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April 11, 2022

Report on the habilitation thesis "WDC-sets" by Dr. Dušan Pokorný

I first want to state that I met Dr. Pokorný at different conferences and have seen some of his talks, but that we have no past or ongoing collaboration.

The habilitation thesis of Dr. Dušan Pokorný consists of 5 papers, 4 of which are published and one preprint; and an introduction/overview of the results in these papers. The central subject are WDC sets. WDC stands for "weakly DC", while DC means "difference of two convex functions".

Before giving my evaluation, I will comment on the papers separately.

1 Fu, Pokorný, Rataj: Kinematic formulas for sets defined by differences of covex functions

The first paper included in the habilitation thesis was published in the prestigous journal *Advances in Mathematics* in 2017.

This beautiful paper contains two main results (Theorems A and B). The first one is a sharper version of the well-known theorem by Ewald-Larman-Rogers (ELR) on the boundary structure of convex sets. The ELR-theorem states that the set of directions of line segments contained in the boundary of a convex body in $\mathbb{R}^d$ is of Hausdorff dimension at most $d-2$. The present theorem A shows that even the Minkowski dimension is at most $d-2$, more precisely the $(d-2)$-dimensional Minkowski content is finite. Moreover, another set which takes into account not only the directions of line segments but also the supporting hyperplanes containing them is introduced. It is shown that it has finite $(d-2)$-dimensional Minkowski content. Since the ELR-set is a projection of this new set, the ELR-theorem follows from Theorem A by the observation that Minkowski content behaves well under Lipschitz maps.

The proof of Theorem A uses a cap covering lemma by ELR and several clever packing arguments. The statement and the proof are of independent interest (in particular to researchers in convex geometry).

The second main result, and probably the motivation for the authors to prove Theorem A, extends the kinematic formulas of Federer from sets of positive reach to so called WDC-sets.

The principal kinematic formula was first shown for compact convex sets. Since then, many variants of this formula were obtained. The kind of questions emerging from this formula can be divided into an algebraic and an analytic part.

In the algebraic part one aims at a deeper understanding of the constants appearing in this formula and similar formulas on isotropic Riemannian manifolds. These constants turn out to be the structure constants of some algebra which is related to Alesker's product of valuations. The first author has several influential papers in this direction.

The analytic side of the theory is to find a precise (or as precise at possible) description of the type of sets to which the principal kinematic formula applies. One major step in this direction was done by Federer, who proved the principal kinematic formula for sets of positive reach. Zähle showed later that integral currents (in the sense of geometric measure theory) can be successfully applied in this theory. More precisely, to each set of positive reach one can associate a certain current, called normal cycle. The quantities appearing in the kinematic formula are given by integration of some specific differential forms over the normal cycle. The proof of the kinematic formula then uses some operations (fiber products, push-forwards and so on) on the level of these currents. It therefore should work (modulo some techical difficulties) for all sets admitting normal cycles. This includes sets of positive reach, compact convex sets, submanifolds with boundary, polyhedral sets, subanalytic sets and certain finite unions of such sets. Fu introduced in the 1990's a class of "geometric" sets as those sets admitting a normal cycle. However, the proof of kinematic formulas in this generality was not possible and it is also not clear how to describe the class of geometric sets in a convenient way. Therefore the problem remained open to find some large class of sets for which the principal kinematic formula holds.

In the present paper, a very nice answer is given to this problem. The authors prove the kinematic formula (or rather the existence of such formulas) on isotropic Riemannian manifolds for the class of WDC-sets. A WDC-set is the sublevel of the difference of two convex functions (DC) at a weakly regular level (the definition of weakly regular level requires the Clarke differential). It was shown before by Pokorný and Rataj that DC-functions are Monge-Ampère functions, which is more or less equivalent to saying that WDC-sets admit normal cycles.

The proof of Theorem B is divided into two parts. In the first part, it is shown that for WDC-sets A, B in an isotropic Riemannian manifold (M, G), the set $A \cap \gamma B$ is again WDC for almost all $\gamma \in G$. This is of course a necessary step for the proof of the kinematic formulas. In this part one needs some estimates on the size of some version of the normal cycle, and this estimate is provided by Theorem A.

The second part of the proof of Theorem A is an adaption of some machinery (using fiber bundles and operations on currents) developed by Fu in other contexts (subanalytic sets).

This paper paper is a major breakthrough in the analytic part of integral geometry.

2 Pokorný, Rataj, Zajíček: On the structure of WDC-sets

This paper was published in 2019 in *Mathematische Nachrichten.*

Given the importance of WDC sets for kinematic formulas, it is a natural question to understand better their structure. The present paper gives several interesting results in this direction.

It is shown that the boundary of a compact WDC-set can be covered by finitely many DC surfaces. Then an analogue of a theorem of Federer is shown: lower dimensional WDC-sets are already DC manifolds. In the case of planar WDC-sets, a complete characterization is given. The authors also state a conjecture concerning a certain natural stratification of WDC-sets, and prove some results supporting it. The paper also contains a complete proof of the Chern-Gauss-Bonnet theorem for Monge-Ampère functions.

3 Pokorný, Zajíček: On sets in $\mathbb{R}^d$ with DC distance functions

This short paper was published in *J. Math. Anal. Appl.* in 2020.

The distance function d_F of a closed subset F is studied. It is known that d_F^2 is a DC function, but d_F in general is not a DC function. This raises the question whether d_F is still a DC function for some large class of reasonably tame susbsets F. The authors formulate a question in this spirit, namely if this holds true for the graph of a DC-function. An affirmative answer is given in the two-dimensional case. The class of sets for which the distance function is a DC function is further studied in the last section, where several structural results for this class are shown.

4 Pokorný, Zajíček: Remarks on WDC sets

This short paper was published in *Comment. Math. Univ. Carolin.* in 2021. It is a continuation of the research from the previous paper. The main result is that the distance function of a WDC set in the plane is a DC aura. This implies that closed locally WDC sets are already WDC sets (still in the plane). The proof uses the structural results on two-dimensional WDC sets obtained in previous papers. Moreover, it is shown that compact WDC sets in the plane form a Borel subset of the space of all compact subsets. In higher dimensions, they still form an analytic subset.

5 Pokorný: Curvatures for unions of WDC sets

This is a preprint. It introduces a generalization of the theory of WDC sets, namely it studies the class $\mathcal{U}_{WDC}$ of finite unions (subject to some condition) of WDC sets. This is similar to the study (by Rataj-Zähle) of the class $\mathcal{U}_{PR}$ of finite unions of sets with positive reach. It is shown that such sets are locally contractible and that their Euler characteristic is a valuation. The main theorem is that each compact set in $\mathcal{U}_{WDC}$ admits a normal cycle. Extending the approach of the first paper, the author is able to formulate the kinematic formulas in $\mathbb{R}^d$ for compact sets in $\mathcal{U}_{WDC}$. The proofs use several tools from geometric measure theory, e.g. the Clarke differential and Legendrian cycles.

The paper then studies compact sets in $\mathbb{R}^2$ more closely and gives a geometric characterization of $\mathcal{U}_{WDC}$. More precisely, it is shown that a compact subset in $\mathbb{R}^2$ is in $\mathcal{U}_{WDC}$ if and only if its complement has finitely many connected components and if the boundary is a union of finitely many DC graphs.

6 Originality check done by Turnitin system

I had a close look at the outcome of the Turnitin originality check. There are many coloured places in the outcome, but none of them comes by any means close to plagiarism. The thesis gives, in all is parts, precise and complete references and satisfies high scientific standards.

7 Evaluation

The scientific quality of the habilitation thesis is very high. All proofs are given in a complete and - as far as I can judge - correct way. Most of the papers are with coauthors, but it is evident that Dr. Pokorný is a driving force of the new theory of WDC-sets and has contributed a lot to it. He has a clear vision on what the important questions and problems are. He knows the relevant techniques and results - coming from different areas such as geometric measure theory, analysis, topology and integral geometry - and can apply them in the study of WDC-sets.

In my point of view, the first paper and the preprint will have the largest impact. The class of WDC-sets, or even more generally the class $\mathcal{U}_{WDC}$ of finite unions of WDC-sets, is very well suited for analytic integral geometry, in particular for the statement of very general kinematic formulas. I am less sure how much impact the other three papers will have. It seems to me that they contain nice results, but that they are intermediate steps towards a more general structure theory of WDC-sets. In general, it is not yet clear to me how and where the theory of WDC-sets will be used in research areas different from convex and integral geometry. The thesis meets the requirements of a habilitation thesis in mathematics.

Besides the papers included in the habilitation thesis, Dr. Pokorný has many other contributions to mathematics. According to MathSciNet, he has 13 other published papers, mainly in geometric measure theory. He is an internationally known mathematician with a high potential for the future.

To summarize, I can **recommend without hesitation to accept the habilitation thesis of Dr. Dušan Pokorný**.

Andreas Bernig

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