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Beyond symmetric solutions in General Relativity

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Contents

Abstract

Most of the attention when analyzing the consequences of general relativity (and even more so in modified theories of gravity) has been focused on highly symmetric solutions. In the realm of black hole spacetimes this lead to static spherically symmetric solutions (Schwarzschild geometry) or stationary solutions (Kerr geometry). In the realm of cosmology most of the attention has been devoted to homogeneous and isotropic solutions represented by FLRW geometries. This attention is natural both from historical perspective (highly symmetric solutions were discovered first) and due to significant technical challenges connected with obtaining and analyzing more general solutions. However, one has to ensure that insights gained using highly symmetric solutions is fundamental to the theory and translates to more general situations as well. This problem is exacerbated by nonlinear character of general relativity. The aim of this thesis is to provide overview of some recent work by the author related to these issues.

Chapter 1

Introduction

One of the most important strategies when investigating the impact of new theories in physics is applying the theory in highly simplified scenarios, such as situations with substantial degree of symmetry and with limited dynamics. This is the case of general relativity as well and even more so of various modified theories of gravity.

This avenue towards understanding of general relativity was started immediately after Albert Einstein finalized the theory when Karl Schwarzschild published the first non-trivial solution at the beginning of 1916 [1]. This static spherically symmetric solution provided both the necessary foundations for deriving experimentally testable predictions and clear comparison with Newtonian picture especially with respect to solar system dynamics. At the same time it shed light on the problem of coordinate vs. spacetime (curvature) singularities, albeit clear resolution of this topic took some time. This was connected to the most important aspect of Schwarzschild solution, namely its interpretation as a black hole — a new object of immense interest in theoretical physics at first but later in astrophysics as well.

It was clear that such a simplified situation might not correspond to generic behavior of the theory. Two parallel routes were followed to provide generalization. One of them led to a famous exact solution that relaxed spherical symmetry by assuming only axisymmetry — the Kerr solution [2] for a rotating black hole. However, even this (in some sense minimal) generalization took almost 50 years clearly showing that the complex nonlinear nature of Einstein equations will not yield easily to our efforts. Moreover, even this did not immediately convince the community that black holes are indeed objects of nature and not only artificially contrived situations (opinion held by Einstein himself). The second approach avoided looking for exact solution and instead employed tools of mathematical relativity to follow geodesic congruences in the collapsing region to establish generic appearance of spacetime singularities. The Penrose theorem [3] thus proved that black holes are indeed occurring generically and was subsequently generalized (together with Hawking [4]) to treat the initial cosmological singularity analogously. In the following, we will concentrate more on the first approach to the problem — the role that exact solutions of reduced symmetry might play in determining robust general relativistic predictions — since we will not predominantly focus on the question of singularities. But the specific areas we want to cover are still black hole physics and cosmology since they represent astrophysically important situations crucially affected by general relativity.

Ideally, the path towards generalizing the black hole solutions would start at the Kerr solution and proceed towards generic twisting (rotating) spacetimes. However, the mathematical and interpretational problems encountered are daunting and although substantial effort has been made over the years the progress is extremely slow. That is why our focus will be on a well-established non-twisting generalization of the Schwarzschild solution derived by Robinson and Trautman [5]. This so-called Robinson–Trautman family of solutions contains other simple important geometries, e.g. Vaidya solution [6] describing pure radiation generalization of Schwarzschild solution or the C-metric [7] representing accelerated black holes. Since the vacuum solutions of this family are governed by a single partial differential equation (albeit of fourth order) for which many advanced mathematical tools could be applied the research produced important analytical results. The obvious criticism for this restricted geometry is that by not allowing rotation it might not correspond to real astrophysical situations. However, a recent investigation of nonlinear stability of the Schwarzschild solution [8] uncovered a special role played by linearized Robinson–Trautman solutions. Furthermore, the presence of exact gravitational waves in a generic Robinson–Trautman solution made it possible to model ring-down phase in the black hole mergers in order to understand the physical nature of so-called anti-kick [9] which was appearing in numerical simulations for pair of black holes of substantially different masses. The dynamical nature of the Robinson–Trautman family provides opportunity to study quasilocal horizons in a situation where event horizon might not be a viable notion.

The Friedmann–Lemaître–Robertson–Walker (FLRW) geometry proved to be extremely successful in providing cosmological model that captures the primary features of our universe extremely well when combined with a cosmological constant and cold dark matter (CDM). Nevertheless, it assumes homogeneity and isotropy which are obviously valid maximally on average over sufficiently large regions while the observed universe is manifestly inhomogeneous on the scales below approximately $200 Mpc$ (however inhomogeneities much bigger than this scale have been observed). This can either be addressed by using linear perturbations or proceeding towards inhomogeneous cosmological models thus accounting for nonlinear effects. The nonlinearity of Einstein equations provides another significant technical obstacle when applying the averaging in order to derive homogeneous geometry from the inhomogeneous one. The averaged geometry does not solve Einstein equations for averaged source distribution due to the appearance of so-called correlation or backreaction terms generated by averaging the Einstein tensor that depends on the geometry in a nonlinear way. But these extra terms provided an opportunity to explain the dark sector of our universe without invoking extra particles or fields. These hopes were reduced when it was shown that any effect of nonlinearities is substantially restricted under certain assumptions [10]. However, the general applicability of such analysis was contested [11]. Nevertheless, in the upcoming era of precision cosmology even effects that are too small to explain dark sector in its entirety will have to be taken into account to properly explain observed data.

Remembering that homogeneous cosmological models were also used for representing the interior geometry of stars it is not surprising that one of the motivations for studying inhomogeneous perfect fluid solutions came from attempts to create more realistic interiors of stars, apart from the obvious utility in cosmology. The most frequently used inhomogeneous cosmological models are spherically symmetric Lemaître–Tolman–Bondi model [12, 13, 14] and a broad Szekeres–Safron family of solutions without symmetries whose dust subfamily was introduced by Szekeres [15] and later generalized to perfect fluid by Szafron [16]. The rich structure of different inhomogeneous cosmologies and relations among their families and other solutions is presented in a book by Krasinski [17].

In the following chapters we show what insights can be gained from using more general exact solutions. We are limiting ourselves here solely to four dimensions and Einstein's general relativity thus ignoring recent substantial interest in Robinson–Trautman solutions in higher dimensions or alternative theories of gravity.

Chapter 2

Black holes without symmetries

As explained in the Introduction, the Robinson–Trautman family of geometries is a useful tool for studying genericity of black hole features. Here we will describe the main properties of this family and show what kind of results one can obtain regarding the properties of black holes and related objects.

The Robinson–Trautman geometry [5, 18, 19] is characterised by the properties of the principle null geodesic congruence which should be shearfree, twistfree and expanding. The first two conditions on the congruence guaranteeing that the resulting spacetime is algebraically special (minimally Petrov type II) and it is non-rotating, therefore this family of geometries cannot contain the Kerr solution. The last condition is hinting at the description of isolated systems and our focus is primarily on black holes.

The overall qualitative picture of a generic dynamical black hole without symmetries belonging to this family is the following: initially we have a (nonlinearly) deformed black hole that emits gravitational radiation which caries away the deviation from sphericity and the black hole settles down to final spherically symmetric state. This interpretation is based on the most important result connected with this family of solutions — its asymptotic evolution — and was derived at the beginning of nineties by Chruściel and Singleton [20, 21, 22]. In this series of papers describing vacuum subfamily of Robinson–Trautman geometries it was shown that given a smooth initial data on characteristic hypersurface this spacetime asymptotically approaches the Schwarzschild geometry. Note that the initial data are not assumed to be small in any sense (e.g., being a small perturbation of Schwarzschild data) or that this analysis is done within a family of exact solutions (as opposed to treating the evolution only in linear order) and therefore it is a truly nonlinear stability result for Schwarzschild solution albeit restricted only to given family of geometries. We will shortly summarize the statement with greater detail.

2.1 Robinson–Trautman solution

The vacuum Robinson–Trautman metric [5, 18, 19] (potentially with a nonzero cosmological constant Λ) can be given in the following form

$$\mathrm{d}s^2 = -2H(u,r,x,y)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{r^2}{P(u,x,y)^2}(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{2.1}$$

with $2H = \Delta(\ln P) - 2r(\ln P)_{,u} - 2m/r - (\Lambda/3)r^2$ and

$$\Delta \equiv P^2(\partial_{xx} + \partial_{yy}) \,, \tag{2.2}$$

where we opted for real coordinates in the two-space spanned by x, y instead of frequently used complex coordinate. This metric is determined by two functions, $P(u, x, y)$ and $m(u)$. In the vacuum case the function $m(u)$ might be set to a constant by suitable coordinate transformation [18, 19] and we consider this to be satisfied for the coordinates of (2.1). Moreover, we consider this constant to be positive $m > 0$. The Einstein equations for vacuum (potentially with a cosmological constant) then reduce to a single nonlinear PDE — the Robinson–Trautman equation

$$\Delta\Delta(\ln P) + 12\, m(\ln P)_{,u} = 0\,. \tag{2.3}$$

These spacetimes are then generally of algebraic type II while the most important member of this family, the Schwarzschild solution, is only type D.

The coordinates are adapted to the distinguished null congruence whose properties are determined by the definition of the Robinson–Trautman family. This geodesic, shearfree, twistfree and expanding null congruence is generated by $\mathbf{l} = \partial_r$ with r being an affine parameter along this congruence, u is then a retarded time and $u = const$ hypersurfaces are null. Spatial coordinates x, y span the cross-section of the congruence and the Gaussian curvature of these 2-spaces is given by (for $r = 1$)

$$K(x, y, u) \equiv \Delta(\ln P)\,. \tag{2.4}$$

The asymptotic evolution result we want to describe is valid when the transversal 2-spaces have spherical topology which leads to a subclass containing the Schwarzschild solution (considering vanishing cosmological constant) corresponding to $K = 1$.

2.1.1 Robinson–Trautman equation and Ricci flow

Since the parabolic Robinson–Trautman equation (2.3) essentially describes evolution of a two-dimensional metric on a compact connected manifold it is natural to compare it and the methods of its analysis to the Ricci flow equation as noted already by Chruściel [20]. He concluded that although it is an equation of higher order and therefore some tools relevant in the study of the Ricci flow are not directly applicable the resulting analysis of the Robinson–Trautman equation was more straightforward. Let us show the comparison between these two equations more explicitly.

Ricci flow describes evolution of geometry on a two-dimensional manifold with one-parametric family of Riemannian metrics $q^{[t]}$ (with parameter denoted by t) driven by the equation (here we use abstract tensors as usual in the mathematical literature related to Ricci flow)

$$\frac{\partial}{\partial t} q^{[t]} = -2\,\mathrm{Ricc}(q^{[t]}) \,, \tag{2.5}$$

where $\mathrm{Ricc}(q^{[t]})$ is a Ricci tensor of $q^{[t]}$. The two-dimensional metric in question is in our case the transversal part of (2.1) (for $r = 1$) and the evolution parameter is the retarded time u

$$q^{[u]}_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{P(u,x,y)^2}(\mathrm{d}x^2+\mathrm{d}y^2)\ , \tag{2.6}$$

while the Ricci tensor can easily be determined

$$\mathrm{Ricc}(q^{[u]}) = \frac{1}{2}\mathrm{RicciSc}(q^{[u]})\,q^{[u]} = K(u,x,y)\;q^{[u]} \tag{2.7}$$

and we have used the relation between the Ricci scalar (denoted as $\mathrm{RicciSc}(\mathrm{q}^{[\mathrm{u}]})$) and the Gaussian curvature K (2.4). The tensorial Ricci flow equation for the family of metrics (2.6) parametrized by u with the Ricci tensor (2.7) simplifies into the following scalar equation

$$(\ln P)_{,u} = \Delta \ln P\ . \tag{2.8}$$

Thus we see that the two equations ((2.3) and (2.8)) differ by application of an additional Laplacian, apart from unimportant constants. The heat-type equation (2.8) has also seemingly different sign compared to (2.3) but one should not forget that Laplacian Δ has a negative spectrum on compact manifolds.

There is another significant type of geometric flow — Calabi flow — proposed as a tool for construction of extremal Kähler geometries (manifolds with mutually compatible complex, Riemannian and symplectic structures). This flow is equivalent to Robinson–Trautman equation in 2-dimensional case and the results achieved by Chruściel [20] are still of fundamental importance in the study of Calabi flow.

2.1.2 Asymptotic behavior

Now let us concentrate on the equation (2.3). For analysis of the behavior of the function P which determines the dynamics of the Robinson–Trautman solution it is useful to consider the transversal 2-spaces as being deformations of spherical geometry and thus introduce the following notation

$$P = f(x,y,u)\,P_0\,, \tag{2.9}$$

where f is a function on a 2-sphere S^2 equipped with the standard metric on a sphere determined by $P_0 = 1 + \frac{1}{4}(x^2+y^2)$ (leading to $K = 1$). Using advanced tools of PDE analysis on the equation (2.3) together with the decomposition (2.9) Chruściel and Singleton [20, 21, 22] proved that for arbitrary smooth initial data $f(x,y,u_0)$ on an initial hypersurface $u = u_0$, the Robinson–Trautman vacuum spacetimes (2.1) exist globally for all $u \geq u_0$. Moreover, they asymptotically converge to the Schwarzschild metric with the corresponding mass m as $u \to +\infty$. This convergence proceeds exponentially fast since f has the following asymptotic expansion

$$f = \sum_{i,j\geq 0} f_{i,j}u^j \mathrm{e}^{-2iu/m} \tag{2.10}$$

where $f_{i,j}(x,y)$ are smooth functions on S^2 and $f_{i,j} = 0$ for $i \leq 14$ and $j > 0$. This shows that for large retarded times u, the function P approaches P_0 exponentially fast and therefore the geometry corresponds to Schwarzschild solution.

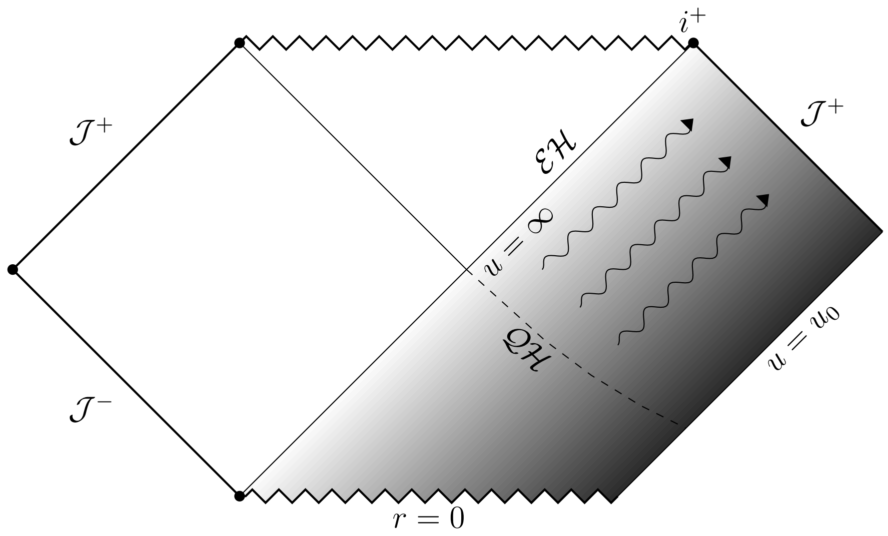

Figure 2.1: **Conformal diagram of Robinson–Trautman spacetime with $\Lambda = 0$ attached to the inner Schwarzschild solution.** Initial data are given on $u = u_0$ hypersurface and the solution radiates away the deviation from sphericity towards (incomplete) future null infinity $\mathcal{J}^+$. At $u = \infty$ the geometry approaches Schwarzschild geometry close to the future event horizon $\mathcal{EH}$ and we can attach inner Schwarzschild solution beyond this hypersurface. The past horizon in Robinson–Trautman spacetime can only be localized as quasilocal horizon $\mathcal{QH}$.

The parabolic PDE (2.3) has evidently very nice behavior in the positive retarded time direction but not in the opposite direction similarly to the behavior of the canonical parabolic PDE — the heat equation. This means that the solution cannot be extended up to past null infinity $\mathcal{J}^-$. We can schematically summarize the situation on the conformal diagram in figure 2.1 where the canonical extension beyond $u = \infty$ is included. This extension can only be made with finite level of smoothness for generic Robinson–Trautman geometries.

This overall picture stays the same upon including cosmological constant [23, 24] (with obvious changes to conformal boundary) or outgoing pure radiation [25] (where the spacetime approaches the Vaidya solution [6]). In these situations the extensions, with Schwarzschild–(anti-)de Sitter and Minkowski respectively, can be made with varying levels of smoothness and in special cases even smooth but not analytic.

2.1.3 Horizons

The impossibility of extending the Robinson–Trautman (RT) solution towards past null infinity $\mathcal{J}^-$ means that the past horizon can only be localized as a quasilocal horizon $\mathcal{QH}$ as indicated on figure 2.1. For vacuum solutions this was studied using the so-called Penrose–Tod equation which encodes the demand of vanishing expansion of a null congruence that is normal to the surface which should be a slice of the full horizon hypersurface. In the case of vacuum RT and a horizon hypersurface described by equation $r = \mathcal{R}(u, x, y)$ the Penrose–Tod equation has the following form for the slice determined by $u = u_1$

$$\Delta \ln \mathcal{R} + \frac{2m}{\mathcal{R}} - K = 0 \ , \tag{2.11}$$

where all the dependence on retarded time was replaced by the fixed value u_1. This is a quasilinear elliptic PDE and its analysis was performed in [26] where the existence, uniqueness and character of the horizon was shown. This analysis was subsequently generalized to RT spacetimes with cosmological constant [27] where some of the mathematical tools from the vacuum case were not applicable. The results were influenced by the value of cosmological constant in a natural way.

Considering both the event horizon (indicated as $\mathcal{EH}$ in figure 2.1, where the inner Schwarzschild solution is attached) and quasilocal horizon ($\mathcal{QH}$ in figure 2.1) we have nice explicit examples for more recent specific definitions of horizons that employ quasilocal characterization. Namely, the hypersurface $\mathcal{EH}$ is satisfying conditions of being an isolated horizon [28] since it is in equilibrium situation while retaining nontrivial radiation close to it (indicated by the wavy lines in figure 2.1) thus being non-stationary there [29]. The past horizon ($\mathcal{QH}$) is an example of horizon in fully dynamical situation and satisfies conditions of a dynamical horizon [30], namely it is spacelike. Both of them satisfy conditions of an earlier definition of a trapping horizon [31] which contains useful local criteria for classifying the horizons into inner/outer and future/past.

2.1.4 Relation to stability of symmetric solutions

What the overall picture of evolution of RT solution exactly tells us regarding the stability of Schwarzschild solution? We can consider fully nonlinear (smooth) deformation away from Schwarzschild solution that fits into the (vacuum) Robinson–Trautman class. This means that we can describe it by smooth initial data for Robinson–Trautman geometry on some initial hypersurface $u = u_0$. Then the asymptotic results mentioned in section 2.1.2 show that such a deformation will settle exponentially fast to the Schwarzschild solution by radiating away the nonsphericity via gravitational waves. This is valuable non-perturbative result, but it is restricted only within non-rotating family of solutions. Obviously, rotating generalization would represent more realistic situation but mathematical problems involved are so far insurmountable on this route that employs exact solutions.

Let us note that recently an alternative path showing nonlinear stability of Schwarzschild solution was successfully completed [32]. This approach uses the results regarding linear stability of the Kerr family around Schwarzschild geometry and newly developed mathematical techniques for controlling nonlinearities in the full set of Einstein equations.

Using the generalizations of Robinson–Trautman solution including cosmological constant [23, 24] or outgoing null radiation [25] appropriate stability results can be formulated in these situations. Moreover, as mentioned in these works, these results show the validity of no-hair conjecture as well, namely the final states of the evolution are simple symmetric geometries described by mass and cosmological constant.

Interesting case arises when we consider ingoing null radiation within RT family as briefly analyzed in [25]. This can represent a situation when radially incoming null fluid (radiation) which is not spherically symmetric forms a black hole or a naked singularity (being a generalization of spherically symmetric situation represented by ingoing Vaidya geometry). Since this represents a time reversed version of the outgoing radiation case the future development is essentially described by the Robinson–Trautman equation (2.3) with an opposite sign in front of the u-derivative term. This leads to the blow up of the solution by gradually increasing deviations from sphericity beyond any bound and the solution can not be extended towards future null infinity. Therefore spherically symmetric collapse of a shell of null dust is not stable within RT family of solutions. The only possibility how to prevent such instability is then necessarily an inclusion of rotating modes that would absorb the divergent behavior.

2.2 Recent developments

Here we will summarize recent developments regarding Robinson–Trautman (RT) solutions with other sources or in non-standard setups as recently obtained by the author and collaborators.

2.2.1 RT with scalar field

In [33] an explicit four-parametric family of solutions in Robinson–Trautman class with free massless minimally coupled scalar field was derived and presence of quasilocal horizons was analyzed for this geometry of generic algebraic type II. Subsequently, asymptotic behavior, energy content of the spacetime and special cases of this geometry were analyzed in [34].

Although scalar field source is usually easily handled in highly symmetric situations and an aligned null radiation solution in RT class exists it proved nontrivial to derive first pure scalar field solution in RT family of geometries. The main reason being that aligned scalar field (its gradient pointing in the principal null direction of the geometry) is incompatible with the scalar field equation of motion in RT geometry. Most matter sources were primarily included to RT solution by considering their aligned setup and this turned out to be impossible for scalar field. Moreover, to accommodate the nonaligned scalar field source the Robinson–Trautman line element had to be modified to admit sources of more general Ricci type as subsequently confirmed in the classification of RT geometries [35].

The generic solution derived in [33] has some surprising properties. Unlike the vacuum RT solutions it can be extended to both positive and negative infinities of retarded time. Furthermore, although in early retarded times the singularity is present there is no quasilocal horizon covering it. Only later it becomes covered by such horizon. This behavior is similar to an appearance of naked singularities in the Vaidya spacetime for certain intensity of null radiation flux [36]. Even these singularities are later covered. The analysis of the horizon performed in [33] heavily relied on a mathematical procedure developed in [37].

As shown in [34] the generic RT solution with scalar field asymptotically settles to Minkowski geometry since all the scalar field content is radiated away. This picture is confirmed by analyzing the Bondi mass which contains only the scalar field contribution and asymptotically vanishes. This shows that the no-hair theorem for scalar field is satisfied since the scalar field is present only in the dynamical phase of the spacetime while it vanishes when approaching the final stationary state. At the same time the results provide additional information towards understanding the limits of cosmic censorship hypothesis of Roger Penrose due to the appearance of naked singularity.

Ideally, one would like to understand behavior of this solution in light of the numerical approaches to the scalar field spacetimes. These studies show that based on the initial data the solution evolves either towards black hole as a final state or the scalar field disperses leading to the Minkowski geometry. Our RT scalar field solution seemingly belongs to the second alternative suggesting that it automatically contains only weaker scalar field. But other properties concerning singularity and quasilocal horizon hint at much more complex situation which deserves dedicated attention in the future.

Special subcases of the general solution arising when some of the parameters vanish were shown to correspond to previously known spherically symmetric solutions, the dynamical Roberts solution [38] and limiting case of the Janis–Newman–Winicour solution [39] (which was originally discovered by Fisher much earlier [40]). This shows that the discovered RT-scalar field solution generalizes previous symmetric solutions. Finally, a phantom scalar field version was analyzed briefly in [34] and it was shown to represent a dynamical appearance and disappearance of a wormhole throat.

This research was published in the following papers:
[33] T. Tahamtan and O. Svítek: Robinson-Trautman solution with scalar hair, Phys. Rev. D 91(10):104032, 2015
`doi: https://doi.org/10.1103/PhysRevD.91.104032`

[34] T. Tahamtan and O. Svítek: Properties of Robinson–Trautman solution with scalar hair, Phys. Rev. D 94(6):064031, 2016
`doi: https://dx.doi.org/10.1103/PhysRevD.94.064031`

2.2.2 RT with nonlinear electrodynamics

Apart from the inclusion of Maxwell field (see e.g. [18]) Robinson–Trautman geometry can accommodate also nonlinear electrodynamics (NE) as a source as shown in [41] (additionally a cosmological constant was included). The main focus has been on traditional NE models (e.g. Born–Infeld) that were developed in order to deal with the singular field of a point charge. New RT solutions were obtained from the spherically symmetric ones by the use of a newly developed generating technique. The resulting algebraic type II solutions asymptotically approach the corresponding spherically symmetric geometries which can be again interpreted as showing the nonlinear stability of such solutions within these RT generalizations. Although the electromagnetic field is non-divergent the spacetime contains curvature singularity. Therefore the presence of horizons was analyzed using previously developed tools [37, 33] leading to natural conditions on the existence of horizon that depend on the value of cosmological constant and charge.

Another interesting class of nonlinear electrodynamics models considered recently are those removing curvature singularity while preserving horizon thus leading to so-called regular black holes. These models use magnetic charge to achieve this behavior. However for these models the generating method developed in [41] fails which might indicate that they may be restricted to spherically symmetric situations only.

Although the NE solutions in RT geometry obtained in [41] are of algebraic type II they do not contain electromagnetic radiation and so there appears to be further room for generalization. This together with the problem of NE models generating regular black holes prompted further study of the Robinson–Trautman spacetimes coupled with NE [42] where a well-posed RT solution with NE was as well derived (which is not possible in the Maxwell case).

This research was published in the following paper:
[41] T. Tahamtan and O. Svítek: Robinson–Trautman solution with nonlinear electrodynamics, Eur. Phys. J. C 76(6):335, 2016
doi: `https://dx.doi.org/10.1140/epjc/s10052-016-4175-9`

2.2.3 Thin-shell wormhole in RT

The investigation of a bridge-type wormhole within Robinson–Trautman class supported by the continuously distributed phantom scalar field performed in [34] motivated our interest in using RT geometry for investigating the other type of wormhole — a thin-shell one. In [43] two identical copies of vacuum RT solutions were glued together in a mirror setup to produce a wormhole throat. The junction condition were shown to produce two streams of a perfect fluid on the throat whose densities are negative since the matter supporting the wormhole cannot satisfy energy conditions.

The asymptotic behavior of the vacuum RT solutions is employed to show that this wormhole asymptotically settles down to thin-shell Schwarzschild wormhole with a throat approaching the position of horizon from above. This can again be interpreted as showing a nonlinear stability of such spherically symmetric wormhole within the RT geometry. Since the outer spacetime is standard RT solution usually describing outer geometry of a black hole the gravitational waves produced by this newly constructed dynamically evolving wormhole throat are indistinguishable from the corresponding black hole version of RT spacetime. Finally, the construction is shown to be easily generalized to RT spacetimes with cosmological constant, Maxwell field or nonlinear electrodynamics sources.

This research was published in the following paper:
[43] O. Svítek and T. Tahamtan: Nonsymmetric dynamical thin-shell wormhole in Robinson–Trautman class, Eur. Phys. J. C 78(2):167, 2018
doi: `https://dx.doi.org/10.1140/epjc/s10052-018-5628-0`

Chapter 3

Inhomogeneous cosmologies and averaging

In this part we will comment on two main aspects that one has to tackle when deviating from strictly homogeneous isotropic cosmology as modeled by FLRW solution which is still part of the most successful picture of our universe — the ΛCDM model — consistent with observations. One concerns the question of averaging the inhomogeneities in both matter content and geometry consistently in order to explain the overall situation in terms of the averaged homogeneous model. The other is the application of exact inhomogeneous cosmological models and their proper analysis. Such models can provide insight into structure formation on non-perturbative level.

We are leaving out one complicated issue from our following discussions although it has attracted attention recently. When the inhomogeneous geometry is involved we have to account for the position of the observer and understand how the local geometry might influence our interpretation of the observed data. Evidently even this problem can best be addressed via exact inhomogeneous models.

In the upcoming sections we will comment on specific inhomogeneous cosmological models and the averaging techniques that can be applied in cosmology.

3.1 Inhomogeneous cosmological models and their properties

We first introduce the main models used in the literature and point out their most interesting features. We start with the spherically symmetric LTB model since it is easily understandable while it still contains crucial novel features stemming from inhomogeneity. These features are carried over to more complex geometries lacking symmetry.

3.1.1 LTB solution

One of the first and most extensively studied inhomogeneous cosmological models is the Lemaître–Tolman–Bondi solution [12, 13, 14] which preserves spherical symmetry and is sourced by a dust. One can imagine this cosmological model as being composed of concentric spherical shells of dust with varying density that evolve under their own gravity. This makes it obviously ideal starting point for models of a star interior as well.

Potential role of this model in describing general structure formation was studied in [44, 45, 46] and with attention to formation of voids in [47] (see as well [48] for broader context).

The Lemaître–Tolman–Bondi (LTB) metric can be described by the line element in comoving coordinates

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E(r)}dr^2 + R^2[d\theta^2 + \sin^2(\theta)d\phi^2] \ , \tag{3.1}$$

where $E(r)$ is an arbitrary function and by prime we denote partial derivative with respect to r. Einstein equations with dust source imply the following equation for the function $R(t,r)$ which describes the dynamics of the model

$${R_{,t}}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3}R^2 \ , \tag{3.2}$$

here $M = M(r)$ is another arbitrary function resulting from integration. The dust source energy density ρ satisfies

$$\kappa\rho = \frac{2M'}{R'R^2} \ , \tag{3.3}$$

with κ being a gravitational constant.

The arbitrary function $E(r)$ has a geometrical interpretation, it determines a local curvature of the $t = const$ hypersurfaces (e.g., they have flat geometry for $E(r) = 0$) which can vary for different values of r. The function $M(r)$ can be interpreted as a gravitational mass contained within the comoving spherical shell at any given r. The density defined in equation (3.3) can evidently diverge if either $R = 0$ or $R' = 0$. The condition $R = 0$ corresponds generically to a curvature singularity which should be absent for all times other than potential Big Bang and/or Big Crunch if the model describes cosmology. Special behavior of function M can render the singularity absent and density finite even for $R = 0$. The other possibility ($R' = 0$ when $M' \neq 0$) is an effect of inhomogeneity and the associated divergence of density (and Kretschmann scalar) is driven by collision of neighboring dust shells leading to the name shell-crossing singularity. This type of singularity is highly undesirable because density additionally changes sign in this location. Ensuring the absence of shell-crossing singularities constitutes crucial part of deriving useful inhomogeneous models for cosmology.

If we formally integrate the equation (3.2) we obtain

$$\int_0^R \frac{d\tilde{R}}{\sqrt{2E + \frac{2M}{\tilde{R}} + \frac{1}{3}\Lambda\tilde{R}^2}} = t - t_B(r) \ , \tag{3.4}$$

where $t_B(r)$ is the third free function of r arising as an integration "constant" (called the bang time function). This brings another aspect missing in the FLRW picture, in the LTB model the Big Bang does not need to happen simultaneously but rather the timing depends on the radial coordinate r. This claim is of course made with respect to the comoving synchronous coordinates used in (3.1). The above formulas are invariant under transformation $\tilde{r} = g(r)$. We can use this freedom to fix one of the functions $E(r), M(r)$ and $t_B(r)$.

3.1.2 Lemaître model

This is a generalization of the LTB model to admit perfect fluid source which was described already by Lemaître [12]. The metric in the comoving coordinates takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{C(t,r)}\mathrm{d}t^2 + \mathrm{e}^{A(t,r)}\mathrm{d}r^2 + R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right), \tag{3.5}$$

where $R(t,r)$ is an areal radius of a sphere $\{t = \mathrm{const}, r = \mathrm{const}\}$ as was the case in LTB model. The normalized 4-velocity of the fluid is $u^\mu = \mathrm{e}^{-\frac{C}{2}}\delta^\mu_0$. The Einstein equations including the cosmological constant term read

$$\kappa p = -\frac{2M_{,t}}{R^2 R_{,t}}, \tag{3.6}$$

$$\kappa\rho = \frac{2M_{,r}}{R^2 R_{,r}}, \tag{3.7}$$

where the newly introduced function $M(t,r)$ satisfies

$$2M(t,r) = R + \mathrm{e}^{-C}R{R_{,t}}^2 - \mathrm{e}^{-A}R{R_{,r}}^2 - \frac{1}{3}\Lambda R^3, \tag{3.8}$$

p is the pressure and ρ the energy density. Generally, the pressure $p(t,r)$ and the energy density $\rho(t,r)$ are functions of both coordinates t and r. The function $M(t,r)$ is referred to as the Misner–Sharp mass [49] and describes energy content within a sphere of given radius on a given time-slice. The limit case of zero pressure leads to the LTB solution.

Since one of the possibilities of curing the shell-crossing singularity would be the pressure holding the shells of matter apart the Lemaître solution is ideal tool to study such scenario. However, only limited results are available so far due to the nonlinearity of the above system of PDEs. Addition of purely homogeneous pressure to LTB model was thoroughly discussed in [50].

3.1.3 Szekeres solution

The Szekeres spacetime is an exact dust solution of Einstein equations without any symmetries (no Killing vectors exist). It was found by Szekeres [15] and generalized by Szafron [16] for an energy momentum tensor describing a perfect fluid.

The general metric in the comoving coordinates takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\beta}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \mathrm{e}^{2\alpha}\mathrm{d}z^2, \tag{3.9}$$

with both α and β being functions of all the coordinates. The comoving synchronous coordinates imply $u^\mu = \delta^\mu_0$ as a four-velocity describing the motion of the dust. There are two major subfamilies of these solutions depending on whether $\beta_{,z} = 0$ or $\beta_{,z} \neq 0$. Since the second case ($\beta_{,z} \neq 0$) includes the LTB spacetime as a spherically symmetric limit it has attracted more attention and it will be our main focus here as well.

The metric of the $\beta_{,z} \neq 0$ subfamily of Szekeres spacetimes can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\left(R' - R\frac{\mathcal{E}'}{\mathcal{E}}\right)^2}{\epsilon + 2E(r)}\mathrm{d}r^2 + \frac{R^2}{\mathcal{E}^2}\left(\mathrm{d}p^2 + \mathrm{d}q^2\right), \tag{3.10}$$

where

$$\mathcal{E}\left(r,p,q\right) \equiv \frac{S}{2}\left[\left(\frac{p-P}{S}\right)^2 + \left(\frac{q-Q}{S}\right)^2 + \epsilon\right], \tag{3.11}$$

$$\mathcal{E}' = \frac{S'}{2}\left[1 - \frac{\left(p-P\right)^2}{S^2} - \frac{\left(q-Q\right)^2}{S^2}\right] - \frac{P'}{S}\left(p-P\right) - \frac{Q'}{S}\left(q-Q\right) \tag{3.12}$$

and f, P, Q, S are arbitrary functions of r and prime denotes a derivative with respect to r. The parameter ϵ takes three discrete values $-1, 0, 1$ and determines the geometry of the two-spaces of constant t and r which can be hyperbolic, planar or spherical. The resulting type of Szekeres spacetime is then quasi-hyperbolic, quasi-planar or quasi-spherical. A dynamical equation for the function $R(t,r)$ follows from Einstein equations

$$R_{,t}{}^2 = \frac{2M\left(r\right)}{R} + 2E\left(r\right) + \frac{\Lambda R^2}{3} \tag{3.13}$$

and an equation describing the density profile has the following form

$$\kappa\rho = 2\frac{M' - 3M\frac{\mathcal{E}'}{\mathcal{E}}}{R^2\left(R' - R\frac{\mathcal{E}'}{\mathcal{E}}\right)}\ , \tag{3.14}$$

where $M(r)$ is again an integration function and Λ is a cosmological constant. Integration of (3.13) leads to the appearance of a bang time function $t_B\left(r\right)$ similarly to the LTB case. Indeed one clearly sees that the line element (3.1) was generalized into (3.10) by introducing the function $\mathcal{E}$. So it is not surprising how similarly the equations describing dynamics look, compare (3.2) resp. (3.3) with (3.13) resp. (3.14). This similarity immediately raises the question whether shell-crossing singularities can appear here as well. They can indeed appear if $R' = R\frac{\mathcal{E}'}{\mathcal{E}}$ which corresponds to the vanishing of the denominator in (3.14). Therefore when using Szekeres solution in cosmology the avoidance of shell-crossing singularities represents important aspect which is harder to ensure than in spherically symmetric LTB solution due to increased complexity.

If one selects $\epsilon = 1$ the geometry of two-spaces spanned by p, q becomes spherical and the spacetime is filled with spherical shells of dust which do not have common center and the distribution of the energy density on a given shell is not homogeneous but rather has a dipolar character. That is why the overall geometry is called quasi-spherical. Similar geometrical picture can be built for quasi-planar case $\epsilon = 0$ with planar foliations and quasi-hyperboloidal (sometimes called quasi-pseudospheric) case $\epsilon = -1$ with hyperboloidal foliations [51].

Although this solution has generally no spacetime symmetries the three-spaces of constant t (spatial sections) are conformally flat. The metric (3.10) is invariant with respect to rescaling of the radial coordinate r (followed by suitable redefinitions of functions) so the overall number of unspecified functions reduces from 6 to 5.

The Szafron solution [16] providing a perfect fluid generalization can be described using the same line element but the dynamical equation (3.13) would contain one more term containing a time integral of an expression involving pressure. As was the case for the Szekeres spacetime there are two branches of the solution determined by $\beta_{,z} = 0$ or $\beta_{,z} \neq 0$. Since the branch corresponding to

$\beta_{,z} \neq 0$ contains Lemaître solution as a spherically symmetric limit it deserves more interest and in the following we will have this branch on our mind when mentioning Szafron solution (frequently called Szekeres-Szafron solution because it contains the dust version as well).

3.2 Recent developments related to inhomogeneous cosmological models

Here we will comment on two important aspects related to inhomogeneous cosmological models briefly described above. The first one is the proper description of these cosmologies using initial data that do not lead to shell-crossing singularities. The second is the analysis of the presence and properties of quasilocal horizons in these models.

3.2.1 Initial data for Szekeres spacetime

In [52] we have carried out an investigation of the possibility to provide useful characterization of the Szekeres spacetime using only the initial data. This work was motivated by previous research [53] providing description in terms of initial and final data. This work concentrated on the quasi-spherical Szekeres spacetime and the initial data were composed from the function E (see (3.10)), radial density profile and the value of the maximum it attains on each shell (but not the position of the maximum). This reflects the fact that the density distribution has dipolar character on each spherical shell. Note that in this way only three functions were specified compared to 5 physical degrees of freedom of the Szekeres spacetime. Nevertheless, this limited set was sufficient for our analysis. The sign of function E determines the character of the evolution of the spacetime in the sense that when $E < 0$ the universe first expands and then collapses — being referred to as an elliptic evolution type. Similarly for $E = 0$ we have a parabolic and for $E > 0$ a hyperbolic evolution. Thus specifying the initial value of function E enables one to control the type of evolution that results.

The prime concern in [52] was to derive conditions that the selected set of functions should satisfy in order to avoid shell-crossing singularity. Furthermore, in the special case of $E = 0$ it was explicitly proven that if the conditions are satisfied on the initial spatial slice they will automatically be satisfied throughout the evolution. In this case we also developed useful approximation scheme which simplifies the understanding of the evolution when the initial inhomogeneity is small and can be helpful for estimating the results in complex scenarios without explicitly solving the nonlinear equations.

This research was published in the following paper:
[52] D. Vrba and O Svítek: Modelling inhomogeneity in Szekeres spacetime, Gen. Rel. Grav. 46(10):1808, 2014
doi: `https://dx.doi.org/10.1007/s10714-014-1808-x`

3.2.2 Horizons in inhomogeneous cosmologies

The existence and properties of horizons in inhomogeneous perfect fluid cosmologies was studied in [54] using their quasilocal characterization. Trapping horizon definition by Hayward [31] was employed to study horizons in Lemaître and Szafron spacetimes.

Due to spherical symmetry of the Lemaître solution the research led to deeper results for this case. In this spacetime an explicit equation governing both future (in the collapsing phase) and past horizons (in the expanding phase) was derived as well as conditions for being outer or inner horizons. In the special case when Misner–Sharp mass [49] is constant along the horizon hypersurface it was shown that such horizon has a null character and the perfect fluid on the horizon has negative pressure. Interestingly, the intrinsic and extrinsic geometry of the horizon in Lemaître spacetime is the same as in the LTB solution.

In the general case of the Szafron spacetime only conditions on the existence of the horizon were derived. These conditions take into account on which side of the shell-crossing condition $R' \lesseqgtr R\frac{\mathcal{E}'}{\mathcal{E}}$ we stay, if we are in collapsing or expanding phase and whether the neighboring shells are approaching each other or vice versa. Simplified conditions were derived for special cases of spacetime or horizon geometry when the symmetry is enhanced. Evidently, avoidance of shell-crossing singularities has direct effect on the structure of horizons within the cosmological model.

This research was published in the following paper:

[54] E. Polášková and O. Svítek: Quasilocal horizons in inhomogeneous cosmological models, Class. Quant. Grav. 36(2):025005, 2019

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3.3 Averaging in cosmology

Standard homogeneous cosmology is so far extremely successful when combined with the cosmological constant and the cold dark matter. Any alternative models are best compared when translated into the homogeneous setting which can be achieved by averaging. Even when we take homogeneous cosmology as a best model we know that matter distribution is highly inhomogeneous and therefore homogeneous description can work only as a model describing the average distribution (with suitably large averaging scale on the order of hundreds of megaparsecs).

However, averaging is nontrivial in general relativity for two main reasons. First, the theory is described in terms of tensorial fields and dynamical curved geometry making it difficult to develop consistent averaging procedure. And second, Einstein equations are highly nonlinear resulting in the appearance of additional nontrivial terms arising due to these nonlinearities and potentially giving rise to new phenomena.

Let us schematically show how to treat the averaging of Einstein equations. Let us assume that $g_{\mu\nu}$ is a metric describing inhomogeneous cosmology. We will denote by $\langle\cdot\rangle$ a suitable averaging procedure applicable to tensors, therefore $\langle g_{\mu\nu}\rangle$ denotes the averaged metric that describes slowly varying geometry or ideally a completely homogeneous one. Since Einstein tensor is nonlinear in metric we have in general

$$G_{\alpha\beta}(\langle g_{\mu\nu}\rangle) \neq \langle G_{\alpha\beta}(g_{\mu\nu})\rangle \ . \tag{3.15}$$

So when we average the Einstein equations for the inhomogeneous metric $g_{\mu\nu}$

$$G_{\alpha\beta}(g_{\mu\nu}) = \kappa\, T_{\alpha\beta} \tag{3.16}$$

and we want to interpret it in terms of the averaged (potentially homogeneous) metric $\langle g_{\mu\nu}\rangle$ we can rewrite the result in the following way

$$G_{\alpha\beta}(\langle g_{\mu\nu}\rangle) = \kappa\,\langle T_{\alpha\beta}\rangle + \underbrace{\left(G_{\alpha\beta}(\langle g_{\mu\nu}\rangle) - \langle G_{\alpha\beta}(g_{\mu\nu})\rangle\right)}_{C_{\alpha\beta}}\ , \tag{3.17}$$

where we denoted by $C_{\alpha\beta}$ a so-called correlation (or backreaction) term that effectively provides an additional source term for the averaged metric. The importance of the correlation term in cosmology was already noted in 80' by Ellis [55]. Originally it was hoped that such term can help explain the dark sector of cosmology but, as mentioned in the Introduction, this is now heavily disputed. We have not commented about the averaging of the energy momentum tensor $T_{\mu\nu}$ but it is usually much simpler, especially for perfect fluid.

The other problem mentioned above concerns correct averaging when tensorial fields are involved. Some of the first proposals applied averaging of tensors naively (we consider tensor Q in an abstract form and omit indices)

$$\langle Q(x)\rangle = \frac{\int\limits_{\Omega} Q(x+x')\sqrt{-g}\,\mathrm{d}x'^4}{\int\limits_{\Omega} \sqrt{-g}\,\mathrm{d}x'^4} \tag{3.18}$$

resulting in objects that could not be interpreted as tensors since the procedure adds together tensors at different points of a curved manifold. One of the first attempts to correct this issue was due to Brill and Hartle [56] who employed a so-called bilocal operator $b^{\alpha'}_{\beta}(x, x')$ that facilitated parallel transport along geodesic from arbitrary point x' to a reference point x where the integration of tensorial objects defined at the same point can be performed. Primed indices refer to components with respect to basis at point x'. The averaging prescription for a second rank covariant tensor $Q_{\mu\nu}$ is thus given by

$$\langle Q_{\mu\nu}(x)\rangle = \frac{1}{V_\Omega} \int\limits_{\Omega} b^{\alpha'}_{\mu}(x, x')\, b^{\beta'}_{\nu}(x, x')\, Q_{\alpha'\beta'}(x')\, \sqrt{-g(x')}\, \mathrm{d}x'^4 \; . \tag{3.19}$$

Brill and Hartle used this prescription when studying gravitational geon. Later it was used by Isaacson [57, 58] when studying high-frequency approximation for gravitational waves on curved background. More recently this averaging approach was further developed, e.g. in the theory of Macroscopic gravity [59, 60] where the averaging is extended to Cartan structure equations as well. These tensorial averaging schemes have attained high level of mathematical rigor, but due to their complexity they were successfully applied mostly in simplified situations.

Substantial simplification can in principle be achieved by concentrating on averaging of scalar quantities which will be our main topic in the next section.

3.4 Recent developments in averaging methods

Here we will shortly describe two new approaches to averaging in cosmology and previous research that motivated them.

3.4.1 Averaging using Cartan scalars

In paper [61] we have developed an averaging scheme based on Cartan scalars. This work was motivated by the averaging method using curvature scalars developed in [62] and based on the classification of geometries by such scalars [63].

The averaging scheme developed in [62] is based on characterizing the spacetime geometry using a set of scalars built from Riemann tensor and a finite number of its covariant derivatives

$$\mathcal{I} = \left\{ R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\delta}R^{\mu\nu\alpha\beta;\delta}, R_{\mu\nu\alpha\beta;\delta\gamma}R^{\mu\nu\alpha\beta;\delta\gamma}, \cdots \right\} . \quad (3.20)$$

It was shown in [63] that such characterization is locally determining the geometry of spacetime unless the geometry belongs to the Kundt class (characterized by shear-free, twist-free and non-expanding geodesic congruence) which exhibits degeneracy. Averaging the set of scalars $\mathcal{I}$ we obtain a new set $\langle \mathcal{I} \rangle$. However, the nonlinearities inherent in the definition of the set $\mathcal{I}$ mean that we are not guaranteed that there is a metric associated to the averaged set $\langle \mathcal{I} \rangle$ since relations among the members of $\mathcal{I}$ will not be preserved upon averaging. To deal with this problem all the elements of $\mathcal{I}$ that are not algebraically independent or can be derived using so-called syzygies (e.g. relations characterizing algebraic types) are removed to obtain reduced set $\mathcal{I}_R$. This set is then averaged $\langle \mathcal{I}_R \rangle$ and the relations that enabled the reduction are subsequently used to generate the full set of scalars. This set automatically satisfies the same algebraic relations and syzygies as the original spacetime and describes large-scale geometry.

Motivated by this result, we have employed Cartan scalars to average a spacetime geometry [61]. These scalars were originally developed by Cartan [64] to solve the problem of equivalence of geometries and adapted to general relativistic context by Karlhede [65]. Cartan scalars are essentially tetrad projections of Riemann tensor and a finite number of its covariant derivatives and therefore they are a true scalars only with respect to the frame bundle of the manifold. However, fixing the tetrad we have scalars on the manifold as well. As was the case for the curvature scalars even for Cartan scalars one can reduce the set to functionally independent quantities using relations stemming from Cartan structure equations on the frame bundle. This reduced set can be averaged and subsequently expanded back by applying the relations. The Cartan-Karlhede algorithm then guarantees the existence of geometry described by this new set of scalars. Although explicit construction of the metric from the full set of Cartan scalars is possible the procedure is quite difficult. Apart from geometry itself, one is usually interested in averaging Einstein equations as well. Fortunately, Einstein tensor can be written as a sum of certain Cartan scalars and therefore its average is constructed trivially and the correlation term is vanishing by this construction.

Due to the technically difficult construction of a metric out of Cartan scalars we have proposed an alternative approach to the problem. Compare the generated set of scalars for averaged geometry with Cartan scalars of known (potentially homogeneous) cosmological models and determine the best fit. Although this seems by no means unambiguous procedure, when applied in cosmology we are anyway led towards homogeneous models thus making the task much clearer. This approach additionally enables one to easily identify the correlation term. In the examples presented in [61] the correlation term gives rise to effective positive cosmological constant or spatial curvature term.

This research was published in the following paper:
[61] P. Kašpar and O. Svítek: Averaging in cosmology based on Cartan scalars, Class. Quant. Grav. 31:095012, 2014
doi: `https://dx.doi.org/10.1088/0264-9381/31/9/095012`

3.4.2 Averaging in LRS class of spacetimes

One of the most influential recent methods for averaging specifically in cosmological context was developed by Buchert [66, 67]. It is based on 3+1 splitting of spacetime which is well defined by the assumed perfect fluid flow and involves only spatial averaging. It focuses on averaging of certain projections of Einstein equations only thus describing backreaction on fluid flow expansion but not its shear. The spatial averaging does not commute with the evolution of quantities in time which has to be taken care of.

The appeal of such restricted averaging approach led us to consider averaging limited to the (class II dust-filled) locally rotationally symmetric (LRS) spacetimes [68]. LRS spacetimes are characterized by existence (in the neighborhood of each point) of nontrivial subgroup of Lorentz group which leaves the Riemann tensor and its covariant derivatives up to third order invariant. This can be understood as a presence of local axis of symmetry at each point providing additional vector field on the spacetime apart from the fluid flow. The presence of additional structure allows the description of these spacetimes in terms of scalars constructed by suitable projections. These can be spatially averaged similarly to Buchert's method. However, apart from nontrivial commutation relations between time-evolution and averaging we have additional commutation relations between averaging and evolution along the additional vector field representing the local symmetry of LRS spacetimes. Subsequently, all the Einstein equations including constraints can be averaged and the backreaction terms for all the quantities appear explicitly. Importantly, the averaged constraints are shown to be preserved during evolution. The full set of averaged equations is not closed but we provide suggestions for resolving this issue — an infinite hierarchy of equations that can be truncated when only finite precision is sufficient or when invoking additional relations for higher order correlation terms. The averaging method is applied to approximate LTB (so-called onion) model showing that backreaction in the equation for shear (optical scalar of the fluid flow) is dominant over the backreaction in the equation for expansion. This would suggest that at least in this specific case the Buchert approach might omit significant effect.

This research was published in the following paper:
[68] P. Kašpar and O. Svítek: Averaging in LRS class II spacetimes, Gen. Rel. Grav. 47(2):4, 2015
doi: `https://dx.doi.org/10.1007/s10714-014-1844-6`

Chapter 4

The relation between black hole solutions and cosmological models

Black hole solutions containing pure radiation have a preferred null direction along which the null matter radiates. The simplest example is the spherically symmetric Vaidya spacetime. Removing all the symmetries while keeping twistfree geometry we have a pure radiation generalization of Robinson–Trautman solution where the pure radiation direction coincides with the principle null congruence defining the geometry.

On the other hand, dust-filled cosmological models naturally have a well defined timelike vector field of the dust flow — we have spherically symmetric LTB models or the Szekeres class generally lacking any symmetries.

Obvious question then arises whether null dust black hole solutions can be obtained by pushing the speed of the timelike dust flow in cosmological models towards the speed of light in some well-defined limiting procedure. There is indeed such possibility. Gleiser [69] found that quasi-spherical Szekeres model turns to pure radiation Robinson–Trautman solution and Hellaby [70] later identified this limit as a generalization of the Kinnersley rocket solution [71]. Vaidya spacetime was shown to be a null limit of certain LTB solutions by Lemos [72].

4.1 Recent developments

Motivated by the above mentioned research we have described the limiting process in greater detail generalizing it to non-zero cosmological constant and proposed how to treat the reverse process (going from pure radiation to timelike dust) in [73]. The extra information needed for the reversal helped to clarify how to treat functional dependencies correctly during the limiting procedure.

Although the limiting procedure is not unique we have shown using results of Geroch [74] that considering the geometrical and physical properties of the spacetimes under consideration the arbitrariness of the process is largely removed. Additionally, employing general arguments about spacetime limits investigated in [74] we have shown that since the Szekeres solution is of algebraic type D its null limit can maximally be of type D as well [73]. This clearly restricts the resulting geometries to a subclass of Robinson–Trautman family that lacks gravitational radiation. Such limitation seems quite natural since one would need very specific property of the geometry of the cosmological model that would transform into exact gravitational waves upon taking the null limit. Therefore, attempts to generalize this relation between geometries towards type II Robinson–Trautman solution are unlikely to succeed.

This research was published in the following paper:
[73] C. Hellaby and O. Svítek: Reversing the null limit of the Szekeres metric, Class. Quant. Grav. 38(3):035004, 2021
doi: `https://dx.doi.org/10.1088/1361-6382/abcc0c`

Chapter 5

Conclusion

In the previous chapters we have shortly described recent research of the author and its background focusing on black hole spacetimes and cosmology. In both areas the presented research extended description of more generic situations lacking symmetries and/or describing inhomogeneous matter distribution. These are important both from astrophysical and mathematical point of view since they can uncover which features are robust and which are merely a coincidence of highly symmetric models. This is especially true in nonlinear theory such as general relativity. At the same time certain important aspects might be missing in highly symmetric situations.

In chapter 2 the main results concerned scalar field generalization of the Robinson–Trautman family of solutions which helped in better understanding the applicability of no-hair theorem and cosmic censorship hypothesis. Asymptotic behavior of Robinson–Trautman equation was successfully used when studying nonlinear electrodynamics generalization of Robinson–Trautman geometry and nonlinear stability of thin-shell wormhole.

In chapter 3 new results concerning Szekeres spacetime were provided. Its characterization using initial data was developed which avoids the shell-crossing singularity (undesirable feature absent in homogeneous cosmology) and existence of horizons for the perfect fluid generalization was analyzed. Averaging problem in cosmology was tackled by focusing on averaging of scalar quantities. Two new approaches were developed, one using averaging of Cartan scalars and the second employing scalar quantities characterizing LRS class II dust cosmologies.

Finally, in chapter 4 a bridge between solutions covered in previous two chapters was presented consisting from limiting procedure taking dust-filled cosmological models to pure radiation filled black hole spacetimes.

Appendix A

List of summarized papers

In the order of appearance in the text:

- T. Tahamtan and O. Svítek: Robinson-Trautman solution with scalar hair, Phys. Rev. D 91(10):104032, 2015
  `doi: https://doi.org/10.1103/PhysRevD.91.104032`

- T. Tahamtan and O. Svítek: Properties of Robinson–Trautman solution with scalar hair, Phys. Rev. D 94(6):064031, 2016
  `doi: https://dx.doi.org/10.1103/PhysRevD.94.064031`

- T. Tahamtan and O. Svítek: Robinson–Trautman solution with nonlinear electrodynamics, Eur. Phys. J. C 76(6):335, 2016
  `doi: https://dx.doi.org/10.1140/epjc/s10052-016-4175-9`

- O. Svítek and T. Tahamtan: Nonsymmetric dynamical thin-shell wormhole in Robinson–Trautman class, Eur. Phys. J. C 78(2):167, 2018
  `doi: https://dx.doi.org/10.1140/epjc/s10052-018-5628-0`

- D. Vrba and O Svítek: Modelling inhomogeneity in Szekeres spacetime, Gen. Rel. Grav. 46(10):1808, 2014
  `doi: https://dx.doi.org/10.1007/s10714-014-1808-x`

- E. Polášková and O. Svítek: Quasilocal horizons in inhomogeneous cosmological models, Class. Quant. Grav. 36(2):025005, 2019
  `doi: https://dx.doi.org/10.1088/1361-6382/aaf77e`

- P. Kašpar and O. Svítek: Averaging in cosmology based on Cartan scalars, Class. Quant. Grav. 31:095012, 2014
  `doi: https://dx.doi.org/10.1088/0264-9381/31/9/095012`

- P. Kašpar and O. Svítek: Averaging in LRS class II spacetimes, Gen. Rel. Grav. 47(2):4, 2015
  `doi: https://dx.doi.org/10.1007/s10714-014-1844-6`

- C. Hellaby and Otakar Svítek: Reversing the null limit of the Szekeres metric, Class. Quant. Grav. 38(3):035004, 2021
  `doi: https://dx.doi.org/10.1088/1361-6382/abcc0c`

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