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Exact spacetimes in theories beyond general relativity

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HABILITATION THESIS

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CONTENTS

CHAPTER

1

INTRODUCTION

Even though gravity corresponds to the weakest interaction identified in the surrounding nature up to date, its various practical as well as theoretical aspects have attracted incredible attention over thousands of years. The enormously efficient kinematic description of the planetary motion by Ptolemy in ancient Greece, its precise Keplerian analysis employing clear geometric language 1500 years later, and its dynamical description in terms of the Newton gravitation law, are real gemstones in the treasury of humankind's knowledge. Beyond all doubt, another highlight on the pathway to a deeper understanding of the laws of nature is the *General theory of relativity* (GR) outlined by Albert Einstein more than a century ago and in its final form presented to the Royal Prussian Academy of Sciences in 1915 [1]. Together with preceding Einstein's special relativity [2], it has completely changed our understanding of such basic and intuitive concepts like space and time. The mathematical description of gravity via the newly introduced dynamical arena of the world — *curved spacetime* governed by the Einstein field equations — has shed light on unsuspected horizons in observational astronomy together with predictions about the past, presence, and anticipated future of the whole Universe itself. The theory immediately elucidated situations where the effects of a strong gravitational field are crucial such as bending of light-rays that pass massive objects, or precise explanation of a small discrepancy in the Mercury precession [3, 4]. These have been followed by various other astrophysical predictions and even practical applications affecting our everyday life such as correction applied to GPS. Here let me mention at least two most recent and fascinating discoveries induced by the GR theoretical predictions. The first one is the direct detection of gravitation waves by the LIGO facility in 2015 [5], where the measured ripples in the spacetime curvature corresponded to the pair of black holes merging more than one billion light-years away. This has been followed by many others observations slowly revealing mysteries of the stellar graveyard. In the case of merging neutron stars [6], the supplementary observations in the whole range of electromagnetic spectrum have opened a completely new field of the 'multi-messenger' astronomy. The second observational achievement directly related to the GR prediction is the imaging of the Messier 87 galaxy center with the shadow of a supermassive black hole [7] followed by the Milky Way black hole image released by the EHT collaboration a few days ago. These extremely successful experimental events are based on a brilliant knowledge of various parts of physics and engineering, enormous technical effort, sophisticated numerical simulations, and observational skills. However, the analytical understanding of the Einstein gravitational law accompanied by the description and interpretation of *exact solutions* to Einstein's field equations stand very deep in their center. In particular, the existence and principal properties of gravitational waves were deduced by Einstein himself almost immediately after the theory release from its weak field limit [8, 9]. Interestingly, the identification of this phenomenon within the full theory (using the exact models and a discussion of the test particles' behavior) took another forty years. These explicit theoretical results started the gravitational-wave hunt which resulted in their indirect observation via analysis of the binary pulsar PSR 1913+16 discovered by Taylor and Hulse in 1974 [10], and finally, in the already mentioned direct LIGO detection seven years ago. Moreover, the interferometric observations show that the gravitational-wave prominent sources are mergers of rotating black holes, where their initial (separated) states, as well as the final remnant, are almost precisely described by the Kerr exact solution — spacetime representing axially symmetric stationary conformally flat black-hole geometry [11]. The same type of object, however with an incredibly higher mass of billions of Suns, is also located in the M87 and Milky Way centers, respectively. These two situations are significant examples of the importance of an exact analytical approach to the field equations and they should indicate and motivate the utility of results derived using explicit simplified models.

These most recent GR achievements, accompanied by many other experimental tests, see e.g. [12], proved Einstein's general relativity to be the excellent theory of gravity without any discrepancies between its predictions and observational data found so far. However, there are some theoretical loopholes and/or pure 'curiosity' reasons to think about theories going beyond GR. In particular, one class of 'practical' questions to be answered is related to the cosmology, e.g., the inflation phase of the Universe or presence and unknown origin of the dark matter and energy (effectively represented by the cosmological constant) do not suitably fit into the GR models. The second, and conceptually even more important, branch of uncertainty arises from the incompatibility of GR with the quantum field theory approach and techniques corresponding to its non-renormalizability. The quantum implications seem to be crucial in the spacetime regions with extreme curvature, in particular singularities, which stand in the heart of the above-mentioned exact black-hole spacetimes or at the beginning of the Universe. To solve at least some aspects of these problems on the classic level of metric theory, various modifications of the famous Einstein–Hilbert action have appeared, see e.g. [13–16] for more details. The prominent and very straightforward extension of GR seems to be so-called *quadratic gravity* (QG) corresponding to the presence of additional quadratic curvature terms in the action, see [17–20] and [22] for a comprehensive review, which in the higher dimensional case includes also the Einstein–Gauss–Bonnet theory as the first non-trivial representative of the Lovelock theories [23, 24]. One can think about QG both as our desired final theory or as the first correction to the Einstein–Hilbert Lagrangian (including the second-order curvature terms) resulting from the expansion of some more involved final theory action, which at least partially solves the above-mentioned imperfections of GR. Simultaneously, any conceptually brilliant final theory, which we may hope for, has to be compatible with the results, predictions, and experimental tests of Einstein's general relativity in its low energy limit.

Besides all these conceptual issues, my main motivation for the presented research is primarily *curiosity*. In particular, GR respects the Occam razor principle where the simplest possibility is chosen. It is thus natural to ask whether more complicated scenarios would lead to significantly different predictions in comparison with those of GR, or whether their implications are qualitatively the same, but simultaneously, the additional degrees of freedom bring some potential for improvements of the GR weaknesses. Dozens of works are studying specific observable situations within modified theories employing both exact models as well as sophisticated approximations and numerical methods. Typically, such 'fine-tuned' models can solve a specific problem, however, they may fail elsewhere. Therefore, the aim is to keep the calculations and discussion as generic and systematic as possible. This approach naturally requires adjustment and development of suitable methods and tools providing control over the physical and mathematical properties of the resulting spacetime. To be able to compare mutually corresponding situations both in GR and any modified gravity it is natural to start from the theory-independent geometric assumptions, e.g., symmetries, algebraic structure, or specific behavior of geodesics. Then the implications and compatibility of consequent exact solutions and identification of various qualitative differences, and in principle, observable effects can be studied.

In the thesis and my related works, the attention is paid solely to spacetimes which are *exact solutions* of Einstein's GR or its various extensions. Neither perturbative techniques nor numerical simulations are employed although they may allow one to study more realistic situations and provide a relevant quantitative perspective. On the other hand, such approximative methods may hide principal aspects of a specific theory and we would like to avoid such flaws. The traditional exact solution studies represent clear model results that may provide direct undistorted insight into the pure theory structure and its specific properties. Moreover, the exact analysis of simplified situations has been extremely important during the whole history of Einstein's general relativity and brought many surprising results which have been further adjusted to the real astrophysical processes. So far, hundreds of various solutions to GR have been found and analyzed. Their comprehensive description and physical interpretation can be found for example in [25–27]. Here, let us only very briefly list at least the most important and influential of them without any ambition to present any details, namely

- The *Minkowski*, *de Sitter*, and *anti-de Sitter spacetimes* [28], see [26] for their various parametrizations, represent maximally symmetric solutions with zero, positive, and negative constant curvature, respectively. The particular cases correspond to the specific sign of the cosmological constant entering the Einstein field equations. These geometries typically represent backgrounds for more involved scenarios within GR or even other branches of physics.
- The *Friedmann–Lemaître–Robertson–Walker cosmological models* [29–33] allow to predict the beginning and the future of our Universe, which at least partially answers the questions as old as humankind itself. In mathematical terms, they are homogeneous and isotropic perfect fluid solutions where Einstein's gravitational law governs the time scaling of the particular model.
- The *Schwarzschild black hole* is the first non-trivial solution to the Einstein field equations [34] found shortly after the final theory was released. However, the understanding of its properties and global structure took other decades. It describes the vacuum spherically symmetric geometry which is unique in GR due to Birkhoff's theorem. From the physical viewpoint, it is the simplest GR black-hole solution characterized by one free parameter interpreted as mass.
- The *Kerr solution* [11] adds one more parameter to the static Schwarzschild geometry which encodes the angular momentum. It describes an axially symmetric stationary rotating black hole that is asymptotically flat. To find such a one-parameter extension took almost fifty years. It is also worth mentioning that this spacetime miraculously well reflects astrophysical reality and plays thus a crucial role in the various experimental tests of GR.
- The most interesting models of *exact gravitational waves* belong to wider classes of the Robinson–Trautman and Kundt geometries [35–38]. In particular, the simplest case of *pp*-waves, originally identified in a different context [39] and abbreviating plane waves with parallel rays, serves as a testbed in miscellaneous applications requiring dynamical propagation of gravitational degrees of freedom.

Except for the cosmological models, all the above classes of exact spacetimes in GR are at least partially related to my past or future research.

Until now, it would seem that the exact solutions are synonyms for something 'trivial' which could be easy to obtain. However, solving the field equations of any reasonable theory of gravity is, in fact, a very difficult task. There were developed various approaches to do so and the above examples are excellent illustrations. After physical specification of the situation which should be analyzed, typically, the essential step has to follow — a reasonable geometric simplification of the problem. Some of the very natural additional assumptions one can think about are

- *symmetries*: it is very common that symmetries, and their extensions (see e.g. [40]), reduce the number of degrees of freedom irrespectively whether the black-hole spacetimes or any other problems formulated in the language of differential equations are studied,
- *algebraically special structure of the curvature tensors*: on the contrary, the effect of abstract algebraic properties on the physical nature of the resulting spacetime is quite surprising, however, for example, the Kerr solution was discovered in this way,
- *geometric behavior of null congruences*: it is interesting how simply the deformations of null geodesic congruences can be described and how important classes of spacetimes can be distinguished with respect to that.

This list is not complete and the categories are not disjunctive. It rather reflects specific directions from which the particular problems can be approached. Simultaneously, these assumptions may introduce the *theory-independent starting points* for extending particular well-known classes of solutions constructed in GR into the modified theories of gravity which is subsequently the solid base for comparison of such theories with GR on the level of their exact solutions.

How to read the thesis

The first part of the present thesis provides the summary of basic concepts, which are important to the original works' better understanding, and which are typically explained only in a very condensed way within the journal papers. I have no ambition to rephrase the papers' original results, but I would like to emphasize the main ingredients entering the game. These are chapters 2, 3, and partially 4. The aim here is to provide an understandable guideline even for a reader who is not too familiar with the field. The rest of chapter 4, related to the Newman–Penrose formalism, consists of results that have been finished recently, and their publication is a matter of near future. The thesis is primarily formed by the original research papers [41–49] which are listed in appendix A. For the readers' convenience, the bibliographic data and very brief comments, mainly pointing out various interesting observations based on the obtained results, are also included.

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CHAPTER

2

THEORIES OF GRAVITATIONAL FIELD

This chapter summarizes the theories of gravity of my interest in terms of the corresponding Lagrange densities, least action principle, and resulting field equations. The aim here is not to present a comprehensive review including all necessary technical details. However, all the gravity descriptions employed in the subsequent chapters and the directions in which they extend classical Einstein's theory will be introduced.

2.1 General relativity

There is no better way how to begin than with general relativity. Apart from thousands of research papers analyzing its most suitable formulation and miscellaneous implications, many pedagogical texts were written explaining the mysteries of the theory in the most understandable manner as possible. At least a pair of canonical books must be quoted here [50, 51]. Sooner or later, the reader of any textbook finds that there are the field equations in Einstein's gravitation theory heart, namely

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \,, \tag{2.1}$$

where R_{ab} is the Ricci tensor given as a contraction of the Riemann tensor $R_{ab} = g^{cd} R_{acbd}$, symbol R encodes the scalar curvature corresponding to the trace of R_{ab}, constant Λ is the cosmological term, T_{ab} stands for the energy-momentum tensor, and finally, c is the speed of light in vacuum and G represents the Newtonian gravitational constant[1]. From the mathematical perspective, in four dimensions (2.1) represents the second-order system of ten non-linear PDEs for the metric tensor components g_{ab}. Even though to find its solution is typically highly non-trivial problem, the fundamental principle is clear, namely the presence of matter and energy (RHS) tells the spacetime geometry (LHS) how to curve itself, and simultaneously, the spacetime curvature forces its sources to move. This again emphasize the inherent non-linearity of such a description.

Already in 1915 David Hilbert introduced an elegant way how to formulate the Einstein gravitation law — least action principle,

$$\delta S = 0 \,, \tag{2.2}$$

[1] From now on the geometric units with $c = 1 = G$ are used.

with the action S given as a spacetime integral of the Lagrangian density $\mathcal{L}$,

$$S = \int \mathcal{L}\, \mathrm{d}^4\mathrm{x}\,, \qquad \text{where} \qquad \mathcal{L} = \left[\frac{1}{16\pi}\,(R - 2\Lambda) + L_M\right] \sqrt{-g}\,. \tag{2.3}$$

Here, quantity g is the metric determinant and L_M denotes matter contribution to the Lagrangian density. It introduces the energy-momentum tensor via its variation as

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta(L_M \sqrt{-g})}{\delta g^{ab}}\,. \tag{2.4}$$

Simultaneously, the variation of scalar curvature and cosmological term combined with the square root of metric determinant provides the geometric contribution. Performing straightforward calculations, one thus find that least action principle (2.2) with the Einstein–Hilbert action (2.3) leads to the field equations (2.1).

However, it is natural to ask whether the Einstein–Hilbert action is the only possibility leading to the field equations (2.1), and vice versa, whether metric g_{ab} solving (2.1) can be also a solution to some more complicated theory. The first answer is negative, but possible extensions of (2.3) preserving the field equations are well factorized. In particular, the uniqueness of Einstein's theory is formulated in terms of the famous *Lovelock's theorem* [23, 24]. In $D = 4$ the Einstein gravitational law (2.1) represents the most general scenario described by the second-order field equations, however, the action can be supplemented by specific combinations of curvature terms which are only topological in four dimensions. These additional terms become non-trivial for $D > 4$ leading to the classes of Lovelock's theories preserving the second-order field equations. In the sense of equation order, the Lovelock theories thus represent a very natural extension of GR into higher dimensions. The first non-trivial example corresponds to the Einstein–Gauss–Bonnet theory which also arises, for example, as the low energy limit of heterotic string theory, see [52, 53]. To answer the second question concerning the uniqueness of GR solutions, one can find various theories which can be solved by the Einstein spacetimes, see e.g., a subsequent section discussing the case of quadratic gravity, or on a more general level, recent results about so-called universal spacetimes [54, 55].

The least action principle formulation of GR provides another great advantage, namely guidelines on how the potential extensions of Einstein's theory could look like. In particular, three main ingredients could be directly modified:

- The first one is the *spacetime dimension*. Einstein's theory is naturally formulated in four dimensions. Straightforwardly, the spacetime dimension can be considered as a free parameter D. The Einstein–Hilbert action in such a case gives the same form of the field equations (2.1), and its uniqueness is still described by Lovelock's results. Surprisingly, it is even reasonable to study scenarios with $D = 3$, see e.g. [56–58], where the curvature is fully determined by the field equations. However, theories within $D \geq 4$ scenarios are studied in the thesis.

- As the second option, it would be possible to adjust the *geometric part* of the action (2.3) beyond adding up only the topological terms. The price to pay in four dimensions is that the second-order of field equations is lost. In higher dimensions, the Lovelock gravities or again some higher-order theories can be considered.

- The last natural possibility is to assume *peculiar matter content* and its coupling to the spacetime curvature represented by the contribution L_M into the overall Lagrange density.

Of course, all the above possibilities can be combined. In the presented research, the vacuum spacetimes are typically studied under the assumption of a generic dimension D and/or extended geometric component of the action. In particular, I have been interested in arbitrary dimensional GR, four-dimensional quadratic gravity, or higher-dimensional Einstein–Gauss–Bonnet theory, see the next sections for more details.

2.2 Modified theories of gravity

In this section, the constraints of a generic quadratic gravity in D dimensions are summarized. This theory corresponds to the natural extension of Einstein's general relativity defined in terms of the Lagrange density, where the most general case is introduced via an arbitrary function of quadratic curvature invariants appearing in the action. Subsequently, the important subclass of such theories simply arises including only a linear combination of the quadratic curvature contractions in addition to the Einstein-Hilbert action.

Generic quadratic theories

In the general D-dimensional case, the theory Lagrange density is given in terms of an arbitrary differentiable function f depending on the Ricci scalar, the quadratic curvature invariants, and L_M encoding the matter contribution. The action can thus be written as

$$S = \int \Big[f\left(R,\, R_{cd}R^{cd},\, R_{cdef}R^{cdef}\right) + L_M \Big] \sqrt{-g}\,\mathrm{d}^D\mathrm{x}\,. \tag{2.5}$$

The shorthands for the curvature squares will be further used, namely

$$R_{cd}R^{cd} \equiv R^2_{cd} \equiv \Psi\,, \qquad \text{and} \qquad R_{cdef}R^{cdef} \equiv R^2_{cdef} \equiv \Omega\,.$$

To derive the field equations of such theory least action principle (2.2) has to be employed, i.e., constraints implied by the condition $\delta S = 0$ found. The first step in the δS calculation is

$$\delta S = \int \left[\big(f_R\,\delta R + f_\Psi\,\delta\Psi + f_\Omega\,\delta\Omega \big) \sqrt{-g} + f\,\delta\sqrt{-g} + \frac{\delta(\sqrt{-g}\,L_M)}{\delta g^{ab}}\,\delta g^{ab} \right] \mathrm{d}^D\mathrm{x}\,, \tag{2.6}$$

where f_R, f_Ψ, and f_Ω are derivatives of the generic function f with respect to the curvature invariants R, Ψ, and Ω, respectively. To resolve the $\delta S = 0$ constraint it is necessary to rewrite all variations in (2.6) in terms of the metric variation δg^{ab}. The aim is not to go through all technical details, which can be found e.g. in the handwritten lecture notes [61] or the introductory parts of the diploma thesis [62, 63]. However, the most important final results should be presented. The matter fields density variation remains the same as in the case of GR and can thus be related to the energy-momentum tensor T_{ab} directly via (2.4). All curvature variations δR, $\delta\Psi$, and $\delta\Omega$ can be expressed using a variation of the Riemann tensor and its specific contractions. To eliminate derivatives of the metric variation the Leibniz rule and the Gauss theorem have to be used analogously as in the GR case.

Evaluating least action principle (2.2) the resulting field equations of the generic quadratic theory (2.5) explicitly become

$$\begin{aligned} &f_R\,R_{ab} - \frac{1}{2} f g_{ab} + (g_{ab}\,\Box - \nabla_a\nabla_b)\, f_R + 2\left(f_\Psi\,R_{ac}R_b{}^c + f_\Omega\,R_{cdea}R^{cde}{}_b\right) \\ &\quad + \Box\,(f_\Psi\,R_{ab}) + g_{ab}\nabla_c\nabla_d\left(f_\Psi\,R^{cd}\right) - 2\nabla_c\nabla_d\left(f_\Psi\,\delta^d_a R^c{}_b + 2 f_\Omega\,R^c{}_{ab}{}^d\right) = \frac{1}{2}\,T_{ab}\,, \end{aligned} \tag{2.7}$$

where box stands for the d'Alembert operator, i.e., $\Box \equiv g^{cd}\nabla_c\nabla_d$. This is the fourth-order system of $\frac{1}{2}D(D+1)$ equations for the spacetime metric components g_{ab}. By prescribing free function f the particularly interesting subclasses can be simply obtained.

$f(R)$ theories

This type of theories is obtained as a generalization of the Einstein–Hilbert action linear dependence on the Ricci scalar R to allow its arbitrary function. Even though these theories are not of the thesis prime interest, except the case $f = \frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha\, R^2$, they should be mentioned since they play the important role in various studies going beyond the scheme of GR, see e.g. [14] for more details and further references. Moreover, writing down the field equations of $f(R)$ theories becomes almost trivial using the above general expression (2.7). In particular, it leads to the constraints

$$f_R\, R_{ab} - \frac{1}{2} f g_{ab} + (g_{ab}\,\Box - \nabla_a\nabla_b)\, f_R = \frac{1}{2}\, T_{ab}\,, \tag{2.8}$$

obtained simply by setting $f_\Psi = 0 = f_\Omega$ in (2.7) since the function f does not depend on the Ricci and Riemann tensor squares, respectively.

Einstein–Gauss–Bonnet theory

The Einstein–Gauss–Bonnet gravity belongs to the already mentioned wider class of Lovelock's theories and represents their first non-trivial GR extension relevant within higher dimensions starting from $D \geq 5$. The generating function f takes the form

$$f(R,\Psi,\Omega) = \kappa^{-1}\,(R - 2\Lambda_0) + \gamma\, L_{GB}\,, \tag{2.9}$$

where the first classic combination of the Ricci scalar R and the theory constants Λ_0 and κ encodes the Einstein part, while the second part coupled via constant γ represents the lowest order Lovelock correction given by the Gauss–Bonnet term L_{GB},

$$L_{GB} \equiv R^2_{cdef} - 4\, R^2_{cd} + R^2 = \Omega - 4\Psi + R^2\,. \tag{2.10}$$

The complete action, which provides the second-order field equations, reads

$$S = \int \left[\kappa^{-1}\,(R - 2\Lambda_0) + \gamma\, L_{GB} + L_M\right] \sqrt{-g}\; \mathrm{d}^D\mathrm{x}\,. \tag{2.11}$$

In particular, the field equations of the Einstein–Gauss–Bonnet theory are given by evaluating (2.7) and employing the substitution for f from (2.9), implying $f_\Psi = -4\gamma$, $f_\Omega = \gamma$ and $f_R = \frac{1}{\kappa} + 2\gamma R$, namely

$$\frac{1}{\kappa}\left(R_{ab} - \frac{1}{2}R\, g_{ab} + \Lambda_0\, g_{ab}\right) + 2\gamma\, H_{ab} = \frac{1}{2}\, T_{ab}\,. \tag{2.12}$$

Here the tensorial quantity H_{ab} denotes the Gauss–Bonnet contribution,

$$H_{ab} \equiv R\, R_{ab} - 2R_{acbd}\, R^{cd} + R_{acde}\, {R_b}^{cde} - 2R_{ac}\, {R_b}^{c} - \frac{1}{4}\, g_{ab}\, L_{GB}\,, \tag{2.13}$$

which trivially vanishes in four dimensions in agreement with the Lovelock theorem.

Pure quadratic gravity in any dimension

As an important special case of the quadratic action (2.5), the simplest generic possibility can be considered, which corresponds to the function f containing the standard Einstein part and the only linear combination of the additional curvature squares. Moreover, it is useful to explicitly separate the Gauss–Bonnet contribution by a suitable re-labeling of coupling constants, which effectively replaces one of the squared terms in the linear combination, e.g., R^2_{cdef}. The generating function f can thus be written in the form

$$f = \frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha\, R^2 + \beta\, R^2_{cd} + \gamma\, L_{GB}\,, \tag{2.14}$$

which, inserted into the action definition, gives

$$S = \int \left[\frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha R^2 + \beta R^2_{cd} + \gamma\left(R^2_{cdef} - 4R^2_{cd} + R^2\right) + L_M\right]\sqrt{-g}\,\mathrm{d}^D\mathrm{x}\,, \tag{2.15}$$

where Λ_0, κ, α, β, and γ are the theory constants. The derivatives of f with respect to the curvature terms are

$$f_R = \frac{1}{\kappa} + 2\alpha R + 2\gamma R\,, \qquad f_\Psi = \beta - 4\gamma\,, \qquad f_\Omega = \gamma\,, \tag{2.16}$$

and their substitution into the general expression (2.7) leads[2] to the *D-dimensional pure quadratic gravity field equations*, namely

$$\begin{aligned}
&\frac{1}{\kappa}\left(R_{ab} - \frac{1}{2}R g_{ab} + \Lambda_0 g_{ab}\right) + 2\alpha R\left(R_{ab} - \frac{1}{4}R g_{ab}\right) + (2\alpha + \beta)\big(g_{ab}\Box - \nabla_a\nabla_b\big)R \\
&\quad + 2\gamma\left[R R_{ab} - 2R_{acbd}R^{cd} + R_{acde}{R_b}^{cde} - 2R_{ac}{R_b}^{c} - \frac{1}{4}\,g_{ab}\left(R^2_{cdef} - 4R^2_{cd} + R^2\right)\right] \\
&\quad + \beta\Box\left(R_{ab} - \frac{1}{2}R g_{ab}\right) + 2\beta\left(R_{acbd} - \frac{1}{4}\,g_{ab}R_{cd}\right)R^{cd} = \frac{1}{2}\,T_{ab}\,.
\end{aligned} \tag{2.17}$$

Pure quadratic gravity in $D = 4$

In *four dimensions* the Gauss–Bonnet contribution becomes trivial by the definition and the parameter γ in (2.17) can be effectively set to zero. Moreover, another constant re-labeling can be used to rewrite the four-dimensional quadratic gravity action as

$$S = \int \left[\frac{1}{\mathsf{k}}(R - 2\Lambda) - \mathfrak{a}\, C_{abcd}C^{abcd} + \mathfrak{b}\, R^2 + L_M\right]\sqrt{-g}\,\mathrm{d}^4\mathrm{x}\,, \tag{2.18}$$

where C_{abcd} is the Weyl tensor, and k, $\mathfrak{a}$ and $\mathfrak{b}$ are coupling constants[3], see again, e.g., [19–22]. The field equations (2.17) with proper constant identification take the form

$$\frac{1}{\mathsf{k}}\left(R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab}\right) - 4\mathfrak{a}\, B_{ab} + 2\mathfrak{b}\left(R_{ab} - \frac{1}{4}R g_{ab} + g_{ab}\Box - \nabla_a\nabla_b\right)R = \frac{1}{2}\,T_{ab}\,, \tag{2.19}$$

[2] Moreover, it is necessary to use the Bianchi identities and their contractions together with the commutator of covariant derivatives to rearrange several terms to obtain exactly this form of the field equations which is similar to that presented, e.g., in [59, 60].

[3] Here the constants k, $\mathfrak{a}$ and $\mathfrak{b}$ are denoted a bit inconsistently with respect to the previous paragraphs to avoid misunderstanding in the context of the $D = 4$ Newman–Penrose formalism discussed in section 4.1, where the Greek letters traditionally stand for the spin coefficients.

where B_{ab} stands for the Bach tensor defined as

$$B_{ab} = \left(\nabla^c \nabla^d + \frac{1}{2} R^{cd}\right) C_{acbd} \,. \tag{2.20}$$

In four dimensions, the Bach tensor can be also specified via its properties, namely, it is symmetric, trace-less, covariantly constant, and conformally re-scaled,

$$B_{ab} = B_{ba} \,, \quad B_{ab} g^{ab} = 0 \,, \quad B_{ab;c}\, g^{bc} = 0 \,, \quad \tilde{g}_{ab} = \Omega^2 g_{ab} \;\Rightarrow\; \tilde{B}_{ab} = \Omega^{-2} B_{ab} \,. \tag{2.21}$$

One of the goals has been to formulate the quadratic gravity using the Newman–Penrose-like approach, see section 4.1. Following the GR strategy, where the field equations provide the algebraic constraint between the Ricci curvature and matter contribution. Therefore, it is beneficial to separate the Ricci tensor in the system (2.19) to get

$$\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b} R\right) R_{ab} - 2\mathfrak{a} R^{cd} C_{acbd} + Z_{ab} = \frac{1}{2}\, T_{ab} \,, \tag{2.22}$$

where the tensorial quantity Z_{ab} is a shorthand for

$$Z_{ab} = -\frac{1}{\mathsf{k}} \left(\frac{1}{2} R g_{ab} - \Lambda g_{ab}\right) - 4\mathfrak{a}\, \nabla^c \nabla^d C_{acbd} - 2\mathfrak{b} \left(\frac{1}{4} R g_{ab} - g_{ab} \Box + \nabla_a \nabla_b\right) R \,. \tag{2.23}$$

Finally, it is worth mentioning the important subclass of solutions to (2.19), namely, the *vacuum spacetimes with constant scalar curvature* $R = \text{const}$. In general, the field equations imply

$$R + \frac{1}{2}\, \mathsf{k}\, T = 6\mathfrak{b}\mathsf{k} \Box R + 4\Lambda \,, \tag{2.24}$$

where T stands for the trace of T_{ab}. In the vacuum case with $T = 0$ and $R = \text{const}$ it thus becomes

$$R = 4\Lambda \,, \tag{2.25}$$

and the field equations (2.19) can be rewritten as

$$\left(\frac{1}{\mathsf{k}} + 8\mathfrak{b}\Lambda\right) (R_{ab} - \Lambda\, g_{ab}) = 4\mathfrak{a}\, B_{ab} \,. \tag{2.26}$$

Their analysis has to be further performed concerning the specific value of the constant combination $\frac{1}{\mathsf{k}} + 8\mathfrak{b}\Lambda$ and $\mathfrak{a}$. The results in subsection A.1.3 discuss these possibilities in detail and emphasize case $\frac{1}{\mathsf{k}} + 8\mathfrak{b}\Lambda = 0$, while in section A.2, the generic case is applied to the spherical geometries.

CHAPTER

3

GEOMETRIES ADMITTING PRIVILEGED NULL CONGRUENCES

The extreme importance of various additional assumptions restricting the studied geometry has been already mentioned in the introduction. These should simplify the problem of solving field equations so that there is a prospect of finding their exact solution. Specific properties can be required, for example, from the null vector fields admitted in the spacetime. At this place, the two significant situations, where the null congruences take the crucial role, are briefly summarized. In particular, the first one is related to the algebraic structure of tensors and the second one lies in the geometric behavior of geodesics generated by a particular null vector field.

3.1 Algebraic structure of the Weyl tensor

Analysis of the Weyl tensor algebraic structure has become a powerful tool within Einstein's GR and seems to be very important within the modified theories as well. The Weyl tensor C_{abcd} is defined as a traceless part of the Riemann tensor. In D dimensions, it reads

$$C_{abcd} = R_{abcd} - \frac{1}{D-2}\left(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}\right) + \frac{R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)}{(D-1)(D-2)}\,. \tag{3.1}$$

In GR, it represents the component of spacetime curvature with subtracted direct influence of the matter due to its coupling to the geometry via Einstein's field equations. In particular, the Ricci scalar is replaced by the trace of Einstein's equations (2.1), i.e., $R = \frac{2}{D-2}(D\,\Lambda - 8\pi\,T)$ in arbitrary dimension D. Then using the field equations, the Ricci tensor can be rewritten into the form

$$R_{ab} = \frac{2}{D-2}\Lambda\, g_{ab} + 8\pi\left(T_{ab} - \frac{1}{D-2}T\, g_{ab}\right)\,. \tag{3.2}$$

Therefore, the Weyl tensor in Einstein's theory is interpreted as a free gravitational field that reflects the inherent properties of the spacetime, subsequently related for example to the tidal deformations and asymptotic structure. Moreover, important classes of GR solutions, e.g., black-hole spacetimes or exact models of gravitational waves, can be identified in terms of the Weyl tensor. The key concept within such description is the Weyl tensor *algebraic type* which can be introduced as a purely geometric property irrespective of any particular theory and spacetime dimension.

Here follows a brief comment on the algebraic classification scheme based on the boost-weight decomposition, see [64–66], which applies to a general tensor in D dimensions. In the four-dimensional Weyl case, it becomes similar to other approaches listed e.g. in [25].

The main ingredient is a real[1] null frame $\{\boldsymbol{k},\, \boldsymbol{l},\, \boldsymbol{m}_i\}$ satisfying the normalization

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1\ , \qquad \boldsymbol{m}_i\cdot\boldsymbol{m}_j = \delta_{ij}\ , \tag{3.3}$$

with other combinations vanishing, which corresponds to the metric expressed as

$$g_{ab} = -2k_{(a}l_{b)} + \delta_{ij}m^i_a m^j_b\,. \tag{3.4}$$

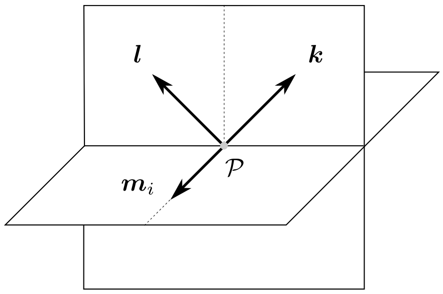

Figure 3.1: In D-dimensional spacetime a real null frame consists of a pair of null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ supplemented by $D-2$ spacelike vectors $\boldsymbol{m}_i$.

The mutual relation between two different null frames is given in terms of the Lorentz transformations, namely

- *null rotation with* $\boldsymbol{k}$ *fixed* which is parametrized by $D-2$ real parameters L^i

$$\boldsymbol{k}' = \boldsymbol{k}\,, \qquad \boldsymbol{l}' = \boldsymbol{l} + \sqrt{2}\,L^i\boldsymbol{m}_i + |L|^2\,\boldsymbol{k}\,, \qquad \boldsymbol{m}'_i = \boldsymbol{m}_i + \sqrt{2}\,L_i\,\boldsymbol{k}\,, \tag{3.5}$$

- *null rotation with* $\boldsymbol{l}$ *fixed* which is parametrized by $D-2$ real parameters K^i

$$\boldsymbol{k}' = \boldsymbol{k} + \sqrt{2}\,K^i\boldsymbol{m}_i + |K|^2\,\boldsymbol{l}\,, \qquad \boldsymbol{l}' = \boldsymbol{l}\,, \qquad \boldsymbol{m}'_i = \boldsymbol{m}_i + \sqrt{2}\,K_i\,\boldsymbol{l}\,, \tag{3.6}$$

- *boost in the* $\boldsymbol{k}-\boldsymbol{l}$ *plane* which is parametrized by a real number B

$$\boldsymbol{k}' = B\,\boldsymbol{k}\,, \qquad \boldsymbol{l}' = B^{-1}\,\boldsymbol{l}\,, \qquad \boldsymbol{m}'_i = \boldsymbol{m}_i\,, \tag{3.7}$$

- *spatial rotation in the space of* $\boldsymbol{m}_i$ which is parametrized by an orthonormal matrix $\Phi_i{}^j$

$$\boldsymbol{k}' = \boldsymbol{k}\,, \qquad \boldsymbol{l}' = \boldsymbol{l}\,, \qquad \boldsymbol{m}'_i = \Phi_i{}^j\,\boldsymbol{m}_j\,. \tag{3.8}$$

The parameters satisfy $L^i = L_i$, $K^i = K_i$ due to (3.3) with $|L|^2 \equiv L^iL_i$, $|K|^2 \equiv K^iK_i$.

[1]In four dimensions the complex null frame $\{\boldsymbol{k},\, \boldsymbol{l},\, \boldsymbol{m},\, \bar{\boldsymbol{m}}\}$ is typically employed, see the end of this paragraph and section 4.1 of the subsequent chapter.

To fully characterize the Weyl tensor and its algebraic type the following frame projections can be introduced,

$$
\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}\, k^a\, m_i^b\, k^c\, m_j^d\,, & & \\
\Psi_{1^{ijk}} &= C_{abcd}\, k^a\, m_i^b\, m_j^c\, m_k^d\,, & \Psi_{1T^i} &= C_{abcd}\, k^a\, l^b\, k^c\, m_i^d \\
\Psi_{2^{ijkl}} &= C_{abcd}\, m_i^a\, m_j^b\, m_k^c\, m_l^d\,, & \Psi_{2S} &= C_{abcd}\, k^a\, l^b\, l^c\, k^d\,, \\
\Psi_{2^{ij}} &= C_{abcd}\, k^a\, l^b\, m_i^c\, m_j^d\,, & \Psi_{2T^{ij}} &= C_{abcd}\, k^a\, m_i^b\, l^c\, m_j^d\,, \\
\Psi_{3^{ijk}} &= C_{abcd}\, l^a\, m_i^b\, m_j^c\, m_k^d\,, & \Psi_{3T^i} &= C_{abcd}\, l^a\, k^b\, l^c\, m_i^d\,, \\
\Psi_{4^{ij}} &= C_{abcd}\, l^a\, m_i^b\, l^c\, m_j^d\,, & &
\end{aligned}
\tag{3.9}
$$

and moreover, also their irreducible components

$$
\begin{aligned}
\tilde{\Psi}_{1^{ijk}} &= \Psi_{1^{ijk}} - \frac{2}{D-3}\delta_{i[j}\Psi_{1T^{k]}}\,, \\
\tilde{\Psi}_{2T^{(ij)}} &= \Psi_{2T^{(ij)}} - \frac{1}{D-2}\delta_{ij}\Psi_{2S}\,, \\
\tilde{\Psi}_{2^{ijkl}} &= \Psi_{2^{ijkl}} - \frac{2}{D-4}\big(\delta_{ik}\tilde{\Psi}_{2T^{(jl)}} + \delta_{jl}\tilde{\Psi}_{2T^{(ik)}} - \delta_{il}\tilde{\Psi}_{2T^{(jk)}} - \delta_{jk}\tilde{\Psi}_{2T^{(il)}}\big) \\
&\quad - \frac{4\,\delta_{i[k}\delta_{l]j}}{(D-2)(D-3)}\Psi_{2S}\,, \\
\tilde{\Psi}_{3^{ijk}} &= \Psi_{3^{ijk}} - \frac{2}{D-3}\delta_{i[j}\Psi_{3T^{k]}}\,,
\end{aligned}
\tag{3.10}
$$

which will identify specific algebraic subtypes. These frame components are sorted with respect to their boost weights. In particular, any quantity T has a boost weight w when it transforms under the Lorentz boost (3.7) as $T' = B^w\, T$. Obviously, the components $\Psi_{0^{ij}}$ has the boost weight 2 while $\Psi_{4^{ij}}$ is of the boost weight -2.

Finally, the Weyl tensor classification scheme is built upon the specific values of the boost weights present in the frame decomposition (3.9), for detailed invariant description and technical nuances see [66]. For practical purposes the frame null vector $\boldsymbol{k}$ would be considered as *aligned* with the Weyl tensor algebraic structure, i.e., it corresponds to a (multiple) Weyl aligned null direction (WAND). If such a vector exists then such identification can be always achieved by a suitable Lorentz transformation, see appendix of section A.1.1. With $\boldsymbol{k}$ being WAND the principal algebraic types and subtypes are directly defined in terms of (non-)vanishing Weyl scalars (3.9) and (3.10), respectively, see table 3.1. Moreover, the secondary alignment types can be discussed concerning the null vector $\boldsymbol{l}$. For example, when multiplicities of both $\boldsymbol{k}$ and $\boldsymbol{l}$ are two, i.e., the only present boost weight is zero, the Weyl tensor is of algebraic type D.

This classification can be also applied to the *traceless Ricci tensor*, i.e., $\mathcal{R}_{ab} \equiv R_{ab} - \frac{1}{D}\, R g_{ab}$. However, it is not crucial at this moment, and therefore only its components Φ_{AB} with respect to

type	**vanishing components**										
G	$\nexists$ frame $\Rightarrow \Psi_{0^{ij}} = 0$ for all i, j (possible only in $D > 4$)										
I	$\Psi_{0^{ij}}$										
I(a)	$\Psi_{0^{ij}}$	Ψ_{1T^i}									
I(b)	$\Psi_{0^{ij}}$		$\tilde{\Psi}_{1^{ijk}}$								
II	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$								
II(a)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}							
II(b)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	$\tilde{\Psi}_{2T^{(ij)}}$							
II(c)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	$\tilde{\Psi}_{2^{ijkl}}$							
II(d)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	$\Psi_{2^{ij}}$							
III	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}	$\tilde{\Psi}_{2T^{(ij)}}$	$\tilde{\Psi}_{2^{ijkl}}$	$\Psi_{2^{ij}}$				
III(a)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}	$\tilde{\Psi}_{2T^{(ij)}}$	$\tilde{\Psi}_{2^{ijkl}}$	$\Psi_{2^{ij}}$	Ψ_{3T^i}			
III(b)	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}	$\tilde{\Psi}_{2T^{(ij)}}$	$\tilde{\Psi}_{2^{ijkl}}$	$\Psi_{2^{ij}}$	$\tilde{\Psi}_{3^{ijk}}$			
N	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}	$\tilde{\Psi}_{2T^{(ij)}}$	$\tilde{\Psi}_{2^{ijkl}}$	$\Psi_{2^{ij}}$	Ψ_{3T^i}	$\tilde{\Psi}_{3^{ijk}}$		
O	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$	Ψ_{2S}	$\tilde{\Psi}_{2T^{(ij)}}$	$\tilde{\Psi}_{2^{ijkl}}$	$\Psi_{2^{ij}}$	Ψ_{3T^i}	$\tilde{\Psi}_{3^{ijk}}$	$\Psi_{4^{ij}}$	
D	$\Psi_{0^{ij}}$	Ψ_{1T^i}	$\tilde{\Psi}_{1^{ijk}}$					Ψ_{3T^i}	$\tilde{\Psi}_{3^{ijk}}$	$\Psi_{4^{ij}}$	

Table 3.1: The Weyl tensor principal algebraic types and subtypes defined via null alignment of the frame vectors. The direct relation between the Weyl scalars and particular algebraic types corresponds to the frame aligned with the Weyl tensor algebraic structure. The algebraic subtypes can be further combined.

the null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$ are defined, namely

$$
\begin{aligned}
\Phi_{00} &= \frac{1}{2}\,\mathcal{R}_{ab}\,k^a\,k^b\,, \\
\Phi_{01^i} &= \frac{1}{\sqrt{2}}\,\mathcal{R}_{ab}\,k^a\,m_i^b\,, \\
\Phi_{11} &= \mathcal{R}_{ab}\,k^a\,l^b\,, \qquad\qquad \Phi_{02^{ij}} = \mathcal{R}_{ab}\,m_i^a\,m_j^b\,, \\
\Phi_{12^i} &= \frac{1}{\sqrt{2}}\,\mathcal{R}_{ab}\,l^a\,m_i^b\,, \\
\Phi_{22} &= \frac{1}{2}\,\mathcal{R}_{ab}\,l^a\,l^b\,,
\end{aligned}
\tag{3.11}
$$

which come in useful later. They are again sorted by their boost weights ranging from 2 to -2.

All the above frame projections are adjusted onto arbitrary D-dimensional geometry. In four dimensions, it is canonical to replace a pair of the real spatial vectors $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ by their complex combinations $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, respectively,

$$
\boldsymbol{m} \equiv \frac{1}{\sqrt{2}}\,(\boldsymbol{m}_2 - \mathrm{i}\,\boldsymbol{m}_3)\,, \qquad \bar{\boldsymbol{m}} \equiv \frac{1}{\sqrt{2}}\,(\boldsymbol{m}_2 + \mathrm{i}\,\boldsymbol{m}_3)\,, \tag{3.12}
$$

and define equivalent expressions for the Weyl tensor projections as

$$
\begin{aligned}
\Psi_0 =& C_{abcd} k^a m^b k^c m^d \,, \\
\Psi_1 =& C_{abcd} k^a l^b k^c m^d \,, \\
\Psi_2 =& C_{abcd} k^a m^b \bar{m}^c l^d = \frac{1}{2} C_{abcd} k^a l^b (k^c l^d - m^c \bar{m}^d) \,, \\
\Psi_3 =& C_{abcd} l^a k^b l^c \bar{m}^d \,, \\
\Psi_4 =& C_{abcd} l^a \bar{m}^b l^c \bar{m}^d \,,
\end{aligned}
\tag{3.13}
$$

and for the Ricci tensor (or equivalently its traceless part $\mathcal{R}_{ab} = R_{ab} - \frac{1}{4} R g_{ab}$) as

$$
\begin{aligned}
&\Phi_{00} = \frac{1}{2} R_{ab} k^a k^b \,, \\
&\Phi_{01} = \frac{1}{2} R_{ab} k^a m^b \,, && \Phi_{10} = \frac{1}{2} R_{ab} k^a \bar{m}^b \,, \\
&\Phi_{11} = \frac{1}{4} R_{ab} \left(k^a l^b + m^a \bar{m}^b \right) , && \Phi_{02} = \frac{1}{2} R_{ab} m^a m^b \,, && \Phi_{20} = \frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b \,, \\
&\Phi_{12} = \frac{1}{2} R_{ab} l^a m^b \,, && \Phi_{21} = \frac{1}{2} R_{ab} l^a \bar{m}^b \,, \\
&\Phi_{22} = \frac{1}{2} R_{ab} l^a l^b \,.
\end{aligned}
\tag{3.14}
$$

This has proved to be useful in various applications within $D = 4$ general relativity, see e.g. [25, 67, 68]. In the thesis, these quantities will be also naturally employed within the generalized Newman–Penrose formalism discussed in section 4.1.

In four dimensions both definitions have to be compatible, again see appendix in section A.1.1. In particular, the Weyl tensor (3.9) has only two real independent components of each boost weight which thus correspond to the real and imaginary parts of (3.13), namely

$$
\begin{aligned}
\Psi_0 &= \Psi_{0^{22}} - \mathrm{i}\, \Psi_{0^{23}} \,, \\
\Psi_1 &= \frac{1}{\sqrt{2}} (\Psi_{1T^2} - \mathrm{i}\, \Psi_{1T^3}) \,, \\
\Psi_2 &= -\frac{1}{2} (\Psi_{2^{2323}} + \mathrm{i}\, \Psi_{2^{23}}) \,, \\
\Psi_3 &= \frac{1}{\sqrt{2}} (\Psi_{3T^2} + \mathrm{i}\, \Psi_{3T^3}) \,, \\
\Psi_4 &= \Psi_{4^{22}} + \mathrm{i}\, \Psi_{4^{23}} \,.
\end{aligned}
\tag{3.15}
$$

The analogous identification can be made between the Ricci tensor components (3.11) and (3.14).

The Weyl tensor components analysis is crucial in A.1.2 where the algebraic classification of non-twisting and shear-free geometries is performed. Moreover, it enters also all other results since, e.g., in A.1.1 the Weyl scalars encode relative deformations of the geodesic congruence, or in section A.2 where the Weyl type D black-hole geometries are investigated.

3.2 Non-twisting geometries and adapted coordinates

In this section, the geometries admitting a non-twisting null affinely parametrized geodesic congruence in any dimension D are described. These manifolds are defined without any assumption on a particular theory, i.e., they are specified in purely geometric terms of the *optical scalars*. The non-twisting assumption inevitably results in the induced null foliation of the manifold and it introduces the natural parametrization using adapted coordinates. This is extremely important for studying various situations where the explicit knowledge of the spacetime null structure is appreciated. As an example, one can think about radiative processes including exact models of gravitational waves, or quite recently about the concept of isolated horizons [69] which description also naturally fits into the form of non-twisting line element in adapted coordinates. A detailed list of explicit applications within GR can be found in [25]. Interestingly, even the Kerr rotating black hole was identified within the non-twisting class [70–72]. Finally, it is also worth introducing two important subclasses of the non-twisting geometries with additionally vanishing shear, namely the expanding Robinson-Trautman class and the non-expanding Kundt class [35–38]. Within the presented research the particular members of non-twisting geometries typically correspond to the metric ansatz which is further restricted by the field equations of the gravity theory.

Generic non-twisting geometries

In the previous section, the null frame $\{\boldsymbol{k},\,\boldsymbol{l},\,\boldsymbol{m}_i\}$ composed by a pair of null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ and $D-2$ orthonormal spacelike vectors $\boldsymbol{m}_i$ was employed to study the algebraic properties of tensors in D dimensions. Moreover, it is also very natural to describe the behavior of geodesic in terms of the null frame, see e.g. [66, 73, 74]. The key quantities entering such analysis are so-called optical scalars. In particular, the covariant derivative of frame vector $\boldsymbol{k}$ can be decomposed as

$$k_{a;b} = K_{11}k_ak_b + K_{10}k_al_b + K_{1i}k_am^i_b + K_{i1}m^i_ak_b + K_{i0}m^i_al_b + K_{ij}m^i_am^j_b\,, \tag{3.16}$$

where the coefficients[2] are inversely given by

$$\begin{aligned} K_{11} &= k_{a;b}l^al^b\,, & K_{10} &= k_{a;b}l^ak^b\,, & K_{1i} &= -k_{a;b}l^am^b_i\,, \\ K_{i1} &= -k_{a;b}m^a_il^b\,, & K_{i0} &= -k_{a;b}m^a_ik^b\,, & K_{ij} &= k_{a;b}m^a_im^b_j\,. \end{aligned} \tag{3.17}$$

The condition on $\boldsymbol{k}$ to generate affinely parameterized geodesic, i.e., to satisfy $k_{a;b}k^b=0$, corresponds to $K_{i0}=0=K_{10}$, while the coefficients K_{ij} remain non-trivial and encode geometric properties of the integral curves generated by $\boldsymbol{k}$, see figure 3.2.

In particular, the antisymmetric part, traceless symmetric part, and the trace of the optical matrix K_{ij} can be simply identified as

$$A_{ij} = K_{[ij]}\,, \qquad \sigma_{ij} = K_{(ij)} - \frac{\mathrm{Tr}K_{ij}}{D-2}\delta_{ij}\,, \qquad \Theta = \frac{\mathrm{Tr}K_{ij}}{D-2}\,, \tag{3.18}$$

corresponding to the twist matrix, shear matrix, and expansion scalar, respectively. These objects become trivial if, and only if, the related scalars expressed using derivatives of $\boldsymbol{k}$ are vanishing as well,

$$A^2 = -k_{[a;b]}k^{a;b}\,, \qquad \sigma^2 = k_{(a;b)}k^{a;b} - \frac{1}{D-2}(k^a_{\ ;a})^2\,, \qquad \Theta = \frac{1}{D-2}k^a_{\ ;a}\,. \tag{3.19}$$

These are the classic optical scalars, i.e., *twist*, *shear*, and *expansion*. Their geometric effect on the null congruence deformation is qualitatively illustrated in figure 3.3.

[2]There are various notations for these coefficients. In $D=4$ they represent the standard NP scalars, see (4.4), while for $D>4$ they were defined and extensively used in [74].

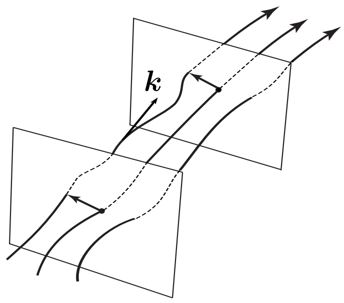

Figure 3.2: The affinely parametrized null geodesic vector field $\boldsymbol{k}$ generates congruence of integral curves whose geometric behavior is described by the coefficient K_{ij}, more precisely, by its irreducible parts corresponding to the optical scalars.

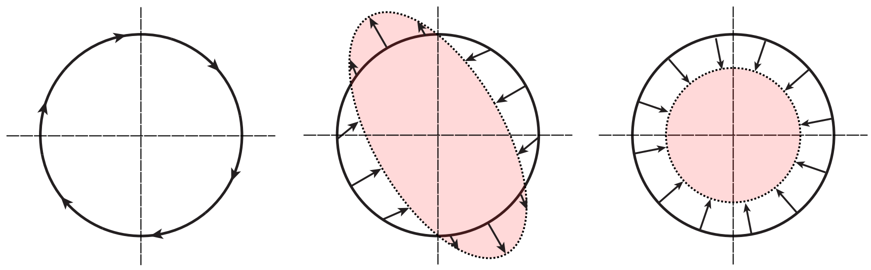

Figure 3.3: Schematic visualization of the optical scalars effects on a null geodesic congruence in terms of the twist (left), shear (middle), and expansion (right).

The very important property of the non-twisting geometries is the existence of the natural null foliation of a manifold. This is related to the Frobenius theorem, see e.g. [25], which states that a null congruence generated by $\boldsymbol{k}$ is (locally) hypersurface-orthogonal if, and only if,

$$k_{[a;b}k_{c]} = 0\,. \tag{3.20}$$

This condition applied in the case of null geodesics can be expressed using the twist matrix A_{ij}, namely

$$k_{[a;b}k_{c]} = \frac{1}{3}A_{ij}m^i_{[a}m^j_b k_{c]}\,, \tag{3.21}$$

and the contraction with any null vector l^c gives $k_{[a;b}k_{c]}l^c = -\frac{1}{3}A_{ij}m^i_a m^j_b$ which immediately implies that the vanishing twist matrix leads to the fulfillment of the Frobenius condition (3.20).

Moreover, in the case of non-twisting D-dimensional geometries it is natural to introduce the adapted coordinates $(r,\, u,\, x^p)$, where $u = \text{const}$ labels null hypersurfaces normal to $\boldsymbol{k}$, coordinate r is the affine parameter along null geodesics generated by $\boldsymbol{k}$, and x^p cover the transverse $(D-2)$-dimensional space with $u = \text{const}$ and $r = \text{const}$, see figure 3.4.

The adapted coordinates can be also introduced on a quite intuitive level. Begin with any coordinates x^a, where $a = 0, \ldots, D-1$, and arbitrary worldline $x^a(u)$. Then a null hypersurface can be constructed at every point of $x^a(u)$. These surfaces are thus uniquely identified by the value $u(x^a) = \text{const}$ and the parameter u is taken as a new coordinate. It can be shown that tangent (and simultaneously) normal to these hypersurfaces, denoted as $\boldsymbol{k}$, generates a null geodesic congruence.

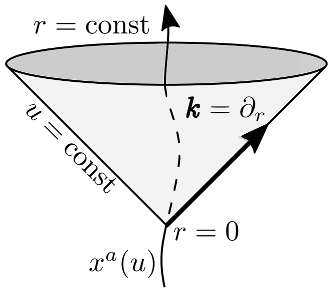

Figure 3.4: The non-twisting geometry is described in terms of adapted coordinates $(r,\, u,\, x^p)$. The coordinate u determines a null hypersurface with $\boldsymbol{k}$ being normal, r is the affine parameter along congruence generated by $\boldsymbol{k}$, and the transverse space with fixed $u = \mathrm{const}$ and $r = \mathrm{const}$ is spanned by $D-2$ coordinates x^p.

The affine parameter r along such a congruence represents the second coordinate, i.e., $\boldsymbol{k} = \partial_r$. Finally, there remain $D-2$ coordinates x^p of the original D-dimensional set x^a. In technical terms, this construction restricts the non-twisting line element to the form

$$\mathrm{d}s^2 = g_{pq}(r,u,x)\,\mathrm{d}x^p\mathrm{d}x^q + 2g_{uq}(r,u,x)\,\mathrm{d}x^q\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \tag{3.22}$$

with $p,\, q = 2, \ldots,\, D-1$ and g_{pq} describing the transverse Riemann space geometry. Notice that the affine character of r corresponds to $g_{ur} = -1$ and can be achieved by a suitable transformation. Since the coefficients K_{ij}, see (3.17), take the form

$$K_{ij} = \frac{1}{2} g_{pq,r}\, m_i^p m_j^q \qquad \text{with} \qquad p,\, q = 2, \ldots,\, D-1\,, \tag{3.23}$$

the optical scalars (3.19) for the privileged vector field $\boldsymbol{k}$ become

$$A^2 = 0\,, \qquad \sigma^2 = \frac{1}{4} g^{pm} g^{qn} g_{pq,r} g_{mn,r} - (D-2)\Theta^2\,, \qquad \Theta = \frac{1}{2(D-2)} g^{pq} g_{pq,r}\,, \tag{3.24}$$

which confirms the initial constraint on $\boldsymbol{k}$ being non-twisting. A fully general form of the curvature tensors for the line element 3.22 was calculated in the doctoral thesis [75] which allowed straightforward analysis of its particular (shear-free) subclasses within my subsequent works.

Moreover, assuming the vanishing shear σ the two important subclasses defined with respect to the value of expansion Θ are distinguished, see subsequent paragraphs. The shear-free condition $\sigma_{ij} = 0$ also enables to express r-derivative of the transverse metric g_{pq} in terms of the expansion,

$$g_{pq,r} = 2\Theta\, g_{pq}\,, \tag{3.25}$$

which can be simply integrated to get

$$g_{pq}(r,u,x) = \exp\left(2\int \Theta(r,u,x)\,\mathrm{d}r\right) h_{pq}(u,x)\,, \tag{3.26}$$

and thus to fully separate the r-dependence of the transverse metric.

Finally, it is useful to introduce two realizations of the suitable null frame $\{\boldsymbol{k},\, \boldsymbol{l},\, \boldsymbol{m}_i\}$ supplementing the non-twisting vector field $\boldsymbol{k}$ and satisfying the normalization conditions (3.3), namely

- possibility with simple $\boldsymbol{l}$

$$\boldsymbol{k} = \partial_r\,, \qquad \boldsymbol{l} = \frac{1}{2} g_{uu}\partial_r + \partial_u\,, \qquad \boldsymbol{m}_i = g_{up} m_i^p \partial_r + m_i^p \partial_p\,, \tag{3.27}$$

- possibility with simple $\boldsymbol{m}_i$

$$\boldsymbol{k} = \partial_r\,, \qquad \boldsymbol{l} = -\frac{1}{2} g^{rr}\partial_r + \partial_u - g^{rp}\partial_p\,, \qquad \boldsymbol{m}_i = m_i^p \partial_p\,, \tag{3.28}$$

which are obviously related by the null rotation with $\boldsymbol{k}$ fixed (3.5) and parameters $L_i = \frac{1}{\sqrt{2}} g_{up} m_i^p$ leading from (3.28) to (3.27).

Kundt class

The Kundt family of geometries admits *all the optical scalars vanishing.* The condition (3.26) thus implies that the transverse space metric g_{pq} has to be r-independent, i.e., the complete line element (3.22) becomes

$$\mathrm{d}s^2_{\mathrm{Kundt}} = h_{pq}(u,x)\,\mathrm{d}x^p\mathrm{d}x^q + 2g_{uq}(r,u,x)\,\mathrm{d}x^q\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,. \tag{3.29}$$

Spacetimes of this form play important role in standard GR as well as in its extensions allowing both $D > 4$ and $D = 3$ [56, 58, 76–79], or in modified theories, e.g., on a very general level of so-called universal and almost universal spacetimes [54, 55]. To be more explicit the GR Kundt solutions contain various direct product geometries, non-expanding gravitational waves, or spacetimes with constant or vanishing scalar curvature invariants, their comprehensive list can be found in [25, 26].

Robinson–Trautman class

This is defined as geometries admitting the non-twisting, shear-free, but *expanding* null geodesic congruence. From (3.22) and (3.26) it follows that the Robinson–Trautman line element takes the form

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{RT}} = &\exp\left(2\int \Theta(r,u,x)\,\mathrm{d}r\right) h_{pq}(u,x)\,\mathrm{d}x^p\mathrm{d}x^q \\ &+ 2g_{uq}(r,u,x)\,\mathrm{d}x^q\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,.\end{aligned} \tag{3.30}$$

It contains various particularly interesting solutions within classic four-dimensional general relativity such as, for example, the Schwarzschild black hole, its Reissner–Nordström or Vaidya generalizations, accelerating black holes described by the C-metric, or spherical type N sandwich gravitational waves and their impulsive limits, see again [25, 26]. This class has been extensively studied also in the GR higher-dimensional extension [80–82], where, surprisingly, the solution space becomes much smaller admitting only the Weyl type D solutions with absence of the C-metric.

At the end of this paragraph it is worth mentioning the alternative metric form to (3.30) which relates the Robinson–Trautman and Kundt class in terms of the suitable conformal rescaling. In particular, the line element

$$\mathrm{d}s^2 = \Omega^2(r,u,x)\,\mathrm{d}s^2_{\mathrm{Kundt}} \tag{3.31}$$

can be considered which belongs to the Robinson–Trautman class if the conformal factor Ω depends on the r coordinate, or it remains the Kundt one if Ω is r-independent. All the details can be found in section A.1.3. This can be illustrated in the case of static spherically symmetric geometries, see section A.2, which are typically introduced as

$$\mathrm{d}s^2 = -h(\bar{r})\,\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{f(\bar{r})} + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{3.32}$$

This line element can be put into the Robinson–Trautman form,

$$\mathrm{d}s^2_{\mathrm{RT}} = \Omega^2(\tilde{r})\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) - 2\,\mathrm{d}u\,\mathrm{d}\tilde{r} + H(\tilde{r})\,\mathrm{d}u^2\,, \tag{3.33}$$

applying the transformation

$$\bar{r} = \Omega(\tilde{r})\,, \qquad t = u - \int \frac{\mathrm{d}\tilde{r}}{H(\tilde{r})}\,, \tag{3.34}$$

related to the original metric functions in (3.32) by

$$H = -h(\bar{r})\,, \qquad \Omega_{,\tilde{r}} = \sqrt{\frac{f(\bar{r})}{h(\bar{r})}}\,. \tag{3.35}$$

Finally, pulling out the factor Ω in (3.33) followed by the simple transformation $r = \int \Omega^{-2}(\tilde{r})\,\mathrm{d}\tilde{r}$, and identification $\mathcal{H} \equiv \Omega^{-2}\,H$, lead to the spherically symmetric geometries (3.32) in the conformal-to-Kundt form,

$$\mathrm{d}s^2_{\mathrm{RT}} = \Omega^2(r)\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 - 2\,\mathrm{d}u\,\mathrm{d}r + \mathcal{H}(r)\,\mathrm{d}u^2\right], \tag{3.36}$$

which has the form of relation (3.31) announced at the beginning. Surprisingly, this line element allows analytical discussion of the spherical solutions to quadratic gravity enormously simplifying the field equations, see section A.2.

CHAPTER

4

ANALYSIS OF EXACT SPACETIMES

This chapter presents two methods suitable for exact spacetime analysis. Its first section describes original results discussing the Newman–Penrose formalism in quadratic gravity which should be published in near future and then would fit into section A.1. The second part of this chapter then unifies invariant description of the geodesic deviation irrespectively of a specific gravity theory and serves as a starting point for various analyses within the subsequent papers.

4.1 Newman–Penrose formalism in quadratic gravity

The Newman–Penrose formalism in GR is an excellent application of the general concept of tetrad description using the advantages of null frames. It naturally interplays with methods of algebraic classification and characterization of null geodesic congruences mentioned in the previous chapter. Therefore, it proves to be useful in various situations within GR and its extension into higher dimensions, see e.g. [25, 74, 83, 84]. The aim here is to adapt this approach also for the *quadratic gravity in four dimensions.* The preliminary form of these results is a part of the diploma thesis [85] and the extended version would be published as [86].

The key ingredient of the Newman–Penrose approach is the null orthonormal frame $\{\boldsymbol{k}\,,\boldsymbol{l}\,,\boldsymbol{m}\,,\bar{\boldsymbol{m}}\}$ already mentioned in section 3.1. In the convention[1] of [25], it is normalized as

$$\boldsymbol{k}\cdot\boldsymbol{l}=-1\,,\qquad \boldsymbol{m}\cdot\bar{\boldsymbol{m}}=1\,, \tag{4.1}$$

while other combinations are vanishing. This is equivalent to the metric form

$$g_{ab}=-2k_{(a}l_{b)}+2m_{(a}\bar{m}_{b)}\,. \tag{4.2}$$

Freedom in the frame choice is described by the Lorentz transformations (3.5)–(3.8) adjusted to the complex vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, see [25]. To decompose covariant derivatives of the frame vectors, it is useful to define specific symbols for the covariant derivative projected into the frame vector directions, namely

$$\mathrm{D}=k^a\nabla_a\,,\quad \Delta=l^a\nabla_a\,,\quad \delta=m^a\nabla_a\,,\quad \bar{\delta}=\bar{m}^a\nabla_a\,. \tag{4.3}$$

[1] Beware of other possibilities typically varying in particular signs, e.g. [68, 87].

To characterize the action of the above NP derivatives again onto the frame vectors the spin coefficients are defined as

$$\begin{aligned}
\kappa &= -k_{a;b}m^a k^b\,, & \nu &= l_{a;b}\bar{m}^a l^b\,, & \epsilon &= \frac{1}{2}\left(m_{a;b}\bar{m}^a k^b - k_{a;b}l^a k^b\right),\\
\rho &= -k_{a;b}m^a \bar{m}^b\,, & \mu &= l_{a;b}\bar{m}^a m^b\,, & \beta &= \frac{1}{2}\left(m_{a;b}\bar{m}^a m^b - k_{a;b}l^a m^b\right),\\
\sigma &= -k_{a;b}m^a m^b\,, & \lambda &= l_{a;b}\bar{m}^a \bar{m}^b\,, & \gamma &= \frac{1}{2}\left(l_{a;b}k^a l^b - \bar{m}_{a;b}m^a l^b\right),\\
\tau &= -k_{a;b}m^a l^b\,, & \pi &= l_{a;b}\bar{m}^a k^b\,, & \alpha &= \frac{1}{2}\left(l_{a;b}k^a \bar{m}^b - \bar{m}_{a;b}m^a \bar{m}^b\right).
\end{aligned} \tag{4.4}$$

The remaining crucial NP quantities are the independent frame components of the Weyl tensor (3.13) and the projections of the Ricci tensor (3.14). Then the quadratic gravity field equations (2.22), expressed in terms of the null frame $\{\boldsymbol{k}\,,\boldsymbol{l}\,,\boldsymbol{m}\,,\bar{\boldsymbol{m}}\}$, take the form

$$\begin{aligned}
\frac{1}{2}T_{(0)(0)} =& -4\mathfrak{a}\left[\Phi_{20}\Psi_0+\Phi_{02}\bar{\Psi}_0-2\Phi_{10}\Psi_1-2\Phi_{01}\bar{\Psi}_1+\Phi_{00}(\Psi_2+\bar{\Psi}_2)\right]\\
&+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{00}+Z_{(0)(0)}\,,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\frac{1}{2}T_{(0)(1)} =& -4\mathfrak{a}\left[\Phi_{21}\Psi_1+\Phi_{12}\bar{\Psi}_1-2\Phi_{11}(\Psi_2+\bar{\Psi}_2)+\Phi_{01}\Psi_3+\Phi_{10}\bar{\Psi}_3\right]\\
&+\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\left(2\Phi_{11}-\frac{R}{4}\right)+Z_{(0)(1)}\,,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\frac{1}{2}T_{(0)(2)} =& -4\mathfrak{a}\left[\Phi_{21}\Psi_0-2\Phi_{11}\Psi_1+\Phi_{02}\bar{\Psi}_1+\Phi_{01}(\Psi_2-2\bar{\Psi}_2)+\Phi_{00}\bar{\Psi}_3\right]\\
&+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{01}+Z_{(0)(2)}\,,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\frac{1}{2}T_{(1)(1)} =& -4\mathfrak{a}\big(\Phi_{22}(\Psi_2+\bar{\Psi}_2)-2\Phi_{12}\Psi_3-2\Phi_{21}\bar{\Psi}_3+\Phi_{02}\Psi_4+\Phi_{20}\bar{\Psi}_4\big)\\
&+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{22}+Z_{(1)(1)}\,,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\frac{1}{2}T_{(1)(2)} =& -4\mathfrak{a}\left[\Phi_{22}\Psi_1+\Phi_{12}(-2\Psi_2+\bar{\Psi}_2)+\Phi_{02}\Psi_3-2\Phi_{11}\bar{\Psi}_3+\Phi_{10}\bar{\Psi}_4\right]\\
&+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{12}+Z_{(1)(2)}\,,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\frac{1}{2}T_{(2)(2)} =& -4\mathfrak{a}\left[\Phi_{22}\Psi_0-2\Phi_{12}\Psi_1+\Phi_{02}(\Psi_2+\bar{\Psi}_2)-2\Phi_{01}\bar{\Psi}_3+\Phi_{00}\bar{\Psi}_4\right]\\
&+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{02}+Z_{(2)(2)}\,,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\frac{1}{2}T_{(2)(3)} =& -4\mathfrak{a}\left[\Phi_{21}\Psi_1+\Phi_{12}\bar{\Psi}_1-2\Phi_{11}(\Psi_2+\bar{\Psi}_2)+\Phi_{01}\Psi_3+\Phi_{10}\bar{\Psi}_3\right]\\
&+\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\left(2\Phi_{11}+\frac{R}{4}\right)+Z_{(2)(3)}\,,
\end{aligned} \tag{4.11}$$

where the symbols $T_{(c)(d)}$ and $Z_{(c)(d)}$ stand for the components of the energy-momentum tensor T_{ab} and the tensor Z_{ab}, see (2.23), projected onto the null frame $\{\boldsymbol{k}\,,\boldsymbol{l}\,,\boldsymbol{m}\,,\bar{\boldsymbol{m}}\}$, e.g., $Z_{(0)(0)} = Z_{ab}k^a k^b$ and $Z_{(1)(2)} = Z_{ab}l^a m^b$ etc. In principle, the above system of equations can be understood as *algebraic constraints on the Ricci tensor components* Φ_{AB}. Such constraints have to be further combined with the standard geometric conditions, i.e., the *Ricci* and *Bianchi identities*, see [25] for their list compatible with the assumed convention. In fact, the same approach is applied within Einstein's general relativity, where the Ricci tensor components are also directly restricted by the field equations. The GR case is given by $\mathfrak{a} = 0 = \mathfrak{b}$, the scalar curvature becomes $R = 4\Lambda - \frac{\mathsf{k}}{2}T$, and the above constrains are significantly simplified.

Finally, to be fully explicit all relevant projections of the Z_{ab} tensor have to be expressed, i.e.,

$$Z_{(0)(0)} = -4\mathfrak{a}B^Z_{(0)(0)} + 2\mathfrak{b}\left[(\epsilon+\bar{\epsilon})\mathrm{D}R - \mathrm{D}\mathrm{D}R - \bar{\kappa}\delta R - \kappa\bar{\delta}R\right], \tag{4.12}$$

$$\begin{aligned} Z_{(0)(1)} = &-4\mathfrak{a}B^Z_{(0)(1)} + \frac{1}{2\mathsf{k}}(R-2\Lambda) + 2\mathfrak{b}\Big[\frac{1}{4}R^2 - (\gamma+\bar{\gamma}-\mu-\bar{\mu})\mathrm{D}R \\ &- (\rho+\bar{\rho})\Delta R + \Delta\mathrm{D}R + (\alpha-\bar{\beta}+\bar{\tau})\delta R - \delta\bar{\delta}R \\ &+ (\bar{\alpha}-\beta+\tau)\bar{\delta}R - \bar{\delta}\delta R\Big], \end{aligned} \tag{4.13}$$

$$Z_{(0)(2)} = -4\mathfrak{a}B^Z_{(0)(2)} + 2\mathfrak{b}\left[\bar{\pi}\mathrm{D}R - \mathrm{D}\delta R - \kappa\Delta R + (\epsilon-\bar{\epsilon})\delta R\right], \tag{4.14}$$

$$Z_{(1)(1)} = -4\mathfrak{a}B^Z_{(1)(1)} + 2\mathfrak{b}\left[-(\gamma+\bar{\gamma})\Delta R - \Delta\Delta R + \nu\delta R + \bar{\nu}\bar{\delta}R\right], \tag{4.15}$$

$$Z_{(1)(2)} = -4\mathfrak{a}B^Z_{(1)(2)} + 2\mathfrak{b}\left[\bar{\nu}\mathrm{D}R - \tau\Delta R - \Delta\delta R + (\gamma-\bar{\gamma})\delta R\right], \tag{4.16}$$

$$Z_{(2)(2)} = -4\mathfrak{a}B^Z_{(2)(2)} + 2\mathfrak{b}\left[\bar{\lambda}\mathrm{D}R - \sigma\Delta R + (-\bar{\alpha}+\beta)\delta R - \delta\delta R\right], \tag{4.17}$$

$$\begin{aligned} Z_{(2)(3)} = &-4\mathfrak{a}B^Z_{(2)(3)} - \frac{1}{2\mathsf{k}}(R-2\Lambda) + 2\mathfrak{b}\Big[-\frac{1}{4}R^2 + (\gamma+\bar{\gamma}-\bar{\mu})\mathrm{D}R \\ &- \mathrm{D}\Delta R + (\rho-\epsilon-\bar{\epsilon})\Delta R - \Delta\mathrm{D}R + (-\alpha+\bar{\beta}+\pi-\bar{\tau})\delta R \\ &+ (\bar{\pi}-\tau)\bar{\delta}R + \bar{\delta}\delta R\Big], \end{aligned} \tag{4.18}$$

where $B^Z_{(c)(d)}$ represents the Ricci-independent part of the Bach tensor corresponding to the second covariant derivative of the Weyl tensor, namely $B^Z_{ab} = \nabla^c\nabla^d C_{acbd}$. After quite straightforward and *very long calculation* this explicitly gives[2]

- the $\boldsymbol{kk}$-projection of B^Z_{ab}, i.e., $B^Z_{(0)(0)} = B^Z_{ab}k^a k^b$:

$$\begin{aligned} B^Z_{(0)(0)} = &\bar{\delta}\bar{\delta}\Psi_0 - \mathrm{D}\bar{\delta}\Psi_1 - \bar{\delta}\mathrm{D}\Psi_1 + \mathrm{D}\mathrm{D}\Psi_2 + \lambda\mathrm{D}\Psi_0 + \bar{\sigma}\Delta\Psi_0 + (2\pi-7\alpha-\bar{\beta})\bar{\delta}\Psi_0 \\ &+ (5\alpha+\bar{\beta}-3\pi)\mathrm{D}\Psi_1 - \bar{\kappa}\Delta\Psi_1 - \bar{\sigma}\delta\Psi_1 + (3\epsilon+\bar{\epsilon}+7\rho)\bar{\delta}\Psi_1 \\ &- (\epsilon+\bar{\epsilon}+6\rho)\mathrm{D}\Psi_2 + \bar{\kappa}\delta\Psi_2 - 5\kappa\bar{\delta}\Psi_2 + 4\kappa\mathrm{D}\Psi_3 \\ &+ \Psi_0[\bar{\kappa}\nu + 4\alpha(3\alpha+\bar{\beta}) - (\epsilon+\bar{\epsilon}+3\rho)\lambda + \pi(\pi-7\alpha-\bar{\beta}) + \bar{\sigma}(\mu-4\gamma) + \mathrm{D}\lambda - 4\bar{\delta}\alpha + \bar{\delta}\pi] \\ &+ 2\Psi_1[2\kappa\lambda + \bar{\kappa}(\gamma-\mu) + \rho(5\pi-9\alpha-2\bar{\beta}) + \bar{\sigma}(\beta+2\tau) + \epsilon(2\pi-4\alpha-\bar{\beta}) + \bar{\epsilon}(\pi-\alpha) \\ &\qquad + \mathrm{D}\alpha - \mathrm{D}\pi + \bar{\delta}\epsilon + 2\bar{\delta}\rho] \\ &+ 3\Psi_2[\kappa(3\alpha+\bar{\beta}-3\pi) - \bar{\kappa}\tau + \rho(\epsilon+\bar{\epsilon}+3\rho) - \sigma\bar{\sigma} - \mathrm{D}\rho - \bar{\delta}\kappa] \\ &+ 2\Psi_3[\kappa(\epsilon-\bar{\epsilon}-5\rho) + \bar{\kappa}\sigma + \mathrm{D}\kappa] + 2\Psi_4\kappa^2 + c.c., \end{aligned} \tag{4.19}$$

[2] The calculation of $B^Z_{(c)(d)}$ was performed independently by each author of [86]. Three different tools for the computer symbolic manipulations were employed. Moreover, the expressions seem to be compatible with those of [88] derived within the GHP formalism and the Weyl gravity.

• the $\boldsymbol{kl}$-projection of B^Z_{ab}, i.e., $B^Z_{(0)(1)} = B^Z_{ab}k^a l^b$:

$$
\begin{aligned}
B^Z_{(0)(1)} =& \bar\delta\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar\delta\delta\Psi_2 + \mathrm{D}\delta\Psi_3 - \lambda\Delta\Psi_0 - \nu\bar\delta\Psi_0 \\
&+ 2\nu\mathrm{D}\Psi_1 + (2\pi - \alpha + \bar\beta)\Delta\Psi_1 + \lambda\delta\Psi_1 + (2\mu - \bar\mu - 2\gamma)\bar\delta\Psi_1 \\
&+ (\bar\mu - 3\mu)\mathrm{D}\Psi_2 + (2\rho - \epsilon - \bar\epsilon)\Delta\Psi_2 + (\alpha - \bar\beta - 2\pi)\delta\Psi_2 + (\bar\pi + 3\tau)\bar\delta\Psi_2 \\
&+ (2\beta - \bar\pi - 2\tau)\mathrm{D}\Psi_3 - \kappa\Delta\Psi_3 + (\epsilon + \bar\epsilon - 2\rho)\delta\Psi_3 - 2\sigma\bar\delta\Psi_3 + \sigma\mathrm{D}\Psi_4 + \kappa\delta\Psi_4 \\
&+ \Psi_0[\lambda(4\gamma - \mu + \bar\mu) + \nu(\alpha - \bar\beta - 2\pi) - \bar\delta\nu] \\
&+ 2\Psi_1[\gamma(\alpha - \bar\beta - 2\pi) - \lambda(\beta + \bar\pi + 2\tau) + \mu(\bar\beta - \alpha + 2\pi) + \bar\mu(\alpha - \pi) + \nu(\epsilon + \bar\epsilon - 2\rho) \\
&\qquad + \mathrm{D}\nu - \bar\delta\gamma + \bar\delta\mu] \\
&+ 3\Psi_2[\kappa\nu + \mu(2\rho - \epsilon - \bar\epsilon) - \bar\mu\rho + \pi\bar\pi + \lambda\sigma + \tau(2\pi - \alpha + \bar\beta) - \mathrm{D}\mu + \bar\delta\tau] \\
&+ 2\Psi_3[\kappa(\bar\mu - 2\mu - \gamma) + \epsilon(\beta - \tau - \bar\pi) + \bar\epsilon(\beta - \tau) + \rho(\bar\pi - 2\beta + 2\tau) + \sigma(\alpha - \bar\beta - 2\pi) \\
&\qquad + \mathrm{D}\beta - \mathrm{D}\tau - \bar\delta\sigma] \\
&+ \Psi_4[\kappa(4\beta - \bar\pi - \tau) + \sigma(\epsilon + \bar\epsilon - 2\rho) + \mathrm{D}\sigma] + c.c.\,,
\end{aligned} \tag{4.20}
$$

• the $\boldsymbol{km}$-projection of B^Z_{ab}, i.e., $B^Z_{(0)(2)} = B^Z_{ab}k^a m^b$:

$$
\begin{aligned}
B^Z_{(0)(2)} =& \bar\delta\Delta\Psi_0 - \mathrm{D}\Delta\Psi_1 - \bar\delta\delta\Psi_1 + \mathrm{D}\delta\Psi_2 \\
&+ \nu\mathrm{D}\Psi_0 + (\pi - 3\alpha + \bar\beta)\Delta\Psi_0 + (\mu - \bar\mu - 4\gamma)\bar\delta\Psi_0 \\
&+ (2\gamma - 2\mu + \bar\mu)\mathrm{D}\Psi_1 + (\epsilon - \bar\epsilon + 3\rho)\Delta\Psi_1 + (3\alpha - \bar\beta - \pi)\delta\Psi_1 + (2\beta + \bar\pi + 4\tau)\bar\delta\Psi_1 \\
&- (\bar\pi + 3\tau)\mathrm{D}\Psi_2 - 2\kappa\Delta\Psi_2 - (\epsilon - \bar\epsilon + 3\rho)\delta\Psi_2 - 3\sigma\bar\delta\Psi_2 + 2\sigma\mathrm{D}\Psi_3 + 2\kappa\delta\Psi_3 \\
&+ \Psi_0[(4\gamma - \mu)(3\alpha - \bar\beta - \pi) + \bar\mu(4\alpha - \pi) + \nu(\bar\epsilon - \epsilon - 3\rho) - \lambda\bar\pi + \mathrm{D}\nu - 4\bar\delta\gamma + \bar\delta\mu] \\
&+ 2\Psi_1[2\kappa\nu + (\mu - \gamma)(\epsilon - \bar\epsilon + 3\rho) - \bar\mu(2\rho + \epsilon) + (\beta + 2\tau)(\pi - 3\alpha + \bar\beta) + \bar\pi(\pi - \alpha) \\
&\qquad + \mathrm{D}\gamma - \mathrm{D}\mu + \bar\delta\beta + 2\bar\delta\tau] \\
&+ 3\Psi_2[\kappa(\bar\mu - 2\mu) + \bar\pi\rho + \sigma(3\alpha - \bar\beta - \pi) + \tau(\epsilon - \bar\epsilon + 3\rho) - \mathrm{D}\tau - \bar\delta\sigma] \\
&+ 2\Psi_3[\kappa(2\beta - \bar\pi - 2\tau) + \sigma(\bar\epsilon - \epsilon - 3\rho) + \mathrm{D}\sigma] + 2\Psi_4\kappa\sigma \\
&+ \delta\delta\bar\Psi_1 - \delta\mathrm{D}\bar\Psi_2 - \mathrm{D}\delta\bar\Psi_2 + \mathrm{D}\mathrm{D}\bar\Psi_3 \\
&- 2\bar\lambda\delta\bar\Psi_0 + 3\bar\lambda\mathrm{D}\bar\Psi_1 + \sigma\Delta\bar\Psi_1 + (4\bar\pi - 3\bar\alpha - \beta)\delta\bar\Psi_1 \\
&+ (\bar\alpha + \beta - 5\bar\pi)\mathrm{D}\bar\Psi_2 - \kappa\Delta\bar\Psi_2 + (\epsilon - \bar\epsilon + 5\bar\rho)\delta\bar\Psi_2 - \sigma\bar\delta\bar\Psi_2 \\
&+ (3\bar\epsilon - \epsilon - 4\bar\rho)\mathrm{D}\bar\Psi_3 - 3\bar\kappa\delta\bar\Psi_3 + \kappa\bar\delta\bar\Psi_3 + 2\bar\kappa\mathrm{D}\bar\Psi_4 \\
&+ \bar\Psi_0[\bar\lambda(5\bar\alpha + \beta - 3\bar\pi) - \bar\nu\sigma - \delta\bar\lambda] \\
&+ 2\bar\Psi_1[\kappa\bar\nu + \bar\alpha(\bar\alpha + \beta) + \bar\pi(2\bar\pi - 3\bar\alpha - \beta) - \bar\lambda(4\bar\rho + \epsilon) + \sigma(\bar\mu - \bar\gamma) + \mathrm{D}\bar\lambda - \delta\bar\alpha + \delta\bar\pi] \\
&+ 3\bar\Psi_2[2\bar\kappa\bar\lambda - \kappa\bar\mu + \bar\pi(\epsilon - \bar\epsilon) + \bar\rho(4\bar\pi - \bar\alpha - \beta) + \sigma\bar\tau - \mathrm{D}\bar\pi + \delta\bar\rho] \\
&+ 2\bar\Psi_3(\kappa(\bar\beta - \bar\tau) + \bar\kappa(\beta - 4\bar\pi) - \sigma\bar\sigma + (\bar\rho - \bar\epsilon)(\epsilon - \bar\epsilon + 2\bar\rho) + \mathrm{D}\bar\epsilon - \mathrm{D}\bar\rho - \delta\bar\kappa) \\
&+ \bar\Psi_4[\bar\kappa(5\bar\epsilon - \epsilon - 3\bar\rho) + \kappa\bar\sigma + \mathrm{D}\bar\kappa]\,,
\end{aligned} \tag{4.21}
$$

- the $\boldsymbol{ll}$-projection of B^Z_{ab}, i.e., $B^Z_{(1)(1)} = B^Z_{ab} l^a l^b$:

$$
\begin{aligned}
B^Z_{(1)(1)} =& \Delta\Delta\Psi_2 - \Delta\delta\Psi_3 - \delta\Delta\Psi_3 + \delta\delta\Psi_4 \\
& - 4\nu\Delta\Psi_1 + (\gamma + \bar{\gamma} + 6\mu)\Delta\Psi_2 + 5\nu\delta\Psi_2 - \bar{\nu}\bar{\delta}\Psi_2 \\
& + \bar{\nu}\mathrm{D}\Psi_3 + (3\tau - \bar{\alpha} - 5\beta)\Delta\Psi_3 - (3\gamma + \bar{\gamma} + 7\mu)\delta\Psi_3 + \bar{\lambda}\bar{\delta}\Psi_3 \\
& - \bar{\lambda}\mathrm{D}\Psi_4 - \sigma\Delta\Psi_4 + (\bar{\alpha} + 7\beta - 2\tau)\delta\Psi_4 \\
& + 2\Psi_0\nu^2 + 2\Psi_1[\nu(\gamma - \bar{\gamma} - 5\mu) + \lambda\bar{\nu} - \Delta\nu] \\
& + 3\Psi_2[\mu(\gamma + \bar{\gamma} + 3\mu) + \nu(\bar{\alpha} + 3\beta - 3\tau) - \lambda\bar{\lambda} - \bar{\nu}\pi + \Delta\mu + \delta\nu] \\
& + 2\Psi_3[\bar{\nu}(\epsilon - \rho) + \bar{\lambda}(\alpha + 2\pi) + \gamma(2\tau - \bar{\alpha} - 4\beta) + \bar{\gamma}(\tau - \beta) + \mu(5\tau - 2\bar{\alpha} - 9\beta) + 2\nu\sigma \\
& \qquad - \Delta\beta + \Delta\tau - \delta\gamma - 2\delta\mu] \\
& + \Psi_4[\kappa\bar{\nu} + \bar{\lambda}(\rho - 4\epsilon) - \sigma(\gamma + \bar{\gamma} + 3\mu) + 4\beta(3\beta + \bar{\alpha}) + \tau(\tau - \bar{\alpha} - 7\beta) \\
& \qquad - \Delta\sigma + 4\delta\beta - \delta\tau] + c.c. \,,
\end{aligned} \tag{4.22}
$$

- the $\boldsymbol{lm}$-projection of B^Z_{ab}, i.e., $B^Z_{(0)(1)} = B^Z_{ab} l^a m^b$:

$$
\begin{aligned}
B^Z_{(1)(2)} =& \Delta\Delta\Psi_1 - \Delta\delta\Psi_2 - \delta\Delta\Psi_2 + \delta\delta\Psi_3 \\
& - 2\nu\Delta\Psi_0 + (4\mu - 3\gamma + \bar{\gamma})\Delta\Psi_1 + 3\nu\delta\Psi_1 - \bar{\nu}\bar{\delta}\Psi_1 \\
& + \bar{\nu}\mathrm{D}\Psi_2 + (5\tau - \bar{\alpha} - \beta)\Delta\Psi_2 + (\gamma - \bar{\gamma} - 5\mu)\delta\Psi_2 + \bar{\lambda}\bar{\delta}\Psi_2 \\
& - \bar{\lambda}\mathrm{D}\Psi_3 - 3\sigma\Delta\Psi_3 + (\bar{\alpha} + 3\beta - 4\tau)\delta\Psi_3 + 2\sigma\delta\Psi_4 \\
& + \Psi_0[\nu(5\gamma - \bar{\gamma} - 3\mu) + \lambda\bar{\nu} - \Delta\nu] \\
& + 2\Psi_1[\nu(\bar{\alpha} - 4\tau) + \bar{\nu}(\alpha - \pi) - \lambda\bar{\lambda} + (\gamma - \mu)(\gamma - \bar{\gamma} - 2\mu) - \Delta\gamma + \Delta\mu + \delta\nu] \\
& + 3\Psi_2[\mu(4\tau - \bar{\alpha} - \beta) + \bar{\lambda}\pi - \bar{\nu}\rho + 2\nu\sigma + \tau(\bar{\gamma} - \gamma) + \Delta\tau - \delta\mu] \\
& + 2\Psi_3[\kappa\bar{\nu} - \sigma(\bar{\gamma} + 4\mu) + \tau(2\tau - \bar{\alpha} - 3\beta) + \beta(\bar{\alpha} + \beta) + \bar{\lambda}(\rho - \epsilon) - \Delta\sigma + \delta\beta - \delta\tau] \\
& + \Psi_4[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + 5\beta - 3\tau) + \delta\sigma] \\
& - \Delta\mathrm{D}\bar{\Psi}_3 + \Delta\delta\bar{\Psi}_2 + \bar{\delta}\mathrm{D}\bar{\Psi}_4 - \bar{\delta}\delta\bar{\Psi}_3 \\
& - 2\bar{\lambda}\Delta\bar{\Psi}_1 - 2\bar{\nu}\delta\bar{\Psi}_1 + 2\bar{\nu}\mathrm{D}\bar{\Psi}_2 + (3\bar{\pi} + \tau)\Delta\bar{\Psi}_2 + (\bar{\gamma} - \gamma + 3\bar{\mu})\delta\bar{\Psi}_2 + 3\bar{\lambda}\bar{\delta}\bar{\Psi}_2 \\
& + (\gamma - \bar{\gamma} - 3\bar{\mu})\mathrm{D}\bar{\Psi}_3 + (2\bar{\rho} - \rho - 2\bar{\epsilon})\Delta\bar{\Psi}_3 + (\alpha - 3\bar{\beta} + \bar{\tau})\delta\bar{\Psi}_3 - (2\bar{\alpha} + 4\bar{\pi} + \tau)\bar{\delta}\bar{\Psi}_3 \\
& + (3\bar{\beta} - \alpha - \bar{\tau})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\Delta\bar{\Psi}_4 + (4\bar{\epsilon} + \rho - \bar{\rho})\bar{\delta}\bar{\Psi}_4 \\
& + 2\bar{\Psi}_0\bar{\lambda}\bar{\nu} + 2\bar{\Psi}_1[\bar{\lambda}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \bar{\nu}(2\bar{\alpha} - 2\bar{\pi} - \tau) - \Delta\bar{\lambda}] \\
& + 3\bar{\Psi}_2[\bar{\lambda}(3\bar{\beta} - \bar{\tau} - \alpha) + \bar{\pi}(3\bar{\mu} - \gamma + \bar{\gamma}) + \bar{\nu}(\rho - 2\bar{\rho}) + \bar{\mu}\tau + \Delta\bar{\pi} + \bar{\delta}\bar{\lambda}] \\
& + 2\bar{\Psi}_3[2\bar{\kappa}\bar{\nu} + (\bar{\epsilon} - \bar{\rho})(\gamma - \bar{\gamma} - 3\bar{\mu}) - \rho(\bar{\gamma} + 2\bar{\mu}) + \tau(\bar{\tau} - \bar{\beta}) + (\bar{\alpha} + 2\bar{\pi})(\alpha - 3\bar{\beta} + \bar{\tau}) \\
& \qquad - \Delta\bar{\epsilon} + \Delta\bar{\rho} - \bar{\delta}\bar{\alpha} - 2\bar{\delta}\bar{\pi}] \\
& + \bar{\Psi}_4[\bar{\kappa}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \rho(4\bar{\beta} - \bar{\tau}) + \bar{\rho}(\alpha - 3\bar{\beta} + \bar{\tau}) + 4\bar{\epsilon}(3\bar{\beta} - \bar{\tau} - \alpha) - \bar{\sigma}\tau \\
& \qquad - \Delta\bar{\kappa} + 4\bar{\delta}\bar{\epsilon} - \bar{\delta}\bar{\rho}] \,,
\end{aligned} \tag{4.23}
$$

• the $\boldsymbol{mm}$-projection of B^Z_{ab}, i.e., $B^Z_{(2)(2)} = B^Z_{ab}m^a m^b$:

$$
\begin{aligned}
B^Z_{(2)(2)} =& \Delta\Delta\Psi_0 - \Delta\delta\Psi_1 - \delta\Delta\Psi_1 + \delta\delta\Psi_2 \\
&+ (2\mu - 7\gamma + \bar{\gamma})\Delta\Psi_0 + \nu\delta\Psi_0 - \bar{\nu}\bar{\delta}\Psi_0 \\
&+ \bar{\nu}\mathrm{D}\Psi_1 + (7\tau - \bar{\alpha} + 3\beta)\Delta\Psi_1 + (5\gamma - \bar{\gamma} - 3\mu)\delta\Psi_1 + \bar{\lambda}\bar{\delta}\Psi_1 \\
&- \bar{\lambda}\mathrm{D}\Psi_2 - 5\sigma\Delta\Psi_2 + (\bar{\alpha} - \beta - 6\tau)\delta\Psi_2 + 4\sigma\delta\Psi_3 \\
&+ \Psi_0[\mu(\mu - 7\gamma + \bar{\gamma}) + \nu(\bar{\alpha} - \beta - 3\tau) + \bar{\nu}(4\alpha - \pi) + 4\gamma(3\gamma - \bar{\gamma}) - \lambda\bar{\lambda} \\
&\qquad - 4\Delta\gamma + \Delta\mu + \delta\nu] \\
&+ 2\Psi_1[2\nu\sigma - \bar{\nu}(\epsilon + 2\rho) + \bar{\lambda}(\pi - \alpha) + (\bar{\gamma} - 2\gamma)(\beta + 2\tau) + (\mu - \gamma)(5\tau - \bar{\alpha} + 2\beta) \\
&\qquad + \Delta\beta + 2\Delta\tau + \delta\gamma - \delta\mu] \\
&+ 3\Psi_2[\kappa\bar{\nu} + \bar{\lambda}\rho + \sigma(3\gamma - \bar{\gamma} - 3\mu) + \tau(3\tau - \bar{\alpha} + \beta) - \Delta\sigma - \delta\tau] \\
&+ 2\Psi_3[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + \beta - 5\tau) + \delta\sigma] + 2\Psi_4\sigma^2 \\
&+ \mathrm{D}\mathrm{D}\bar{\Psi}_4 - \mathrm{D}\delta\bar{\Psi}_3 - \delta\mathrm{D}\bar{\Psi}_3 + \delta\delta\bar{\Psi}_2 \\
&- 4\bar{\lambda}\delta\bar{\Psi}_1 + 5\bar{\lambda}\mathrm{D}\bar{\Psi}_2 + \sigma\Delta\bar{\Psi}_2 + (\bar{\alpha} - \beta + 6\bar{\pi})\delta\bar{\Psi}_2 \\
&+ (\beta - 3\bar{\alpha} - 7\bar{\pi})\mathrm{D}\bar{\Psi}_3 - \kappa\Delta\bar{\Psi}_3 + (\epsilon - 5\bar{\epsilon} + 3\bar{\rho})\delta\bar{\Psi}_3 - \sigma\bar{\delta}\bar{\Psi}_3 \\
&+ (7\bar{\epsilon} - \epsilon - 2\bar{\rho})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\delta\bar{\Psi}_4 + \kappa\bar{\delta}\bar{\Psi}_4 \\
&+ 2\bar{\Psi}_0\bar{\lambda}^2 + 2\bar{\Psi}_1[\bar{\lambda}(\bar{\alpha} + \beta - 5\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] \\
&+ 3\bar{\Psi}_2[\kappa\bar{\nu} + \bar{\lambda}(3\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \bar{\mu}\sigma + \bar{\pi}(\bar{\alpha} - \beta + 3\bar{\pi}) + \mathrm{D}\bar{\lambda} + \delta\bar{\pi}] \\
&+ 2\bar{\Psi}_3[2\bar{\kappa}\bar{\lambda} - \kappa(2\bar{\mu} + \bar{\gamma}) + \sigma(\bar{\tau} - \bar{\beta}) + (\bar{\rho} - \bar{\epsilon})(2\bar{\alpha} - \beta + 5\bar{\pi}) + (\epsilon - 2\bar{\epsilon})(2\bar{\pi} + \bar{\alpha}) \\
&\qquad - \mathrm{D}\bar{\alpha} - 2\mathrm{D}\bar{\pi} - \delta\bar{\epsilon} + \delta\bar{\rho}] \\
&+ \bar{\Psi}_4[\kappa(4\bar{\beta} - \bar{\tau}) + \bar{\kappa}(\beta - \bar{\alpha} - 3\bar{\pi}) + (\bar{\rho} - 4\bar{\epsilon})(\epsilon - 3\bar{\epsilon} + \bar{\rho}) - \sigma\bar{\sigma} \\
&\qquad + 4\mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}] \,.
\end{aligned} \tag{4.24}
$$

The shorthand *c.c.* in the above expressions denotes the complex conjugation. Finally, the complete Bach tensor projections can be constructed as

$$B_{(0)(0)} = B^Z_{(0)(0)} + \Phi_{20}\Psi_0 + \Phi_{02}\bar{\Psi}_0 - 2\Phi_{10}\Psi_1 - 2\Phi_{01}\bar{\Psi}_1 + \Phi_{00}(\Psi_2 + \bar{\Psi}_2)\,, \tag{4.25}$$

$$B_{(0)(1)} = B^Z_{(0)(1)} + \Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3\,, \tag{4.26}$$

$$B_{(0)(2)} = B^Z_{(0)(2)} + \Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{01}(\Psi_2 - 2\bar{\Psi}_2) + \Phi_{02}\bar{\Psi}_1 + \Phi_{00}\bar{\Psi}_3\,, \tag{4.27}$$

$$B_{(1)(2)} = B^Z_{(1)(2)} + \Phi_{22}\Psi_1 + \Phi_{12}(-2\Psi_2 + \bar{\Psi}_2) + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{10}\bar{\Psi}_4\,, \tag{4.28}$$

$$B_{(2)(2)} = B^Z_{(2)(2)} + \Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4\,, \tag{4.29}$$

$$B_{(1)(1)} = B^Z_{(1)(1)} + \Phi_{22}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{12}\Psi_3 - 2\Phi_{21}\bar{\Psi}_3 + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4\,. \tag{4.30}$$

It may seem that some components are missing above. However, since $\boldsymbol{m}$ is a complex vector, there is, e.g., $\bar{B}_{(0)(2)} = B_{(0)(3)}$, and since the Bach tensor is traceless, it holds $B_{(0)(1)} = B_{(2)(3)}$ and actually, also $B^Z_{(0)(1)} = B^Z_{(2)(3)}$.

This formulation of QG constraints, accompanied by the geometric identities, is especially useful to prove general propositions like, e.g., a vacuum solution to QG (2.19) with the Ricci tensor of the form $R_{ab} = \Lambda g_{ab} + s\, k_a k_b$, where $s \neq 0$ and $k^a k_a = 0$, and aligned Weyl tensor of any Petrov type is necessarily the Kundt spacetime. In A.1.3 this result was proved in a complicated way in comparison with the simple substitution of initial assumptions into the above expressions.

4.2 Geodesic deviation

The geodesic deviation is a typical textbook example that demonstrates how the gravitational field inhomogeneities, encoded in the Riemann curvature tensor $R^a{}_{bcd}$, tidally deform congruences of freely falling observers. Technically, this is described by the equation of geodesic deviation which serves as an important source of information about the gravitational field, namely

$$\frac{\mathrm{D}^2 Z^a}{\mathrm{d}\tau^2} = R^a{}_{bcd}\, u^b u^c Z^d\,, \tag{4.31}$$

with u^b representing components of the time-like reference observer velocity, which moves along a geodesic $\gamma(\tau)$ with τ being its proper time. In this linear approximation Z^a stand for components of the vector connecting the reference observer with another nearby one moving along geodesic $\bar{\gamma}(\tau)$. This illustrates figure 4.1.

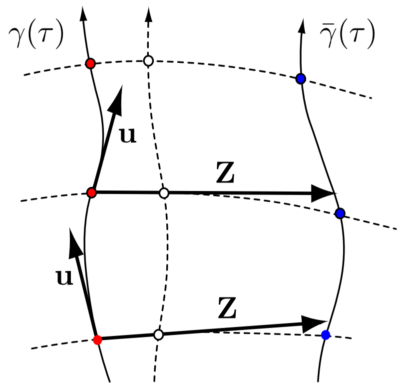

Figure 4.1: Schematic description of the kinematic quantities entering the geodesic deviation. Evolution of the connecting vector Z^a along the geodesic $\gamma(\tau)$ is given by the spacetime curvature.

In D-dimensional spacetime, all the above indices range from 0 to $D-1$ and particular results are coordinate-dependent which makes the extraction of any information more difficult. To do so, it is very natural to calculate corresponding frame projections where the frame is associated with the fiducial reference observer. Subsequently, specific components of the curvature effects can be identified and connected with the matter distribution via particular theory of gravity and its field equations, see, e.g., works in classic GR [89–92], the $D>4$ extension in section A.1.1 and its application in the case of Kundt spacetimes [93], or specific examples in sections A.2 and A.4. It is useful to briefly describe this approach on the generic geometric level without any assumption on the particular gravity theory.

Consider the orthonormal frame $\{\boldsymbol{e}_{(0)}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(i)}\}$, i.e.,

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab}\,. \tag{4.32}$$

It is natural to associate this frame with the reference test observer in such a way that the observer D-velocity corresponds to the frame time-like vector, i.e., $\boldsymbol{e}_{(0)} \equiv \boldsymbol{u} = \dot{x}^a \partial_a$. Projecting the equation (4.31) onto such a frame immediately gives

$$\ddot{Z}^{(a)} = R^{(a)}{}_{(0)(0)(b)}\, Z^{(b)}\,, \tag{4.33}$$

where $\ddot{Z}^{(a)} \equiv e^{(a)}_b \, \frac{\mathrm{D}^2 Z^b}{\mathrm{d}\,\tau^2}$ and $Z^{(b)} \equiv e^{(b)}_a Z^a$ with $a, b = 0, 1, \ldots, D-1$. The Riemann tensor antisymmetry implies $\ddot{Z}^{(0)} = 0$ and, without loss of generality, $Z^{(0)} = 0$ can be fixed which corresponds to the location of test observers on the same space-like surface synchronized by the proper time τ. Moreover, standard decomposition of the Riemann tensor [51] is used to separate its traceless Weyl part. Then the projection of the equation of geodesic deviation (4.33) takes the form

$$\ddot{Z}^{(\mathrm{i})} = \left[C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \frac{1}{D-2} \left(R_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}} \, R_{(0)(0)} \right) - \frac{R \, \delta_{\mathrm{ij}}}{(D-1)(D-2)} \right] Z^{(\mathrm{j})} , \tag{4.34}$$

with $\mathrm{i}, \mathrm{j} = 1, 2, \ldots, D-1$. Finally, to connect the above expression with the Newman–Penrose-like quantities introduced in section 3.1, and to analyze their particular contributions to the overall deformation of a test geodesic congruence, the orthonormal frame vectors are combined into the *null interpretation frame*[3] as

$$\boldsymbol{k}^{\mathrm{int}} = \frac{1}{\sqrt{2}} \left(\boldsymbol{u} + \boldsymbol{e}_{(1)} \right) , \qquad \boldsymbol{l}^{\mathrm{int}} = \frac{1}{\sqrt{2}} \left(\boldsymbol{u} - \boldsymbol{e}_{(1)} \right) , \qquad \boldsymbol{m}_i^{\mathrm{int}} = \boldsymbol{e}_{(i)} , \tag{4.35}$$

see figure 4.2 as an illustration.

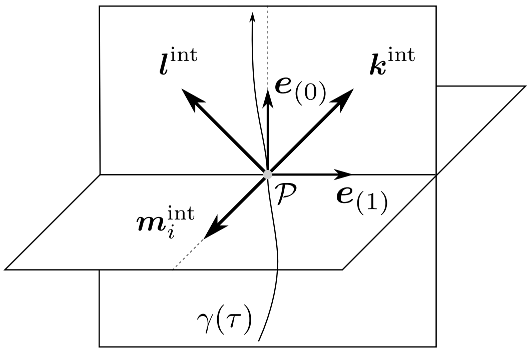

Figure 4.2: The orthonormal frame vectors associated with the fiducial test observer moving along $\gamma(\tau)$ are combined into the null interpretation frame $\{\boldsymbol{k}^{\mathrm{int}}, \boldsymbol{l}^{\mathrm{int}}, \boldsymbol{m}_i^{\mathrm{int}}\}$. This prefers the time-like direction $\boldsymbol{e}_{(0)}$ and one of the space-like directions, w.l.o.g., $\boldsymbol{e}_{(1)}$ referred as a longitudinal direction. Remaining $D-2$ vectors $\boldsymbol{e}_{(i)} \equiv \boldsymbol{m}_i^{\mathrm{int}}$ cover the transverse space.

Subsequently, the Weyl tensor projections $C_{(\mathrm{i})(0)(0)(\mathrm{j})}$ can be decomposed in terms of the null interpretation frame and scalars defined by (3.9), namely

$$\begin{aligned} C_{(1)(0)(0)(1)} &= \Psi^{\mathrm{int}}_{2S} , \\ C_{(1)(0)(0)(j)} &= \frac{1}{\sqrt{2}} (\, \Psi^{\mathrm{int}}_{1T^j} - \Psi^{\mathrm{int}}_{3T^j}) , \\ C_{(i)(0)(0)(1)} &= \frac{1}{\sqrt{2}} (\, \Psi^{\mathrm{int}}_{1T^i} - \Psi^{\mathrm{int}}_{3T^i}) , \\ C_{(i)(0)(0)(j)} &= -\frac{1}{2} (\, \Psi^{\mathrm{int}}_{0^{ij}} + \Psi^{\mathrm{int}}_{4^{ij}}) - \Psi^{\mathrm{int}}_{2T^{(ij)}} , \end{aligned} \tag{4.36}$$

[3] At this place the natural null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$, adapted to the algebraic structure of the Weyl tensor and thus simplifying the NP equations, and the null interpretation frame $\{\boldsymbol{k}^{\mathrm{int}}, \boldsymbol{l}^{\mathrm{int}}, \boldsymbol{m}_i^{\mathrm{int}}\}$, associated with a particular geodesic observer, are distinguished. Their mutual relation is given by the Lorentz transformations (3.5)–(3.8), see the example at the end of this section.

and the relevant Ricci tensor components $R_{\rm (i)(j)}$ and $R_{(0)(0)}$ can be expressed using the definitions (3.11) as

$$\begin{aligned} R_{(0)(0)} &= \Phi^{\rm int}_{00} + \Phi^{\rm int}_{22} + \Phi^{\rm int}_{11} - \frac{R}{D}\,, \\ R_{(1)(1)} &= \Phi^{\rm int}_{00} + \Phi^{\rm int}_{22} - \Phi^{\rm int}_{11} + \frac{R}{D}\,, \\ R_{(1)(j)} &= \Phi^{\rm int}_{01^j} - \Phi^{\rm int}_{12^j}\,, \\ R_{(i)(j)} &= \Phi^{\rm int}_{02^{ij}} + \frac{R}{D}\,\delta_{ij}\,, \end{aligned} \tag{4.37}$$

where the i, j indices cover the $D-2$-dimensional transverse space orthogonal to the privileged longitudinal direction $\boldsymbol{e}_{(1)}$, i.e., $i,j = 2,\ldots,D-1$. The influence of particular terms on the overall deformation is affected by their inherent symmetries, see A.1.1. The typical example corresponds to the transverse traceless effect of $\Psi_{4^{ij}}$ related to the gravitational waves in GR.

Until now the discussion of geodesic deviation was on a completely general level. However, in the context of this thesis, it is important to introduce the interpretation frame in the particular case of non-twisting geometries (3.22) parametrized by coordinate set $(r,\, u,\, x^p)$. The observer velocity thus becomes $\boldsymbol{e}_{(0)} \equiv \boldsymbol{u} = \dot{r}\partial_r + \dot{u}\partial_a + \dot{x}^p\partial_p$ and a combination of the normalization (4.32) with the definition (4.35) then leads to the frame

$$\begin{aligned} \boldsymbol{k}^{\rm int} &= \frac{1}{\sqrt{2}\,\dot{u}}\,\partial_r\,, \\ \boldsymbol{l}^{\rm int} &= \Big(\sqrt{2}\,\dot{r} - \frac{1}{\sqrt{2}\,\dot{u}}\Big)\partial_r + \sqrt{2}\,\dot{u}\,\partial_u + \sqrt{2}\,\dot{x}^p\partial_p\,, \\ \boldsymbol{m}^{\rm int}_i &= \frac{1}{\dot{u}}\,m^p_i\,(g_{up}\dot{u} + g_{pq}\,\dot{x}^q)\,\partial_r + m^p_i\,\partial_p\,. \end{aligned} \tag{4.38}$$

The above frame, adapted onto the motion of the arbitrary timelike observer, can be obtained from the simple natural frame (3.27) by a boost (3.7) with $B = \frac{1}{\sqrt{2}\,\dot{u}}$ followed by the null rotation with $\boldsymbol{k}$ fixed (3.5) taking parameters $L_i = g_{pq}\,m^p_i\,\dot{x}^q$. In the opposite way, the natural frame (3.27) corresponds to the transversely static observer, i.e., $\dot{x}^p = 0$, with $\sqrt{2}\,\dot{u} = 1$ and $\dot{r}$ given by the normalization of $\boldsymbol{u}$.

This general concept was employed within the discussion of possibly observable effects distinguishing Schwarzschild and Schwarzschild–Bach black holes in QG, see section A.2, or within the analysis of the Kundt solutions to the Einstein–Gauss–Bonnet theory in A.4.

CHAPTER

5

CONCLUSIONS AND OUTLOOK

The present thesis can be classified as theoretical research in the field of gravitational theories and their exact solutions. From the scientific point of view, my main interests have been related to the geometrically characterized exact solutions and methods of their studies in the case of Einstein's general relativity and its higher-dimensional and/or quadratic extensions including, e.g., the Einstein–Weyl gravity or the Gauss-Bonnet theory. Generic tools suitable for particular solution analysis as well as its physical interpretation have been derived, namely the generalized Newman–Penrose formalism in the case of four-dimensional quadratic gravity, see chapter 4, the invariant form of the geodesic deviation in higher dimensional GR [41] (A.1.1), or description of the non-twisting and shear-free geometries in terms of their algebraic structure [42] (A.1.2). Discussion of generic solution structure to the quadratic gravity was presented in [43] (A.1.3). On the level of specific geometries, we have been mainly interested in the static spherically symmetric solutions to the quadratic gravity where we have presented a comprehensive discussion of admitted solution types [44–47] (A.2) since the space of spherical geometries is much richer than in classic GR. The entropy behavior related to the Vaydia-like scenarios within a specific subclass of such theories was discussed in [48] (A.3). Finally, a complete non-twisting, shear-free, and non-expanding Kundt class of geometries within the Einstein–Gauss–Bonnet theory was analyzed in [49] (A.4). The performed research has been motivated by the questions about the uniqueness of Einstein's general relativity in comparison with its extensions, looking for the answer on the level of exact solutions. In the above-mentioned works, various situations and qualitative differences related to particular theories were identified and analyzed.

Let me finish the thesis with a brief outlook for possible further research topics naturally extending the previous studies. Their expected results should deepen our knowledge about various theories of gravity in terms of admitted exact spacetimes and their implications which most likely emphasize the GR uniqueness. This should be understood more as a medium-term prospect than a detailed fixed plan.

One of the goals would be to better understand the non-twisting geometries irrespectively of any particular gravity theory. This extends studies of the Robinson–Trautman and Kundt families to allow, in addition, a non-vanishing shear. Although these geometries have been studied in various contexts for a very long time, there is still a lack of their more suitable parametrization and compact explicit presentation of all geometric quantities. This would enable to discuss the algebraic structure of the Weyl tensor with respect to the privileged non-twisting null frame which, in general, does not coincide with the spacetime principal null directions (PNDs). Subsequently, specifying an appropriate Lorentz transformation to the PNDs frame the conditions on particular algebraic types can be determined in full generality. The suitability of parametrization is closely related to the metric form and values of the shear and expansion. In the case of non-twisting and shear-free geometries, we have already shown that the specific value of expansion can be encoded in an artificial conformal transformation of the seed non-expanding line element. In general, this generates the Robinson–Trautman geometry from a simpler non-expanding Kundt one, simultaneously preserving the algebraic Weyl type and structure of the null directions. This is extremely important in scenarios with quantities nicely behaving under the conformal rescaling such as the Bach tensor which appears in the quadratic gravity field equations. The question arises if there is an analogous procedure accessible also in the case of non-trivial shear. More precisely, whether the shear could be generated by some analogous formal approach from a simpler seed non-shearing geometry. Such a procedure has to be surely more involved than a simple conformal transformation. This could also clarify the foundation of the Newman–Janis algorithm on a purely geometric level.

The second large group of problems is related to the geodesic motion and, in particular, geodesic deviation. This should support physical interpretation and discussion of, in principle, observable effects within investigated exact spacetimes. It would be useful to extend the description of section 4.2 and derive the invariant frame form of the geodesic deviation equation, where the curvature decomposition appears in terms of the Newman–Penrose quantities, for the generic non-twisting geometries without any preceding assumptions on the theory. This can serve in both directions. Naturally, it can be used as a tool for analysis of a known solution to the specific theory, where the field equations imply constraints between particular curvature contributions. In such a case the test particle relative motion is fixed. However, we can proceed vice versa. That is not to analyze existing solutions, but to primarily prescribe specific behavior of test particles which should impose additional restrictions on the initial metric functions, e.g., black-hole tidal deformations or purely transverse traceless deformations, classically interpreted as gravitational waves far from the source. The geometry has to be further constrained by the field equations. However, these supplementary conditions may simplify their integration. For example, using such an approach in the context of quadratic gravity and assuming that the relative deformations of the test body are compatible with the known GR spacetime should allow to explore a specific part of the quadratic gravity solution space mimicking the implications of Einstein's theory. Simultaneously, such solutions may provide extra free parameters arising from the QG higher-order nature. Surprisingly, in the case of spherical symmetry with constant scalar curvature, this type of behavior is not possible.

Finally, with the Newman–Penrose formulation of generic four-dimensional QG in hand, more ambitious and even more realistic situations could be discussed. In particular, the entire class of Robinson–Trautman solutions possibly including non-trivial matter fields, e.g., an extension of the pure radiation Vaidya-like spacetimes is a very natural candidate. However, the ultimate goal within the quadratic gravity is to study the rotating black holes generalizing the Kerr solution. Besides these particular problems, the Newman–Penrose like analysis should be extended even for the quadratic gravity within the higher dimensional scenarios where the Gauss–Bonnet term enters the game and the Bach tensor is no more consistently defined.

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APPENDIX

A

ORIGINAL PAPERS

This chapter contains nine of my co-authored research works, which represent studies in the field of exact spacetimes in various extensions of classic four-dimensional Einstein's general relativity.

Section A.1 begins with results on the geodesic deviation in higher-dimensional general relativity [41] which can be simply formulated also without any assumption on a particular gravity theory, see section 4.2, and then straightforwardly applied, e.g., in the case of quadratic gravity or the Einstein–Gauss–Bonnet theory. In the subsequent work [42], the Weyl tensor algebraic properties of non-twisting shear-free geometries are analyzed and all curvature quantities are calculated in the unified form. The last part of this section investigates solutions to the quadratic gravity on a general level [43] and, as its byproduct, derives the conformal relation between the Kundt and Robinson–Trautman geometries. The unpublished results on the Newman–Penrose formalism in quadratic gravity, see section 4.1, would also naturally belong to this section.

The second part A.2, formed by four papers [44–47], studies spherically symmetric vacuum solutions to quadratic gravity with a constant scalar curvature. The previous general results are extensively used to formulate the field equations and to analyze the solution properties.

Finally, the last sections A.3 and A.4 correspond to the papers [48, 49] and investigate specific model situations within the critical gravity and the Einstein–Gauss–Bonnet theory, respectively.

A.1 Mathematical methods and general concepts

A.1.1 Interpreting spacetimes of any dimension using geodesic deviation

Authors: Jiří Podolský and Robert Švarc

Journal reference: Physical Review D **85** 044057 (2012)

Abstract: We present a general method that can be used for geometrical and physical interpretation of an arbitrary spacetime in four or any higher number of dimensions. It is based on the systematic analysis of relative motion of free test particles. We demonstrate that the local effect of the gravitational field on particles, as described by the equation of geodesic deviation with respect to a natural orthonormal frame, can always be decomposed into a canonical set of transverse, longitudinal and Newton–Coulomb-type components, isotropic influence of a cosmological constant, and contributions arising from specific matter content of the Universe. In particular, exact gravitational waves in Einstein's theory always exhibit themselves via purely transverse effects with $D(D-3)/2$ independent polarization states. To illustrate the utility of this approach, we study the family of pp-wave spacetimes in higher dimensions and discuss specific measurable effects on a detector located in four spacetime dimensions. For example, the corresponding deformations caused by generic higher-dimensional gravitational waves observed in such physical subspace need not be trace-free.

DOI: 10.1103/PhysRevD.85.044057

arXiv: arXiv:1201.4790

Interesting to know: In opposite way to the abstract last sentence, the hypothetical discovery of the non-traceless, however still gravitation-wave-like, deformation of the detector could be interpreted in terms of the higher-dimensional GR where the trace-free wave generically deforms the whole D-dimensional space whereas we observe its non-traceless effect as a result of restriction on its four-dimensional subspace.

A.1.2 Algebraic structure of Robinson–Trautman and Kundt geometries in arbitrary dimension

Authors: Jiří Podolský and Robert Švarc

Journal reference: Classical and Quantum Gravity **32** 015001 (2015)

Abstract: We investigate the Weyl tensor algebraic structure of a fully general family of D-dimensional geometries that admit a non-twisting and shear-free null vector field $\mathbf{k}$. From the coordinate components of the curvature tensor we explicitly derive all Weyl scalars of various boost weights. This enables us to give a complete algebraic classification of the metrics in the case when the optically privileged null direction $\mathbf{k}$ is a (multiple) Weyl aligned null direction (WAND). No field equations are applied, so the results are valid not only in Einstein's gravity, including its extension to higher dimensions, but also in any metric gravitation theory that admits non-twisting and shear-free spacetimes.
We prove that all such geometries are of type I(b), or more special, and we derive surprisingly simple necessary and sufficient conditions under which $\mathbf{k}$ is a double, triple or quadruple WAND. All possible algebraically special types, including the refinement to subtypes, are thus identified, namely II(a), II(b), II(c), II(d), III(a), III(b), N, O, $\mathrm{II_i}$, $\mathrm{III_i}$, D, D(a), D(b), D(c), D(d), and their combinations. Some conditions are identically satisfied in four dimensions.
We discuss both important subclasses, namely the Kundt family of geometries with the vanishing expansion ($\Theta = 0$) and the Robinson–Trautman family ($\Theta \neq 0$, and in particular $\Theta = 1/r$). Finally, we apply Einstein's field equations and obtain a classification of all Robinson–Trautman vacuum spacetimes. This reveals fundamental algebraic differences in the $D > 4$ and $D = 4$ cases, namely that in higher dimensions there only exist such spacetimes of types $\mathrm{D(a)} \equiv \mathrm{D(abd)}$, $\mathrm{D(c)} \equiv \mathrm{D(bcd)}$ and O.

DOI: 10.1088/0264-9381/32/1/015001

arXiv: arXiv:1406.3232

Interesting to know: In the case of non-twisting and shear-free geometries, it is useful to employ the adapted coordinates. These allow to express derivatives along the privileged null vector field in terms of its expansion. Moreover, it seems that its specific values typically identify the subclass of Einstein's spacetimes within various extended theories, see e.g., the subsequent section.

A.1.3 Exact solutions to quadratic gravity

Authors: Vojtěch Pravda, Alena Pravdová, Jiří Podolský, and Robert Švarc

Journal reference: Physical Review D **95** 084025 (2017)

Abstract: Since all Einstein spacetimes are vacuum solutions to quadratic gravity in four dimensions, in this paper we study various aspects of non-Einstein vacuum solutions to this theory. Most such known solutions are of traceless Ricci and Petrov type N with a constant Ricci scalar. Thus we assume the Ricci scalar to be constant which leads to a substantial simplification of the field equations. We prove that a vacuum solution to quadratic gravity with traceless Ricci tensor of type N and aligned Weyl tensor of any Petrov type is necessarily a Kundt spacetime. This will considerably simplify the search for new non-Einstein solutions. Similarly, a vacuum solution to quadratic gravity with traceless Ricci type III and aligned Weyl tensor of Petrov type II or more special is again necessarily a Kundt spacetime. Then we study the general role of conformal transformations in constructing vacuum solutions to quadratic gravity. We find that such solutions can be obtained by solving one nonlinear partial differential equation for a conformal factor on any Einstein spacetime or, more generally, on any background with vanishing Bach tensor. In particular, we show that all geometries conformal to Kundt are either Kundt or Robinson–Trautman, and we provide some explicit Kundt and Robinson–Trautman solutions to quadratic gravity by solving the above mentioned equation on certain Kundt backgrounds.

DOI: 10.1103/PhysRevD.95.084025

arXiv: arXiv:1606.02646

Interesting to know: The conformal relation between Kundt and Robinson–Trautman classes of geometries discussed in this paper has become even more important in the subsequent series of works analyzing the spherically symmetric solutions to quadratic gravity. In general, the complexity of the Bach tensor, and thus the QG field equations, within the Robinson–Trautman setting can be simply reduced in terms of the unphysical Kundt seed metric and its suitable conformal rescaling.

A.2 Spherically symmetric solutions to quadratic gravity

A.2.1 Explicit black hole solutions in higher-derivative gravity

Authors: Jiří Podolský, Robert Švarc, Vojtěch Pravda, and Alena Pravdová

Journal reference: Physical Review D **98** 021502(R) (2018)

Abstract: We present, in an explicit form, the metric for all spherically symmetric Schwarzschild-Bach black holes in Einstein-Weyl theory. In addition to the black hole mass, this complete family of spacetimes involves a parameter that encodes the value of the Bach tensor on the horizon. When this additional "non-Schwarzschild parameter" is set to zero, the Bach tensor vanishes everywhere, and the "Schwa-Bach" solution reduces to the standard Schwarzschild metric of general relativity. Compared with previous studies, which were mainly based on numerical integration of a complicated form of field equations, the new form of the metric enables us to easily investigate geometrical and physical properties of these black holes, such as specific tidal effects on test particles, caused by the presence of the Bach tensor, as well as fundamental thermodynamical quantities.

DOI: 10.1103/PhysRevD.98.021502

arXiv: arXiv:1806.08209

Interesting to know: In general, it is not surprising that a suitable parametrization simplifies the problem formulation and its solution. However, the way how to find such a parametrization is not clear at all. Here, the conformal-to-Kundt metric form of the spherical geometry allows writing the quadratic gravity field equations in just one line, while the standard Schwarzschild-like line element leads to the one-page expressions.

A.2.2 Black holes and other exact spherical solutions in quadratic gravity

Authors: Jiří Podolský, Robert Švarc, Vojtěch Pravda, and Alena Pravdová

Journal reference: Physical Review D **101** 024027 (2020)

Abstract: We study static, spherically symmetric vacuum solutions to quadratic gravity, extending considerably our previous rapid communication [Phys. Rev. D **98**, 021502(R) (2018)] on this topic. Using a conformal-to-Kundt metric ansatz, we arrive at a much simpler form of the field equations in comparison with their expression in the standard spherically symmetric coordinates. We present details of the derivation of this compact form of two ordinary differential field equations for two metric functions. Next, we apply analytical methods and express their solutions as infinite power series expansions. We systematically derive all possible cases admitted by such an ansatz, arriving at six main classes of solutions, and provide recurrent formulas for all the series coefficients. These results allow us to identify the classes containing the Schwarzschild black hole as a special case. It turns out that one class contains only the Schwarzschild black hole, three classes admit the Schwarzschild solution as a special subcase, and two classes are not compatible with the Schwarzschild solution at all since they have strictly nonzero Bach tensor. In our analysis, we naturally focus on the classes containing the Schwarzschild spacetime, in particular on a new family of the Schwarzschild-Bach black holes which possesses one additional non-Schwarzschild parameter corresponding to the value of the Bach tensor invariant on the horizon. We study its geometrical and physical properties, such as basic thermodynamical quantities and tidal effects on free test particles induced by the presence of the Bach tensor. We also compare our results with previous findings in the literature obtained using the standard spherically symmetric coordinates.

DOI: 10.1103/PhysRevD.101.024027

arXiv: arXiv:1907.00046

Interesting to know: The solution space for the spherically symmetric metric ansatz is much richer within the quadratic gravity than in classic GR, where Birkhoff's theorem identifies the Schwarzschild spacetime as the only possibility.

A.2.3 Exact black holes in quadratic gravity with any cosmological constant

Authors: Robert Švarc, Jiří Podolský, Vojtěch Pravda, and Alena Pravdová

Journal reference: Physical Review Letters **121** 231104 (2018)

Abstract: We present a new explicit class of black holes in general quadratic gravity with a cosmological constant. These spherically symmetric Schwarzschild–Bach–(anti-)de Sitter geometries, derived under the assumption of constant scalar curvature, form a three-parameter family determined by the black-hole horizon position, the value of the Bach invariant on the horizon, and the cosmological constant. Using a conformal to Kundt metric ansatz, the fourth-order field equations simplify to a compact autonomous system. Its solutions are found as power series, enabling us to directly set the Bach parameter and/or cosmological constant equal to zero. To interpret these spacetimes, we analyze the metric functions. In particular, we demonstrate that for a certain range of positive cosmological constant there are both black-hole and cosmological horizons, with a static region between them. The tidal effects on free test particles and basic thermodynamic quantities are also determined.

DOI: 10.1103/PhysRevLett.121.231104

arXiv: arXiv:1806.09516

Interesting to know: To include the cosmological constant into the field equations is straightforward even within the conformal-to-Kundt metric form. However, it non-trivially affects the spacetime geometry and causes various qualitative changes, e.g., within the thermodynamic quantities.

A.2.4 Black holes and other spherical solutions in quadratic gravity with a cosmological constant

Authors: Vojtěch Pravda, Alena Pravdová, Jiří Podolský, and Robert Švarc

Journal reference: Physical Review D **103** 064049 (2021)

Abstract: We study static spherically symmetric solutions to the vacuum field equations of quadratic gravity in the presence of a cosmological constant Λ. Motivated by the trace no-hair theorem, we assume the Ricci scalar to be constant throughout a spacetime. Furthermore, we employ the conformal-to-Kundt metric ansatz that is valid for all static spherically symmetric spacetimes and leads to a considerable simplification of the field equations. We arrive at a set of two ordinary differential equations and study its solutions using the Frobenius-like approach of (infinite) power series expansions. While the indicial equations considerably restrict the set of possible leading powers, careful analysis of higher-order terms is necessary to establish the existence of the corresponding classes of solutions. We thus obtain various non-Einstein generalizations of the Schwarzschild, (anti-)de Sitter [or (A)dS for short], Nariai, and Plebański-Hacyan spacetimes. Interestingly, some classes of solutions allow for an arbitrary value of Λ, while other classes admit only discrete values of Λ. For most of these classes, we give recurrent formulas for all series coefficients. We determine which classes contain the Schwarzschild-(A)dS black hole as a special case and briefly discuss the physical interpretation of the spacetimes. In the discussion of physical properties, we naturally focus on the generalization of the Schwarzschild-(A)dS black hole, namely the Schwarzschild-Bach-(A)dS black hole, which possesses one additional Bach parameter. We also study its basic thermodynamical properties and observable effects on test particles caused by the presence of the Bach tensor. This work is a considerable extension of our Letter [Phys. Rev. Lett. **121**, 231104 (2018)].

DOI: 10.1103/PhysRevD.103.064049

arXiv: arXiv:2012.08551

Interesting to know: This work ultimately summarizes and simultaneously provides all technical details on spherical solutions to the vacuum quadratic gravity with any cosmological constant. The landscape of admitted spacetimes is surprisingly rich, especially in comparison with classic GR where the Schwarzschild–(anti-)de Sitter spacetime is the only allowed possibility. Interestingly, in quadratic gravity, various peculiar geometries arise due to the coupling with a cosmological constant. Even the standard cases are in principle distinguishable from those of GR in terms of the geodesic deviation.

A.3 Decreasing entropy of dynamical black holes in critical gravity

Authors:	Hideki Maeda, Robert Švarc, and Jiří Podolský
Journal reference:	Journal of High Energy Physic **06** 118 (2018)
Abstract:	Critical gravity is a quadratic curvature gravity in four dimensions which is ghost-free around the AdS background. Constructing a Vaidya-type exact solution, we show that the area of a black hole defined by a future outer trapping horizon can shrink by injecting a charged null fluid with positive energy density, so that a black hole is no more a one-way membrane even under the null energy condition. In addition, the solution shows that the Wald–Kodama dynamical entropy of a black hole is negative and can decrease. These properties expose the pathological aspects of critical gravity at the non-perturbative level.
DOI:	doi.org/10.1007/JHEP06(2018)118
arXiv:	arXiv:1805.00026
Interesting to know:	This work corresponds to the case, where discussion of the explicit analytical model shows the peculiarity of the whole theory standing behind.

A.4 Kundt spacetimes in the Einstein–Gauss–Bonnet theory

Authors: Robert Švarc, Jiří Podolský, and Ondřej Hruška

Journal reference: Physical Review D **102** 084012 (2020)

Abstract: We systematically investigate the complete class of vacuum solutions in the Einstein–Gauss–Bonnet (EGB) gravity theory which belong to the Kundt family of nonexpanding, shear-free, and twist-free geometries (without gyratonic matter terms) in any dimension. The field equations are explicitly derived and simplified, and their solutions classified into three distinct subfamilies. Algebraic structures of the Weyl and Ricci curvature tensors are determined. The corresponding curvature scalars directly enter the invariant form of the equation of geodesic deviation, enabling us to understand the specific local physical properties of the gravitational field constrained by the EGB theory. We also present and analyze several interesting explicit classes of such vacuum solutions, namely, the Ricci type-III spacetimes, all geometries with constant-curvature transverse space, and the whole *pp*-wave class admitting a covariantly constant null vector field. These exact Kundt EGB gravitational waves exhibit new features which are not possible in Einstein's general relativity.

DOI: 10.1103/PhysRevD.102.084012

arXiv: arXiv:2007.06648

Interesting to know: Surprisingly, there exists a specific coupling between the transverse space geometry and the theory constants such that the crucial metric function g_{uu}, encoding for example transverse wave-like deformations, remains free. Preliminary results in the Robinson–Trautman class, restricted by the EGB theory, exhibit the same behavior. This would imply that the non-existence of the Weyl type N Robinson–Trautman solutions within higher-dimensional GR is much more the unique property of Einstein's theory than the geometric constraint implied by the Robinson–Trautman class in higher dimensions itself.