Report on thesis by Jan Scherz entitled: *Weak Solutions to Mathematical Models of the Interaction between Fluids, Solids and Electromagnetic Fields*

December 20, 2023

The PhD thesis, "Weak Solutions to Mathematical Models of the Interaction between Fluids, Solids, and Electromagnetic Fields" by Jan Scherz, investigates the mathematical analysis of systems of nonlinear partial differential equations that model various physical phenomena. The thesis focuses on two primary classes of problems: the interaction of an electrically conducting fluid with an insulating rigid body and the evolution of a magnetoelastic material. The main achievements of the thesis lie in establishing the existence of weak solutions for each of these studied problems.

Organized into six chapters and an appendix, the thesis begins with an Introduction in the first chapter. This section presents the motivation and methodology, along with a review of the state-of-the-art literature. Furthermore, the mathematical models to be analyzed in the thesis are precisely formulated.

The second chapter provides a detailed mathematical derivation of the fluid-rigid body models with an electrically conducting fluid. Although this chapter does not introduce original results, it serves to offer additional clarifying details about the physical derivation and interpretation of the models. The derivation starts by coupling the Maxwell equations with the Navier-Stokes equations and the equations governing the motion of rigid bodies. From this general system, the author derives simplified systems that are used in the subsequent analysis. The chapter clearly distinguishes between physically and mathematically motivated simplifications. Physically, simplifications are achieved by reformulating the system in non-dimensional form and neglecting small terms. Mathematically, simplifications are introduced to render rigorous analysis viable, and the reasons for these assumptions are discussed in detail.

In the third chapter, the author analyzes the interaction between an incompressible electrically conducting fluid and an insulating rigid body. The geometrical configuration is such that the fluid is contained in a container with a floating rigid body. As the fluid domain depends on the position of the rigid body, it is unknown a priori. Consequently, the problem falls into the category of moving boundary fluid-structure interaction problems, an active area of research. The primary result of this chapter is the establishment of the existence of a weak solution. This weak solution persists until the moment when the rigid body comes into contact with the container's boundary. The main technical challenge arises from geometric non-linearity, given that test functions in induction equation depend on the position of the rigid body. To overcome this difficulty, the author employs a modification of the Rothe method, specifically a hybrid discretization scheme. In this scheme, the transport equation, which tracks the position of the rigid body, remains continuous, while other equations are discretized in time. The rigid body is approximated using the Brinkman penalization, allowing it to be permeable on the approximate level. This approach yields suitable approximate solutions, and a weak solution is obtained in the limit.

In the fourth chapter, a similar problem is explored, but this time with a compressible fluid and the allowance for a finite number of rigid bodies. The main result is the existence of a global-in-time weak solution. Notably, the constructed solutions permit contact between rigid bodies or between a rigid body and the container boundary. However, the description of contact and post-contact dynamics seem non-physical. The challenges parallel those in the previous chapter, with an additional difficulty in constructing a discretization scheme that preserves the non-negativity of density. The author addresses this concern by discretizing only the induction equation in time, keeping other equations continuous. Several additional levels of approximation are introduced, a common practice in the mathematical theory of compressible fluids. Once again, a weak solution is obtained as the limit of approximate solutions.

In the fifth chapter, the thesis explores the evolution of magnetoelastic materials, characterized by the interaction between magnetization and deformation. The analyzed materials fall into the category of so-called generalized standard materials and exhibit dissipation. Specifically, they possess coercive elastic energy in $W^{2,p}, p > 3$, and the determinant of the deformation gradient is penalized to remain bounded away from zero. It's crucial to note the non-convex nature of the energy, which seems to preclude the use of standard PDE techniques. However, this challenge is addressed through the application of De Giorgi's minimizing movement scheme. In this approach, the problem is semi-discretized in time, and instead of solving a PDE, a minimization problem is solved at each step. This strategy allows for the utilization of calculus of variations techniques, well-suited for such problems. Additionally, solutions immediately satisfy the energy inequality. The main result of this chapter is the establishment of the existence of a local-in-time weak solution. Furthermore, potential reasons for solutions not being global-in-time are identified.

In the concluding chapter, the main results are succinctly summarized, and potential avenues for further research are outlined. The Appendix recalls important results and concepts employed in the thesis, and presents some technical results.

Conclusion The original and correct results presented in this thesis contribute to significant advancements in the field, opening new avenues for research. The work demonstrates a high level of technical proficiency and is excellently written, presenting details in a clear and precise manner while highlighting and explaining conceptual difficulties comprehensively. Notably, two papers related to chapters three and four have already been published in prestigious mathematical journals. In my opinion, the author has showcased the ability to conduct creative scientific work. Considering these factors, I strongly recommend the thesis for acceptance, suggesting a grade of *summa cum laude.*

Zagreb, December 20, 2023

Boris Muha
Department of Mathematics
Faculty of Science
University of Zagreb