\perp}$ is enveloping. However, $\mathcal{SF}$ is not closed under direct limits (the closure of $\mathcal{SF}$ under direct limits is the class of all flat modules $\mathcal{F}$, since every projective module is strongly flat and every flat module is a direct limit of projective modules). Specifically, if R is a Prüfer domain, then R not being almost perfect implies that R is not Dedekind (cf. [10, Corollary 2.3 and 2.4]).

Problem 1.1.50. *Does the converse of Proposition 1.1.45 hold, meaning, is every perfect cotorsion pair closed?*

Remark 4. Notice that, while $\mathcal{A}$ is special precovering, if and only if $\mathcal{B}$ is special preenveloping for any cotorsion pair $(\mathcal{A}, \mathcal{B})$ by Lemma 1.1.43, there exist cotorsion pairs $(\mathcal{A}, \mathcal{B})$, such that $\mathcal{B}$ is enveloping, but $\mathcal{A}$ is not covering (see Example 1.1.49).

The question whether the converse implication of Theorem 1.1.33 holds as well is known as the *Enochs' conjecture*:

Conjecture 1.1.51 (Enochs)**.** Every covering class of modules is closed under direct limits.

This conjecture has been verified for various special types of classes. However, the general case of the conjecture remains an open question. In this section, we present notable examples to provide more context. For any unexplained terminology, readers are encouraged to refer to the respective theorem references accompanying each result.

We recall that a $\mathcal{TF}$-cover (precover) of M is called a *torsion-free cover* (*precover*) of M. Note that if $\varphi : T \to M$ is a torsion-free precover, then φ is always surjective.

We start with the following result due to Enochs in 1963, showing that the class $\mathcal{TF}$ is covering over any integral domain.

Theorem 1.1.52. *[20, Theorem 1] Every module over an integral domain has a torsion-free cover.*

The *flat cover conjecture* below was proved in 2001 by Bican, El Bashir and Enochs [13].

Theorem 1.1.53. *[13, Theorem 3] Over any associative ring R, every R-module has a flat cover.*

In 2017, a positive answer was given by L. Angeleri Hügel, J. Šaroch and J. Trlifaj: if $\mathcal{A}$ fits in a cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}$ closed under direct limits. In particular, when $(\mathcal{A}, \mathcal{B})$ is any tilting cotorsion pair.

Recall that the *tilting cotorsion pair* induced by a tilting module T is the cotorsion pair $({}^{\perp}\mathcal{B}, \mathcal{B})$, where

$$\mathcal{B} = T^{\perp_\infty} = \{B \in \text{Mod–}R \mid \text{Ext}^i_R(T, B) = 0 \text{ for all } i \geq 1\}.$$

Theorem 1.1.54. *[6, Corollary 5.5] Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair with $\mathcal{B} = \varinjlim \mathcal{B}$. Let K be a module such that* $\text{Ker}\,(\mathfrak{C}) = \text{Add}\,(K)$, *where* $\text{Add}\,(K)$ *denotes all direct summands of arbitrary direct sums of copies of K. Then the following conditions are equivalent:*

(i) $\mathcal{A} = \varinjlim \mathcal{A}$;

(ii) every module (in $\mathcal{B}$) has an $\mathcal{A}$-cover;

(iii) every module in $\text{Ker}(\mathfrak{C})$ *has a semiregular endomorphism ring;*

(iv) K is Σ-pure-split;

(v) every module (in $\mathcal{B}$) has a $\text{Ker}\,(\mathfrak{C})$*-cover.*

It's important to mention another significant advancement towards proving the Enochs' conjecture namely a paper by Bazzoni-Positselski-Šťovíček in 2020 [9], which gives the following application:

Theorem 1.1.55. *[9, Application 8.3] Let $\mathcal{A}$ be an Ab 5-category and $(\mathscr{L}, \mathscr{E})$ be a cotorsion pair in $\mathcal{A}$. Assume that the class of objects $\mathscr{E} \subset \mathcal{A}$ is closed under direct limits, and let $M \in \mathscr{L} \cap \mathscr{E}$ be an object of the kernel. Let $D \in \varinjlim$* $\text{Add}\,(M)$ *be an object having an* $\text{Add}\,(M)$*-cover in $\mathcal{A}$. Then $D \in$* $\text{Add}\,(M)$.

We conclude this section with a more recent result due to S. Bazzoni and J. Šaroch in 2023, assuming the following additional set-theoretic hypothesis consistent with ZFC (for more details, see [11]):
Assumption (*). For each infinite regular cardinal θ, there is a proper class of cardinals κ such that:

(i) There exists a non-reflecting stationary set $E \subseteq \kappa^+$ consisting of ordinals with cofinality θ, and

(ii) $\kappa^{<\theta} = \kappa$.

Theorem 1.1.56. *[11, Corollary 3.5] Assume (*). Let $\mathcal{S} \subseteq \text{Mod–}R$ be a set. Put $\mathcal{A} = \text{Filt}(\mathcal{S})$. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ is a covering class of modules;

(ii) $\mathcal{A}$ is closed under direct limits;

(iii) $\mathcal{A}$ is closed under direct summands and under taking direct limits of ξ-continuous well-ordered directed systems of modules (and monomorphisms between them) for an infinite regular cardinal ξ.

1.2 Mittag-Leffler modules

Mittag-Leffler modules can be seen in several equivalent ways that may initially appear unrelated. In this section, we will see different facets of the same notion.

1.2.1 Mittag-Leffler conditions and modules

Grothendieck introduced the Mittag-Leffler condition for countable inverse systems as a sufficient condition for the exactness of the inverse limit functor [29]. We define the Mittag-Leffler conditions as stabilization criteria for decreasing chains of images of inverse system maps:

Definition 1.2.1. Let $\mathcal{D} = (D_i, g_{ij} \,|\, i \leq j \in I)$ be an inverse system of modules and let $D = \varprojlim \mathcal{D} = (D_i, g_i \,|\, (i \in I))$ be its inverse limit.

(i) $\mathcal{D}$ is *Mittag-Leffler*, provided that for each $i \in I$ there exists $i \leq j \in I$, such that $\operatorname{Im} g_{ij} = \operatorname{Im} g_{ik}$ for each $j \leq k \in I$. That is, the terms of the decreasing chain $(\operatorname{Im} g_{ik} \mid i \leq k \in I)$ of submodules of D_i stabilize.

(ii) $\mathcal{D}$ is *strict Mittag-Leffler*, provided that for each $i \in I$ there exists $i \leq j \in I$, such that $\operatorname{Im} g_{ij} = \operatorname{Im} g_i$.

We recall a well-known characterization by Lenzing that concerns finitely generated and finitely presented modules M. This characterization is based on the properties of the tensor product functor $M \otimes_R -$.
For each module M and each sequence $\mathcal{Q} = (Q_i \,|\, i \in I)$ of left R-modules, we let $\varphi_{M,\mathcal{Q}}$ denote the abelian group homomorphism

$$\varphi_{M,\mathcal{Q}} : M \otimes_R \textstyle\prod_{i\in I} Q_i \to \prod_{i\in I} M \otimes_R Q_i$$

defined by $\varphi_{M,\mathcal{Q}}(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$.

Lemma 1.2.2. *[27, Lemma 3.8] Let R be a ring and M be a module.*

(i) M is finitely generated, if and only if $\varphi_{M,\mathcal{Q}}$ is an epimorphism for all sequences $\mathcal{Q}$ of left R-modules, if and only if $\varphi_{M,\mathcal{Q}}$ is an epimorphism for the sequence $\mathcal{Q} = (Q_i \,|\, i \in I)$, such that $I = M$ and $Q_i = R$ for all $i \in I$.

(ii) M is finitely presented, if and only if $\varphi_{M,\mathcal{Q}}$ is an isomorphism for all sequences $\mathcal{Q}$ of left R-modules.

Now, let's consider the case where $\varphi_{M,\mathcal{Q}}$ in Lemma 1.2.2 is a monomorphism. In fact, it introduces the central notion of this section:

Definition 1.2.3. Let R be a ring and M be a module. Then M is *Mittag-Leffler*, provided that the homomorphism $\varphi_{M,\mathcal{Q}}$ is a monomorphism for all sequences $\mathcal{Q}$ of R-modules.

Remark 5. (i) A module M is *pure-projective*, provided that M is projective with respect to pure epimorphisms, that is, $\mathrm{Hom}_R(g, M) : \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is surjective for each pure epimorphism $g : B \to C$. Since each pure-projective module is a direct summand of a direct sum of finitely presented modules (see [27, Example 2.10]), Lemma 1.2.2(ii) implies that each pure-projective module is Mittag-Leffler.

(ii) From Definition 1.2.3, we also immediately see that the class of all Mittag-Leffler modules is closed under pure submodules and pure extensions. That is, if $0 \to M' \to M \to M'' \to 0$ is a pure-exact sequence and M is Mittag-Leffler (both M' and M'' are Mittag-Leffler), then M' is Mittag-Leffler (M is Mittag-Leffler, respectively).

To investigate the structure of Mittag-Leffler modules, an additional lemma is needed. This lemma establishes a local condition for a module M to be Mittag-Leffler when M is already expressed as a direct limit of Mittag-Leffler modules:

Lemma 1.2.4. *[27, Lemma 3.11] Let R be a ring and M a module, such that $M = \varinjlim_{\alpha\in\Lambda} M_\alpha$ where $(M_\alpha, f_{\beta\alpha} \,|\, \alpha < \beta \in \Lambda)$ is a direct system of Mittag-Leffler modules. Assume that $M' = \varinjlim_{n<\omega} M_{\alpha_n}$ is Mittag-Leffler for each countable chain $\alpha_0 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ in Λ. Then M is Mittag-Leffler.*

Theorem 1.2.6 below unifies various approaches to the notion of a Mittag-Leffler module. With the exception of parts (2) and (3), these perspectives were already discovered in the seminal work of Raynaud and Gruson [38]. The $\aleph_1$-pure-projectivity approach employed in parts (2) and (3) is provided by [31]. It appears as a special case of $\kappa = \aleph_1$ in the following definition:

Definition 1.2.5. Let R be a ring, κ a regular uncountable cardinal, and M an R-module. We say that M is *κ-pure-projective* if there exists a set $\mathcal{S}$ consisting of $< \kappa$-generated pure-projective submodules of M with the following properties: $0 \in \mathcal{S}$, every subset of M with cardinality $< \kappa$ is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of well-ordered chains of length $< \kappa$.

Mittag-Leffler modules are closely related to Mittag-Leffler inverse systems of modules:

Theorem 1.2.6. *[27, Theorem 3.14] Let R be a ring and M be a module. Then the following conditions are equivalent:*

(1) Each finite (or countable) subset of M is contained in a countably generated pure-projective and pure submodule of M.

(2) M is $\aleph_1$-pure-projective.

(3) M is the directed union of a set $\mathcal{S}$ consisting of Mittag-Leffler submodules of M, such that $0 \in \mathcal{S}$, each countable subset of M is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of countable chains.

(4) M is Mittag-Leffler.

(5) For each finitely presented module P and each $f \in \mathrm{Hom}_R(P, M)$, there exist a finitely presented module P' and $f' \in \mathrm{Hom}_R(P, P')$, such that $\mathrm{Ker}(f \otimes_R N) = \mathrm{Ker}(f' \otimes_R N)$ for each left R-module N.

(6) If $M = \varinjlim_{i \in I} M_i$, for a direct system $(M_i, f_{ji} \mid i \leq j \in I)$ of finitely presented modules, then for each module N, the induced inverse system

$$(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ji}, N) \mid i \leq j \in I)$$

satisfies the Mittag-Leffler condition.

Remark 6. (1) Mittag-Leffler modules were first introduced by Grothendieck, Raynaud, and Gruson [38], using condition (5) of Theorem 1.2.6 as definition.

(2) A variant of this notion, introduced by Rothmaler in [39], is the relative version, which allows sequences $\mathcal{Q}$ to include only elements from a specified class of left R-modules. Angeleri and Herbera [5] extensively studied this relative version. Additionally, [39] unveiled another facet of Mittag-Leffler modules from a model-theoretic perspective: they are positively atomic modules, meaning that the *pp*-type of each tuple from these modules is finitely generated.

We end this section by giving several results that follow from Theorem 1.2.6:

Corollary 1.2.7. *[27, Corollary 3.16] Let R be a ring and denote by $\mathcal{ML}$ the class of all Mittag-Leffler modules. Then the following statements hold:*

(a) $\mathcal{ML}$ contains all pure-projective modules, and it is closed under pure submodules, pure extensions, and unions of pure chains.

(b) If $M \in \mathrm{Mod}\text{–}R$ is countably generated, then $M \in \mathcal{ML}$, if and only if M is pure-projective.

(c) Let κ be an infinite cardinal. Then each $\leq \kappa$-generated Mittag-Leffler module is $\leq \kappa$-presented.

(d) Let κ be an infinite cardinal, $M \in \mathcal{ML}$, and N be a $\leq \kappa$-generated submodule of M. Then N is contained in a pure and $\leq \kappa$-presented submodule P of M.

(e) Let κ be an uncountable cardinal, and M be a κ-pure-projective module. Then M is $\aleph_1$-pure-projective.

1.2.2 Flat Mittag-Leffler modules

In this paragraph, we restrict the implications of Theorem 1.2.6 to the setting of flat modules. We shall observe that the correspondence between Mittag-Leffler modules and $\aleph_1$-pure-projective modules can be restricted to a correspondence between the flat Mittag-Leffler modules and the $\aleph_1$-projective ones.

It is worth noting that $\aleph_1$-projective (or almost free) modules represent an important topic of research in general module theory, cf. [16, 25]. Let's start by recalling the more general setting:

Definition 1.2.8. Let R be a ring, κ a regular uncountable cardinal, and M an R-module. We say that M is *κ-projective* if there exists a set $\mathcal{S}$ consisting of $< \kappa$-generated projective submodules of M with the following properties: $0 \in \mathcal{S}$, any subset of M with cardinality $< \kappa$ is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of well-ordered chains of length $< \kappa$.

The connection for $\kappa = \aleph_1$ is established by the following straightforward lemma.

Lemma 1.2.9. *[27, Lemma 3.18] Let R be a ring, and M be an R-module. Then M is $\aleph_1$-projective, if and only if M is flat and $\aleph_1$-pure projective.*

Theorem 1.2.6 readily specializes to the case of flat modules:

Corollary 1.2.10. *[27, Corollary 3.19] Let R be a ring and M be a module. Then the following conditions are equivalent:*

(i) Each finite (or countable) subset of M is contained in a countably generated projective and pure submodule of M.

(ii) M is $\aleph_1$-projective.

(iii) M is a directed union of a set $\mathcal{S}$ consisting of flat Mittag-Leffler submodules of M, such that $0 \in \mathcal{S}$, each countable subset of M is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of countable chains.

(iv) M is flat and Mittag-Leffler.

(v) M is flat, and if $M = \varinjlim_{i\in I} M_i$ of a direct system $(M_i, f_{ji} \mid i \leq j \in I)$ consisting of finitely generated free modules, then the induced inverse system

$$(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ji}, R) \mid i \leq j \in I)$$

satisfies the Mittag-Leffler condition.

Similarly, we obtain

Corollary 1.2.11. *[27, Corollary 3.20] Let R be a ring.*

(i) The class $\mathcal{FM}$ of all flat Mittag-Leffler modules is closed under pure submodules, extensions, unions of pure chains and transfinite extensions.

(ii) If M is a countably generated R-module, then $M \in \mathcal{FM}$, if and only if M is projective.

(iii) *Let κ be an infinite cardinal, and let M be a $\leq \kappa$-generated flat Mittag-Leffler module. Then, M admits a projective resolution consisting of $\leq \kappa$-generated modules.*

(iv) *Each subset of cardinality $\leq \kappa$ in a flat Mittag-Leffler module M is contained in a pure and $\leq \kappa$-presented submodule of M.*

(v) *Let κ be an uncountable cardinal, and let M be a κ-projective module. Then M is $\aleph_1$-projective.*

1.3 Quasi-coherent sheaves and Zariski locality

We assume the reader has a certain level of familiarity with fundamental concepts related to schemes and quasi-coherent sheaves of modules over a scheme. For additional details, we refer to [28, 30].

All rings in this section are commutative. We will denote by CRing the category of commutative rings.

1.3.1 Quasi-coherent sheaves as quasi-coherent representations

The purpose of this section is to recall how the category of quasi-coherent sheaves over a given scheme can be characterized in terms of quasi-coherent modules over a flat ring representation of a poset.

We adopt the approach presented in [42, §2], which is an adjusted version of [22, §2].

Definition 1.3.1. For a partially ordered set $(I, \leq)$, a *representation* $\mathcal{R}$ of I in the category CRing is defined as follows:

(1) for each $i \in I$, we have a ring $\mathcal{R}(i)$,

(2) for each $i \leq j$, we have a ring homomorphism $f^i_j : \mathcal{R}(i) \to \mathcal{R}(j)$, and

(3) we require that for each triple $i \leq j \leq k$, $f^i_k = f^j_k \circ f^i_j$, and also that $f^i_i = 1_{\mathcal{R}(i)}$.

Remark 7. Viewing I as a thin category, $\mathcal{R}$ can simply be regarded as a covariant functor $\mathcal{R} : I \to$ CRing.

After defining representations of I in the category CRing, one can easily define modules over these representations as follows.

Definition 1.3.2. Let $\mathcal{R}$ be a representation of a poset I in CRing. A *right $\mathcal{R}$-module* is

(1) a collection $(M(i))_{i\in I}$, where $M(i) \in \text{Mod–}\mathcal{R}(i)$ for each $i \in I$,

(2) together with morphisms of the additive groups $g^i_j : M(i) \to M(j)$ for each $i \leq j$,

(3) satisfying the compatibility conditions $g_k^i = g_k^j \circ g_j^i$ and $g_i^i = 1_{M(i)}$ for every triple $i \leq j \leq k$, and such that,

(4) the ring actions are respected in the following manner: given $x \in \mathcal{R}(i)$ and $m \in M(i)$ for $i \in I$, then for any $j \geq i$ we have the equality

$$g_j^i(m \cdot x) = g_j^i(m) \cdot f_j^i(x).$$

Throughout the rest of this section, all our modules will be considered as right modules. To form a category, the only remaining step is to define morphisms of $\mathcal{R}$-modules.

Definition 1.3.3. Let $\mathcal{R}$ be a representation of a poset I in CRing, and let M and N be $\mathcal{R}$-modules. A morphism $f : M \to N$ is a collection $(f(i) : M(i) \to N(i))_{i \in I}$, where $f(i)$ is a morphism of $\mathcal{R}(i)$-modules for every $i \in I$, and the square

$$\begin{array}{ccc} M(i) & \xrightarrow{f(i)} & N(i) \\ \downarrow f_j^i & & \downarrow g_j^i \\ M(j) & \xrightarrow{f(j)} & N(j) \end{array}$$

commutes for every $i < j$.

Let Mod–$\mathcal{R}$ denote the category of all $\mathcal{R}$-modules. Then, we have:

Proposition 1.3.4. *[42, Proposition 2.8] Let $(I, \leq)$ be a poset and let $\mathcal{R}$ be a representation of I in* CRing*. Then* Mod–$\mathcal{R}$ *is a Grothendieck category. Furthermore, limits and colimits of diagrams of modules are computed component wise—we compute the corresponding (co)limit in* Mod–$\mathcal{R}(i)$ *for each $i \in I$ and connect these by the (co)limit morphisms.*

The categories Mod–$\mathcal{R}$ discussed previously are examples of Grothendieck categories, but they are not our central focus in this section. To fulfill the promise of describing categories of quasi-coherent sheaves, we need to consider specific full subcategories. To achieve this, we require an additional condition on $\mathcal{R}$:

Definition 1.3.5. Let $\mathcal{R}$ be a representation of a poset I in CRing. We say that $\mathcal{R}$ is a *flat* representation if for each pair $i < j$ in I, the ring homomorphism $f_j^i : \mathcal{R}(i) \to \mathcal{R}(j)$ gives $\mathcal{R}(j)$ the structure of a flat left $\mathcal{R}(i)$-module. That is,

$$- \otimes_{\mathcal{R}(i)} \mathcal{R}(j) : \text{Mod–}\mathcal{R}(i) \to \text{Mod–}\mathcal{R}(j)$$

is an exact functor.

As we will observe later, the representations arising from the structure sheaves of schemes satisfy this condition. In the case of such an $\mathcal{R}$, we can explicitly define the modules of interest:

Definition 1.3.6. Let $\mathcal{R}$ be a flat representation of a poset I in CRing. A module $M \in$ Mod–$\mathcal{R}$ is called *quasi-coherent* if, for every $i < j$, the $\mathcal{R}(j)$-module homomorphism

$$M(i) \otimes_{\mathcal{R}(i)} \mathcal{R}(j) \to M(j)$$

$$m \otimes x \mapsto g^i_j(m) \cdot x$$

is an isomorphism.

Denote the full subcategory of Mod–$\mathcal{R}$ consisting of quasi-coherent $\mathcal{R}$-modules by Qcoh($\mathcal{R}$). Again, we obtain a Grothendieck category:

Proposition 1.3.7. *[42, Theorem 2.11] Let $(I, \leq)$ be a poset and $\mathcal{R}$ be a flat representation of I in* CRing *. Then* Qcoh($\mathcal{R}$) *is a Grothendieck category. Moreover, colimits of diagrams and limits of finite diagrams are computed component wise–that is, for each $i \in I$ separately.*

We present a specific example of a flat representation for a poset with geometric origins and the quasi-coherent modules associated with it.

Example 1.3.8. [42, Example 2.13] Consider the three element poset be given by the Hasse diagram

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

and a representation in CRing of the form

$$\mathcal{R}: \qquad k[x] \xhookrightarrow{\subseteq} k[x, x^{-1}] \xhookleftarrow{\supseteq} k[x^{-1}],$$

where k is an arbitrary commutative ring. Clearly $\mathcal{R}$ is a flat representation since the inclusions are localization morphisms.

For each $n \in \mathbb{Z}$, we have a quasi-coherent $\mathcal{R}$-module

$$\mathcal{O}(n): \qquad k[x] \xhookrightarrow{\subseteq} k[x, x^{-1}] \xhookleftarrow{x^n \cdot -} k[x^{-1}].$$

By direct computation, we have $\mathrm{Hom}_{\mathcal{R}}(\mathcal{O}(m), \mathcal{O}(n)) = 0$ for $m > n$. Consequently, it follows that $\mathcal{O}(m) \ncong \mathcal{O}(n)$ whenever $m \neq n$.

In fact, the category Qcoh($\mathcal{R}$) is equivalent to the category of quasi-coherent sheaves over $\mathbb{P}^1_k$, the projective line over k.

Let $(X, \mathcal{O}_X)$ be a scheme, that is, a ringed space which is locally isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for a commutative ring R (cf. [28, Definition 3.1]). With this data in hand, our first step is to construct a representation of a poset in the category of commutative rings.

Construction 1.3.9. [42, Construction 2.15] Let $\mathfrak{U}$ be a collection of open affine sets of X satisfying the following two conditions:

(1) $\mathfrak{U}$ covers X; that is $X = \bigcup \mathfrak{U}$.

(2) Given $U, V \in \mathfrak{U}$, then $U \cap V = \bigcup \{W \in \mathfrak{U} \mid W \subseteq U \cap V\}$.

Remark 8. It's usually safe to choose the collection of all affine open sets, although frequently, much smaller sets $\mathfrak{U}$ suffice. For projective schemes for instance, we can always choose $\mathfrak{U}$ to be finite.

Now $\mathfrak{U}$ forms a poset with respect to inclusion, and we define $I = \mathfrak{U}^{op}$ as the opposite poset. Since $\mathcal{O}_X$ is a sheaf of commutative rings, we consequently obtain a functor

$$\mathfrak{U}^{op} \to \text{CRing}$$

which sends a pair $U \supseteq V$ of sets in $\mathfrak{U}$ to the restriction $\text{res}^U_V : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. By defining I in this way, it is equivalent to say that there exists a covariant functor $\mathcal{R} : I \to \text{CRing}$ such that, following the notation in Definition 1.3.1, we have $\mathcal{R}(U) = \mathcal{O}_X(U)$ and $f^U_V = \text{res}^U_V$.

A well-known fact is that the representation of I obtained in this manner is flat:

Lemma 1.3.10. *[42, Lemma 2.16] Let $\mathcal{R}$ be the representation of I in* CRing *as in Construction 1.3.9. Then $\mathcal{R}$ is flat.*

It is now straightforward to construct a functor from the category $\text{Qcoh}(X)$ of quasi-coherent sheaves on X to the category $\text{Qcoh}(\mathcal{R})$ of quasi-coherent modules over $\mathcal{R}$.

Construction 1.3.11. [42, Construction 2.17] Consider the notation introduced in Construction 1.3.9. For a quasi-coherent sheaf $\mathcal{M}$ on X and two affine open sets $U \supseteq V$, there exists a canonical isomorphism

$$\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \cong \mathcal{M}(V).$$

This isomorphism is obtained by applying [28, Remark 7.23 and Proposition 7.24(2)] to the open immersion

$$(\text{Spec}(\mathcal{R}(V)), \mathcal{O}_{\text{Spec}(\mathcal{R}(V))}) \cong (V, \mathcal{O}_{X|V}) \to (U, \mathcal{O}_{X|U}) \cong (\text{Spec}(\mathcal{R}(U)), \mathcal{O}_{\text{Spec}(\mathcal{R}(U))})$$

and the corresponding ring homomorphism $f^U_V : \mathcal{R}(U) \to \mathcal{R}(V)$.

Therefore, we can restrict the functor to $I = \mathfrak{U}^{op}$ by viewing the sheaf $\mathcal{M}$ as a contravariant functor from the poset of open affine sets of X to the category of abelian groups Ab. In doing so, we assign an $\mathcal{R}$-module $F(\mathcal{M})$ to $\mathcal{M} \in \text{Qcoh}(X)$, and the discussion above implies that $F(\mathcal{M})$ is quasi-coherent. Since this assignment is clearly functorial, we obtain an additive functor:

$$F : \text{Qcoh}(X) \to \text{Qcoh}(\mathcal{R}).$$

Compared to $F(\mathcal{M})$, there appears to be significantly more structure in $\mathcal{M} \in \text{Qcoh}(X)$. The latter is just a set of modules that meet a specific coherence condition, whereas the former is a sheaf of modules over a potentially complex topological space X. However, the fact that $\mathcal{M}$ is quasi-coherent is very restrictive in itself, yielding the following crucial result; see [22, §2].

Theorem 1.3.12. *[42, Theorem 2.18] The functor F from Construction 1.3.11 (which depends on the choice of $\mathfrak{U}$ in Construction 1.3.9) is an equivalence of categories.*

1.3.2 Zariski locality

Consider $\mathfrak{P}$ to be a property of modules. For any commutative ring R, let $\mathfrak{P}_R$ denote the class of all R–modules that satisfy $\mathfrak{P}$. In the following discussion, we assume that $\mathfrak{P}$ is *compatible with ring direct products* which means: if $n < \omega$, $R = \prod_{i<n} R_i$ is a ring direct product, and $M_i \in \mathfrak{P}_{R_i}$ for each $i < n$, then

$M = \prod_{i<n} M_i \in \mathfrak{P}_R$.

The global notion for quasi-coherent sheaves corresponding to a property of R-modules $\mathfrak{P}_R$ is said to be *Zariski local* in case the following holds true: If X is a scheme with the structure sheaf $\mathcal{O}_X$, $X = \bigcup_{i\in I} \operatorname{Spec}(R_i)$ is an open affine covering of X, and $\mathcal{M}$ is a quasi-coherent sheaf on X such that the R_i-module of sections $\mathcal{M}(\operatorname{Spec}(R_i))$ satisfies $\mathfrak{P}_{R_i}$ for each $i \in I$, then the $\mathcal{O}_X(U)$-module of sections $\mathcal{M}(U)$ satisfies $\mathfrak{P}_{\mathcal{O}_X(U)}$ for all open affine subsets U of X (see [23, §3] and [46, §5.3]).

In order to prove Zariski locality, it suffices to verify the assumptions of the following "Affine Communication Lemma" for the given setting (cf. [23, 3.5]).

Lemma 1.3.13. *[32, Lemma 2.1] Let R be a commutative ring, $M \in$ Mod–R, and $\mathfrak{P}_R$ be a property of R-modules such that*

(i) if M satisfies property $\mathfrak{P}_R$, then $M[f^{-1}] = M \otimes_R R[f^{-1}]$ satisfies property $\mathfrak{P}_{R[f^{-1}]}$ for each $f \in R$.

(ii) if $R = \sum_{j<m} f_j R$, and the $R[f_j^{-1}]$-modules $M[f_j^{-1}] = M \otimes_R R[f_j^{-1}]$ satisfy property $\mathfrak{P}_{R[f_j^{-1}]}$ for all $j < m$, then M satisfies property $\mathfrak{P}_R$.

Then the global notion for quasi-coherent sheaves corresponding to $\mathfrak{P}_R$ is Zariski local.

Note that for each $f \in R$, the localization in f, denoted by $\varphi_f : R \to R[f^{-1}]$, is a flat ring homomorphism. That is, φ_f makes $R[f^{-1}]$ into a flat R-module. Furthermore, the ring homomorphism $\varphi_{f_0,\cdots,f_{m-1}} : R \to \prod_{i<m} R[f_i^{-1}]$ is faithfully flat when $R = \sum_{j<m} f_j R$, which means that $\varphi_{f_0,\cdots,f_{m-1}}$ makes $\prod_{i<m} R[f_i^{-1}]$ into a faithfully flat R-module. Consequently, the assumptions of the Affine Communication Lemma hold in case $\mathfrak{P}$ ascends along flat ring homomorphisms and descends along faithfully flat ring homomorphisms, as per the following definition:

Definition 1.3.14. Let $\varphi : R \to S$ be a flat ring homomorphism, and $\mathfrak{P}$ be a property of modules.

(i) $\mathfrak{P}$ is said to *ascend* along φ if for each R-module M with the property $\mathfrak{P}_R$, the S-module $M \otimes_R S$ has the property $\mathfrak{P}_S$.

(ii) Assume φ is a faithfully flat ring homomorphism. Then $\mathfrak{P}$ is said to *descend* along φ if for each R-module M, M has the property $\mathfrak{P}_R$ whenever the S-module $M \otimes_R S$ has the property $\mathfrak{P}_S$.

If $\mathfrak{P}$ ascends along all flat ring homomorphisms, and descends along all faithfully flat ring homomorphisms, then $\mathfrak{P}$ is called an *ad-property*.

In view of Lemma 1.3.13, in order to prove Zariski locality of a property $\mathfrak{P}$ of quasi-coherent sheaves on a scheme X, it suffices to verify that $\mathfrak{P}$ is an ad-property. This is the way we will proceed in Chapter 3 for the properties arising from Mittag-Leffler conditions.

1.4 The Vopěnka's Principles

While graphs may appear to be simple objects at first sight, the category of all graphs is complex. This is reflected by the fact that some of its properties are independent of standard set theory axioms. This necessitates exploration of additional principles beyond those axioms.

One notable principle is *Vopěnka's Principle* (VP), which asserts that the category of graphs has no large discrete full subcategory. In other words, for any proper class of graphs, there exists a non-identity homomorphism among the graphs within that class.

It is called a large-cardinal principle because it implies the existence of measurable cardinals while its consistency follows from the existence of huge cardinals [3, Theorem A.6 and Theorem A.18].

Vopěnka's Principle applies not only to graphs but also to many other categories that can be fully embedded in the category of graphs, such as locally presentable categories (e.g., [3, Theorem 2.65]). The Vopěnka's Principle and its weak forms discussed below thus allow for equivalent statements in the context of locally presentable categories (for more details we refer to [3]).

In [1], Adámek, Rosický, and Trnková observed that Vopěnka's principle is equivalent to the statement that the category of all ordinals, Ord, cannot be fully embedded into the category of graphs. This means that there is no sequence of graphs $(G_\alpha : \alpha \in \mathrm{Ord})$ that satisfies both of the following properties:

- for each $\alpha \leq \alpha'$, there is a unique homomorphism $G_\alpha \to G_{\alpha'}$, and
- for each $\alpha < \alpha'$, there is no homomorphism $G_{\alpha'} \to G_\alpha$.

They introduced the *Weak Vopěnka's Principle* (WVP) as the dual statement: the opposite category Ord^{op} cannot be fully embedded into the category of graphs. This means that there is no sequence of graphs $(G_\alpha : \alpha \in \mathrm{Ord})$ that satisfies both of the following properties:

- for each $\alpha \leq \alpha'$, there is a unique homomorphism $G_{\alpha'} \to G_\alpha$, and
- for each $\alpha < \alpha'$, there is no homomorphism $G_\alpha \to G_{\alpha'}$.

They additionally proved that (VP) implies (WVP) [1, Lemma 2] and raised the question of whether the two principles are equivalent.

Removing the uniqueness of homomorphisms from the (WVP) statement yields the *Semi-Weak Vopěnka's Principle* (SWVP), as introduced by Adámek and Rosický [2]. This means that there is no sequence of graphs $(G_\alpha : \alpha \in \mathrm{Ord})$ that satisfies both of the following properties:

- for each $\alpha \leq \alpha'$, there is a homomorphism $G_{\alpha'} \to G_\alpha$, and
- for each $\alpha < \alpha'$, there is no homomorphism $G_\alpha \to G_{\alpha'}$.

It is clear from the definitions that Semi-Weak Vopěnka's Principle implies Weak Vopěnka's Principle. The following recent result, proved in ZFC by Wilson, greatly simplified matters:

Theorem 1.4.1. *[47, Theorem 1.4] Weak Vopěnka's Principle is equivalent to Semi-Weak Vopěnka's Principle.*

Moreover, Wilson proved that assuming the existence of supercompact cardinals, (SWVP) does not imply (VP); for more details, see [47, Theorem 1.2 and 1.3]. Hence, under the latter assumption, (WVP) does not imply (VP).

Bibliography of Chapter 1

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Flat relative Mittag-Leffler modules and approximations

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Abstract. The classes $\mathcal{D}_Q$ of flat relative Mittag-Leffler modules are sandwiched between the class $\mathcal{FM}$ of all flat (absolute) Mittag-Leffler modules, and the class $\mathcal{F}$ of all flat modules. Building on the works of Angeleri Hügel, Herbera, and Šaroch, we give a characterization of flat relative Mittag-Leffler modules in terms of their local structure, and show that Enochs' Conjecture holds for all the classes $\mathcal{D}_Q$. In the final section, we apply these results to the particular setting of f-projective modules.

Chapter 2

Flat relative Mittag-Leffler modules and approximations

2.1 Introduction

For a ring R, denote by $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{FM}$ the classes of all projective, flat, and flat Mittag-Leffler (right R-) modules, respectively. We always have the inclusions $\mathcal{P} \subseteq \mathcal{FM} \subseteq \mathcal{F}$. The equality $\mathcal{P} = \mathcal{FM} = \mathcal{F}$ holds, if and only if R is a right perfect ring. By a classic result of Bass, in this case $\mathcal{P}$ is a covering class consisting of modules isomorphic to direct sums of (indecomposable projective) modules generated by primitive idempotents of the ring R.

If R is not right perfect, then $\mathcal{P} \subsetneq \mathcal{FM} \subsetneq \mathcal{F}$. In fact, though the classes $\mathcal{P}$ and $\mathcal{FM}$ contain the same countably generated modules, there always exist $\aleph_1$-generated modules in $\mathcal{FM}$ that are not projective, cf. [3, §VII.1]. Moreover, there exist countably presented modules $N \in \mathcal{F} \setminus \mathcal{FM}$. Each such module N is called a Bass module [20].

By a classic theorem of Kaplansky, each projective module is a direct sum of countably generated projective modules, so the class $\mathcal{P}$ is $\aleph_1$-decomposable. If $\kappa = \operatorname{card} R + \aleph_0$, then each flat module is known to be a transfinite extension of $\leq \kappa$-presented flat modules, so the class $\mathcal{F}$ is κ^+-deconstructible. The class $\mathcal{P}$ is easily seen to be precovering, while $\mathcal{F}$ is a covering class by [2] (see Section 2.2 for unexplained terminology).

The intermediate class $\mathcal{FM}$ can be described as the class of all 'locally projective', or better $\aleph_1$-projective modules [9]. Its global structure over non-right perfect rings is known to be quite complex: there is no cardinal λ such that $\mathcal{FM}$ is λ-deconstructible [9]; moreover, the class $\mathcal{FM}$ is not precovering [18].

In this note, we will deal with classes of flat relative Mittag-Leffler modules, or more precisely, flat $\mathcal{Q}$-Mittag-Leffler modules for a class of left R-modules $\mathcal{Q}$.

The notion of an (absolute) Mittag-Leffler module was introduced already in the seminal paper by Raynaud and Gruson [14], and studied in a number of sequel works revealing its many facets. Relative Mittag-Leffler modules appeared much later, in the Habilitationsschrift of Rothmaler [15]. Rothmaler has further pursued the model theoretic point of view in [16], where he proved that if $\mathcal{Q}$ is a definable class of left R-modules and $D(\mathcal{Q})$ is its dual definable class of (right R-) modules, then $\mathcal{Q}$-Mittag-Leffler modules are exactly the $D(\mathcal{Q})$-atomic modules, [16, Theorem 3.1]. For a very recent application of relative atomic modules to a description of Ziegler spectra of tubular algebras, we refer to [12].

A detailed algebraic study of relative Mittag-Leffler modules was performed in [1] (see also [8] and [9]); notably, it discovered their role in (infinite dimensional) tilting theory.

Following [9], we denote the class of all flat $\mathcal{Q}$-Mittag-Leffler modules by $\mathcal{D}_{\mathcal{Q}}$. Thus $\mathcal{FM} \subseteq \mathcal{D}_{\mathcal{Q}} \subseteq \mathcal{F}$. Notice that $\mathcal{D}_{R\text{–Mod}} = \mathcal{FM}$ and $\mathcal{D}_{\{0\}} = \mathcal{F}$, while $\mathcal{D}_{\{R\}}$ is the class of all f-projective modules studied by Goodearl et al. in [4], [6], etc.

Our goal here is to investigate the structure and approximation properties of the class $\mathcal{D}_{\mathcal{Q}}$ in dependence on $\mathcal{Q}$. In Theorem 2.3.5, we prove that the classes $\mathcal{D}_{\mathcal{Q}}$ are determined by their countably presented modules, while Theorem 2.3.6 shows that approximation properties of $\mathcal{D}_{\mathcal{Q}}$ depend completely on whether there exists a Bass module $N \notin \mathcal{D}_{\mathcal{Q}}$. In the final part, we apply these results to the particular setting of $\mathcal{Q} = \{R\}$, i.e., to the f-projective modules.

2.2 Preliminaries

For a ring R, we denote by Mod–R the class of all (right R-) modules, and by R–Mod the class of all left R-modules.

2.2.1 Filtrations and deconstructible classes

Let R be a ring, M a module, and $\mathscr{C}$ a class of modules. A family of submodules, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of M is called a *continuous chain* in M, provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \sigma$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.
A continuous chain $\mathcal{M}$ in M is a *$\mathscr{C}$-filtration* of M, provided that $M = M_\sigma$, and each of the modules $M_{\alpha+1}/M_\alpha$ ($\alpha < \sigma$) is isomorphic to an element of $\mathscr{C}$.
M is called *$\mathscr{C}$-filtered*, provided that M possesses at least one $\mathscr{C}$-filtration. We will use the notation Filt ($\mathscr{C}$) for the class of all $\mathscr{C}$-filtered modules. The modules $M \in$ Filt ($\mathscr{C}$) are also called *transfinite extensions* of the modules in $\mathscr{C}$. A class $\mathscr{A}$ is said to be *closed under transfinite extensions* provided that $\mathscr{A} =$ Filt ($\mathscr{A}$). Clearly, this implies that $\mathscr{A}$ is closed under extensions and arbitrary direct sums. Given a class $\mathscr{C}$ and a cardinal κ, we use $\mathscr{C}^{\leq\kappa}$ and $\mathscr{C}^{<\kappa}$ to denote the subclass of $\mathscr{C}$ consisting of all $\leq \kappa$-presented and $< \kappa$-presented modules, respectively.
Let κ be an infinite cardinal. A class of modules $\mathscr{C}$ is *κ-deconstructible* provided that $\mathscr{C} \subseteq$ Filt ($\mathscr{C}^{<\kappa}$). If moreover each module $M \in \mathscr{C}$ is a direct sum of modules from $\mathscr{C}^{<\kappa}$, then $\mathscr{C}$ is called *κ-decomposable.* For example, the class $\mathscr{P}$ of all projective modules is $\aleph_1$-deconstructible by a classic theorem of Kaplansky. A class $\mathscr{C}$ is *deconstructible* in case it is κ-deconstructible for some infinite cardinal κ.

2.2.2 Approximations

A map $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathscr{C}$ is a *$\mathscr{C}$-precover* of M, if the abelian group homomorphism $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathscr{C}$.
A $\mathscr{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ of M is called a *$\mathscr{C}$-cover* of M, provided that f is right minimal, that is, provided $fg = f$ implies that g is an automorphism for

each $g \in \text{End}_R(\mathscr{C})$.
$\mathscr{C} \subseteq$ Mod-R is a *precovering class* (*covering class*) provided that each module has a $\mathscr{C}$-precover ($\mathscr{C}$-cover).

2.2.3 (Relative) Mittag-Leffler modules

Let R be a ring. A module M is *Mittag-Leffler* provided that the canonical group homomorphism

$$\varphi : M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$$

defined by

$$\varphi(m \otimes_R (n_i)_{i \in I}) = (m \otimes_R n_i)_{i \in I}$$

is monic for each family $(N_i \mid i \in I)$ of left R-modules.
Let $M \in$ Mod–R and $\mathcal{Q} \subseteq R$–Mod. Then M is $\mathcal{Q}$*-Mittag-Leffler*, provided that the canonical morphism $M \otimes \prod_{i \in I} Q_i \to \prod_{i \in I} M \otimes_R Q_i$ is injective for any family $(Q_i \mid i \in I)$ consisting of elements of $\mathcal{Q}$.

2.2.4 Direct limits and add(M)

Let $\mathscr{C}$ be any class of modules and $\mathscr{D} = (C_i, f_{ji} \mid i \leq j \in I)$ a direct system of modules in $\mathscr{C}$. Viewing $\mathscr{D}$ as a diagram in the category Mod–R, we can form its colimit, $(M, f_i \mid i \in I)$. In particular, M is a module, and $f_i \in \text{Hom}_R(M_i, M)$ satisfy $f_i = f_{ij} f_j$ for all $i \leq j \in I$.
This colimit (or sometimes just the module M itself) is called the *direct limit* of the direct system $\mathscr{D}$. It is denoted by $\varinjlim_{i \in I} M_i$ (or just $\varinjlim \mathscr{D}$).
Let $\mathcal{Q} \subseteq R$–Mod. We denote by $\varinjlim \mathcal{Q}$ the class of all modules N such that $N = \varinjlim Q_i$ where $(Q_i, f_{ji} \mid i \leq j \in I)$ is a direct system of modules from $\mathcal{Q}$.
Let R be a ring, M be a module. We define add (M) to be the class of all modules isomorphic to direct summands of finite direct sums of copies of M.

2.2.5 Bass modules

Given a class $\mathscr{C}$ of finitely generated free modules, we call a module M a *Bass module* over $\mathscr{C}$, provided that M is the direct limit of a direct system

$$C_0 \xrightarrow{f_0} C_1 \to \cdots \xrightarrow{f_{n-1}} C_n \xrightarrow{f_n} C_{n+1} \xrightarrow{f_{n+1}} \cdots$$

where $C_n \in \mathscr{C}$ for each $n < \omega$.
If $\mathscr{C}$ is the class of all finitely generated free modules, then the Bass modules over $\mathscr{C}$ are just called the (unadorned) *Bass modules*; they are exactly the countably presented flat modules.

For basic properties of the notions defined above, we refer the reader to [5].

2.3 Flat relative Mittag-Leffler modules

We record the following well-known properties of the class $\mathcal{D}_\mathcal{Q}$ of all flat $\mathcal{Q}$-Mittag-Leffler modules (cf. [1, SS1 and 5], [9, §4] or [5, 3.20(a)]):

Lemma 2.3.1. *Let $\mathcal{Q} \subseteq R$–*Mod*.*

(i) The class $\mathcal{D}_\mathcal{Q}$ is closed under pure submodules, extensions, and unions of pure chains. Hence $\mathcal{D}_\mathcal{Q}$ is closed under transfinite extensions.

(ii) $\mathcal{D}_\mathcal{Q}$ is a resolving subcategory of Mod–R *(i.e., $\mathcal{D}_\mathcal{Q}$ contains all projective modules, and it is closed under extensions and kernels of epimorphisms).*

Remark 9. Clearly, $\mathcal{D}_\mathcal{Q}$ is closed under direct limits, iff $\mathcal{D}_\mathcal{Q} = \mathcal{F}$. This case will be examined in more detail in Theorems 2.3.5(ii) and 2.3.6 below. The closure of the class $\mathcal{D}_\mathcal{Q}$ under products was studied in [9, §4]: if $\mathcal{Q}$ is the limit closure of a class of finitely presented left R-modules, then $\mathcal{D}_\mathcal{Q}$ is closed under products, iff $R^R \in \mathcal{D}_\mathcal{Q}$ (see [9, Theorem 4.6]).

Another basic property of the classes $\mathcal{D}_\mathcal{Q}$ is that in their study, one can restrict to definable classes of left R-modules. Recall that a class of modules is *definable* provided that it is closed under direct limits, direct products and pure submodules. For each class of left R-modules $\mathcal{Q}$ there is a least definable class $\mathrm{Def}\,(\mathcal{Q})$ in R–Mod containing $\mathcal{Q}$; it is obtained by closing $\mathcal{Q}$ first by direct products, then direct limits, and finally by pure submodules, cf. [8, Lemma 2.9 and Corollary 2.10].

Lemma 2.3.2. *Let $\mathcal{Q} \subseteq R$–*Mod*. Then $\mathcal{D}_\mathcal{Q} = \mathcal{D}_{\mathrm{Def}\,(\mathcal{Q})}$.*

Definable classes are parametrized by the subset of the set of all indecomposable pure-injective modules which they contain. So though R–Mod is a proper class, there is only a set of classes of relative Mittag-Leffler modules. Note however, that it may still happen that $\mathcal{D}_{\mathrm{Def}\,(\mathcal{Q})} = \mathcal{D}_{\mathrm{Def}\,(\mathcal{Q}')}$ even if $\mathrm{Def}\,(\mathcal{Q}) \neq \mathrm{Def}\,(\mathcal{Q}')$: just take a right noetherian ring R which is not completely reducible, and consider the following two definable classes of left R-modules: $\mathcal{Q} = \{0\}$ and $\mathcal{Q}' = \mathcal{F}'$ (the class of all flat left R-modules). Then $\mathcal{D}_\mathcal{Q} = \mathcal{D}_{\mathcal{Q}'} = \mathcal{F}$ by Proposition 2.4.7(i) below. In Theorem 2.3.5, we will give a different parametrization of the classes $\mathcal{D}_\mathcal{Q}$, by their countably presented modules.

Our next prerequisite was proved in [9, 2.2] (see also [5, 3.11]):

Lemma 2.3.3. *Let R be a ring, M be a module, and $\mathcal{Q}$ be a class of left R-modules. Assume that $M = \varinjlim_{\alpha \in L} M_\alpha$ where $(M_\alpha, f_{\beta\alpha} \mid \alpha < \beta \in L)$ is a direct system of $\mathcal{Q}$-Mittag-Leffler modules. Moreover, assume that $M' = \varinjlim_{n<\omega} M_{\alpha_n}$ is $\mathcal{Q}$-Mittag-Leffler for each countable chain $\alpha_0 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ in L.*

Then M is $\mathcal{Q}$-Mittag-Leffler.

For all rings, flat relative Mittag-Leffler modules include the flat (absolute) Mittag-Leffler modules, and for some rings, even all the flat modules (see Section 2.4 below). So the following description of the local structure of flat relative Mittag-Leffler modules extends simultaneously the 'local projectivity' of flat Mittag-Leffler modules from [9, Theorem 2.10(i)] and the deconstructibility, and hence abundance of small pure flat submodules, of flat modules from [5, Lemma 6.17 and Theorem 7.10] (cf. also [1, Theorem 5.1], [9, Theorem 2.6], [16, Corollary 6.5], and [19, Lemma 2.5]):

Proposition 2.3.4. *Let R be a ring, M be a module, and $\mathcal{Q}$ be a class of left R-modules. Let $\kappa = \mathrm{card}\,(R) + \aleph_0$. Then the following conditions (i)-(iv) are equivalent:*

(i) M is a flat $\mathcal{Q}$-Mittag-Leffler module.

(ii) For each subset C of M of cardinality $\leq \kappa$, there exists a pure flat $\mathcal{Q}$-Mittag-Leffler submodule N of M such that $C \subseteq N$, and N has cardinality $\leq \kappa$.

(iii) There exists a system $\mathcal{S}$ consisting of pure flat $\mathcal{Q}$-Mittag-Leffler submodules of M of cardinality $\leq \kappa$, such that for each subset C in M of cardinality $\leq \kappa$ there is $N \in \mathcal{S}$ containing C, and $\mathcal{S}$ is closed under unions of well-ordered chains of length $\leq \kappa$.

(iv) M is a directed union of a direct system $\mathcal{T}$ consisting of flat $\mathcal{Q}$-Mittag-Leffler submodules of M, such that $\mathcal{T}$ is closed under unions of countable chains.

Consider also the following condition:

(v) There exists a system $\mathcal{U}$ consisting of countably presented flat $\mathcal{Q}$-Mittag-Leffler submodules N of M such that the inclusion $N \hookrightarrow M$ remains injective when tensored by any left R-module $Q \in \mathcal{Q}$, and satisfying the following two conditions: (a) for each countable subset C in M there is $N \in \mathcal{U}$ containing C, and (b) $\mathcal{U}$ is closed under unions of countable chains.

Then (v) implies (i), and if $R \in \mathcal{Q}$, then (v) is equivalent to (i).

Proof. (i) $\Rightarrow$ (ii). Let C be a subset of M with $\operatorname{card} C \leq \kappa$. Then there is a pure submodule $P \subseteq^* M$ such that $C \subseteq P$ and $\operatorname{card} P \leq \kappa$ (see e.g. [5, 2.25(i)]). By Lemma 2.3.1, P is flat and $\mathcal{Q}$-Mittag-Leffler, whence (ii) holds.
(ii) $\Rightarrow$ (iii). We will prove that the set $\mathcal{S}$ consisting of all pure flat $\mathcal{Q}$-Mittag-Leffler submodules of M of cardinality $\leq \kappa$ has the required two properties. The first one is just a restatement of (ii). For the second (closure under unions of well-ordered chains of length $\leq \kappa$), let $(N_\alpha \mid \alpha < \kappa)$ be a such a chain in $\mathcal{S}$. Let $N = \bigcup_{\alpha<\kappa} N_\alpha$. Since N_α is pure in M for each $\alpha < \kappa$, N is pure in M, too, by [5, Lemma 2.25(d)]. Since $\operatorname{card} N \leq \kappa$, (ii) yields existence of $N' \in \mathcal{S}$ such that $N \subseteq N'$. Finally, $N \subseteq^* M$ implies $N \subseteq^* N'$. As N' is flat and $\mathcal{Q}$-Mittag-Leffler, Lemma 2.3.1(i) gives $N \in \mathcal{S}$.
(iii) $\Rightarrow$ (iv) This is clear - just take $\mathcal{T} = \mathcal{S}$.
(iv) $\Rightarrow$ (i) First, M, being a directed union of flat modules, is flat. In view of (iv), we can apply Lemma 2.3.3 to the presentation of M as the directed union of the elements of $\mathcal{T}$; thus, M is $\mathcal{Q}$-Mittag-Leffler.

Assume (v). Then M is a directed union of the modules in $\mathcal{U}$, and Lemma 2.3.3 applies, showing that M is a flat $\mathcal{Q}$-Mittag-Leffler module.

Finally, let $R \in \mathcal{Q}$. Assume M is a flat $\mathcal{Q}$-Mittag-Leffler module, and let $\mathcal{U}$ be the set consisting of all countably presented flat $\mathcal{Q}$-Mittag-Leffler submodules N of M such that the inclusion $N \hookrightarrow M$ remains injective when tensored by any left R-module $Q \in \mathcal{Q}$. Since $R \in \mathcal{Q}$, the implication (1) $\Rightarrow$ (4) of [1, Theorem 5.1] (for $\mathcal{S}$ = the class of all finitely generated free modules) yields condition (a). Let N' be the union of a countable chain $(N_i \mid i < \omega)$ of modules from $\mathcal{U}$. Then N' is flat, and each finite subset of N' is contained in some term of the chain, so by the implication (4) $\Rightarrow$ (1) of [1, Theorem 5.1], N' is a $\mathcal{Q}$-Mittag-Leffler module. Since the inclusion $\nu : N' \hookrightarrow M$ is a direct limit of the inclusions $\nu_i : N_i \hookrightarrow M$ $(i < \omega)$, $\nu \otimes_R Q$ is the direct limit of the monomorphisms $\nu_i \otimes_R Q : N_i \otimes_R Q \hookrightarrow M \otimes_R Q$ $(i < \omega)$, hence $\nu \otimes_R Q$ is injective, for each $Q \in \mathcal{Q}$. Thus $N' \in \mathcal{U}$, and condition (b) holds, too. □

Remark 10. 1. If $R \notin \mathcal{Q}$, then (i) need not imply (v). For a simple counterexample, consider a von Neumann regular ring R such that there exists a simple module M which is not countably presented (i.e., $M = R/I$ where I is a maximal right ideal of R which is not countably generated). Examples of such rings R include infinite products of fields, or endomorphism rings of infinite dimensional linear spaces. Let $\mathcal{Q} = \{0\}$. Then M is flat (= flat $\mathcal{Q}$-Mittag-Leffler), as all modules over von Neumann regular rings are flat, but (v) fails, because the only countably presented submodule of M is 0.

2. If we remove the assumption of flatness from conditions (i)-(v), then the result still holds true, cf. [9, Theorem 2.6].

3. The system $\mathcal{S}$ in (iii) consists of 'big' (= of cardinality $\leq \kappa$) pure submodules of M and it is closed under unions of 'long' (= of length $\leq \kappa$) well-ordered chains, while the system $\mathcal{U}$ in (v) consists of 'small' (= countably presented), but possibly non-pure, submodules of M, and it is closed under unions of 'short' (= countable) chains.

It may even happen that no non-zero module in the system $\mathcal{U}$ is pure in M: for an example, consider the polynomial ring $R = \mathbb{C}[x]$, let $\mathcal{Q} = \{R\}$, and let M be the quotient field of R viewed as an (uncountably generated) R-module. Then $M \in \mathcal{D}_{\mathcal{Q}}$, but M has no non-zero countably generated pure submodules. So in this setting, there are only the trivial choices for a system $\mathcal{S}$ satisfying condition (iii) (namely, $\mathcal{S} = \{M\}$, and $\mathcal{S} = \{0, M\}$), while $\mathcal{U}$ from (v) must be uncountable - e.g., $\mathcal{U}$ can be taken as the set of all countably generated submodules of M.

We arrive at a simple test of coincidence for various classes $\mathcal{D}_{\mathcal{Q}}$ – one only has to check their countably presented modules:

Theorem 2.3.5. *Let R be a ring.*

(i) Let $\mathcal{Q}$ and $\mathcal{Q}'$ be classes of left R-modules containing R. Then $\mathcal{D}_{\mathcal{Q}} = \mathcal{D}_{\mathcal{Q}'}$, iff $\mathcal{D}_{\mathcal{Q}}$ and $\mathcal{D}_{\mathcal{Q}'}$ contain the same countably presented modules.

(ii) Let $\mathcal{Q}$ be an arbitrary class of left R-modules. Then $\mathcal{D}_{\mathcal{Q}} = \mathcal{F}$, iff each countably presented flat module is $\mathcal{Q}$-Mittag-Leffler.

(iii) Let $\mathcal{Q}$ be a class of left R-modules containing R. Then $\mathcal{D}_{\mathcal{Q}} = \mathcal{FM}$, iff each countably presented flat $\mathcal{Q}$-Mittag-Leffler module is projective.

Proof. (i) Assume there is a module $M \in \mathcal{D}_{\mathcal{Q}} \setminus \mathcal{D}_{\mathcal{Q}'}$. Consider the system $\mathcal{U} \subseteq \mathcal{D}_{\mathcal{Q}}$ provided by Proposition 2.3.4(v) for the class $\mathcal{Q}$. Then M is the directed union of the modules in $\mathcal{U}$, but $M \notin \mathcal{D}_{\mathcal{Q}'}$. By Lemma 2.3.3, there is a (countably presented) module $N \in \mathcal{U}$ such that $N \notin \mathcal{D}_{\mathcal{Q}'}$.

(ii) Assume there is a module $M \in \mathcal{F} \setminus \mathcal{D}_{\mathcal{Q}}$. Being flat, M is a direct limit of a direct system $\mathfrak{D}$ of finitely generated free modules. By Lemma 2.3.3, there exists a Bass module N over $\mathfrak{D}$ such that $N \notin \mathcal{D}_{\mathcal{Q}}$.

(iii) is a special instance of (i) for $\mathcal{Q}' = R$–Mod, since countably presented flat Mittag-Leffler modules are projective. □

The relation of countably generated $\mathcal{Q}$-Mittag-Leffler modules to $D(\mathrm{Def}\,(\mathcal{Q}))$-pure chains of finitely presented modules was examined in [16, §7] (see [17] for corrections and further results).

Our next theorem says that precovers (right approximations) by modules in the class $\mathcal{D}_\mathcal{Q}$ exist only in the threshold case of $\mathcal{D}_\mathcal{Q} = \mathcal{F}$. The theorem also confirms Enochs' Conjecture (that covering classes of modules are closed under direct limits) for all the classes $\mathcal{D}_\mathcal{Q}$:

Theorem 2.3.6. *Let R be a ring and $\mathcal{Q}$ be a class of left R-modules. Then the following conditions are equivalent:*

(i) Each Bass module (= countably presented flat module) is $\mathcal{Q}$-Mittag-Leffler.

(ii) $\mathcal{D}_\mathcal{Q} = \mathcal{F}$.

(iii) $\mathcal{D}_\mathcal{Q}$ is covering.

(iv) $\mathcal{D}_\mathcal{Q}$ is precovering.

(v) $\mathcal{D}_\mathcal{Q}$ is deconstructible.

(vi) $\mathcal{D}_\mathcal{Q}$ is closed under direct limits.

Proof. (i) ⇒ (ii) by Theorem 2.3.5(ii).

(ii) ⇒ (iii), (v), and (vi): This follows from the fact that for any ring R, the class of all flat modules is a deconstructible covering class closed under direct limits, cf. [2].

(iii) ⇒ (iv) is trivial.

(iv) ⇒ (i): Assume (i) fails, so there is a Bass module $N \in \mathcal{F} \setminus \mathcal{D}_\mathcal{Q}$. Let $f : A \to N$ be a (surjective) $\mathcal{D}_\mathcal{Q}$-precover of N and $M = \ker f$. Let κ be an infinite cardinal such that $\mathrm{card}\, R \leq \kappa$ and $\mathrm{card}\, M \leq 2^\kappa = \kappa^\omega$. By [20, Lemma 5.6], there are a 'tree module' L and an exact sequence $0 \to D \to L \to N^{(2^\kappa)} \to 0$, where D is a direct sum of κ finitely generated free modules and L is flat and Mittag-Leffler. Proceeding as in the proof of [18, Lemma 3.2], we infer that f splits, whence $N \in \mathcal{D}_\mathcal{Q}$, a contradiction.

(v) ⇒ (i): This has been proved in [9, Corollary 7.2(ii)].

(vi) ⇒ (i): This holds since $\mathcal{F} = \varinjlim \mathcal{P}$, whence $\varinjlim \mathcal{D}_\mathcal{Q} = \mathcal{F}$. □

Remark 11. If $\mathcal{Q}$ is a class of left R-modules such that $\mathcal{D}_\mathcal{Q} = \mathcal{F}$, then $\mathcal{D}_\mathcal{Q} = \mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa}$ for any infinite cardinal $\kappa \geq \mathrm{card}\, R$.

In constrast, if $\mathcal{Q}$ is a class of left R-modules such that $\mathcal{D}_\mathcal{Q} \neq \mathcal{F}$, then the classes $\mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa}$, where κ runs over all infinite cardinals $\geq \mathrm{card}\, R$, form a strictly increasing 'chain' – a proper class of subclasses of $\mathcal{D}_\mathcal{Q}$ – consisting of classes closed under transfinite extensions, whose union is $\mathcal{D}_\mathcal{Q}$.

Indeed, the existence of a Bass module $N \in \mathcal{F} \setminus \mathcal{D}_\mathcal{Q}$ makes it possible to construct for each infinite cardinal $\kappa \geq \mathrm{card}\, R$ a κ^+-generated flat Mittag-Leffler module M_{κ^+} such that M_{κ^+} is not $\mathcal{D}_\mathcal{Q}^{\leq\kappa}$-filtered, cf. [9, §5]. Thus $\mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa} \subsetneq \mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa^+} \subsetneq \mathcal{D}_\mathcal{Q}$, and $\mathcal{D}_\mathcal{Q} = \bigcup_{\kappa \geq \mathrm{card}\, R} \mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa}$. Moreover, though $\mathcal{FM} \subseteq \mathcal{D}_\mathcal{Q}$, there is no cardinal κ such that $\mathcal{FM} \subseteq \mathrm{Filt}\, \mathcal{D}_\mathcal{Q}^{\leq\kappa}$.

2.4 f-projective modules

In this section, we will consider a particular kind of relative Mittag-Leffler modules, the f-projective ones. Their original definition is as follows:

Definition 2.4.1. A module M is said to be *f-projective* if for every finitely generated submodule C of M, the inclusion map factors through a (finitely generated) free module F.

$$\begin{array}{ccc} C & \hookrightarrow & M \\ & \searrow & \uparrow \\ & & F \end{array}$$

So every projective module is f-projective, and every finitely generated f-projective module is projective. Since flat modules are characterized as the direct limits of finitely generated free modules, each f-projective module is flat by the following lemma due to Lenzing, cf. [5, Lemma 2.13]:

Lemma 2.4.2. *Let R be a ring and $\mathcal{C}$ be a class of finitely presented modules closed under finite direct sums. Then the following are equivalent for a module M.*

(i) Every homomorphism $\varphi : G \to M$, where G is finitely presented, has a factorisation through a module in $\mathcal{C}$.

(ii) $M \in \varinjlim \mathcal{C}$.

The fact that f-projective modules are a particular kind of flat relative Mittag-Leffler modules goes back to Goodearl [6], see also [4, Proposition 2.7]:

Proposition 2.4.3. *A module M is f-projective if and only if it is flat and $\{R\}$-Mittag-Leffler.*

In particular, each countably generated f-projective module is countably presented, and hence of projective dimension ≤ 1.

Proof. Let $\mathcal{F}'$ denote the class of all flat left R-modules. By Lemma 2.3.2 (or [6, Theorem 1]), $\mathcal{D}_{\{R\}} = \mathcal{D}_{\mathcal{F}'}$. By [6, Theorem 1], for each module M, $M \in \mathcal{D}_{\mathcal{F}'}$, iff M is flat and for each finitely generated submodule F of M, the inclusion $F \hookrightarrow M$ factors through a finitely presented module. By Lemma 2.4.2, this is further equivalent to the f-projectivity of M.

The final claim follows from [1, Corollary 5.3]. □

We also note the following corollary of [1, Theorem 5.1] (and Theorem 2.3.4):

Corollary 2.4.4. *Let R be a ring and $M \in$ Mod–R. Then M is f-projective, if and only if M possesses a system of submodules, $\mathcal{U}$, consisting of countably presented f-projective modules such that each countable subset of M is contained in an element of $\mathcal{U}$ (and $\mathcal{U}$ is closed under unions of countable chains).*

We will denote by $\mathcal{FP}$ the class of all f-projective modules. So $\mathcal{FP} = \mathcal{D}_{\{R\}} = \mathcal{D}_{\mathcal{F}'}$.

There is an interesting relation between f-projectivity and coherence. Here, we will call a module M *coherent*, if all its finitely generated submodules are finitely presented. (Thus a ring R is right coherent, if the regular right module is coherent.)

Lemma 2.4.5. *Let R be a ring.*

(i) Let M be a flat coherent module. Then M is f-projective.

(ii) The ring R is right coherent, iff $\mathcal{FP}$ coincides with the class of all flat coherent modules.

Proof. (i) Since M is flat, $M = \varinjlim F_i$ where the modules F_i are finitely generated and free. If C is a finitely generated submodule of M, then C is finitely presented, so the inclusion $C \subseteq M$ factors through some F_i by Lemma 2.4.2, whence M is f-projective.

(ii) In view of part (i) and Lemma 2.4.2, in order to prove the only-if part, we have to prove that f-projectivity implies coherence for any module M. Let F be be a finitely generated submodule of M. By f-projectivity, F is a submodule of a finitely generated free module. Since R is right coherent, F is finitely presented.

The if part is clear, since the regular module R is always f-projective. □

Note that the situation simplifies for coherent domains:

Lemma 2.4.6. *Let R be a coherent domain. Then $\mathcal{FP} = \mathcal{F}$.*

Proof. Let M be a flat module. By Lemma 2.4.5(i), it suffices to prove that M is coherent. Let F be a finitely generated submodule of M. Then M and F are torsion-free, so by a classic result of Cartan and Eilenberg [5, 16.1], F embeds into a finitely generated free module. By the coherence of R, F is finitely presented, proving that M is coherent. □

Further instances of the coincidence $\mathcal{FP}$ with $\mathcal{F}$ (i.e., of the case when $\mathcal{FP}$ is a covering class, see Theorem 2.3.6) appear in part (i) of the following proposition:

Proposition 2.4.7. *Let R be a ring.*

(i) Assume that R is right noetherian or R is right perfect. Then $\mathcal{FP} = \mathcal{F}$ is a covering class.

(ii) If R is right non-singular, then all f-projective modules are non-singular.

(iii) If R is von Neumann regular, then $\mathcal{FP} = \mathcal{FM}$. Hence $\mathcal{FP}$ is covering, only if R is completely reducible.

(iv) Assume R is von Neumann regular and right self-injective. Then $\mathcal{FP}$ coincides with the class of all non-singular modules.

Proof. (i) If R is right noetherian, then each finitely generated module is finitely presented, so Lemma 2.4.2 yields that all flat modules are f-projective. If R is right perfect, then all relative Mittag-Leffler modules are projective (= flat).

(ii) Let M be an f-projective module. By Definition 2.4.1, each finitely generated submodule C of M embeds into a finitely generated free module F. By assumption, F is non-singular, hence so are C and M.

(iii) This is clear, since von Neumann regularity of R is equivalent to the property that all left R-modules are flat.

(iv) The non-singularity of all f-projective modules follows from (ii). Since R is right self-injective, [7, 9.2] shows that all finitely generated submodules of non-singular modules are projective, hence all non-singular modules are f-projective by Definition 2.4.1. Alternatively, we can use (iii) and the fact that under the assumptions of (iv), flat Mittag-Leffler modules coincide with the non-singular ones by [9, 6.8]. □

For semihereditary rings, we have a fully ring theoretic characterization:

Proposition 2.4.8. *(i) The following conditions are equivalent for a ring R:*

(1) R is right semihereditary.

(2) $\mathcal{FP}$ coincides with the class of all modules M such that each finitely generated submodule of M is projective.

(3) The class $\mathcal{FP}$ is closed under submodules.

(ii) Assume R is right semihereditary. Then the following conditions are equivalent:

(1) $\mathcal{FP}$ is a covering class.

(2) Each finitely generated flat module is projective.

(3) For each $n > 0$, the full matrix ring $M_n(R)$ contains no infinite sets of non-zero pairwise orthogonal idempotents.

Proof. (i) By [10, Theorem 2.29], R is right semihereditary, iff all finitely generated submodules of projective modules are projective. So the implication (1) implies (2) is immediate from Definition 2.4.1, (2) implies (3) is trivial, and (3) implies (1) because projective modules are f-projective, and finitely generated f-projective modules are projective.

(ii) The implication (1) $\Rightarrow$ (2) holds for any ring: By Theorem 2.3.6, (1) implies $\mathcal{FP} = \mathcal{F}$, so each finitely generated flat module is f-projective, hence projective, and (2) holds. If R is right semihereditary, then R has flat dimension ≤ 1, i.e., submodules of flat modules are flat. So if each finitely generated flat module is projective, then by part (i), each flat module is f-projective. This proves (2) $\Rightarrow$ (1).

Assume (2) holds, and there is an $n > 0$ such that the full matrix ring $S = M_n(R)$ contains an infinite set $\{e_i \mid i < \omega\}$ of non-zero pairwise orthogonal idempotents. Then $M = S/\oplus\sum_{i<\omega} e_i S$ is a direct limit of the projective modules $S/\oplus\sum_{i\in X} e_i S$, where X runs over all finite subsets of ω. So M is a cyclic flat right S-module which is not projective. Further, S is Morita equivalent to R. If (F, G) is a pair functors realizing this equivalence, then $F(M)$ is a finitely generated non-projective flat module, in contradiction with (2).

The implication (3) $\Rightarrow$ (2) is proved e.g. in [13, Proposition 4.10]. □

On the one hand, if $\mathcal{FP} \subsetneq \mathcal{F}$, then there is always a finitely generated projective module $M \in \mathcal{FP}$ such that $\varinjlim \operatorname{add} M \nsubseteq \mathcal{FP}$ (just take $M = R$). On the other hand, we have the following result that applies, e.g., to any simple projective module M:

Proposition 2.4.9. *Let R be a ring and M be an f-projective module. Let $S = \operatorname{End} M$. Assume that S is right noetherian and M is a flat left S-module. Then $\varinjlim \operatorname{add} M \subseteq \mathcal{FP}$.*

Proof. By [11, Theorem 3.3], $\varinjlim \operatorname{add} M = \{F \otimes_S M \mid F \text{ a flat right } S\text{-module }\}$. So we have to prove that $F \otimes_S M$ is a flat and $\{R\}$-Mittag-Leffler module, for each flat right S-module F. Flatness of $F \otimes_S M$ is clear since M is f-projective, hence flat.

Let I be a set. By assumption, the canonical map $M \otimes_R R^I \to M^I$ is an injective homomorphism of left S-modules, whence the map $(F \otimes_S M) \otimes_R R^I \to F \otimes_S M^I$ is monic. Since S is right noetherian, F is an f-projective right S-module by Proposition 2.4.7(i). Since M is a flat left S-module, the canonical map $F \otimes_S M^I \to (F \otimes_S M)^I$ is monic by Proposition 2.4.3. Thus, the module $F \otimes_S M$ is $\{R\}$-Mittag-Leffler. □

Bibliography of Chapter 2

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Flat relative Mittag-Leffler modules and Zariski locality

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Abstract. The ascent and descent of the Mittag-Leffler property were instrumental in proving Zariski locality of the notion of an (infinite dimensional) vector bundle by Raynaud and Gruson in [25]. More recently, relative Mittag-Leffler modules were employed in the theory of (infinitely generated) tilting modules and the associated quasi-coherent sheaves, [1], [21]. Here, we study the ascent and descent along flat and faithfully flat homomorphisms for relative versions of the Mittag-Leffler property. In particular, we prove the Zariski locality of the notion of a locally f-projective quasi-coherent sheaf for all schemes, and for each $n \geq 1$, of the notion of an n-Drinfeld vector bundle for all locally noetherian schemes.

Chapter 3

Flat relative Mittag-Leffler modules and Zariski locality

3.1 Introduction

Relative Mittag-Leffler modules were introduced by Rothmaler in [26]. His approach was model theoretic: Mittag-Leffler modules were shown to be the counterparts of pure-injective modules in the sense that the former are atomic (i.e., they realize only the finitely generated pp-types) while the latter are saturated (i.e., they realize all pp-types). The adjective 'relative' referred to restricting to theories of modules induced by definable subclasses of Mod–R. Much later, the important role of relative Mittag-Leffler modules for (infinite dimensional) tilting theory was recognized by Angeleri and Herbera [1]; this in turn led to a proof of finite type of all 1-tilting modules in [4].

Flat Mittag-Leffler modules played a key role in proving Zariski locality of the notion of an (infinite dimensional) vector bundle in the classic work of Raynaud and Gruson, [25, Seconde partie]. The locality follows by the Affine Communication Lemma (see e.g. [30, 5.3.2]), whose assumptions are guaranteed by the ascent and descent of projectivity along flat ring homomorphisms, and faithfully flat ring homomorphisms, respectively.

Once a structure theory of tilting modules over commutative rings was developed in [2] and [20], it was possible to generalize the classic results to proving Zariski locality for various notions of quasi-coherent sheaves associated with tilting, [21]. Another generalization, employing the notion of a restricted flat Mittag-Leffler module, proved the Zariski locality of restricted Drinfeld vector bundles in [14].

Our goal here is to refine the classic result on the ascent and descent of flat Mittag-Leffler modules to the relative setting. The main technical tools needed for this purpose are presented in Section 3.3. In Section 3.4, we apply these tools and prove Zariski locality of the corresponding notions of flat quasi-coherent sheaves. In particular, we prove the Zariski locality of the notion of a locally f-projective quasi-coherent sheaf for all schemes, and for each $n \geq 1$, of the notion of an n-Drinfeld vector bundle for all locally noetherian schemes.

3.2 Preliminaries

Let R be an (associative, unital) ring and Mod–R the category of all (unitary right R-) modules. The elements of Mod–R will often be referred to simply as *modules*. Further, R–Mod will denote the category of all (unitary) left R-modules.

Let $n \geq 0$. A module M is an *FP$_n$ module* provided that M possesses a projective resolution $\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$ such that all the modules P_i ($i \leq n$) are finitely generated. So FP$_0$ modules are just the finitely generated modules, FP$_1$ modules are the finitely presented ones, etc. Notice that the ring R is right noetherian, iff the classes of FP$_n$ modules coincide for all $n \geq 0$, while R is right coherent, iff the classes of FP$_n$ modules coincide for all $n \geq 1$.

We will denote by $\mathcal{P}_n$, $\mathcal{F}_n$, and $\mathcal{I}_n$ the classes of all modules of projective, weak, and injective dimension $\leq n$, respectively.

Let $\mathcal{B}$ be a class of modules. Then ${}^{\perp}\mathcal{B}$ denotes the class of all modules A such that $\mathrm{Ext}^1_R(A,B) = 0$ for each $B \in \mathcal{B}$. Similarly, $\mathcal{B}^{\perp}$ is the class of all modules C such that $\mathrm{Ext}^1_R(B,C) = 0$ for all $B \in \mathcal{B}$. Further, $\mathcal{B}^{\intercal}$ denotes the class of all left R-modules D such that $\mathrm{Tor}^R_1(B,D) = 0$ for all $B \in \mathcal{B}$. Similarly, for a class of left R-modules $\mathcal{D}$, ${}^{\intercal}\mathcal{D}$ denotes the class of all modules C such that $\mathrm{Tor}^R_1(C,D) = 0$ for all $D \in \mathcal{D}$.

For a class of modules $\mathcal{C}$ we denote by $\varinjlim \mathcal{C}$ the class of all modules that are direct limits of direct systems consiting of modules from $\mathcal{C}$. For example, $\mathcal{F}_0 = \varinjlim \mathcal{P}_0$ for any ring R. Also, $\mathcal{PI}$ will denote the class of all pure-injective modules.

We will need the following consequence of [15, Theorem 8.40 and Corollary 8.42]:

Lemma 3.2.1. *Let R be a ring and $\mathcal{C}$ be a class of FP$_2$-modules closed under extensions, direct summands and containing R. Let $\mathcal{B} = \mathcal{C}^{\perp}$. Then $\varinjlim \mathcal{C} = {}^{\perp}(\mathcal{B} \cap \mathcal{PI})$.*

We also recall the following identities satisfied by the Tor bifunctor.

Lemma 3.2.2. *Let $\varphi : R \to S$ be a flat ring homomorphism of commutative rings.*

(1) For all modules A and B, there is an S-isomorphism $\mathrm{Tor}^R_1(A,B) \otimes_R S \cong \mathrm{Tor}^S_1(A \otimes_R S, B \otimes_R S)$.

(2) If A is a module and B is an S-module, then there is an S-isomorphism $\mathrm{Tor}^S_1(A \otimes_R S, B) \cong \mathrm{Tor}^R_1(A,B)$.

Proof. (1) is a particular instance of [12, Theorem 2.1.11], and (2) a particular instance of [9, Proposition VI.4.1.2]. □

The central notion of our paper is that of a relative Mittag-Leffler module:

Definition 3.2.3. Let R be an arbitrary ring, $M \in$ Mod–R and $\mathcal{Q} \subseteq R$–Mod. Then M is *$\mathcal{Q}$-Mittag-Leffler* (or *Mittag-Leffler relative to* $\mathcal{Q}$), provided that the canonical morphism $\psi_M : M \otimes_R \prod_{i\in I} Q_i \to \prod_{i\in I} M \otimes_R Q_i$ defined by $\psi_M(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$ is injective for any family $(Q_i \mid i \in I)$ consisting of elements of $\mathcal{Q}$.

As mentioned above, relative Mittag-Leffler modules were introduced in [26]. Further results on these modules were proved in [1], and in the more recent papers [17] and [27]. Following [19], we will denote by $\mathcal{D}_{\mathcal{Q}}$ the class of all flat $\mathcal{Q}$-Mittag-Leffler modules.

The two borderline cases of Definition 3.2.3 occur for $\mathcal{Q} = \emptyset$, when $\mathcal{D}_{\mathcal{Q}} = \mathcal{F}_0$, and for $\mathcal{Q}_0 = R$–Mod, when $\mathcal{D}_{\mathcal{Q}_0} = \mathcal{FM}$ is the class of all flat Mittag-Leffler modules. As $\mathcal{Q}_0 = (\mathcal{F}_0)^\intercal$, the latter setting can be extended as follows: for each $n \geq 0$, we let $\mathcal{Q}_n = (\mathcal{F}_n)^\intercal$. We will call the modules $M \in \mathcal{D}_{\mathcal{Q}_n}$ *flat n-Mittag-Leffler*.

Another case of interest is when $\mathcal{Q} = \{R\}$, or equivalently, $\mathcal{Q}$ is the class of all flat left R-modules. Then the flat $\mathcal{Q}$-Mittag-Leffler modules coincide with the *f-projective* modules, that is, the modules M such that each homomorphism from a finitely generated module to M factorizes through a free module, see [16] or [5, §3].

Denoting the class of all f-projective modules by $\mathcal{FP}$, we have the following chain of classes of modules

$$(*) \quad \mathcal{P}_0 \subseteq \mathcal{FM} = \mathcal{D}_{\mathcal{Q}_0} \subseteq \cdots \subseteq \mathcal{D}_{\mathcal{Q}_n} \subseteq \mathcal{D}_{\mathcal{Q}_{n+1}} \subseteq \cdots \subseteq \mathcal{FP} \subseteq \mathcal{F}_0.$$

The inclusions in the chain $(*)$ need not be strict in general. For example, if R has weak global dimension $\leq n$, then $\mathcal{D}_{\mathcal{Q}_n} = \mathcal{D}_{\mathcal{Q}_{n+1}} = \cdots = \mathcal{FP}$. If R is a right perfect ring, then all the classes in the chain $(*)$ coincide.

Remark 12. 1. Other variants of the notion of a flat Mittag-Leffler module, called *restricted flat Mittag-Leffler* modules, were introduced in [14]. Their classes form a chain located between the classes $\mathcal{P}_0$ and $\mathcal{FM}$.

2. The following generalization of the notion of an f-projective module goes back to Simson [28]: given a cardinal $\kappa \geq \aleph_0$, a module M is *κ-projective* if each homomorphism from a $< \kappa$-generated module to M factorizes through a free module. Denote by $\mathcal{C}_\kappa$ the class of all κ-projective modules. Since $\mathcal{C}_{\aleph_0} = \mathcal{FP} = \mathcal{D}_{\{R\}}$ one may wonder whether the classes $\mathcal{C}_\kappa$ fit in the setting of flat relative Mittag-Leffler modules also for $\kappa > \aleph_0$.

It is easy to see that $\mathcal{FM} \subseteq \mathcal{C}_{\aleph_1} \subseteq \mathcal{FP}$ for any ring R, and that $\mathcal{C}_{\aleph_1} = \mathcal{FM}$ when R is right hereditary or von Neumann regular (cf. [15, 3.19] and [5, 3.7(iii)]). Also, for each $\kappa \geq \aleph_0$, all $< \kappa$-generated modules in the class $\mathcal{C}_\kappa$ are projective. In particular, for $\kappa = \aleph_1$, the classes $\mathcal{C}_{\aleph_1}$ and $\mathcal{FM}$ contain the same countably presented modules (namely the projective ones), so if $\mathcal{C}_{\aleph_1} \neq \mathcal{FM}$, then $\mathcal{C}_{\aleph_1} \neq \mathcal{D}_{\mathcal{Q}}$ for any class of left R-modules $\mathcal{Q}$ by [5, 2.5(i)]. If R is not right perfect, then the class $\mathcal{FM}$ contains $\aleph_1$-generated non-projective modules (cf. [15, 3.19] and [11, VII.1.3]), so for each $\kappa > \aleph_1$, $\mathcal{FM} \nsubseteq \mathcal{C}_\kappa$, whence again $\mathcal{C}_\kappa \neq \mathcal{D}_{\mathcal{Q}}$ for any class of left R-modules $\mathcal{Q}$.

Definition 3.2.4. Let R be a commutative ring.

(1) Let $\mathfrak{P}$ a property of modules. Then $\mathfrak{P}(\text{Mod-}R)$ denotes the class of all modules satisfying the property $\mathfrak{P}$.

(2) Let $\mathfrak{R}$ be a class of commutative rings. Let X be a scheme and $(\mathcal{R}(u) \mid u \subseteq X, u \text{ open affine })$ be its structure sheaf. Then X is a *locally* $\mathfrak{R}$*-scheme* provided that $\mathcal{R}(u) \in \mathfrak{R}$ for each open affine set u of X.

The main properties $\mathfrak{P}$ of modules that we will be interested in here are the flatness, projectivity, and various properties related to Mittag-Leffler conditions that are in general weaker than projectivity, but stronger than flatness. We will work with general schemes, but in our final application, we will restrict ourselves to *locally noetherian* schemes, that is, the locally $\mathfrak{R}$-schemes where $\mathfrak{R}$ is the class of commutative noetherian rings.

Recall that given two commutative rings R and S, a ring homomorphism $\varphi : R \to S$ is *flat*, provided that S is a flat R-module (where the R-module structure on S is induced by φ), that is, the functor $F = - \otimes_R S$ is exact. Moreover, φ is *faithfully flat* provided that φ is flat, and $N \otimes_R S \neq 0$, whenever $0 \neq N \in$ Mod-R. Faithful flatness of φ is equivalent to the following property of the functor F: for each complex $\mathcal{C}$ of R-modules, $\mathcal{C}$ is exact in Mod-R, if and only if $F(\mathcal{C})$ is exact in Mod-S, [23, Theorem 7.2].

A useful characterization of faithfully flat ring homomorphisms of commutative rings goes back to [6, Chap. I, §3, Proposition 9] (see also [3, Lemma 2]):

Lemma 3.2.5. *A flat ring homomorphism $\varphi : R \to S$ of commutative rings is faithfully flat, if and only if φ – viewed as an R-homomorphism – is a pure monomorphism.*

Next, we recall the classic notions of ascent and descent, cf. [22, 10.82] or [24].

Definition 3.2.6. Let $\mathfrak{P}$ be a property of modules, and $\mathfrak{R}$ a class of commutative rings.

(1) $\mathfrak{P}$ is said to *ascend* along flat morphisms in $\mathfrak{R}$, provided that for each flat ring homomorphism $\varphi : R \to S$, such that $R, S \in \mathfrak{R}$, and each $M \in \mathfrak{P}(\text{Mod-}R)$, also $M \otimes_R S \in \mathfrak{P}(\text{Mod-}S)$.

(2) $\mathfrak{P}$ is said to *descend* along faithfully flat morphisms in $\mathfrak{R}$, provided that for each faithfully flat ring homomorphism of commutative rings, $\varphi : R \to S$, such that $R \in \mathfrak{R}$ and S is a finite direct product of rings from $\mathfrak{R}$, and for each $M \in$ Mod-R, such that $M \otimes_R S \in \mathfrak{P}(\text{Mod-}S)$, also $M \in \mathfrak{P}(\text{Mod-}R)$.

(3) $\mathfrak{P}$ is an *ad-property* in $\mathfrak{R}$, provided that $\mathfrak{P}$ ascends along flat morphisms in $\mathfrak{R}$, descends along faithfully flat morphisms in $\mathfrak{R}$, and, moreover, $\mathfrak{P}$ is *compatible with finite ring direct products* in the following sense: if $R = \prod_{i<n} R_i$ is a finite ring direct product of rings with $R_i \in \mathfrak{R}$ for each $i < n$, and $(M_i \mid i < n)$ satisfy $M_i \in \mathfrak{P}(\text{Mod-}R_i)$ for each $i < n$, then $M = \prod_{i<n} M_i \in \mathfrak{P}(\text{Mod-}R)$.

In the case when $\mathfrak{R}$ is the class of all commutative rings, we will omit the attribute 'in $\mathfrak{R}$' and say simply that $\mathfrak{P}$ ascends, descends, and $\mathfrak{P}$ is an ad-property.

Let $\mathfrak{P}$ be a property of modules. If X is an affine scheme, i.e., $X = \text{Spec}(R)$ for a commutative ring R, then $\text{Qcoh}(X) = \text{Mod–}R$, so $\mathfrak{P}$ is at the same time a property of quasi-coherent sheaves on X. For general schemes X, one can extend $\mathfrak{P}$ to a property of quasi-coherent sheaves $\mathcal{M}$ on X algebraically, by requiring property $\mathfrak{P}$ to hold for each module of sections of $\mathcal{M}$:

Definition 3.2.7. Let $\mathfrak{P}$ be a property of R-modules, X a scheme, and $(\mathcal{R}(u) \mid u \subseteq X, u \text{ open affine })$ be its structure sheaf. A quasi-coherent sheaf $\mathcal{M}$ on X is a *locally $\mathfrak{P}$-quasi-coherent sheaf on X* in the case when for each open affine set u of X, the $\mathcal{R}(u)$-module of sections $\mathcal{M}(u)$ satisfies $\mathfrak{P}$. That is, $\mathcal{M}(u) \in \mathfrak{P}(\mathcal{R}(u))$.

If $\mathfrak{P}$ is the property of being a projective module, then the locally $\mathfrak{P}$-quasi-coherent sheaves are the (infinite dimensional) vector bundles, see [10]. When $\mathfrak{P}$ denotes the property of being a flat Mittag-Leffler module (a restricted flat Mittag-Leffler module) then by [13], the locally $\mathfrak{P}$-quasi-coherent sheaves are called *Drinfeld vector bundles* (*restricted Drinfeld vector bundles*). Extending this notation to $n \geq 0$, we will call a quasi-coherent sheaf $\mathcal{M}$ an *n-Drinfeld vector bundle* in case it is a locally $\mathfrak{P}_n$-quasi-coherent sheaf where $\mathfrak{P}_n$ is the property of being a flat n-Mittag-Leffler module. Thus, 0-Drinfeld vector bundles are just the Drinfeld vector bundles from [13].

A basic question concerning the various algebraic notions of locally $\mathfrak{P}$-quasi-coherent sheaves defined above is whether these notions are also geometric, independent on a particular choice of affine coordinates on X, that is, whether the notions are Zariski local:

Definition 3.2.8. Let $\mathfrak{R}$ be a class of commutative rings, and $\mathfrak{C}$ be the class of all locally $\mathfrak{R}$-schemes.

The notion of a locally $\mathfrak{P}$-quasi-coherent sheaf is *Zariski local on* $\mathfrak{C}$ provided that for each $X \in \mathfrak{C}$, each open affine covering $X = \bigcup_{v \in V} v$ of X, and each quasi-coherent sheaf $\mathcal{M}$ on X, the following implication holds true: if $\mathcal{M}(v) \in \mathfrak{P}(\mathcal{R}(v))$ for all $v \in V$, then $\mathcal{M}$ is locally $\mathfrak{P}$-quasi-coherent.

ad-properties of modules are important, because they guarantee Zariski locality:

Lemma 3.2.9. *Let $\mathfrak{R}$ be a class of commutative rings. Let $\mathfrak{P}$ be an ad-property in $\mathfrak{R}$. Then the notion of a locally $\mathfrak{P}$-quasi-coherent sheaf is Zariski local on the class of all locally $\mathfrak{R}$-schemes.*

Proof. This is proved via the Affine Communication Lemma [30, 5.3.2], see [22, 27.21.2] or [14, Lemma 3.5]. □

It is well-known that the properties of being a projective, flat, flat Mittag–Leffler, and restricted flat Mittag–Leffler module, are ad-properties in the class of all commutative rings. Thus the corresponding notions of an (infinite dimensional) vector bundle, flat quasi-coherent sheaf, Drinfeld vector bundle, and restricted Drinfeld vector bundle, are Zariski local on the class of all schemes (see [25, Seconde partie], [24, §§8-9], and [14]). Further instances of ad-properties, related to tilting and silting, have recently been introduced in [7] and [21].

Our goal here is to investigate the ascent and descent for flat relative Mittag-Leffler modules, i.e., the flat $\mathcal{Q}$-Mittag-Leffler modules where $\mathcal{Q}$ is a subclass of R–Mod. Then we will apply the results obtained to proving Zariski locality for the corresponding notions of quasi-coherent sheaves.

For further unexplained terminology, we refer to [12] and [15].

3.3 The algebraic background of ascent and descent for flat relative Mittag-Leffler modules

First, we recall some connections between the Mittag-Leffler property and stationarity.

Definition 3.3.1. Let R be an arbitrary ring and B be a module.

(1) Let $(I, \leq)$ be an upper directed poset. A direct system $(M_i, f_{ji} \mid i \leq j \in I)$ of modules is said to be B-*stationary* provided that the induced inverse system
$$(\mathrm{Hom}_R(M_i, B), \mathrm{Hom}_R(f_{ji}, B) \mid i \leq j \in I)$$
satisfies the *Mittag-Leffler condition*, that is, for each $i \in I$ there exists $i \leq j \in I$ such that $\mathrm{Im}\ \mathrm{Hom}_R(f_{ki}, B) = \mathrm{Im}\ \mathrm{Hom}_R(f_{ji}, B)$ for all $j \leq k \in I$.

(2) A module M is said to be B-*stationary* if there exists a B-stationary direct system of finitely presented modules $(M_i, f_{ji} \mid i \leq j \in I)$ such that $M = \varinjlim M_i$.

(3) Let $\mathcal{B}$ be a class of right R-modules. We say that a direct system $(M_i, f_{ji} \mid i \leq j \in I)$, or a right R-module M, is $\mathcal{B}$-*stationary*, if it is B-stationary for all $B \in \mathcal{B}$.

Recall that a class of modules is said to be *definable* provided that it is closed under direct limits, direct products and pure submodules. For each class of modules $\mathcal{Q}$ there is the least definable class of modules containing $\mathcal{Q}$, called the *definable closure* of $\mathcal{Q}$ and denoted by $\mathrm{Def}\,\mathcal{Q}$. It is obtained by closing $\mathcal{Q}$ first by direct products, then direct limits, and finally by pure submodules, cf. [17, Lemma 2.9 and Corollary 2.10]. Note that each definable class is also closed under direct sums, pure extensions, and pure-epimorphic images (see e.g. [15, Lemma 6.9]).

There is a duality between definable classes of left and right R-modules: given a definable class $\mathcal{Q}$ of left (right) R-modules, the *dual definable class* $\mathcal{Q}^\vee$ of $\mathcal{Q}$ is the least definable class of right (left) R-modules containing the character modules $Q^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of all modules $M \in \mathcal{Q}$. Then $\mathcal{Q} = (\mathcal{Q}^\vee)^\vee$ for any definable class of left (right) modules $\mathcal{Q}$, see e.g. [27, §2.5].

FP_2 modules are important sources of mutually dual definable classes of left and right modules:

Example 3.3.2. Let $\mathcal{S}$ be a class of FP_2 modules. Then $\mathcal{S}^\perp$ is a definable class in Mod–R (see [15, Example 6.10]), and $\mathcal{S}^\intercal$ is a definable class of left R-modules. Indeed, $\mathcal{S}^\intercal$ is always closed under direct limits and pure submodules, and since $\mathcal{S}$ consists of FP_2 modules, $\mathcal{S}^\intercal$ is also closed under products (cf. [12, Theorem 3.2.26] and [8, §VIII.5]). Since $M^+ \in \mathcal{S}^\intercal$ for each $M \in S^\perp$, and $N^+ \in \mathcal{S}^\perp$ for each $N \in \mathcal{S}^\intercal$ by [15, Lemma 2.16(b) and (d)], the definable classes $\mathcal{S}^\perp$ and $\mathcal{S}^\intercal$ are mutually dual.

The classes of left R-modules $\mathcal{Q}$ of the form $\mathcal{Q} = \mathcal{S}^\intercal$ for a class $\mathcal{S}$ consisting of FP_2 modules will be called *of finite type*.

For example, when R is a right coherent ring and $\mathcal{S}$ the class of all finitely presented modules, then the class $\mathcal{S}^\perp$ of all absolutely pure modules is definable in Mod–R, and its dual definable class of all flat left R-modules, $\mathcal{S}^{\intercal}$, is of finite type.

Proposition 3.3.3. *[17, Proposition 1.7 and Theorem 2.11] Let R be a ring. Let $\mathcal{Q}$ be a definable class of left R-modules and $\mathcal{B} = \mathcal{Q}^{\vee}$ be its dual definable class. Let M be a right R-module. Then the following conditions are equivalent:*

(1) M is $\mathcal{Q}$-Mittag-Leffler.

(2) M is Q-Mittag-Leffler for all $Q \in \mathcal{Q}$.

(3) M is Q^{+}-stationary for all $Q \in \mathcal{Q}$.

(4) M is $\mathcal{B}$-stationary.

While studying flat $\mathcal{Q}$-Mittag-Leffler modules, one can actually restrict to definable classes of modules $\mathcal{Q}$:

Proposition 3.3.4. *[17, Corollary 2.10] Let $\mathcal{Q}$ be a class of left R-modules. Let M be a $\mathcal{Q}$-Mittag-Leffler module. Then M is also* Def $\mathcal{Q}$*-Mittag-Leffler.*

Now we will turn to the ascent for flat relative Mittag-Leffler modules, so we will again restrict ourselves to commutative rings.

Lemma 3.3.5. *Let $\varphi : R \to S$ be a flat homomorphism of commutative rings and $\mathcal{Q}$ be any class of modules. If M is a flat $\mathcal{Q}$-Mittag-Leffler module, then $M \otimes_R S$ is a flat $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module.*

Proof. Since M is a flat module, the functor $(M \otimes_R S) \otimes_S - : \text{Mod–}S \to \text{Mod–}\mathbb{Z}$ is a composition of two exact functors

$$(M \otimes_R S) \otimes_S - = (M \otimes_R -)(S \otimes_S -).$$

Thus $M \otimes_R S$ is a flat S-module.
Assume that M is a $\mathcal{Q}$-Mittag-Leffler module and let $(Q_i \mid i \in I)$ be a family of elements of $\mathcal{Q}$. First, note that $\mathcal{Q} \otimes_R S \subseteq \text{Def}\,(\mathcal{Q})$ as classes of modules. Indeed, since S is a flat module, we can write it as a direct limit of finitely generated free modules, say $S = \varinjlim_{\alpha} R^{n_\alpha}$. Therefore, $\mathcal{Q} \otimes_R \varinjlim_{\alpha} R^{n_\alpha} \cong \varinjlim_{\alpha} \mathcal{Q}^{n_\alpha} \in \text{Def}\,(\mathcal{Q})$. By our assumption on M and by Proposition 3.3.4, we infer that the canonical map $\psi_M : M \otimes_R \prod_{i\in I}(Q_i \otimes_R S) \to \prod_{i \in I}(M \otimes_R Q_i \otimes_R S)$ is monic.

We have the following commutative diagram whose horizontal maps are isomorphisms:

$$\begin{array}{ccc}
(M \otimes_R S) \otimes_S \prod_{i\in I}(Q_i \otimes_R S) & \xrightarrow{\cong} & M \otimes_R \prod_{i\in I}(Q_i \otimes_R S) \\
{\scriptstyle \psi_{M\otimes_R S}}\downarrow & & {\scriptstyle \psi_M}\downarrow \\
\prod_{i\in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S) & \xrightarrow{\cong} & \prod_{i\in I} M \otimes_R (Q_i \otimes_R S).
\end{array}$$

Here, the left vertical map $\psi_{M\otimes_R S}$ is the canonical morphism $\psi_{M\otimes_R S} : (M \otimes_R S) \otimes_S \prod_{i\in I}(Q_i \otimes_R S) \to \prod_{i\in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S)$. Thus $\psi_{M\otimes_R S}$ is monic. This proves that $M \otimes_R S$ is a $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module. □

The descent of flatness is well-known, we include a proof here for the sake of completeness.

Lemma 3.3.6. *Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings, and let M be a module such that the S-module $M \otimes_R S$ is flat. Then M is a flat module.*

Proof. First, since S is a flat module, also $M \otimes_R S$, viewed as an R-module, is flat. Indeed, the functor $(M \otimes_R S) \otimes_R -$ is a composition of two exact functors as follows: $M \otimes_R (S \otimes_S S) \otimes_R - = ((M \otimes_R S) \otimes_S -)(S \otimes_R -)$. So for each short exact sequence $\mathcal{C}$ of modules, $\mathcal{C} \otimes_R (M \otimes_R S)$ is a short exact sequence of S-modules. Hence, by faithful flatness of φ, $\mathcal{C} \otimes_R M$ is exact in Mod-R, whence M is a flat module. □

Recently, a short proof of the descent of the (absolute) flat Mittag-Leffler property along all pure (and hence all faithfully flat) ring homomorphisms was presented in [3, Lemma 5]. We include this short proof here as it works also in our relative setting. (We refer to [18] for a broader context and further applications.)

Lemma 3.3.7. *Let $\varphi : R \to S$ be a pure monomorphism of commutative rings. Let $\mathcal{Q}$ be a class of modules. Let M be a flat module such that $M \otimes_R S$ is a $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module. Then M is a $\mathcal{Q}$-Mittag-Leffler module.*

Proof. Let $(Q_i \mid i \in I)$ be a family consisting of modules from $\mathcal{Q}$. Since φ is pure, the canonical morphism $g_i : Q_i \cong Q_i \otimes_R R \to Q_i \otimes_R S$ is monic for each $i \in I$, and so is $g = \prod_{i \in I} g_i : \prod_{i \in I} Q_i \to \prod_{i \in I}(Q_i \otimes_R S)$.

Let M be a flat module such that $M \otimes_R S$ is a $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module. Since M is flat, the morphism $M \otimes_R g : M \otimes_R \prod_{i \in I} Q_i \to M \otimes_R \prod_{i \in I}(Q_i \otimes_R S)$ is monic. Moreover, we have the canonical isomorphism $\psi : M \otimes_R \prod_{i \in I}(Q_i \otimes_R S) \cong M \otimes_R (S \otimes_S \prod_{i \in I}(Q_i \otimes_R S)) \cong (M \otimes_R S) \otimes_S \prod_{i \in I}(Q_i \otimes_R S)$. Since $M \otimes_R S$ is a $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module, the canonical morphism $h : (M \otimes_R S) \otimes_S \prod_{i \in I}(Q_i \otimes_R S) \to \prod_{i \in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S)$ is monic. Thus the composite morphism $k = h\psi(M \otimes_R g)$ is monic.

Notice that $k(m \otimes_R (q_i)_{i \in I}) = ((m \otimes_R 1) \otimes_S (q_i \otimes_R 1))_{i \in I}$, so k can also be expressed as the composition of another triple of canonical morphisms: $k = \psi' g' h'$, where $h' : M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I}(M \otimes_R Q_i)$, g' is the monomorphism $\prod_{i \in I}(M \otimes_R Q_i) \to \prod_{i \in I}(M \otimes_R Q_i \otimes_R S)$, and ψ' the isomorphism $\prod_{i \in I}(M \otimes_R Q_i \otimes_R S) \to \prod_{i \in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S)$. Since k is monic, so is h'. The latter says that M is a $\mathcal{Q}$-Mittag-Leffler module. □

Now, we can easily prove the descent for flat relative Mittag-Leffler modules:

Theorem 3.3.8. *Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings. Let $\mathcal{Q}$ be a class of modules. Let M be a module such that $M \otimes_R S$ is a flat $(\mathcal{Q} \otimes_R S)$-Mittag-Leffler S-module. Then M is a flat $\mathcal{Q}$-Mittag-Leffler module.*

Proof. By Lemma 3.3.6, we can assume that M is a flat module. By Lemma 3.2.5, φ is a pure monomorphism, so Lemma 3.3.7 applies and shows that M is a $\mathcal{Q}$-Mittag-Leffler module. □

It is worth noting that for countably presented flat modules, Mittag-Leffler conditions relative to definable classes of modules can be expressed in terms of vanishing of the Ext functor, following [17, §1].

Lemma 3.3.9. *Let R be any ring. Let M be a countably presented flat module, $\mathcal{Q}$ be a definable class of left R-modules, and $\mathcal{B} = \mathcal{Q}^\vee$. Then M is $\mathcal{Q}$-Mittag-Leffler, if and only if $M \in {}^\perp\mathcal{B}$.*

Proof. If M is $\mathcal{Q}$-Mittag-Leffler, then M is $\mathcal{B}$-stationary by Proposition 3.3.3. Since M is a countable direct limit of finitely presented free modules and $\mathcal{B}$ is closed under countable direct sums, we infer from [15, Corollary 2.23] and [17, Lemma 1.11(3)] that $\mathrm{Ext}^1_R(M, B) = 0$ for each $B \in \mathcal{B}$. The converse implication follows by [17, Lemma 1.11(1)] and Proposition 3.3.3. □

Remark 13. Lemma 3.3.9 does not extend to uncountably presented modules in general. Just consider any non-right perfect ring R and let $\mathcal{Q} = R$–Mod. Then $\mathcal{B} = \mathcal{Q}^\vee =$ Mod–R, so ${}^\perp\mathcal{B} = \mathcal{P}_0 \subsetneq \mathcal{FM}$ (though, as correctly claimed by Lemma 3.3.9, the countably presented modules in $\mathcal{P}_0$ and $\mathcal{FM}$ are the same).

Theorem 3.3.8, Proposition 3.3.4, and Lemmas 3.3.5 and 3.3.9 yield the following

Corollary 3.3.10. *Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings. Let $\mathcal{Q}$ be a definable class of modules and $\mathcal{B} = \mathcal{Q}^\vee$. Let $\mathcal{Q}'$ denote the least definable class of S-modules containing $\mathcal{Q} \otimes_R S$, and $\mathcal{B}'$ its dual definable class.*

Let M be a countably presented flat module. Then $M \in {}^\perp\mathcal{B}$, if and only if $M \otimes_R S \in {}^\perp\mathcal{B}'$.

In the particular setting of definable classes arising from kernels of Tor functors (such as the definable classes of finite type from Example 3.3.2), we have the following relation between definable closures:

Lemma 3.3.11. *Let $\varphi : R \to S$ is a flat homomorphism of commutative rings and $\mathcal{C}$ be a class of R-modules. Then* $\mathrm{Def}\,(\mathcal{C} \otimes_R S)^\intercal = \mathrm{Def}\,(\mathcal{C}^\intercal \otimes_R S)$.

In particular, if $\mathcal{C}$ consists of FP_2 modules, then $\mathrm{Def}\,(\mathcal{C}^\intercal \otimes_R S) = (\mathcal{C} \otimes_R S)^\intercal$.

Proof. First, $(\mathcal{C}^\intercal) \otimes_R S \subseteq (\mathcal{C} \otimes_R S)^\intercal$ by Lemma 3.2.2(1), whence $\mathrm{Def}\,(\mathcal{C}^\intercal \otimes_R S) \subseteq \mathrm{Def}\,(\mathcal{C} \otimes_R S)^\intercal$.

For the opposite inclusion, note that by Lemma 3.2.2(2), $(\mathcal{C} \otimes_R S)^\intercal$ is the class of all S-modules N satisfying the following condition: N, viewed as an R-module, is an element of $\mathcal{C}^\intercal$. Then again $N \otimes_R S \in (\mathcal{C} \otimes_R S)^\intercal$ by Lemma 3.2.2(1). Since the canonical homomorphism $f : n \mapsto n \otimes 1$ from N to $N \otimes_R S$ is an S-homomorphism, and the S-homomorphism $g : N \otimes_R S \to N$ defined by $g : n \otimes s \mapsto n.s$ satisfies $gf = 1_N$, we infer that N is isomorphic to a direct summand in $N \otimes_R S$ as an S-module. Thus $(\mathcal{C} \otimes_R S)^\intercal$ consists of S-modules isomorphic to direct summands of the modules from $\mathcal{C}^\intercal \otimes_R S$, whence $(\mathcal{C} \otimes_R S)^\intercal \subseteq \mathrm{Def}\,(\mathcal{C}^\intercal \otimes_R S)$, proving the opposite inclusion.

If $\mathcal{C}$ consists of FP_2 modules, then also $\mathcal{C} \otimes_R S$ consists of FP_2 S-modules, whence $(\mathcal{C} \otimes_R S)^\intercal$ is a definable class by Example 3.3.2. □

3.4 Zariski locality of quasi-coherent sheaves associated with flat relative Mittag-Leffler modules

In this section, we will apply the results of Section 3.3 to prove Zariski locality of flat relative $\mathcal{Q}$-Mittag-Leffler modules in various particular settings.

We start with a direct general application to quasi-coherent sheaves associated with f-projective modules. Recall that a module M is *f-projective* if M is flat and $\{R\}$-Mittag-Leffler, or equivalently, M is a flat $\mathcal{Q}$-Mittag-Leffler module where $\mathcal{Q}$ is the class of all flat left R-modules, [16] (see also Proposition 3.3.4 and [5, §3]). In accordance with our Definition 3.2.7, we call a quasi-coherent sheaf $\mathcal{M}$ on a scheme X *locally f-projective* in case for each open affine set u in X, the $\mathcal{R}(u)$-module of sections $\mathcal{M}(u)$ is an f-projective $\mathcal{R}(u)$-module.

Theorem 3.4.1. *The notion of a locally f-projective quasi-coherent sheaf is Zariski local on the class of all schemes.*

Proof. By Lemma 3.2.9, it suffices to prove that the property of being an f-projective module is an ad-property in the class of all commutative rings. However, its ascent and descent follows for $\mathcal{Q} = \{R\}$ immediately by Lemma 3.3.5 and Theorem 3.3.8, respectively. The compatibility with finite ring direct products is obvious (cf. Definition 3.2.6(3)). □

For the rest of this section, R will denote a commutative ring, $\mathcal{C}_R$ a class of modules, and $\mathcal{Q}_R$ the definable class $\mathcal{Q}_R = \operatorname{Def}\mathcal{C}_R^{\intercal}$. In particular, $\mathcal{Q}_R = \mathcal{C}_R^{\intercal}$ in case $\mathcal{C}_R$ consists of FP_2 modules.

The relevant property $\mathfrak{P}$ of modules is defined as follows: *if M is a module, then $M \in \mathfrak{P}(\text{Mod–}R)$, iff M is a flat $\mathcal{Q}_R$-Mittag-Leffler module.*

In order to prove locality of the induced notions of quasi-coherent sheaves in this setting, we will need compatibility of the properties $\mathfrak{P}$ for commutative rings R and S connected by flat, and faithfully flat, morphisms. More precisely, we will require the following compatibility conditions (C1), (C2) and (C3):

Definition 3.4.2. Let $\mathfrak{R}$ be a class of commutative rings.

(C1) For each flat ring homomorphism $\varphi : R \to S$ with $R, S \in \mathfrak{R}$, $\mathcal{C}_R \otimes_R S \subseteq \mathcal{C}_S$.

(C2) For each faithfully flat ring homomorphism $\varphi : R \to S$ where $R \in \mathfrak{R}$ and S is a finite direct product of rings in $\mathfrak{R}$, $\operatorname{Def}\mathcal{C}_S^{\intercal} = \operatorname{Def}(\mathcal{C}_R \otimes_R S)^{\intercal}$.

(C3) If $S = \prod_{i<n} R_i$ where $R_i \in \mathfrak{R}$ for each $i < n$, then $\mathcal{C}_S = \prod_{i<n} \mathcal{C}_{R_i}$.

Notice that (C1) implies the inclusion $\mathcal{C}_S^{\intercal} \subseteq (\mathcal{C}_R \otimes_R S)^{\intercal}$, and hence $\operatorname{Def}\mathcal{C}_S^{\intercal} \subseteq \operatorname{Def}(\mathcal{C}_R \otimes_R S)^{\intercal}$.

Lemma 3.4.3. *Let $\mathfrak{R}$ be a class of commutative rings such that condition (C1) holds. Then the property $\mathfrak{P}$ ascends along flat morphisms in $\mathfrak{R}$.*

Proof. Let $\varphi : R \to S$ be a flat ring homomorphism with $R, S \in \mathfrak{R}$ and M be a flat $\mathcal{Q}_R$-Mittag-Leffler module. By Lemma 3.3.5, $M \otimes_R S$ is a flat $(\mathcal{Q}_R \otimes_R S)$-Mittag-Leffler S-module, and hence a flat $\operatorname{Def}(\mathcal{Q}_R \otimes_R S)$-Mittag-Leffler S-module by Proposition 3.3.4. Condition (C1) and Lemma 3.3.11 give

$$\mathcal{Q}_S = \operatorname{Def} \mathcal{C}_S^{\intercal} \subseteq \operatorname{Def}(\mathcal{C}_R \otimes_R S)^{\intercal} = \operatorname{Def}(\mathcal{Q}_R \otimes_R S).$$

Thus, $M \otimes_R S$ is a flat $\mathcal{Q}_S$-Mittag-Leffler S-module. □

Lemma 3.4.4. *Let $\mathfrak{R}$ be a class of commutative rings such that condition (C2) holds. Then the property $\mathfrak{P}$ descends along faithfully flat morphisms in $\mathfrak{R}$.*

Proof. Let $\varphi : R \to S$ be a faithfully flat ring homomorphism, where $R \in \mathfrak{R}$ and S is a finite direct product of rings in $\mathfrak{R}$. Let M be a module such that $M \otimes_R S$ is a flat $\mathcal{Q}_S$-Mittag-Leffler S-module. Condition (C2) and Lemma 3.3.11 yield

$$\mathcal{Q}_S = \operatorname{Def} \mathcal{C}_S^{\intercal} = \operatorname{Def}(\mathcal{C}_R \otimes_R S)^{\intercal} = \operatorname{Def}(\mathcal{Q}_R \otimes_R S),$$

so $M \otimes_R S$ is a flat $(\mathcal{Q}_R \otimes_R S)$-Mittag-Leffler S-module. By Theorem 3.3.8, M is a flat $\mathcal{Q}_R$-Mittag-Leffler module. □

Thus, we obtain

Theorem 3.4.5. *Let $\mathfrak{R}$ be a class of commutative rings such that conditions (C1), (C2) and (C3) hold. Then $\mathfrak{P}$ is an ad-property in $\mathfrak{R}$, whence the notion of a locally $\mathfrak{P}$-quasi-coherent sheaf is Zariski local on the class of all locally $\mathfrak{R}$-schemes.*

Proof. By condition (C3), $\mathfrak{P}$ is compatible with finite ring direct products, so the ad-property of $\mathfrak{P}$ follows by Lemmas 3.4.3 and 3.4.4. The final claim follows by Lemma 3.2.9. □

We finish this section by noting several applications of Theorem 3.4.5:

3.4.1 Applications

1. Let $\mathcal{R}$ be the class of all commutative rings and $\mathcal{C}_R = \{0\}$, so $\mathcal{Q}_R = R$–Mod. In this case, Theorem 3.4.5 yields the Zariski locality of the notion of a Drinfeld vector bundle (= locally flat Mittag-Leffler quasi-coherent sheaf) proved in [14].

2. Let $\mathfrak{R}$ be the class of all commutative rings and $\mathcal{C}_R$ the class of all finitely presented modules. Then $\mathcal{Q}_R = \operatorname{Def} \mathcal{C}_R^{\intercal} = \operatorname{Def} \mathcal{F}_0$. By Proposition 3.3.4, a module M has property $\mathfrak{P}$, iff M is f-projective. Conditions (C1) and (C3) clearly hold true.

Condition (C2) holds even in the stronger form of $\mathcal{C}_S^{\intercal} = (\mathcal{C}_R \otimes_R S)^{\intercal}$ whenever $\varphi : R \to S$ is a faithfully flat homomorphism of commutative rings. Indeed, $\mathcal{C}_S^{\intercal}$ is the class of all flat S-modules. Let $M \in (\mathcal{C}_R \otimes_R S)^{\intercal}$. By Lemma 3.2.2(2), $\operatorname{Tor}_1^R(\mathcal{C}_R, M) = 0$, whence M is a flat R-module. Then $M \otimes_R S$ is a flat S-module, by (the proof of) Lemma 3.3.5. However, the S-module M is isomorphic to a direct summand in $M \otimes_R S$ (cf. the proof of Lemma 3.3.11), whence M is a flat S-module. This proves the inclusion $(\mathcal{C}_R \otimes_R S)^{\intercal} \subseteq \mathcal{C}_S^{\intercal}$; the other inclusion is a consequence of condition (C1).

Thus, Theorem 3.4.1 is just a particular instance of Theorem 3.4.5 for $\mathcal{C}_R =$ the class of all finitely presented modules.

3. A more involved application of Theorem 3.4.5 concerns the case when $\mathcal{C}_R = \mathcal{F}_n$ for some $n \geq 1$. In this case, we will verify conditions (C1) – (C3) for $\mathfrak{R} =$ the class of all noetherian rings.

Condition (C1) holds since $\mathcal{C}_R \otimes_R S \subseteq \mathcal{C}_S$ when S is a flat module, and (C3) is obvious. As in Application 2, it only suffices to prove the inclusion $(\mathcal{C}_R \otimes_R S)^\intercal \subseteq \mathcal{C}_S^\intercal$ for each faithfully flat homomorphism of commutative noetherian rings $\varphi : R \to S$.

Recall that for an S-module M, $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denotes the S-module of characters of M, and for a class of S-modules $\mathcal{E}$, $\mathcal{E}^+ = \{M^+ \mid M \in \mathcal{E}\}$. We claim that $\mathcal{F}_n^+ = \mathcal{I}_n \cap (\text{Mod–}S)^+$. Since character modules of flat modules are injective, the $\subseteq$ inclusion holds. Conversely, let $N = M^+ \in \mathcal{I}_n$. Since S is noetherian, character modules of injective modules are flat (e.g., by [15, Lemma 2.16(d)]), so $N^+ = M^{++} \in \mathcal{F}_n$. As the class $\mathcal{F}_n$ is closed under pure submodules and the embedding $M \hookrightarrow M^{++}$ is pure, $M \in \mathcal{F}_n$ and the claim is proved. Using [15, Lemma 2.16(b)] and the fact that the pure embedding $M \hookrightarrow M^{++}$ splits for any pure-injective module M, we get $(\mathcal{F}_n)^\intercal = {}^\perp(\mathcal{F}_n^+) = {}^\perp(\mathcal{I}_n \cap \mathcal{PI})$.

Let $\mathcal{S}_{n,R}$ denote the class of all finitely generated modules that appear as nth syzygies in some projective resolution, $\mathcal{P}$, of a finitely generated module such that $\mathcal{P}$ consists of finitely generated modules. Then $R \in \mathcal{S}_{n,R}$, and since R is noetherian, $\mathcal{S}_{n,R}^\perp = \mathcal{I}_n$ by the Baer Test of Injectivity and by dimension shifting. Let $\mathcal{D}_{n,R}$ denote the class of all modules M that are isomorphic to direct summands of finite extensions of the modules from $\mathcal{S}_{n,R}$. Then the class $\mathcal{D}_{n,R}$ is closed under extensions, direct summands, and contains R. Moreover, $\mathcal{D}_{n,R}^\perp = \mathcal{I}_n$. By Lemma 3.2.1, $\varinjlim \mathcal{D}_{n,R} = {}^\perp(\mathcal{I}_n \cap \mathcal{PI})$.

Finally, let $M \in (\mathcal{C}_R \otimes_R S)^\intercal$. Then Lemma 3.2.2(2) gives $\mathrm{Tor}_1^R(\mathcal{C}_R, M) = 0$. By the above, M, viewed as an R-module, is an element of $\varinjlim \mathcal{D}_{n,R}$. Since S is a flat module, $\mathcal{S}_{n,R} \otimes_R S \subseteq \mathcal{S}_{n,S}$, whence also $\mathcal{D}_{n,R} \otimes_R S \subseteq \mathcal{D}_{n,S}$. Moreover, the tensor product commutes with direct limits, so $M \otimes_R S \in \varinjlim \mathcal{D}_{n,S} = \mathcal{C}_S^\intercal$. As M is isomorphic to a direct summand in $M \otimes_R S$ as an S-module, also $M \in \mathcal{C}_S^\intercal$, and the inclusion $(\mathcal{C}_R \otimes_R S)^\intercal \subseteq \mathcal{C}_S^\intercal$ is proved.

Recall that if $n \geq 0$ and $\mathcal{Q}_n = (\mathcal{F}_n)^\intercal$, then the flat $\mathcal{Q}_n$-Mittag-Leffler modules are called flat n-Mittag-Leffler, and the corresponding quasi-coherent sheaves are the n-Drinfeld vector bundles. Thus, we have the following consequence of Theorem 3.4.5 for $\mathcal{C}_R = \mathcal{F}_n$:

Theorem 3.4.6. *For each $n \geq 1$, the notion of an n-Drinfeld vector bundle is Zariski local on the class of all locally noetherian schemes.*

Remark 14. If R is a non-right perfect ring (e.g., a commutative noetherian ring of Krull dimension ≥ 1), then there is a gap between the classes $\mathcal{FM}$ of all flat Mittag-Leffler modules and $\mathcal{F}$ of all flat modules. In fact, for each class $\mathcal{Q}$ of left R-modules we have $\mathcal{FM} \subseteq \mathcal{D}_\mathcal{Q} \subseteq \mathcal{F}$. Since $\mathcal{D}_\mathcal{Q} = \mathcal{D}_{\mathrm{Def}(\mathcal{Q})}$ by Proposition 3.3.4 and there is only a set of definable classes of modules, there is also only a set of such intermediate classes $\mathcal{D}_\mathcal{Q}$ between $\mathcal{FM}$ and $\mathcal{F}$ (see also [5, Theorem 3.5(i)]).

Of course, the variety of classes of modules between $\mathcal{FM}$ and $\mathcal{F}$ translates directly into the same variety of classes of locally $\mathfrak{P}$-quasi-coherent sheaves in the class of all flat quasi-coherent sheaves on the affine scheme $X = \mathrm{Spec}(R)$, where

R is any commutative non-perfect ring (since in this case, $\mathrm{Qcoh}(X)$ is equivalent to Mod–R). Moreover, all these classes contain a flat generator, as they contain all vector bundles on X.

However, the picture for non-affine schemes may be different, depending on further properties of the schemes. For example, by [29], if X is a quasi-compact and quasi-separated scheme, then $\mathrm{Qcoh}(X)$ contains a flat generator, if and only if X is semiseparated (i.e., the intersection of any two open affine sets is affine).

Bibliography of Chapter 3

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Dualizations of approximations, $\aleph_1$-projectivity, and Vopěnka's Principles

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Abstract. The approximation classes of modules that arise as components of cotorsion pairs are tied up by Salce's duality. Here we consider general approximation classes of modules and investigate possibilities of dualization in dependence on closure properties of these classes. While some proofs are easily dualized, other dualizations require large cardinal principles, and some fail in ZFC, with counterexamples provided by classes of $\aleph_1$-projective modules over non-perfect rings. For example, we show that Vopěnka's Principle implies that each covering class of modules closed under homomorphic images is of the form Gen (M) for a module M, and that the latter property restricted to classes generated by $\aleph_1$-free abelian groups implies Weak Vopěnka's Principle.

Chapter 4

Dualizations of approximations, $\aleph_1$-projectivity, and Vopěnka's Principles

4.1 Introduction

Cotorsion pairs were introduced by Salce in [21] as analogs of the well-known torsion pairs where the Hom functor was replaced by Ext. A formal replacement was certainly not the main point: Salce proved the remarkable fact that though there is no duality available in the category Mod–R, for each cotorsion pair $(\mathcal{A}, \mathcal{B})$, the classes $\mathcal{A}$ and $\mathcal{B}$ are tied up by a duality: $\mathcal{A}$ is a special precovering class, if and only if $\mathcal{B}$ is a special preeveloping class, cf. [14, Salce's Lemma 5.20].

For general classes $\mathscr{C}$ of modules that do not necessarily fit in cotorsion pairs, one still has the formally dual notions of a $\mathscr{C}$-preenvelope (or a left $\mathscr{C}$-approximation) and a $\mathscr{C}$-precover (or a right $\mathscr{C}$-approximation). However, there is no general tool for dualization like Salce's Lemma at hand. In the present paper, we consider general approximation classes of modules and investigate if, and how, dualizations are possible assuming extra closure properties of these classes.

While some results can easily be dualized simply by employing dual arguments, other require the use of large cardinal principles. On the one hand, we prove that Vopěnka's Principle implies that each covering class of modules closed under homomorphic images is of the form Gen (M) for a module M (Proposition 4.3.7). On the other hand, we show that the latter property restricted to classes of abelian groups generated by $\aleph_1$-free groups implies Weak Vopěnka's Principle (Theorem 4.3.11).

In several cases, the class $\mathcal{FM}$ of all $\aleph_1$-projective modules (= flat Mittag-Leffler modules) over a non-right perfect ring R serves as a barrier for dualization in ZFC. While Weak Vopěnka's Principle is known to guarantee that each class of modules closed under direct products and direct summands is preenveloping (Lemma 4.3.1), the dual statement is not true in ZFC: by Example 4.3.4, $\mathcal{FM}$ is a class of modules closed under direct sums and direct summands which is not precovering. Also, in contrast with the claim of Proposition 4.3.7 mentioned above, if R is a Dedekind domain with a countable spectrum which is not a complete discrete valuation domain (e.g., when $R = \mathbb{Z}$), then $\mathcal{FM}$ is an enveloping class of modules closed under submodules, but $\mathcal{FM}$ is not of the form Cog (M)

for any module M (Example 4.3.6).

4.2 Preliminaries

For a ring R, we denote by Mod–R the class of all (right R-) modules, and by R–Mod the class of all left R-modules.

4.2.1 Approximations

A map $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathscr{C}$ is a *$\mathscr{C}$-preenvelope* of M, if the abelian group homomorphism $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathscr{C}$.
A $\mathscr{C}$-preenvelope $f \in \mathrm{Hom}_R(M, C)$ of M is called a *$\mathscr{C}$-envelope* of M, provided that f is *left minimal*, that is, provided $f = gf$ implies that g is an automorphism for each $g \in \mathrm{End}_R(\mathscr{C})$.
$\mathscr{C} \subseteq$ Mod–R is a *preenveloping class* (*enveloping class*) provided that each module has a $\mathscr{C}$-preenvelope ($\mathscr{C}$-envelope).

Dually, a map $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathscr{C}$ is a *$\mathscr{C}$-precover* of M, if the abelian group homomorphism $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathscr{C}$.
A $\mathscr{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ of M is called a *$\mathscr{C}$-cover* of M, provided that f is *right minimal*, that is, provided $fg = f$ implies that g is an automorphism for each $g \in \mathrm{End}_R(\mathscr{C})$.
$\mathscr{C} \subseteq$ Mod–R is a *precovering class* (*covering class*) provided that each module has a $\mathscr{C}$-precover ($\mathscr{C}$-cover).

Let $\mathscr{C}$ be a class of R-modules. We define

$$\mathscr{C}^{\perp} = \{N \in R\text{–Mod} \mid \mathrm{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathscr{C}\}$$

$$^{\perp}\mathscr{C} = \{N \in R\text{–Mod} \mid \mathrm{Ext}^1_R(N, C) = 0 \text{ for all } C \in \mathscr{C}\}$$

Let $\mathcal{A}, \mathcal{B} \subseteq$ Mod–R. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* [14, §5.2] (or a *cotorsion theory*, [21]), if $\mathcal{A} =^{\perp} \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

A module M is called *Enochs cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat modules F. We denote by $\mathcal{EC}$ the class of all Enochs cotorsion modules, and $\mathcal{F}_0$ the class of all flat modules. By [14, Lemma 5.17], $(\mathcal{F}_0, \mathcal{EC})$ forms a cotorsion pair, known as the *Enochs cotorsion pair*.

A $\mathscr{C}$-preenvelope f is called *special* if f is monic and its cokernel is an element of $^{\perp}\mathscr{C}$. A $\mathscr{C}$-precover g is called *special* if g is surjective and its kernel is an element of $\mathscr{C}^{\perp}$. A class $\mathscr{C}$ is *special preenveloping* (*special precovering*) in case each module has a special $\mathscr{C}$-preenvelope (special $\mathscr{C}$-precover).

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *complete* if $\mathcal{A}$ is a special precovering class. By Salce's Lemma mentioned in the Introduction, this is equivalent to $\mathcal{B}$ being a special preenveloping class.

Throughout our paper, we stick to the terminology introduced for modules by Enochs [10]. Notice that a different, though equivalent, terminology has been used in representation theory by the Auslander school (see e.g.[4]): $\mathscr{C}$-preenvelopes and $\mathscr{C}$-envelopes are called *left $\mathscr{C}$-approximations* and *minimal left $\mathscr{C}$-approximations*.

Dually, $\mathscr{C}$-precovers and $\mathscr{C}$-covers are called *right $\mathscr{C}$-approximations* and *minimal right $\mathscr{C}$-approximations*.

In category theory, a still different terminology has been used, cf. [1]: Preenveloping classes closed under direct summands are called *weakly reflective*, while precovering classes closed under direct summands are *weakly coreflective*.

It is easy to see that all enveloping (covering) classes of modules are closed under direct summands, so they are weakly reflective (weakly coreflective). An example of a precovering class that is not coreflective is provided by the class of all free modules over any ring possessing projective modules that are not free (such as a path algebra of a non-trivial quiver). Examples of preenveloping classes of modules that are not weakly reflective arise from pure-injective modules that are not dual modules:

Example 4.2.1. Let R be a ring, $\mathcal{D}$ be the class of all *dual modules* (= the class of all modules isomorphic to the character modules $N^+ = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ of all left R-modules N), and let $\mathcal{PI}$ be the class of all *pure-injective modules* (= modules injective w.r.t. all pure embeddings). By [14, Theorem 2.27], $\mathcal{PI}$ is the class of all direct summands of the modules from $\mathcal{D}$. Both $\mathcal{PI}$ and $\mathcal{D}$ are preenveloping classes of modules, since the canonical embedding of any module M into its double dual module M^{++} is pure, cf. [14, Corollary 2.21(b)]. However, $\mathcal{D} \subsetneq \mathcal{PI}$ in general:

For an example, let $R = \mathbb{Z}$. Let $\mathcal{E}$ denote the class of all torsion-free divisible groups, that is, the underlying groups of $\mathbb{Q}$-linear spaces. By [14, Corollary 2.18(a)], $\mathcal{E}^+ = \mathcal{D} \cap \mathcal{E}$. However, $\mathbb{Q} \in \mathcal{E} \setminus \mathcal{E}^+$, so $\mathbb{Q}$ is a pure-injective abelian group which is not dual. The point is that duals of non-zero torsion-free divisible groups are uncountable, as $\mathbb{Q}^+$ is uncountable. Indeed, $\mathbb{Q}^+$ fits in the short exact sequence $0 \to (\mathbb{Q}/\mathbb{Z})^+ \to \mathbb{Q}^+ \to \mathbb{Z}^+ \to 0$ where $(\mathbb{Q}/\mathbb{Z})^+ \cong \prod_{p \in \mathbb{P}} \mathbb{J}_p$ is an uncountable group. Here, $\mathbb{P}$ denotes the set of all prime integers and $\mathbb{J}_p$ the group of all p-adic integers, for $p \in \mathbb{P}$.

4.2.2 Modules

Let R be a ring, M a module, and $\mathscr{C}$ a class of modules. A family of submodules, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of M is called a *continuous chain* in M, provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \sigma$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.
A continuous chain $\mathcal{M}$ in M is a *$\mathscr{C}$-filtration* of M, provided that $M = M_\sigma$, and each of the modules $M_{\alpha+1}/M_\alpha$ ($\alpha < \sigma$) is isomorphic to an element of $\mathscr{C}$.
M is called *$\mathscr{C}$-filtered* (or a *transfinite extension* of modules in $\mathscr{C}$), provided that M possesses at least one $\mathscr{C}$-filtration. A class $\mathscr{C}$ is *closed under transfinite extensions* provided that $M \in \mathscr{C}$ for each $\mathscr{C}$-filtered module M.

Notice that each class of modules closed under transfinite extensions is closed under extensions and (arbitrary) direct sums.

Let $\mathscr{C}$ be a class of modules. A module M is said to be *generated* by $\mathscr{C}$ if there exists a set Λ, a family $(C_\lambda)_\Lambda$ of elements of $\mathscr{C}$ and an epimorphism $\bigoplus_{\lambda\in\Lambda} C_\lambda \to M$.

Dually, let $\mathscr{C}$ be a class of modules. A module M is said to be *cogenerated* by $\mathscr{C}$ if there exists a set Λ, a family $(C_\lambda)_\Lambda$ of elements of $\mathscr{C}$ and a monomorphism $M \to \prod_{\lambda\in\Lambda} C_\lambda$.

The class of all modules generated and cogenerated by $\mathscr{C}$ is denoted by $\mathrm{Gen}(\mathscr{C})$ and $\mathrm{Cog}(\mathscr{C})$, respectively. If $\mathscr{C}$ consists of a single module C, we say that C generates (cogenerates) M. We also use the notations $\mathrm{Gen}(C)$ and $\mathrm{Cog}(C)$ instead of $\mathrm{Gen}(\mathscr{C})$ and $\mathrm{Cog}(\mathscr{C})$.

We recall the following easy facts (see e.g. [24, 13.4 and 14.4])

Lemma 4.2.2. *(i) The class* $\mathrm{Gen}(\mathscr{C})$ *is closed under homomorphic images and direct sums, and it contains* $\mathscr{C}$*. Moreover if* $\mathcal{X}$ *is a subclass of* Mod–R *which contains* $\mathscr{C}$*, and is closed under epimorphic images and direct sums, then* $\mathrm{Gen}(\mathscr{C}) \subseteq \mathcal{X}$*.*

(ii) The class $\mathrm{Cog}(\mathscr{C})$ *is closed under submodules and direct products, and it contains* $\mathscr{C}$*. Moreover if* $\mathcal{X}$ *is a subclass of* Mod–R *which contains* $\mathscr{C}$*, and is closed under submodules and direct products, then* $\mathrm{Cog}(\mathscr{C}) \subseteq \mathcal{X}$*.*

Let M be a module. We will denote by $\mathrm{sum}\, M$ the class of all finite direct sums of copies of M, and by $\mathrm{add}\, M$ the class of all direct summands of all modules in $\mathrm{sum}\, M$.

For a class of modules $\mathscr{C}$, we denote $\mathrm{Tr}_N(\mathscr{C})$ the *trace* of $\mathscr{C}$ in N, that is, the sum of images of all homomorphisms from modules in $\mathscr{C}$ to N. Dually, $\mathrm{Rej}_N(\mathscr{C})$ denotes the *reject* of $\mathscr{C}$ in N, that is, the intersection of kernels of all homomorphisms from N to modules in $\mathscr{C}$, cf. [3, p.109].

Proposition 4.2.3. *[3, Proposition 8.12] Let* $\mathscr{C}$ *be a class of modules, and let* N *be a module. Then:*

(i) $\mathrm{Tr}_N(\mathscr{C})$ *is the unique largest submodule* L *of* N *generated by* $\mathscr{C}$*;*

(ii) $\mathrm{Rej}_N(\mathscr{C})$ *is the unique smallest submodule* U *of* N *such that* N/U *is cogenerated by* $\mathscr{C}$*.*

For a module M, $\sigma[M]$ denotes the class of all modules *subgenerated* by the module M, that is, the submodules of all modules generated by M. This class is closed under submodules and homomorphic images, and it is the smallest Grothendieck subcategory of Mod–R containing the module M, cf. [24, §15]. Dually, $\pi[M]$ will denote the class of all homomorphic images of modules cogenerated by M. Also this class is closed under homomorphic images and submodules.

For a class of modules $\mathscr{C}$, we will denote by $\varinjlim \mathscr{C}$ the class of all modules that are direct limits of directed systems consisting of modules from $\mathscr{C}$.

4.2.3 $\aleph_1$-projectivity

Let R be a ring and M be an R-module. We say that M is $\aleph_1$-*projective* in case there exists a set $\mathcal{S}$ consisting of countably generated projective submodules of M with the following properties: $0 \in \mathcal{S}$, any countable subset of M is contained in an element of $\mathcal{S}$, and $\mathcal{S}$ is closed under unions of well-ordered chains of countable length.

Notice that if R is a right hereditary ring, then a module is $\aleph_1$-projective, iff each of its countably generated submodules is projective. In particular, $\aleph_1$-projective abelian groups (i.e., the abelian groups all of whose countable subgroups are free) are called $\aleph_1$-*free*.

Let R be a ring. A module M is *Mittag-Leffler* provided that the canonical group homomorphism

$$\varphi : M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$$

defined by

$$\varphi(m \otimes_R (n_i)_{i \in I}) = (m \otimes_R n_i)_{i \in I}$$

is monic for each family $(N_i \mid i \in I)$ of left R-modules.
Let $M \in \text{Mod}–R$ and $\mathcal{Q} \subseteq R–\text{Mod}$. Then M is *$\mathcal{Q}$-Mittag-Leffler*, provided that the canonical morphism $M \otimes \prod_{i \in I} Q_i \to \prod_{i \in I} M \otimes_R Q_i$ is injective for any family $(Q_i \mid i \in I)$ consisting of elements of $\mathcal{Q}$. So a module M is Mittag-Leffler, iff it is $\mathcal{Q}$-Mittag-Leffler for $\mathcal{Q} = R–\text{Mod}$.

We will be interested in flat Mittag-Leffler modules, and more in general, flat $\mathcal{Q}$-Mittag-Leffler modules. Following [15], we will denote the class of all flat Mittag-Leffler modules by $\mathcal{FM}$, and the class of all flat $\mathcal{Q}$-Mittag-Leffler modules by $\mathcal{D}_{\mathcal{Q}}$.

The key relation between these notions goes back to [15, Corollary 2.14(i)]: If R is any ring and M any module, then M is $\aleph_1$-projective, if and only if M is flat Mittag-Leffler. In particular, abelian groups are flat Mittag-Leffler, iff they are $\aleph_1$-free.

4.2.4 Vopěnka's Principles

We will employ two large cardinal principles. The first one is due to Petr Vopěnka, cf. [2, p. 278]. It is now called *Vopěnka's Principle*, and one of its equivalent renderings says that there exist no large rigid systems in the category $\mathcal{G}$ of all graphs. That is, there exists no proper class of graphs $\{G_\alpha \mid \alpha \in \text{Ord}\}$ such that $\text{Hom}_{\mathcal{G}}(G_\alpha, G_\beta) = \emptyset$ for all ordinals $\alpha \neq \beta$ and $\text{Hom}_{\mathcal{G}}(G_\alpha, G_\alpha) = \{\text{id}_{G_\alpha}\}$ for each ordinal α.

The second principle, called *Weak Vopěnka's Principle*, says that there exists no proper class of graphs $(G_\alpha \mid \alpha \in \text{Ord})$ such that for all ordinals α, β, $\text{Hom}_{\mathcal{G}}(X_\alpha, X_\beta) \neq \emptyset$, iff $\alpha \geq \beta$ [1].

It is known that Vopěnka's Principle implies Weak Vopěnka's Principle ([1, Observation I.12]), but the converse fails under the assumption of existence of supercompact cardinals, see [23, Theorem 1.2]. A recent application of Vopěnka's Principle to approximation theory has appeared in [6]: If Vopěnka's Principle is consistent, then it is also consistent that each cotorsion pair over any right hereditary ring is complete. However, by [8], it is consistent with ZFC that the Whitehead cotorsion pair is not complete (the *Whitehead cotorsion pair* is the cotorsion pair of abelian groups $({}^{\perp}\mathbb{Z}, ({}^{\perp}\mathbb{Z})^{\perp})$). We refer to [2, Appendix] and [23] for more details on the large cardinal strength of Vopěnka's principles.

Also, we refer to [14, Part II] for basics of approximation theory of modules, to [7, Chap. IV and VII] for properties of $\aleph_1$-projective modules, and to [3] for basics of general theory of modules.

[1] This is not the original formulation of Weak Vopěnka's Principle, but it is equivalent to it by [23, Theorem 1.4]

4.3 Closure properties, and enveloping and covering classes of modules

We start with a characterization of preenveloping classes in terms of their closure properties:

Lemma 4.3.1. *Let R be a ring, and $\mathscr{C} \subseteq \mathrm{Mod}\text{–}R$ a class of modules closed under direct summands. Consider the following two conditions:*

(i) $\mathscr{C}$ is preenveloping.

(ii) $\mathscr{C}$ is closed under direct products.

Then (i) implies (ii). If Weak Vopěnka's Principle holds, then (ii) implies (i).

Proof. Assume (i) and let $(E_i \mid i \in I)$ be a family of modules from $\mathscr{C}$. Let $f : P \to C$ be a $\mathscr{C}$-preenvelope of the module $P = \prod_{i\in I} E_i$. Denote by $\pi_i : P \to E_i$ the canonical projection ($i \in I$). Then there exist homomorphisms $g_i : C \to E_i$ such that $g_i f = \pi_i$ for each $i \in I$. Define a homomorphism $g : C \to P$ by $\pi_i g(c) = g_i(c)$ for all $c \in C$ and $i \in I$. Then $gf(x) = (g_i(f(x)) \mid i \in I) = x$ for all $x \in P$. Thus P is isomorphic to a direct summand in C, and $P \in \mathscr{C}$ by our assumption on the class $\mathscr{C}$, so (ii) holds.

The implication (ii) implies (i) holds under Weak Vopěnka's Principle by [1, Theorem 1.9 and Remark 1.10] and [23, Theorem 1.4]. □

Next, we consider classes of modules closed under submodules. Under this additional assumption, the conditions (i) and (ii) of Lemma 4.3.1 become equivalent in ZFC:

Lemma 4.3.2. *Let R be a ring, and $\mathscr{C} \subseteq \mathrm{Mod}\text{–}R$ a class of modules. Consider the following conditions*

(i) $\mathscr{C}$ is (pre-) enveloping and closed under submodules.

(ii) $\mathscr{C}$ is closed under submodules and direct products.

(iii) $\mathscr{C} = \mathrm{Cog}\,(M)$ for a module M.

Then (i) is equivalent to (ii), and it is implied by (iii).

Proof. (i) implies (ii) by Lemma 4.3.1. Assume (ii) and let $N \in \mathrm{Mod}\text{–}R$. Let $U = \mathrm{Rej}_N(\mathscr{C})$. We claim that the canonical projection $\pi_U : N \to N/U$ is a $\mathscr{C}$-envelope of N.

First, $N/U \in \mathscr{C}$ by Proposition 4.2.3(ii). Let $f \in \mathrm{Hom}_R(N, C)$ where $C \in \mathscr{C}$. Since $\ker f \supseteq U$, the Homomorphism Theorem implies that f factorizes through π_U, and that the only factorization of π_U through itself is by the identity automorphism $\mathrm{id}_{N/U}$.

Therefore, $\mathscr{C}$ is an enveloping class and (i) holds. Finally, (iii) trivially implies (ii). □

Let us try to dualize Lemma 4.3.1. First, dualizing the proof of the implication (i) $\implies$ (ii) in Lemma 4.3.1, we easily obtain

Lemma 4.3.3. *Let R be a ring, and $\mathscr{C} \subseteq \mathrm{Mod}\text{–}R$ a class of modules closed under direct summands. Consider the following two conditions:*

(i) $\mathscr{C}$ is precovering.

(ii) $\mathscr{C}$ is closed under direct sums.

Then (i) implies (ii).

The following two examples show that flat relative Mittag-Leffler modules yield a barrier in ZFC for proving both reverse implications, i.e., both (ii) $\Longrightarrow$ (iii) in Lemma 4.3.2, and (ii) $\Longrightarrow$ (i) in Lemma 4.3.3. We start with the second implication:

Example 4.3.4. Let R be a ring, $\mathcal{Q}$ a class of left R-modules, and $\mathcal{D}_{\mathcal{Q}}$ the class of all flat $\mathcal{Q}$-Mittag-Leffler modules So $\mathcal{D}_{\mathcal{Q}} = \mathcal{FM}$ is the class of all flat Mittag-Leffler ($= \aleph_1$-projective) modules in the particular case when $\mathcal{Q} = R\text{–Mod}$, and $\mathcal{D}_{\mathcal{Q}} = \mathcal{FP}$ the class of all f-projective modules in the particular case when $\mathcal{Q} = \{R\}$, cf. [5, §4]. By [5, Lemma 3.1(i)], the class $\mathcal{D}_{\mathcal{Q}}$ is always closed under transfinite extensions and pure submodules (and hence under direct sums and direct summands).

By [5, Theorem 3.6] (see also [22, Theorem 3.3]), for any class $\mathcal{Q}$ of left R-modules, the class $\mathcal{D}_{\mathcal{Q}}$ is precovering, iff $\mathcal{D}_{\mathcal{Q}}$ coincides with the class $\mathcal{F}_0$ of all flat modules.

For $\mathcal{Q} = R\text{–Mod}$, it is well-known that $\mathcal{FM} = \mathcal{F}_0$, only if the ring R is right perfect, cf. [3, §28]. So (ii) does not imply (i) in Lemma 4.3.3 whenever R is any non-right perfect ring and $\mathscr{C} = \mathcal{FM}$.

A different kind of examples arises for $\mathcal{Q} = \{R\}$: by [5, Proposition 4.8(ii)] and [20, Proposition 4.10], if R is a right semihereditary ring such that $\mathcal{FP} = \mathcal{F}_0$, then R left semihereditary. So if R is (the opposite ring of) the Chase ring from [16, Chap. 1,§2F,2.34], then (ii) does not imply (i) in Lemma 4.3.3 for $\mathscr{C} = \mathcal{FP}$.

Before presenting our second example showing that in general (ii) does not imply (iii) in Lemma 4.3.2, we need a lemma generalizing a construction from [12, Theorem 5.8].

Lemma 4.3.5. *Let R be a right hereditary ring, $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in* $\mathrm{Mod}\text{–}R$. *Let $\mathscr{C} \subseteq \mathcal{A}$ be such that $\mathscr{C}$ is closed under submodules and transfinite extensions, $\mathscr{C}^{\perp} = \mathcal{B}$, and $\mathscr{C} \cap \mathcal{B} = 0$.*

Then for each $0 \neq C \in \mathscr{C}$ there exists $0 \neq D \in \mathscr{C}$ such that $\mathrm{Hom}_R(D, C) = 0$. *In particular, $\mathscr{C} \nsubseteq \mathrm{Cog}\,(C)$ for any module $C \in \mathscr{C}$.*

Proof. Let $0 \neq C \in \mathscr{C}$ and $\kappa = \mathrm{card}\, C + \mathrm{card}\, R + \aleph_0$. Let $\mathcal{S}$ denote a representative set of all non-zero modules in $\mathscr{C}$ of cardinality $\leq \kappa$ such that $C \in \mathcal{S}$. Since $\mathscr{C}^{\perp} = \mathcal{B}$ and $\mathscr{C} \cap \mathcal{B} = 0$, for each $S \in \mathcal{S}$ there exists $C_S \in \mathscr{C}$ such that $\mathrm{Ext}^1_R(C_S, S) \neq 0$. Let $E = \bigoplus_{S \in \mathcal{S}} C_S \in \mathscr{C}$. Then $\mathrm{Ext}^1_R(E, S) \neq 0$ for each $S \in \mathcal{S}$. Let $\lambda = \mathrm{card}\, E (\geq \kappa)$. Since R is right hereditary, there is a projective resolution of E of the form $0 \to K \overset{\eta}{\hookrightarrow} F \to E \to 0$ where F is free of rank λ.

The module D will be constructed as the last term of a $\mathscr{C}$-filtration $(D_\alpha \mid \alpha \leq \tau)$ for some $\tau \leq \lambda^+$ by induction as follows: $D_0 = C$; if D_α has already been constructed and $\mathrm{Hom}_R(D_\alpha, C) = 0$, we let $\tau = \alpha$ and $D = D_\tau$, and finish the construction.

Otherwise we proceed by putting $H_\alpha = \mathrm{Hom}_R(D_\alpha, C) \setminus \{0\}$. For each $h \in H_\alpha$, we let $I_h = \mathrm{Im}\, h \subseteq C$. Since $\mathscr{C}$ is closed under submodules, $I_h \in \mathscr{C}$, and $\mathrm{card}\, I_h \le \kappa$ implies $\mathrm{Ext}^1_R(E, I_h) \neq 0$ by our definition of E. Using the projective resolution of E above, we infer that there exists $\phi_h \in \mathrm{Hom}_R(K, I_h)$ that cannot be extended to a homomorphism from F to I_h. Since K is projective and $h : D_\alpha \to I_h$ is surjective, there exists $\psi_h \in \mathrm{Hom}_R(K, D_\alpha)$ such that $\phi_h = h\psi_h$.

We have the exact sequence $0 \to K^{(H_\alpha)} \overset{\eta_\alpha}{\hookrightarrow} F^{(H_\alpha)} \to E^{(H_\alpha)} \to 0$ where $E^{(H_\alpha)} \in \mathscr{C}$ because $\mathscr{C}$ is closed under transfinite extensions. For each $h \in H_\alpha$, let ν_h be the hth canonical inclusion of K into $K^{(H_\alpha)}$, and μ_h hth canonical inclusion of F into $F^{(H_\alpha)}$. Then $\mu_h \eta = \eta_\alpha \nu_h$ for each $h \in H_\alpha$. We define $\Psi_\alpha \in \mathrm{Hom}_R(K^{(H_\alpha)}, D_\alpha)$ by $\Psi_\alpha \nu_h = \psi_h$ for each $h \in H_\alpha$.

The module $D_{\alpha+1}$ is defined by the pushout of η_α and Ψ_α,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{(H_\alpha)} & \xrightarrow{\eta_\alpha} & F^{(H_\alpha)} & \longrightarrow & E^{(H_\alpha)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Psi_\alpha} & & \Big\downarrow{\scriptstyle \rho} & & \Big\| & & \\
0 & \longrightarrow & D_\alpha & \xrightarrow{\subseteq} & D_{\alpha+1} & \longrightarrow & E^{(H_\alpha)} & \longrightarrow & 0.
\end{array}$$

Then $D_{\alpha+1} \in \mathscr{C}$ because $\mathscr{C}$ is closed under extensions. For a limit ordinal α, we put $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$.

We claim that our construction terminates at some $\tau \le \lambda^+$, whence for $D = D_\tau$, we have $\mathrm{Hom}_R(D, C) = 0$.

If not, then D_{λ^+} is defined and satisfies $D_{\lambda^+} = \bigcup_{\alpha<\lambda^+} D_\alpha$, and there exists $0 \neq g \in \mathrm{Hom}_R(D_{\lambda^+}, C)$. Let $\beta < \lambda^+$ be the least ordinal such that $g \restriction D_\beta \neq 0$.

We will prove that for each $\beta \le \alpha < \lambda^+$, $\mathrm{Im}\,(g \restriction D_\alpha)$ is a proper submodule of $\mathrm{Im}\,(g \restriction D_{\alpha+1})$. Then $(\mathrm{Im}\,(g \restriction D_\alpha) \mid \beta \le \alpha < \lambda^+)$ is a strictly increasing continuous chain of submodules of C of cardinality λ^+, in contradiction with $\mathrm{card}\, C = \kappa \le \lambda$.

Assume $\mathrm{Im}\,(g \restriction D_\alpha) = \mathrm{Im}\,(g \restriction D_{\alpha+1})$ for some $\beta \le \alpha < \lambda^+$. Using the notation from the non-limit step of our construction for $h = g \restriction D_\alpha$, we define $f_h = (g \restriction D_{\alpha+1})\rho\mu_h \in \mathrm{Hom}_R(F, I_h)$.

We claim that f_h extends ϕ_h from K to F, in contradiction with our definition of ϕ_h. Indeed, $f_h\eta = (g \restriction D_{\alpha+1})\rho\eta_\alpha\nu_h$. Denoting the inclusion $D_\alpha \subseteq D_{\alpha+1}$ by σ_α and using the pushout diagram above, we obtain $f_h\eta = (g \restriction D_{\alpha+1})\sigma_\alpha\Psi_\alpha\nu_h$. The latter map is equal to $h\Psi_\alpha\nu_h$, and hence to $h\psi_h = \phi_h$. Thus, $f_h\eta = \phi_h$, q.e.d. □

Example 4.3.6. Let R be a Dedekind domain with a countable spectrum which is not a complete discrete valuation domain (e.g., let $R = \mathbb{Z}$). We claim that the class of all $\aleph_1$-projective modules $\mathcal{FM}$ is an enveloping class closed under submodules, but $\mathcal{FM}$ is not of the form $\mathrm{Cog}\,(M)$ for any module M.

To verify this claim, we apply Lemma 4.3.5 to the Enochs cotorsion pair, i.e., $\mathcal{A} = \mathcal{F}_0$ and $\mathcal{B} = \mathcal{EC}$, and let $\mathscr{C} = \mathcal{FM}$. This is possible, since R is hereditary, whence $\mathscr{C}$ is closed under submodules by [15, Corollary 2.10(i)]. Moreover $\mathscr{C}$ is closed under transfinite extensions by [14, Corollary 3.20(i)]. Since $\mathrm{Spec}(R)$ is countable, the quotient field Q is countably presented, whence $Q \in {}^\perp(\mathscr{C}^\perp)$ by [14, Lemma 10.19]. Then ${}^\perp(\mathscr{C}^\perp)$ contains all torsion-free modules, whence ${}^\perp(\mathscr{C}^\perp) = \mathcal{F}_0$, and $\mathscr{C}^\perp = \mathcal{B}$.

Moreover, if $0 \neq M \in \mathscr{C} \cap \mathcal{B}$, then M is a flat cotorsion module, so $M \cong Q^{(\alpha)} \oplus \prod_{0\neq p\in\mathrm{Spec}(R)} \widehat{R_p^{(\alpha_p)}}$ for some cardinals α, α_p ($0 \neq p \in \mathrm{Spec}(R)$), where

R_p denotes the localization of R at p and $\widehat{\ }$ the p-adic completion, see [11, Theorem 5.3.28]. Since $Q \notin \mathscr{C}$, M has a direct summand isomorphic to $\widehat{R_p}$, whence $\widehat{R_p} \in \mathscr{C}$. We have an exact sequence $0 \to R_p \to \widehat{R_p} \to D \to 0$ where D is torsion free and divisible, hence a direct sum of copies of Q. Since R_p is a countably generated pure submodule of $\widehat{R_p}$, it is projective. This implies that $R = R_p$ is a (discrete) valuation domain. If R is not complete, then there is an exact sequence $0 \to R \to N \to Q \to 0$ with $N \subseteq \widehat{R_p}$ countably generated, and hence free, a contradiction. Thus, if R is not a complete discrete valuation domain, then $\mathscr{C} \cap \mathcal{B} = 0$.

Finally, since R is noetherian, $\mathscr{C}$ is closed under direct products by [15, Proposition 4.3], so by Lemmas 4.3.2 and 4.3.5, $\mathscr{C}$ is an enveloping class of modules closed under submodules, but $\mathscr{C}$ is not of the form $\operatorname{Cog}(M)$ for any module M.

We have seen that only one implication from Lemma 4.3.1 can be dualized. However, the dual version of Lemma 4.3.2 does hold true, even in an extended form:

Proposition 4.3.7. *Let R be a ring, and $\mathscr{C} \subseteq \text{Mod–}R$ a class of modules. Consider the following conditions*

(i) $\mathscr{C}$ is (pre-) covering and closed under homomorphic images.

(ii) $\mathscr{C}$ is closed under homomorphic images and direct sums.

(iii) $\mathscr{C} = \operatorname{Gen}(M)$ for a module M.

Then (i) is equivalent to (ii), and it is implied by (iii). If Vopěnka's Principle holds, then (ii) implies (iii).

Proof. (i) implies (ii) by Lemma 4.3.3. Assume (ii) and let $N \in \text{Mod–}R$. Let $T = T_{\mathscr{C}}(N)$. By (ii), $T \in \mathscr{C}$, and basic properties of the trace yield that the monomorphism $T \subseteq N$ is a $\mathscr{C}$-cover of N. So (i) holds. The implication (iii) $\implies$ (ii) is trivial.

Finally, assume (ii). Since $\mathscr{C}$ is closed under direct sums and homomorphic images, it is also closed under direct limits. Then Vopěnka's Principle gives $\mathscr{C} = \varinjlim \mathcal{S}$ for a subset $\mathcal{S} \subseteq \mathscr{C}$ (cf. [9, Theorem 3.3] or [2, Corollary 6.18]). Let $M = \bigoplus_{S \in \mathcal{S}} S$. Then $M \in \mathscr{C}$, and the closure properties of $\mathscr{C}$ yield that $\operatorname{Gen}(M) \subseteq \mathscr{C}$. Moreover, by [17, Lemma 1.1], $\operatorname{Gen}(M) \supseteq \varinjlim \operatorname{sum} M = \varinjlim \mathcal{S} = \varinjlim \operatorname{add} M$, so $\operatorname{Gen}(M) = \mathscr{C}$, proving (iii). □

Adding further closure properties of the class $\mathscr{C}$ allows a complete characterization of the dual setting in ZFC in terms of the Grothendieck categories $\sigma[M]$.

Lemma 4.3.8. *Let R be a ring, and $\mathscr{C} \subseteq \text{Mod–}R$ a class of modules. Then the following conditions are equivalent:*

(i) $\mathscr{C}$ is (pre-) covering and closed under submodules and homomorphic images.

(ii) $\mathscr{C} = \sigma[M]$ $(= \operatorname{Gen}(M))$ for a module M.

Proof. Assume (i). Denote by $\mathcal{D}$ a representative set of all finitely generated modules in the class $\mathscr{C}$. Let $M = \bigoplus_{D\in\mathcal{D}} D$. Since $\mathscr{C}$ is precovering and closed under direct summands, $\mathscr{C}$ is closed under direct sums by Lemma 4.3.3. So $M \in \mathscr{C}$. Then $\operatorname{Gen}(M) \subseteq \sigma[M] \subseteq \mathscr{C}$ as $\mathscr{C}$ is also closed under homomorphic images and submodules. Conversely, let $N \in \mathscr{C}$. Then N is a directed union of copies of some modules from $\mathcal{D}$, say $N = \bigcup_{F\in\mathcal{E}} F$ for some $\mathcal{E} \subseteq \mathcal{D}$. Thus $N \in \varinjlim \operatorname{add} M = \varinjlim \operatorname{sum} M$ by [17, Lemma 1.1], whence (ii) holds because $\varinjlim \operatorname{sum} M \subseteq \operatorname{Gen}(M) \subseteq \sigma[M]$.

Assume (ii). Clearly, $\sigma[M]$ is closed under submodules and homomorphic images. For a module N, let $T = T_{\sigma[M]}(N)$. Then $T \in \sigma[M]$ and the inclusion $T \hookrightarrow N$ is a $\sigma[M]$-cover of N, whence (i) holds true. □

Let's turn again to the setting of preenveloping classes. For a module M, we will denote by $\pi[M]$ the class of all homomorphic images of all modules cogenerated by M.

Lemma 4.3.9. *Let R be a ring, and $\mathscr{C} \subseteq$ Mod–R a class of modules. Then the following conditions are equivalent:*

(i) $\mathscr{C}$ is (pre-) enveloping and closed under submodules and homomorphic images.

(ii) $\mathscr{C} = \pi[M]$ for a module M.

(iii) $\mathscr{C} =$ Mod–(R/I) for a two-sided ideal I in R.

Proof. Assume (i). For each $M \in \mathscr{C}$, let $I_M = \operatorname{Ann}(M)$, and let $I = \bigcap_{M\in\mathscr{C}} I_M$. By Lemma 4.3.1, $\mathscr{C}$ is closed under direct products, so $I = I_N$ for a module $N \in \mathscr{C}$, and moreover, there is a cyclic module $C = xR \in \mathscr{C}$ such that $I = \operatorname{Ann}(x)$. Thus $R/I \in \mathscr{C}$, and Mod–$(R/I) \subseteq \mathscr{C}$ by the closure properties of $\mathscr{C}$. However, $I \subseteq \operatorname{Ann}(M)$ for each module $M \in \mathscr{C}$, whence $\mathscr{C} \subseteq$ Mod–(R/I) and (iii) holds.

Assume (iii). Then (ii) holds for $M = R/I \in$ Mod–R. The proof that (ii) implies (i) is dual to the one given in Lemma 4.3.8, replacing the trace by the reject. □

Remark 15. Since for each two-sided ideal I in R, Mod–$(R/I) = \sigma[R/I]$, Lemmas 4.3.8 and 4.3.9 imply that for each class $\mathscr{C}$ of modules closed under submodules and homomorphic images, $\mathscr{C}$ is covering whenever $\mathscr{C}$ is preenveloping. The converse fails in general as witnessed, for example, by the class $\mathscr{C}$ of all torsion abelian groups, cf. [23, 15.10].

Notice also that for each module M, the class $\sigma[M]$ is determined by the filter $\mathcal{F}_M = \{I \leq R_R : R/I \in \sigma[M]\}$ consisting of right ideals of R (see e.g. [18, §1]). The condition $\sigma[M] =$ Mod–(R/I) for a two-sided ideal I was characterized in [18, Proposition 1.5]: $\sigma[M] =$ Mod–(R/I), iff the filter $\mathcal{F}_M$ is principal.

We finish by showing that the proof of the implication (ii) implies (iii) in Proposition 4.3.7 does in general require a large cardinal assumption, namely the Weak Vopěnka's Principle. The extra tool that we will need for this purpose is due to Przeździecki, [19, Theorem 3.14]:

Lemma 4.3.10. *There exists a functor G from the category $\mathcal{G}$ of all graphs to* Mod–$\mathbb{Z}$ *which induces for all $X, Y \in \mathcal{G}$ a group isomorphism $\mathbb{Z}^{(\operatorname{Hom}_{\mathcal{G}}(X,Y))} \cong \operatorname{Hom}_{\mathbb{Z}}(G(X), G(Y))$ natural in both variables.*

Remark 16. By [13, Corollary 4.11], we can moreover assume that the functor G from Lemma 4.3.10 takes its values in the class of all $\aleph_1$-free groups.

Theorem 4.3.11. *Assume that each covering class $\mathscr{C}$ of abelian groups which is closed under homomorphic images and is generated by a class of $\aleph_1$-free groups, satisfies $\mathscr{C} = \mathrm{Gen}\,(M)$ for an abelian group M. Then Weak Vopěnka's Principle holds true.*

Proof. Assume that Weak Vopěnka's Principle fails, that is, there exists a proper class of graphs $(X_\alpha \mid \alpha \in \mathrm{Ord})$ such that for all ordinals α, β, $\mathrm{Hom}_{\mathcal{G}}(X_\alpha, X_\beta) \neq \emptyset$, iff $\alpha \geq \beta$.

Let $\mathscr{C}$ be the subclass of Mod–$\mathbb{Z}$ generated by the groups $G(X_\alpha)$ ($\alpha \in \mathrm{Ord}$). By Remark 16, we can w.l.o.g. assume that $G(X_\alpha)$ is $\aleph_1$-free for each $\alpha \in \mathrm{Ord}$. Since $\mathscr{C}$ is closed under direct sums and homomorphic images, $\mathscr{C}$ is covering by Proposition 4.3.7. We will show that there is no abelian group $M \in \mathscr{C}$ such that $\mathscr{C} = \mathrm{Gen}\,(M)$.

Assume that such a group M does exist. Let α be the least ordinal such that M is generated by the groups $G(X_\beta)$ ($\beta < \alpha$). Then M is a homomorphic image of a direct sum of copies of those groups. Since $G(X_\alpha) \in \mathrm{Gen}\,(M)$, $G(X_\alpha)$ a homomorphic image of a direct sum of copies of M. Thus, there is a non-zero homomorphism from from $G(X_\beta)$ to $G(X_\alpha)$ for some $\beta < \alpha$. Then $\mathrm{Hom}_{\mathcal{G}}(X_\beta, X_\alpha) \neq \emptyset$ by Lemma 4.3.10, a contradiction. □

Corollary 4.3.12. *If Vopěnka's Principle holds, then each covering class of modules closed under homomorphic images is of the form $\mathrm{Gen}\,(M)$ for a module M. The latter property restricted to classes of abelian groups generated by $\aleph_1$-free groups implies Weak Vopěnka's Principle.*

Proof. By Proposition 4.3.7 and Theorem 4.3.11. □

Bibliography of Chapter 4

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- A. Ben Yassine, J. Trlifaj, *Flat relative Mittag-Leffler modules and approximations*, J. Algebra and Its Appl., DOI: 10.1142/S0219498824502190, arXiv:2110.06792v2.
- A. Ben Yassine, J. Trlifaj, *Flat relative Mittag-Leffler modules and Zariski locality*, submitted to J. of Pure and Applied Algebra, arXiv:2208.00869v2.
- A. Ben Yassine, J. Trlifaj, *Dualizations of approximations,* $\aleph_1$*-projectivity, and Vopěnka's Principles*, submitted to Applied Categorical Structures, arXiv:2401.11979v1.