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Advisor evaluation: Jakub Krásenský

Vítězslav Kala

The thesis

As explained in a well-written, detailed Introduction, the primary area of the Ph.D. thesis of Jakub Krásenský are universal quadratic forms over number fields and corresponding topics, such as the quadratic Waring's problem and additively indecomposable integers. This is an internationally active area with significant contributions by giants such as Fermat and Gauss – but also recently Bhargava, the laureate of Fields medal 2014.

While my past work led to a fairly good understanding of universal forms and indecomposables over quadratic fields $\mathbb{Q}(\sqrt{D})$, it relied on the periodic continued fraction for $\sqrt{D}$, which is no longer available in number fields of higher degree.

The author's dissertation work contributed to several advances that managed to break through this barrier and to obtain strong results in higher degrees, which is a significant achievement.

The most impressive results are contained in Chapter 4 of the thesis (based on a joint paper with Pavlo Yatsyna, published in Proc. Amer. Math. Soc. 2023) that considers the g-invariant $g_{\mathcal{O}}(r)$ of the ring of integers $\mathcal{O}$ of a totally real number field K, i.e., the smallest number of squares of linear forms that is required to represent all the quadratic forms of rank r that are representable as the sum of any number of squares. This topic naturally fits in the study of Waring's problem and its variants, and also on the specific goal of estimating $g_{\mathcal{O}}(r)$ there have been numerous recent works.

The proof is based on author's realization that one can relate $g_{\mathcal{O}}(r)$ to $g_{\mathbb{Z}}(rd)$, where d is the degree of K over $\mathbb{Q}$ (and more generally, the g-invariant of a ring to its suitable subring). This then leads to very nice estimates for $g_{\mathcal{O}}(r)$, sidestepping the need to deal with the delicate arithmetic of $\mathcal{O}$ and reducing the estimates to the ring of integers $\mathbb{Z}$. This not only establishes stronger results than Chan–Icaza (Trans. Amer. Math. Soc. 2021), but also gives a simpler proof for them.

Chapters 1 and 2 of the thesis focus on the arithmetic of totally real biquadratic number fields. Chapter 1, based on a joint work of the author together with Tinková and Zemková (published in Proc. Edinb. Math. Soc. 2020), proves Kitaoka's conjecture on the non-existence of ternary universal quadratic forms over biquadratic fields.

Chapter 2 is based on a joint paper with Raška and Sgallová (published in Expo. Math. 2022) and focuses on Pythagoras numbers P of orders $\mathcal{O}$ in totally real biquadratic fields, i.e., on determining the least number P such that if an element of $\mathcal{O}$ is the sum of any number of squares, then it is the sum of P squares. This is an extremely classical and well-studied topic in algebraic number theory, and Chapter 2 establishes (bounds on) the Pythagoras number in many cases.

Both of these chapters are quite impressive: They are technically very involved and long (42 and 40 pages) – and of course required that the authors come up with a number of clever ideas. In both cases, the candidate significantly contributed to the results (at least corresponding to the number of authors).

Chapters 3 and 5 are solo works of the author, the first of which appeared in Arch. Math. (Basel) 2022, and the second one should be submitted for publication soon. Both concern Pythagoras numbers of orders of number fields: Chapter 3 shows that the ring of integers of the cubic field of discriminant 49 has Pythagoras number 4 (which is the smallest possible value).

Chapter 5 then determines Pythagoras number of various non-maximal orders in biquadratic fields.

Not only does the thesis contain a number of impressive original results, but the author also makes great effort to understand things thoroughly, and to communicate his ideas very clearly. His thesis is thus written in a very reader-friendly way.

The candidate

Let me also make some general comments on the Ph.D. candidate and his other achievements.

Overall, he has a broad range of interests. Besides the algebraic number theory problems contained in the thesis, he has also worked on topics that are closer to combinatorics or theoretical computer science, especially on generalized number systems and cellular automata. He is currently preparing several manuscripts on these topics.

His research has been supported by a 3-year grant from Charles University (2020–2022). These are competitive grants for Ph.D. students that give nice funding for stipends and travels. The candidate was the PI of a successful team of 3 other students that, among other positives, also provided him with opportunities for further developing his leadership skills.

The candidate is a very enthusiastic teacher. In the last five years (starting already while he was a Master's student), he has taught 8 exercise classes for Calculus and Linear Algebra courses, and 2 classes for an elective Number Theory Proseminar (all for first year undergraduate students), as well as 2 exercises for a second year Number Theory course. He takes great care of his students and makes sure that they understand the material well, and so he did a very good job in all of his courses, as evidenced for example by the fact that he received *Dean's teaching award* in 2020.

He has given many seminar and conference talks that were entertaining and well-prepared, including invited talks at the ALGAR summer school 2022 in Belgium and at a recent summer school in Germany, as well as several invited seminar talks.

He has also been very active in the popularization of mathematics. For six years, he organized a Czech Mathematical correspondence seminar for high school students (with over 100 participants each year) – he prepared and graded problem sheets for the students, organized camps for them full of math lectures as well as games, and served for one year as the main organizer of the whole seminar.

The candidate also has been an active member of our department, always willing to help with whatever was necessary. In particular, he for example served as the referee for four different Bachelor's theses.

Summary

Overall, I think that the candidate clearly demonstrated his potential as an independent researcher with an established international network of collaborators, as well as his commitment to teaching and organizational work. Therefore I strongly recommend the acceptance of his thesis and the award of the Ph.D. title.

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