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Elementary axiomatic theories over intuitionistic logic
(Elementární axiomatické teorie nad intuicionistickou logikou)

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstract: This thesis studies basic properties of intuitionistic logic and several elementary theories over it. We choose three theories to explore: the theory of equality, the theory of linear order, and the theory of apartness. We do not work with the last theory in classical logic and we will study it in connection with the other two theories, especially in relation to conservativity. This thesis draws mainly from the results of Dirk van Dalen, Richard Statman, and Craig Smorynski.

Keywords: intuitionistic logic, elementary theories, apartness, conservativity

Abstrakt: Tato práce se zabývá základními vlastnostmi intuicionistické logiky a některými elementárními teoriemi v ní. Ke zkoumání jsme vybrali následující teorie: teorie ekvivalence, teorie lineárního uspořádání a teorie mimolehlosti. Poslední teorie není známá v klasické logice a my ji budeme zkoumat ve spojení se zbylými dvěma teoriemi a to zejména v souvislosti s konzervativitou. Tato práce čerpá především z prací Dirka van Dalena, Richarda Statmana a Craiga Smorynského.

Klíčová slova: intuicionistická logika, elementární teorie, mimolehlost, konzervativita

Contents

1 Introduction

This thesis aims to study some elementary theories over intuitionistic logic. We will investigate the theory of equality and the theory of linear order along with the theory of apartness which is a theory that is not well known in the classical logic since it blends with inequality there.

The equality relation is not stable in the intuitionistic logic as opposed to the classical version, therefore the intuitionistic theory is weaker. We will introduce sequence of inequalities and the stability axioms to prove for every n, the stability axiom S_n is stronger than the stability axioms for every smaller n. We will then outline a proof that if we add those axioms to the theory of equality, the theory of apartness will be conservative over it. That is a result of D. van Dalen and R. Statman which was redone by C. Smorynski. We will reproduce the later proof.

The theory of linear order has several axiomatization in classical logic that are equivalent. However, these axiomatization are not equivalent in intuitionistic logic. We will investigate those formulations and chose one to work with. We will show the law of trichotomy does not hold in the intuitionistic theory of linear order. Then we will reproduce a proof of C. Smorynski of implementing a linear ordering on a model with the apartness relation, only we will do it for a stronger axiomatization.

This thesis is divided into six chapters. After the introduction, you are now reading, we will present some basic properties of intuitionistic logic. We will show how it works and prove that some classically valid schemes do not hold in this logic. In this chapter, we will also present three elementary theories with which we will work in the three upcoming chapters. The third chapter is focused on the differences between the elementary theories based on the underlying logic. The fourth chapter studies the theory of equality. We will define stability axioms and show that we can find a theory over which the theory of apartness is conservative. The fifth chapter investigates the theory of linear order and its axiomatization. We will also prove two theorems of conservativity concerning the theory of apartness. After that, we finish with the conclusion.

The main sources that we used are works and results of D. van Dalen, R. Statman, and C. Smorynski[vDS79], [Smo73b], and [Smo77].

2 Basics

In this section, we introduce the basics of intuitionistic logic and explain how it works. To elucidate intuitionistic logic further, we show several proofs and counterexamples to selected classical schemes. At the end of this chapter, we will introduce three elementary theories that we will examine later.

2.1 Intuitionistic logic

Intuitionistic logic is a non-classical logic where the law of the excluded middle and the principle of double negation elimination are not valid. The law of the excluded middle says that a statement must be true or false ($p \vee \neg p$) and the double negation elimination principle says that if a statement is not false, it has to be true ($\neg\neg p \rightarrow p$). In this chapter, we will present essential properties of this logic and show number of basic classically valid schemes and investigate whether they hold in this logic or not. To those that are not valid, we will construct models as counterexamples. We will also present three elementary theories. The purpose of this chapter is to introduce the logic and the theories to the reader so that we can work with both later.
Intuitionistic logic originated as a response to several paradoxes in set theory that showed classical logic can contain problems. The founder of this logic is considered to be L. E. J. Brouwer, a Dutch mathematician and philosopher. If the reader wants to know more about the history of intuitionistic logic, we recommend him [Mos22].

2.1.1 Propositional intuitionistic logic

The language of propositional logic consists of four symbols: $\neg$, $\rightarrow$, $\&$, $\vee$. As opposed to classical logic, none of the symbols can be expressed by the others, therefore we necessarily need all of them. We do not consider equivalence $\equiv$ as a basic symbol because as well as in classical logic, it can be defined as a conjunction of two implications. We denote atoms by p or q, formulas by uppercase letters $A, B, C, \ldots$, and nodes of a model by Greek letters from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots$

We will now establish three terms for Kripke models that we will use in this thesis.

Accessibility: We say a node β is accessible from α for $\beta \geq \alpha$. We will also use expression α sees β.
Root: We say α is a root if there is no α_0, such that α is accessible from α_0.
Leaf: We say α is a leaf if α does not see any other node.

Now let us define the basic terms of propositional intuitionistic logic [Šve02].

Definition 2.1 A Kripke frame is a pair $\langle W, \leq \rangle$ where W is a nonempty set and the relation $\leq$ is reflexive, weakly antisymmetric, and transitive between the nodes of W.

Definition 2.2 The triple $K = \langle W, \leq, \Vdash \rangle$ where $\Vdash$ is a relation between nodes and formulas is a Kripke model if for an arbitrary atom p and arbitrary formulas A and B the following holds:

(i) $\alpha \Vdash p$ and $\alpha \leq \beta$, then $\beta \Vdash p$,

(ii) $\alpha \Vdash A \ \&\ B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$,

(iii) $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$,

(iv) $\alpha \Vdash A \rightarrow B$ iff for all $\beta \geq \alpha$, if $\beta \Vdash A$, then $\beta \Vdash B$,

(v) $\alpha \Vdash \neg A$ iff for all $\beta \geq \alpha$, $\beta \nVdash A$.

Condition (i) is called persistence. We understand conjunction and disjunction the same way we do in classical logic. The difference here is in the implication and negation for which, if we want to verify their validity in α, we need to look at all the nodes accessible from α and see whether the conditions for these symbols are satisfied there.

2.1.2 Predicate intuitionistic logic

Formulas in predicate logic are built up using four symbols: $\neg$, $\rightarrow$, $\&$, $\vee$, and two quantifiers $\forall$ and $\exists$. The nodes of Kripke structures are denoted by Greek letters from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots$, the elements of those structures are denoted by small Latin letters $a, b, c, \ldots$ and for formulas we will use Greek letters φ and ψ.

We borrow the two upcoming definitions and lemma from Logic: Incompleteness, Complexity, and Necessity [Šve23].

Definition 2.3 A triple $\langle W, \leq, J \rangle$ is a Kripke structure for a language L if W is a nonempty set, $\leq$ is a reflexive, antisymmetric, and transitive relation on W, and J is a function defined on W, satisfying following conditions:

(i) All values $J(\alpha)$ of function J are structures for language L,

(ii) If A and B are domains of structures $J(\alpha)$ and $J(\beta)$ and $\alpha \leq \beta$, then $A \subseteq B$,

(iii) If $s^{J(\alpha)}$ and $s^{J(\beta)}$ are realizations of an arbitrary function or predicate symbol s in structures $J(\alpha)$ and $J(\beta)$ and $\alpha \leq \beta$, then $s^{J(\alpha)} \subseteq s^{J(\beta)}$

Structures mentioned in condition (i) are meant as a structures in the classical sense.

The condition (ii) states that if we have an element in a node α then it will be in every node β that is above α. In other words, an element will not disappear in accessible nodes but an element can be added there.

The condition (iii) states that if an element is in the realization of a function or predicate symbol, it will remain there. But again, the realization can extend in accessible nodes.

Definition 2.4 If $\langle W, \leq, J\rangle$ is a Kripke structure for a language L, the function J is defined on W as follows:

(i) If φ is an atomic formula of language L then $\alpha \Vdash \varphi[e]$ iff $J(\alpha) \models \varphi[e]$,

(ii) $\alpha \Vdash (\varphi \ \& \ \psi)[e]$ iff $\alpha \Vdash \varphi[e]$ and $\alpha \Vdash \psi[e]$,

(iii) $\alpha \Vdash (\varphi \vee \psi)[e]$ iff $\alpha \Vdash \varphi[e]$ or $\alpha \Vdash \psi[e]$,

(iv) $\alpha \Vdash (\varphi \rightarrow \psi)[e]$ iff for all $\beta \geq \alpha$, if $\beta \Vdash \varphi[e]$, then $\beta \Vdash \psi[e]$,

(v) $\alpha \Vdash (\neg\varphi)[e]$ iff $\forall\beta \geq \alpha, \beta \nVdash \varphi[e]$

(vi) $\alpha \Vdash (\exists x\varphi)[e]$ iff there exists an element a of a domain of $J(\alpha)$ such that $\alpha \Vdash \varphi[e(x/a)]$

(vii) $\alpha \Vdash (\forall x\varphi)[e]$ iff for all $\beta \geq \alpha$ and all elements a of a domain $J(\beta)$ we have $\beta \Vdash \varphi[e(x/a)]$

From these conditions it is clear the existential quantifier works the way it does in classical logic. On the other hand, if we want to verify the validity of the universal quantifier in α, we need to look not only at all the elements of α but at all elements of all the nodes that are accessible from α.

Lemma 2.5 *Let* $\langle W, \leq, \Vdash\rangle$ *be a Kripke model for language L. Let* α *be a node of the structure and let* e *be a valuation of variables in* $J(\alpha)$ *and an arbitrary formula* φ *then, if* $\alpha \Vdash \varphi[e]$*, then* $\beta \Vdash \varphi[e]$ *for every* β *accessible from* α*.*

Proof. By induction on the complexity of φ. QED

2.2 Validity of basic schemes

In this section, we look at several essential schemes that hold in classical logic and we will examine whether they are valid in intuitionistic logic.

First, we will show a counterexample for the two laws we mentioned at the very beginning.

Lemma 2.6 *The law of excluded middle and double negation elimination are not valid in intuitionistic logic.*

Proof. We can picture a simple model with only two nodes that refutes the validity of the two laws. We only need to consider one atom p.

$\alpha_0 \nVdash p$

$\alpha_1 \Vdash p$

Where $\alpha_1 \geq \alpha_0$

We can see that the statement p is not satisfied in α_0. To prove the negation of this statement is also not satisfied here, we have to look at the accessible node where the atom p is satisfied which does not meet the condition for the validity of negation. We get $\alpha_0 \nVdash p \vee \neg p$. We have verified that *tertium non datur*, a classically valid principle, is not intuitionistically valid.

As counterexample for the double negation elimination, we use the same model. We know that in α, the statement p is not satisfied. To refute the implication we now need to prove that $\neg\neg p$ is not satisfied. We want to show that $\neg p$ is not satisfied in any $\alpha \geq \alpha_0$. We have already shown that $\neg p$ is not satisfied in α_0 and it is obvious that the negation is neither satisfied in α_1 since $\alpha_1 \Vdash p$. Hence the conclusion of the implication $\neg\neg p$ is not satisfied in α_0 and double negation elimination principle is not valid in intuitionistic logic. QED

Lemma 2.7 *The sentence $\neg\exists x\varphi(x)$ is intuitionistically equivalent to $\forall x\neg\varphi(x)$.*

Proof. First let us prove the implication from left to right.

We have $\alpha \Vdash \neg\exists x\varphi(x)$ and we want to prove $\alpha \Vdash \forall x\neg\varphi(x)$. The condition for the validity of $\neg\exists x\varphi(x)$ is $\forall\beta \geq \alpha$, $\beta \nVdash \exists x\varphi(x)$. To put it in words, in every node accessible from α, there is no element that would meet the condition φ. Therefore, $\forall\beta \geq \alpha$, $\beta \nVdash \varphi(x)$ for all elements x. That is equivalent to $\forall x\neg\varphi(x)$.

Now from right to left.

Our assumption is $\alpha \Vdash \forall x\neg\varphi(x)$ and we want to get $\alpha \Vdash \neg\exists x\varphi(x)$. The universal quantifier gives us $\forall\beta \geq \alpha$, $\beta \Vdash \neg\varphi(x)$. The negation gives us for every γ accessible from β, $\gamma \nVdash \varphi(x)$. There is no node above α in which there would exist an element complying the condition φ. That is the conclusion we wanted: $\alpha \Vdash \neg\exists x\varphi(x)$. QED

The other situation differs from classical logic.

Lemma 2.8 *The sentence $\neg\forall\varphi(x)$ is not intuitionistically equivalent to $\exists x\neg\varphi(x)$.*

Proof. We will construct a simple model to refute this equivalence. Let $\varphi(x)$ be the formula $P(x)$ for an arbitrary unary predicate symbol P.

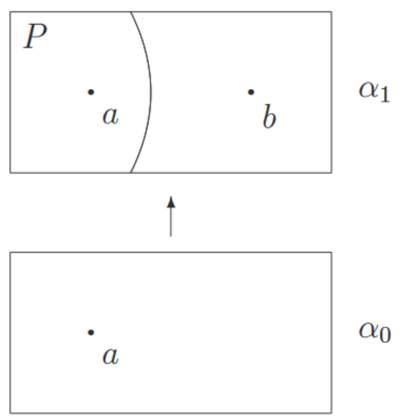

Figure 1: Model violating $\neg\forall xP(x) \rightarrow \exists x\neg P(x)$

The predicate has an empty realization in α_0. But in α_1, the element a is added to the realization. In both of those nodes, we can find an element that is not in the realization of P which is the condition for the validity of the premise of the implication. Therefore $\alpha_0 \Vdash \neg\forall xP(x)$.
On the other hand, in α_0 there does not exist an element that would meet the condition $\neg P(x)$ because the element a is added to the realization in α_1 and for negation, we need to look at all the nodes above. Therefore $\neg P(a)$ is not satisfied and since α_0 has only one element, it is safe to say $\alpha_0 \nVdash \exists x\neg P(x)$. QED

Definition 2.9 DNS scheme: $\forall \overline{y}(\forall x \neg\neg\varphi(x, \overline{y}) \rightarrow \neg\neg\forall x \varphi(x, \overline{y}))$. The DNS stands for *double negation shift.*

Lemma 2.10 *The double negation shift scheme is not valid in intuitionistic logic.*

Proof. We reproduce a model from [Šve23] as a counterexample.

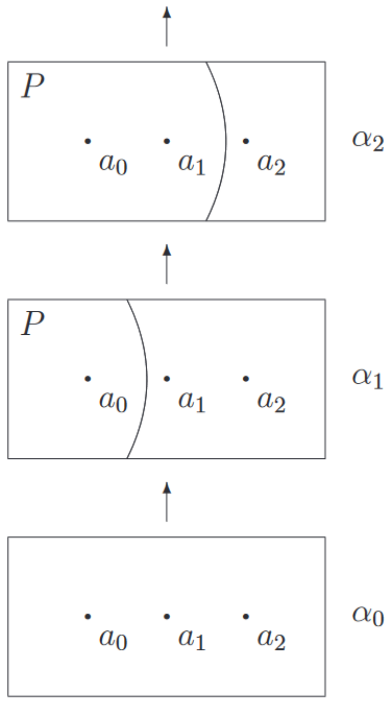

Figure 2: Model violating DNS

We have an infinite number of elements a in α_0. In every node α_{n+1}, we add the element a_n to the realization of the predicate P. We are violating the sentence $\exists x \neg\varphi(x)$ as we did in the proof before but we do it in every node. Hence in α_0, the formula $\neg\exists x \neg P(x)$ is satisfied. This formula is equivalent to $\forall x \neg\neg P(x)$. It is obvious $\neg\neg\forall x P(x)$ is not satisfied in α_0 since $\alpha_0 \Vdash \neg\forall x P(x)$ as is shown in the previous proof. Therefore the conclusion is not satisfied: $\alpha_0 \nVdash \neg\neg\forall x P(x)$. QED

The model we constructed is infinite. That can lead us to a question if DNS is valid in finite models. We will show an even stronger claim.

Lemma 2.11 *DNS is valid in all models in which every node α sees a node β, such that β is a leaf.*

Proof. Let us assume that we have such a model and that $\forall x \neg\neg\varphi(x)$ is satisfied in the root α. That means for all $\beta \geq \alpha$, $\beta \Vdash \neg\neg\varphi(b)$ for all elements $b \in \beta$. We want to prove that for all $\beta \geq \alpha$ there exists $\gamma \geq \beta$, such that $\gamma \Vdash \forall x \varphi(x)$.
Let us fix $\beta \geq \alpha$. From our assumption, we know that there is a node (γ) accessible from β that does not have any nodes above it. In γ the formula $\neg\neg\varphi(c)$ holds for every $c \in \gamma$ from the sentence we have in our assumption. The double

negation means that there is a node accessible from γ from which we can go to a node where $\varphi(c)$ is met for all $c \in \gamma$. But since γ is a leaf, we know that $\varphi(c)$ for all elements c has to be satisfied in γ which is exactly the conclusion we wanted because it gives us $\gamma \Vdash \forall x \varphi(x)$.
Now we need to prove this holds for all nodes accessible from the root α. That is obvious because from our assumption we know that for all nodes $\beta \geq \alpha$ there exists a node $\gamma \geq \beta$ that is a leaf. In the leafs, we will always get the desired conclusion $\forall x \varphi(x)$ we want.
This proof is based on the fact that in the end nodes, all tautologies from classical logic are satisfied. QED

Corollary 2.12 *DNS does hold in all finite models.*

Since the double negation shift scheme does not have a finite model as a counterexample we know that the finite model property that states if we have a model as a counterexample to a formula then there exist a finite model as a counterexample, is not valid in the predicate intuitionistic logic.

2.3 Elementary theories

In this section, we introduce three elementary theories in intuitionistic logic which we will then explore further.

The theory of equality
This theory is denoted by EQ. It has a language $\{=\}$ and the following axioms [Šve02]:

E1: $\forall x(x = x)$,
E2: $\forall x \forall y(x = y \rightarrow y = x)$,
E3: $\forall x \forall y \forall z(x = y \ \& \ y = z \rightarrow x = z)$

and two schemes for functional and predicate symbols:

E4: $\forall \underline{x} \forall \underline{y}(x_1 = y_1 \ \& \ .. \ \& \ x_n = y_n \rightarrow F(x_1, .., x_n) = F(y_1, .., y_n))$

for any functional symbol F

E5: $\forall \underline{x} \forall \underline{y}(x_1 = y_1 \ \& \ .. \ \& \ x_n = y_n \rightarrow (P(x_1, .., x_n) \equiv P(y_1, .., y_n)))$

for any predicate symbol P
We will not need the axiom E4 in this thesis but it belongs to the axiomatization.

The theory of apartness
This theory is denoted by AP. It has a language $\{\#\}$ and the following axioms [vDS79]:

A1: $\forall x \forall y (\neg(x \# y) \equiv x = y)$,

A2: $\forall x \forall y (x \# y \rightarrow y \# x)$,

A3: $\forall x \forall y \forall z (x \# y \rightarrow z \# x \vee z \# y)$

This theory is not as well known as the others we are mentioning in this thesis since it is not well developed in classical logic. It allows us to express if two elements are not equivalent. As mentioned in the beginning, in intuitionistic logic the principle of excluded middle is not valid, therefore we can not assume that if $x = y$ is not satisfied then $x \neq y$ is.

The theory of linear order

This theory is denoted by LO. It has a language $\{=, <\}$, the axioms E1–E3, E5 plus the following axioms [vD86]:

LO1: $\forall x \forall y \forall z (x < y \ \& \ y < z \rightarrow x < z))$,

LO2: $\forall x \forall y \forall z (x < y \rightarrow z < y \vee x < z)$,

LO3: $\forall x \forall y \forall z (x = y \equiv \neg(x < y) \ \& \ \neg(y < x))$

These axioms are a bit different than in classical logic. Instead of the trichotomy axiom we would use in classical logic $\forall x \forall y (x < y \vee x = y \vee y < x)$, we chose a weaker one (LO2). It is because the stronger one does not hold in the intuitionistic theory of linear order for example for some real numbers, see [vD86]. We will not show the proof that for some real numbers the trichotomy is not satisfied but we will show a model of LO where trichotomy is not valid in the fifth chapter.
We will also show another axiomatization but we will work with the one listed above.

3 Differences between the two logics

We will now prove that the two theories that we work with in classical logic (EQ and LO) do not trivialize intuitionistic logic, *i.e.* they are not the same in intuitionistic logic as they are in classical logic since they have different underlying logic. What we mean by the word *trivialize* is that the axioms of the theory are so strong that all of the classically valid formulas are provable. We will show this does not happen with our theories and therefore we can find formulas (in our case it will be atomic formulas) that are undecidable in the intuitionistic version of the theories [Bra10].

3.1 The theory of equality

In classical logic, this theory is incomplete. As an example of an independent sentence, we have $\forall x \forall y (x = y)$ [Mon12]. In intuitionistic logic, we even have an atomic formula that is undecidable.

Lemma 3.1 *The atomic formula $x = y$ is an undecidable formula in the theory of equality.*

Proof. Let us show a model of EQ where $x = y \lor \neg(x = y)$ is not valid.

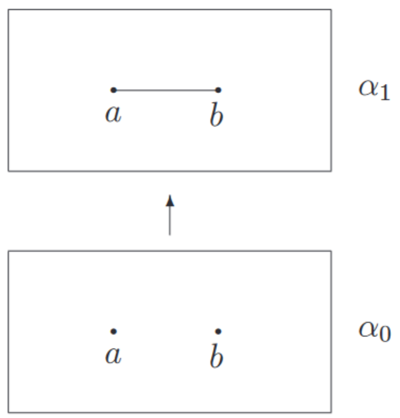

Figure 3: Model violating $x = y \lor \neg(x = y)$ in EQ

It is clear that a and b are not equal in α_0, therefore $\alpha_0 \not\Vdash a = b$. Since we have the formula $a = b$ satisfied in α_1, the negation is also not satisfied in the root, hence $\alpha_0 \not\Vdash \neg(a = b)$. QED

Corollary 3.2 *The theory of equality does not trivialize intuitionistic logic.*

3.2 The theory of linear order

The theory of linear order is also incomplete in classical logic. As an example of an independent sentence, we can choose $\forall x \exists y (x < y)$. In intuitionistic logic, we again have atomic formulas that are undecidable.

Lemma 3.3 *The atomic formula $x = y$ is not decidable in the theory* LO.

Proof. Let us again construct a simple model where $x = y \lor \neg(x = y)$ is violated.

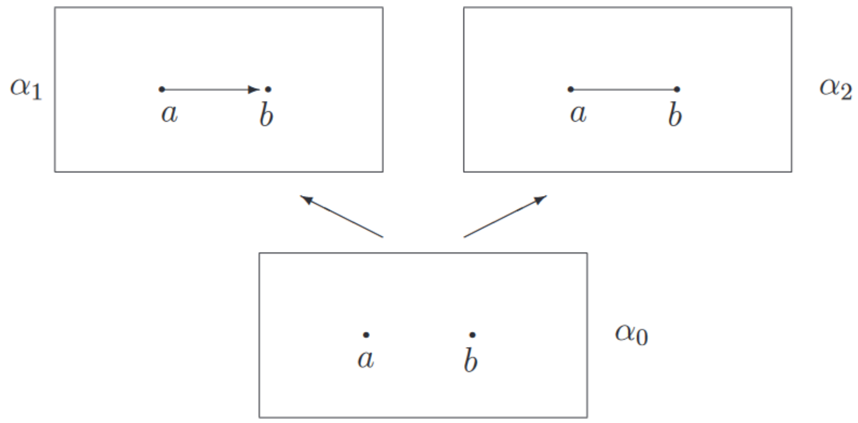

Figure 4: Model violating $x = y \lor \neg(x = y)$ in LO

The arrow in α_1 indicates the order relation.

$\alpha_0 \not\Vdash \neg(a = b)$

because $\alpha_2 \Vdash a = b$ but also

$\alpha_0 \not\Vdash a = b$

We have two nodes above the root. In α_2 we have $a = b$ to refute the negation as is explained above and in α_1 we have $a < b$ to refute the premise of LO3. If this node was missing, we would get $\neg(a < b)$ & $\neg(b < a)$ valid and it would give us the conclusion $a = b$. QED

We have another atomic formula that is undecidable in the theory of linear order. Let us prove it.

Lemma 3.4 *The atomic formula $x < y$ is not decidable in the theory* LO.

Proof. Let us construct another Kripke model.

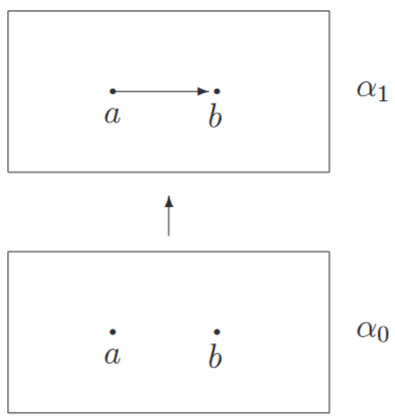

Figure 5: Model violating $x < y \lor \neg(x < y)$ in LO

It is easy to see that the formula $a < b \lor \neg(a < b)$ is violated in this model. QED

Corollary 3.5 *The theory of linear order does not trivialize intuitionistic logic.*

This claim is not surprising since this theory even has different axioms than the classical version.

4 Equality and apartness

4.1 Stability

In this section, we introduce stability axioms for every n and we will show they grow in strength for every additional grade. Before we prove this claim, we examine a sequence of inequalities and show that they also grow in strength for every other grade. We prove two implications concerning the inequalities and refute the converse of them. We will use both later in the proof of the stability axioms.
First, let us define the inequalities and the stability axioms [vDS79].

Definition 4.1 The sequence of inequalities is defined as follows:
$x \neq_0 y := \neg(x = y)$
$x \neq_{n+1} y := \forall z(z \neq_n x \ \vee \ z \neq_n y)$

Definition 4.2 The sequence of stability axioms is defined as follows:
$S_n := \forall x \forall y(\neg(x \neq_n y \ \rightarrow \ x = y))$

We will now show that every inequality on grade $n + 1$ implies the inequality on the lower grade n.

Lemma 4.3 EQ $\Vdash \forall x \forall y(x \neq_{n+1} y \ \rightarrow \ x \neq_n y)$

Proof. Let us assume that $x \neq_{n+1} y$ holds. We can rewrite the inequality $x \neq_{n+1} y$ as $\forall z(x \neq_n z \ \vee \ y \neq z)$. Since it holds for all elements z, we can choose z to be the element x. That gives us $x \neq_n x \ \vee \ x \neq_n y$. We know that $x \neq_n x$ is not true (we could break it down the same way we used here and always choose x to be the z and in the end, we would get $x \neq_0 x \ \vee \ x \neq_0 x$ which we know is not true). It leaves us with $x \neq_n y$ which is the conclusion of the implication we wanted. QED

Lemma 4.4 EQ $\Vdash \forall x \forall y(\neg(x \neq_n y) \ \rightarrow \ \neg(x \neq_{n+1} y))$

Proof. Contraposition from the previous Lemma. QED

Corollary 4.5 EQ $\Vdash S_{n+1} \ \rightarrow \ S_n$ *for all* n

Proof. Our assumptions are $\forall x \forall y(\neg(x \neq_{n+1} y) \ \rightarrow \ x = y)$ and the premise from the axiom S_n which is $\neg(x \neq_n y)$. We want to prove the conclusion of S_n which is $x = y$. Let x and y be given. From Lemma 4.4 and the premise from S_n we know that $\neg(x \neq_{n+1} y)$ holds. Since $\neg(x \neq_{n+1} y)$ is the premise from S_{n+1} and this stability axiom is our assumption, we get the conclusion $x = y$. QED

Now we will prove that the inequality on a lower grade n does not imply the stronger inequality on grade $n + 1$. It will then help us prove that the contraposition from this implication, which is a weaker claim, also does not hold there.

Lemma 4.6 EQ $\nVdash \forall x \forall y(x \neq_n y \ \rightarrow \ x \neq_{n+1} y)$

Proof. Let us start with the case when $n = 0$. For this proof, we will use a Kripke model:

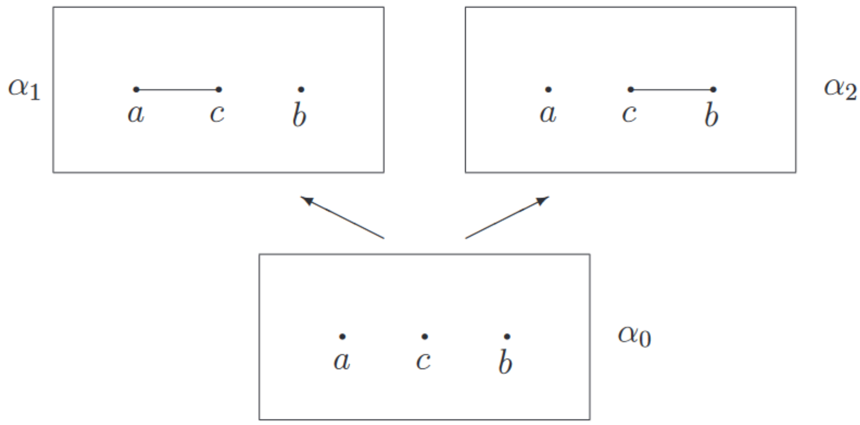

Figure 6: Model violating $x \neq_0 y \rightarrow x \neq_1 y$

In the node α_0, the elements a and b are not equal so $a \neq_0 b$ is satisfied in the root. But we can find an element (the element c) that is sometimes equal to a (in α_1) and sometimes to b (in α_2). Let us call it the middle element. Therefore the inequality on grade one is violated since it states $\forall z(z \neq_0 x \ \vee \ z \neq_0 y)$. We then get $\alpha_0 \Vdash a \neq_0 b$ and $\alpha_0 \not\Vdash a \neq_1 b$. We have a finite model with a constant domain consisting of a finite number of elements.

For $n = 1$, we again have a finite counterexample:

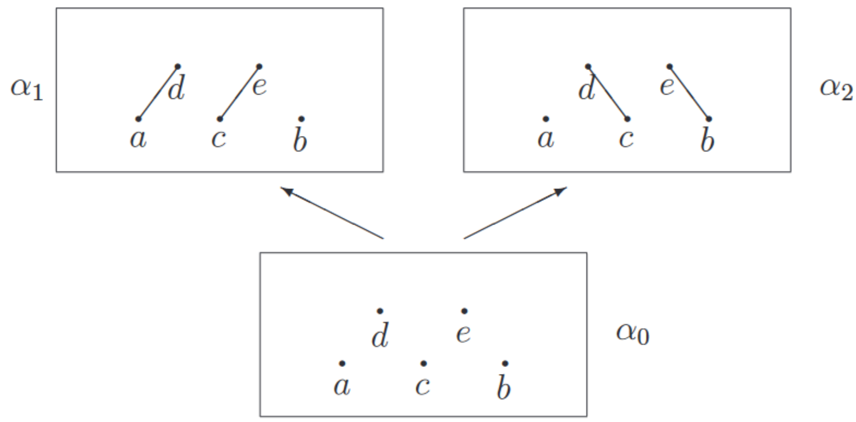

Figure 7: Model violating $x \neq_1 y \rightarrow x \neq_2 y$

This time we needed to use two more elements. There is no element that would be equal to a as well as to b so the inequality on grade 1 is valid. We can rewrite the inequality on grade 2 as $\forall c(a \neq_1 c \ \vee \ b \neq_2 c)$. To refute it we need a middle element (the element d) that is equal to a as well as to c and a middle element (the element e) that equals b as well as it equals to c. It is easy to confirm those elements exist in the model and that they comply this condition. Therefore we have $\alpha_0 \Vdash a \neq_1 b$ and $\alpha_0 \not\Vdash a \neq_2 b$.

Now for $n = 2$:

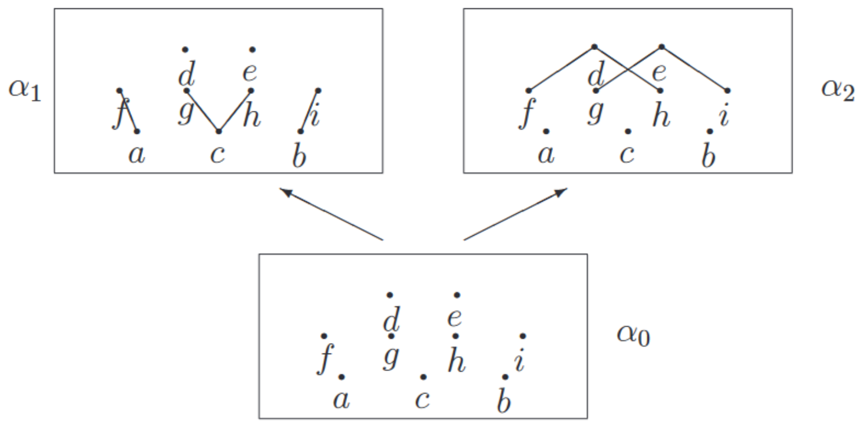

Figure 8: Model violating $x \neq_2 y \to x \neq_3 y$

We need four more elements than in the previous model. What is happening here is we looked at what elements are equal in Figure 7 and we put another element between them in such a way that in α_1 the new element will be equal to one of the previously equal elements and in α_2 it would become equal to the other. That way we assured the validity of the inequality on grade 2 and violated the validity of the inequality on grade 3. We have $\alpha_0 \Vdash a \neq_2 b$ and $\alpha_0 \nVdash a \neq_3 b$.

We will not construct any more models for higher n but the principle is still the same. For $n + 1$ we look at what elements become equal in α_1 and α_2 of the previous model for n and we put one more element between them the same way we described above. The number of elements will grow exponentially for each additional model. That way we can have a model with only three nodes and with a finite number of elements for every $n \in \omega$.

For every n we have a model that complies the condition of a constant domain. It also complies the double negation shift scheme (DNS) since every model we have is finite.

If we would want only one model as counterexample for all n, we would need a model with an infinite number of elements. The model would look like follows.

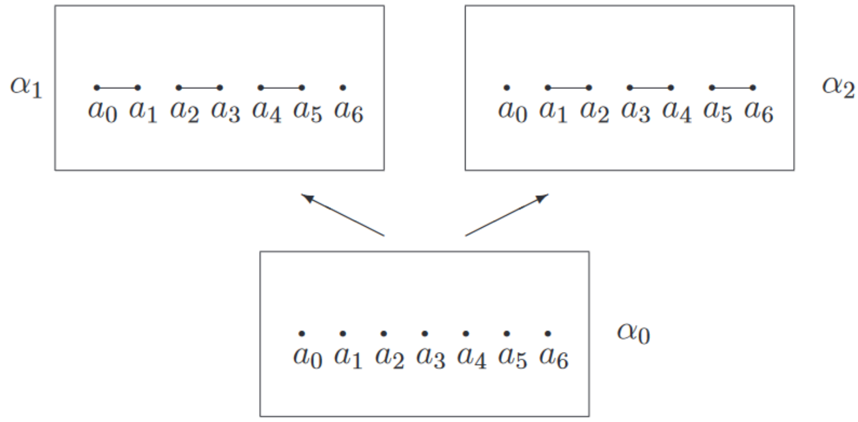

Figure 9: Model violating $x \neq_n y \rightarrow x \neq_{n+1} y$

We have infinite number of the elements a. For each n, we have to choose different elements. For the first step when $n = 0$ we will choose the elements a_0 and a_2 (or any other elements that have one other element between them). Then we get $\alpha_0 \Vdash a_0 \neq_0 a_2$ and $\alpha_0 \nVdash a_0 \neq_1 a_2$. We can see there exist a middle element a_1 that is equal to a_0 in α_1 and is equal to a_2 in α_2.

If we wanted to refute it for $n = 1$, we would have to choose the elements that have two other elements between them, for example, a_0 and a_3. The number of elements between the ones we choose grows exponentially. For $n = 0$, we need one element between. For $n = 1$, we need two of them. For $n = 3$, we would choose elements that have four elements between and so on. For n we need $2n$ elements between. In this sense, the relation of inequality on grade $n + 1$ is a composition of the relation of inequality on grade n with itself.

Since every element a_n where n is an even number is equal to an element with an index number $n+1$ in the left node and every element a_{n+1} where $n+1$ is an odd number is equal to an element with an index number $n+2$ in the right node we know, the inequality on the lower grade will always be satisfied if we choose the right elements as is explained above.

This way we constructed a model with a finite number of nodes but with an infinite number of elements. We need an infinite number of elements because we have an infinite number of the inequality grades. It is clear that this model has a constant domain and, because it has only three nodes, it complies the double negation shift scheme. QED

One may ask, why have we shown all the other models if the general model is sufficient? It is because for every n we can have a model with a finite number of nodes and elements. Also, all the models above will help us prove the following Lemma which is a weaker version of the implication we have just proven. Therefore we will have more complex models.

Lemma 4.7 EQ $\nVdash \forall x \forall y(\neg(x \neq_{n+1} y) \rightarrow \neg(x \neq_n y))$

Proof. Again we will start with $n = 0$. Let us construct a Kripke model to show that the implication is not valid. The beginning of the model is the model in

Figure 6 in proof of Lemma 4.6 so we will start from α_2 and continue to the right side:

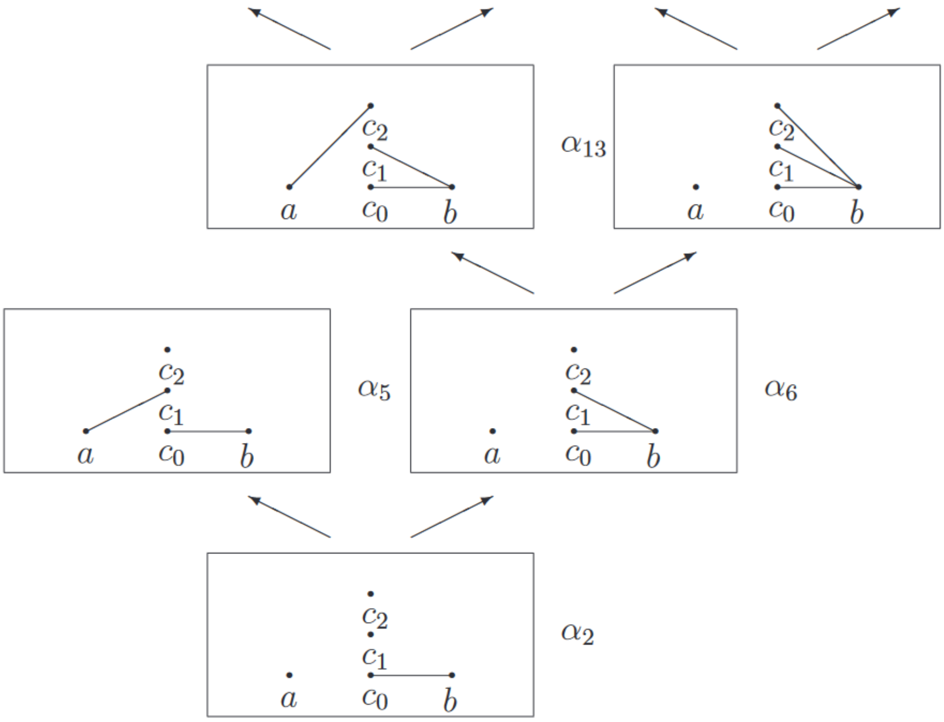

Figure 10: Model violating $\neg(x \neq_1 y) \rightarrow \neg(x \neq_0 y)$

This model has to be infinite because from the validity of $\neg(a \neq_1 b)$ in the root we know that $a \neq_1 b$ has to be violated in every node. Therefore every node needs to see two more nodes to satisfy this condition. We also need an infinite number of elements so that we can ensure the validity of $a \neq_0 b$. If we only used the element c_0 as the middle element, from transitivity, we would get $a = b$. The left side from α_1 looks the same, only $a = c_0$ and not $b = c_0$.

In this model, double negation shift scheme does not hold. We can choose φ to be the formula $x = a \vee x = b$ and it is easy to verify that the whole implication $\forall x \neg\neg(x = a \vee x = b) \rightarrow \neg\neg\forall x(x = a \vee x = b)$ is violated. Let us explain it.

In the root α_0, we want $\alpha_0 \Vdash \forall x \neg\neg(x = a \vee x = b)$. Therefore, for all nodes α_1 accessible from α_0 and all elements x, we can go from all β accessible from α_1 to a node $\gamma \geq \beta$, such that $\gamma \Vdash x = a \vee x = b$. We know that for all elements c we can go to a node where they will be equal to a or to b. To verify it for c_0, we can go to α_2 (or anywhere higher) where $c_0 = b$. For c_1, we can go to a node, for example, α_5 where $c_1 = a$. Every element will become equal to a or to b at some point, therefore for all x we can go to a node (from any α) where φ is satisfied.

To satisfy the conclusion $\neg\neg\forall x(x = a \vee x = b)$ in the root, we would have to be able to, from every β accessible from α_0, go to a node $\gamma \geq \beta$, such that in γ the sentence $\forall x(x = a \lor x = b)$ is satisfied. But since we have an infinite model with an infinite number of elements, we know that in every node, there are elements that are not equal to anything (except themselves) yet. Hence $\forall x(x = a \lor x = b)$ is not satisfied anywhere in this model and the conclusion is refuted.

For $n = 1$ the model will look like follows:

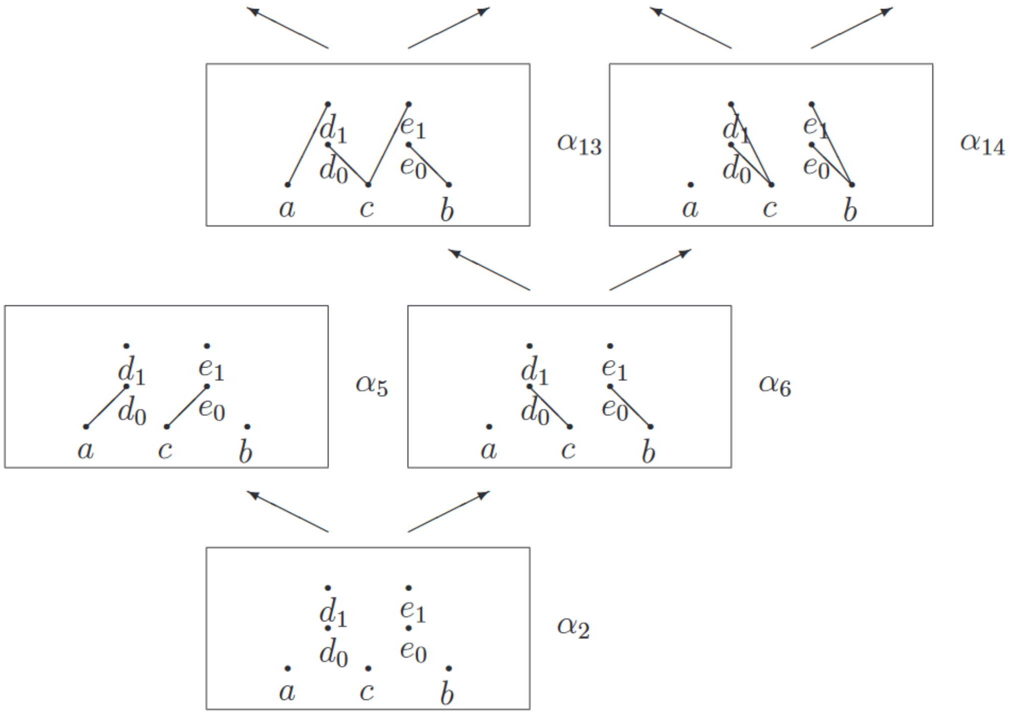

Figure 11: Model violating $\neg(x \neq_2 y) \to \neg(x \neq_1 y)$

We also have an infinite model because, above every node, there are two more nodes to refute the inequality on grade 2. In every node (except for the root α_0) we need to use two more elements than in the one below him. That will leave us with an infinite number of middle elements (d) between a and c and an infinite number of middle elements (e) between c and b. We know $\neg(a \neq_1 b)$ is violated since there is no element that would be equal to a as well as to b. This model is similar to the one we had in Figure 7 in Lemma 4.6, the first tree nodes are even the same. The only difference there is, from the negation we get $\alpha \not\Vdash a \neq_2 b$ for every node α.

As in the previous model in Figure 10, the left side of this model would look the same, only instead of $d_0 = c$ and $e_0 = b$, we have $d_0 = a$ and $e_0 = c$.

Here the double negation shift scheme is also not valid. If we chose φ to be the formula $x = a \lor x = b \lor x = c$, it is clear that $\forall x \neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$ is violated. The proof would proceed the same way it did in the previous model, only we had to add the element c in the formula because otherwise it would not meet the premise.

The construction for every other n would proceed the same way. All of the models would have an infinite number of nodes and elements which is caused by the negation in the formula we want to satisfy. The first tree nodes would look like the model for the same n in Lemma 4.6 and then we would only make use of more elements that would act in the same manner as in the two nodes below.

The double negation shift scheme would not be valid in any of those models. The formula to violate it would be the same, only we would be adding more elements that would not meet it if they were not in the formula (the same way we did it in the last model for element c). We always have a finite number of those elements so the formula would also be finite.

We will now construct a model to refute the general implication for all n. The first three nodes look like the model for all n in Figure 12 in the proof of the previous Lemma so we will start from α_2.

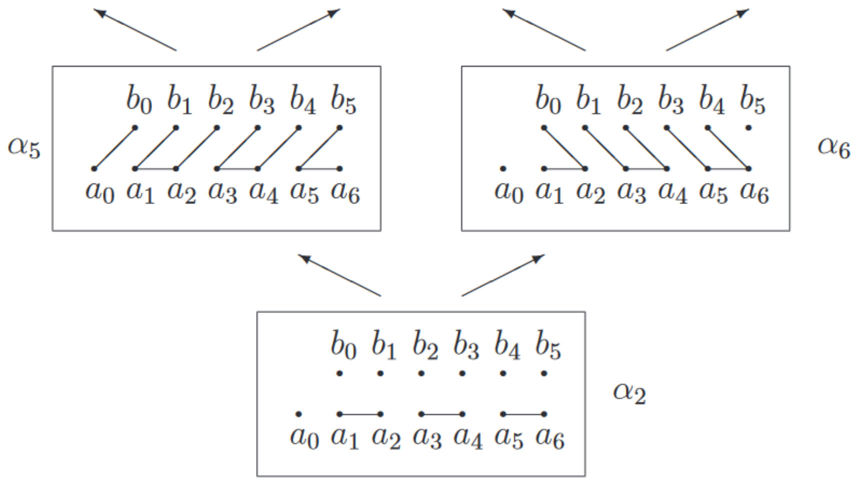

Figure 12: Model violating $\neg(x \neq_{n+1} y) \rightarrow \neg(x \neq_n y)$

As opposed to the model in Figure 9, we are dealing with negation here. Therefore this model will have an infinite number of nodes.

Since in α_0, we have the formula $\neg(a \neq_{n+1} b)$ satisfied, we need to violate the formula $a \neq_{n+1} b$ in every node accessible from α_0. Therefore the elements $a_0, \ldots, a_n$ are not enough and we need to use another sequence $b_0, \ldots, b_n$ in the nodes accessible from α_1 and α_2, such that every b_n would be equal to the corresponding element a_n in the left node and every b_n would be equal to a_{n+2} in the right node. In the nodes that α_5 and α_6 see, we need another sequence that would behave the same way the the sequence $b_0, \ldots, b_n$ did. In the accessible nodes, we would need more sequences, hence we will end up with an infinite number of sequences that have an infinite number of elements.

To refute the implication, we need to choose different elements for different n same as we did in Figure 9. Therefore to refute the implication for $n = 0$, we choose, for example, the elements a_0 and a_2. First, we use a_1 as the middle element that should be equal to both of the elements. In the nodes we see in this model, we have b_0 as the middle element. In the accessible nodes it would always be the element with the index number 0.

For $n = 1$, we use a_0 and a_3 (or any other elements distant two other elements from each other). In this situation, we need a middle element for a_0 and a_2 and another middle element for a_3 and a_1. We always choose the one in the middle: $a_1, b_0, c_0, \ldots$ for the first case and $a_2, b_1, c_1, \ldots$ for the other.

For higher n it works the same way.

QED

We could now prove the stability axiom S_n does not imply the axiom S_{n+1}. Instead we will prove the stronger stability axiom is equivalent to the formula we proved in the previous lemma if we have S_n in our assumption.

Lemma 4.8 *Assuming S_n, the stronger stability axiom S_{n+1} is equivalent with the sentence (φ): $\forall x \forall y(\neg(x \neq_{n+1} y) \to \neg(x \neq_n y))$*

Proof. First $S_{n+1} \to \varphi$

Let S_{n+1} and $\neg(x \neq_{n+1} y)$ be our assumptions. Because the premise from φ is valid and the axiom S_{n+1} has the same premise, we get the conclusion $x = y$. We know $x = y \to \neg(x \neq y)$. From $x \neq_n y \to x \neq y$, which we have proven in Lemma 4.3, we get $\neg(x \neq y) \to \neg(x \neq_n y)$

Now $\varphi \to S_{n+1}$

Let φ and $\neg(x \neq_{n+1} y)$ be our assumptions. Again because S_{n+1} and φ have the same premises, we have the conclusion $\neg(x \neq_n y)$ from φ. Then from S_n, we get the conclusion $x = y$. QED

Now that we now those two sentences are equivalent, we can prove the stability axiom S_n does not imply the formula φ. From that it will be clear the axiom S_n does not imply S_{n+1} and therefore those axioms grow in strength for every other grade.

Lemma 4.9 EQ $\not\Vdash S_n \to \varphi$

Proof. Let us picture a model where φ is violated. In the root, we have:

$\alpha_0 \Vdash \neg(x \neq_{n+1} y)$

$\alpha_0 \not\Vdash \neg(x \neq_n y)$

The invalidity of $\neg(x \neq_n y)$ tells us we can go to a node α accessible from α_0, such that $\alpha \Vdash (x \neq_n y)$. Let us choose α as our new root of the model. We have the formula $\neg(x \neq_{n+1} y)$ satisfied in α from persistence and it is obvious $\alpha \not\Vdash \neg(x \neq_n y)$ since the formula behind the negation holds in α. We can then be sure φ is violated in this new model.

Now we need to show the stability axiom S_n is valid in this model. Since the formula $x \neq_n y$ is satisfied in α, we get from persistence For every β accessible from $\alpha, \beta \Vdash x \neq_n y$. Therefore, $\neg(x \neq_n y)$ is not satisfied in any node accessible from α. Since this formula is the premise of S_n we know the whole implication meets the condition for validity in this model and $\alpha \Vdash S_n$. QED

Corollary 4.10 EQ $\not\Vdash S_n \to S_{n+1}$ *for all n*

Proof. Immediate QED

We have shown that the stability axioms are increasing in strength for every additional grade.

4.2 Equality with apartness

We will now prove several small lemmas concerning the stability of equivalence in the theory of apartness that we will need for the proof of conservativity.

Lemma 4.11 *The stability of equality is valid in the theory of apartness [vD04].* AP $\Vdash \forall x \forall y \neg\neg(x = y) \rightarrow x = y$

Proof. $\neg\neg(x = y) \Leftrightarrow \neg\neg\neg(x \# y) \Leftrightarrow \neg(x \# y) \Leftrightarrow x = y$ QED

Corollary 4.12 *The theory of apartness is not conservative over the theory of equality because the relation of equality is not stable in* EQ*.*

We want to find a theory containing the theory of equality that AP is conservative over. First, let us prove a small Lemma.

Lemma 4.13 AP $\Vdash S_n$ *for all* n*.*

Proof. We will prove $\forall x \forall y (x \# y \rightarrow x \neq_n y)$ by induction on n.
$\forall x \forall y (x \# y \rightarrow x \neq_0 y)$ holds from the first axiom of the theory of apartness.
Our induction assumption is $\forall x \forall y (x \# y \rightarrow x \neq_n y)$. From that, we want to prove $\forall x \forall y (x \# y \rightarrow x \neq_{n+1} y)$:

$\forall x \forall y (x \# y \rightarrow x \neq_n y) \rightarrow \forall x \forall y (x \# y \rightarrow x \neq_{n+1} y)$
$\forall x \forall y (x \# y \rightarrow x \neq_n y) \rightarrow \forall x \forall y (\forall z (z \# x \vee z \# y) \rightarrow (x \neq_{n+1} y))$ From the third axiom of apartness.
$\forall x \forall y (x \# y \rightarrow x \neq_n y) \rightarrow \forall x \forall y (\forall z (z \neq_n x \vee z \neq_n y) \rightarrow x \neq_{n+1} y)$ Using the induction assumption.
$\forall x \forall y (x \# y \rightarrow x \neq_n y) \rightarrow \forall x \forall y (\forall z (z \neq_n x \vee z \neq_n y) \rightarrow \forall z (z \neq_n x \vee z \neq_n y))$
From the definition of inequalities. QED

4.2.1 Conservativity

In this section, we outline the prove that the theory of apartness is conservative over the ω-stable theory of equality $\mathrm{EQ} + \{Sn | n \in \omega\}$. Before we can start with the proof itself, we need to show a few essential definitions and lemmas that we will borrow from an article On Axiomatizing Fragments [Smo77]. We will not use all of them but if the reader wants to dig deeper in the proof in the mentioned article then they are necessary. We will only show the main idea of it. We will denote theory as a pair (Γ, Δ) where Γ is the set of derivable sentences and Δ the set of underivable sentences.

Definition 4.14 A theory (Γ, Δ) is consistent iff there does not exist a sentence $\bigwedge \Gamma_0 \rightarrow \bigvee \Delta_0$ derivable in the predicate calculus for intuitionistic logic where Γ_0 is a finite subset of Γ and Δ_0 is a finite subset of Δ.

Definition 4.15 A theory (Γ, Δ) is complete iff for any sentence φ, $\varphi \in \Gamma$ or $\varphi \in \Delta$.

Definition 4.16 (Γ', Δ') extends the theory (Γ, Δ) if Γ is a subset of Γ'.

Definition 4.17 (Γ', Δ') strongly extends the theory (Γ, Δ) if Γ is a subset of Γ' and Δ is a subset of Δ'.

Definition 4.18 A theory (Γ, Δ) is a C-saturated theory iff

(i) (Γ, Δ) is consistent,

(ii) (Γ, Δ) is complete,

(iii) If the sentence $\exists x\varphi(x) \in \Gamma$, then $\varphi(c) \in \Gamma$ for some $c \in C$ where C is a set of constants.

Lemma 4.19 *(Henkin lemma) Let (Γ, Δ) be a consistent theory. Let C be an infinite but countable set of constants c that are not in the language of Γ. Then there exists a strong C-saturated extension (Γ', Δ') of (Γ, Δ).*

Theorem 4.20 *(Completeness theorem) $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$ [Smo73a].*

Now we will show the idea of the proof of conservativity itself.

Theorem 4.21 *(van Dalen and Statman) The theory of apartness is conservative over the ω-stable theory of equality SEQ^{ω} ($\mathrm{EQ} + \{Sn|n \in \omega\}$).*

Proof. We will outline the proof from On axiomatizing fragments [Smo77].
We want to prove that if a sentence φ in the language of SEQ^{ω} is valid in the theory of apartness then it is already valid in SEQ^{ω}.
Let (Γ, Δ) be a consistent extension of SEQ^{ω}. Let K be the model of SEQ^{ω}, such that every underivable φ is not satisfied in some node of the model. For our purposes, we can call K an universal model. For all $\alpha \in K$ and all elements a, b of the domain of the structure of α, we define $\alpha \Vdash a \# b \Leftrightarrow \forall n(\alpha \Vdash a \neq_n b)$.
Now let us show that $\alpha_0 \Vdash \forall x \forall y(x = y \Leftrightarrow \neg(x \# y))$.

The left-to-right implication is easy but we will show it anyways.
We have defined $\alpha \Vdash x \# y \ \rightarrow \ x \neq_n y$ for all elements. If we use contraposition, we get $\alpha \Vdash x = y \ \rightarrow \ \neg(x \# y)$.

For the right-to-left implication we need to prove a small lemma.

Lemma 4.22 *Let a, b such that $\Gamma \not\Vdash \neg(a \neq_n b)$ for any n. That means the set $\{\neg(a \neq_0 b), \neg(a \neq_1 b), \ldots\}$ is a subset of Δ. Then Γ united with the set $\{a \neq_0 b, a \neq_1 b, \ldots\}$ is consistent.*

Proof. Let us assume for contradiction that $\Gamma + a \neq_0 b + \ldots + a \neq_n b \Vdash \varphi \ \& \ \neg\varphi$ for an arbitrary n and φ. But since $\mathrm{SEQ}^{\omega} \Vdash a \neq_i b \ \rightarrow \ a \neq_j b$ for any $i > j$, we can reduce the set to $\Gamma + a \neq_n b$. Then we get $\Gamma \Vdash \neg(a \neq_n b)$. That is a contradiction of our assumption. QED

Now let us prove the implication $\neg(x \# y) \ \rightarrow \ x = y$ itself. Let (Γ, Δ) and some elements a, b such that $(\Gamma, \Delta) \Vdash \neg(a \# b)$. And let $(\Gamma, \Delta) \not\Vdash a = b$ for contradiction. Since K is a model of SEQ^{ω}, we know from the stability axioms that if $a = b$ is not valid, the formula $\neg(a \neq_n b)$ also can not be valid for any n. From the previous lemma we now know there exists an extension (Γ', Δ') such that $\Gamma' \Vdash a \neq_n b$ for all n. But we defined $\alpha \Vdash a \# b \Leftrightarrow \forall n(\alpha \Vdash a \neq_n b)$. Therefore, we would get $(\Gamma', \Delta') \Vdash a \# b$ which is a contradiction.

We can now see that we can choose $(\Gamma, \Delta) = (\mathrm{SEQ}^{\omega}, \{\varphi\})$ for any sentence φ that is underivable in SEQ^{ω} and see that we constructed a model of $(AP, \{\varphi\})$. QED

To summarize this chapter, we have defined the sequence of inequalities and proved that every other grade is stronger than the previous one. Using this knowledge we then showed the stability axioms also grow in strength. Then we proved several small lemmas about stability in the theory of apartness so that we could proceed with the outline of the proof of conservativity over SEQ^{ω}.

5 Linear order and apartness

In this chapter, we consider the theory of linear order in the presence of apartness. First, we will show there exist more axiomatization of LO and then we will show that LO + AP is conservative over the theory of linear order as well as over the theory of apartness.

5.1 Axiomatization

There exist two axiomatization of the theory of linear order.
The first axiomatization looks as follows [vD86]:

LO1: $\forall x \forall y \forall z (x < y \ \& \ y < z \rightarrow x < z)$,

LO2: $\forall x \forall y \forall z (x < y \rightarrow z < y \vee x < z)$,

LO3: $\forall x \forall y \forall z (x = y \equiv \neg(x < y) \ \& \ \neg(y < x))$

We can replace the equivalence in LO3 with implication only from right to left but then we would need to add another axiom stating antireflexivity of the ordering.

The second formulation of axioms looks like follows [Smo77]:

LO1: $\forall x \forall y \forall z (x < y \ \& \ y < z \rightarrow x < z)$,

LO2: $\forall x \forall y \forall z (x < y \rightarrow z < y \vee x < z)$,

LO3: $\forall x \neg(x < x)$

We do not have the antisymmetry axiom in any of those formulations but it is easy to show it holds. Since the first axiom is valid for all z, we choose x to be z in LO1 and we get $x < y \ \& \ y < x \rightarrow x < x$ and from antireflexivity we know this can not happen.

What differs in these formulations is the third axiom. It is clear that from $\forall x \forall y \forall z (x = y \Leftrightarrow \neg(x < y) \ \& \ \neg(y < x))$ we get the antireflexivity since for every x we have $x = x$. The question is if we can prove the axiom in the first formulation $\forall x \forall y \forall z (x = y \Leftrightarrow \neg(x < y) \ \& \ \neg(y < x))$ from the second formulation. The left-to-right implication is provable since we have the antireflexivity axiom in the second formulation. Now, let us construct a model where all axioms from the second axiomatization are valid but the formula $\neg(x < y) \ \& \ \neg(y < x) \rightarrow x = y$ is not.

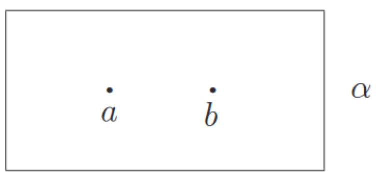

Figure 13: Model showing the second formulation is weaker

We need only one node where $a < b$ and $b < a$ is violated but a is not equal to the element b. Antireflexivity is satisfied here and since $a < b$ nor $b < a$ are not valid, the premises from LO1 and LO2 are violated. Therefore all axioms from the second axiomatization are valid here.

It is possible the axiom $\forall x \forall y \forall z(\neg(x < y)\ \&\ \neg(y < x) \to x = y)$ is only missing in the second formulation. Then the two axiomatization would be equivalent. Either way we will work with the first axiomatization and refer to it as LO.

Lemma 5.1 *Trichotomy is not provable in* LO*.*

Proof. We will construct a simple model where all the axioms are valid but trichotomy is not.

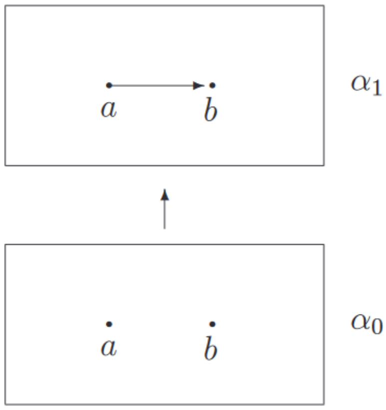

Figure 14: Model of LO where trichotomy does not hold

The arrow in α_1 denotes $a < b$. We can see a and b are not equal in the root. We also have $a < b$ and $b < a$ violated in α_0.
We need to have another node above α_0 where $a < b$ or $b < a$. otherwise we would get $a = b$ using LO3. It is clear all the axioms of LO are valid in this model. Transitivity is valid since the premise is not satisfied, LO2 is valid because we only have two elements, and the third axiom also does not satisfy the premises. QED

Trichotomy is stronger than the axiom LO2. It states that every two elements have to be either equal or have an arrow between, whereas the axiom we have states that if at least two elements have an arrow between themselves then every two elements have a relation (equality or the arrow) among themselves. But the premise does not always have to be valid so the whole implication is.
Since we do not have trichotomy, the third axiom does not induce the other possible case if $y < x$ and $x = y$ are not satisfied anywhere, we do not automatically get $x < y$. Let us prove it.

Lemma 5.2 *The sentence* $\forall x \forall y \neg(y < x)\ \&\ \neg(x = y) \to x < y$ *is not valid in* LO*.*

Proof. We can have the model from Figure 14 as a counterexample. We can see the formulas $\neg(b < a)$ and $\neg(a = b)$ are satisfied in α_0 since the formulas after the negations are not satisfied anywhere in this model. But we also have $\alpha_0 \not\Vdash a < b$. The node α_1 again serves the purpose to maintain the validity of LO3. QED

5.2 Conservativity

In this section, we investigate conservativity theorems concerning the theory of apartness and the theory of linear order. First, we create a model with two nodes where $\alpha_0 \not\Vdash x \# y$ but $\alpha_1 \Vdash x \# y$. It will help visualize a part of the proof of an upcoming lemma.

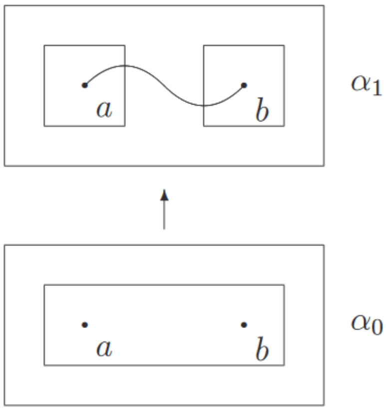

Figure 15: A model where $\alpha_0 \not\Vdash x \# y$ but $\alpha_1 \Vdash x \# y$

The rectangles symbolize equivalence classes with which we will work in the next lemma. The curved line between a and b in α_1 symbolizes $a \# b$.

Lemma 5.3 *Linear ordering is applicable on any model that has the apartness relation.*

Proof. This proof is a result from C. Smorynski [Smo77] who did it for the second axiomatization. We will do it for the stronger axiomatization we are working with. Let K be a model for the theory of apartness. Then for all nodes α, we have $\alpha \not\Vdash x \# y$ as an equivalence relation. Let us prove it.

(i) Reflexivity: $\alpha \Vdash x = x \Leftrightarrow \neg x \# x$ from the first apartness axiom. Because we know $\alpha \Vdash x = x$ then $\alpha \Vdash \neg x \# x$, hence $\alpha \not\Vdash x \# x$.

(ii) Symmetry: $x \# y \Leftrightarrow y \# x$, from the second apartness axiom. From this, it is obvious that $\alpha \not\Vdash x \# y \Leftrightarrow \alpha \not\Vdash y \# x$.

(iii) Transitivity: Let $\alpha \not\Vdash x \# y$ and $\alpha \not\Vdash y \# z$ and let us assume $\alpha \Vdash x \# z$ for contradiction. We get $\alpha \Vdash x \# y \vee \alpha \Vdash y \# z$ from the third apartness axiom which contradicts our assumption. Hence $\alpha \not\Vdash x \# z$.

In every node, we have the domain split into equivalence classes, such that if $\alpha_0 \not\Vdash x \# y$, those elements belong to the same equivalence class. We can arbitrarily linearly order those equivalence classes. But in the accessible nodes, we can have new elements and therefore possibly new equivalence classes. And even if we had a constant domain if $\alpha_0 < \alpha_1$ and $\alpha_0 \not\Vdash x \# y$ it does not necessarily mean $\alpha_1 \not\Vdash x \# y$ as shown in Figure 15. We linearly order the domain of the root and then continue to the accessible nodes. If the domain and the classes remain the same in all the nodes, we are done. Let us explore three possibilities that could be a bit problematic.

(i) If we get a new element a in the accessible node α_1, such that for any previously existing element b, $\alpha_1 \not\Vdash a \# b$, we will put a in the same equivalence class b is in.

(ii) If we get a new element that will create his new equivalence class, we will put his class above all the existing classes. If there are more new elements and more new equivalent classes, we will again arbitrarily linearly order them between themselves and then put them all above the others previously existing equivalence classes.

(iii) The third possibility is probably the most difficult one because we have to be careful to preserve persistence. Let us assume for some elements a, b, we have $\alpha_0 \not\Vdash a \# b$, therefore $a, b \in [a]$. But let us assume $\alpha_1 \Vdash a \# b$. That means that one of the element (let us choose the element b) have to "leave" the equivalence class $[a]$ and create his own $[b]$. This is the case we have shown in Figure 15. To preserve persistence, we need to check that if $b < c$ in the root for any element c then $[b] < [c]$ in all $\alpha \geq \alpha_0$. This will put the new equivalence class $[b]$ right next to the former one $[a]$. We can arbitrarily choose if $[a] < [b]$ or $[b] < [a]$. We know that if an element b "leaves" his equivalence class, it will create a new one. If b would join an existing class, it would not preserve persistence between b and all the elements belonging to the already existing class.

We will show later that the sentence $\forall x \forall y (x \# y \Leftrightarrow x < y \ \vee \ y < x)$ holds in this model.
Now we need to check we created a linear ordering.

(i) *Transitivity.* Let us assume $\alpha \Vdash a < b$ and $\alpha \Vdash b < c$. We then know from the way we constructed the ordering that $[a] < [b]$ and $[b] < [c]$. But since this ordering is a linear ordering, we get $[a] < [c]$. Therefore $\alpha \Vdash a < c$

(ii) *Weak linearity.* Let $\alpha \Vdash a < b$, hence $[a] < [b]$. Let us consider an arbitrary equivalence class $[c]$. Since the ordering on the equivalence classes is linear we get $[c] < [b] \ \vee \ [a] < [c]$ from LO2. Therefore $\alpha \Vdash c < b \ \vee \ a < c$.

(iii) *Antisymmetry.* Let us assume $\alpha \Vdash a = b$ then $[a] = [b]$. From linearity of the ordering we get from the third axiom that $\neg([a] < [b]) \ \& \ \neg([b] < [a])$. Hence $\alpha \Vdash \neg(a < b) \ \& \ \neg(b < a)$.

QED

Now that we know we can implement linear ordering on any model having the apartness relation, we can prove the two theorems about conservativity of AP + LO. Let us start with conservativity over AP.

Theorem 5.4 *(van Dalen-Statman) The theory* AP + LO *is conservative over the theory of apartness [vD04].*

Proof. Let us assume that AP $\Vdash\!\!\!\!/\ \varphi$ for a formula φ of the language of apartness and we want to show that there exists a model of AP+LO that is a counterexample for φ. From AP $\Vdash\!\!\!\!/\ \varphi$ we know that there exists a model K of AP, such that K is a counterexample of φ, hence in the root of this model we have $k_0 \Vdash\!\!\!\!/\ \varphi$. We can implement linear order in this model. We get a model K^* of the theory AP + LO. And because the formula φ was in the language of apartness we know, it can not contain the symbol $>$, hence the root of the new model $k_0^* \Vdash\!\!\!\!/\ \varphi$. We now have AP + LO $\Vdash \varphi \rightarrow$ AP $\Vdash \varphi$ for any formula φ in the language of apartness. QED

Now the conservativity of AP + LO over LO. For that we need to prove the following lemma.

Lemma 5.5 AP + LO $\Vdash x < y \vee y < x \Leftrightarrow x \# y$ *[Smo77].*

Proof. Immediate from Lemma 5.1. If $x \# y$, we know they are in different equivalence classes. As we order all of the equivalence classes between themselves, we have $[x] < [y] \vee [y] < [x]$. Therefore $x < y \vee y < x$. The direction from left to right is the same. QED

Theorem 5.6 *The theory* AP + LO *is conservative over the theory of linear order.*

Proof. Immediate from the previous Lemma. QED

To sum this chapter up, we have shown two different axiomatization of the theory of linear order and we proved that one is stronger than the other. We then reproduced the proof of implementing a linear ordering on a model with the apartness relation from Smorynski [Smo77] but with the stronger axiomatization from van Dalen [vD86]. Using this claim, we then proved the theory AP + LO is conservative over the theory of apartness but also over the theory of linear order.

6 Conclusion

The main aim of this thesis was to explore three theories in intuitionistic logic. We chose two theories we have in classical logic: the theory of equality and the theory of linear order and one theory that is not well known in classical logic: the theory of apartness. We constructed new models to prove the stability axioms are increasing in strength. We then showed the theorem of the conservativity of the theory of apartness over the ω-stable theory of equality and proved the conservativity of $\mathrm{AP} + \mathrm{LO}$ over the theory of apartness and the theory of linear order.

First we have proven several lemmas to show to the reader the system of this logic. We also proved neither of the two theories trivializes intuitionistic logic, hence they are weaker than their classical versions.

We explored the theory of equality where the relation is not stable. We introduced the sequence of inequalities and stability axioms and constructed several models as counterexamples to show the implications $x \neq_n y \rightarrow x \neq_{n+1} y$ and the contraposition $\neg(x \neq_{n+1} y) \rightarrow \neg(x \neq_n y)$ are not valid in the theory. Then we proved the stability axioms hold in the theory of apartness so that we could show the conservativity theorem.

We then continued to the theory of linear order where we have shown two possible axiomatizations. In neither of this axiomatizations the law of trichotomy holds. We have proven we can implement linear ordering on any model of the theory apartness. This lemma helped us prove the conservativity theorems.

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