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Report on
the thesis "Universal quadratic forms over orders in number fields" by Jakub Krásenský

The thesis comprises four published papers and one preprint, the preprint and one of the papers are in single authorship. The presentation of all these papers is appealing to a broad mathematical audience; they have been accepted in very good non-specialised international mathematical journals. With this publication record, Jakub Krásenský proves that he can find new relevant research results and present them in a broad mathematical context. The thesis itself contains a very well written long introduction, where Jakub Krásenský exposes these new results. Therefore I strongly recommend to accept the thesis.

The subject of the thesis is classical within the area of number theory and in particular the arithmetic theory of quadratic forms. It has its roots in the classical problem of representability of integers as sums of squares. Already Lagrange has shown that every non-negative integer is the sum of four squares. Over $\mathbf{Q}(\sqrt{5})$ every totally non-negative integer is the sum of three squares, a result due to Maaß. In modern language: $x^2+y^2+z^2+t^2$ is a universal quadratic form over $\mathbf{Z}$ and $x^2+y^2+z^2$ is a universal quadratic form over the ring of integers $\mathbf{Z}[\alpha]$ in $\mathbf{Q}(\sqrt{5})$ where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio. In fact this is one of the rare universal ternary quadratic forms.

Siegel showed that over totally real number fields other than $\mathbf{Q}$ or $\mathbf{Q}(\sqrt{5})$ the sum of any number of squares cannot be universal. So one needs a notion for the set of all sums of squares.

For a ring R, let ΣR^2 be the subset of R that consists of all sums of squares. Then $\Sigma\mathbf{Z}^2 = \mathbf{Z}_{\geq 0}$ and $\Sigma\mathbf{Z}[\alpha]^2$ is the set of all totally positive elements in this ring of integers of $\mathbf{Q}(\sqrt{5})$.

In general it is not easy to give a nice description of ΣR^2. In the paper number three (a single author paper that appeared in the Archiv der Mathematik) Jakub Krásenský investigates ΣO^2 for the ring of integers O in the smallest totally real cubic field, the maximal real subfield of the seventh cyclotomic field. He shows that all totally positive elements of O are sums of squares, except for the ones of norm 7. This allows him to conclude the main result of this paper: any element of O that is the sum of squares is already the sum of four squares. It is well known (by Hilbert reciprocity) that four variables are necessary for number fields of odd degree. One interesting side result is that the diagonal form $x^2 + y^2 + z^2 + t^2 + \beta s^2$ is universal over O for any totally postive $\beta \in O$ of norm 7. The previously known universal forms over this ring were either not classical or of higher dimension.

The Pythagoras number $P(R)$ is the minimum ℓ such that every elements in ΣR^2 is a sum of ℓ squares. So the above results imply that $P(\mathbf{Z}) = 4$, $P(\mathbf{Z}[\alpha]) = 3$, and $P(O) = 4$.

The second chapter of the thesis, based on the paper of Krásenský with two younger coauthors, deals with Pythagoras numbers of orders in biquadratic number fields K. They show that for all but possibly seven such fields the Pythagoras number is at least 5. The paper also exhibits an infinite family of maximal orders in totally real biquadratic fields with Pythagoras number 7, which is the largest possible Pythagoras number by a result of Kala and Yatsyna. Building upon these ideas Krásenský treats nore general non-maximal orders R in biquadratic number fields in the last chapter. If R contains $\sqrt{2}$ or the golden ratio α, then $P(R) \leq 5$. Also the infinite family from the second chapter can be generalised to a much larger class of orders.

A totally positive definite integral quadratic form is universal, if it represents all totally positive integral elements. The famous 15-theorem by Conway and Schneeberger shows that a quadratic form over $\mathbf{Z}$ is universal, if and only if it represents all positive integers in $\{1, \dots 15\}$. The first chapter is based on a long paper with two more experienced coauthors published already in 2020 and deals with the question of the existence of ternary universal forms. There are no such forms over the rational integers, and Chan, Kim and Raghavan exhibited all such forms over real quadratic fields: they only exist over $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, and $\mathbf{Q}(\sqrt{5})$. It is conjectured that over all other totally real number fields there are no totally positive universal ternary forms. This is proven, if the field has odd degree over the rationals.

Chapter 1 contributes to this research by showing the non-existence of universal ternary quadratic forms over biquadratic fields.

Chapter 4 is the highlight of this thesis. It deals with the analogue of the Pythagoras number for quadratic forms, the so called quadratic Waring's problem in totally real number fields. For a number ring R the g-invariant of R is defined as $g_R(k) := \min\{\ell \mid$ any k-variable totally positive semidefinite quadratic form is the sum of ℓ squares of linear forms $\}$. In particular $g_R(1) = P(R)$ is the Pythagoras number of R. As R might not be a principal ideal domain, there is a similar notion of $G_R(k)$ where "quadratic forms" are replaced by "lattices". Then Chapter 4 proves the following result that has a nice application of Waring's problem to Pythagoras numbers: For a number field extension with $[K : F] = d$ and any order R in K that contains the ring of integers O_F of F one gets $g_R(k) \leq g_{O_F}(kd + 1)$ and $G_{O_K}(k) \leq G_{O_F}(kd)$. In particular $P(O_K) \leq G_{O_F}(d)$. One important consequence is that all g-invariants of orders in number fields are finite, previously only known for maximal orders. The authors also obtain the lattice g-invariant $G(2)$ of all real quadratic fields K; $G_{O_K}(2) = 7$ for all $K \neq \mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, and $\mathbf{Q}(\sqrt{5})$, the three fields admitting ternary universal forms.

Signed

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