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pp-Elimination of Quantifiers in Module Theories

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To my loving wife Anastasia, dear Mother, Brother, and Father, and my good friends Thomas Cassar and Marek Makovec who have all valiantly supported me throughout the years, as well as to my learned professors and my brilliant-yet-patient supervisor Jan Šaroch; to them I humbly present this work with heartfelt thanks.

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Abstract: The aim of this thesis is to prove the Baur–Monk Theorem and thereby show complete module-theories admit an elimination of quantifiers down to (Boolean combinations of) existential formulæ.

To achieve this, following a brief introduction in Chapter 1, the reader is familiarised in Chapter 2 with the notion of a positive-primitive formula in the language of right $\boldsymbol{R}$-modules, and its close relationship with commutative groups, their cosets, and lattices.

Chapter 3 first lays the technical groundwork for the proof of the Baur–Monk Theorem, presented in Section 3.3, in its opening two subsections which contain the needed combinatorial and group-theoretical results, namely the Neumann Lemma and a variation on the Inclusion-Exclusion Principle.

Chapter 4 concludes the mathematical work contained herein with a brief overview of some immediate corollaries of the the Baur–Monk Theorem and earlier results.

Keywords: pp-Formula, Quantifier Elimination, pp-Definable Subgroup

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1. Introduction

1.1 A Historical Note

The intellectual foremother of the present work is arguably Wanda Szmielew who in her article *Elementary Properties of Abelian Groups* (see [12]) showed a version of Baur–Monk Theorem for Abelian groups.

Building upon further work of Kenneth Jon Bairwise, Paul Eklof, and Gabriel Sabbagh between 1955 and 1976, Walter Baur was the first man to prove this result in full in his article *Elmination of Quantifiers for Modules* (see [2]). Independently of Mr Baur, James Donald Monk gave a proof for Abelian groups which 'works just as well over any ring' in the words of Mike Prest in [8].

Indeed, it is a variation on his proof, as given by Martin Ziegler, and Alessandro Achille with Luca Ghidelli in *Model Theory of Modules* (see [13]) and *Modules in Model Theory* (see [1]) respectively, presented in Chapter 3.

1.2 Sources

The bulk of the present work is largely based on [8], [13], [1], and, in passing, on [9], and [4] (listed in decreasing order of significance), while the more elementary treatment of mathematical logic made use of the remarks and definitions found in [7], [3], and [10]. Some elementary results from other fields, such as combinatorics and group theory, have been sourced from [6] and [11] respectively.

1.3 Motivation

During his first encounter with Real Analysis, the young student imagines a typical real-valued univariate function (or simply *a function* in his mind) as a sequence of smooth hills and valleys over an uninterrupted interval. It is only with further instruction this idealistic, human view is altered by introducing into his thought-horizon the notions of functions so fantastically counterintuitive as those of Weierstrass and Cantor.

By the time his education in the subject concludes, he will have had to accept such functions are not only well-defined but, in fact, make up so much of the function landscape it is nigh miraculous there is any room left for the 'human' ones.

Much like one may imagine the requirement a function be *continuous everywhere yet differentiable nowhere* or *infinitely-differentiable everywhere yet analytic nowhere* is too strong to hold for any function, one may at first be doubtful if the requirement a first-order theory should yield a quantifier-free equivalent for any formal writing of arbitrary complexity is likewise too strong.

By way of analogy to Real Analysis, almost every theory has this property, or, more precisely, every theory may be extended to do so:

Let $\boldsymbol{T}$ be any theory in the language $\mathcal{L}$. Enrich both systems: $\mathcal{L}$ by adding an n-ary relation symbol R_ϕ for every $\mathcal{L}$-formula ϕ in n free variables, and $\boldsymbol{T}$ by adding the axiom

$$(\forall x_1, \ldots, x_n)\, \phi(x_1, \ldots, x_n) \leftrightarrow R_\phi(x_1, \ldots, x_n) \quad \text{for each formula } \phi.$$

The new, enriched theory on the extended language admits a full elimination of quantifiers. Sadly, this clumsy procedure has bloated it to such a degree nothing sensible may be said about the quantifier-free formulæ thereof.

The language and theory of modules is stark opposite: They are poor with meagre vocabulary and all expression limited to systems of linear equations[1].

They, nevertheless, satisfy the heavy requirement at least halfway. Baur–Monk Theorem 3.3.1, whose proof in its fullness is the aim of most pages to follow, shows every formula in the language of modules is equivalent *modulo some complete theory of modules* to a Boolean combination of existential formulæ of a very specific form. Plainly put, they admit a partial elimination of quantifiers.

1.4 Preliminaries

Herein we fix conventions used in the sections to follow; they are to be kept at the back of one's mind while allowing for their explicit contradiction should it be convenient to do so.

I acknowledge the following pages give leeway to some informal treatment and abuse of notation. For example, the simple *if* is used in definitions[2] as opposed to the more logically correct *iff*.

We shall, moreover, speak of *sets of formulæ* and even iterate over such sets even though the logic assumed throughout this paper is of first order. We permit these instances to accommodate tradition in the former and enhance brevity in the latter. Another repeated instance of abuse will be speaking of *subgroups of modules*; by this we will have in mind the *subgroups of the module's underlying commutative group*.

Notation 1.4.1 (Boldface)**.** Given an algebraic structure $\boldsymbol{A}$, be it a ring, a module, etc., we denote it using boldface while its underlying set is left without any special formatting. For example, given some ring $\boldsymbol{R}$, its underlying set is denoted R.

Convention 1.4.2 (Ring $\boldsymbol{R}$)**.** We fix the symbol $\boldsymbol{R}$ to stand for a fixed ring $(R, +, \cdot, -, 0, 1)$ on a (nonempty) set R.

Notation 1.4.3 (Variables)**.** We fix the variables' set

$$\mathrm{Var} = \{x, y, z, u, v, v_0, v_1, \ldots, w, w_0, w_1, \ldots\}\,.$$

[1]See Remark 1.4.6

[2]e.g. 'An integer is said to be even *if* it is a multiple of 2'

Convention 1.4.4 (Language $\mathcal{L}$)**.** Our principal language[3] shall be that of right $\boldsymbol{R}$-modules $\mathcal{L} = (+, -, 0, r)_{r \in R} \cup \text{Var} \cup \{=\}$. Namely, $\mathcal{L}$ contains a binary function-letter $+$, a unary function-letter $-$, a constant-letter 0, and a unary function-letter r for each element $r \in R$, applied using the postfix notation '$\_r$' as a matter of convention.

Whenever we speak of well-formed formulæ, or simply formulæ, we mean $\mathcal{L}$-formulæ unless contradicted.

Notation 1.4.5 (Theory $\mathbf{Mod}_{\mathrm{R}}$)**.** We fix the symbol $\mathbf{Mod}_{\mathrm{R}}$ to stand for the theory of right $\boldsymbol{R}$-modules: the theory of commutative groups enriched by the following set of axioms for each pair $(r, s) \in R^2$:

$$(\forall u, v)\,(u + v)r = ur + vr \qquad (\forall v)\,v(r + s) = vr + vs$$
$$(\forall v)\,v(r \cdot s) = (vr)s \qquad (\forall v)\,v1 = v$$

Remark 1.4.6. $\mathcal{L}$-terms are precisely $\boldsymbol{R}$-linear combinations; since $\mathcal{L}$ contains only 0 as a constant-letter and its only predicate-letter is $=$, any atomic $\mathcal{L}$-formula is equivalent (modulo $\mathbf{Mod}_{\mathrm{R}}$) to the homogeneous $\boldsymbol{R}$-linear equation:

$$v_1 r_1 + v_2 r_2 + \cdots + v_n r_n = 0.$$

A nonzero RHS may be obtained either by enriching $\mathcal{L}$ with constant-letters or by a partial substitution of parameters (see Notation 2.0.6).

Convention 1.4.7 (Logical Symbols)**.** We shall consider the standard collection of logical symbols

$$\wedge, \vee, \neg, \leftarrow, \rightarrow, \leftrightarrow, \forall, \exists.$$

To cut down on needless brackets when composing formulæ, we assign them descending priority in the order we have listed them.

For formal purposes, we shall consider the reduced collection $\wedge, \neg, \forall$ and treat the remaining symbols as derived therefrom.

Notation 1.4.8 (Overlines)**.** Whenever we write $\phi(\overline{v})$ for a formula ϕ and any sequence $\overline{v} = \{x_\alpha\}_{\alpha \in I}$ of distinct variables from Var, we indicate that all variables occurring free in ϕ are amongst those in $\overline{v}$, though we do not require any of them actually occur free in ϕ. By $\operatorname{len} \overline{v}$ we understand the length of the sequence $\overline{v}$.

We shall often simply write $\phi(\overline{v})$, $\phi(\overline{w})$, and so forth, without giving an explicit enumeration of such sequences; it will be sufficient to suppose they are long enough to contain all those variables in Var that do occur free in ϕ in concordance with the foregoing convention

An analogous convention for sets of formulæ $\Phi(\overline{v}), \Psi(\overline{v}), \ldots$ shall be used.

Notation 1.4.9 ($\equiv$)**.** Whenever we wish to assign a specific formula a name, the designation will be indicated by the binary operator $\equiv$; e.g. if we wish to refer to the formula $(\forall x)\,x0 = 0$ by ϕ in some subsequent writing, the fact is denoted by $\phi \equiv (\forall x)\,x0 = 0$.

[3]We use the term *language* as defined in [3]; i.e. to designate a disjoint union of relation- and function-letters with assigned arities. The term used on `en.wikipedia.org` `encyclopediaofmath.org`, as of June 2023, is *signature*.

2. Positive-Primitive Formulæ

In this section we present an elementary introduction into the theory of *positive-primitive formulæ*, or sometimes more briefly *pp-fomulæ*, in the language of right $\boldsymbol{R}$-modules. We, nevertheless, deviate from the standard definition of *positive-primitive*, as found in [8], [9], [5] and others, where such formulæ are defined outright and not as formulæ *positive* and *primitive* (variations on the respective definitions of which may be found in in [3] and [4]), which shall be our approach[1].

Definition 2.0.1 (Positive, Primitive, Positive-Primitive Formula)**.** Let $\mathcal{A}$ be a first-order language. An $\mathcal{A}$-formula ϕ is said to be:

- *positive* if $\vdash \phi \leftrightarrow \chi$ where χ contains no occurrences of $\neg$.
- *primitive* if $\vdash \phi \leftrightarrow \exists \overline{w}\, \psi$ for some $\mathcal{A}$-formula ψ, which itself is a conjunction of atomic and negations of atomic $\mathcal{A}$-formulæ.
- *positive-primitive* if $\vdash \phi \leftrightarrow \exists \overline{w}\, \omega(\overline{v}, \overline{w})$ where ω is a conjunction of atomic $\mathcal{A}$-formulæ.

Definition 2.0.2 (Positive-Primitive $\mathcal{L}$-Formula & Sentence, Prefix, Matrix)**.** Due to the poverty of our language $\mathcal{L}$, recalling Remark 1.4.6, an $\mathcal{L}$-formula of n free variables is positive-primitive if it is equivalent to a formula of the form

$$\overbrace{(\exists w_1, w_2, \ldots, w_l)}^{\text{Prefix}} \underbrace{\bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} \right) = 0}_{\text{Matrix}},$$

where for each i, j, k, we have $r_{ij}, s_{kj} \in R$. The terms *prefix* and *matrix* of such a formula are defined as indicated[2]; the latter term will be shown to be especially fitting in Remark 2.0.5. Naturally, a closed positive-primitive formula is said to be a *positive-primitive sentence.*

Example 2.0.3. Suppose we have two positive-primitive formulæ ϕ, ψ. Then due to our relaxed definition of primitiveness[3] $\phi \wedge \psi$ is likewise positive-primitive (relabel the variables to forestall conflicts if necessary, join up their conjunctions and move the existential quantifiers to a common prefix). This fact shall be used a number of times, including in the proof of Baur–Monk Theorem 3.3.1 in Chapter 3.

Notation 2.0.4 ($\mathfrak{pp}$)**.** We fix the symbol $\mathfrak{pp}$ as shorthand for the positive-primitive formula specified in Definition 2.0.2. One may think of it as a general positive-primitive formula with familiar components that may be readily referenced (such as r_{ij}, s_{ij}, etc).

[1] We do so for reasons of etymology.

[2] These two terms have been passed down from the given formula being in the prenex normal form.

[3] Compared to [7], for example.

Remark 2.0.5. The $\mathfrak{pp}$ formula may be rewritten using matrix-notation as

$$(\exists w_1, \ldots, w_l)(v_1, \ldots, v_n, w_1, \ldots, w_l) \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1m} \\ r_{21} & r_{22} & \ldots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \ldots & r_{nm} \\ s_{11} & s_{12} & \ldots & s_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{l1} & s_{l2} & \ldots & s_{lm} \end{bmatrix} = (0, \ldots, 0),$$

or more compactly as

$$(\exists \overline{w})\,(\overline{vw}) \begin{bmatrix} R \\ S \end{bmatrix} = \overline{0},$$

where $[R\,S]^T$ is a matrix with a block-decomposition into matrices $R = (r_{ij})_{i,j=1}^{n,m}$ and $S = (s_{ij})_{i,j=1}^{n,m}$. This gives us a new way of viewing $\mathfrak{pp}$. Finally, if one prefers, the above may be further rewritten to obtain

$$\exists \overline{w}\, \overline{v} R = \overline{w}(-S).$$

This suggests one may think of positive-primitive formulæ as generalised divisibility statements.

Notation 2.0.6. Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. By $\mathcal{L}_A$ for some $A \subseteq M$ we understand the enriched language obtained from $\mathcal{L}$ by adding a constant-letter for each $a \in A$. Of special interest may be the largest such language for any one $\boldsymbol{M}$, denoted $\mathcal{L}_c$, especially if $\boldsymbol{M}$ is understood only tacitly.

Remark 2.0.7. It is clear from Remark 2.0.5 that positive-primitive formulæ pertain to the solubility of $\boldsymbol{R}$-linear equations. In the case of positive-primitive sentences, they contain the binary information whether a system defined thereby *is* soluble.

We shall now list two elegant yet trivial examples of positive-primitive formulæ.

Example 2.0.8.

(1) The positive-primitive $\mathcal{L}$-formula $vr = 0$, where $r \in R$, is satisfied by precisely those elements a of a right $\boldsymbol{R}$-module $\boldsymbol{M}$ annihilated by r.

(2) The positive-primitive $\mathcal{L}_{\{a\}}$-sentence $(\exists w)a = wr$, where $r \in R$ and $a \in M$ captures the divisibility of a by r (see Remark 4.1.4).

By working with tuples $\overline{v}, \overline{w}, \overline{a}$ we may generalise these conditions as conjunctions of entrywise equalities analogous to those above. Such generalisations are clearly other examples of positive-primitive formulæ and sentences respectively.

2.1 pp-Definable Subgroups & Cosets

One of the staples of Linear Algebra is the subspace-oriented view of solution-sets of systems of linear equations.

In view of Remark 2.0.5, an analogy lends itself to consideration: We shall introduce the notion of a subspace defined by a system of linear equations (represented by a positive-primitive formula). As we shall see, these subspaces are closed under addition and, if a few concessions on the generality of $\boldsymbol{R}$ are granted, also closed under scalar multiplication.

Definition 2.1.1 (Positive-Primitive-Definable Set)**.** Let $\phi(\overline{v})$ be a positive-primitive formula and $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. Then those $\overline{a} \in M^{\operatorname{len} \overline{v}}$ for which $\boldsymbol{M} \models \mathfrak{pp}(\overline{a})$ define a set $\phi(\boldsymbol{M})$ termed *positive-primitive-definable* or more briefly *pp-definable*. An analogous notion for a set Φ of positive-primitive formulæ follows:

$$\phi(\boldsymbol{M}) \stackrel{\text{Def}}{=} \left\{\overline{a} \in \boldsymbol{M}^{\operatorname{len}\overline{v}} \,\middle|\, \boldsymbol{M} \models \phi(\overline{a})\right\}$$
$$\Phi(\boldsymbol{M}) \stackrel{\text{Def}}{=} \left\{\overline{a} \in \boldsymbol{M}^{\operatorname{len}\overline{v}} \,\middle|\, (\forall \phi \in \Phi)\, \boldsymbol{M} \models \phi(\overline{a})\right\}.$$

Example 2.1.2. A real plane P in $\mathbb{R}^3$ may be represented by an equality of the form $ax + by + cz = d$, where a, b, c, d are real coefficients. In this context, planes are pp-definable sets over the module $\mathbb{R}_{\mathbb{R}}$.

More concretely, $P = \phi(\mathbb{R}_{\mathbb{R}})$, where $\phi(x, y, z) \equiv xa + yb + zc = d$ is a positive-primitive $\mathcal{L}_c$-formula.

Definition 2.1.3 (Witness)**.** Let $\mathcal{A}$ be a first-order language, $\boldsymbol{A}$ an $\mathcal{A}$-structure, and $\phi(\overline{v})$ a quantifier-free $\mathcal{A}$-formula. Then $\overline{a} \in A^{\operatorname{len}\overline{v}}$ is termed an $\boldsymbol{A}$*-witness*, or simply a *witness* supposing $\boldsymbol{A}$ is clear from context, of the existential formula $\exists \overline{v}\, \phi(\overline{v})$ if $\boldsymbol{A} \models \phi(\overline{a})$ (and hence $\boldsymbol{A} \models \exists \overline{v}\, \phi(\overline{v})$).

A set Φ of existential $\mathcal{A}$-formulæ defined as above is said to have $\boldsymbol{A}$*-witnesses*, or simply *witnesses*, in $\boldsymbol{A}$ if there exists an $\boldsymbol{A}$-witness for every $\phi \in \Phi$.

Example 2.1.4. In school-algebra over natural numbers[4] a number n is said to be *composite* if there exists a pair of two smaller non-unit numbers p, q such that $n = p \cdot q$. Then the pair $(2, 3)$ is a witness for 6 being composite.

Theorem 2.1.5 (Linearity of Positive-Primitive Formulæ)**.** *If* $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$*, then*

(1) $\boldsymbol{M} \models \mathfrak{pp}(\overline{0})$.

(2) $\boldsymbol{M} \models \mathfrak{pp}(\overline{a})$ *and* $\boldsymbol{M} \models \mathfrak{pp}(\overline{b})$ *implies* $\boldsymbol{M} \models \mathfrak{pp}(\overline{a} \pm \overline{b})$.

(3) If $r \in R$ *commutes with all* r_{ij}, s_{kj} *in* $\mathfrak{pp}$*, then* $\boldsymbol{M} \models \mathfrak{pp}(\overline{a})$ *implies* $\boldsymbol{M} \models \mathfrak{pp}(\overline{a}r)$*, where* $\overline{a}r$ *denotes entrywise multiplication.*

Proof. Let $\psi(\overline{v}, \overline{w})$ be the matrix of

$$\mathfrak{pp}(\overline{v}) \equiv (\exists w_1, w_2, \ldots, w_l) \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} = 0 \right).$$

[4] Formally the commutative semiring $(\mathbb{N}, +, \cdot, 0, 1)$.

(1) The element $(\overline{0}, \overline{0})$ is a witness for $(\exists \overline{v}, \overline{w})\, \psi(\overline{v}, \overline{w})$; in particular $\overline{0}$ is a witness for $(\exists \overline{w})\, \psi(\overline{0}, \overline{w}) \equiv \mathfrak{pp}(\overline{0})$. Hence $\boldsymbol{M} \models \mathfrak{pp}(\overline{0})$.

(2) Suppose $\boldsymbol{M} \models \mathfrak{pp}(\overline{a})$ and $\boldsymbol{M} \models \mathfrak{pp}(\overline{b})$. Then there are some $\overline{\alpha}$ and $\overline{\beta}$ such that $(\overline{a}, \overline{\alpha})$ and $(\overline{b}, \overline{\beta})$ are witnesses for $(\exists \overline{v}, \overline{w})\, \psi(\overline{v}, \overline{w})$. We observe this implies that $(\overline{a} \pm \overline{b}, \overline{\alpha} \pm \overline{\beta})$ is also a witness since

$$\boldsymbol{M} \models \psi(\overline{a} \pm \overline{b}, \overline{\alpha} \pm \overline{\beta}) \equiv \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} (a_i \pm b_i) r_{ij} + \sum_{k=1}^{l} (\alpha_k \pm \beta_k) s_{kj} = 0 \right)$$

$$\Leftrightarrow \boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\underbrace{\left(\sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} \alpha_k s_{kj} \right)}_{0} \pm \underbrace{\left(\sum_{i=1}^{n} b_i r_{ij} + \sum_{k=1}^{l} \beta_k s_{kj} \right)}_{0} = 0 \right).$$

Then in particular, $(\overline{\alpha} \pm \overline{\beta})$ is a witness for $(\exists \overline{w})\, \psi(\overline{a} \pm \overline{b}, \overline{w}) \equiv \mathfrak{pp}(\overline{a} \pm \overline{b})$, whence $\boldsymbol{M} \models \mathfrak{pp}(\overline{a} \pm \overline{b})$.

(3) Suppose $\boldsymbol{M} \models \mathfrak{pp}(\overline{a})$ and that r commutes with all r_{ij}, s_{kj} in $\mathfrak{pp}(\overline{v})$; there is some $\overline{b}$ such that $(\overline{a}, \overline{b})$ is a witness for $(\exists \overline{v}, \overline{w})\, \psi(\overline{v}, \overline{w})$. Then so is $(\overline{a}r, \overline{b}r)$ since

$$\boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} b_k s_{kj} = 0 \right)$$

$$\Rightarrow \boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} a_i r_{ij} r + \sum_{k=1}^{l} b_k s_{kj} r = 0r \right)$$

$$\Rightarrow \boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} (a_i r) r_{ij} + \sum_{k=1}^{l} (b_k r) s_{kj} = 0 \right).$$

QED

Remark 2.1.6. Observe Theorem 2.1.5 does not generally hold for $\mathcal{L}_c$-formulæ.

Definition 2.1.7 (Full $\boldsymbol{R}$-Characteristicity)**.** Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. It is said a subgroup $\boldsymbol{N}$ of $\boldsymbol{M}$ is fully $\boldsymbol{R}$-characteristic if, given any $\boldsymbol{R}$-endomorphism[5] f of $\boldsymbol{M}$, the subgroup $\boldsymbol{N}$ is invariant under f.

Lemma 2.1.8 (pp-Definable Subgroup Existence)**.** *Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. Then $\mathfrak{pp}(\boldsymbol{M})$ is a fully $\boldsymbol{R}$-characteristic subgroup of $\boldsymbol{M}^{\mathrm{len}(\overline{v})}$. In particular, it is an $\boldsymbol{R}$-submodule if $\boldsymbol{R}$ is commutative.*

Proof. One deduces that $\mathfrak{pp}(\boldsymbol{M})$ is a subgroup immediately from Theorem 2.1.5. It remains to show full $\boldsymbol{R}$-characteristicity.

Let $\overline{a} \in \mathfrak{pp}(\boldsymbol{M})$ and f be an $\boldsymbol{R}$-endomorphism of $\boldsymbol{M}$. Then there exists some $\overline{b}$

[5]By this term we mean a function $f : M \to M$ such that $f(ar + bs) = f(a)r + f(b)s$.

such that

$$\boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} b_k s_{kj} \right) = 0 \Rightarrow \boldsymbol{M} \models \bigwedge_{j=1}^{m} f\left(\sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} b_k s_{kj} \right) = f(0)$$

$$\Leftrightarrow \boldsymbol{M} \models \bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} f(a_i) r_{ij} + \sum_{k=1}^{l} f(b_k) s_{kj} \right) = 0.$$

Hence $f(\overline{a}) \in \mathfrak{pp}(\boldsymbol{M})$, proving the desired closure. If $\boldsymbol{R}$ is commutative (or rather if all r_{ij}, s_{ij} lie in the centre of $\boldsymbol{R}$) $\_r$ is an $\boldsymbol{R}$-endomorphism for each $r \in R$. QED

Definition 2.1.9 (Positive-Primitive-Definable Subgroup)**.** Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. The group $\mathfrak{pp}(\boldsymbol{M})$ is termed a *positive-primitive-definable subgroup*, or a *pp-definable (fully $\boldsymbol{R}$-characteristic) subgroup.* More accurately, it is a subgroup of $\boldsymbol{M}^{\mathrm{len}(\overline{v})}$ pp-definable in $\boldsymbol{M}$.

Remark 2.1.10. In view of Lemma 2.1.8, note $\mathfrak{pp}$ defines (in the categorical sense) a functor F from **Mod-$\boldsymbol{R}$** to **Ab** with $F(\boldsymbol{M}) = \mathfrak{pp}(\boldsymbol{M})$ for any object $\boldsymbol{M} \in \mathrm{Obj}(\mathbf{Mod}\text{-}\boldsymbol{R})$ and $F(f)$ defined as the restriction-map $\mathfrak{pp}(\boldsymbol{M}) \to \mathfrak{pp}(\boldsymbol{N})$ for any $\boldsymbol{R}$-homomorphism $f : \boldsymbol{M} \to \boldsymbol{N}$.

Example 2.1.11. Let $\boldsymbol{R}$ be the ring of 2×2 matrices over some field $\boldsymbol{K}$ and $\boldsymbol{M}$ be the regular module over $\boldsymbol{R}$. The ring is clearly noncommutative, and hence does not satisfy the assumptions of the latter part of Lemma 2.1.8. We define the positive-primitive formula

$$\phi(v) \equiv (\exists w) \left(v = w \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right). \text{ Clearly } \phi(\boldsymbol{M}) = \begin{bmatrix} \boldsymbol{K} & 0 \\ \boldsymbol{K} & 0 \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \middle| \, a, b \in K \right\}.$$

Observe $\phi(\boldsymbol{M})$ is a left but not a right ideal of $\boldsymbol{R}$ and consequently not a right submodule of the regular module $\boldsymbol{M}$.

Corollary 2.1.11.1 (Generalised pp-Definable Subgroup Existence)**.** *Let $\Phi(\overline{v})$ be a set of positive-primitive $\mathcal{L}$-formulæ and $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. Then $\Phi(\boldsymbol{M})$ is a fully $\boldsymbol{R}$-characteristic subgroup of $\boldsymbol{M}^{\mathrm{len}(\overline{v})}$.*

Proof. Since $\Phi(\boldsymbol{M}) = \bigcap \{\phi(\boldsymbol{M}) \,|\, \phi \in \Phi\}$, the claim immediately follows from Lemma 2.1.8. QED

Lemma 2.1.12 (pp-Definable Coset Existence)**.** *Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$ and consider $\mathfrak{pp}(\overline{v})$. Suppose we substitute specific values $a_{s+1}, a_{s+2}, \ldots, a_n \in M$ for the last $n - s$ variables in $\overline{v}$. Then the set*

$$\mathfrak{pp}(\boldsymbol{M}, \overline{a}) = \{\overline{c} \in M^s \,|\, \boldsymbol{M} \models \mathfrak{pp}(\overline{c}, \overline{a})\},$$

defined by the resultant positive-primitive formula, is either empty or is a coset of the subgroup $\mathfrak{pp}(\boldsymbol{M}, \overline{0})$ of $\boldsymbol{M}^s$.

Proof. Observe $\phi(\boldsymbol{M}, \overline{0})$ indeed is a group. This easily follows from Lemma 2.1.8 and the fact $\boldsymbol{M} \models \phi(\overline{c}, \overline{0})$ iff $\boldsymbol{M} \models \chi(\overline{c})$ for the positive-primitive formula χ derived from ϕ by omission of all $v_i r_{ij}$ for $i > s$.

Suppose $\phi(\boldsymbol{M}, \overline{a})$ is nonempty and let $\overline{c}, \overline{c}'$ be two, possibly nondistinct, elements therein; then $\boldsymbol{M} \models \phi(\overline{c}, \overline{a}), \phi(\overline{c}', \overline{a})$. Consequently, $\boldsymbol{M} \models \phi(\overline{c} - \overline{c}', \overline{a} - \overline{a}) = \phi(\overline{c} - \overline{c}', 0)$ by Theorem 2.1.5 whence we, in addition, infer that for any $c_0 \in \phi(\boldsymbol{M}, \overline{0})$, $\boldsymbol{M} \models \phi(\overline{c} + \overline{c}_0, \overline{a})$. Hence $\phi(\boldsymbol{M}, \overline{a})$ is a coset of $\phi(\boldsymbol{M}, \overline{0})$. QED

Definition 2.1.13 (pp-Definable Coset)**.** Let $\overline{a}$, ϕ, $\boldsymbol{M}$ be as in Lemma 2.1.12. The resultant coset $\phi(\boldsymbol{M}, \overline{a})$ is said to be *positive-primitive-definable* or more briefly *pp-definable*.

2.2 The Lattice of pp-Definable Groups

Theorem 2.2.1 (pp-Definable Group Arithmetic)**.** *Let $\phi(\overline{v}), \psi(\overline{v})$ be some pp-formulæ and $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. Then*

$$\phi(\boldsymbol{M}) \cap \psi(\boldsymbol{M}) = (\phi \wedge \psi)(\boldsymbol{M}) \qquad \phi(\boldsymbol{M}) + \psi(\boldsymbol{M}) = (\phi + \psi)(\boldsymbol{M})$$

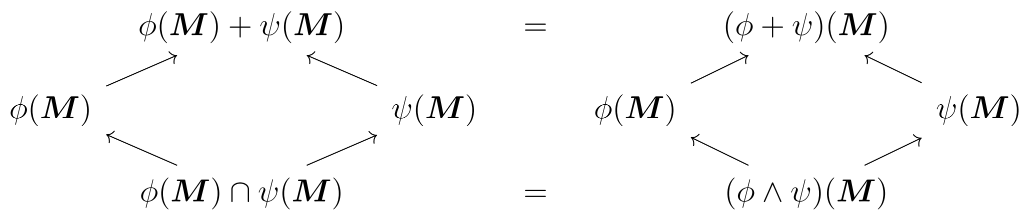
where

$$(\phi + \psi)(\overline{v}) \overset{\text{Def}}{\equiv} \exists \overline{u}, \overline{w}\, (\phi(\overline{u}) \wedge \psi(\overline{w}) \wedge \overline{v} = \overline{u} + \overline{w}).$$

Proof.

$(\phi \wedge \psi)$: Clearly, $\overline{a} \in \phi(M) \cap \psi(M) \Leftrightarrow \overline{a} \in \phi(M) \wedge \overline{a} \in \psi(M) \Leftrightarrow a \in (\phi \wedge \psi)(M)$.

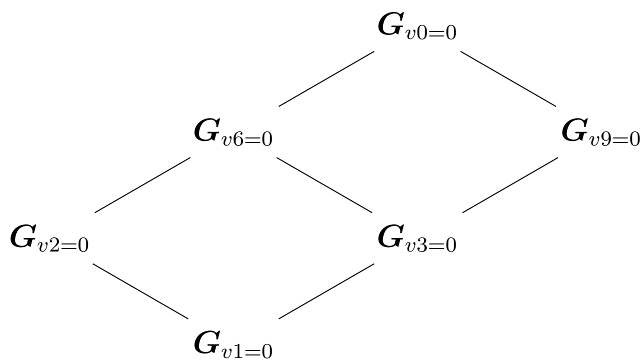
$(\phi + \psi)$: We have $\overline{a} \in \phi(M) + \psi(M)$ iff there are some $\overline{b} \in \phi(M)$ and $\overline{c} \in \psi(M)$ such that $\overline{a} = \overline{b} + \overline{c}$. This is equivalent with $\overline{a} \in (\phi + \psi)(M)$.



QED

Remark 2.2.2. It follows from Theorem 2.2.1 that given some $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$, the partially ordered set of all pp-definable subgroups of $\boldsymbol{M}^n$ forms a (modular) sublattice of the lattice of all subgroups of $\boldsymbol{M}^n$.

Example 2.2.3. Suppose $\boldsymbol{R} = \mathbb{Z}$ and let $(\mathbb{Z}_2 \oplus \mathbb{Z}_9)_{\mathbb{Z}} \models \mathbf{Mod}_{\mathrm{R}}$. The lattice of all pp-definable subgroups of $(\mathbb{Z}_2 \oplus \mathbb{Z}_9)_{\mathbb{Z}}$ is shown below.



For every group $\boldsymbol{G}_\phi$ in the lattice, ϕ is a positive-primitive formula defining it. In greater detail,

$$\begin{array}{rrr} \mathbb{Z}_1 \simeq \boldsymbol{G}_{v1=0} & \mathbb{Z}_2 \simeq \boldsymbol{G}_{v2=0} & \mathbb{Z}_3 \simeq \boldsymbol{G}_{v3=0} \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \boldsymbol{G}_{v6=0} & \mathbb{Z}_9 \simeq \boldsymbol{G}_{v9=0} & \mathbb{Z}_2 \times \mathbb{Z}_9 \simeq \boldsymbol{G}_{v0=0} \end{array}$$

We see that the lattice of all pp-definable subgroups coincides with the lattice of *all* subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_9$.

This is not always the case as evidenced by $\mathbb{Q}_\mathbb{Z}$; it can be proven that not only are not all subgroups of $\mathbb{Q}$ pp-definable in $\mathbb{Q}_\mathbb{Z}$, but that only the trivial ones are.

3. Elimination of Quantifiers

3.1 Combinatorial Lemmata

Notation 3.1.1. Let $0 < n \in \mathbb{N}$. Then $[n] := \{1, 2, \ldots, n\}$.

Theorem 3.1.2 (Inclusion-Exclusion Principle)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq \Delta \subseteq [n]} (-1)^{|\Delta|+1} \left|\bigcap_{i \in \Delta} A_i\right|.$$

This is a well-known result. For the proof, see Theorem 3.6.2 in [6] amongst other sources.

Corollary 3.1.2.1. *Let $A_0, A_1, A_2, \ldots, A_n$ be sets and A_0 be finite. Then*

$$A_0 \subseteq \bigcup_{i=1}^{n} A_i \Leftrightarrow \sum_{\Delta \subseteq [n]} (-1)^{|\Delta|} \left|A_0 \cap \bigcap_{i \in \Delta} A_i\right| = 0.$$

Proof. By application of the Inclusion-Exclusion Principle:

$$\begin{aligned}
\sum_{\Delta \subseteq [n]} (-1)^{|\Delta|} \left|A_0 \cap \bigcap_{i \in \Delta} A_i\right| = 0 &\Leftrightarrow \sum_{\emptyset \neq \Delta \subseteq [n]} (-1)^{|\Delta|} \left|A_0 \cap \bigcap_{i \in \Delta} A_i\right| + |A_0| = 0 \\
&\Leftrightarrow \sum_{\emptyset \neq \Delta \subseteq [n]} (-1)^{|\Delta|+1} \left|A_0 \cap \bigcap_{i \in \Delta} A_i\right| - |A_0| = 0 \\
&\Leftrightarrow \sum_{\emptyset \neq \Delta \subseteq [n]} (-1)^{|\Delta|+1} \left|\bigcap_{i \in \Delta} A_i \cap A_0\right| = |A_0| \\
&\overset{3.1.2}{\Leftrightarrow} \left|\bigcup_{i=1}^{n} A_i \cap A_0\right| = |A_0| \\
&\Leftrightarrow A_0 \subseteq \bigcup_{i=1}^{n} A_i.
\end{aligned}$$

QED

3.2 Group-Theoretical Lemmata

It is perhaps well-reflective of the origins of Baur–Monk Theorem in group theory nearly seventy years ago, that a good deal of mathematical labour needed for its proof remains with groups.

Since the requirement groups be commutative is not needed in this section, we discard it for generality's sake. This will be reflected in the newly adopted product-notation for groups (though the additive notation will be readopted in the subsequent sections).

Definition 3.2.1 (Cover, Thin Cover)**.** Let $A, A_1, A_2, \ldots, A_n$ be sets. It is said $A_1, A_2, \ldots, A_n$ form a *cover* of A if $A \subseteq \bigcup_{i=1}^n A_i$. Moreover, the cover is said to be *thin* if removing any one union-component would unmake it.

Lemma 3.2.2 (Coset Disjointness)**.** *Let $\boldsymbol{H} \leq \boldsymbol{G}$ be groups. Then for each a, b in G, either $aH = bH$ or $aH \cap bH = \emptyset$.*

Proof. Lemma 14.7 in [11]. QED

Lemma 3.2.3 (Index Frolics)**.** *Let $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n \leq \boldsymbol{H} \leq \boldsymbol{G}$ be groups (permissibly indistinct) and $a_1, a_2, \ldots, a_n \in G$ be coefficients such that they form the thin cover (even equality)*

$$H = \bigcup_{i=1}^n a_i H_i.$$

Then

(1) Every index $[\boldsymbol{H} : \boldsymbol{H}_i]$ is finite.

(2) $[\boldsymbol{H} : \boldsymbol{H}_i] \leq n$ for at least one $i \in [n]$.

(3) $[\boldsymbol{H} : \boldsymbol{H}_i] \leq n!$ for every $i \in [n]$.

Proof. Denote by r the number of distinct groups amongst the $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n$.

(1) The proof is by induction on r.

Base Case. Suppose $r = 1$. Then $H = a_1 H_1 \cup a_2 H_1 \cup \cdots \cup a_n H_1$. Since the cosets are disjoint (as per Lemma 3.2.2) we have

$$[\boldsymbol{H} : \boldsymbol{H}_1] = |\{aH_1 \mid a \in H\}| \leq n < \infty.$$

Note we actually did not need not have assumed the cover's thinness to show the foregoing.

Induction Step. Fix $j \in [n]$. Since $r \geq 2$ and the cover is thin, $H \setminus \bigcup \{a_i H_i \mid H_i = H_j\} \neq \emptyset$; let $g \in H \setminus \bigcup \{a_i H_i \mid H_i = H_j\}$.

We claim

$$gH_j \subseteq H \setminus \bigcup \{a_i H_i \mid H_i = H_j\}.$$

Towards a contradiction, suppose $gH_j = a_i H_j (= a_i H_i)$ (recall any two cosets are either disjoint or equal). Then there are some $h_1, h_2 \in H_j$ such that

$$gh_1 = a_i h_2 \Rightarrow g = a_i \underbrace{h_2 h_1^{-1}}_{\in H_j} \in a_i H_j. \qquad \text{A contradiction.}$$

Hence the inclusion holds and thus

$$gH_j \subseteq H \setminus \bigcup \{a_i H_i \mid H_i = H_j\} \subseteq \bigcup \{a_i H_i \mid H_i \neq H_j\},$$

whence

$$H_j \subseteq \bigcup \left\{ g^{-1} a_i H_i \,\middle|\, H_i \neq H_j \right\} \subseteq H.$$

We have effectively constructed a way to cover H without using H_j in any of the union-members of $\bigcup_{i=1}^n a_i H_i$ — any such member $a_k H_k$, with $H_k = H_j$, may be replaced:

$$a_k H_k = a_k H_j \subseteq \bigcup \left\{ a_k g^{-1} a_i H_i \,\middle|\, H_i \neq H_j \right\}.$$

The new equality only uses $r-1$ distinct groups, whereby the induction hypothesis shows $[H : H_i] < \infty$ for all $i \neq j$. But since j was arbitrary, we can repeat the procedure and reapply the induction-hypothesis to get $[H : H_j] < \infty$; concluding the induction-step.

(2) Consider the group $\boldsymbol{K} = \bigcap \boldsymbol{H}_i$. By (1), the index $[\boldsymbol{H} : \boldsymbol{K}] = m < \infty$. Towards a contradiction, suppose $[\boldsymbol{H} : \boldsymbol{H}_i] > n$ for each i. By Lagrange Theorem (Theorem 14.9 in [11])

$$n < [\boldsymbol{H} : \boldsymbol{H}_i] = \frac{[\boldsymbol{H} : \boldsymbol{K}]}{[\boldsymbol{H}_i : \boldsymbol{K}]}.$$

Then $[\boldsymbol{H}_i : \boldsymbol{K}] = [\boldsymbol{H} : \boldsymbol{K}]/[\boldsymbol{H} : \boldsymbol{H}_i] < m/n$. It follows

$$[\boldsymbol{H} : \boldsymbol{K}] = \left[\bigcup_{i=1}^n a_i \boldsymbol{H}_i : \boldsymbol{K} \right] \leq \sum_{i=1}^n [\boldsymbol{H}_i : \boldsymbol{K}] < n \cdot \frac{m}{n} = m = [\boldsymbol{H} : \boldsymbol{K}].$$

A contradiction.

(3) We proceed by induction on r.

Base Case. Suppose $r = 1$. Then the claim holds by (2).

Induction Step. By (2) we know there exists some coset H_k such that $[H : H_k] \leq n$. Without any loss of generality, assume it is H_1. Fix i; we will show the claim for an arbitrary $\boldsymbol{H}_i$. If $H_1 = H_i$, then the claim holds trivially. Suppose then $H_1 \neq H_i$.

Let

$$g \in H \setminus \bigcup \{ a_j H_j \mid j \neq i \}.$$

We reuse the cover from (1) to obtain

$$g H_1 \subseteq \bigcup \{ a_k H_k \mid H_k \neq H_1 \}.$$

To apply the induction-hypothesis, the cover need first be made thin and we make it such by removing any redundant elements (and for the sake of convenience keep the original indexing). Doing so retains $a_i H_i$ in the new (thin) cover because by our choice of g, it is the only coset containing g (which lies in $g H_1$).

We have obtained a new thin cover which is readily made into an equality:

$$H_1 = \bigcup \left\{ g^{-1} a_k H_k \cap H_1 \,\middle|\, H_k \neq H_1 \right\}.$$

There are at most $n-1$ cosets and $r-1$ subgroups involved. By the induction-hypothesis,

$$[\boldsymbol{H}_1 : \boldsymbol{H}_1 \cap \boldsymbol{H}_i] \leq (n-1)!$$

Lagrange Theorem yields:

$$\begin{aligned}[\boldsymbol{H} : \boldsymbol{H}_i] &\leq [\boldsymbol{H} : \boldsymbol{H}_1 \cap \boldsymbol{H}_i] \\ &= [\boldsymbol{H} : \boldsymbol{H}_1] \cdot [\boldsymbol{H}_1 : \boldsymbol{H}_1 \cap \boldsymbol{H}_i] \\ &\leq n \cdot (n-1)! \\ &= n!\end{aligned}$$

QED

Lemma 3.2.4 (Neumann). *Let $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n \leq \boldsymbol{H} < \boldsymbol{G}$ be groups (permissibly indistinct) with coefficients $a, a_1, a_2, \ldots, a_n \in G$ such that*

$$aH \subseteq \bigcup_{i=1}^{n} a_i H_i.$$

Then the cover of aH is preserved if one omits all cosets $a_i H_i$ with $n! < [\boldsymbol{H} : \boldsymbol{H} \cap \boldsymbol{H}_i]$. In particular, all cosets of infinite index in aH may be omitted.

Proof. A twofold simplification is possible from the outset:

(1) It may be supposed $a = 1$ since

$$aH \subseteq \bigcup_{i=1}^{n} a_i H_i \quad \Leftrightarrow \quad H \subseteq \bigcup_{i=1}^{n} \underbrace{a^{-1} a_i}_{a'_i} H_i.$$

(2) The equality $H = \bigcup_{i=1}^{n} a_i H_i$ may be considered instead since

$$H \subseteq \bigcup_{i=1}^{n} a_i H_i \quad \Leftrightarrow \quad H = \bigcup_{i=1}^{n} \underbrace{a_i H_i \cap H}_{a_i H'_i}.$$

Suppose the cover is thin; were it not so, we would remove all redundant members (and keep the original indexing for convenience's sake as before). By Lemma 3.2.3, $[\boldsymbol{H} : \boldsymbol{H}_i] \leq n!$ holds for every component of this slimmed-down cover; hence all cosets H_k such that $[\boldsymbol{H} : \boldsymbol{H}_i] > n!$ indeed may have been omitted. QED

Remark 3.2.5. It is clear from the proof of Neumann Lemma 3.2.4 that for the thin cover

$$aH \subseteq \bigcup_{i=1}^{n} a_i H_i,$$

the inequality $n! \geq [\boldsymbol{H} : \boldsymbol{H} \cap \boldsymbol{H}_i]$ is satisfied for all cover-members $\boldsymbol{H}_i$ from the outset.

3.3 Baur–Monk Theorem

Theorem 3.3.1 (Baur–Monk)**.** *For every* $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$*, every* $\mathcal{L}$*-formula is equivalent to a Boolean combination of positive-primitive formulæ:*

Given an $\mathcal{L}$*-formula* $\phi(\overline{v})$*, there exists a Boolean combination* $\psi(\overline{v})$ *of positive-primitive formulæ such that*

$$\boldsymbol{M} \models \psi(\overline{v}) \leftrightarrow \phi(\overline{v}).$$

Proof. We proceed by induction on ϕ. Atomic formulæ are positive-primitive as we have hinted at in Remark 1.4.6. If ϕ and ψ are equivalent to Boolean combinations of positive-primitive formulæ, then so are $\phi \wedge \psi, \phi \vee \psi, \neg\phi, \neg\psi$.

It remains to show the induction-step for $\forall$. Suppose then $\phi(\overline{v})$ indeed admits such a combination. We need to verify $\psi(x, \overline{v}) \equiv \forall x\, \phi(x, \overline{v})$ likewise has this property.

Simplifying ϕ

Recall positive-primitive formulæ are closed under conjunction[1], so whatever form ϕ may have, wherever there may be conjunctions of positive-primitive formulæ, we merge them. This yields (after a possible rearrangement of the disjuncts)

$$\phi \equiv \neg\phi_0^1 \vee \neg\phi_0^2 \vee \cdots \vee \neg\phi_0^{k'} \vee \phi_1 \vee \cdots \vee \phi_k \quad \text{for some pp } \phi_0^i, \phi_i.$$

Since $\neg\chi_1 \vee \neg\chi_2 \leftrightarrow \neg(\chi_1 \wedge \chi_2)$ for any formulæ χ_1, χ_2, we have

$$\neg\phi_0^1 \vee \neg\phi_0^2 \vee \cdots \vee \neg\phi_0^{k'} \leftrightarrow \neg(\phi_0^1 \wedge \phi_0^2 \wedge \cdots \wedge \phi_0^{k'}) \leftrightarrow \neg(\phi_0) \quad \text{for some pp } \phi_0.$$

This leads to the final simplification below. Note that we may assume without any loss of generality all ineffective disjuncts (such as duplicate ϕ_i) have been omitted.

$$\phi \equiv \neg\phi_0 \vee \phi_1 \vee \cdots \vee \phi_k \quad \text{which is equivalent to} \quad \phi_0 \to \phi_1 \vee \cdots \vee \phi_k. \tag{3.1}$$

Note

$$\boldsymbol{M} \models \forall x\, \phi \leftrightarrow \forall x\, (\phi_0 \to \phi_1 \vee \cdots \vee \phi_k). \tag{3.2}$$

The latter form is particularly useful, for it tells us

$$\boldsymbol{M} \models \psi(\overline{b}) \quad \text{iff} \quad \phi_0(\boldsymbol{M}, \overline{b}) \subseteq \bigcup_{i=1}^{k} \phi_i(\boldsymbol{M}, \overline{b}) \qquad \text{for all } \overline{b} \in M^{\operatorname{len} v}. \tag{3.3}$$

[1] i.e. if ϕ, ψ are positive-primitive then so is $\phi \wedge \psi$.

Translating into Language of Groups

Put $\boldsymbol{H}_i = \phi_i(\boldsymbol{M}, 0)$ and fix some arbitrary $\bar{b} \in M^{\operatorname{len} v}$. As we have observed at the beginning of the proof of Lemma 2.1.12, $\boldsymbol{H}_i$ is a group. By the same lemma, $\phi_i(\boldsymbol{M}, \bar{b})$ is either empty or a coset of $\boldsymbol{H}_i$ (for any i). Rewriting (3.3) using this notation, we obtain

$$a_0 + H_0 \subseteq \bigcup_{i=1}^{k} a_i + H_i \quad \text{for some } a_i \text{ in } M \text{ dependent on } \bar{b}.$$

By Neumann Lemma 3.2.4 (or rather Remark 3.2.5) the cover above is thin[2] with $[\boldsymbol{H}_0 : \boldsymbol{H}_i \cap \boldsymbol{H}_0] < \infty$. Then (by Lagrange's Theorem) $H_0/(H_i \cap H_0)$ is a finite set and, consequently, so is $H_0/\bigcap_{i=0}^{k} H_i$.

Denoting by π the natural projection $H \to H/\bigcap_{i=0}^{k} H_i$, the above yields the following inclusion with a finite LHS:

$$\pi(a_0) + \pi(H_0) \subseteq \bigcup_{i=1}^{k} \pi(a_i) + \pi(H_i). \tag{3.4}$$

Corollary 3.1.2.1 becomes applicable to (3.4) yielding the following from (3.2)

$$\boldsymbol{M} \models \psi(\bar{b}) \quad \text{iff} \quad \sum_{\Delta \in [k]} (-1)^{|\Delta|} \underbrace{\left| \pi(a_0) + \pi(H_0) \cap \bigcap_{i \in \Delta} \pi(a_i) + \pi(H_i) \right|}_{N_\Delta} = 0. \tag{3.5}$$

Eliminating the $\boldsymbol{a}_i$

Since the $\boldsymbol{H}_i$ are commutative,

$$\begin{aligned} N_\Delta &= \left| \pi(a_0) + \pi(H_0) \cap \bigcap_{i \in \Delta} \pi(a_i) + \pi(H_i) \right| \\ &= \left| \pi\left((a_0 + H_0 \cap \bigcap_{i \in \Delta} (a_k + H_i) \right) \right|. \end{aligned} \tag{3.6}$$

It follows from (3.6) and Lemma 3.2.2 that either

$$N_\Delta = |\emptyset| = 0 \qquad \text{or} \qquad N_\Delta = \left| \pi\left(H_0 \cap \bigcap_{\in \Delta} H_i \right) \right|, \qquad \text{depending on } \bar{b}.$$

Restated: The latter equality occurs iff the coset in question[3] is nonempty. By definition

$$\begin{aligned} a_0 + H_0 \cap \bigcap_{i \in \Delta} (a_i + H_i) &= \phi_0(\boldsymbol{M}, \bar{v}) \cap \bigcap_{i \in \Delta} (\phi_i(\boldsymbol{M}, \bar{b})) \\ &\overset{2.2.1}{=} \left(\phi_0 \wedge \bigwedge_{i \in \Delta} \phi_i \right) (\boldsymbol{M}, \bar{b}). \end{aligned}$$

[2] This is due to our assumption of ineffective disjuncts having been omitted.
[3] Meaning, the one whose cardinality N_Δ captures.

Therefore, if we define

$$\phi_\Delta(\overline{v}) \equiv (\exists x) \left(\phi_0 \wedge \bigwedge_{i\in\Delta} \phi_i \right) (x, \overline{v}),$$

then

$$N_\Delta \neq 0 \qquad \text{iff} \qquad \boldsymbol{M} \models \phi_\Delta(\overline{b})$$

whence the following is the set of precisely those $\Delta \subseteq [k]$ for which $N_\Delta \neq 0$:

$$\mathcal{N} = \left\{ \Delta \subseteq [k] \,\middle|\, \boldsymbol{M} \models \phi_\Delta(\overline{b}) \right\}.$$

Thus, (3.5) is reduced to

$$\begin{aligned} \boldsymbol{M} \models \forall x\, \phi \quad &\text{iff} \quad \sum_{\Delta\in\mathcal{N}} (-1)^{|\Delta|} N_\Delta = 0 \\ \boldsymbol{M} \models \forall x\, \phi \quad &\text{iff} \quad \sum_{\Delta\in\mathcal{N}} (-1)^{|\Delta|} \left| \pi \left(H_0 \cap \bigcap_{i\in\Delta} H_i \right) \right| = 0. \end{aligned} \tag{3.7}$$

The Boolean Combination Constructed

We have freed ourselves from any dependence on the a_i (which themselves were dependent on $\overline{b}$). It remains to show the RHS of the equivalence above may be characterised by a Boolean combination of positive-primitive formulæ. We shall now construct such a formula.

For a fixed $\mathcal{M} \subseteq \mathcal{P}([k])$, where $\mathcal{P}$ denotes the powerset, define

$$\phi_\mathcal{M} \equiv \bigwedge_{\Delta\in\mathcal{M}} \phi_\Delta \;\wedge\; \bigwedge_{\Delta\notin\mathcal{M}} \neg\phi_\Delta$$

and

$$\mathcal{O} = \left\{ \mathcal{M} \subseteq \mathcal{P}([k]) \,\middle|\, \sum_{\Delta\in\mathcal{M}} (-1)^{|\Delta|} \left| \pi \left(H_0 \cap \bigcap_{i\in\Delta} H_i \right) \right| = 0 \right\}.$$

Then the sought Boolean combination of positive-primitive formulæ is

$$\phi_\mathcal{O} \equiv \bigvee_{\mathcal{M}\in\mathcal{O}} \phi_\mathcal{M}.$$

In conclusion,

$$\boldsymbol{M} \models \forall x\, \phi(x, \overline{b}) \leftrightarrow \phi_\mathcal{O}(\overline{b}) \qquad \text{for all } \overline{b}$$

and thus

$$\boldsymbol{M} \models \forall x\, \phi \leftrightarrow \phi_\mathcal{O}.$$

Verifying the Construction

→ Assume $\boldsymbol{M} \models \forall x\, \phi(x, \bar{b})$. Then by (3.7), $\mathcal{O}$ contains $\mathcal{N}$ and $\boldsymbol{M} \models \phi_{\mathcal{N}}(\bar{b})$. Then $\boldsymbol{M} \models \phi_{\mathcal{O}}(\bar{b})$.

← Assume $\boldsymbol{M} \models \phi_{\mathcal{O}}$. This means there exists some $\mathcal{M} \subseteq \mathcal{P}([k])$ with

$$\sum_{\Delta \in \mathcal{M}} (-1)^{|\Delta|} \left| \pi \left(H_0 \cap \bigcap_{i \in \Delta} H_i \right) \right| = 0 \qquad \text{and} \qquad \boldsymbol{M} \models \phi_{\mathcal{M}}(\bar{b}).$$

By definition of $\phi_{\mathcal{M}}$, the latter implies $\mathcal{M} = \left\{ \Delta \subseteq [k] \;\middle|\; \boldsymbol{M} \models \phi_{\Delta}(\bar{b}) \right\}$, whence by definition of $\mathcal{N}$, $\mathcal{M} = \mathcal{N}$. But then

$$\sum_{\Delta \in \mathcal{N}} (-1)^{|\Delta|} \left| \pi \left(H_0 \cap \bigcap_{i \in \Delta} H_i \right) \right| = 0$$

which coupled with (3.7) finally yields

$$\boldsymbol{M} \models \forall x\, \phi(x, \bar{b}).$$

QED

3.3.1 Immediate Corollaries

Observe the elimination-procedure shown in the proof of the Baur–Monk Theorem depends very much upon the chosen module $\boldsymbol{M}$. This is easily overcome by considering some complete theory $\boldsymbol{T}$ of right $\boldsymbol{R}$-modules whereby all instances of '$\boldsymbol{M} \models$' may be replaced by '$\boldsymbol{T} \models$' in the proof. This train of thought leads to the following corollary.

Corollary 3.3.1.1 (Baur–Monk for Complete Theories)**.** *Given a complete theory $\boldsymbol{T}$ of right $\boldsymbol{R}$-modules, every $\mathcal{L}$-formula is equivalent (modulo $\boldsymbol{T}$) to a Boolean combination of positive-primitive formulæ.*

Corollary 3.3.1.2. *Any Boolean combination of positive-primitive formulæ is equivalent to a formula of the form*

$$\neg\phi_0 \vee \phi_1 \vee \cdots \vee \phi_k \quad \textit{or equivalently} \quad \phi_0 \to \phi_1 \vee \cdots \vee \phi_k \quad \textit{for some pp } \phi_i.$$

Proof. See (3.1) in the proof of Baur–Monk Theorem 3.3.1. QED

4. Corollaries of Baur–Monk Theorem

4.1 Types

There exists a dual way of defining types, as we shall see. One may view them either as sets of formulæ or as sets definable thereby; indeed they may be defined as filters (or ultrafilters) on the Boolean algebra of definable sets.

Definition 4.1.1 (Type, pp-Type, Partial Types)**.** Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$, $B \subseteq M$, and $\overline{a}$ be a tuple in M.

(1) The following set is termed the *(complete) type* of $\overline{a}$ in M over B.

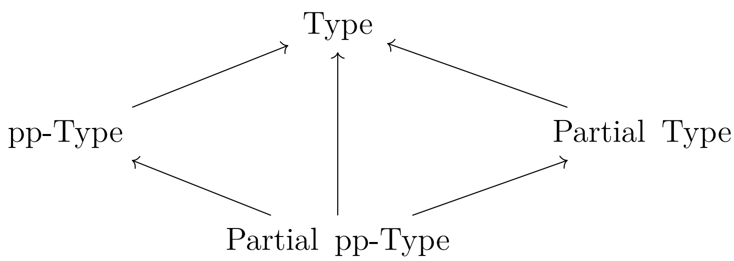
$$\mathrm{Type}^{\boldsymbol{M}}(\overline{a}/B) := \{\phi(\overline{v}) \in \mathcal{L}_B \mid \boldsymbol{M} \models \phi(\overline{a})\}.$$

(2) The *positive-primitive type* of $\overline{a}$ in M over B (or the *primitive-positive part* of $\mathrm{Type}^{\boldsymbol{M}}(\overline{a}/B)$) is defined by

$$\text{pp-Type}^{\boldsymbol{M}}(\overline{a}/B) := \left\{\phi(\overline{v}) \in \mathrm{Type}^{\boldsymbol{M}}(\overline{a}/B) \,\middle|\, \phi \text{ is positive-primitive}\right\}.$$

(3) By a *partial positive-primitive type* we understand any set of pp-formulæ consistent modulo $\mathbf{Mod}_{\mathrm{R}}$ (or possibly any other theory if specified; e.g. the complete theory $\mathrm{Th}(\boldsymbol{M})$ for some $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$). By relaxing the requirement only positive-primitive formulæ be considered, we obtain the definition of a *partial type.*

If $B = \emptyset$ or $B = \{0\}$, we may simply write $\mathrm{Type}^{\boldsymbol{M}}(\overline{a})$ and $\text{pp-Type}^{\boldsymbol{M}}(\overline{a})$ respectively. Likewise, if $\boldsymbol{M}$ is clear from context, it may be omitted.



Remark 4.1.2. It is clear a (positive-primitive) type is a maximal element in the partially ordered sets of partial (positive-primitive) types. Conversely, every partial (primitive-positive) type may be extended to a (primitive-positive) type by the Zorn Lemma.

Lemma 4.1.3. *It is clear* $p = \text{pp-Type}^{\boldsymbol{M}}(\overline{a}/B)$ *is always infinite regardless of any specification of* $\boldsymbol{M}$*, a, or B; note the formula* $\phi \equiv \bigwedge_{j=1}^{m} \sum_{i=1}^{\operatorname{len}\overline{a}} v_i 0 = 0$ *lies in p trivially for any natural m.*

Remark 4.1.4. Let $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$. Given any $m \in M$, its annihilator may be reconstructed from pp-Type$^{\boldsymbol{M}}(m)$. Namely,

$$\mathrm{Ann}_{\boldsymbol{R}}(m) = \left\{ r \in R \,\middle|\, (vr = 0) \in \text{pp-Type}^{\boldsymbol{M}}(m) \right\}.$$

Proof. Denote the set in question A.

$\supseteq$ Suppose $r \in A$. Then $(vr = 0) \in$ pp-Type$^{\boldsymbol{M}}(m)$, whence $\boldsymbol{M} \models mr = 0$ and therefore r annihilates m.

$\subseteq$ Suppose $r \in \mathrm{Ann}_{\boldsymbol{R}}(m)$. Then $\boldsymbol{M} \models (mr = 0) \equiv \phi(m)$ for some $\phi \equiv (vr = 0)$. Since ϕ is also clearly positive-primitive, $\phi \in$ pp-Type$^{\boldsymbol{M}}(m)$, whence $\phi \in A$.

QED

Types as Sets Defined Thereby

We could have defined positive-primitive types not as sets of positive-primitive formulæ but as the sets of subgroups these formulæ define.

Definition 4.1.5 (Filter, Ultrafilter)**.** Let S be a nonempty set and $F \subsetneq S$.

(1) F is said to be a *filter* on S if F is nonempty and

$$(\forall x, y \in F\, \exists z \in F)\, z \leq x \wedge z \leq y \quad \text{and} \quad (\forall x \in F, y \in S)\, x \leq y \to y \in F.$$

(2) A filter F is said to be an *ultrafilter* on S if it is a maximal (proper) filter.

Lemma 4.1.6.

(1) Type$^{\boldsymbol{M}}(\overline{a}/B)$ *forms an ultrafilter in the Boolean algebra consisting of all subsets of* $M^{\operatorname{len} a}$ *definable over* B *ordered by inclusion.*

(2) pp-Type$^{\boldsymbol{M}}(\overline{a}/B)$ *is the filter comprising precisely those subsets which are pp-definable over* B *in which* $\overline{a}$ *lies.*

(3) A partial positive-primitive type is a filter in the partially-ordered set of pp-definable cosets ordered by inclusion

Proof.

(1) Maximality and upwards closure are clear from definition. Indeed, given some two definable sets $\phi(\boldsymbol{M})$, $\psi(\boldsymbol{M})$ over B, their intersection (given by $\phi \wedge \psi$ as per Theorem 2.2.1) lies in Type$^{\boldsymbol{M}}(\overline{a}/B)$.

(2) By analogy (especially if one considers it is, in fact, an ultrafilter on the Boolean algebra of pp-definable subsets over B).

(3) Since partial pp-types are restrictions of (complete) pp-types, and pp-types form ultrafilters, the claim follows trivially.

QED

4.1.1 Correspondence of Types & pp-Types

Having familiarised ourselves with pp-types somewhat, we present two interesting corollaries of Baur–Monk Theorem 3.3.1.

Theorem 4.1.7. *Let* $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$ *and* $\overline{a}, \overline{b}$ *be in* M. *Then* $\overline{a}$ *and* $\overline{b}$ *have the same type iff they have the same pp-type:*

$$\mathrm{Type}^{\boldsymbol{M}}(\overline{a}) = \mathrm{Type}^{\boldsymbol{M}}\left(\overline{b}\right) \qquad \textit{iff} \qquad \text{pp-Type}^{\boldsymbol{M}}(\overline{a}) = \text{pp-Type}^{\boldsymbol{M}}\left(\overline{b}\right).$$

Proof.

- $\Rightarrow$ Clear, since pp-types are restrictions of types to positive-primitive formulæ.
- $\Leftarrow$ Let ϕ be an $\mathcal{L}$-formula with $\boldsymbol{M} \models \phi(\overline{a})$ (i.e. $\phi \in \mathrm{Type}^{\boldsymbol{M}}(\overline{a})$). By Baur–Monk Theorem 3.3.1, there exists a Boolean combination of pp-formulæ ψ such that $\boldsymbol{M} \models \phi \leftrightarrow \psi$. Denote by $\phi_0, \phi_1, \ldots, \phi_n$ the positive-primitive formulæ occurring in ψ.

  To prove the claim, it suffices to show $\boldsymbol{M} \models \psi(\overline{b})$ (whence $\boldsymbol{M} \models \phi(\overline{b})$ and thus $\phi \in \mathrm{Type}^{\boldsymbol{M}}(\overline{b})$). By assumption, $\boldsymbol{M} \models \phi_i(\overline{a})$ iff $\boldsymbol{M} \models \phi_i(\overline{b})$ for any i; it follows (by induction) the same must hold for any Boolean combination of ϕ_i. Then in particular $\boldsymbol{M} \models \psi(\overline{a})$ iff $\boldsymbol{M} \models \psi(\overline{b})$, and hence $\boldsymbol{M} \models \psi(\overline{b})$.

QED

Notation 4.1.8. Let $p = \mathrm{Type}^{\boldsymbol{M}}(\overline{a}/B)$. Then

$$p^+ = \{\phi \in \mathcal{L}_B \,|\, \phi \in p \text{ is pp}\} \quad p^- = \{\phi \in \mathcal{L}_B \,|\, \phi \notin p \text{ is pp}\} \quad \neg p^- = \left\{\neg\phi \,\middle|\, \phi \in p^-\right\}$$

Theorem 4.1.9. *Let* $\boldsymbol{M} \models \mathbf{Mod}_{\mathrm{R}}$ *with* $B \subseteq M$ *and* $p = \mathrm{Type}^{\boldsymbol{M}}(\overline{a}/B)$. *Then* $p^+ \cup \neg p^-$ *proves* p*; i.e.* $p^+ \cup \neg p^- \vdash \phi$ *for every* $\phi \in p$.

Proof. Let $\phi \in p$ be arbitrary. Baur–Monk Theorem 3.3.1 gives

$$\boldsymbol{M} \vdash \phi \leftrightarrow \beta \tag{4.1}$$

for some Boolean combination β of positive-primitive formulæ. In fact, we know from Corollary 3.3.1.2 that

$$\beta \equiv \neg\phi_0 \vee \phi_1 \vee \cdots \vee \phi_k \quad \text{for some pp } \phi_i.$$

By assumption, $\boldsymbol{M} \models \phi(\overline{a})$ whereby (4.1) gives $\boldsymbol{M} \models (\neg\phi_0 \vee \phi_1 \vee \cdots \vee \phi_k)(\overline{a})$. In particular then, either $\boldsymbol{M} \models \neg\phi_0(\overline{a})$ or $\boldsymbol{M} \models \phi_i(\overline{a})$ for some $i > 0$. Whichever case occurs, denote the (truthful) formula ψ. Clearly then if the former holds, $\psi \in \neg p^-$ and if the later does $\psi \in p^+$. Either way, $\psi \in p^+ \cup \neg p^-$. Altogether then,

$$p^+ \cup \neg p^- \vdash \psi \to \beta \leftrightarrow \phi \overset{\mathrm{MP}}{\vdash} \phi.$$

QED

4.2 A Reflection on Model-Completeness

We conclude this thesis with a nonexample: A fairly elementary notion in model theory is that of *model-completeness* as given below. The significance, and shortcomings, of the Baur–Monk 3.3.1 admitting an elimination down to a *Boolean combination* of (a kind of) existential formulæ as opposed to *an* existential formula will be illustrated.

Definition 4.2.1 (Model-Complete Theory). A theory $\boldsymbol{T}$ is model-complete if whenever $\boldsymbol{M}$, $\boldsymbol{N}$ are models of $\boldsymbol{T}$ and $\boldsymbol{M} \leq \boldsymbol{N}$, then $\boldsymbol{M}$ is an elementary substructure of $\boldsymbol{N}$: $\boldsymbol{M} \preccurlyeq \boldsymbol{N}$.

Lemma 4.2.2. *If a theory $\boldsymbol{T}$ admits a (full) elimination of quantifiers, it is model-complete.*

Proof. Let $\boldsymbol{M}, \boldsymbol{N}$ be models of $\boldsymbol{T}$ with $\boldsymbol{M} \leq \boldsymbol{N}$. By assumption, there exists a quantifier-free formula ψ for any formula ϕ with

$$\boldsymbol{T} \models \phi(\overline{v}) \leftrightarrow \psi(\overline{v}).$$

Suppose $\boldsymbol{M} \models \phi(\overline{a})$ for some $\overline{a}$ in M. Then $\boldsymbol{M} \models \psi(\overline{a})$. Since ψ is merely a Boolean combination of relations (ψ is quantifier-free) altogether satisfied by $\overline{a}$, $\boldsymbol{N} \models \psi(\overline{a})$ whence $\boldsymbol{N} \models \phi(\overline{a})$. The converse implication follows from symmetry. Overall

$$\boldsymbol{M} \models \phi(\overline{a}) \quad \text{iff} \quad \boldsymbol{N} \models \phi(\overline{a}).$$

QED

As we have already discussed in Section 1.3, the requirement a theory should admit a full elimination of quantifiers is easy to impose so long as we permit ourselves unbounded enrichment of languages and theories.

For any sensibly-sized theory, however, it is rather strong, but luckily may be weakened to admission of a partial quantifier-elimination as shown by Abraham Robinson:

Theorem 4.2.3 (Robinson). *A theory $\boldsymbol{T}$ is model-complete iff it admits elimination of quantifiers down to existential formulæ.*

Corollary 4.2.3.1. *Complete module-theories are not model-complete.*

Proof. For the counterexample, consider the complete theory $\mathrm{Th}(\mathbb{Z}_{\mathbb{Z}})$ and its models $2\mathbb{Z} \subseteq \mathbb{Z}$. Setting $\phi(y) \equiv (\exists x)\, x + x = y$, we have $\mathbb{Z} \models \phi(2)$ and $2\mathbb{Z} \not\models \phi(2)$. Hence $2\mathbb{Z}$ is not an elementary substructure of $\mathbb{Z}$. QED

5. Appendix

5.1 Fixed Symbols

A list of symbols fixed throughout the document whose meaning is invariant under the change of mathematical context. One may think of these as ‘global variables’.

(1) $\boldsymbol{R}$, (1.4.2)

(2) $\mathcal{L}$, (1.4.4)

(3) Var, (1.4.3)

(4) $\mathbf{Mod}_{\mathrm{R}}$, (1.4.5)

(5) $\overline{v}$, (1.4.8)

(6) $\mathrm{len}(\overline{v})$, (1.4.8)

(7) $\mathcal{L}_c$, $\mathcal{L}_A$ (2.0.6)

(8) $\equiv$, (1.4.9)

(9) $\mathfrak{pp}$, (2.0.4)

5.2 Bibliography

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