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Hyperintensional Modal Logic: Motivation, Semantic Frameworks, and Basic Theory.

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Prague, 10th of May 2023 Ivana Dragonová

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Abstract: A modal operator is hyperintensional if it does not respect the Equivalence Rule (RE), according to which if two formulas are logically equivalent, then so are the results of applying the modal operator to them. Typically, this happens when dealing with topics finer-grained than propositions, such as the notions of knowledge and belief. This thesis discusses the class of modal logics not closed under (RE) called hyperintensional modal logics and gives an overview of the semantic approaches one can use to give a suitable interpretation for this class of logics. We discuss a state-based approach first introduced by Rantala(1982) and later developed by Wansing(1990) and a structuralist approach proposed by Cresswell(1975). In the final part, we discuss a recent approach by Sedlár (2021), Pascucci and Sedlár (2023), and show that the above-mentioned state-based and structuralist approaches can both be modeled within Sedlár's hyperintensional models. We prove completeness results for the discussed hyperintensional semantic frameworks - all of them are sound and complete with respect to the smallest (hyperintensional) modal logic.

Keywords: epistemic logic, hyperintensions, logical semantics, modal logic

Contents

Introduction

The hyperintensional paradox, as Cresswell (Cresswell [1975]) coined the term, is the problem where there are two equivalent propositions, yet a person may believe one and not the other. Cresswell chooses the word 'paradox', since in the classical intensional semantics, which understands propositions as sets of possible worlds, it is not possible to have a modal operator that does not express a function on the set of possible worlds. But it seems that when considering human knowledge, it is possible to have a pair of logically equivalent propositions, and believe one without believing also the other.

Expressed in the language of modal logic, it may happen that $\varphi \leftrightarrow \psi$ is valid, but $\Box\varphi \leftrightarrow \Box\psi$ is not. The inference rule $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$ is known in the literature as the Equivalence Rule (RE). Classical and normal modal logics are closed under this rule, but because of the hyperintensional paradox, it seems that epistemic logic, the logic of knowledge and belief should not be.

While the origins of epistemology, the philosophical discipline, can be traced back to ancient Greece, a formal approach to epistemic logic dates back only to 1950s. The work of Von Wright [1951] is acknowledged as having initiated the formal study of epistemic logic as we know it today. His work has been extended by Jaako Hintikka in the first book on the topic, *Knowledge and Belief* (Hintikka [1962]). Other notable early contributions to the study of epistemic logic have been made by R. Carnap, J. Łoś, A. Prior, N. Rescher and others. The major interest of epistemic logic was (and still is) in capturing inherent properties of knowledge, and finding the most suitable axioms for modeling knowledge. In the 1980s and 1990s, the focus shifted to studying the properties of logical systems formalizing groups of knowers, which led to systems containing multiple modalities. Later, dynamic epistemic logic extended traditional epistemic logic by introducing the dynamic process of knowledge acquisition and belief revision.

In this work, we are concerned with the traditional branch of epistemic logic restricted to a single modal operator for simplicity. The modal operator in modal logics may have various readings, but since our pursuit is motivated by problems arising in epistemology, we will primarily read $\Box$ as 'agent knows that'. This leads us to the conclusion that the $\Box$ must have different properties than in the traditional reading.

The class of modal logics that is not closed under (RE) is called hyperintensional, and it is the complement of classical modal logics in the class of all modal logics. Our goal in the thesis is to find fitting semantics for this whole class of modal logics. We present Pascucci and Sedlár's (Pascucci and Sedlár [2023]) algebraic hyperintensional models as the solution of the problem. Algebraic hyperintensional models are models based on set-theoretic hyperintensional models (Sedlár [2021]), which deal with the problem of hyperintensionality by introducing the 'semantic content' of formulas. In these models, it is supposed that two formulas may have different semantic contents, but still express the same proposition, i.e. the set of worlds in which they hold. These models are motivated by the semantic triangle, according to which a formula (symbol) has a semantic content (reference) which in turn determines the proposition expressed by the formula (referent). We find this to be a nice and intuitive approach to semantics, which not only in a way reflects what we know about natural language but also has the potential to be a universal semantic framework for hyperintensional modal logic, as it does not specify the nature of the semantic contents of the formulas.

From the formal point of view, hyperintensional models are a generalization of neighborhood models. They are complete with respect to the smallest hyperintensional logic, and they can embed various approaches present in the literature. We show two of such translations, that is, the translation of Rantala models (Wansing [1990]), and the translation of Cresswell models, which we define as a propositional modal fragment of Cresswell semantics presented in the famous paper Cresswell [1975]. We prove completeness with respect to the smallest modal logic for both of these approaches. Each of these approaches also resolves the hyperintensional paradox in some way, though the approaches may seem completely different at the first sight. Wansing generalizes relational (Kripke) semantics by adding one more set of worlds into the models - the set of *non-normal* or *impossible* worlds. This is a simple and efficient solution. However, impossible worlds may be seen as problematic entities. As traditional relational semantics is motivated by the fact that the set of worlds, or states, is the set of possible ways things can be, or (in epistemic logic) the set of consistent bodies of information, the existence of inconsistent, non-compositional worlds clashes with this intuition. On the other hand, Cresswell resolves the hyperintensional paradox by introducing a hyperintensional operator into the framework of Lewis [1970], which allows two formulas to have the same intension (i.e. proposition), but different meanings. The hyperintensional operator then alters how the formula at hand is interpreted. This approach is called structuralist, because it supposes that the meaning of a sentence can be represented by its internal structure.

The thesis is structured as follows. The first chapter defines modal logic and some of its classes, and overviews the common approaches to semantics of modal logics. We show that the relational models are complete with respect to the smallest normal modal logic, and that the neighborhood models are complete with respect to the smallest classical modal logic. The second chapter reviews semantic approaches from literature that accommodate hyperintensional modal operator.[1] We study a state-based approach called Rantala models and a structuralist approach derived from Cresswell semantics in greater detail, and prove that they are complete with respect to the smallest (hyperintensional) modal logic. In the final chapter of the thesis we introduce hyperintensional models, show that they are also complete with respect to the smallest (hyperintensional) modal logic, and show that both Rantala models, and Cresswell models (which we derive based on the Cresswell's structuralist approach in the second chapter), are special cases of hyperintensional models.

Notably, we provide some original contributions, which were developed in collaboration with the supervisor of the thesis, I. Sedlár. The contributions are a joint effort and we believe they will be a valuable contribution to the field.

[1] In this thesis, we are concerned with a hyperintensional modal operator in the sense that it does not respect logical equivalence. There is, however, a second alternative notion of a hyperintensional operator (Berto and Nolan [2021]), according to which $\Box$ is hyperintensional in case there are two formulas φ, ψ such that φ and ψ are necessarily equivalent, but $\Box\varphi$ and $\Box\psi$ are not necessarily equivalent. To accommodate this 'metaphysical notion' of hyperintensionality, we would need to specify what is meant by 'necessarily equivalent', however, it is not very clear what is meant by this notion. We stick to the simpler 'logical notion' of hyperintensionality.

Specifically, we: 1) Formulate Cresswell models based on Cresswell [1975] and show that their class is sound and complete with respect to the basic hyperintensional modal logic (2.2.2 Cresswell Models), 2) We provide a novel embedding of Rantala models into algebraic hyperintensional models, which improves the construction in Pascucci and Sedlár [2023]. (3.2.1 Translation of Rantala Models), 3) And finally, we provide an embedding of Cresswell models from the subsection 2.2.2 into algebraic hyperintensional models. (3.2.2 Translation of Cresswell Models).

1. Modal Logic and Hyperintensionality

In this chapter, we introduce the basic theory of modal logic and some of the interpretations of the modal box operator present in current literature. Thereafter we describe the most common semantic frameworks. In the last part of the chapter, we discuss the motivation to study hyperintensional modal logic in greater detail. In the definitions and notation, we mostly follow Blackburn et al. [2001] and Chellas [1980], in the case of neighbourhood semantics Pacuit [2017].

1.1 Modal Logics

First, we define the basic modal language. The definition can be expanded into a definition of a modal language, for example into one with multiple modalities [1], but in this work, we will not need to go to such an extent. Therefore, in the rest of the work, unless stated otherwise, we are working in the basic modal language.

Definition 1 (Basic Modal Language)**.** *Let $At := \{p, q, r, ...\}$ be a countable set of propositional variables representing atomic statements. We define a well-formed formula φ inductively using the Backus-Naur Form as follows:*

$$\varphi, \psi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$$

where $p \in At$. We define the rest of the usual classical connectives, constant falsum (bottom), constant true (top), and the dual operator to $\Box$, $\Diamond$ in the usual way:

- $\varphi \rightarrow \psi := \neg(\varphi \wedge \neg\psi)$,
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$,
- $\varphi \leftrightarrow \psi := \neg(\varphi \wedge \neg\psi) \wedge \neg(\neg\varphi \wedge \psi)$,
- $\bot := \varphi \wedge \neg\varphi$,
- $\top := \neg(\varphi \wedge \neg\varphi)$,
- $\Diamond\varphi := \neg\Box\neg\varphi$.

We will refer to the basic modal language as $\mathcal{L}_0$. Moreover, we denote the set of all well-formed formulas in the basic modal language as Fm, and the set of all classical propositional tautologies as $Taut$.

[1] Multiple modalities can be useful in various contexts. For example, as in this work we are interested in epistemic modal logic, it seems desirable to have multiple modalities in order to capture knowledge and belief of multiple agents. However, in many cases, it would complicate the notation. Therefore, for the sake simplicity, we decided to work with a single modality (only one reasoning agent). There is no reason why the same could not be formulated for a modal language with multiple modalities.

The inseparable part of the basic modal language $\mathcal{L}_0$, which makes it a *modal* language, is the modal operator $\Box$ (and its dual $\Diamond$). In the traditional (alethic) reading, $\Box$ would be understood as 'necessarily' (or 'it is necessary that'), and its dual $\Diamond$ would be 'possibly' (or 'it is possible that'). Over time, however, the expressive power of modal logic has been adapted by various related systems. Modal operators have different readings in these systems, such as the one of tense 'it will (always) be the case that' or 'it has (always) been the case that' formalized in temporal logics, reasoning about obligations such as 'it is obligaory that' or 'it is forbidden that' in deontic logic, reasoning about beliefs such as 'it is believed that' in doxastic logic, reasoning about computer systems and other transition systems such as 'after a program/computation/action finishes the program enables that' in dynamic logic, and finally reasoning about knowledge such as 'it is known that' or 'it is consistent with the knower's current information that' studied in epistemic logic, which is of our main interest in this work.

Now that we have a language, we aim to define a modal logic. What is the motivation to define logical systems? We are interested to see what facts have which consequences, i.e., which arguments are correct. As we will see, in this view, we understand a modal logic as a set of formulas; a set of formulas can be seen as *representing* an argument with premises $\varphi_1, \ldots, \varphi_n$ and conclusion ψ as *valid* iff it contains the formula $\varphi_1 \wedge \ldots \wedge \varphi_n \to \psi$.

To be able to define a modal logic, however, first, we need to define what we mean by a (uniform) substitution:

Definition 2 (Uniform Substitution)**.** *Substitution σ is a function $\sigma : At \to Fm$. A substitution σ then induces a function $(\cdot)^\sigma : Fm \to Fm$ defined recursively as follows:*

- $p^\sigma = \sigma(p)$,
- $(\neg\varphi)^\sigma = \neg\varphi^\sigma$,
- $(\varphi \wedge \psi)^\sigma = \varphi^\sigma \wedge \psi^\sigma$,
- $(\Box\varphi)^\sigma = \Box\varphi^\sigma$.

We say that ψ is a substitution instance of φ if there is some substitution τ such that $\psi = \varphi^\tau$. Let Σ be the set of all substitutions $\sigma : At \to Fm$, and for a set $\Gamma \subseteq Fm$, let $\Sigma(\Gamma)$ be the set of all substitution instances of elements of Γ: $\Sigma(\Gamma) = \{\varphi^\sigma \mid \sigma \in \Sigma \text{ and } \varphi \in \Gamma\}$. In the rest of the thesis, we will use σ and $(\cdot)^\sigma$ interchangeably.

Definition 3 (Modal logic)**.** *A Modal Logic Λ is a set of well-formed formulas in the basic modal language over the set of propositional variables, such that Λ satisfies the following conditions:*

1) $Taut \subseteq \Lambda$,

2) Λ is closed under Modus Ponens: $\varphi, \varphi \to \psi / \psi$,

3) Λ is closed under uniform substitution: $\varphi / \sigma(\varphi)$ for $\sigma \in \Sigma$.

Note that the definition does not postulate any requirements on the properties of the modal box. Moreover, it is easily seen that the class of modal logics is closed under arbitrary intersections, and so the smallest modal logic exists. As the definition of modal logic does not impose any restrictions on the behavior of $\Box$, in the smallest modal logic $\Box$ will not have any specific properties.

Remark. $\Sigma(Taut)$, the set of all substitution instances of classical propositional tautologies over the modal language $\mathcal{L}_0$, is a modal logic. $\Sigma(Taut)$ is the smallest modal logic.

Having defined a general notion of modal logic, we may ask how the various logical operators should behave to become an accurate representation of what we want to model. Here we are concerned with the modal box $\Box$. Surely, in the alethic reading it should have different properties than in the epistemic or doxastic reading? After all, consider some simple formula such as $\Box\varphi \to \varphi$, and the following two different readings depending on the meaning of the modal box: '*What is necessary is so (in the world),*' vs '*What I believe is so (in the world).*' Obviously, the latter is untrue, and therefore an undesirable principle within a logic that should model belief. Another interesting example can be $\neg\Box\varphi \to \Box\neg\Box\varphi$. In the alethic reading, we would understand it as *'What is not necessary, is necessarily not necessary.'* Although this sounds as a somewhat unnatural statement, it appears to be perfectly reasonable. On the other hand, '*What I don't know, I know that I don't know,*' seems to be often untrue when considering human knowledge.

In this thesis, we are primarily concerned not with individual modal logics, but rather with classes of modal logics. We start by defining the most well known class of modal logics, the class of normal modal logics.

Definition 4 (Normal Modal Logic)**.** *A normal modal logic is* $\Lambda \subseteq Fm$ *such that* Λ *is a modal logic and:*

- Λ *contains all instances of axiom schema K:* $\Box(p \to q) \to (\Box p \to \Box q)$*,*
- Λ *is closed under the Necessitation rule (Nec):* $\varphi/\Box\varphi$*.*

The smallest logic satisfying these conditions is known as the logic **K**. Depending on the axioms satisfied by the normal modal logic, we may get various extensions of **K**, such as the logic **T** (obtained by the adding axiom schema **T**: $\Box\varphi \to \varphi$ to **K**), the logic **K4** (obtained by the adding axiom schema **4**: $\Box\varphi \to \Box\Box\varphi$ to **K**), the logic **S4** (obtained from **K** by adding the axiom schemas **T, 4**), and the logic **S5** (obtained from the logic **S4** by also adding the axiom schema **5**: $\Diamond\varphi \to \Box\Diamond\varphi$). Normal modal logics have perhaps been studied the most in the literature. Specific extensions of **K** are not our topic in the thesis, but a good overview of them is given in for example Blackburn et al. [2001] or Chellas [1980].

In any case, not logic **K**, nor any of its extensions, are considered suitable to model epistemic logic. If we want to understand $\Box$ as 'agent x knows/believes that', we obtain some undesirable consequences. For one, under *Nec*, we get that the agent knows at least all logical tautologies, which considering human reasoning capacities, seems implausible. Also as a result of schema K in combination with *the Necessitation rule*, in every normal logic it holds that from $\Box\varphi$ and $\varphi \to \psi$ follows $\Box\psi$. This is known as *full omniscience*, i.e., that the reasoning agent is aware of all consequences of their beliefs. Which also, considering the human reasoning capacities, is an unwanted feature. Therefore, normal modal logic does not seem suitable to model knowledge and belief.

Classical modal logic can be considered a generalization of normal modal logic; it imposes fewer requirements on the on the properties of the modal operator:

Definition 5 (Classical Modal Logic)**.** *A classical modal logic is* $\Lambda \subseteq Fm$ *such that* Λ *is a modal logic and:*

- Λ *is closed under equivalence rule (RE):* $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$*.*

The smallest classical modal logic is called **E**.

But, classical modal logic still preserves the equivalence rule. While this rule is obviously desirable for $\Box$ read as 'it is necessary that' (if it is necessary that φ and φ expresses the same proposition as ψ, it is clearly also necessary that ψ), for $\Box$ read as 'agent knows that' it might pose a problem. If I know φ and φ is equivalent to ψ, it does not mean that me, as a fallible human agent, I have to know ψ.

Consider the following example: There is an apple in front of me. I know the apple is red (I can see its color). And according to physics, an object being red is the same as that it reflects light of the wavelength of approximately 625–740 nanometres from its surface; so being red can be considered equivalent to reflecting light at a specific wavelength. But the fact that I see its color does not mean that I'm aware of the physics underlying my experience. Therefore, if for example nobody taught me that and I didn't think to look it up, I do not know both of these facts.

This is not a limiting example. In the imperfect reasoning capacities of humans, contexts which do not adhere to (*RE*), are not rare. Cresswell [1975] calls such contexts *hyperintensional*, because finer (than intensional) distinctions need to be drawn to differentiate between two propositions. Typically, epistemic modalities give rise to such contexts. But they are by far not the only context, in which hyperintensionality arises.

In the next theorem, we show that normal modal logics are a subset of classical modal logics, and therefore normal modal logics (in addition to the the other problems with normal modal logics in epistemic contexts we presented earlier), are also closed under (*RE*).

Theorem 1. *Every normal modal logic is a classical modal logic.*

Proof. Fix an arbitrary Λ that is a normal modal logic. We will show that then Λ is also closed under the equivalence rule $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

Suppose that then $\varphi \leftrightarrow \psi \in \Lambda$ for some $\varphi, \psi \in Fm$. That means that $\varphi \to \psi \in \Lambda$ and $\psi \to \varphi \in \Lambda$. By *Nec*, it follows that $\Box(\varphi \to \psi) \in \Lambda$, $\Box(\psi \to \varphi) \in \Lambda$. By *axiom schema K*, and *modus ponens*, it holds that $\Box\varphi \to \Box\psi \in \Lambda$, $\Box\psi \to \Box\varphi \in \Lambda$, and so $\Box\varphi \leftrightarrow \Box\psi \in \Lambda$.

$\Box$

It seems that human reasoners cannot be adequately modelled by classical modal logics due to hyperintensional contexts, which require the violation of the equivalence rule (RE). We turn to a class of logics called hyperintensional (or non-congruential), which violate (RE). In the rest of this work, we will be interested in these logics.

Definition 6 (Hyperintensional Modal Logic)**.** *A hyperintensional modal logic is* $\Lambda \subseteq Fm$ *such that* Λ *is a modal logic and:*

- Λ *is not closed under the equivalence rule (RE):* $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

It follows from the definitions that the class of hyperintensional modal logics is a complement of the class of classical modal logics. Figure 1.1 represents the relationship between the classes of modal logic we defined.

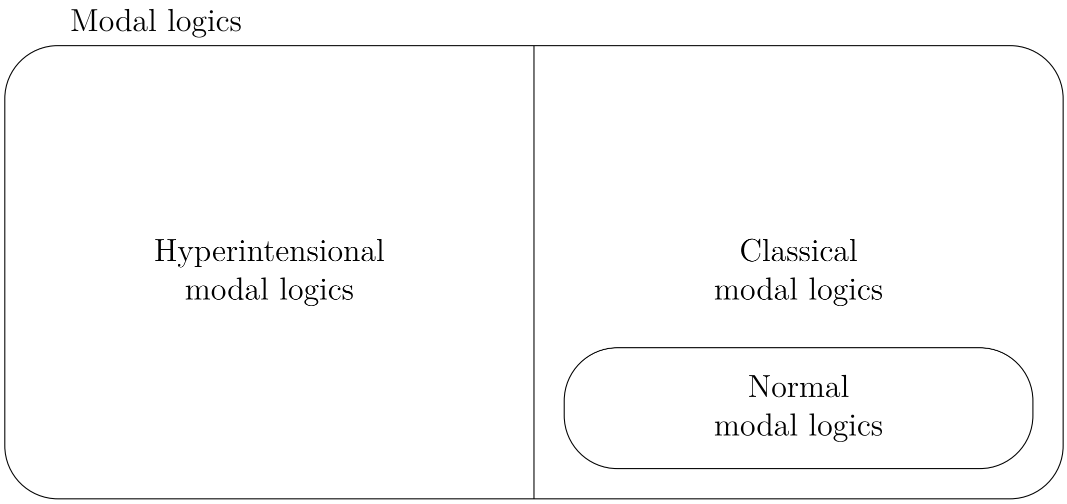

Figure 1.1: Some classes of modal logics

Remark. The smallest hyperintensional logic is $\Sigma(Taut)$. This easily follows from the fact that $\Sigma(Taut)$ is the smallest modal logic and that it is not closed under (RE). For example, $(p \wedge q) \leftrightarrow (q \wedge p) \in \Sigma(Taut)$, but $\Box(p \wedge q) \leftrightarrow \Box(q \wedge p) \notin \Sigma(Taut)$. This follows the fact that the second formula is of the form $\Box\varphi \leftrightarrow \Box\psi$, which is not a substitutional instance of a proportional tautology. This will become clear when we prove Theorem 4.

We close this section by providing some definitions and claims about modal logics that will be useful in the subsequent parts of the work.

Definition 7 (Deducibility, Λ-theories)**.** φ *is deducible in logic* Λ *from set of formulas* Γ *(*$\Gamma \vdash_\Lambda \varphi$*), if and only if* $((\psi_1 \wedge ... \wedge \psi_n) \to \varphi) \in \Lambda$ *for some* $\psi_1, ..., \psi_n \in \Gamma$. *A set of formulas* Γ *is a* Λ*-theory if* $\Gamma \vdash_\Lambda \varphi$ *implies* $\varphi \in \Gamma$, *and* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$ *imply that* $\varphi \wedge \psi \in \Gamma$.

Definition 8 (Consistency, Maximal Λ-consistency)**.** *A set of formulas* Γ *is* Λ*-consistent if* $\Gamma \nvdash_\Lambda \bot$, *and* Λ*-inconsistent otherwise. A* Λ*-theory* Γ *is maximal* Λ*-consistent if* Γ *is* Λ*-consistent* Λ*-theory, and any set properly containing* Γ *is* Λ*-inconsistent. (Equivalently,* $\varphi \notin \Gamma$ *implies* $\Gamma \vdash_\Lambda \neg\varphi$*).*

Lemma 2 (Lindenbaum's Lemma). *If* Γ *is a* Λ*-consistent set of formulas, then there is a maximal* Λ*-consistent theory* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$.

Proof. The proof is provided in Blackburn et al. [2001] (p. 199).

□

Lemma 3 (Properties of Maximal Λ-consistent Theories). *If* Γ *is a maximal* Λ*-consistent theory for some* Λ*, then it holds that:*

- $\neg\varphi \in \Gamma \iff \varphi \notin \Gamma$,
- $\varphi \wedge \psi \in \Gamma \iff \{\varphi, \psi\} \subseteq \Gamma$,
- $\varphi \in \Lambda$ *iff* $\varphi \in \Gamma$ *for all maximal* Λ*-consistent theories* Γ.

Proof. It cannot be the case that $\varphi \in \Gamma$ and $\neg\varphi \in \Gamma$, because then $\Gamma \vdash \varphi \wedge \neg\varphi$, which would mean that Γ is inconsistent. If neither $\varphi \in \Gamma$ nor $\neg\varphi \in \Gamma$, then the set would not be maximal. Therefore, either $\neg\varphi \in \Gamma$ or $\varphi \in \Gamma$, but not both. It follows that $\neg\varphi \in \Gamma \iff \varphi \notin \Gamma$.

Similarly, $\varphi \wedge \psi \in \Gamma$, therefore, $\neg\varphi$ nor $\neg\psi$ cannot be in Γ, as then Γ would be inconsistent. And since Γ is maximal, it follows that both $\varphi \in \Gamma$ and $\psi \in \Gamma$.

For the last property, we show both directions separately. First, suppose that $\varphi \notin \Gamma$ for some Γ a maximal Λ-consistent theory. By the definition of Λ-consistency, then $\Lambda \cup \{\neg\varphi\}$ is a consistent set of formulas. That means that $\varphi \notin \Lambda$, because otherwise $\Lambda \cup \{\neg\varphi\}$ would be inconsistent. In the other direction, if we suppose that $\varphi \in \Gamma$ for all maximal Λ-consistent theories Γ, then it must also be the case that $\varphi \in \bigcap\{\Gamma \mid \Gamma \text{ is a maximal } \Lambda\text{-consistent theory}\}$. Since $\bigcap\{\Gamma \mid \Gamma \text{ is a maximal } \Lambda\text{-consistent theory}\} = \Lambda$ from the properties of maximal Λ-consistent theories, it follows that $\varphi \in \Gamma$ for every maximal Λ-consistent theory Γ.

□

In the rest of the thesis, we will discuss various semantic frameworks for logics over the basic modal language and their relationships, with the goal to find a fitting semantics for hyperintensional modal logic.

1.2 Semantics

In the beginnings of modal logic (around the first half of the 20th century), the work done on modal logic has been largely syntactic. According to Blackburn et al. [2001], the year 1918 might be considered for as the year of birth of modern modal logic thanks to the publication of Lewis' *Survey of Symbolic Logic* (Lewis [1918]), which sparked the interest in the idea of 'modalizing' propositional logic. However, the addition of the modal operator $\Box$ as primitive happened only later by the work of K. Gödel. In the following years, a lot of work has been done in modal logics, but without a semantic interpretation, the research ran into problems. When missing semantics, it's not only difficult to compare different approaches, but also proving that two systems differ or knowing whether all relevant possibilities are considered, become problems.

That is why, when Kripke semantics has been introduced (Kripke [1963a], Kripke [1963b]), the work that could be done on modal logics has been revolutionized. Kripke semantics offered a clear and intuitive way to talk about modalities. It allowed to also build canonical models for various logics, and so lead to many soundness and completeness results. Soon variations arose, such as the neighborhood semantics (Montague [1968],Scott [1970], Segerberg [1971]) discussed in this chapter.

In the recent years, modal logic has found many applications in computer science, economics, philosophy in other disciplines. Continually, new modal logics arise and attempt to solve various problems in these and other disciplines.

Before discussing the established semantic approaches to modal logic, let us first discuss one intuitive way in which one could attempt to provide semantics for (or at least define the valuation of) modal formulas.

Consider how propositional formulas are evaluated in classical propositional logic. As modal language is theoretically just an extension of the propositional language, we may be well tempted just to define an extended valuation $e : At \cup \{\Box\varphi \mid \varphi \in Fm\} \to \{0, 1\}$, where the satisfaction of a formula $\varphi \in Fm$ under a given e ($e \vDash \varphi$) is defined as expected:

- $e \vDash p \iff e(p) = 1$ for an $p \in At$,
- $e \vDash \neg\varphi \iff e \nvDash \varphi$,
- $e \vDash \varphi \wedge \psi \iff e \vDash \varphi$ and $e \vDash \psi$,
- $e \vDash \Box\varphi \iff e(\Box\varphi) = 1$.

However, such a definition does not tell us anything about what kind of properties should the modal box exhibit. Of course, by restricting valuations to various classes it would be possible to capture various contexts, but it is unintuitive in a way that it does not tell us anything about the relations between the formulas and the modal formulas. Therefore, it feels unsatisfactory to settle with such semantics.

Nevertheless, the following theorem shows that the class of all extended valuations e generates the logic $\Sigma(Taut)$. That is, $\Sigma(Taut)$ is complete with respect to the class of all valuations e. It will be useful when proving theorems in the latter parts of the thesis.

Theorem 4. *$\varphi \in \Sigma(Taut)$ iff $e \vDash \varphi$ for every extended evaluation e.*

Proof. We will show the theorem by showing both implications separately. ($\implies$) *Soundness.* We want to show that if $\varphi \in \Sigma(Taut)$ then $e \vDash \varphi$.

First, we need to show that every tautology is satisfied by every extended valuation e. But this follows directly from the definition of what it means for a formula to be satisfied by e - on formulas in classical propositional language, satisfaction by e behaves as valuation in classical propositional logic.

Secondly, we will show that if a formula φ is satisfied by every valuation e, then $\sigma(\varphi)$ is satisfied by every valuation e. Fix a valuation e. We define the evaluation e' so that for $p \in At$:

$$e'(p) \begin{cases} 1 & \text{if } e \vDash \sigma(p), \\ 0 & \text{if } e \nvDash \sigma(p), \end{cases}$$

and for $\varphi \in \{\Box\varphi \mid \varphi \in Fm\}$:

$$e'(\Box\varphi) \begin{cases} 1 & \text{if } e(\Box(\sigma(\varphi))) = 1, \\ 0 & \text{if } e(\Box(\sigma(\varphi))) = 0. \end{cases}$$

Now all that is left to do is to show that $e' \vDash \varphi \iff e \vDash \sigma(\varphi)$. We do so by induction on the complexity of formulas.

If $\varphi = p$ for some $p \in At$, the equivalence follows from the definition of e'.

If $\varphi = \neg\psi$ for some ψ for which the induction hypothesis holds, then: $e' \vDash \neg\psi \iff e' \nvDash \psi \iff e \nvDash \sigma(\varphi) \iff e \vDash \neg\sigma(\varphi) \iff e \vDash \sigma(\neg\varphi)$.

If $\varphi = \psi \wedge \xi$ for some ψ, ξ for which the induction hypothesis holds, then: $e' \vDash \psi \wedge \xi \iff e' \vDash \psi$ and $e' \vDash \xi \iff e \vDash \sigma(\psi)$ and $e \vDash \sigma(\varphi) \iff e \vDash \sigma(\psi) \wedge \sigma(\xi) \iff e \vDash \sigma(\psi \wedge \xi)$.

If $\varphi = \Box\psi$ for some ψ for which the induction hypothesis holds, then: $e' \vDash \Box\psi \iff e'(\Box\psi) = 1 \iff e(\Box(\sigma(\psi)) = 1 \iff e(\sigma(\Box\psi)) = 1 \iff e \vDash \sigma(\Box\varphi)$.

Therefore we showed that if $\varphi \in \Sigma(Taut)$, then $e \vDash \varphi$.

($\Longleftarrow$) *Completeness.* We will show that if $\varphi \notin \Sigma(Taut)$, then there is some valuation e, such that $e \nvDash \varphi$. If $\varphi \notin \Sigma(Taut)$, then by Lemma 3, there is some maximal $\Sigma(Taut)$-consistent theory Γ, such that $\varphi \notin \Gamma$. We define the extended valuation $e_\Gamma(\psi) = 1 \iff \psi \in \Gamma$ for $\psi \in At \cup \{\Box\psi \mid \psi \in Fm\}$. The satisfaction $e_\Gamma \vDash \varphi$ for any formula $\varphi \in Fm$ is defined as usual. For e_Γ and arbitrary formula φ holds that:

$$\varphi \in \Gamma \iff e_\Gamma \vDash \psi.$$

This is shown by induction on the complexity of formulas:

If $\varphi = p$ for some $p \in At$, or $\varphi = \Box\psi$ for some formula ψ, the equivalence holds by definition of an extended valuation.

If $\varphi = \neg\psi$ for some ψ for which the induction base holds, then $e_\Gamma \vDash \neg\psi \iff e_\Gamma \nvDash \psi \iff \psi \notin \Gamma \iff \neg\psi \in \Gamma$.

If $\varphi = \psi \wedge \xi$ for some ψ, ξ for which the induction base holds, then $e_\Gamma \vDash \psi \wedge \xi \iff e_\Gamma \vDash \psi$ and $e_\Gamma \vDash \xi \iff \psi \in \Gamma$ and $\xi \in \Gamma \iff \psi \wedge \xi \in \Gamma$.

Therefore, we found an extended valuation e_Γ, such that if $\varphi \notin \Gamma$, then $e_\Gamma \nvDash \varphi$.

$\Box$

In the rest of the chapter, we introduce the relational (Kripke) semantics and neighborhood semantics, offer some basic definitions, and show that they are sound and complete with respect to the basic normal modal logic, and the basic classical modal logic, respectively.

1.2.1 Relational Semantics

Definition 9 (Relational frame)**.** *A relational frame $\mathcal{F}$ for the basic modal language is a pair $\mathcal{F} = (W, R)$ such that $W \neq \emptyset$ and R is a binary relation on W.*

Informally, we may think of $w \in W$ as possible worlds, points or states. R is sometimes also called accessibility relation. To be able to talk about the validity or the truth of the formulas at various points of W, we need to expand the notion of a frame into the one of a model:

Definition 10 (Model)**.** *A model $\mathcal{M}$ for the basic modal language is a triple $\mathcal{M} = (W, R, V)$, such that (W, R) is a frame for the basic modal language, and V is a function $V : At \to \mathcal{P}(W)$.*

Intuitively, V assigns to each propositional atom $p \in At$ a subset of W. We want to think of this $V(p) \subseteq W$ as a set of worlds in which p is true, also called *the proposition* expressed by p. Therefore, we will call V *a valuation function.* Let us now define the notion of truth for any formula φ in a state $w \in W$:

Definition 11 (Truth, Validity)**.** *Let $\mathcal{M} = (W, R, V)$ be a model, and $w \in W$. We inductively define the notion of a formula φ to be satisfied (or true) in $\mathcal{M}$ at the state w as follows:*

- *$\mathcal{M}, w \Vdash p$ iff $w \in V(p)$, where $p \in At$,*
- *$\mathcal{M}, w \Vdash \neg\varphi$ iff not $\mathcal{M}, w \Vdash \varphi$,*
- *$\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$,*
- *$\mathcal{M}, w \Vdash \Box\varphi$ iff for every $u \in W$ such that Rwu, we have $\mathcal{M}, u \Vdash \varphi$.*

A formula φ is valid (universally true) in a model $\mathcal{M} = (W, R, V)$ if it is satisfied at all points $w \in W$. φ is a valid in a frame $\mathcal{F} = (W, R)$ ($\mathcal{F} \Vdash \varphi$) if it is valid in every model based on the frame. φ is valid on a class of frames F if it is valid on every frame $\mathcal{F}$ in F, and it is valid if it is valid in the class of all frames. The set of all formulas that are valid in a class of frames F is called the logic of F (denoted Λ_F).

It follows from the definition that $\mathcal{M}, w \Vdash \Diamond\varphi$ if and only if there is some $u \in W$ such that Rwu and $\mathcal{M}, u \Vdash \varphi$. If $\mathcal{M}$ does not satisfy φ at w, we write $\mathcal{M}, w \nVdash \varphi$.

Note that in relational semantics we want to intuitively understand the relation R as the accessibility relation, i.e., it should represent which points of W are 'accessible' or 'visible' from any given point $w \in W$. On epistemic reading, for example, the accessibility relation represents which worlds, i.e. states of possibilities, are consistent with the agent's current knowledge. Say that at a world w, it holds that $w \Vdash \Box\varphi$. Then, this means that the set $R(w) = \{u \in W \mid Rwu\}$, representing the set of worlds consistent with the agent's knowledge in w, has to be a subset of all worlds in which φ holds, so in all the worlds the agent can consider as possible in w, it is true that φ. Therefore, the evaluation of the formula $\Box\varphi$, is defined by the truth of the formula φ in the accessible worlds from w.

The following figure is an example of a relational frame $\mathcal{F} = (W, R, V)$, where $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $R = \{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_2, w_5), (w_2, w_6), (w_3, w_6), (w_4, w_4), (w_5, w_5)\}$. Now suppose we have some model $\mathcal{M}$ over this frame. If we were interested in evaluating a formula $\Box\varphi$ at w_1, we would have to check whether φ hold in $\{w_2, w_3, w_4\}$. If we wanted to evaluate $\Box\varphi$ at w_5, φ would need to hold only in w_5. And if we wanted to evaluate it at w_6, it is automatically satisfied, because there are no accessible worlds from w_6.

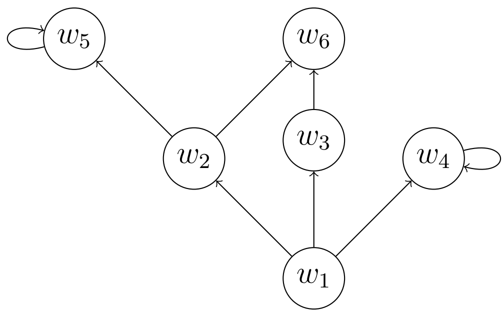

Figure 1.2: An example of a relational frame

Claim 5. *In the relational models, $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$ is a valid inference rule.*

Proof. Suppose that $\mathcal{M} = (W, R, V)$ is a relational model, and that $\mathcal{M} \Vdash \varphi \leftrightarrow \psi$. We want to show that $\mathcal{M} \Vdash \Box\varphi \leftrightarrow \Box\psi$, which means that $\mathcal{M}, w \Vdash \Box\varphi$ iff $\mathcal{M}, w \Vdash \Box\psi$. Suppose that $\mathcal{M}, w \Vdash \Box\varphi$ for some $w \in W$. From the definition, $\mathcal{M}, w \Vdash \Box\varphi$ for any $w \in W$ iff φ holds in all u such that Rwu. But from the assumption that $\mathcal{M} \Vdash \varphi \leftrightarrow \psi$, it must be the case that in all these $u \in W$, $\mathcal{M}, u \Vdash \psi$, and so $\mathcal{M}, w \Vdash \Box\psi$. The proof that $\mathcal{M} \Vdash \varphi \leftrightarrow \psi$ entails $\mathcal{M} \Vdash \Box\psi \rightarrow \Box\varphi$ is analogous.

$\Box$

Theorem 6. *Every class of relational frames generates a normal modal logic.*

Proof. We will show that every class of relational frames generates a normal modal logic, that is, if F is a class of frames, then Λ_F is a normal modal logic.

To see that every class of relational frames generates a normal modal logic, fix an arbitrary class F of such frames. Consider the set

$$\Lambda_F = \{\varphi \mid \mathcal{F} \Vdash \varphi \text{ for all } \mathcal{F} \in F\}.$$

Now to show that Λ_F is a normal modal logic, we need to show that:

1) $Taut \subseteq \Lambda_F$,

2) Λ_F is closed under MP,

3) Λ_F is closed under substitution,

4) Λ_F contains all instances of schema K,

5) Λ_F is closed under the *Nec* rule.

The conditions 1) and 2) are fairly obvious. Λ_F is a superset of $Taut$, since all tautologies are true at every frame for some modal logic. It's also easy to see that Λ_F is closed under *modus ponens*: if $\varphi \in \Lambda_F$ and $\varphi \to \psi \in \Lambda_F$ for some formulas φ, ψ, then also $\mathcal{F} \Vdash \varphi$ and $\mathcal{F} \Vdash \varphi \to \psi$ for every frame $\mathcal{F}$. Therefore for every frame $\mathcal{F}$ we have $\mathcal{F} \Vdash \psi$, and so $\psi \in \Lambda_F$.

Ad 3), $\varphi \in \Lambda_F$ if and only if $\mathcal{M}, w \Vdash \varphi$ for every model $\mathcal{M} = (W, R, V)$ and world $w \in W$ (from the definition of validity on a frame $\mathcal{F}$). We show that for all substitutions $\sigma \in \Sigma$, also $\mathcal{M}, w \Vdash \sigma(\varphi)$, and so $\sigma(\varphi) \in \Lambda_F$.

Fix σ a substitution function, such that for a φ consisting of atomic propositions $p_1, ..., p_n$, $\sigma(p_1) = \psi_1, ..., \sigma(p_1) = \psi_n$ for some $\psi_1, ..., \psi_n \in Fm$. We construct a model $\mathcal{M}'$, such that $V'(p_i) = V(\psi_i)$, and show that,

$$\mathcal{M}', w \Vdash \varphi \iff \mathcal{M}, w \Vdash \sigma(\varphi)$$

We do so by induction on the complexity of formulas.

If $\varphi = p$ for some $p \in At$, then $\mathcal{M}', w \Vdash p \iff w \in V'(p) \iff w \in V(\psi) \iff \mathcal{M}, w \Vdash \psi \iff \mathcal{M}, w \Vdash p[p/\psi] \iff \mathcal{M}, w \Vdash \sigma(\varphi)$.

If $\varphi = \neg\psi$ for some ψ for which the induction hypothesis holds, then $\mathcal{M}', w \Vdash \neg\psi \iff \mathcal{M}', w \nVdash \psi \iff \mathcal{M}, w \nVdash \sigma(\psi) \iff \mathcal{M}, w \Vdash \sigma(\neg\psi)$.

If $\varphi = \psi \wedge \xi$ for some ψ, ξ for which the induction hypothesis holds, then $\mathcal{M}', w \Vdash \psi \wedge \xi \iff \mathcal{M}', w \Vdash \psi$ and $\mathcal{M}', w \Vdash \xi \iff \mathcal{M}, w \Vdash \sigma(\psi)$ and $\mathcal{M}, w \Vdash \sigma(\xi) \iff \mathcal{M}, w \Vdash \sigma(\psi \wedge \xi)$.

And finally, if $\varphi = \Box\psi$ for some ψ for which the induction hypothesis holds, then: $\mathcal{M}'_0, w_0 \Vdash \Box\psi \iff \forall v(Rwv \to \mathcal{M}'_0, v \Vdash \psi) \iff \forall v(Rwv \to \mathcal{M}_0, v \Vdash \sigma(\psi)) \iff \mathcal{M}_0, w_0 \Vdash \Box(\sigma(\psi)) \iff \mathcal{M}_0, w_0 \Vdash \sigma(\Box\psi) \iff \mathcal{M}_0, w_0 \Vdash \sigma(\varphi)$.

And so we have shown that if $\mathcal{M}, w \Vdash \varphi$ for all $\mathcal{M}$, then it also must be the case that $\mathcal{M}, w \Vdash \sigma(\varphi)$ for all models $\mathcal{M}$ and $\sigma \in \Sigma$.

We show 4) by contradiction: if Λ_F does not contain all instances of schema K, then there is a formula φ such that $\varphi := \Box(\psi \to \xi) \to (\Box\psi \to \Box\xi)$ for some ψ, ξ, and that φ is not in Λ_F. Suppose then, that there is some $\mathcal{F} \in F$, such that $\mathcal{F} \nVdash \varphi$. That means that there is some $w \in W$, and a model $\mathcal{M}$ of $\mathcal{F}$, such that $\mathcal{M}, w \Vdash \Box(\psi \to \xi)$ and $\mathcal{M}, w \Vdash \Box\psi$, but $\mathcal{M}, w \nVdash \Box\xi$. It follows that for every u such that Rwu, $\mathcal{M}, u \Vdash \psi \to \xi$, and $\mathcal{M}, u \Vdash \psi$, so by the properties of implication also $\mathcal{M}, u \Vdash \xi$. However, by the definition from $\mathcal{M}, w \nVdash \Box\xi$ follows that there is some $u \in W$, such that Rwu, such that $\mathcal{M}, u \nVdash \xi$, which is a contradiction.

And finally, for 5) also by contradiction, suppose that Λ_F is not closed under the *Nec* rule. Then there is some φ in Λ_F such that $\Box\varphi$ is not in Λ_F. Then there is some $\mathcal{F}$ for which we have $\mathcal{F} \Vdash \varphi$ and $\mathcal{F} \nVdash \Box\varphi$, i.e. there is some model $\mathcal{M}$ and some $w \in W$, such that $\mathcal{M} \Vdash \varphi$ and $\mathcal{M}, w \nVdash \Box\varphi$. But $\mathcal{M}, w \nVdash \Box\varphi$ means that there is some $u \in W$ such that Rwu, so that $\mathcal{M}, u \nVdash \varphi$. At the start we assumed that $\mathcal{M} \Vdash \varphi$, so we have a contradiction. Therefore, Λ_F has to be closed also under *Nec*.

□

Theorem 7. *The basic normal modal logic* **K** *is the logic of the class of all relational frames.*

Proof. ($\Longrightarrow$) *Soundness.* We need to show that if φ is in **K**, then it is valid in the class of all relational frames. We show this by contraposition, that is, if there is a relational frame in which φ does not hold, then φ is not a theorem of **K**.

However, this follows from the previous theorem. If we consider a class of all relational frames $\Lambda_F = \{\varphi \mid \mathcal{F} \Vdash \varphi \text{ for all } \mathcal{F} \in F\}$, and there is a frame $\mathcal{F}$ such that $\mathcal{F} \nVdash \varphi$, then $\varphi \notin \Lambda_F$. We showed that Λ_F is a normal modal logic, and since **K** is the smallest modal logic, $\mathbf{K} \subseteq \Lambda_F$, therefore, $\varphi \notin \mathbf{K}$.

($\Longleftarrow$) *Completeness.* The proof of completeness will be slightly more complicated. We need to show that if φ is valid in the class of all relational frames, then φ is a theorem of normal modal logic **K**. We also show this by contraposition. We suppose that φ is not a theorem of **K**, and show that then there is some model $\mathcal{M}$ in which φ does not hold.

To this end, we construct a canonical model $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, R^{\mathcal{C}}, V^{\mathcal{C}})$, such that:

- $W^{\mathcal{C}}$ is the set of all maximal **K**-consistent theories,
- $R^{\mathcal{C}}$ is a relation defined so that $R^{\mathcal{C}}wu \iff \forall\varphi(\Box\varphi \in w \to \varphi \in u)$,
- $V^{\mathcal{C}}$ is a valuation function so that $V^{\mathcal{C}}(p) = \widehat{p}$, where $\widehat{p} = \{w \in W^{\mathcal{C}} \mid p \in w\}$.

It is clear that $\mathcal{F}^{\mathcal{C}} = (W^{\mathcal{C}}, R^{\mathcal{C}})$ is a relational frame and that $\mathcal{M}^{\mathcal{C}}$ is a model based on the frame.

Lemma 8 (TL[2])**.** *For any formula $\varphi \in \mathbf{K}$, and $w \in W^{\mathcal{C}}$, $\mathcal{M}^{\mathcal{C}}, w \Vdash \varphi \iff \varphi \in w$.*

Proof. We show the lemma by induction on the complexity of φ.

If $\varphi = p$ for some $p \in At$, $\mathcal{M}^{\mathcal{C}}, w \Vdash p \iff w \in V^{\mathcal{C}}(p) \iff w \in \widehat{p} \iff p \in w$.

If $\varphi = \neg\psi$ for some $\psi \in Fm$ for which the induction hypothesis holds, then $\mathcal{M}^{\mathcal{C}}, w \Vdash \neg\psi \iff \mathcal{M}^{\mathcal{C}}, w \nVdash \psi \iff \psi \notin w \iff \neg\psi \in w$.

If $\varphi = \psi \wedge \xi$ for some $\psi, \xi \in Fm$ for which the induction hypothesis holds, then $\mathcal{M}^{\mathcal{C}}, w \Vdash \psi \wedge \xi \iff \mathcal{M}^{\mathcal{C}}, w \Vdash \psi$ and $\mathcal{M}^{\mathcal{C}}, w \Vdash \xi \iff \psi \in w$ and $\xi \in w \iff \psi \wedge \xi \in w$.

If $\varphi = \Box\psi$ for some $\psi \in Fm$, we show the both directions separately. The $\Longrightarrow$ direction is shown by contraposition, so suppose that $\Box\psi \notin w$. To show that $\mathcal{M}^{\mathcal{C}}, w \nVdash \Box\psi$, we need to show that there is some $u \in W^{\mathcal{C}}$, such that $R^{\mathcal{C}}wu$ and $\varphi \notin u$. Consider the set $u_0 = \{\varphi \mid \Box\varphi \in w\}$. Clearly, $\psi \notin u_0$, and so $u_0 \cup \{\neg\psi\}$ is **K**-consistent. By Lindenbaum's Lemma, there is some $u \supseteq u_0$, such that u is maximal **K**-consistent and $\psi \notin u$. Therefore, we found $u \in W^{\mathcal{C}}$, such that $R^{\mathcal{C}}wu$ and $\psi \notin u$. The converse direction follows from the definition, because if we suppose that $\Box\varphi \in w$, and $\mathcal{M}^{\mathcal{C}}, w \Vdash \Box\psi \iff \forall u(R^{\mathcal{C}}wu \to \psi \in u)$ and $R^{\mathcal{C}}wu$ and relation is defined so that $\forall\varphi(\Box\psi \in w \to \psi \in u)$, then it is clear that if $\Box\psi \in w$, then it must also be the case that $\psi \in u$.

□

It follows that if φ is not a theorem of **K**, then there is some $w \in W^{\mathcal{C}}$ such that $\varphi \notin w$, and so $\mathcal{M}^{\mathcal{C}}, w \nVdash \varphi$. So there is a relational model, in which φ does not hold.

□

[2] We use 'TL' as a short to mark that this is commonly referred to as a truth lemma.

1.2.2 Neighborhood Semantics

Neighborhood models generalize the concept of relational models. To be precise, in neighborhood models, instead of a relation R on the set of worlds W, there is a function N that generalizes R. As a result, the logic of every class of neighborhood models is a classical modal logic, which we will show in the last part of the section.

Definition 12 (Neighbourhood Frame, Neighbourhood Model). *A neighborhood frame is a tuple $\mathcal{F} = (W, N)$, where W is a non-empty set, and N is a neighborhood function $N : W \to \mathcal{P}(\mathcal{P}(W))$.*

A neighborhood model is a triple $\mathcal{M} = (W, N, V)$, where (W, N) is a frame, and V is a valuation function $V : At \to \mathcal{P}(W)$.

The set W is again understood as a set of worlds or states, the valuation function V intuitively assigns each propositional atom a set of worlds in which it is true as before, and N is called a neighborhood function.

Definition 13 (Truth, Validity). *Let $\mathcal{M} = (W, N, V)$ be a neighbourhood model and $w \in W$. The truth of a formula φ at w is defined recursively as follows:*

- $\mathcal{M}, w \Vdash p$ *iff* $w \in V(p)$, *where* $p \in At$,
- $\mathcal{M}, w \Vdash \neg\varphi$ *iff not* $\mathcal{M}, w \Vdash \varphi$,
- $\mathcal{M}, w \Vdash \varphi \wedge \psi$ *iff* $\mathcal{M}, w \Vdash \varphi$ *and* $\mathcal{M}, w \Vdash \psi$,
- $\mathcal{M}, w \Vdash \Box\varphi$ *iff* $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$,

where $[\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$ *is the truth set of* φ.

We say φ is valid in $\mathcal{M}$ ($\mathcal{M} \Vdash \varphi$) if $\mathcal{M}, w \Vdash \varphi$ for every $w \in W$. φ is valid in $\mathcal{F}$, it is valid in every model $\mathcal{M}$ based on $\mathcal{F}$. φ is valid on a class of frames F if $\mathcal{F} \Vdash \varphi$ for every frame $\mathcal{F} \in F$.

The truth set $[\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$ for some formula φ denotes the set of all worlds in which φ is true. For every model $\mathcal{M}$, $[\![\cdot]\!]$ may be thought of as a function $[\![\cdot]\!]_{\mathcal{M}} : Fm \to \mathcal{P}(W)$. Sometimes we also refer to $[\![\varphi]\!]_{\mathcal{M}}$ as 'the proposition of φ (in $\mathcal{M}$)'.

The idea is that the neighborhood function N lists which propositions are 'necessary' for each state w. Then, $\Box\varphi$ is true at w if the truth set of φ is a member of the list at that state. So instead of w being related to a single set of worlds set of worlds (the set $R(w)$ in relational models), the intuition here is that the neighborhood function provides a set of such '$R(w)$', in the sense that to w it assigns multiple sets of states. This is the reason why in neighborhood semantics, it might be the case that $w \Vdash \Box\varphi$ and $w \Vdash \Box\psi$, but not $w \Vdash \Box\varphi \wedge \Box\psi$ - there might be one neighborhood of w such that $[\![\varphi]\!] \in N(w)$, and at the same time $[\![\psi]\!] \in N(w)$, but not $[\![\varphi \wedge \psi]\!] \in N(w)$. In relational models, the truth set of a formula is always compared to the single set $R(w)$, so we would get $R(w) \subseteq [\![\varphi]\!]$ and $R(w) \subseteq [\![\psi]\!]$, and therefore it must hold that $[\![\varphi \wedge \psi]\!] \subseteq R(w)$. which is consistent with the fact that $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ holds in relational frames but not in neighborhood frames.

The notion of neighborhoods might be slightly difficult to grasp, therefore, we provide an illustration of a neighborhood model in the following figure:

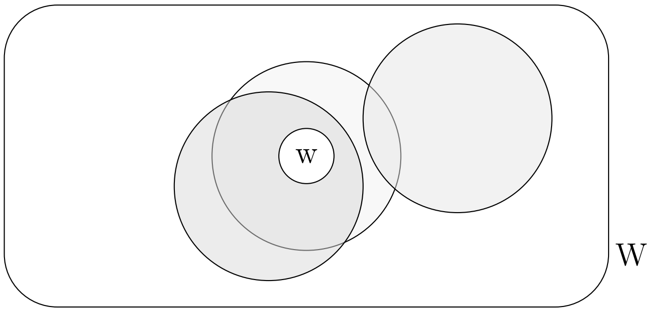

Figure 1.3: Visualization of neighborhood models

W is the space of all worlds W. The grey circles represent neighborhoods of w. Note that in a general case, such as this one, there is no requirement on the neighborhoods to contain w. For $\Box\varphi$ to be true at w, it is enough that $[\![\varphi]\!]$ is a superset of one of these neighborhoods.

A natural association between neighborhood models and relational models is easily observed, as every relational model can be seen as a special case of a neighborhood model. A neighborhood model $\mathcal{M}^N = (W^N, N, V^N)$ generalizing a relational model $\mathcal{M}^R = (W^R, R, V^R)$ is a model where $W^N = W^R, V^N = V^R$, and $N(X) = \{X \mid R(w) \subseteq X\}$. (In the relational models it holds that $w \Vdash \Box\varphi \iff R(w) \subseteq [\![\varphi]\!]$, so the neighborhood models just generalize this notion.)

Sometimes it might be useful to consider a dual function N' to N defined as as $N' : \mathcal{P}(W) \to \mathcal{P}(W)$ such that for $X \subseteq W$:

$$N'(X) = \{w \mid X \in N(W)\}.$$

If we define N' in this way, it follows that for a model $\mathcal{M} = (W, N, V)$:

- $[\![p]\!]_{\mathcal{M}} = p$, for $p \in At$,
- $[\![\neg\varphi]\!]_{\mathcal{M}} = W \backslash [\![\varphi]\!]_{\mathcal{M}}$,
- $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$,
- $[\![\Box\varphi]\!]_{\mathcal{M}} = N'([\![\varphi]\!]_{\mathcal{M}})$.

However, as we will now show, neighbourhood models still preserve the equivalence rule, and therefore they are not general enough to avoid the problem of hyperintensional operators.

Claim 9. *In the neighborhood models, $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$ is a valid inference rule.*

Proof. Suppose that $\varphi \leftrightarrow \psi$ holds for some formulas φ, ψ in $\mathcal{L}_0$ in a model $\mathcal{M}$. Then by the definition of a truth set it must hold that their truth sets are equal: $[\![\varphi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}}$. By applying the function N' to both sides we get that $N'[\![\varphi]\!]_{\mathcal{M}} = N'[\![\psi]\!]_{\mathcal{M}}$, however, due to the equivalence of N' this is then equal to $[\![\Box\varphi]\!]_{\mathcal{M}} = [\![\Box\psi]\!]_{\mathcal{M}}$, and so we obtain $\Box\varphi \leftrightarrow \Box\psi$.

$\Box$

Finally, we will prove that not only the equivalence rule holds in the neighborhood models, but that by taking any class of such frames, we can obtain a classical modal logic.

Theorem 10. *Every class of neighborhood frames generates a classical modal logic.*

Proof. The proof is similar to the analogous theorem for relational models in the subsection 1.2.1. Again, to show that every class of neighborhood frames gives a classical modal logic, we fix an arbitrary class of neighborhood frames F and show that

$$\Lambda_F = \{\varphi \mid \mathcal{F} \Vdash \varphi \text{ for all } \mathcal{F} \in F\}$$

is a classical modal logic.

To show that, we will need to show the following three requirements hold:

1) $Taut \subseteq \Lambda_F$,

2) Λ_F is closed under MP,

3) Λ_F is closed under substitution.

Again, 1) is quite obvious, since all tautologies hold in every neighborhood frame. 2) holds also quite intuitively, since all frames are closed under MP, also Λ_F has to be.

The proof of 3) is almost the same as the proof of the analogous claim in the relational models. Given an arbitrary neighborhood model $\mathcal{M} = (W, N, V)$, we define a model $\mathcal{M}' = (W, N, V')$, where V' is defined as in the Theorem 4. Similarly as in the proof of that theorem, we can show by induction on the complexity of formulas that $\mathcal{M}', w \Vdash \varphi \iff \mathcal{M}, w \Vdash \sigma(\varphi)$. The only difference is in the case of $\varphi = \Box\psi$, which we give in full detail here:

If $\varphi = \Box\psi$, for some ψ for which the induction hypothesis holds, then $\mathcal{M}', w \Vdash \Box\psi \iff [\![\psi]\!]_{\mathcal{M}'} \in N'(w) \iff [\![\sigma(\psi)]\!]_{\mathcal{M}} \in N(w) \iff \mathcal{M}, w \Vdash \Box\sigma(\psi) \iff \mathcal{M}, w \Vdash \sigma(\Box\psi)$.

And so it follows that the logic of all neighborhood frames is closed under substitution. This concludes the proof.

□

Note that, however, a stronger claim that *every class of neighbourhood frames gives a normal modal logic* would not hold. Consider the schema $K : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. For example, if we take a following neighbourhood model: $\mathcal{M} = (W, N, V)$, where $W = \{a, b, c\}, N(a) = \{\{a\}, \{a, b, c\}\}$, and the valuation function defined so that for atoms p, q we have: $V(p) = \{a\}, V(q) = \{a, b\}$, we obtain $[\![p]\!]_{\mathcal{M}} = \{a\}$, $[\![q]\!]_{\mathcal{M}} = \{a, b\}$, $[\![p \to q]\!]_{\mathcal{M}} = (\neg p \vee q)^{\mathcal{M}} = \{a, b, c\}$. That means that $\mathcal{M}, a \Vdash \Box(p \to q), \mathcal{M}, a \Vdash \Box p$, but $\mathcal{M}, a \nVdash \Box q$.

Theorem 11. *The smallest classical modal logic* **E** *is the logic of the class of all neighborhood frames.*

Proof. ($\Longrightarrow$) *Soundness.* We need to show that if φ is in **E**, then it is valid in the class of all neighborhood frames. We show this by contraposition, that is, if there is a neighborhood frame in which φ does not hold, then φ is not a theorem of **E**. However, this is clear from the previous theorem.

($\Longleftarrow$) *Completeness.* We prove completeness using the construction of a canonical model. Suppose that φ is not a theorem of **E**, we will show that there is some model $\mathcal{M}$ (the canonical model), in which φ is not valid. Let $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, N^{\mathcal{C}}, V^{\mathcal{C}})$ be a neighborhood model, such that:

- $W^{\mathcal{C}}$ is the set of all maximal **E**-consistent theories,
- $N^{\mathcal{C}}$ is a neighborhood function so that $N^{\mathcal{C}}(w) = \{\widehat{\varphi} \mid \Box\varphi \in w\}$,
- $V^{\mathcal{C}}(p) = \widehat{p}$,

where $\widehat{\varphi} = \{w \in W^{\mathcal{C}} \mid \varphi \in w\}$. It is clear that $\mathcal{F}^{\mathcal{C}} = (W^{\mathcal{C}}, N^{\mathcal{C}})$ is a neighborhood frame and that $\mathcal{M}^{\mathcal{C}}$ is a model based on the frame.

Lemma 12 (TL)**.** *For any formula $\varphi \in$ **E**, and $w \in W^{\mathcal{C}}$, $\mathcal{M}^{\mathcal{C}}, w \Vdash \varphi \iff \varphi \in w$.*

Proof. The truth lemma is shown by induction on the complexity of φ.

For $\varphi = p$ for $p \in At$, it follows from the definition of $V^{\mathcal{C}}$. The cases for $\neg, \wedge$ are the same as those in Lemma 8.

We show the case for $\varphi = \Box\psi$ for some $\psi \in Fm$:

$\mathcal{M}^{\mathcal{C}}, w \Vdash \Box\psi \iff [\![\varphi]\!]_{\mathcal{M}^{\mathcal{C}}} \in N^{\mathcal{C}}(w) \iff \widehat{\varphi} \in N^{\mathcal{C}}(w) \iff \Box\varphi \in w$.

$\Box$

It follows that of φ is not a theorem of **E**, then there is some $w \in W^{\mathcal{C}}$, such that $\varphi \notin w$, and so $\mathcal{M}^{\mathcal{C}}, w \nVdash \varphi$. So there is some neighborhood model, in which φ does not hold.

$\Box$

2. Older Approaches

In this chapter we discuss some prominent approaches from the literature that provide semantics for hyperintensional modal logic. These approaches share the assumption that modal operators express properties of entities that are more *finely-grained* than propositions (sets of possible worlds) or even entities that are not propositions at all. As a consequence of the fine-grained nature of these entities, the equivalence rule (RE) may fail in these frameworks - two fine-grained entities may have different properties, but still share the exact same set of worlds in which they hold.

The approaches, however, differ on the question of *how* to deal with this task. It seems that we could split these approaches from the literature into three major groups: *state-based, structuralist*, and *syntactic*. State-based, as the name suggests, deal with the problem of hyperintensional contexts in terms of additional states(worlds). They tend to resemble relational semantics, but with the addition of *non-normal* or *impossible* worlds. These are states, in which no compositionality or logical laws hold. A good overview of the motivations and logics working within this framework is Berto and Jago [2022]. Structuralist approaches assume there is not only the proposition of a formula, but also some kind of 'structure'. Therefore, for a formula φ there is not only the *intension of* φ, but also the *meaning of* φ, which are two separate entities. We discuss both of these approaches in greater detail in this chapter. We outline the state-based approach proposed in Wansing [1990][1] , and the structuralist approach in Cresswell [1975], and show that both of these approaches produce models that are complete with respect to the smallest hyperintensional logic $\Sigma(Taut)$.

The last class of approaches which we do not discuss in detail here, is the syntactic approach. On the syntactic approach, modalities are not taken to explicitly express properties of semantic contents in sentences, rather they refer to the sentences themselves. This is the approach of, for example, **?** and their framework of logic of awareness. In this approach, agent may know some propositions without knowing all their logical consequences, because the agent simply does not have to be aware of them. So it might hold, supposing that all mathematical truths are equivalent, that an agent knows that $2+3=5$, but not know that 3301 is a prime number, because he lacks the concept of a prime or never thought of the number 3301. Another syntactic approach is justification logic due to Artemov [2008]. On this approach, agent would have to not only *know* that 3301 is a prime , but also justify it to himself.

Nevertheless, the relationship between syntactic and state-based approaches is well-understood, for example, Wansing [1990] shows that some syntactic approaches can be embedded into his framework. Therefore, we decided not to study them in a greater detail in the thesis.

[1]Another interesting state-based approach, popular especially in the knowledge representation literature, was introduced by Levesque [1984]. It is a logic of implicit and explicit belief, which splits agent's beliefs into those he is consciously aware of those that are consequences of the held beliefs, though the agent might not hold them explicitly. Wansing [1990] shows that the approach can be translated into his approach.

2.1 State-based Approaches

State-based approaches, as the name suggests, deal with the problem of hyperintensional contexts in terms of states. To be exact, they deal with the problem by introducing *non-normal* or sometimes also called *impossible* worlds. The idea is that in these worlds, not all logical rules have to apply, as it is in the case of normal worlds. By allowing such worlds to exist, they manage to evade the problem created by the equivalence rule, which does not hold in these logics anymore.

On the other hand, a philosophical problem of the existence and nature of such worlds arises. As they essentially defy logic, they diverge from the intuition of the relational semantics that formulas that hold at a given world/state $w \in W$ represent the facts true in the said world/represent a consistent state of knowledge (information state) of some agent. It is not clear how a world, where say, $p \wedge q$ holds, but neither does p nor q, should look.

Rantala [1982] proposed to deal with the problem by restricting the set of formulas Ω to which the necessitation rule Nec applies. In this way, the agent's logical omniscience is restricted to the set $\Omega \subseteq Fm$, thus avoiding the aforementioned problems. By varying the set Ω, one can get a range of sublogics with various properties. Semantically, the models are 4-tuples $\mathcal{M} = \langle W, W^*, R, V \rangle$, extending the relational models with the set W^* of non-normal worlds. R is defined on the union of normal and non-normal worlds, and V is defined so that not much except *modus ponens* has to hold in the valuations of non-normal worlds.

The ideas proposed in the mentioned Rantala's article have been generalized by H. Wansing in Wansing [1990]. In the article, Wansing shows a translation of Rantala's non-normal worlds semantics into his generalized semantics, so we will introduce only the generalized semantics here. As it is a generalized version of Rantala's approach, the models are named after him.

2.1.1 Rantala Models

Definition 14 (Rantala Model)**.** *A Rantala model* $\mathcal{M} = \langle W, W^*, R, V \rangle$[2] *is obtained from a relational model in such a way, that* W *is non-empty set,* W^* *is any set,* R *is a binary relation on* $W \cup W^*$*, and* V[3] *is a function* $V : Fm \to (W \cup W^*)$ *such that for* $\forall w \in W$*:*

- $w \in V(\neg\varphi) \iff w \notin V(\varphi)$,
- $w \in V(\varphi \wedge \psi) \iff w \in V(\varphi)$ *and* $w \in V(\psi)$,
- $w \in V(\Box\varphi) \iff (\forall u \in W \cup W^*)(Rwu \to u \in V(\varphi))$.

An $\mathcal{L}_0$*-formula* φ *is true in* $\mathcal{M}$ *at* $w \in (W \cup W^*)$ $(\mathcal{M}, w \Vdash \varphi)$ *iff* $w \in V(\varphi)$. φ *is valid in* $\mathcal{M}$ $(\mathcal{M} \Vdash \varphi)$ *iff* $\forall w \in W, w \in V(\varphi)$. φ *is valid iff for every model* $\mathcal{M}, \mathcal{M} \Vdash \varphi$.

[2] Wansing formulates the models for multi-modal propositional languages. For the sake of simplicity, we reduce the models to our basic modal language with a single modality.

[3] To be exact, Wansing formulates the function V as $V : Fm \times (W \cup W^*) \to \{0, 1\}$, where $V(\varphi, w) = 1$ in his definition, when $w \in V(\varphi)$ in our definition. We make this change to simplify our work with Rantala models later.

W is thought of as a set of *normal* worlds, and W^* as the set of *non-normal* worlds. In the non-normal worlds, there is no compositional structure imposed on the evaluation of formulas. Moreover, note that for a formula φ to be valid in a model $\mathcal{M}$, it is enough that φ holds only in all the normal worlds.

Claim 13. *The logic of Ranatala models is not closed under* (RE)*.*

Proof. Consider the following model $\mathcal{M} = (W, W^*, R, V')$, such that $W = \{a\}$ $W^* = \{b, c\}$, $R = \{(a, b), (a, c)\}$, and V is defined so that $V(\varphi) = \{a, b, c\}$, $V(\psi) = \{a, b\}$.

It is easy to see that $\mathcal{M} \Vdash \varphi$ iff $\mathcal{M} \Vdash \psi$ because they hold in the same normal worlds (i.e. the world a). However $V(\Box\varphi) = \{a\}$ because Rab and Rac, and in both b, c φ holds, but $a \notin V(\Box\psi)$, since $V(\psi) = \{a, b\}$, and so $\exists u \in W \cup W^*$ s.t. Raw' and $u \notin V(\psi)$, and that is the state $c \in W^*$.

□

Wansing proposed this semantics with the goal to serve as the basic, general, and possibly unifying framework for epistemic logic. That it is basic follows from the following theorem:

Theorem 14. $\Sigma(Taut)$ *is the logic of all Rantala models.*[4]

Proof. Let us denote the logic of all Rantala models $Log(Ra)$. We want to show that $\Sigma(Taut) = Log(Ra)$

$(\implies)$ *Soundness.* We will show that if $\varphi \in \Sigma(Taut)$ then $\varphi \in Log(Ra)$. We show this by contraposition, that is, if there is some Rantala model $\mathcal{M} = (W, W^*, R, V)$ such that $\mathcal{M} \nVdash \varphi$ then $\varphi \notin \Sigma(Taut)$. $\mathcal{M} \nVdash \varphi \iff \exists w \in W, w \notin V(\varphi)$. We want to find an extended valuation $e : At \cup \{\Box\varphi \mid \varphi \in Fm\} \to \{0, 1\}$ so that $e \nvDash \varphi$. We do so simply by setting

$$e(\psi) = (1) \iff w \in V(\psi)$$

for the world $w \in W$ in $\mathcal{M}$, and for $\psi \in At \cup \{\Box\psi \mid \psi \in Fm\}$. We define satisfaction by e as usual. We need to verify that $e \vDash \varphi \iff w \in V(\varphi)$. However, similarly as in the proof of Theorem 4 this follows straight from the definition of e for atomic formulas and formulas in $\psi \in At \cup \{\Box\psi \mid \psi \in Fm\}$, and for the rest of the formulas it follows from the definition of satisfaction by extended valuation e. So e is an extended valuation such that $\varphi \notin V(\varphi)$ at w, then $e \nvDash \varphi$. By Theorem 4, $\varphi \notin \Sigma(Taut)$.

$(\impliedby)$ *Completeness.* The converse direction will be shown again through a construction of the canonical model. Let $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, {W^*}^{\mathcal{C}}, R^{\mathcal{C}}, V^{\mathcal{C}})$ be a model such that:

- $W^{\mathcal{C}}$ is a set of all maximal $\Sigma(Taut)$-consistent theories,
- ${W^*}^{\mathcal{C}}$ is $\mathcal{P}(Fm) \setminus W^{\mathcal{C}}$ (set of sets of all formulas in $\mathcal{L}_0$ without the maximal $\Sigma(Taut)$-consistent theories),

[4] Wansing states a similar claim in his paper, and that the 'The classical propositional logic is VAL-determined by the class of all Rantala frames,' where VAL is the class of all valuations according to Rantala models. For the whole definition see Wansing [1990] (p. 526).

- $R^{\mathcal{C}}$ is a relation on $W \cup W^{*^{\mathcal{C}}}$ such that $Rwu \iff (\forall\varphi)\Box\varphi \in w \rightarrow \varphi \in u$,
- $V^{\mathcal{C}}$ is a valuation function $Fm \rightarrow W \cup W^{*^{\mathcal{C}}}$, such that $V(\varphi) = \widehat{\varphi}$, where $\widehat{\varphi} = \{w \in W \cup W^{*^{\mathcal{C}}} \mid \varphi \in w\}$.

To clarify, we want $W \cup W^{*^{\mathcal{C}}}$ to be the powerset of the set of formulas.

We need to verify that $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, W^{*^{\mathcal{C}}}, R^{\mathcal{C}}, V^{\mathcal{C}})$ is indeed a Rantala model: Obviously, $W^{\mathcal{C}} \neq \emptyset$, $W^{*^{\mathcal{C}}}$ is a set, and $R^{\mathcal{C}}$ is a binary relation on $W^{\mathcal{C}} \cup W^{*^{\mathcal{C}}}$.

$V^{\mathcal{C}}$ is defined as it should be: it is clearly a function from Fm to $W^{\mathcal{C}} \cup W^{*^{\mathcal{C}}}$. We just need to show that the properties we expect from the valuation function hold for every $w \in W^{\mathcal{C}}$:

$w \in V^{\mathcal{C}}(\neg\varphi) \iff w \in \widehat{\neg\varphi} \iff \neg\varphi \in w \iff \varphi \notin w \iff w \notin \widehat{\varphi} \iff w \notin V^{\mathcal{C}}(\varphi)$,

$w \in V^{\mathcal{C}}(\varphi \wedge \psi) \iff w \in \widehat{\varphi \wedge \psi} \iff w \in \widehat{\varphi}$ and $w \in \widehat{\psi} \iff w \in V^{\mathcal{C}}(\varphi)$ and $w \in V^{\mathcal{C}}(\psi)$,

For the last case, we show both implications separately:

$w \in V^{\mathcal{C}}(\Box\varphi) \implies \forall u \in W^{\mathcal{C}} \cup W^{*^{\mathcal{C}}}(Rwu \rightarrow u \in V^{\mathcal{C}}(\varphi))$ follows from the definition. (Because $Rwu \iff \Box\varphi \in w \rightarrow \varphi \in u$ for $u \in W^{\mathcal{C}} \cup W^{*^{\mathcal{C}}}$, and $\Box\varphi \in w$.)

In the converse direction, we proceed by contraposition: $w \notin V^{\mathcal{C}}(\Box\varphi) \implies \exists u(Rwu \wedge u \in V^{\mathcal{C}}(\varphi))$. $w \notin V^{\mathcal{C}}(\Box\varphi) \iff \Box\varphi \notin w$. We need to find $u \in W^{\mathcal{C}} \cup W^{*^{\mathcal{C}}}$, such that Rwu and $\varphi \notin u$. It follows that such u exists, as it is enough to consider the set $u_0 = \{\psi \mid \Box\psi \in w\}$. Obviously, u_0 exists and $u_0 \in W^{*^{\mathcal{C}}}$, as $W^{*^{\mathcal{C}}}$ contains all sets of formulas (except those in $W^{\mathcal{C}}$). Rwu_0 follows from the definition of R and $\varphi \notin u_0$ is clear from the definition of u_0.

It follows that $V^{\mathcal{C}}$ is well-defined.

Lemma 15 (TL)**.** *For all $\varphi \in Fm$ and $w \in W^{\mathcal{C}}$, $\mathcal{M}^{\mathcal{C}}, w \Vdash \varphi \iff \varphi \in w$.*

Proof. The proof is immediate from the definition of Rantala models (and the properties of $V^{\mathcal{C}}$).

□

And so we have shown that if $\varphi \notin \Sigma(Taut)$, then $\varphi \notin Log(Ra)$ by constructing a model, in which φ does not hold.

□

In the rest of his paper, Wansing shows that some approaches which do not deal with non-possible worlds can also be translated into special cases of Rantala models, and he shows this for the Levesque's logic of implicit and explicit belief, Fagin and Halpern's logic of awareness, the logic of general awareness, and for the logic of local reasoning. Obviously, also original Rantala's Impossible worlds semantics is a special case. By these transformations, he hopes to show that his framework is unifying (of the alternative approaches). That it is general ought to mean that various classes of the modal logic can be embedded into the framework. Normally (in other semantical approaches), this would be done by restricting the class of frames (models) in some way. For Rantala Models, Wansing achieves this by defining classes of valuations, and limiting the acceptable valuations to obtain classes of logics ($\Sigma(Taut)$, minimal classical logic, minimal normal logic, and KD45).

2.2 Structuralist Approaches

In the structuralist approach, the semantic content reflects the structure of sentences, though it is not considered syntactical in nature. At the same time, it is not understood as a set of states, therefore it is not the intension of a sentence, but it does determine it.

The structuralist approach evades the hyperintensional paradox by allowing multiple structured contents to determine the same intension, while the contents themselves can be distinct from each other. What the structured content itself is, however, may differ depending on the theory.

M. Cresswell in Cresswell [1975] modifies D. Lewis's account postulated in Lewis [1970] to accommodate hyperintensional functors. The main idea is that intensions and meanings are two different entities, and while intensions *determine* sets of possible worlds, meanings capture the structure of the sentence - and they are composed of meanings of its parts. This results in a semantics where tautologies, though all with the same intension, have different meanings. Into this system proposed by Lewis, Cresswell introduces an operator θ that allows reasoning about hyperintensional contexts. We will first explain a (simplified) version of Cresswell's account here, and then we will consider a restricted version to include only what's needed for accomodating propositional modal logics.

2.2.1 Original Hyperintensional Semantics by Cresswell

Let Syn be the smallest set such that $\mathbb{N} \subseteq Syn$ and if $\tau, \sigma_1, ..., \sigma_n \in Syn$, then $\langle \tau, \sigma_1, ..., \sigma_n \rangle \in Syn$. *Syn* will be the set of syntactic types (or categories). Cresswell mentions that it seems to him that only two basic types are needed - 0 for sentences and 1 for names. As in this work, we are interested in *propositional* modal logic, the only basic type we will be interested in are 0 - sentences. We will also want to accommodate only a limited number of logical operators (the operators from $\mathcal{L}_0$), so it will suffice to consider only the syntactic the types $\{0, \langle 0, 0\rangle, \langle 0, 0, 0\rangle\} \in Syn$ and their combinations. Type 0 will correspond to formulas, $\langle 0, 0\rangle$ to functions from formulas to formulas, so to negation and the modal box [5], and finally $\langle 0, 0, 0\rangle$ will correspond to a conjunction.

A θ-categorial language $\mathcal{L}$ is a pair $\langle F, E\rangle$, where F is a function from *Syn* to finite sets such that if $\sigma \neq \tau$ then $F_\sigma \cap F_\tau = \emptyset$. $F_\sigma \subseteq E_\sigma$ for every syntactic type σ. For E_σ holds the following: $E_\sigma = F_\sigma \cup E^f_\sigma \cup E^\theta_\sigma$. E^f_σ will consist of functorial expressions and E^θ_σ of θ-expressions. Both E^f_σ and E^θ_σ are defined by the following two conditions: If $\delta \in E_{\langle \tau, \sigma_1, ..., \sigma_n\rangle}$ and $\alpha_1 \in E_{\sigma_1}, ..., \alpha_n \in E_{\sigma_n}$, then $\langle \delta, \alpha_1, ..., \alpha_n\rangle \in E^f_\sigma$. And if $\alpha \in E^f_\sigma$ then $\langle \theta, \alpha\rangle \in E^\theta_\sigma$. In our case, this would mean that F_0 contains atomic propositions, $F_{\langle 0,0\rangle}$ contains negation and the modal box, and $F_{\langle 0,0,0\rangle}$ contains a conjunction. E_0 is a set of all well-formed formulas. However, for the same reason as to why we limited the number of syntactic types, we limit the $\neg, \wedge$ to always occur only in functorial expressions, and $\Box$ on the other hand to always occur in θ-expressions.

[5] One could argue that some compositional types, such as $\langle\langle 0, 0\rangle, 0\rangle$, which eventually reduce to $\langle 0, 0\rangle$ should be considered here as well. In $\mathcal{L}_0$, such a function would be a conjunction with one place already occupied. We decide to omit this interpretation, as then it would be necessary to have a separate function for each formula (i.e $\varphi \wedge \_$ for every φ). Therefore, we consider a conjunction to always take both arguments at the same time. We do so to preserve simplicity.

This deserves further elaboration. Cresswell [1975] differentiates two types of expressions - functorial expressions and θ-expressions. According to the formation rules, the operator θ can be written in front of essentially any functorial expression to form a θ-expression. θ alters how the expression at hand is interpreted. However, in the logic we are trying to model, there is no need to look for an interpretation of $\neg$ and $\wedge$ beyond classical truth-functional semantics, though in the case of $\Box$, for the reasons introduced in the previous parts, we require a hyperintensional interpretation.

Cresswell himself acknowledges[6] the possibility to take a different approach to the problem, by differentiating all functors (and expressions) into intensional and hyperintensional. So instead of $\neg, \wedge, \Box$, we would have $\neg^f, \wedge^f, \Box^f$ and $\neg^\theta, \wedge^\theta, \Box^\theta$. We take inspiration from this suggestion, but again in order to simplify and accomodate only the extent necessary for our work, we limit ourselves to the following operators: $\neg^f, \wedge^f, \Box^\theta$. In the course of the work we will omit the superscripts, and we will consider $\neg, \wedge$ to be always intensional, and $\Box$ to be hyperintensional. This is in accord with our intuition that $\Box$ is the operator introducing hyperintensional contexts into the modal logic.

Now, to define a system $\mathcal{A}$ of admissible meanings, we need first to define the system of intensional domains $\mathcal{D}$. $\mathcal{D}$ is a function from Syn, such that $\mathcal{D}_0$ consists of sets of possible worlds from some fixed set W ($\mathcal{D}_0 \subseteq \mathcal{P}(W)$)[7], and for every $\sigma = \langle \tau, \sigma_1, ..., \sigma_n \rangle$, $\mathcal{D}_\sigma$ is a set of partial functions from $\mathcal{D}_{\sigma_1} \times ... \times \mathcal{D}_{\sigma_n} \rightarrow \mathcal{D}_\tau$. Every $\mathcal{D}_\sigma \subseteq \mathcal{A}_\sigma$. Similarly to the set E, we have that if $\sigma = \langle \tau, \sigma_1, ..., \sigma_n \rangle$, and $d \in \mathcal{A}_{\langle \tau, \sigma_1, ..., \sigma_n \rangle}$ and $a_1 \in \mathcal{A}_{\sigma_1}, ..., a_n \in \mathcal{A}_{\sigma_n}$, then $\langle d, a_1, ..., a_n \rangle \in \mathcal{A}_\tau$. So $\mathcal{D}_0$ are sets of possible worlds, $\mathcal{D}_{\langle 0,0 \rangle}$ is a subset of functions from possible worlds into possible worlds ($\mathcal{P}(W) \rightarrow \mathcal{P}(W)$), and $\mathcal{D}_{\langle 0,0,0 \rangle}$ is a subset of $\mathcal{D}_0 \times \mathcal{D}_0 \rightarrow \mathcal{D}_0$. Elements of $\mathcal{A}_0$ are admissible meanings, that is, all entities contained in $\mathcal{D}_0$, but also the tuples reflecting the internal structure of all expressions in the language. An example will be provided after defining the functions of intension and meaning.

A value assignment is the following: if σ is a *basic* category and $\alpha \in F_\sigma$, then $V(\alpha) \in \mathcal{D}_\sigma$. In our case this will just mean that if $\sigma = 0$ and $\alpha \in F_0$ (so α is an atomic proposition), then $V(\alpha) \in \mathcal{D}_0$ (the valuation of α is a set of possible worlds). If σ is a composite, i.e $\sigma = \langle \tau, \sigma_1, ..., \sigma_n \rangle$ and $\delta \in F_\sigma$, then $V(\delta)$ is a partial function $V(\delta) : \mathcal{A}_{\sigma_1} \times ... \times \mathcal{A}_{\sigma_n} \rightarrow D_\tau$. However, if δ is an ordinary intensional functor, then the function $V(\delta)$ is just $V(\delta) : \mathcal{D}_{\sigma_1} \times ... \times \mathcal{D}_{\sigma_n} \rightarrow D_\tau$. Presumably, for our case, the only time when $V(\delta)$ will be taken from the domain of all admissible meanings is the case of the hyperintensional functor $\Box$. So $V(\Box) : \mathcal{A}_0 \rightarrow \mathcal{D}_0$. For now, we do not specify which properties we want to impose on $V(\Box)$. On the other hand, we will expect $V(\neg) : \mathcal{D}_0 \rightarrow \mathcal{D}_0$ to behave as negation (i.e., assigns to a set of possible worlds its complement), and $V(\wedge) : \mathcal{D}_0 \times \mathcal{D}_0 \rightarrow \mathcal{D}_0$ to behave as conjunction (for every two propositions, it assigns them their intersection).

Finally, we now define how intension $I(\alpha)$ and meaning $M(\alpha)$ are characterized for every expression α. Firstly, if $\alpha \in F_\sigma$, then $I(\alpha) = M(\alpha) = V(\alpha)$. This

[6]Cresswell [1975], chapter 4, note 11: '[...] we might try to dispense with θ altogether by classifying all expressions (and not merely the symbols) into intensional and hyperintensional, and suppose that any expression whose functor is hyperintensional is evaluated as if it were a θ-expression.'

[7]$\mathcal{D}_1$ would be a set of entities or things.

is the case for atomic propositions, negation, conjunction, and the modal box. If $\alpha \notin F_\sigma$, then it must be either in E^f_σ or E^θ_σ (The 'outermost' operator is either intensional or θ/hyperintensional). If $\alpha \in E^f_\sigma$, then $\alpha = \langle \delta, \alpha_1, ..., \alpha_n \rangle$ for some $\delta, \alpha_1, ..., \alpha_n$. Then we have

$$I(\alpha) = I(\delta)(I(\alpha_1), ..., I(\alpha_n)),$$

$$M(\alpha) = \langle M(\delta), M(\alpha_1), ..., M(\alpha_n) \rangle.$$

If $\alpha \in E^\theta_\sigma$, then α has the form $\langle \theta, \beta \rangle$ for some functorial expression $\beta = \langle \delta, \beta_1, ..., \beta_n \rangle$. In this case:

$$I(\alpha) = I(\delta)(M(\alpha_1), ..., M(\alpha_n)),$$

$$M(\alpha) = I(\alpha).$$

By postulating $M(\alpha) = I(\alpha)$ it is ensured that the meaning of a θ-expression is always admissible. [8]

The intension of a functorial expression is an intension applied to intensions; while meaning of a functorial expression is a tuple of objects. In the case of a θ-expression, the meaning is identical to the intension.

Example. Let's say we're interested in meaning and intension of a formula $\neg\varphi \wedge \psi$. This is a functorial expression. Written in Cresswell's notation, this would be $\langle \wedge, \langle \neg, \varphi \rangle, \psi \rangle$.

We can check that this is indeed a formula, that is $\langle \wedge, \langle \neg, \varphi \rangle, \psi \rangle \in E_0$. According to the rules we introduced, $\neg \in E_{\langle 0,0 \rangle}$ and $\varphi \in E_0$, so $\langle \neg, \varphi \rangle \in E_0$. A similiar argument applies to the whole expression - $\wedge \in E_{\langle 0,0,0 \rangle}$ and $\langle \neg, \varphi \rangle \in E_0$ and $\psi \in E_0$, so it holds that $\langle \wedge, \langle \neg, \varphi \rangle, \psi \rangle \in E_0$. Therefore $\neg\varphi \wedge \psi$ is a formula that determines a set of possible worlds.

Now to find the intension of said formula:

$$\begin{aligned} I(\neg\varphi \wedge \psi) &= I(\wedge)(I(\neg\varphi), I(\psi)) \\ &= V(\wedge)((I(\neg)I(\varphi)), I(\psi)) \\ &= V(\wedge)((V(\neg)I(\varphi))I(\psi)). \end{aligned}$$

On the other hand, meaning is:

$$M(\neg\varphi \wedge \psi) = \langle M(\wedge), M(\neg\varphi), M(\psi) \rangle.$$

If we consider an equivalent logical formula $\psi \wedge \neg\varphi$ by similiar manipulation we obtain

$$I(\psi \wedge \neg\varphi) = V(\wedge)(I(\psi), V(\neg)I(\varphi))$$

and

$$M(\psi \wedge \neg\varphi) = \langle M(\wedge), M(\psi), M(\neg\varphi) \rangle.$$

[8] Cresswell gives a rather thorough discussion on the motivation for postulating that $M(\alpha) = I(\alpha)$ for θ-expressions α. We won't repeat his arguments here, but the main motivation is that an alternative approach (postulating that the meaning of a hyperintensional expression is just a tuple of meanings of its parts) becomes problematic in the context of nested hyperintensional modalities. Since to reject the iteration or nesting of modalities is undesirable, Cresswell chooses the presented version of the rule.

Since the valuation $V(\wedge)$ is a function, (and we have it defined in a way that it acts like a conjunction), it is perfectly feasible to have $I(\neg\varphi \wedge \psi) = I(\psi \wedge \neg\varphi)$ as we would expect. On the other hand, in the case of meanings, no matter the valuation of $\wedge$, $\neg\varphi \wedge \psi$ and $\psi \wedge \neg\varphi$ will not be equivalent, as the meaning of the expression is just a tuple of objects reflecting its syntactic structure.

Example. A different situation ensues when we consider a formula that contains a hyperintensional functor $\Box$. Consider a hyperintensional formula $\Box(\neg\varphi \wedge \psi)$, based on the functorial expression from the previous example

$$\begin{aligned} M(\Box(\neg\varphi \wedge \psi)) &= I(\Box(\neg\varphi \wedge \psi)) \\ &= I(\Box)M(\neg\varphi \wedge \psi) \\ &= I(\Box)\langle M(\wedge), M(\neg\varphi), M(\psi)\rangle. \end{aligned}$$

Theorem 16. *In the Cresswell semantics, (RE) fails for some* $\xi, \zeta \in Fm$.

Proof. Consider the formulas $\xi := \neg\varphi \wedge \psi$ and $\zeta := \psi \wedge \neg\varphi$ from the previous examples. It is clear that $\xi \leftrightarrow \zeta$, because they have the same intension. But if we calculate the intensions of $\Box\xi$ and $\Box\zeta$, we get that:

$$I(\Box(\neg\varphi \wedge \psi)) = I(\Box)\langle M(\wedge), M(\neg\varphi), M(\psi)\rangle$$

and

$$I(\Box(\psi \wedge \neg\varphi)) = I(\Box)\langle M(\wedge), M(\psi), M(\neg\varphi)\rangle.$$

However, $\langle M(\wedge), M(\neg\varphi), M(\psi)\rangle \neq \langle M(\wedge), M(\psi), M(\neg\varphi)\rangle$. These are two different admissible meanings that may get assigned different intensions in $\mathcal{D}_0$ (depending on the intension of the modal box).

□

Interestingly enough, note that the set of valid formulas is closed under:

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}\ (Ep)$$

for $p, q \in At$, and:

$$\frac{\Box\varphi \leftrightarrow \Box\psi}{\Box^2\varphi \leftrightarrow \Box^2\psi}\ (EE).$$

for arbitrary formulas φ, ψ. This is easily seen as

$I(\Box p) = I(\Box)M(p) = I(\Box)I(p) = I(\Box)I(q) = I(\Box)M(q) = I(\Box q)$, and

$I(\Box\Box\varphi)) = I(\Box)M(\Box\varphi) = I(\Box)M(\Box\psi) = I(\Box\Box\psi)$.

In what follows in this section, we show that the fragment of Cresswell semantics we specified so far can be equivalently described in a form of what we will call a 'Cresswell model', and that the logic of all such models is $\Sigma(Taut)$.

2.2.2 Cresswell Models

The semantics described in the original paper by Cresswell is very different from the other approaches we are working with. This is because Cresswell is offering a formal language, that ought to be suitable for modeling natural languages, which requires a different kind of tools than a logical system would. Nevertheless, it is a very broad system, and in any way it does not mean it cannot be used to model logics[9].

In what follows, we transform a fragment of the Cresswell semantics with limited syntactic categories we specified in this section to Cresswell models. It is easily seen that Cresswell models 'correspond' to the segment of Cresswell semantics.

Definition 15 (Cresswell Model). *A Cresswell model is a 5-tuple* $\mathcal{M} = (W, D, A, N, v)$*, such that:*

- $W \neq \emptyset$,
- $D \subseteq \mathcal{P}(W)$ *such that* $W \in D$ *and* D *is closed under complement and (finite) intersection,*
- A *is the smallest set such that* $D \subseteq A$ *and* $a_1, a_2 \in A \implies \langle \neg^D, a_1 \rangle \in A$ *and* $\langle \wedge^D, a_1, a_2 \rangle \in A$*, where:*
  $\neg^D : D \to D$*, such that* $\neg^D(X) = W \backslash X$,
  $\wedge^D : D \times D \to D$*, such that* $\wedge^D(X, Y) = X \cap Y$,
- N *is a function* $N : A \to D$,
- v *is a function* $v : At \to D$.

W is as usual a set of possible worlds, D and A intuitively correspond to $\mathcal{D}_0$ and $\mathcal{A}_0$ introduced earlier, N to the intension of the $\Box$, and v is a valuation function that assigns atomic formulas their intentsions (the sets of possible worlds where they are true).

Given a model $\mathcal{M}$, we define the functions of intension (I) and meaning (M) as expected:

- $I(\neg) = M(\neg) = \neg^D$,
- $I(\wedge) = M(\wedge) = \wedge^D$,
- $I(\Box) = M(\Box) = N$,
- $I(p) = M(p) = V(p)$ for all $p \in At$,
- $I(\neg\varphi) = I(\neg)(I(\varphi))$ and $M(\neg\varphi) = \langle M(\neg), M(\varphi) \rangle$,
- $I(\varphi \wedge \psi) = I(\wedge)(I(\varphi), I(\psi))$ and $M(\varphi \wedge \psi) = \langle M(\wedge), M(\varphi), M(\psi) \rangle$,
- $I(\Box\varphi) = I(\Box)M(\varphi) = M(\Box\varphi)$.

Definition 16 (Validity). *An* $\mathcal{L}_0$*-formula* φ *is true in* $\mathcal{M}$ *at* $w \in W$ $(\mathcal{M}, w \Vdash \varphi)$ *iff* $w \in I(\varphi)$*.* φ *is valid in* $\mathcal{M}$ $(\mathcal{M} \Vdash \varphi)$ *iff* $I(\varphi) = W$*.* φ *is valid iff for every model* $\mathcal{M}$*,* $\mathcal{M} \Vdash \varphi$*.*

[9]Cresswell [1975](p.35) 'Obviously it would be possible to introduce, e.g., truth functors and modal operators, and provide an axiom system and completeness proof. Such a task may even shed illumination on the nature of θ-categorial languages. It is however a task that will be left to others.')

Claim 17. *Functions I and M are well-defined for all $\varphi \in Fm$.*

Proof. We show that $I(\varphi) \in D$ and $M(\varphi) \in A$ for every $\varphi \in Fm$. We do so by induction on the complexity of formulas:

If $\varphi = p$ for some atomic proposition p, $I(p) = M(p) = V(p) \in D$.

If $\varphi = \neg\psi$ for some formula ψ for which the induction hypothesis holds, then $I(\varphi) = I(\neg\psi) = I(\neg)I(\psi)$, but we assumed that $I(\psi) \in D$ and $I(\neg)$ is a function from D to D, therefore also $I(\neg)I(\psi) \in D$. $M(\varphi) = M(\neg\psi) = \langle M(\neg), M(\psi)\rangle$ from the definition of A in Cresswell models, since $M(\neg) = \neg^D$ and $M(\psi) \in A$, then it also must be the case that $\langle M(\neg), M(\psi)\rangle \in A$.

The argument is analogous for the case where $\varphi = \psi \wedge \xi$.

If $\varphi = \Box\psi$ for some formula ψ for which the induction hypothesis holds, then: $I(\varphi) = I(\Box\psi) = M(\Box\psi) = I(\Box)(M(\psi))$. We know that $M(\psi) \in A$ from the induction hypothesis, so $I(\Box)(M(\psi)) \in D$. Since $D \subseteq A$, we also have that $M(\Box\psi) \in A$.

□

Now we will show that the logic of all such models (so of the fragment of Cresswell semantics) is $\Sigma(Taut)$.

Theorem 18. $\Sigma(Taut)$ *is the logic of all Cresswell models.*

Proof. Let us denote the logic of all Cresswell models *Log*(Cr). We need to show that $\Sigma(Taut) = Log(Cr)$.

($\Longrightarrow$) *Soundness.* First, we show that $\Sigma(Taut) \subseteq Log(Cr)$. We show by contraposition that if $I(\varphi) \neq W$ in some $\mathcal{M}$, then $\varphi \notin \Sigma(Taut)$, because in Cresswell models $\mathcal{M} \nVdash \varphi \iff I(\varphi) \neq W$. Suppose that $I(\varphi) \neq W$ in some $\mathcal{M} = (W, D, A, N, v)$, and define extended valuation

$$e(\psi) = 1 \iff I(\psi) = W$$

for $\psi \in At \cup \{\Box\psi \mid \psi \in Fm\}$. We define the satisfaction under e as usual. Similarly to the argument in Theorem 4, we need to verify that $e \vDash \varphi \iff I(\varphi) = W$. The equivalence holds for atomic formulas and for formulas in $\{\Box\psi \mid \psi \in Fm\}$ directly from the definition, for the rest of the formulas from the definition of satisfaction by extended valuation. It follows that $I(\varphi) \neq W$ implies $e(\varphi) \nvDash \varphi$. Therefore, we found an extended valuation e, such that $e \nvDash \varphi$, which by the Theorem 4 means that $\varphi \notin \Sigma(Taut)$.

($\Longleftarrow$) *Completeness.* The converse direction is as usually shown through a canonical model construction. We show that if $\varphi \notin \Sigma(Taut)$ then there is some model $\mathcal{M}$, namely the canonical model $\mathcal{M}^{\mathcal{C}}$, in which φ does not hold.

Let $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, D^{\mathcal{C}}, A^{\mathcal{C}}, N^{\mathcal{C}}, v^{\mathcal{C}})$ such that:

- $W^{\mathcal{C}}$ is a set of all maximal $\Sigma(Taut)$-consistent theories
- $D^{\mathcal{C}} = \{\widehat{\varphi} \mid \varphi \in Fm\}$, where $\widehat{\varphi} := \{w \in W^{\mathcal{C}} \mid \varphi \in w\}$,
- $A^{\mathcal{C}}$ is defined on the basis of $D^{\mathcal{C}}$ as in the definition of a Cresswell model,
- the definition of $N^{\mathcal{C}}$ uses the function $S : Fm \to A^{\mathcal{C}}$ defined as follows:

1. $S(p) = \widehat{p}$,
2. $S(\neg\varphi) = \langle \neg^{D^{\mathcal{C}}}, S(\varphi)\rangle$,
3. $S(\varphi \wedge \psi) = \langle \wedge^{D^{\mathcal{C}}}, S(\varphi), S(\psi)\rangle$,
4. $S(\Box\varphi) = \widehat{\Box\varphi}$.

$N^{\mathcal{C}} : A^{\mathcal{C}} \to D^{\mathcal{C}}$ such that

$$N^{\mathcal{C}}(x) = \begin{cases} \widehat{\Box\varphi} & \text{if } \exists\varphi(x = S(\varphi)), \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

- $v^{\mathcal{C}}(p) = \widehat{p}$.

First, we verify that in $\mathcal{M}^{\mathcal{C}}$ it holds that $\varphi \in \Sigma(Taut) \iff \varphi \in w$ for all $w \in W^K$. But this is obvious by the last point in Lemma 3.

We check that the function $N^{\mathcal{C}}$ is well-defined: It is sufficient to show that for all φ and all ψ, $S(\varphi) = S(\psi)$ only if $\varphi = \psi$.

For the induction base, suppose that $S(p) = S(\psi)$ for some formula ψ. By the definition of S, $\psi = q$ for some $q \in At$ or $\psi = \Box\chi$ for some $\chi \in Fm$. In the former case, from $S(p) = S(q)$ we obtain $\widehat{p} = \widehat{q}$, which means that $p \leftrightarrow q \in \Sigma(Taut)$, and so by Theorem 4, $p = q$. In the latter case we obtain that $\widehat{p} = \widehat{\Box\chi}$, and so $p \leftrightarrow \Box\chi \in \Sigma(Taut)$, which is impossible, and therefore $S(p) = S(\Box\chi)$ will never happen.

If $S(\varphi) = S(\neg\varphi_1)$, then $S(\varphi) = S(\neg\varphi_1) = \langle \neg^{D^{\mathcal{C}}}, S(\varphi_1)\rangle$, and so $S(\varphi)$ also has to be of the form $S(\varphi) = S(\neg\varphi_0)$ for some $\varphi_0 \in Fm$. And so, $\varphi = \neg\varphi_1$.

An analogous argument would be given for $S(\varphi) = S(\varphi_1 \wedge \varphi_2)$.

If $S(\varphi) = S(\Box\varphi_1)$, then since $S(\Box\varphi_1) = \widehat{\Box\varphi_1}$, it must be the case that either $S(\varphi) = \widehat{p}$ for some $p \in At$ or $S(\varphi) = \widehat{\Box\varphi_0}$ for some $\varphi_0 \in Fm$. Since $\widehat{p} = \widehat{\Box\varphi_1}$ is impossible by the same argument as in the induction base, it follows that it must be the case that $\widehat{\Box\varphi_0} = \widehat{\Box\varphi_1}$, but that means that $\Box\varphi_0 = \Box\psi_1$.

Finally, we verify that $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, D^{\mathcal{C}}, A^{\mathcal{C}}, N^{\mathcal{C}}, v^{\mathcal{C}})$ is a Cresswell model.

$W^{\mathcal{C}} \neq \emptyset$ is obvious, and that $A^{\mathcal{C}}$, $N^{\mathcal{C}}$, and $v^{\mathcal{C}}$ are as in Cresswell models, follows directly from their definitions. We show that $D^{\mathcal{C}} \subseteq \mathcal{P}(W^{\mathcal{C}}), W^{\mathcal{C}} \in D^{\mathcal{C}}$ and that $D^{\mathcal{C}}$ is closed under a finite intersection and complement.

The elements in $D^{\mathcal{C}}$ are subsets of $W^{\mathcal{C}}$ by definition, and so $D^{\mathcal{C}} \subseteq \mathcal{P}(W^{\mathcal{C}})$ is obvious. For $W^{\mathcal{C}} \in D^{\mathcal{C}}$ to be true, it must be the case that $W^{\mathcal{C}} = \widehat{\varphi} = \{w \in W^{\mathcal{C}} \mid \varphi \in w\}$. But this is true for all $\varphi \in \Sigma(Taut)$. Finally, we need to show that $D^{\mathcal{C}}$ is closed under complement and finite intersection.

Let $\widehat{\varphi} \in D^{\mathcal{C}}$, we need to show that also $D^{\mathcal{C}} \backslash \widehat{\varphi} \in D^{\mathcal{C}}$. But $D^{\mathcal{C}} \backslash \widehat{\varphi} = \{w \in W^{\mathcal{C}} \mid \varphi \notin w\} = \{w \in W^{\mathcal{C}} \mid \neg\varphi \in w\} = \widehat{\neg\varphi}$, for $\neg\varphi \in Fm$.

Let $\widehat{\varphi_1}, ..., \widehat{\varphi_n} \in D^{\mathcal{C}}$ for some $n \in \mathbb{N}$. We show that also $\bigcap_{i \leq n} \widehat{\varphi_i} \in D^{\mathcal{C}}$. But since $\widehat{\varphi_1}, ..., \widehat{\varphi_n}$ are sets describable by formulas $\varphi_1, ..., \varphi_n$, then the intersection $\bigcap_{i \leq n} \widehat{\varphi_i}$ can be described by a formula $\bigwedge_{i \leq n} \varphi_i$. That means that $\bigcap_{i \leq n} \widehat{\varphi_i} = \{w \in W^{\mathcal{C}} \mid \bigwedge_{i \leq n} \varphi_i \in w\}$, which is in $D^{\mathcal{C}}$. So $M^{\mathcal{C}} = (W^{\mathcal{C}}, D^{\mathcal{C}}, A^{\mathcal{C}}, N^{\mathcal{C}}, v^{\mathcal{C}})$ as defined is indeed a Cresswell model. Therefore, we may define functions $I^{\mathcal{C}}$ and $M^{\mathcal{C}}$ as before.

All that is left to show to see that if $\varphi \notin \Sigma(Taut)$, then φ is not valid in $\mathcal{M}^{\mathcal{C}}$ is the truth lemma.

Lemma 19 (TL)**.** *The following hold for all $\varphi \in Fm$:*

1. $I^{\mathcal{C}}(\varphi) = \widehat{\varphi}$,
2. $M^{\mathcal{C}}(\varphi) = S(\varphi)$.

Proof. We prove both claims of the truth lemma by simultaneous induction on the complexity of φ.

If $\varphi = p$ for some $p \in At$, $I^{\mathcal{C}}(p) = M^{\mathcal{C}}(p) = v^{\mathcal{C}}(p) = \widehat{p} = S(p)$.

If $\varphi = \neg\psi$ for some ψ for which the induction hypothesis holds, then $I^{\mathcal{C}}(\varphi) = I^{\mathcal{C}}(\neg\psi) = I^{\mathcal{C}}(\neg)I^{\mathcal{C}}(\psi) = \neg^{D^{\mathcal{C}}}\widehat{\psi} = \widehat{\neg\psi}$, and $M^{\mathcal{C}}(\varphi) = M^{\mathcal{C}}(\neg\psi) = \langle \neg^{D^{\mathcal{C}}}, M^{\mathcal{C}}(\psi)\rangle = \langle \neg^{D^{\mathcal{C}}}, S(\psi)\rangle = S(\neg\psi)$.

If $\varphi = \psi \wedge \xi$ for some ψ, ξ for which the induction hypothesis holds, then $I^{\mathcal{C}}(\varphi) = I^{\mathcal{C}}(\psi \wedge \xi) = I^{\mathcal{C}}(\wedge)(I^{\mathcal{C}}(\psi), I^{\mathcal{C}}(\xi)) = \wedge^{D^{\mathcal{C}}}(\widehat{\psi}, \widehat{\xi}) = \widehat{(}\psi \wedge \xi)$, and $M^{\mathcal{C}}(\psi \wedge \xi) = \langle \wedge^{D^{\mathcal{C}}}, M^{\mathcal{C}}(\psi), M^{\mathcal{C}}(\xi)\rangle = \langle \wedge^{D^{\mathcal{C}}}, S(\psi), S(\xi)\rangle = S(\psi \wedge \xi)$.

If $\varphi = \Box\psi$ for some ψ for which the induction base holds, then simultaneously for $M^{\mathcal{C}}, I^{\mathcal{C}}$: $M^{\mathcal{C}}(\Box\psi) = I^{\mathcal{C}}(\Box\psi) = I^{\mathcal{C}}(\Box)M^{\mathcal{C}}(\psi) = I^{\mathcal{C}}(\Box)S(\psi) = N^{\mathcal{C}}(\psi) = \widehat{\Box\psi}$. □

Therefore, we showed that if $\varphi \notin \Sigma(Taut)$, then there is a Cresswell model in which $I(\varphi) \neq W$, i.e. φ is not valid. This concludes the proof. □

3. Hyperintensional Models

3.1 Definition and properties

We motivate the definition of hyperintensional models by generalizing neighborhood models. To this end, we repeat the definition of neighborhood models, however, we do slightly adjust the notation.

A neighborhood model is a tuple

$$\mathcal{M} = (W, N, [\![\cdot]\!])$$

where W is a non-empty set (a set of possible worlds), N is a function $N : W \to \mathcal{P}(\mathcal{P}(W))$ (intuitively we want to think of it as a function from the possible worlds to sets of propositions on W), and $[\![\cdot]\!]$ is a function $[\![\cdot]\!] : Fm \to \mathcal{P}(W)$. This function assigns every formula φ a proposition $[\![\varphi]\!]$ on W. We remind the reader that a proposition of a formula φ is understood as a set of worlds in which φ holds. This is in accord with the idea that every formula φ is or at least determines a set of worlds in which it is true.

Moreover, we now want to impose an additional requirement on the valuation function (in this context denoted $[\![\cdot]\!]$). We require $[\![\cdot]\!]$ to be defined in such a way, that the Boolean connectives correspond to the usual set-theoretic operations on the set of propositions:

- $[\![\neg\varphi]\!] = W \backslash [\![\varphi]\!]$,
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$,
- $[\![\Box\varphi]\!] = \{w \mid [\![\varphi]\!] \in N(w)\}$.

These models are still neighborhood models, and so they preserve the validity (RE). We will adjust this semantics in order for it to be able to accommodate hyperintensional attitudes. However, before introducing hyperintensional models, we introduce some useful algebraic notions first.

Consider a propositional language $\mathcal{P}$. $\mathcal{P}$ will correspond to the classical propositional fragment of our $\mathcal{L}_0$, that is, $\mathcal{P}$ consists of a countable set of propositional variables At, a set of connectives $Con_{\mathcal{P}} = \{\neg, \wedge\}$ (and round brackets). The modal extension of $\mathcal{P}$ is $\mathcal{L}_0$ as defined in the Chapter 1. Now, we will need the notion of $\mathcal{P}$-algebra and $\mathcal{P}$-homomorphism for $\mathcal{P}$. We give the definition for arbitrary propositional language ξ:

Given a propositional language χ, an χ-type algebra is any algebra $\mathbf{A} = (A, \{c^{\mathbf{A}} \mid c \in Con_\chi\})$, where A is the carrier of $\mathbf{A}$ and $c^{\mathbf{A}}$, and $c \in Con_\chi$ are algerbraic operations on A.

If χ, γ, υ are propositional languages such that $\chi \subseteq \gamma$, $\chi \subseteq \upsilon$, and $\mathbf{A} = (A, \{c_1^{\mathbf{A}}, c_2^{\mathbf{A}}, ...\})$ is a γ-type algebra, $\mathbf{B} = (B, \{c_1^{\mathbf{B}}, c_2^{\mathbf{B}}, ...\}$ is a υ-type algebra, a mapping

$$f : A \to B$$

is called a χ-homomorphism from $\mathbf{A}$ to $\mathbf{B}$ if for all $t_1, ..., t_n \in A$ and every $\nabla \in Con_\chi$ holds the following:

$$f(\nabla^{\mathbf{A}}(t_1, ..., t_n)) = \nabla^{\mathbf{B}}(f(t_1), ..., f(t_n)).$$

Now we have all tools required to define hyperintensional models. There are two ways in which one could view these: there is the set-theoretic definition (Sedlár [2021]) and the algebraic definition (Pascucci and Sedlár [2023]). Currently, it seems that there are some advantages and disadvantages for both approaches. We will introduce both and briefly discuss the difference.

Definition 17 ((Set-theoretic) Hyperintensional Model)**.** *A (set-theoretic) hyperintensional model is a tuple*

$$\mathcal{M} = (W, C, O, N, I),$$

where W and C are non-empty sets, and O, N, I are functions such that $O : Fm \to C$, $N : W \to \mathcal{P}(C)$, $I : C \to \mathcal{P}(W)$.

Moreover, we define a function $[\![\cdot]\!]$ by $[\![\varphi]\!]_{\mathcal{M}} = I(O(\varphi))$ so that it satisfies the semantic conditions for Boolean connectives as in the adjusted neighborhood models, and the following holds:

- $[\![\Box\varphi]\!]_{\mathcal{M}} = \{w \mid O(\varphi) \in N(w)\}$.

Definition 18 ((Algebraic) Hyperintensional Model)**.** *An (algebraic) hyperintensional model is a tuple*

$$\mathcal{M} = (W, \mathbf{C}, O, N, I),$$

where W is a non-empty set, and:

- $\mathbf{C} = (C, \{\neg, \wedge\})$ *is a $\mathcal{P}$-type algebra,*
- $O : Fm \to \mathbf{C}$ *is a $\mathcal{P}$-homomorphism,*
- $N : \mathbf{C} \to \mathbf{2^W}$ *is a generalized neighbourhood function,*
- $I : \mathbf{C} \to \mathbf{2^W}$ *is a $\mathcal{P}$-homomorphism, such that for every $\varphi \in Fm$ and every $w \in W$ the following holds:*

$$I(O(\Box\varphi)) = N(O(\varphi)).$$

$\mathbf{2^W}$ *is a powerset algebra $\mathbf{2^W} = (\mathcal{P}(W), -, \cap)$. We define a function $[\![\cdot]\!] : Fm \to \mathbf{2^W}$, so that $[\![\varphi]\!]_{\mathcal{M}} = I(O(\varphi))$ holds.*

The idea is that W is the set of the possible worlds as usual. C is the set of unspecified entities. In the context of epistemic logic, it might be useful to think of it as a set of 'semantic contents'. However, the models don't have any requirements on the specific nature of the members of this set. O is then a function that assigns a formula φ a semantic content $O(\varphi) \in C$. N is a function that for each world $w \in W$ specifies a set of contents distinguished at this world. Lastly, we would like to think of the function I as a function that assigns 'intensions' to the contents $c \in C$. These intensions are propositions, that is, the subsets of W determined by the content $O(\varphi) \in C$.

The algebraic models slightly complicate this intuitive reading. W is the same as in the set-theoretic models. The functions O and I are almost the same, but they are required to be homomorphisms now. Moreover, I has the additional requirement that for every $\varphi \in Fm$, $I(O(\Box\varphi)) = N(O(\varphi))$. The carrier C of $\mathbf{C}$ is a set of 'semantic contents' as before, but now these semantic contents are required to have an algebraic structure. N is a generalized neighborhood function. It is possible to define a dual function $N' : W \to \mathcal{P}(C)$, which corresponds to the way N is defined in the set-theoretic models.

The function composed function $[\![\cdot]\!]$ is meant to resemble the function with the same denotation in the adjusted neighborhood models. The idea is that $[\![\varphi]\!]$ still represents the proposition of φ, and has certain algebraic properties.

The requirement in set-theoretic models, that $[\![\varphi]\!] = I(O(\varphi))$ acts in such a way that 'Boolean connectives correspond to the usual set-theoretic operations' on sets is equivalently, though more precisely, specified in the algebraic models by the fact that I and O are homomorphisms, and thus their composition is also a homomorphism.

This comes with an (un)wanted consequence that the contents in C in algebraic models are compositional. Therefore, these models force that if $O(\varphi) = O(\psi)$, then it also must be the case that $O(\neg\varphi) = O(\neg\psi)$, and similarly for conjunction. A set-theoretic hyperintensional model, on the other hand, might violate this. It is up for discussion whether this property of algebraic hyperintensional models is to be wanted in hyperintensional semantics, however, currently it does not seem as a problematic feature.

One more interesting point is that the hyperintensional models are just generalized neighborhood models. The only difference is that instead of taking the neighborhood function $N' : \mathcal{P}(W) \to \mathcal{P}(W)$, we generalize the domain to be instead of $\mathcal{P}(W)$ to be any set C (with some some specific properties in the case of algebraic hyperintensional models.). A similiar idea is employed in the Cresswell models, where we took N to be defined from the domain of accessible meanings; and something similiar can be considered for Rantala models as well, as we will see in the next subsection.

The following figure visually represents the structure of hyperintensional models:

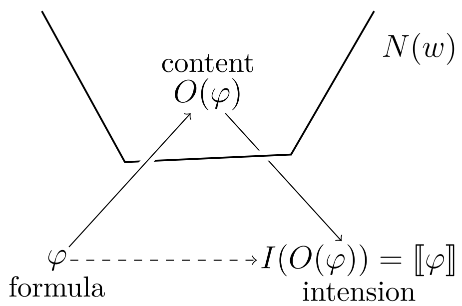

Figure 3.1: The 'semantic triangle' of hyperintensional models.

Remark. The visualization somewhat resembles the semantic triangle. In fact, after consulting semantic or semiotic triangle, one may notice that it in fact, corresponds to it - *symbol* (φ) *symbolizes* (function O) the *thought/reference* ($O(\varphi)$), which in turn *refers to* (function I) *the referent* (the proposition expressed by φ). A symbol (φ) *stands for* ($[\![\cdot]\!]$) the referent ($[\![\varphi]\!]$).

In the rest of the chapter, we will by *hyperintensional models* mean algebraic hyperintensional models, unless stated otherwise. Now we define satisfiability in hyperintensional models, and then in the spirit of introducing previous semantic approaches, we will show that hyperintensional models are not closed under (RE), and then we will show that the logic of all hyperintensional models is the smallest modal logic $\Sigma(Taut)$, as it was the case of Rantala and Cresswell models.

Definition 19 (Truth, Satisfiability, Validity)**.** *A formula φ is true in model $\mathcal{M}$ at a world w ($\mathcal{M}, w \Vdash \varphi$) iff $w \in [\![\varphi]\!]$. A formula φ is satisfiable in a model $\mathcal{M}$ iff $[\![\varphi]\!] \neq \emptyset$. A formula φ is valid in $\mathcal{M}$ ($\mathcal{M} \Vdash \varphi$) iff $[\![\varphi]\!] = W$.*

Theorem 20. *There is some hyperintensional model $\mathcal{M}$, in which* (RE) *fails for some $\varphi, \psi \in Fm$.*

Proof. We need to find a hyperintensional model $\mathcal{M} = (W, \mathbf{C}, O, N, I)$ such that $[\![\varphi]\!] = [\![\psi]\!]$, but $[\![\Box\varphi]\!] \neq [\![\Box\psi]\!]$. We define $\mathcal{M}$ so that: $W = \{w\}$, $C = \{\{c_1, c_2\}, \{\neg^C, \wedge^C\}\}$, $O(p) = c_1$, $O(q) = c_2$, $N(c_1) = \{w\}$, $N(c_2) = \emptyset$, $I(c_1) = I(c_2) = \{w\}$. It is easily seen that $I(O(p)) = I(O(q)) = \{w\}$, while $N(O(p)) = N(c_1) = \{w\}$, but $N(O(q)) = N(c_2) = \emptyset$. Therefore, for $\varphi := p$ and $\psi := q$, $[\![\varphi]\!] = [\![\psi]\!]$ and $[\![\Box\varphi]\!] \neq [\![\Box\psi]\!]$ as we wanted. $\Box$

Theorem 21. $\Sigma(Taut)$ *is the logic of all hyperintensional models.*

Proof. Let us denote the logic of all hyperintensional models $Log(Hyp)$. We will show that $\Sigma(Taut) = Log(Hyp)$.

($\Longrightarrow$) *Soundness.* We show that if $\varphi \in \Sigma(Taut)$ then $\varphi \in Log(Hyp)$. We show this by contraposition, that is, if there is some hyperintensional model $\mathcal{M} = (W, \mathbf{C}, O, N, I)$ such that $\mathcal{M} \nVdash \varphi$ then $\varphi \notin \Sigma(Taut)$. In hyperintensional models, $\mathcal{M} \nVdash \varphi \iff [\![\varphi]\!] \neq W$. Choose the $w \in W$ such that $w \notin [\![\varphi]\!]$. We define extended valuation evaluation

$$e(\psi) = 1 \iff w \in [\![\psi]\!]$$

for the chosen state w and for $\psi \in At \cup \{\Box\psi \mid \psi \in Fm\}$. We define satisfaction by e as usual. We need to verify that $e \vDash \varphi \iff w \in [\![\varphi]\!]$, but this is clear from the definition of e for all formulas in $At \cup \{\Box\varphi \mid \varphi \in Fm\}$, and the definition of satisfaction under e for the rest of the formulas. The argument is again similar to the argument in the Theorem 4. It follows that we found an extended evaluation such that $e \nvDash \varphi$, and so $\varphi \notin \Sigma(Taut)$.

($\Longleftarrow$) *Completeness.* We show that $Log(Hyp)$ is complete with respect to $\Sigma(Taut)$ again by constructing a canonical model.
Let $\mathcal{M}^{\mathcal{C}} = (W^{\mathcal{C}}, \mathbf{C}^{\mathcal{C}}, O^{\mathcal{C}}, N^{\mathcal{C}}, I^{\mathcal{C}})$ be a model such that:

- $W^{\mathcal{C}}$ is a set of all maximal $\Sigma(Taut)$-consistent theories,
- $\mathbf{C}^{\mathcal{C}} = \{Fm, \{\neg, \wedge\}\}$,

- $O^{\mathcal{C}}(\varphi) = \varphi$,
- $N^{\mathcal{C}}(\varphi) = \widehat{\Box\varphi}$,
- $I^{\mathcal{C}}(\varphi) = \widehat{\varphi}$,

where $\widehat{\varphi} = \{w \in W^{\mathcal{C}} \mid \varphi \in w\}$. $[\![\varphi]\!]^{\mathcal{C}}$ for $\varphi \in Fm$ is defined as $I^{\mathcal{C}}(O^{\mathcal{C}}(\varphi))$.

Lemma 22 (TL)**.** *For all $\varphi \in Fm$: $\varphi \in Log(Hyp) \iff \mathcal{M}^{\mathcal{C}}, w \Vdash \varphi$.*

Proof. $[\![\varphi]\!]^{\mathcal{C}} = I^{\mathcal{C}}(O^{\mathcal{C}}(\varphi)) = I^{\mathcal{C}}(\varphi) = \widehat{\varphi}$ from the definition of canonical model. Then $\varphi \in w \iff w \in \widehat{\varphi} \iff w \in [\![\varphi]\!]^{\mathcal{C}} \iff \mathcal{M}^{\mathcal{C}}, w \Vdash \varphi$. □

So if $\varphi \notin \Sigma(Taut)$, then $\widehat{\varphi} \neq W^{\mathcal{C}}$. Because $\Sigma(Taut) \cup \{\neg\varphi\}$ is consistent by the properties of maximal consistent theories, and by Lindenbaum's Lemma, there is a maximal $\Sigma(Taut)$-consistent theory $w \in W^{\mathcal{C}}$, such that $\Sigma(Taut) \cup \{\neg\varphi\} \subseteq w$. It follows that $\widehat{\varphi} \neq W^{\mathcal{C}}$, and therefore $\varphi \notin Log(Hyp)$. □

3.2 Translations of older approaches

In this last section we will show that the approaches introduced in the section 2.1 and in the section 2.2 can be translated into the hyperintensional models. That means that we can define a hyperintensional model for each Rantala or Cresswell model that validates the exactly same formulas.

3.2.1 Translation of Rantala Models

To show that the state-based approaches can be interpreted within the hyperintensional framework, we present a translation of Rantala models as defined in the chapter 2.

In Sedlár [2021], there is shown a translation of Rantala models into set-theoretic hyperintensional models. The translation there is quite simple and intuitive, however, it does not work for algebraic hyperintensional models, because it sets C in the hyperintensional model to be the powerset of the union of normal and non-normal worlds in the translated Rantala model. However, non-normal worlds aren't closed under any logical nor compositional principles, or they even might be inconsistent. Therefore, C defined in this way cannot be an algebra. So, we take a different approach to the translation.

Consider a Rantala model $\mathcal{M}_R = (W_R, W_R^*, R_R, V_R)$. A hyperintensional model based on $\mathcal{M}_R$, is a a model $\mathcal{M}_H = (W_H, \mathbf{C}_H, O_H, N_H, I_H)$ such that:

- $W_H = W_R$,
- $\mathbf{C}_R = (\{\langle \varphi, V_R(\varphi)\rangle \mid \varphi \in Fm\}, \{\neg^{\mathbf{C}}, \wedge^{\mathbf{C}}\})$,
- $O_H(\varphi)$ is defined as a tuple $\langle \varphi, V_R(\varphi)\rangle$,
- $N_H(\langle \varphi, V_R(\varphi)\rangle) = V(\Box\varphi) \cap W_R$,
- $I_H(\langle \varphi, V_R(\varphi)\rangle) = V(\varphi) \cap W_R$.

Moreover, $\neg^{\mathbf{C}}$ is defined as $\neg^{\mathbf{C}}(O_H(\varphi)) = \langle \neg\varphi, V_R(\neg\varphi)\rangle$, and $\wedge^{\mathbf{C}}$ is defined as $\wedge^{\mathbf{C}}(O_H(\varphi), O_H(\psi)) = \langle \varphi \wedge \psi, V_R(\varphi \wedge \psi)\rangle$.

To confirm that this is a well-defined hyperintensional model, we verify that all functions are well-defined, and that O_H, and I_H are $\mathcal{P}$-homomorphisms, and that $\mathbf{C}$ is an algebra.

To see that $\mathbf{C}$ is an algebra, we verify that the operations $\neg^{\mathbf{C}}$, and $\wedge^{\mathbf{C}}$ are well defined.

Suppose that $O_H(\varphi) = O_H(\psi)$ for some φ, ψ. Note that that means $O_H(\varphi) = \langle \varphi, V_R(\varphi)\rangle = \langle \psi, V_R(\psi)\rangle = O_H(\psi)$, which means that it has to hold that $\varphi = \psi$ and $V_R(\varphi) = V_R(\psi)$. Then: $\neg^{\mathbf{C}} O_H(\varphi) = \langle \neg\varphi, V_R(\neg\varphi)\rangle = \langle \neg\psi, V_R(\neg\psi)\rangle = \neg^{\mathbf{C}} O_H(\psi)$, and similarly for the operator $\wedge^{\mathbf{C}}$, suppose that $O_H(\varphi) = O_H(\xi)$, and $O_H(\psi) = O_H(\zeta)$ for $\varphi, \psi, \xi, \zeta \in Fm$. Then by the same reasoning as above, it has to hold that $\varphi = \xi$, and $\psi = \zeta$. Then $\wedge^{\mathbf{C}}(O_H(\varphi), O_H(\psi)) = \langle \varphi \wedge \psi, V_R(\varphi \wedge \psi)\rangle = \langle \varphi \wedge \psi, V_R(\varphi \wedge \psi)\rangle = \wedge^{\mathbf{C}}(O_H(\xi), O_H(\zeta))$.

The function O_H is well-defined: Suppose that $\varphi = \psi$ for some formulas φ, ψ. Then $O_H(\varphi) = \langle \varphi, V_R(\varphi)\rangle = \langle \psi, V_R(\psi)\rangle = O_H(\psi)$ from the presupposition $\varphi = \psi$ and the way V_R is defined in Rantala models.

The function O_H is a homomorphism:

- $\neg^{\mathbf{C}} O_H(\varphi) = \langle \neg\varphi, V_R(\neg\varphi)\rangle = O_H(\neg\varphi)$,
- $\wedge^{\mathbf{C}}(O_H(\varphi), O_H(\psi)) = \langle \varphi \wedge \psi, V_R(\varphi \wedge \psi)\rangle = O_H(\varphi \wedge \psi)$.

The function N_H is well-defined: Suppose that $\langle \varphi, V_R(\varphi)\rangle = \langle \psi, V_R(\psi)\rangle$ for some φ, ψ, then:

$$N_H(\langle \varphi, V_R(\varphi)\rangle) = V_R(\Box\varphi) \cap W_R = V_R(\Box\psi) \cap W_R = N_H(\langle \psi, V_R(\psi)\rangle).$$

The function I_H is well-defined: Suppose that $\langle \varphi, V_R(\varphi)\rangle = \langle \psi, V_R(\psi)\rangle$ for some φ, ψ, then:

$$I_H(\langle \varphi, V_R(\varphi)\rangle) = V_R(\varphi) \cap W_R = V_R(\psi) \cap W_R = I_H(\langle \psi, V_R(\psi)\rangle).$$

The function I_H is a homomorphism:

- $I(\neg^{\mathbf{C}}\langle \varphi, V_R(\varphi)\rangle) = I(\langle \neg\varphi, V_R(\neg\varphi)) = V_R(\neg\varphi) \cap W_R = W_R \backslash (V_R(\varphi) \cap W_R) = W \backslash I(\langle \varphi, V_R(\varphi)\rangle)$,
- $I(\wedge^{\mathbf{C}}(\langle \varphi, V_R(\varphi)\rangle, \langle \psi, V_R(\psi)\rangle) = I(\langle \varphi \wedge \psi, V_R(\varphi \wedge \psi)\rangle) = V_R(\varphi \wedge \psi) \cap W_R = (V_R(\varphi) \cap W_R) \cap (V_R(\psi) \cap W_R) = (V_R(\varphi) \cap V_R(\psi)) \cap W_R$.

In the next claim we show that so defined hyperintensional models are equivalent to Rantala models, that is, they validate exactly the same formulas:

Claim 23. *Rantala models can be embedded into hyperintensional models, that is for all formulas $\varphi \in Fm$ (in $\mathcal{L}_0$) it holds that $V_R(\varphi) \cap W_R = [\![\varphi]\!]_H$.*

Proof. The proof follows from the definition of models $\mathcal{M}_H$. For any formula φ it holds that $V_R(\varphi) \cap W_R = [\![\varphi]\!]_H$, because $[\![\varphi]\!]_H = I(O(\varphi)) = I(\langle \varphi, V_R(\varphi)\rangle) = V_R(\varphi) \cap W_R$. In the case of modal formulas $\varphi = \Box\psi$ for some ψ, then we have $[\![\Box\varphi]\!]_H = I(O(\Box\varphi)) = N(O(\varphi)) = N(\langle \varphi, V_R(\varphi)\rangle) = V_R(\Box\varphi) \cap W_R$. □

From this it follows that Rantala models can be regarded as a special case of hyperintensional models. The translation provided here is our own contribution, and we believe that the approach has an intuitive advantage when compared to the translation from the literature.

The translation has been done in Sedlár [2021] for set-theoretic hyperintensional models, but its intuitive approach of setting $\mathbf{C}$ to be the set of propositions on all worlds, faces problems when we want to do the same for algebraic hyperintensional models. This is because of the fact, that non-normal worlds are *any* sets, and so there can be no algebra built on them. Pascucci and Sedlár [2023] resolve the problem by setting the domain of $\mathbf{C}$ in hyperintensional models to the set of all formulas. $\mathbf{C}$ is then an algebra, however, it seems to be an unsatisfactory translation, as under this approach, the non-normal worlds, which are the essential idea of the state-based approach, completely fall out of the picture.

Our translation on the other hand, preserves a more intuitive connection with former models. Surely enough, we may not use the set of non-normal worlds directly because of the requirement on the algebraic properties, however, by taking $\langle \varphi, V_R(\varphi) \rangle$ for each formula φ we do so in an indirect way. We use the original valuation from Rantala models, which is defined on all the worlds. We look at the function as a set of tuples that assigns every formula a set of *normal and non-normal worlds*, and instead of *applying* V_R (because validity is ultimately defined only on the set of normal worlds) we look at the V_R from the outside as on a set of tuples. By taking this approach, we can ensure that $\mathbf{C}$ is an algebra, while still preserving the original idea of Rantala models.

3.2.2 Translation of Cresswell Models

Analogous claim can be shown for Cresswell models defined as in the subsection 2.2.2.

Consider a Cresswell model $\mathcal{M}_C = (W_C, D_C, A_C, N_C, v_C)$ with defined functions of meaning M_C and intension I_C as usual. A hyperintensional model based on $\mathcal{M}_C$ is a model $\mathcal{M} = (W_H, \mathbf{C}_H, O_H, N_H, I_H)$ such that:

- $W_H = W_C$,
- $\mathbf{C}_H = (A_C, \{\neg^A, \wedge^A\})$,
- O_H is M_C,
- N_H is N_C,
- I_H is defined inductively as:
  - $I_H(x) = x$ for if $x = v_C(p)$ for $p \in At$,
  - $I_H(\neg^A a) = \neg^D I_H(a)$,
  - $I_H(\wedge^A, a_1, a_2) = \wedge^D(I_H(a_1), I_H(a_2))$,
  - $I_H(O_H(\Box\varphi)) = N_H(O_H(\varphi)) = N_C(M_C(\varphi))$.

Moreover, $\neg^A$ is defined as $\neg^A(a) = \langle \neg^D, a_1 \rangle$, and $\wedge^A(a_1, a_2)$ is defined as $\langle \wedge^D, a_1, a_2 \rangle$.▌

We will verify that the functions O_H, N_H and I_H are well-defined, and that the functions O_H and I_H are $\mathcal{P}$-homomorphisms.

The function O_H is well-defined:

Suppose that $\varphi = \psi$. Then $O_H(\varphi) = M_C(\varphi) = M_C(\psi) = O_H(\psi)$.

The function O_H is a homomorphism:

- $\neg^A(O_H(\varphi)) = \langle \neg^D, O_H(\varphi)\rangle = O_H(\neg\varphi)$,
- $\wedge^A(O_H(\varphi), O_H(\psi)) = \langle \wedge^D, O_H(\varphi), O_H(\psi)\rangle = O_H(\varphi \wedge \psi)$.

The function N_H is well-defined: Suppose that $a_1 = a_2$. $N_H(a_1) = N_C(a_1) = N_C(a_2) = N_H(a_2)$.

The function I_H is well-defined by definition.

Additionally, we show that I_H is a homomorphism:

- $I_H(\neg^A(O_H(\varphi)) = I_H(O_H(\neg\varphi)) = W \setminus [\![\varphi]\!]_H$,
- $I_H(\wedge^A(O_H(\varphi), O_H(\psi))) = I_H(O_H(\varphi \wedge \psi))s = [\![\varphi]\!]_H \cap [\![\psi]\!]_H$.

Claim 24. *Cresswell models are equivalent to hyperintensional models. That is, for every $\varphi \in Fm$ in $\mathcal{L}_0$, $\mathcal{M}_C \Vdash \varphi$ if and only if $\mathcal{M}_H \Vdash \varphi$.*

Proof.

$\mathcal{M}_C \Vdash \varphi \iff I_C(\varphi) = W_C \iff [\![\varphi]\!]_H = W_H \iff \mathcal{M}_H \Vdash \varphi$.

We just need to show that $I_C(\varphi) = W_C \iff [\![\varphi]\!]_H = W_H$. Since $W_C = W_H$ from the definition, we will omit the subscript in the rest of the proof, and we will show that $I_C(\varphi) = [\![\varphi]\!]$ by induction on the complexity of φ.

We show the equality by induction on the complexity of formulas:

If $\varphi = p$ for some $p \in At$, then: $[\![p]\!]_H = I_H(O_H(p)) = I_H(M_C(p)) = I_H(v_C(p)) = v_C(p) = I_C(p)$.

If $\varphi = \neg\psi$ for some ψ for which the induction hypothesis holds, then:

$I_H(O_H(\neg\psi)) = I_H(\neg^A(O_H(\psi))) = W \setminus [\![\psi]\!]_H = W \setminus I_C(\psi) = I_C(\neg\psi)$.

If $\varphi = \psi \wedge \xi$ for some ψ, ξ for which the induction hypothesis holds, then: $I_H(O_H(\psi \wedge \xi)) = I_H(\wedge^A(O_H(\varphi), O_H(\psi))) = [\![\varphi]\!]_H \cap [\![\psi]\!]_H = I_C(\varphi) \cap I_C(\psi) = I_C(\varphi \wedge \psi)$.

If $\varphi = \Box\psi$ for some ψ for which the induction hypothesis holds, then:

$I_H(O_H(\Box\psi)) = N_H(O_H(\psi)) = N_C(M_C(\psi)) = I_C(\Box)M_C(\psi) = I_C(\Box\psi)$.

$\square$

Conclusion

In the thesis, we studied and attempted to find a suitable semantics for hyperintensional modal logics over the basic modal language, which we defined as a classical propositional language enriched with a modal operator.

Over this language, we defined a modal logic, and various classes of modal logic, namely normal modal logics, classical modal logics, and hyperintensional modal logics. Normal logic is a subclass of classical modal logics, and though both are more commonly studied in the literature, neither is suitable to model knowledge or belief. This is due to the hyperintensional nature of knowledge, as the equivalence of sentences does not constitute their interchangeability in the context of the agent's knowledge. Classical modal logic is closed under the inference rule (RE), which causes the paradox of the hyperintensional operator to arise in the classical modal logics. That is why we turned to study the class of hyperintensional modal logics, which, as it is the complement of the class of classical modal logics, inherently is not closed under (RE). It was our goal to study and find possible semantics for hyperintensional modal logic.

First we reviewed older approaches to the problem, namely Wansing's Rantala models, which deal with the problem of hyperintensional contexts by extending Kripke semantics with a set of non-normal worlds, and a structuralist approach by Cresswell, which postulates that there is not only the intension (the set of worlds) in which the formula holds, but also a separate meaning of the formula, which exists independently, and is represented by the formula's structure.

In the last part of the thesis, we defined two types of what we call hyperintensional models, which is a recently developed approach by Sedlar and Pascucci. The hyperintensional models resemble the semantic triangle, so there is a formula, the content of the formula, and the intension of the formula. It is a generalized version of neighborhood semantics, where instead of the relationship formula-proposition, we obtain a triangle formula-content-proposition. The models resolve the hyperintensional paradox by allowing two equivalent formulas to have different contents, but the same intension. In the spirit of Wansing, we argue that this is approach might serve as basic, general, and unifying approach to (hyperintensional) modal logic. We show that it is basic, because it is complete to the smallest modal logic, and that it is unifying, because it is possible to embed (not only) the formerly discussed approach into this framework. Hyperintensional models seem suitable to be a unifying approach to the semantics of hyperintensional modal logic.

Our own contribution developed in collaboration with the supervisor of the thesis is the formulation of Cresswell models based on Cresswell's semantics, their translation into hyperintensional models, and also our own version of the translation of Rantala models into hyperintensional models.

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