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OF MATHEMATICS AND PHYSICS
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## BACHELOR THESIS

Michal Medek

# Möbius function of matrix posets 

Computer Science Institute of Charles University

Supervisor of the bachelor thesis: doc. RNDr. Vít Jelínek, Ph.D.
Study programme: Computer Science
Study branch: Programming and software development Bc.

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Title: Möbius function of matrix posets
Author: Michal Medek
Department: Computer Science Institute of Charles University
Supervisor: doc. RNDr. Vít Jelínek, Ph.D., Computer Science Institute of Charles University

Abstract: In this work, we focus on the Möbius function $\mu(X, Y)$ of four variants of containment posets of sparse matrices, for which the Möbius function has not been studied before. A sparse matrix is a binary matrix containing at most one 1-cell in each row and column. We focus mainly on the dominated scattered containment, where $X \leq Y$ if $X$ can be created from $Y$ by removing some rows and columns and by changing some 1 -cells to 0 -cells. We consider this poset to be a generalization of the permutation poset, as for permutations $\sigma$ and $\pi$, if $\sigma \leq \pi$, then the permutation matrices $M_{\sigma}$ and $M_{\pi}$ satisfy $M_{\sigma} \leq M_{\pi}$. For the dominated scattered containment, we study the values of the Möbius function on intervals of the form $[\mathbf{1}, Y]$, where $\mathbf{1}$ is the $1 \times 1$ matrix consisting of a single 1 -cell. We show that the situation when $Y$ contains a zero row or column can be reduced to a situation when $Y$ has no such zero line, that is, $Y$ is a permutation matrix. For a permutation matrix $Y$, we derived a theorem expressing $\mu(\mathbf{1}, Y)$ in terms of the blocks of the sum decomposition of $Y$.

Keywords: Möbius function sparse matrix submatrix

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## 1. Introduction

This chapter will introduce the reader to the Möbius function. Here we state all the definitions and general claims about the Möbius function, which are used throughout this thesis. Feel free to skip the chapter if you are already familiar with the topic.

### 1.1 Möbius function

The Möbius function, denoted by the Greek letter $\mu$ (mu), is a mathematical function named after the German mathematician August Ferdinand Möbius, who introduced it in 1832 [1]. This function arises in number theory, where it has various applications. While it was initially defined for positive integers, the Möbius function has been generalized to partially ordered sets, or posets for short.

The poset Möbius function was introduced by the Italian-American mathematician Gian-Carlo Rota [2]. Rota has proposed a vast generalization of the classical Möbius function with the aim of using it to study the structure of partially ordered sets. The poset Möbius function establishes a profound connection between combinatorics and a wide range of other mathematical fields.

Researchers have since studied the poset Möbius function on various types of posets, such as ranked posets, word posets [3] [4] [5, permutation posets [6] [7], and more. This work focuses on the poset of sparse matrices, which can be seen as a generalization of the permutation poset.

### 1.2 Basic definitions and notation

Let us start with the key definitions of this thesis, which will lead us to the definition of the Möbius function. Let $P$ be a set. Let $\leq$ be a relation on $P$ where for every $X, Y, Z$ from $P$, the following must hold: $X \leq X$ (reflexivity), if $X \leq Y$ and $Y \leq X$, then $X=Y$ (anti-symmetry) and if $X \leq Y$ and $Y \leq Z$, then $X \leq Z$ (transitivity). Then the pair $(P, \leq)$ is called a partially ordered set (or poset for short). Throughout this thesis, we adopt the convention of using the letter $\mathcal{P}$ to denote a poset.

Let $X, Y$ be elements from $\mathcal{P}$. We say that $X$ and $Y$ are comparable if either $X \leq Y$ or $Y \leq X$. If $X<Y$ and there does not exist any element $Z$ from $\mathcal{P}$ where $X<Z<Y$, then $X$ is called a direct predecessor of $Y$ and $Y$ is called a direct successor of $X$.

A closed interval $[X, Y]$ denotes the subset of $\mathcal{P}$ containing all elements $Z$ from $\mathcal{P}$ where $X \leq Z \leq Y$. We let $(X, Y]$ denote the half-open interval $[X, Y] \backslash$ $\{X\}$ and $[X, Y)$ the half-open interval $[X, Y] \backslash\{Y\}$. Finally, the open interval $(X, Y)$ denotes $[X, Y] \backslash\{X, Y\}$. We say that a poset $\mathcal{P}$ is locally finite if for every $X, Y$ from $\mathcal{P}$, the closed interval $[X, Y]$ is finite (has finite number of elements).

For a locally finite poset $\mathcal{P}$, we can define its Hasse diagram as the pair ( $V, E$ ) of vertices and edges (we call this pair a graph), where $V$ is the set of all elements from $\mathcal{P}$ and $E$ is the set containing all pairs of elements $X, Y$ from $\mathcal{P}$ where $X$
is a direct predecessor of $Y$. We can draw the Hasse diagram, where we usually draw the edges from bottom to top. The Hasse diagram uniquely represents $\mathcal{P}$.

Now, we have stated all the definitions to be able to properly define the Möbius function.

Definition 1.1 (Möbius function). Let $\mathcal{P}=(P, \leq)$ be a locally finite poset and let $X, Y$ be elements of $P$. Then the Möbius function of the poset $\mathcal{P}$ is the function $\mu_{\mathcal{P}}: P \times P \rightarrow \mathbb{Z}$ defined by the formula

$$
\mu_{\mathcal{P}}(X, Y)= \begin{cases}0 & \text { if } X \neq Y, \\ 1 & \text { if } X=Y, \\ -\sum_{Z \in[X, Y)} \mu(X, Z) & \text { otherwise } .\end{cases}
$$

We write $\mu$ instead of $\mu_{\mathcal{P}}$ whenever the poset $\mathcal{P}$ is clear from the context. Typically, $X$ will denote the first argument of the Möbius function and $Y$ will denote the second one. The Möbius value will denote the value of the Möbius function.

There is some additional notation that we use in this thesis, and which needs to be introduced. Let $\mathcal{I}$ be a closed interval $[X, Y]$. An element $Z \in(X, Y)$ is called a cut of size one in $\mathcal{I}$ if $Z$ is comparable to every element of $\mathcal{I}$; see Figure 1.1.

A chain is a subset $C$ of $\mathcal{P}$ such that every two elements from $C$ are comparable. The length of the chain $C$, denoted by $|C|$, is defined as the size of the set $C$. An even chain is then a chain of even length and an odd chain is a chain of odd length. We say that a chain $C$ is a chain from $X$ to $Y$ if $C$ contains $X$ and $Y$, and all its elements belong to $[X, Y]$. We usually use fraktur symbols, like $\mathfrak{C}$, to denote sets of chains. In particular, we let $\mathfrak{C}[X, Y]$ denote the set of all chains from $X$ to $Y$. For a set of chains $\mathfrak{C}$, we define $w(\mathfrak{C})$ as $\sum_{C \in \mathfrak{C}}(-1)^{|C|+1}$. For the purposes of the future proofs, a parity-reversing bijection is a bijection which maps odd-sized chains to even-sized ones and vice versa, while a parity-preserving bijection maps odd-size chains to odd-sized chains and even-sized to even-sized.

### 1.3 General claims about the Möbius function

There are some general claims that hold for the Möbius function and we will use them frequently in this thesis. All of them talk about alternative ways how to calculate the Möbius function. Some claims hold only under special conditions, others always. The first one says that if an interval contains a cut of size one, the Möbius function is always zero.

Lemma 1.2. Let $\mathcal{P}$ be a finite poset with Möbius function $\mu$ and let $X$ and $Y$ be elements of $\mathcal{P}$. If the closed interval $[X, Y]$ contains a cut of size one, then $\mu(X, Y)$ is equal to 0 .

Proof. Let $Z$ be the element of the cut, and let $r$ be $\mu(X, Z)$. By Definition 1.1,

$$
\sum_{W \in[X, Z)} \mu(X, W)=-r .
$$

We proceed by induction on the number of elements in $(Z, Y)$. If there is no element, then $Y$ is a direct successor of $Z$ and the value of $\mu(X, Y)$ is then

$$
-\sum_{W \in[X, Y)} \mu(X, W)=-\sum_{W \in[X, Z)} \mu(X, W)-\mu(X, Z)=r-r=0 .
$$

Otherwise, all the elements in $(Z, Y)$ have zero Möbius value by induction, and hence, the value of $\mu(X, Y)$ is then

$$
\begin{aligned}
-\sum_{W \in[X, Y)} \mu(X, W) & =-\sum_{W \in[X, Z)} \mu(X, W)-\mu(X, Z)-\sum_{W \in(Z, Y)} \mu(X, W) \\
& =r-r-0=0 .
\end{aligned}
$$



Figure 1.1: Four examples of the Hasse diagrams of $[X, Y]$ containing a cut of size one, where $Z$ is the element of the cut.

The next lemma says that if some elements in a poset have zero Möbius value, we can omit them from the poset. This can be useful if we combine this fact with the previous lemma and find a cut of size one in the thinner poset.

Lemma 1.3. Let $\mathcal{P}$ be a locally finite poset with Möbius function $\mu_{\mathcal{P}}$ and let $X$ and $Y$ be elements of $\mathcal{P}$. Let $\mathcal{P}^{*}$ be a poset which is created from $\mathcal{P}$ by removing some (possibly all) elements $Z \in[X, Y)$ such that $\mu_{\mathcal{P}}(X, Z)=0$. Then $\mu_{\mathcal{P}}(X, Y)$ is equal to $\mu_{\mathcal{P}^{*}}(X, Y)$.

Proof. We proceed by induction with respect to the number of elements in $(X, Y)$. If there is no or one element, the statement holds trivially.

For the induction step, let us consider some elements $X, Y$ from $\mathcal{P}$. From Definition 1.1 ,

$$
\mu_{\mathcal{P}}(X, Y)=-\sum_{\substack{Z \in \mathcal{P} \\ Z \in X, Y)}} \mu_{\mathcal{P}}(X, Z) .
$$

We can remove any elements $Z$ such that $\mu_{\mathcal{P}}(X, Z)=0$ because they do not contribute to the sum and by induction, all the values $\mu_{\mathcal{P}^{*}}(X, Z)$ are equal to $\mu_{\mathcal{P}}(X, Z)$. In mathematical notation,

$$
-\sum_{\substack{Z \in \mathcal{P} \\ Z \in[X, Y)}} \mu_{\mathcal{P}}(X, Z)=-\sum_{\substack{Z \in \mathcal{P}^{*} \\ Z \in[X, Y)}} \mu_{\mathcal{P} *}(X, Z) .
$$

Again, from Definition 1.1,

$$
-\sum_{\substack{Z \in \mathcal{P}^{*} \\ Z \in[X, Y)}} \mu_{\mathcal{P}^{*}}(X, Z)=\mu_{\mathcal{P}^{*}}(X, Y)
$$

The following observation describes the behavior of the Möbius function when we apply an automorphism on a poset $\mathcal{P}$. First, we need to define the automorphism of posets.

Definition 1.4 (automorphism of posets). Let $\mathcal{P}$ be a poset. An automorphism of $\mathcal{P}$ is a bijection $\psi$ on $\mathcal{P}$ such that for any two elements $X, Y$ of $\mathcal{P}$, it must hold that $X \leq Y$ if and only if $\psi(X) \leq \psi(Y)$.

Observation 1.5. Let $\mathcal{P}$ be a locally finite poset with Möbius function $\mu$. Let $\psi$ be an automorphism of $\mathcal{P}$. Then $\mu(X, Y)$ is equal to $\mu(\psi(X), \psi(Y))$.

This fact is very useful and saves a lot of time when proving some theorems for a trivial element $X$. We can prove just one variant and find an automorphism of the poset which proves the other variants.

Finally, there is an interesting variant of calculating the Möbius function. It says that the Möbius function can we calculated as the difference of the number of all odd chains from $X$ to $Y$ and the number of all even chains from $X$ to $Y$.

Lemma 1.6 (Rota [2). Let $\mathcal{P}$ be a locally finite poset with Möbius function $\mu$ and let $X$ and $Y$ be elements of $\mathcal{P}$. Let $\mathfrak{C}$ be the set of all chains from $X$ to $Y$. Then $\mu(X, Y)$ is equal to $w(\mathfrak{C})$.

Proof. The proof to this lemma can be found in Rota's article [2] about Möbius function.

### 1.4 Previous results for word and permutation posets

As was already stated in the introductory section about the Möbius function, we will focus on posets of sparse matrices, which we consider to be a generalization
of permutation posets. In this section, we will point out several works that have researched word and permutation posets, with a brief description of each work's focus.

A word is a string of elements from an alphabet. A word poset can either have a factor order or a subword order. The factor order defines the relation between words as word $\beta$ is a factor of word $\alpha$ if $\alpha=\gamma \beta \delta$ for some words $\gamma, \delta$. We can call this order consecutive. The subword order can be called non-consecutive, because $\beta$ is a subword of $\alpha$ if $\beta$ can be obtained from $\alpha$ by deleting some elements. The word posets were researched mainly by Anders Björner who presented his results in his articles. He focused on both, the Möbius function of subword order [3] 4] and also on the Möbius function of factor order [5]. He derived a recursive rule for the Möbius function of the factor order and a proof for combinatorial interpretation of the Möbius function of the subword order.

In general, there are two permutation posets, consecutive and non-consecutive. The difference is that for consecutive poset, a permutation $\sigma$ is contained in a permutation $\pi$ if $\sigma$ is a consecutive part of $\pi$. For non-consecutive, $\sigma$ can be a non-consecutive part of $\pi$. The non-consecutive poset is more general and therefore, more researched. These posets were researched by many researchers, but we were mainly inspired by the work of Burstein et al. [6], where the authors focused on the poset of permutations ordered by pattern containment. They explored the Möbius function of an interval $[\sigma, \pi]$ where $\pi$ is a decomposable permutation. They also showed that for any separable permutation $\pi$, the Möbius function of $(1, \pi)$ is either 0,1 or -1 . The permutation posets were also researched in the work by Brignall et al. [7, where the authors focused on the conditions when the value of the Möbius function is equal to zero.

Permutations can be represented by permutation matrices. A permutation matrix is a square matrix containing exactly one 1 -cell in every row and column. We can transform any permutation to a permutation matrix and vice versa. Then the permutation containment can be described by the following: a permutation $\sigma$ is contained in a permutation $\pi$ if the permutation matrix $M_{\sigma}$ is a submatrix of the permutation matrix $M_{\pi}$.

## 2. Sparse matrices

Sparse matrices can be seen as a generalization of permutation matrices as they do not need to contain 1-cell in every row and column. This means that they can contain zero lines anywhere in the matrix and furthermore, they can have a rectangular shape.

The Möbius function has been researched on posets of permutations (as was stated in the first chapter) and it is only natural to try to generalize the poset. One way to do this, is to substitute the permutation poset with the poset of sparse matrices, which is the central focus of this research.

### 2.1 Definition

For a matrix $X$, let $X_{i, j}$ denote the value in the $i^{\prime}$ th row in the $j^{\prime}$ 'th column. The first column of a matrix is the leftmost one and the first row of a matrix is the bottom one. A line of a matrix is a row or a column. A matrix of size $m \times n$ is a matrix of $m$ rows and $n$ columns. If $X_{i, j}$ is equal to 0 , we call it a 0 -cell. Similarly, $X_{i, j}$ equal to 1 is called a 1 -cell. If a matrix contains only 0 -cells and 1-cells, we call it a binary matrix. A sparse matrix is a binary matrix containing at most one 1-cell in every row and column. Typically, we denote sparse matrices by characters $W, X, Y, Z$. We let $S$ denote the set of all sparse matrices. From now on, when we talk about sparse matrix, we will use just the word matrix.

There are some special types of matrices, for which we will introduce our own notation. The smallest possible matrix is called the empty matrix and is a matrix of size $0 \times 0$. We denote it by $\emptyset$. Zero matrices are matrices containing only 0 -cells and are denoted by $0^{i \times j}$ where $i$ denotes the number of rows and $j$ denotes the number of columns. We do not consider $\emptyset$ to be a zero matrix so in the notation $0^{i \times j}$, we always assume that both $i$ and $j$ are at least 1 . Finally, 1 denotes the matrix of size $1 \times 1$ containing a 1 -cell.

A diagonal matrix is a special case of the permutation matrix, where the 1-cell in the first row is in the first column, the 1-cell in the second row in the second column, etc. As the name suggests, the 1-cells are forming a diagonal of the matrix that is visually increasing. We let $I_{n}$ denote the diagonal matrix with $n$ rows and $n$ columns. In this notation, $n$ is always assumed to be positive.

We will frequently use the operation of making a new matrix by adding a zero line to the edge of a given matrix. Let $X$ be an arbitrary non-empty matrix. Then $X^{\boxed{ }}$ will denote the matrix created from $X$ by adding a zero row to the top. For example

$$
X=\left(\begin{array}{l}
0100 \\
0000 \\
0001
\end{array}\right), X^{\boldsymbol{\Sigma}}=\left(\begin{array}{c}
0000 \\
0100 \\
0000 \\
0001
\end{array}\right)
$$

We can see two symbols in the notation: a dot and a line. The dot indicates the matrix we wrap, and the line indicates where we put the zero line. Following this rule, $X^{\mathbf{r}}$ denotes the matrix created from $X$ by adding a zero line to $X$ 's left side, $X^{\boldsymbol{\prime}}$ to $X^{\prime}$ 's right side and $X^{\dot{\bullet}}$ to $X$ 's bottom. If we want to add a zero line
to more than one side, we can combine the notation. For example, $X^{\beth}$ denotes the matrix created from $X$ by adding a zero line to $X$ 's top, bottom and right side. By wrapping, we mean adding zero lines to the sides of a matrix $Y$. We will denote wraps by small Greek letters such as $\alpha$ or $\beta$. The size of a wrap $\alpha$, denoted by $|\alpha|$, is defined as the number of zero lines added to a matrix $Y$. We let $\mathcal{W}$ denote the set of all wraps. Explicitly,
where • denotes the trivial wrap that adds no zero lines, i.e., $Y^{\bullet}=Y$ for any matrix $Y$.

Apart from making new matrices by wrapping them with zero lines, we will be making new matrices by connecting two matrices. Let $X$ be a matrix of size $m \times n$ and $Y$ a matrix of size $m \times o$. Then we let $X \mid Y$ denote a matrix of size $m \times(n+o)$ which is created by putting the matrices $X$ and $Y$ next to each other. An important condition for $X \mid Y$ to be a sparse matrix is that $X$ and $Y$ cannot contain a 1 -cell in the same row. For example,

$$
X=\left(\begin{array}{l}
10 \\
00 \\
00 \\
01
\end{array}\right), Y=\left(\begin{array}{l}
0000 \\
0000 \\
0100 \\
0000
\end{array}\right), X \left\lvert\, Y=\left(\begin{array}{c}
100000 \\
000000 \\
000100 \\
010000
\end{array}\right)\right.
$$

Let $X$ be a matrix of size $m_{1} \times n_{1}$ and $Y$ a matrix of size $m_{2} \times n_{2}$. The direct sum of two matrices $X$ and $Y$, denoted by $X \oplus Y$, is defined as the matrix of size $\left(m_{1}+m_{2}\right) \times\left(n_{1}+n_{2}\right)$ where the matrix $X$ forms the lower left corner of $X \oplus Y$ and $Y$ forms the upper right corner of $X \oplus Y$. The rest of the matrix is filled with 0-cells. For example,

$$
X=\left(\begin{array}{l}
100 \\
001 \\
010
\end{array}\right), Y=\binom{10}{01}, X \oplus Y=\left(\begin{array}{l}
00010 \\
00001 \\
10000 \\
00100 \\
01000
\end{array}\right) .
$$

This operation can be repeated as many times as we want. Also, the direct sum is associative. If we want to form the direct sum of $k$ copies of a matrix $X$, we can use the notation $\oplus_{k} X$ defined as $\underbrace{X \oplus \cdots \oplus X}_{k \text { times }}$. Finally, we say that $X$ is an indecomposable matrix if there do not exist non-empty matrices $Y_{1}$ and $Y_{2}$ such that $X=Y_{1} \oplus Y_{2}$.

### 2.2 Containment

The main topic of this thesis is the study of the Möbius function in the containment poset of sparse matrices. We will in fact study four variants of this poset, corresponding to four different notions of containment of sparse matrices. The containment can be dominated or exact. It can also be consecutive or scattered. This gives us the four variants of containment: dominated consecutive
denoted by DC, dominated scattered denoted by DS, exact consecutive denoted by EC and exact scattered denoted by ES.

Before we formally define the four containment relations, we need some auxiliary terminology. Let $X$ and $X^{\prime}$ be two matrices of the same size $m \times n$. We say that $X$ is dominated by $X^{\prime}$ if for every $i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, n\}$, we have $X_{i, j} \leq X_{i, j}^{\prime}$. Here are several examples:

$$
\left(\begin{array}{l}
0000 \\
0001 \\
0000 \\
0000 \\
0100
\end{array}\right) \leq\left(\begin{array}{l}
1000 \\
0001 \\
0010 \\
0000 \\
0100
\end{array}\right),\left(\begin{array}{l}
010 \\
100 \\
001
\end{array}\right) \leq\left(\begin{array}{l}
010 \\
100 \\
001
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right) \leq\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right),\binom{0000}{1000} \leq\binom{ 0001}{1000} .
$$

We also say that the matrix $X$ of size $m \times n$ is equal to the matrix $Y$ of size $k \times l$ if $m=k, n=l$ and for every $i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, n\}$, we have $X_{i, j}=Y_{i, j}$.

Let $Y$ be a matrix of size $m \times n, R \subseteq\{1, \ldots, m\}$ a set of rows and $C \subseteq$ $\{1, \ldots, n\}$ a set of columns. We let $Y[R \times C]$ denote the scattered submatrix of $Y$ induced by the entries on the intersection of the rows $R$ and columns $C$. An example of such relation is the following:

$$
\begin{gathered}
Y=\left(\begin{array}{l}
10000 \\
00100 \\
01000 \\
00001
\end{array}\right), \\
R=\{2,4\}, C=\{2,3,5\}, \\
R \times C=\{(2,2),(2,3),(2,5),(4,2),(4,3),(4,5)\}, \\
Y[R \times C]=\binom{000}{100} .
\end{gathered}
$$

A consecutive set ${ }^{T}$ is a set $\{i, \ldots, j\}$ for some $i, j$ natural, where $i \leq j$. Here are some examples of consecutive sets:

$$
\{1\},\{4,5,6,7\},\{1,2\},\{9\},\{2, \ldots, 11\} .
$$

If $R$ and $C$ are consecutive sets, then we say that $Y[R \times C]$ is a consecutive submatrix of $Y$ induced by the entries on the intersection of the rows $R$ and columns $C$. An example of such relation is the following:

$$
\begin{gathered}
Y=\left(\begin{array}{c}
10000 \\
00100 \\
01000 \\
00001
\end{array}\right), \\
R=\{1,2\}, C=\{2,3,4,5\}, \\
R \times C=\{(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5)\}, \\
Y[R \times C]=\binom{1000}{0001} .
\end{gathered}
$$

[^0]Now, we proceed to the definition of the containments. Let $Y$ be a matrix of size $m \times n$. For dominated consecutive containment, we say that $X \leq_{\mathrm{DC}} Y$ if there are consecutive sets $R$ and $C$ such that $X$ is dominated by $Y[R \times C]$. For better understanding, see the following examples:

$$
\left(\begin{array}{l}
100 \\
000 \\
001 \\
000
\end{array}\right) \leq_{\mathrm{DC}}\left(\begin{array}{c}
1000 \\
0100 \\
0010 \\
0001
\end{array}\right),(0000) \leq_{\mathrm{DC}}\binom{1000}{0010},(0) \leq_{\mathrm{DC}}(1) .
$$

For dominated scattered containment, we say that $X \leq_{\text {DS }} Y$ if there are sets $R$ and $C$ such that $X$ is dominated by $Y[R \times C]$. Here are a few examples of this relation:

$$
\left(\begin{array}{l}
100 \\
000 \\
000 \\
001
\end{array}\right) \leq_{\mathrm{DS}}\left(\begin{array}{l}
1000 \\
0100 \\
0010 \\
0001
\end{array}\right),\binom{10}{01} \leq_{\mathrm{DS}}\binom{1000}{0010},(0) \leq_{\mathrm{DS}}(1)
$$

For exact consecutive containment, we say that $X \leq_{\text {EC }} Y$ if there are consecutive sets $R$ and $C$ such that $X$ is equal to $Y[R \times C]$. Here are a few examples of this relation:

$$
\left(\begin{array}{l}
100 \\
010 \\
001 \\
000
\end{array}\right) \leq_{\mathrm{EC}}\left(\begin{array}{l}
1000 \\
0100 \\
0010 \\
0001
\end{array}\right),(010) \leq_{\mathrm{EC}}\binom{1000}{0010},\binom{01}{10} \leq_{\mathrm{EC}}\left(\begin{array}{l}
1000 \\
0010 \\
0100 \\
0001
\end{array}\right)
$$

For exact scattered containment, we say that $X \leq_{\text {ES }} Y$ if there are sets $R$ and $C$ such that $X$ is equal to $Y[R \times C]$. Here are a few examples of this relation:

$$
\left(\begin{array}{l}
100 \\
010 \\
000 \\
001
\end{array}\right) \leq_{\mathrm{ES}}\left(\begin{array}{l}
1000 \\
0100 \\
0010 \\
0001
\end{array}\right),\left(\begin{array}{l}
10 \\
00 \\
00
\end{array}\right) \leq_{\mathrm{ES}}\left(\begin{array}{l}
1000 \\
0010 \\
0100
\end{array}\right),\binom{10}{01} \leq_{\mathrm{ES}}\left(\begin{array}{l}
1000 \\
0010 \\
0100 \\
0001
\end{array}\right) .
$$

Now, we can introduce the four posets: $\mathcal{S}_{\mathrm{DC}}=\left(S, \leq_{\mathrm{DC}}\right), \mathcal{S}_{\mathrm{DS}}=\left(S, \leq_{\mathrm{DS}}\right)$, $\mathcal{S}_{\mathrm{EC}}=\left(S, \leq_{\mathrm{EC}}\right)$ and $\mathcal{S}_{\mathrm{ES}}=\left(S, \leq_{\mathrm{ES}}\right)$. For the poset $\mathcal{S}_{\mathrm{DC}}$, the Möbius function on $\mathcal{S}_{\mathrm{DC}}$ will be denoted by $\mu_{\mathrm{DC}}$, a closed interval will be denoted by $[\cdot, \cdot]_{\mathrm{DC}}$ and we will use similar notation for other intervals. We apply the same rules also for the other three posets of sparse matrices. If it is clear from the context which of the four posets is considered, we omit the subscripts DC, DS, EC and ES and write simply $X \leq Y,[X, Y), \mu(X, Y)$ etc.

In general, neither pair of containments has the same results of $\mu(X, Y)$ for a pair of matrices $X, Y$. Also, the interval $[X, Y]$ usually differs for different types of containment. However, for any matrices $X, Y$, if $X \leq_{\mathrm{EC}} Y$, then both $X \leq_{\mathrm{ES}} Y$ and $X \leq_{\mathrm{DC}} Y$ and if either $X \leq_{\mathrm{ES}} Y$ or $X \leq_{\mathrm{DC}} Y$, then $X \leq_{\mathrm{DS}} Y$. We show the differences between the posets in Figure 2.1 and Figure 2.2 on the closed interval $\left[\emptyset,\left(\begin{array}{l}10 \\ 00 \\ 01\end{array}\right)\right]$.

For both exact containments, the layer of a matrix $X$ is defined as the sum of the number of rows and columns and for both dominated containments, it is defined as the sum of the number of rows, columns and 1-cells. There is an exception for $\emptyset$, where we say that $\emptyset$ is always on layer 1. Every two different matrices $X, Y$ on the same layer are incomparable. For a matrix $X$ on layer $n$, each direct predecessor of $X$ is on layer $n-1$ and each direct successor of $X$ is on layer $n+1$.


Figure 2.1: An example of the Hasse diagram of dominated consecutive poset on the left and of dominated scattered poset on the right, with numbered layers.

Now, we will state the relation between the posets of sparse matrices and the poset of permutations. For two permutations $\sigma$ and $\pi$, the following holds: $\sigma \leq \pi$ (in the usual sense of permutation containment) if and only if the permutation matrices $M_{\sigma}$ and $M_{\pi}$ satisfy $M_{\sigma} \leq_{\text {ES }} M_{\pi}$, which is equivalent to $M_{\sigma} \leq_{\text {DS }}$ $M_{\pi}$. However, since the intervals $\left[M_{\sigma}, M_{\pi}\right]_{\mathrm{ES}}$ and $\left[M_{\sigma}, M_{\pi}\right]_{\mathrm{DS}}$ also contain nonpermutation matrices, the values of Möbius function $\mu(\sigma, \pi)$ in the permutation poset will in general not be equal to either $\mu_{\mathrm{ES}}\left(M_{\sigma}, M_{\pi}\right)$ or $\mu_{\mathrm{DS}}\left(M_{\sigma}, M_{\pi}\right)$. Therefore, we cannot use the previous results on the Möbius function of the permutation poset to get results on the Möbius values in the posets of sparse matrices.


Figure 2.2: An example of the Hasse diagram of exact consecutive poset on the left and of exact scattered poset on the right, with numbered layers.

### 2.3 Introductory claims

The following claims are trivial and hold for all the four posets of sparse matrices we introduced in Section 2.2. They serve as a motivation for subsequent results in this thesis.

The first chapter provided some alternative ways how to calculate the Möbius function. One was Observation 1.5, which stated that an automorphism of a poset does not change the result of the Möbius function. Here, we will introduce some automorphisms, which are applicable on all four variants of the containment poset.

First, let us define some operations on matrices. The reverse operation is defined as flipping a matrix by its vertical axis. The complement operation is defined as flipping a matrix by its horizontal axis. The inverse operation is defined as flipping a matrix by its diagonal. For a matrix $X$, let $X^{R}$ denote the reversed matrix $X$, let $X^{C}$ denote the complemented matrix $X$ and let $X^{I}$ denote the inverted matrix $X$. Here are some examples of such operations:

$$
X=\left(\begin{array}{l}
010 \\
100 \\
000 \\
001
\end{array}\right), X^{R}=\left(\begin{array}{l}
010 \\
001 \\
000 \\
100
\end{array}\right), X^{C}=\left(\begin{array}{l}
001 \\
000 \\
100 \\
010
\end{array}\right), X^{I}=\left(\begin{array}{c}
1000 \\
0001 \\
0010
\end{array}\right)
$$

Observation 2.1 (trivial automorphisms). For each of the four posets we consider, if we apply any of the three operations of reverse, complement or inverse on every matrix from the poset, we get an automorphism of the poset.

We might compose these operations and obtain more automorphisms.

Observation 2.2 (symmetries). By composing the three operations, we obtain eight different automorphisms, including the identity mapping. We will call these eight automorphisms the trivial symmetries.

We actually need only two operations, the complement operation and the inverse operation, because for any matrix $X$, the matrix $X^{C}$ can be obtained by combining the previous two operations, in particular $X^{C}=\left(\left(X^{I}\right)^{R}\right)^{I}$.

Observation 2.2 is particularly useful for some trivial matrices $X$. For example, if $X$ is equal to $\emptyset, 0^{1 \times 1}$ or 1 , neither the reverse nor the inverse operation will change the matrix $X$. Let $\mathcal{Y}$ be the set of all symmetries of a matrix $Y$ (created by applying the reverse and the inverse operation on $Y$ multiple times). This means that for every two matrices $Y_{1}, Y_{2}$ from $\mathcal{Y}$, the values $\mu\left(X, Y_{1}\right)$ and $\mu\left(X, Y_{2}\right)$ are equal for each variant of the containment poset. This holds for any matrix $X$ that is not affected by the reverse and the inverse operation. Those are (additionally to $\emptyset, 0^{1 \times 1}$ or $\mathbf{1}$ ), for example, square zero matrices. Throughout this thesis, we will usually omit to explicitly state all the results that follow from our theorems by trivial symmetries.

In the following two lemmas, we state trivial but important results where for almost every zero matrix $Y$, the Möbius value is zero. The first lemma will prove that $\mu\left(\emptyset, 0^{i \times j}\right)$ is equal to zero and the second one will prove the same for $\mu\left(0^{1 \times 1}, 0^{i \times j}\right)$.

Lemma 2.3. For all $i, j$ where either $i \geq 2$ or $j \geq 2$, the values $\mu_{\mathrm{DC}}\left(\emptyset, 0^{i \times j}\right)$, $\mu_{\mathrm{DS}}\left(\emptyset, 0^{i \times j}\right), \mu_{\mathrm{EC}}\left(\emptyset, 0^{i \times j}\right)$ and $\mu_{\mathrm{ES}}\left(\emptyset, 0^{i \times j}\right)$ are all equal to zero.

Proof. For any $i, j$ where $i$ or $j$ is at least 2 , the matrix $0^{1 \times 1}$ is a cut of size one of the interval $\left[\emptyset, 0^{i \times j}\right]$ in all the four posets considered, and the Möbius value is zero by Lemma 1.2 .

Lemma 2.4. For all $i, j$ where either $i \geq 3$ or $j \geq 3$, the values $\mu_{\mathrm{DC}}\left(0^{1 \times 1}, 0^{i \times j}\right)$, $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, 0^{i \times j}\right), \mu_{\mathrm{EC}}\left(0^{1 \times 1}, 0^{i \times j}\right)$ and $\mu_{\mathrm{ES}}\left(0^{1 \times 1}, 0^{i \times j}\right)$ are all equal to zero.

Proof. Let $Y$ be the matrix $0^{i \times j}$. We proceed by induction with respect to $i+j$. It can be easily verified by calculation that the statement holds for $i+j \leq 5$.

For the induction step, let us assume that $i+j \geq 6$. We know that the statement holds for all the matrices $0^{k \times l}$, where $0^{k \times l} \in\left[0^{1 \times 1}, Y\right)$ and where either $k \geq 3$ or $l \geq 3$. Let $\mathcal{P}^{*}$ be a poset which is created from $\left[0^{1 \times 1}, Y\right)$ by removing all the matrices $0^{k \times l}$. By Lemma 1.3, $\mu\left(0^{1 \times 1}, Y\right)$ is equal to $\mu_{\mathcal{P}^{*}}\left(0^{1 \times 1}, Y\right)$. Finally, we find a cut of size one in $\mathcal{P}^{*}$ on exactly one of the matrices $0^{1 \times 2}, 0^{2 \times 1}$ or $0^{2 \times 2}$ and by Lemma 1.2, the value $\mu_{\mathcal{P}^{*}}\left(0^{1 \times 1}, Y\right)$ is equal to zero.

These lemmas will be useful in future proofs because when combined with Lemma 1.3, we can omit almost all zero matrices from the poset and still get the same result. In other words, when calculating $\mu\left(0^{1 \times 1}, Y\right)$ for some non-zero matrix $Y$ of size $m \times n$ where $m+n \geq 3$, we can ignore all zero matrices because the contributions of zero matrices of at most two rows and columns will cancel out themselves and other zero matrices contribute by zero.

These lemmas can be extended to any other matrix $X$. For a zero matrix $X$, we would use similar proof as for Lemma 2.4 and arbitrary matrix $X$ containing a 1-cell is not comparable with zero matrices.

## 3. Results for sparse matrices

This is the main chapter of this thesis. Here, we present the majority of the results we discovered for the Möbius function on the sparse matrix containment posets.

This chapter is split into two sections. The first one will present the results for both exact containments, while the second one will focus primarily on dominated scattered containment.

### 3.1 Exact containments

This section is dedicated to exploring the results of the Möbius function on exact posets. We focus on consecutive containment, but we show a parallel to scattered containment.

We begin with a lemma that states that for the matrix $X$ equal to $\emptyset$, any matrix that starts with two zero columns has zero Möbius value.

Lemma 3.1. Let $Y$ be an arbitrary matrix where the first two columns do not contain 1-cell (they are called zero columns). For such $Y$, the value $\mu_{\mathrm{EC}}(\emptyset, Y)$ is equal to zero.

Proof. We proceed by induction. The base case is the matrix $0^{1 \times 2}$, for which the statement holds by Lemma 2.3 .

Now, consider a matrix $Y$ which satisfies the assumption. Let $Y^{-}$be the matrix created from $Y$ by removing the leftmost column. By induction, all the matrices $W<Y$ not comparable with $Y^{-}$have zero Möbius value. That is because they all contain two zero columns on the left side. Let $\mathcal{P}^{*}$ be the poset which is created from $[\emptyset, Y)$ by removing all the matrices $W$. By Lemma 1.3, the value $\mu_{\mathrm{EC}}(\emptyset, Y)$ is equal to $\mu_{\mathcal{P}^{*}}(\emptyset, Y)$. Finally, we find a cut of size one in $\mathcal{P}^{*}$ on matrix $Y^{-}$and by Lemma 1.2 , the value $\mu_{\mathcal{P}^{*}}(\emptyset, Y)$ is equal to zero.

By symmetries, this lemma can be extended to all the matrices $Y$ starting or ending with two zero lines. It can also be extended to scattered containment, where we no longer require that the two zero lines are at the edge. The proof is similar to the proof of Lemma 3.1, so we will omit it.

Observation 3.2. For a matrix $Y$ containing a pair of adjacent zero lines, the values $\mu_{\mathrm{ES}}(\emptyset, Y)$ and $\mu_{\mathrm{ES}}(\mathbf{1}, Y)$ are both equal to zero.

### 3.1.1 Matrices with at most two 1-cells

This subsection focuses on the values of $\mu_{\mathrm{EC}}(\emptyset, Y)$, where $Y$ is a matrix with at most two 1-cells. We already know how zero matrices behave in these circumstances. Furthermore, there are only a few matrices that have single 1-cell and yield a non-zero Möbius value. We will mention these matrices and their results in this subsection. Finally, we state and prove how matrices with two 1-cells behave.

Let us begin by examining the behavior of matrices that contain two 1-cells situated in opposite corners. Let $\Theta$ denote the set of all matrices with two 1-cells,
which are not in adjacent columns, or they are not in adjacent rows, with the first one located in the upper-left corner and the second one in the lower-right corner.

Lemma 3.3. For a matrix $Y$ from $\Theta$, the value $\mu_{\mathrm{EC}}(\emptyset, Y)$ is equal to -1 .
Proof. Consider a matrix $Y$ from $\Theta$. Let us consider its submatrices. We will divide them into three sets. Let $\mathcal{A}$ denote the first set, which contains only $0^{i \times j}$, where $i \geq 2$ or $j \geq 2$. For every $A$ in $\mathcal{A}$, the value $\mu_{\mathrm{EC}}(\emptyset, A)$ is equal to zero by Lemma 2.3 .

Let $\mathcal{B}$ denote the set containing matrices of size $i \times j$, where $i \geq 3$ or $j \geq 3$, with exactly one 1 -cell. The 1 -cell must be in the corner of the matrix so by Lemma 3.1, $\mu_{\mathrm{EC}}(\emptyset, B)$ is equal to zero for every $B$ in $\mathcal{B}$.

Let $\mathcal{C}$ denote the set of the remaining submatrices of $Y$. In particular,

$$
\mathcal{C}=\left\{\mathbf{1}^{\lrcorner}, \mathbf{1}^{\Gamma}, \mathbf{1}^{\dot{\circ}}, \mathbf{1}^{\top}, \mathbf{1}^{\cdot}, \mathbf{1}^{{ }^{\circ}}, \mathbf{1}, 0^{1 \times 1}, \emptyset\right\} .
$$

The results of matrices in $\mathcal{C}$ are

$$
\begin{gathered}
\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{\bullet}\right)=\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{\bullet}\right)=\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{\bullet}\right)=\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{1}\right)=\mu_{\mathrm{EC}}(\emptyset, \emptyset)=1, \\
\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{\lrcorner}\right)=\mu_{\mathrm{EC}}\left(\emptyset, \mathbf{1}^{\mathbb{}}\right)=\mu_{\mathrm{EC}}(\emptyset, \mathbf{1})=\mu_{\mathrm{EC}}\left(\emptyset, 0^{1 \times 1}\right)=-1 .
\end{gathered}
$$

Finally,

$$
\begin{aligned}
\mu_{\mathrm{EC}}(\emptyset, Y) & =-\sum_{Z \in[\emptyset, Y)} \mu_{\mathrm{EC}}(\emptyset, Z) \\
& =-\sum_{A \in \mathcal{A}} \mu_{\mathrm{EC}}(\emptyset, A)-\sum_{B \in \mathcal{B}} \mu_{\mathrm{EC}}(\emptyset, B)-\sum_{C \in \mathcal{C}} \mu_{\mathrm{EC}}(\emptyset, C) \\
& =-0-0-1 \\
& =-1 .
\end{aligned}
$$

We have shown how matrices containing exactly two 1 -cells in the opposite corners, where the 1-cells are not in adjacent columns, or they are not in adjacent rows, behave. Next, we show what happens if we wrap a matrix $Y$ containing one or two 1-cells with zero lines. We do not have to consider adding more than one zero line to a side of $Y$ because such matrix will have zero Möbius value by Lemma 3.1

Lemma 3.4. Let $Y$ be a matrix containing one or two 1-cells that has a 1-cell in the leftmost row, and if it has two 1-cells, then either they are not in adjacent columns, or they are not in adjacent rows. Then $\mu_{\mathrm{EC}}\left(\emptyset, Y^{\bullet}\right)=-\mu_{\mathrm{EC}}(\emptyset, Y)$.

Proof. By Lemma 1.3, we can ignore all matrices $W \leq Y$ starting or ending with two zero lines because they do not contribute to $\mu_{\mathrm{EC}}(\emptyset, Y)$ by Lemma 3.1.

Let $\mathfrak{C}$ be the set of all chains from $\emptyset$ to $Y^{\mathbf{1}}$. We split $\mathfrak{C}$ into two sets: $\mathfrak{C}_{1}$, the set of chains containing $Y$ and $\mathfrak{C}_{2}$, the set of chains not containing $Y$.

The value $w\left(\mathfrak{C}_{1}\right)$ is equal to $-w(\mathfrak{C}[\emptyset, Y])$. This is because both sets $\mathfrak{C}_{1}$ and $\mathfrak{C}[\emptyset, Y]$ have the same number of chains and by removing $Y^{\bullet}$ from every chain in $\mathfrak{C}_{1}$, we obtain a chain from $\mathfrak{C}[\emptyset, Y]$, and this is a parity-reversing bijection.

The value $w\left(\mathfrak{C}_{2}\right)$ is equal to zero. This can be proven by finding a parityreversing involution between chains in $\mathfrak{C}_{2}$, which will show that there is an equal
number of even and odd chains in $\mathfrak{C}_{2}$. For a chain $C$ from $\mathfrak{C}_{2}$, let $Z$ denote the biggest matrix in $C$ with a 1 -cell in the leftmost column. For $Z=\emptyset$, we define the involution by adding/removing the matrix $0^{1 \times 1}$ to/from $C$. For $Z \neq \emptyset$, we can add/remove the matrix $Z^{\bullet}$. Removing $Z^{\bullet \bullet}$ from a chain trivially produces another chain from $\mathfrak{C}_{2}$, in which $Z$ is again the biggest matrix with a 1-cell in its leftmost column. The difficult part is to show that if a chain $C$ does not contain $Z^{\prime}$, we can add it. Let $M$ be the smallest matrix in $C$ larger than $Z$. We know that $Z<Z^{\cdot}$ and $Z^{\cdot \bullet} \neq M$. By the definition of $Z$, we also know that the leftmost column of $M$ contains no 1-cell. We have to prove that $Z^{\bullet}<M$.

If $Y$ does not contain the two 1-cells in adjacent columns, then there is at least one zero line to the left of every 1-cell in $M$ so we know that $M$ contains $Z^{1}$. Otherwise, $Y$ cannot contain the two 1-cells in adjacent rows so there must be at least one zero row between the 1-cells. Also, if $Z$ contains two 1 -cells, then there is at least one zero line to the left of two leftmost 1 -cell in $M$ so we know that $M$ contains $Z^{1}$. If $Z$ has only one 1 -cell, then $Z$ can have at most one zero row to each side of the 1 -cell. Otherwise, it would contain two adjacent zero rows on its edge and would have been removed from the poset. Therefore, $Z^{\bullet}$ is contained in $M$ because there cannot be another 1-cell in any of the three rows of $M$ that contain an occurrence of $Z^{\cdot}$. Thus, we have found the involution for $Z \neq \emptyset$. Finally,

$$
\begin{aligned}
\mu_{\mathrm{EC}}\left(\emptyset, Y^{\bullet}\right) & =w(\mathfrak{C})=w\left(\mathfrak{C}_{1}\right)+w\left(\mathfrak{C}_{2}\right) \\
& =-w(\mathfrak{C}[\emptyset, Y])+0 \\
& =-\mu_{\mathrm{EC}}(\emptyset, Y) .
\end{aligned}
$$

Thanks to symmetries, we can add a zero line to any other side of $Y$, not only to the left, and the Möbius value will still behave as stated in the lemma. This yields explicit formulas for the Möbius function of all matrices from $\Theta$ wrapped in zero lines.

Additionally, this lemma can be generalized to any matrix $Y$ with no two 1 -cells in adjacent columns and with a 1-cell in the leftmost column. The proof is similar, but we will omit it because it is not relevant for this subsection.

Finally, this lemma also holds for the exact scattered containment where we no longer require the special structure of $Y$. Again, the proof is omitted, because it is similar to the presented proof.

Now, we have proven all necessary lemmas for a general theorem, which explicitly states the results $\mu_{\mathrm{EC}}(\emptyset, Y)$ where $Y$ is a matrix containing at most two 1-cells. We will state only non-symmetrical results. If a matrix $Y$ is symmetrical to any matrix from the lemma, the assumptions also hold for $Y$ by Observation 2.2

Theorem 3.5. The Möbius values $\mu_{\mathrm{EC}}(\emptyset, Y)$ for matrices $Y$ with at most two 1 -cells are as follows:
(a) $\mu_{\mathrm{EC}}(\emptyset, Y)=-1$ for $Y=0^{1 \times 1}$,
(b) $\mu_{\mathrm{EC}}(\emptyset, Y)=0$ for any zero matrix $Y$ other than $0^{1 \times 1}$, and more generally, for any matrix $Y$ that has two adjacent zero lines at one of its edges,
(c) $\mu_{\mathrm{EC}}(\emptyset, Y)=(-1)^{|\alpha|+1}$ for $Y$ of the form $Z^{\alpha}$, where $\alpha \in \mathcal{W}$ and $Z \in \Theta \cup\{\mathbf{1}\}$, and finally,
(d) for $Y=I_{2}^{\alpha}$ where $I_{2}=\binom{10}{01}$ and $\alpha \in \mathcal{W}$,

$$
\mu_{\mathrm{EC}}(\emptyset, Y)=\left\{\begin{aligned}
-3 & \text { when }|\alpha|=0 \\
2 & \text { when }|\alpha|=1 \\
-1 & \text { when }|\alpha|=2, \\
0 & \text { when }|\alpha|=3 \text { or }|\alpha|=4 .
\end{aligned}\right.
$$

Proof. Cases (a) and (d) follow directly from calculation. Case (b) follows from Lemma 2.3 and Lemma 3.1. Case (c) follows from Lemma 3.3 and Lemma 3.4 for $Z \in \Theta$ and follow directly from calculation for $Z=1$.

Note that the four cases of the previous theorem cover all the matrices with at most two 1-cells, up to obvious symmetries.

We have somewhat similar results for exact scattered containment, but with a difference for matrices with two 1-cells. Matrices with zero and one 1-cell behave in the same way as for exact consecutive containment, so we will not repeat their results. Instead, we state an observation putting together consecutive and scattered containments for such matrices.

Observation 3.6. For any two matrices $W$ and $Z$ with at most one 1-cell, we have $W<_{\mathrm{EC}} Z$ if and only if $W<_{\mathrm{ES}} Z$, and consequently, for any two matrices $X$ and $Y$ with at most one 1-cell, $\mu_{\mathrm{EC}}(X, Y)=\mu_{\mathrm{ES}}(X, Y)$.

For exact scattered containment and for a matrix $Y$ with two 1-cells and two adjacent zero lines anywhere in the matrix, $\mu_{\mathrm{ES}}(\emptyset, Y)$ is equal to zero by Observation 3.2. Additionally, matrices with two 1-cells that do not contain two adjacent zero lines behave differently than in exact consecutive containment. We state the exact behavior in the following observation. The proof is omitted, as it can be easily obtained through calculation.

Observation 3.7. Let $X$ denote the matrix $\binom{10}{01}$, $Y$ denote the matrix $\binom{100}{001}$ and $Z$ denote the matrix $\left(\begin{array}{l}100 \\ 000 \\ 001\end{array}\right)$. Then

$$
\begin{gathered}
\mu_{\mathrm{ES}}(\emptyset, X)=-3, \\
\mu_{\mathrm{ES}}(\emptyset, Y)=2, \\
\mu_{\mathrm{ES}}(\emptyset, Z)=-2 .
\end{gathered}
$$

For a wrap $\alpha$ from $\mathcal{W}$, wrapping any of the matrices $X, Y, Z$ with $\alpha$ multiplies the value of $\mu_{\mathrm{ES}}(\emptyset, \cdot)$ by $(-1)^{|\alpha|}$.

The last part of the observation holds also by the generalized version of Lemma 3.4 talking about the exact scattered containment.

### 3.1.2 Diagonal matrices

This subsection focuses on diagonal matrices. Again, we set the matrix $X$ to the empty matrix $\emptyset$. The main result is that for natural $n \geq 3$, the value $\mu_{\mathrm{EC}}\left(\emptyset, I_{n}\right)$ is equal to -4 . This will be proved by analyzing the submatrices of $I_{n}$. We will discover that the majority of the submatrices have zero Möbius value by Lemma 3.1. The rest of them are matrices of type $\mathcal{Z}_{1}=\left\{I_{m}^{\bullet}, I \dot{\stackrel{\circ}{m}}, I_{m}^{\bullet}, I_{m}^{\bullet}\right\}$ and $\mathcal{Z}_{2}=\left\{I_{m}^{\bar{\Xi}}, I_{m}^{\mathbf{l P}^{\prime \prime}}, I_{m}^{\boldsymbol{\Gamma}}, I_{m}^{\stackrel{\rightharpoonup}{\prime}}\right\}$ where $m<n$. Finally, for $Z$ in $\mathcal{Z}_{1}$, the value $\mu_{\mathrm{EC}}(\emptyset, Z)$ is equal to 2 and for $Z$ in $\mathcal{Z}_{2}$, the value $\mu_{\mathrm{EC}}(\emptyset, Z)$ is equal to -1 .

Lemma 3.8. Let $A_{1}$ be the set of wraps $A_{1}=\left\{\mathbf{l}_{\bullet}, \boldsymbol{\bullet}, \boldsymbol{\bullet}, \mathbf{\top}\right\}$ and $A_{2}$ the set of


$$
\mu_{\mathrm{EC}}(\emptyset, Y)=\left\{\begin{aligned}
-4 & \text { when } \alpha=\bullet \\
2 & \text { when } \alpha \in A_{1} \\
-1 & \text { when } \alpha \in A_{2}
\end{aligned}\right.
$$

Actually, for wraps from $A_{1}$, the lemma holds even for $n \geq 2$ and for wraps from $A_{2}$, the lemma holds for any $n$ natural.

Proof. We proceed by induction with respect to $n$. It can be easily shown by calculation that all the statements hold for $n \leq 3$.

For the induction step, let us consider a diagonal matrix $I_{n}$ where $n \geq 4$. We know that the value $\mu_{\mathrm{EC}}\left(\emptyset, I_{n-1}\right)$ is equal to -4 by induction. Let $\mathcal{Y}$ denote the difference of the interval $\left[\emptyset, I_{n}\right)$ and the interval $\left[\emptyset, I_{n-1}\right)$. We split $\mathcal{Y}$ into two subsets. The subset $\mathcal{Y}_{1}$ will contain all matrices from $\mathcal{Y}$ with two zero lines which are on the edge. The rest of the matrices from $\mathcal{Y}$ will be contained in $\mathcal{Y}_{2}$.

For every $Y$ from $\mathcal{Y}_{1}$, the value $\mu_{\mathrm{EC}}(\emptyset, Y)$ is equal to zero by Lemma 3.1. The set $\mathcal{Y}_{2}$ can be listed explicitly. In particular,

The sum $\sum_{Y \in \mathcal{Y}_{2}} \mu_{\mathrm{EC}}(\emptyset, Y)$ is equal to zero by induction. Then

$$
\begin{aligned}
\mu_{\mathrm{EC}}\left(\emptyset, I_{n}\right) & =-\sum_{W \in\left[\emptyset, I_{n}\right)} \mu_{\mathrm{EC}}(\emptyset, W) \\
& =-\sum_{W \in\left[\emptyset, I_{n-1}\right)} \mu_{\mathrm{EC}}(\emptyset, W)-\sum_{Y \in \mathcal{Y}_{1}} \mu_{\mathrm{EC}}(\emptyset, Y)-\sum_{Y \in \mathcal{Y}_{2}} \mu_{\mathrm{EC}}(\emptyset, Y) \\
& =-4-0-0 \\
& =-4 .
\end{aligned}
$$

We use similar proof for showing that $\mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\mathbf{\bullet}}\right), \mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\dot{\dot{ }}}\right), \mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\bullet}\right)$ and $\mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\bar{\bullet}}\right)$ are all equal to 2. Similarly, $\mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\bar{\Xi}}\right), \mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{י \cdot \mathbf{1}}\right), \mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\mathbf{F}}\right)$ and $\mu_{\mathrm{EC}}\left(\emptyset, I_{n}^{\lrcorner}\right)$are all equal to -1 .

There are no similar results for scattered containment. Furthermore, the calculated results indicate that for diagonal matrices, the Möbius function is likely to diverge towards negative infinity. Here are some results:

$$
\begin{gathered}
\mu_{\mathrm{ES}}\left(\emptyset, I_{1}\right)=-1, \mu_{\mathrm{ES}}\left(\emptyset, I_{2}\right)=-3, \mu_{\mathrm{ES}}\left(\emptyset, I_{3}\right)=-10, \\
\mu_{\mathrm{ES}}\left(\emptyset, I_{4}\right)=-36, \mu_{\mathrm{ES}}\left(\emptyset, I_{5}\right)=-137, \mu_{\mathrm{ES}}\left(\emptyset, I_{6}\right)=-543, \ldots
\end{gathered}
$$

This sequence, up to the sign, seems to match A002212 from OEIS [8].

### 3.2 Dominated scattered containment

This section primarily focuses on the results of the Möbius function for dominated scattered poset. However, some of the results also apply to consecutive containment, and we will explicitly mention that fact whenever relevant.

It is natural to start examining the Möbius function on some fixed matrix $X$. Although, the smallest possible matrix is $\emptyset$, we will not focus on the results $\mu_{\mathrm{DS}}(\emptyset, Y)$, because they are equal to zero for almost any $Y$. That follows from the structure of the poset, which contains a cut of size one on the matrix $0^{1 \times 1}$. This fact also applies to dominated consecutive containment and the proof is identical, so we omit it.

Lemma 3.9. For an arbitrary matrix $Y$ except $\emptyset$ and $0^{1 \times 1}$, the value $\mu_{\mathrm{DS}}(\emptyset, Y)$ is equal to zero.

Proof. For any matrix $Y$, we find a size one cut on matrix $0^{1 \times 1}$ and by Lemma 1.2 , $\mu_{\mathrm{DS}}(\emptyset, Y)$ is equal to zero.

Naturally, we would move on to the matrix $X=0^{1 \times 1}$. Instead, we will focus on the matrix 1. We do that, because, as we state in the following lemma, $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to $-\mu_{\mathrm{DS}}\left(0^{1 \times 1}, Y\right)$ almost for every matrix $Y$.
Theorem 3.10. For any matrix $Y$ except for $0^{1 \times 1}, 0^{1 \times 2}, 0^{2 \times 1}, 0^{2 \times 2}$ (which are not comparable with $\mathbf{1})$, the value $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, Y\right)$ is equal to $-\mu_{\mathrm{DS}}(\mathbf{1}, Y)$.

Proof. The lemma holds for any zero matrix $Y$, because if $Y$ is a zero matrix of at least three rows or three columns, then

$$
\mu_{\mathrm{DS}}\left(0^{1 \times 1}, Y\right)=0=\mu_{\mathrm{DS}}(\mathbf{1}, Y),
$$

where the first equality holds by Lemma 2.4, and the second equality holds because 1 is not comparable with zero matrices.

Next, we proceed by induction. The base cases are non-zero matrices of size $i \times j$ where $i+j=3$, for which the statement holds, which can be shown by calculation.

Consider a non-zero matrix $Y$ of size $i \times j$ where $i+j \geq 4$. Let $\mathcal{Z}$ denote the set $\left\{0^{1 \times 1}, 0^{1 \times 2}, 0^{2 \times 1}, 0^{2 \times 2}\right\} \cap\left[0^{1 \times 1}, Y\right)$. For any $Y \notin \mathcal{Z}$, the $\operatorname{sum} \sum_{Z \in \mathcal{Z}} \mu_{\mathrm{DS}}\left(0^{1 \times 1}, Z\right)$ is equal to zero. The set $\mathcal{Z}$ always contains $0^{1 \times 1}$, where $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, 0^{1 \times 1}\right)$ is equal to 1 and it must contain either $0^{1 \times 2}$ or $0^{2 \times 1}$, where both $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, 0^{1 \times 2}\right)$ and $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, 0^{2 \times 1}\right)$ are equal to -1 so the sum is equal to zero. If $\mathcal{Z}$ contains both $0^{1 \times 2}$ and $0^{2 \times 1}$, then it also contains $0^{2 \times 2}$, where $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, 0^{2 \times 2}\right)$ is equal to 1 , and the sum is again equal to zero. For every matrix $Z$ from $\left[0^{1 \times 1}, Y\right) \backslash \mathcal{Z}$, the value $\mu_{\mathrm{DS}}\left(0^{1 \times 1}, Z\right)$ is equal to $-\mu_{\mathrm{DS}}(\mathbf{1}, Z)$ by induction. From the Definition 1.1,

$$
\begin{aligned}
\mu_{\mathrm{DS}}\left(0^{1 \times 1}, Y\right) & =-\sum_{Z \in\left[0^{1 \times 1}, Y\right) \backslash \mathcal{Z}} \mu_{\mathrm{DS}}\left(0^{1 \times 1}, Z\right)-\sum_{Z \in \mathcal{Z}} \mu_{\mathrm{DS}}\left(0^{1 \times 1}, Z\right) \\
& =-\sum_{Z \in[\mathbf{1}, Y)}-\mu_{\mathrm{DS}}(\mathbf{1}, Z)-0 \\
& =\sum_{Z \in[\mathbf{1}, Y)} \mu_{\mathrm{DS}}(\mathbf{1}, Z) \\
& =-\mu_{\mathrm{DS}}(\mathbf{1}, Y) .
\end{aligned}
$$

It can be proven in a similar way that this theorem holds also for consecutive containment. We omit the proof.

The theorem states that all the results of the Möbius function for dominated containment, where $X=0^{1 \times 1}$ are almost the same as for $X=1$, with the only difference being the sign of the result. Any theorem stating something for $X=$ $0^{1 \times 1}$ also holds for $X=1$. From now on, we will mainly focus on $X=1$ and present specific results only for $X=1\left(\right.$ not for $\left.0^{1 \times 1}\right)$.

### 3.2.1 Matrices containing a zero line

As we fixed the matrix $X$ to $\mathbf{1}$, we can now focus on different matrices $Y$. In this subsection, we will focus on matrices $Y$ containing a zero line. The zero line can be either on the edge of the matrix or somewhere in the middle of the matrix.

We start with matrices $Y$ containing a zero line which is not on the edge. For such $Y$, the value $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to zero. This will be very useful in future proofs, as we can omit those matrices from a poset and still get the same result.

Lemma 3.11. Let $Y$ be a matrix of size $m \times n$ where for $i$ natural where $1<i<n$, the $i$ 'th column of $Y$ does not contain any 1-cell. Then $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to zero.

Proof. We proceed by induction. The base cases are matrices of size $1 \times 3$ with a 0 -cell in the middle. For those matrices, the statement holds, which can be shown by calculation.

Now, consider a matrix $Y$ which satisfies the assumption. Let $Y^{-}$be the matrix created from $Y$ by removing the $i$ 'th column. By induction, all the matrices $W<Y$ not comparable with $Y^{-}$have zero Möbius value because they contain a zero column which is not on the edge. Let $\mathcal{P}^{*}$ be the poset which is created from $[\mathbf{1}, Y)$ by removing all such matrices $W$. By Lemma 1.3 , the value $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to $\mu_{\mathcal{P}^{*}}(\mathbf{1}, Y)$. Finally, we find a cut of size one in $\mathcal{P}^{*}$ on matrix $Y^{-}$and by Lemma 1.2 , the value $\mu_{\mathcal{P}^{*}}(\mathbf{1}, Y)$ is equal to zero.

By symmetries, this lemma can be extended to the matrices $Y$ having a zero row which is not on the edge.

Now, we turn to matrices containing a zero line on the edge. Such matrix $Z$ has similar result $\mu_{\mathrm{DS}}(X, Z)$ as $\mu_{\mathrm{DS}}(X, Y)$, where $Y$ is a matrix created from $Z$ by removing the zero line, with the only difference being the sign of the result. In other words, when we add a zero line to the edge of some matrix $Y$, we only change the sign of the result $\mu_{\mathrm{DS}}(X, Y)$. This can be seen as an analogue of Lemma 3.4, but has much less restrictive assumptions.

Furthermore, we derive a much more general theorem that states the relation between two matrices $X$ and $Y$ with a 1-cell in the leftmost column, where we add any number of zero lines to the left side of $X$ respectively $Y$. By symmetries, this theorem also holds for any other edge of the matrices.

For the purpose of this theorem, we introduce a notation for a matrix with zero columns on the left. Let $Z$ be a matrix of size $m \times n$ and $k$ a natural number. Then $0^{k} \mid Z$ will denote the matrix $0^{m \times k} \mid Z$.

Theorem 3.12. Let us have $k, l$ non-negative integers, $l \geq 1$ and $X, Y$ matrices.

$$
\mu_{\mathrm{DS}}\left(0^{k}\left|X, 0^{l}\right| Y\right)= \begin{cases}\mu_{\mathrm{DS}}\left(0^{k-l} \mid X, Y\right)-\mu_{\mathrm{DS}}\left(0^{k-l+1} \mid X, Y\right) & \text { if } k \geq l \\ -\mu_{\mathrm{DS}}(X, Y) & \text { if } k=l-1 \\ 0 & \text { if } k \leq l-2\end{cases}
$$

Proof. Let $X^{0}$ denote $0^{k} \mid X$ and let $Y^{0}$ denote $0^{l} \mid Y$. Let $\mathfrak{C}$ be the set of all chains from $X^{0}$ to $Y^{0}$. Let $\mathfrak{C}(Z)$ denote the subset of chains $C$ whose largest matrix with fewest zero columns on the left is the specific matrix $Z$. Consider the following cases:

- If $Z=Y^{0}$, then $\mathfrak{C}\left(Y^{0}\right)$ is empty when $k<l$ and otherwise has weight $\mu_{\mathrm{DS}}\left(0^{k-l} \mid X, Y\right)$.
- If $Z=0^{l-1} \mid Y$, then $\mathfrak{C}\left(0^{l-1} \mid Y\right)$ is empty when $k<l-1$ and otherwise has weight $-\mu_{\mathrm{DS}}\left(0^{k-l+1} \mid X, Y\right)$. If $k=l-1$, then $k-l+1$ is equal to zero, hence the second case in the theorem.
- For all other $Z$, the weight of $\mathfrak{C}(Z)$ is zero because of the parity-reversing involution on $\mathfrak{C}(Z)$. The involution is defined by adding/removing a matrix $Z^{\boldsymbol{V}}$ to/from a chain $C$, by which we obtain another chain from $\mathfrak{C}(Z)$. We can do that because $Z^{\cdot}$ must exist $(k \leq l-2)$ and $Z^{\bullet} \neq Y^{0}$.

As stated, this theorem is very general. We now introduce some special cases of this theorem, which might be more understandable for the reader. In the following corollaries, we wrap the matrices with zero lines only on the left side. By symmetries, it holds for any other side.
Corollary 3.13. For $X$ with a 1-cell in the leftmost column, $\mu_{\mathrm{DS}}\left(X, 0^{k} \mid X\right)$ is equal to zero for any $k$ natural where $k \geq 2$.

An interesting special case of the theorem is when we set $k=0$ and $l=1$.
Corollary 3.14. For $X, Y$ with a 1-cell in the leftmost column, $\mu_{\mathrm{DS}}\left(X, Y^{\vee \cdot}\right)$ is equal to $-\mu_{\mathrm{DS}}(X, Y)$.

This corollary simply states that adding a zero line to the edge of a matrix changes the sign of the Möbius value. This holds for any matrix $X$ with a 1-cell in the leftmost column, and especially, for 1 . Notice, that if a matrix $Y$ has a 1-cell on every side, we can gradually add a zero line to each side of $Y$ and we will only change the sign of the result $\mu_{\mathrm{DS}}(X, Y)$. An example of such matrices are the permutation matrices.

Now we know, how matrices containing a zero line behave. We move on to the matrices not containing any zero line. These are called permutation matrices.

### 3.2.2 Permutation matrices

We recall that permutation matrices are square matrices containing exactly one 1-cell in every row and column. Such matrices do not contain any zero line.

This subsection will mainly focus on sum decompositions of such matrices. This can be seen as a parallel to sum decompositions of permutations [6], but here,
we present different results with different proofs. We derive a theorem stating that if a permutation matrix can be decomposed to identical indecomposable matrices, its Möbius value will behave regularly, but if it cannot, then its Möbius value is zero.

Theorem 3.15. Let $B_{1}, B_{2}, \ldots, B_{m}$ be indecomposable permutation matrices, and let $Y=B_{1} \oplus B_{2} \oplus \cdots \oplus B_{m}$. If $B_{1}=B_{2}=\cdots=B_{m}$, then $\mu_{\mathrm{DS}}(\mathbf{1}, Y)=\mu_{\mathrm{DS}}\left(\mathbf{1}, B_{1}\right)$, otherwise $\mu_{\mathrm{DS}}(\mathbf{1}, Y)=0$.

First, we introduce some notation. The decomposition of $Y$ are the indecomposable permutation matrices $B_{1}, B_{2}, \ldots, B_{m}$, where $Y=B_{1} \oplus B_{2} \oplus \cdots \oplus B_{m}$. The decomposition of $Y$ is always unique. We say that the decomposition of $Y$ is uniform if $B_{1}=B_{2}=\cdots=B_{m}$. Then the matrix $B_{1}$ is then called the core of $Y$ and $Y$ is uniformly decomposable. We also say that for a wrap $\alpha$ and matrices $W, Y$, the wrap $\alpha$ is admissible for $W$ in $Y$ when $W^{\alpha}$ is a proper submatrix of $Y$.

By this notation, the first part of the lemma says that if $X$ is the core of $Y$, then $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to $\mu_{\mathrm{DS}}(\mathbf{1}, X)$ and the second part says that if $Y$ is not uniformly decomposable, then $\mu_{\mathrm{DS}}(\mathbf{1}, Y)=0$. Also, notice that for a matrix $Y$ with trivial decomposition $\left(Y=B_{1}\right)$, we only state that $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to $\mu_{\mathrm{DS}}\left(\mathbf{1}, B_{1}\right)$, which is trivially true.

Proof. We proceed by induction with respect to the size of the matrix $Y$. The theorem holds for matrices up to size $3 \times 3$, which can be shown by calculation.

For the induction step, let us consider a matrix $Y$ of size $n \times n$ where $n \geq 4$. Let $W$ be a uniformly decomposable matrix with core $Z$ and let it be a submatrix of $Y$. According to the induction, for calculating $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$, it is sufficient to sum up the values $\mu_{\mathrm{DS}}\left(1, W^{\alpha}\right)$ for $\alpha$ from $\mathcal{W}$ admissible for $W$ in $Y$. The main idea of the proof is that for a fixed $Z$, the Möbius values of all the admissible wraps of all uniformly decomposable matrices with core $Z$ sum up to zero, except when $Z=X$.

In other words, let $\mathcal{M}(Z)$ be the set

$$
\begin{aligned}
\{U \in & {[1, Y), U=U_{0}^{\alpha}, U_{0} \text { is a uniformly decomposable } } \\
& \text { permutation matrix with core } Z, \alpha \in \mathcal{W}\}
\end{aligned}
$$

The value $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is then equal to

$$
-\sum_{\substack{Z \in[1, Y) \\ \text { indecomposable }}} \underbrace{\sum_{U \in \mathcal{M}(Z)} \mu_{\mathrm{DS}}(\mathbf{1}, U)}_{\text {equals zero, except when } Z \text { is the core of } Y} .
$$

Let $Z$ be an indecomposable matrix and let $k$ be the largest possible natural number such that the matrix $W=\oplus_{k} Z$ is a submatrix of $Y$. Now, let us consider the following cases.

First, let $W$ be a submatrix of the matrix $B_{2} \oplus \cdots \oplus B_{m}$. Then $Z$ is not a submatrix of $B_{1}$. For $1 \leq j<k$, any wrap $\alpha$ from $\mathcal{W}$ is admissible for $\oplus_{j} Z$ in $Y$. Any wrap $\beta$ from $\{\bullet, \boldsymbol{\bullet}, \bullet \bullet, \boldsymbol{\bullet}\}$ is admissible for $W^{\gamma}$ in $Y$ for $\gamma$ from the subset of $\{\boldsymbol{\bullet}, \mathbf{-}, \boldsymbol{\bullet}, \boldsymbol{\nabla}\}$ where $\gamma$ is admissible for $W$ in $Y$. For fixed wrap $\gamma$, there are always four variants of wraps from $\beta$. There are two odd sized ( $\mathbf{l} \bullet$ and $\boldsymbol{\bullet}$ ) and two even sized ( $\bullet$ and $\boldsymbol{\bullet}$ ) wraps, and so, their contributions will cancel
out themselves. By that, all the matrices $\left(W^{\gamma}\right)^{\beta}$ and $\left(\oplus_{j} Z\right)^{\alpha}$ will contribute to $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ by zero.

Second, let $W$ be a submatrix of the matrix $B_{1} \oplus \cdots \oplus B_{m-1}$. This case is symmetrical to the previous case so by very similar proof, for $\alpha$ from $\mathcal{W}, \beta$ from $\{\bullet, \mathbf{\bullet}, \boldsymbol{\bullet}, \boldsymbol{\nabla}\}$ and $\gamma$ from the subset of $\{\bullet, \boldsymbol{\bullet}, \bullet, \bullet\}$ where $\gamma$ is admissible for $W$ in $Y$, all the matrices $\left(W^{\gamma}\right)^{\beta}$ and $\left(\oplus_{j} Z\right)^{\alpha}$ will again contribute to $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ by zero.

Third, let $Z$ be a submatrix of both matrices $B_{1}$ and $B_{m}$. Let $W_{-1}$ denote the matrix $\oplus_{k-1} Z$. Let $a$ be the largest possible natural number such that the matrix $Z_{a}=\oplus_{a} Z$ is a submatrix of $B_{1}$. Moreover, any of the ma-
 $\{\bullet, \boldsymbol{l}, \bullet, \bullet \bullet\}$. Let $\mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right)$ denote the set of those wraps $\alpha \in \mathcal{W}_{\mathrm{BL}}$ such that $Z_{a}^{\alpha} \leq B_{1}$. Symmetrically to that, let $b$ be the largest possible natural number such that the matrix $Z_{b}=\oplus_{b} Z$ is a submatrix of $B_{m}$. Moreover, any of the matrices $Z_{\bar{b}}^{\overline{5}}, Z_{b}^{\bullet}, Z_{b}^{\mathfrak{7}}$ can be a submatrix of $B_{m}$. Let $\mathcal{W}_{\text {TR }}$ denote the set $\{\bullet, \overline{\mathbf{\bullet}}, \boldsymbol{\bullet}, \boldsymbol{\nabla}\}$. Let $\mathcal{W}_{\text {TR }}\left(Z_{b}\right)$ denote the set of those wraps $\beta \in \mathcal{W}_{\text {TR }}$ such that $Z_{b}^{\beta} \leq B_{m}$. Observe that for any $\alpha \in \mathcal{W}_{\mathrm{BL}}$ and $\beta \in \mathcal{W}_{\mathrm{TR}}$, the matrix $\left(W^{\alpha}\right)^{\beta}$ is contained in $Y$ if and only if $\alpha$ is in $\mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right)$ and $\beta$ is in $\mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)$; such wraps can be identified with the set $\mathcal{W}(\alpha \wedge \beta)=\left\{(\alpha, \beta) \in \mathcal{W}_{\mathrm{BL}} \times \mathcal{W}_{\mathrm{TR}}: \alpha \in \mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right) \wedge \beta \in \mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)\right\}$. Similarly, the matrix $\left(W_{-1}^{\alpha}\right)^{\beta}$ is contained in $Y$ if and only if $\alpha$ is in $\mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right)$ or $\beta$ is in $\mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)$, and these wraps can be identified with the set $\mathcal{W}(\alpha \vee \beta)=$ $\left\{(\alpha, \beta) \in \mathcal{W}_{\mathrm{BL}} \times \mathcal{W}_{\mathrm{TR}}: \alpha \in \mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right) \vee \beta \in \mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)\right\}$.

Suppose now that $W$ is a proper submatrix of $Y$ and let $\mu_{\mathrm{DS}}(\mathbf{1}, W)$ be equal to $r$. By induction, $\mu_{\mathrm{DS}}\left(\mathbf{1}, W_{-1}\right)$ is also equal to $r$. The set $\mathcal{W}(\alpha \wedge \beta)$ will then contribute to $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ by

$$
\sum_{\alpha \in \mathcal{\mathcal { W } _ { \mathrm { BL } }}\left(Z_{a}\right)} \sum_{\beta \in \mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)} r \cdot(-1)^{|\alpha|+|\beta|},
$$

and the set $\mathcal{W}(\alpha \vee \beta)$ by

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{W}_{\mathrm{BL}}} \sum_{\beta \in \mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)} r \cdot(-1)^{|\alpha|+|\beta|}+\sum_{\alpha \in \mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right)} & \sum_{\beta \in \mathcal{W}_{\mathrm{TR}}} r \cdot(-1)^{|\alpha|+|\beta|} \\
& -\sum_{\alpha \in \mathcal{W}_{\mathrm{BL}}\left(Z_{a}\right)} \sum_{\beta \in \mathcal{W}_{\mathrm{TR}}\left(Z_{b}\right)} r \cdot(-1)^{|\alpha|+|\beta|}
\end{aligned}
$$

by the Inclusion-exclusion principle. The first and the last sum will cancel out themselves and the rest is equal to zero. This is because for fixed $\alpha$, there is the same amount of even sized and odd sized $\beta$-s and vice versa. The absolute contribution is then zero.

If $W$ is equal to $Y$ (which then must be uniformly decomposable), then $B_{1}$ is equal to $Z$. The matrix $W$ will then not contribute to $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ and any $\alpha \in\{\bullet, \boldsymbol{\bullet}, \bullet, \bullet \mathbf{I}, \mathbf{\bullet}, \boldsymbol{\bullet}, \boldsymbol{\nabla}\}$ is admissible for $W_{-1}$ in $Y$. By this, the contribution of $W_{-1}$ is $-\mu_{\mathrm{DS}}\left(\mathbf{1}, W_{-1}\right)$ which is $-\mu_{\mathrm{DS}}(1, Z)$ by induction, which is $-\mu_{\mathrm{DS}}\left(1, B_{1}\right)$.

For any $j \leq k-2$, the set of matrices $\oplus_{j} Z$ wrapped with $\alpha \in \mathcal{W}$ will contribute by zero, because the contributions of even sized and odd sized wraps will cancel out themselves.

Note that this procedure can be repeated. Consider a permutation matrix $Y$ with core $X$. If we can simplify $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ to $\mu_{\mathrm{DS}}(\mathbf{1}, X)$, we can do the same for $X$ (more specifically for some symmetry of $X$ ) etc.

Although Theorem 3.15 is general, it is quite simple to state. Despite this, we present some special cases of this theorem which we believe are notable. First, we state how the diagonal matrices behave.
Corollary 3.16. For diagonal matrix $I_{n}$, the value $\mu_{\mathrm{DS}}\left(\mathbf{1}, I_{n}\right)$ is equal to 1 .
Following that, if a matrix contains a 1 -cell in any corner but is not diagonal, its Möbius value is zero.
Corollary 3.17. For a non-diagonal permutation matrix $Y$ with 1-cell in the lower left corner, $\mu_{\mathrm{DS}}(\mathbf{1}, Y)$ is equal to zero.

## 4. Conclusion

In this chapter, we summarize our main results, and also discuss potential areas for future research, which may be useful for readers interested in further investigation in this field. At the end, we state several conjectures that are supported by the results of our computer-assisted experiments.

### 4.1 Summary

The study of the Möbius function has been a subject of extensive research in the mathematical community. Many significant results on the Möbius function of the permutation poset have been discovered over the years, which motivated us to generalize the permutation poset and focus on the more general poset. This generalized poset is the poset of sparse matrices, where we looked at four closely related variants of the containment relation and initiated the study of their Möbius functions, which have not been studied before.

We derived several results for all the containments, but we focused primarily on the poset with dominated scattered containment, which we consider to be the natural generalization of the permutation poset. For this poset, we proved that the results for $X=0^{1 \times 1}$ and $X=1$ are identical, except for the sign of the result. Also, we discovered that matrices containing a zero line behave regularly, depending on the position of the zero line. Finally, we turned to matrices not containing any zero lines, which are permutation matrices. We tried to find a link connecting the permutation poset and the poset of sparse matrices, and we presented a theorem inspired by previous results on the Möbius function of the permutation poset.

### 4.2 Future directions

As was mentioned, the dominated scattered poset is the most general poset, so it would be fitting to continue focusing on this poset. We know how matrices containing a zero line behave in this poset, and also, we know that their results depend on the results of permutation matrices (which are matrices not containing any zero line). By this, we believe that the right way to continue with this research is searching for explicit formulas of the Möbius function for the permutation matrices.

As we consider the poset of sparse matrices to be a generalization of the permutation poset, it would be interesting to try to find a link between the results of both posets. Another possibility is to read some works focusing on the permutation poset and rephrase and prove some results for the poset of sparse matrices.

Alternative area of focus might be searching for more claims that explore the value $\mu(X, Y)$ for more general matrices $X$. This thesis focused primarily on $X \in\left\{\emptyset, 0^{1 \times 1}, \mathbf{1}\right\}$, but some claims were generalized for other matrices $X$, and it might be convenient to try to generalize all the claims.

In the following section, we present some conjectures that might motivate the reader to continue in this work.

### 4.3 Conjectures

The following conjectures were tested by calculation, but we do not have proofs for them.

Conjecture 4.1. Let $X$ be an indecomposable matrix. Let $k, l$ be natural numbers, both greater than 1. Then

$$
\mu_{\mathrm{DS}}\left(\bigoplus_{k} X, \bigoplus_{l} X\right)=\binom{l-1}{k-1} .
$$

Equivalently,

$$
\mu_{\mathrm{DS}}\left(\bigoplus_{k} X, \bigoplus_{l} X\right)=\mu_{\mathrm{DS}}\left(\bigoplus_{k-1} X, \bigoplus_{l-1} X\right)+\mu_{\mathrm{DS}}\left(\bigoplus_{k} X, \bigoplus_{l-1} X\right) .
$$

This conjecture was tested on matrices $X \in\left\{\mathbf{1},\binom{10}{01},\left(\begin{array}{l}100 \\ 010 \\ 001\end{array}\right),\left(\begin{array}{l}100 \\ 001 \\ 010\end{array}\right)\right\}$ for $k \leq 5$ and $l \leq 8$.

Here, we can speculate that if the previous conjecture holds, then a more general conjecture might also hold. It is generalized in a way, where the parameters of the Möbius function can be different matrices.

Conjecture 4.2. Let $X$ and $X^{\prime}$ be indecomposable matrices where $X \oplus X \not \underbrace{}_{\mathrm{DS}} X^{\prime}$. Let $k, l$ be natural numbers, both greater than 1 . Then

$$
\mu_{\mathrm{DS}}\left(\bigoplus_{k} X, \bigoplus_{l} X^{\prime}\right)=\binom{l-1}{k-1} \cdot \mu_{\mathrm{DS}}\left(X, X^{\prime}\right) .
$$

The following conjecture was tested on matrices up to the size $11 \times 11$.
Conjecture 4.3. Let $I_{n}^{R}$ denote a diagonal matrix of size $n \times n$ flipped by its vertical axis and let $k$ be a natural number. The value $\mu_{\mathrm{DS}}\left(\mathbf{1} \oplus I_{2}^{R}, \mathbf{1} \oplus I_{k}^{R}\right)$ is equal to $\mu_{\mathrm{DS}}\left(I_{2}^{R}, I_{k}^{R}\right)$.

To understand this conjecture better, we show an example of such matrices and relation. For matrices

$$
\begin{gathered}
X^{\prime}=\binom{10}{01}, X=\mathbf{1} \oplus X^{\prime}=\left(\begin{array}{l}
010 \\
001 \\
100
\end{array}\right), \\
Y^{\prime}=\left(\begin{array}{l}
10000 \\
01000 \\
00100 \\
00010 \\
00001
\end{array}\right), Y=\mathbf{1} \oplus Y^{\prime}=\left(\begin{array}{l}
010000 \\
001000 \\
000100 \\
000010 \\
000001 \\
100000
\end{array}\right),
\end{gathered}
$$

we say $\mu_{\mathrm{DS}}(X, Y)=\mu_{\mathrm{DS}}\left(X^{\prime}, Y^{\prime}\right)$.
This last conjecture only formally rephrases what we mentioned in the third chapter.

Conjecture 4.4. The sequence $\mu_{\mathrm{ES}}\left(\emptyset, I_{n}\right)$ for natural number $n$ corresponds, up to the sign, to A002212 from OEIS [8].

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[^0]:    ${ }^{1} \mathrm{~A}$ consecutive set is also known as an integer interval. In the definition of scattered submatrix, the word scattered emphasizes that the set of rows respectively columns is not necessarily consecutive.

