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Vojtěch Doleček

Spacetimes generated by an electromagnetic field and perfect fluid

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Supervisor of the bachelor thesis: doc. RNDr. Martin Žofka, Ph.D.
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Author: Vojtěch Doleček

Institute: Institute of Theoretical Physics

Supervisor: doc. RNDr. Martin Žofka, Ph.D., Institute of Theoretical Physics

Abstract: This bachelor thesis deals with the search for sources of curvature of spacetime in general theory of relativity. In particular, it seeks to find a source of curvature that can be confidently said to be composed of a perfect fluid and an electromagnetic field. The problem of finding such physical solutions is first summarized in the first two chapters. Then, for specific spacetimes, it is already trying to find exact solutions that would correspond to such a source.

Keywords: sources of gravitational fields, electromagnetic field, perfect fluid, Einstein's equations, cylindrical symmetry, dynamical spacetimes

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Introduction

This bachelor thesis is motivated by the desire to find an exact solution to Einstein's equations (involving the cosmological term), where the right side of the equations describes a perfect fluid which is in combination electromagnetic field.

When Albert Einstein published for the first time his field equations in 1915, he created the most precise theory of gravity till nowadays. Many revolutionary and in his time unimaginable predictions have been made since then, based purely on the elegant formalism and mathematical pillars of his theory and many of those were experimentally proven only in recent years. Even though solving Einstein's equations is not an easy task, due to non-linearity of the equations, there exist many exact solutions today. We all know Shchwarzschild's solution from 1916 for spacetime outside non-rotating spherical mass without electric charge with zero cosmological constant, or FLRW solution which has deep cosmological meaning. These two solutions are some of the first solutions ever found and also some of the most influential ones during the years, giving first predictions for a black hole and Big Bang.

There, of course, exist many more solutions besides those two already named, there even exist whole classes of solutions like vacuum solutions, electrovacuum solutions or fluid solutions. What all these solutions have in common is, that they usually take into account only one source that is curving the spacetime. For example if we are looking at electrovacuum solutions, we assume that the only thing appearing on the right side of Einstein's equations is stress-energy tensor given simply by electromagnetic field and nothing else. Same thing for fluid solutions and others as well. In our work, we are going to try to merge two sources, namely magnetic field and prefect fluid, into one and try to find such spacetime, for which this source of curvature would make sense.

Of course, there exist solutions specifically for perfect fluids (FRW radiation fluids, Wahlquist fluid,...) and for electromagnetic fields (Reissner–Nordström electrovacuum solution, Kerr–Newman electrovacuum,...). As already stated above, our work is going to be a bit harder here, due to trying to merge both of these cases into one. We are going to try to find a solution, starting from a given spacetime that we deform by introducing new functions into the original metric. We then solve Maxwell's equations and make sure that the resulting stress-energy tensor would satisfy our requirement of perfect fluid plus an electromagnetic field. The two seed spacetimes we decided to look at are namely the Kantowski-Sachs metric and the Minkowski metric. The modifications are done by introducing unknown time-dependent functions. The choice of these spacetimes is based on two facts: firstly, they are both mathematically simple and, secondly, they admit cylindrical symmetry, which is of interest since there is a family of static and stationary cylindrically symmetric spacetimes admitting an electromagnetic field. It is thus reasonable to hope that by perturbing these solutions—applying a metric deformation—one could find a solution admitting a more general form of the source, which would consist of both an electromagnetic field and a perfect fluid. We further assume that the Maxwell field inherits the symmetry of the spacetime and that the fluid is at rest with respect to the (preferred) coordinate system we use. This choice of the electromagnetic field enables us to solve the Maxwell equations, leaving thus only Einstein equations to deal with. As a novel feature, we are particularly interested in time-dependent solutions. We assume, that the resulting stress-energy tensor on the right side of the equations is going to be a sum of electromagnetic stress-energy tensor and of perfect fluid stress-energy tensor.

These assumptions are not without prejudice to the generality. There could possibly exist perfect fluids combined with an electromagnetic field that result in a completely different metric, but our assumptions are huge simplifications, thanks to which we can get corresponding tensor to perfect fluid just from knowledge of metric tensor and Maxwell's tensor. Additionally, resulting equations respectively inequalities won't be that hard to solve, as they could be, if we assumed metric tensor in a different forms. On the other hand, the price we are to pay is that it may easily turn out that there are no satisfactory solutions with our assumptions.

As we have mentioned in previous paragraphs, solving Einstein's equations is hard due to their non-linearity, even for vacuum. Our task here is all the more difficult because we only have one non-linear differential equation and several differential inequalities we need to comply with. Unfortunately, there exists no comprehensive theory of differential inequalities, so the only thing that is left is "blindly" trying to find functions, that would satisfy our inequalities.

In this thesis, we are going to use spacelike metric signature convention, meaning that the signature of our metric tensor is $(-,+,+,+)$.
We also set $c = G = 1$, thus $\kappa = 8\pi$.

1. Einstein's equations

As mentioned in the preface, Albert Einstein came up with his field equations in 1915 and to this day it is the unsurpassed theory of gravity.

It is a total of 10 strongly nonlinear differential equations for the metric field. In other words, as is often stated in popular literature, "Matter tells spacetime how to curve and spacetime tells matter where to move."—this is, in fact, a loosely stated quotation by John Archibald Wheeler, see [Misner et al. [1973]]. Thanks to the elegant formalism of differential geometry, the equations don't look particularly complicated at first glance. Many people not interested in general relativity would hardly notice, that there are several equations involved and not just one. But the reality is, that despite the power of today's computing technology, even more than a century after the publication of Einstein's general theory of relativity, the only known solutions are those that assume many symmetries or great simplifications. The sad truth remains that solving the equations of general relativity in general seems like a nearly, if not completely, impossible task.

So let us now look at the shape of the equations we are talking about.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.1}$$

The $G_{\mu\nu}$ tensor is named the Einstein tensor after its discoverer and can be written in a form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is Ricci tensor and $R = R^{\mu}_{\ \mu}$ is Ricci scalar.

Term $\Lambda g_{\mu\nu}$ is called the cosmological term. Albert Einstein originally added it in his equations in 1917 because he liked the idea of a static, spherical universe [Einstein [1917]]. A move he later called "the biggest blunder" of his life was proved correct when Edwin Hubble demonstrated from redshift observations that the universe was expanding.

The last term on the right side of the equations is called stress-energy tensor and represents the source of spacetime curvature in the equations. In general, spacetime does not have to be curved only by matter. For example, the curvature of spacetime can also be caused by electromagnetic fields. It is also generally assumed that the sources are additive, i.e. the total source of spacetime curvature (total the stress-energy tensor) for multiple sources simultaneously is the sum over the individual sources (i.e. is the sum over the corresponding stress-energy tensors).

So if we now rewrite the Einstein equations for a field of multiple curvature sources, we get equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \sum_{sources} T_{\mu\nu}. \tag{1.3}$$

Depending on the shape of the right side of Einstein's equations, we speak of three or four basic classes of solutions respectively. For $T_{\mu\nu} = 0$ we speak of vacuum solutions, for $T_{\mu\nu}$ given by a pure electromagnetic field we speak of electro-vacuum solutions, and for solutions where the source is a perfect fluid we speak of solutions with a perfect fluid. The fourth case is the pressureless dust, which is a special case of the ideal fluid.

In the present thesis, we investigate an interesting option of combining two different source terms on the right-hand side of Einstein equations. Namely, we add the electromagnetic field and perfect fluid. In fact, we shall further restrict the electromagnetic field, specifying it completely, and only look for the remaining stress-energy contribution due to the perfect fluid.

More about stress-energy tensors is written in the next section, here we now just derive a useful identity for Ricci scalar for the case where the stress-energy tensor on the right side of Einstein's equations is traceless. We start with Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T^{mag}_{\mu\nu}. \tag{1.4}$$

Now, we multiply the whole equation system by $g^{\mu\nu}$, which gives us one "trace equation" and use a fact, that trace of electromagnetic stress-energy tensor is equal to 0 (as has been shown in chapter 1.2).

$$\begin{aligned} &g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + \Lambda g^{\mu\nu}g_{\mu\nu} = 8\pi g^{\mu\nu}T_{\mu\nu} \\ &R - 2R + 4\Lambda = 0 \\ &R = 4\Lambda. \end{aligned} \tag{1.5}$$

This identity states for any traceless stress-energy tenor. Example of an traceless stress-energy tensor is for example any electro-magnetic stress-energy tensor.

2. Stress-energy tensor

In Newton's theory of gravity is generated by mass density, where equation

$$\Delta\Phi = 4\pi G\rho. \tag{2.1}$$

When Albert Einstein was discovering his new theory of gravity, he needed source for his gravitational field (or as we know source of curvature). In his theory, this place is represented by the stress energy tensor $T_{\mu\nu}$. This tensor describes, how does density and flow of energy and momentum in spacetime and is analogue to mass density in Newtonian theory.

2.1 Electromagnetic stress-energy tensor

Important form of stress-energy tensor in our work is the one of electromagnetic field [Misner et al. [1973]]. Electromagnetic field stress energy tensor is defined by equation

$$T_{\text{elmag}}{}^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}g_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right), \tag{2.2}$$

where $F^{\mu\nu}$ is corresponding Maxwell tensor and $g_{\mu\nu}$ is metric tensor. We also set $\mu_0 = 4\pi$ in accordance with the CGS system of units.

What is interesting about electro-magnetic stress-energy tensor in four dimensions is, that it is traceless. Statement can be simply proven from definition by following steps

$$\begin{aligned} T_{\text{elmag}}{}^{\mu\nu} &= F^{\mu\alpha}g_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\gamma}F^{\delta\gamma} \\ T_{\text{elmag}}{}^{\mu}{}_{\mu} = g_{\mu\nu}T_{\text{elmag}}{}^{\mu\nu} &= g_{\mu\nu}\left(F^{\mu}_{\ \beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right) \\ T_{\text{elmag}}{}^{\mu}{}_{\mu} &= F_{\nu\beta}F^{\nu\beta} - F_{\delta\gamma}F^{\delta\gamma} = 0, \end{aligned}$$

where we have used that $g_{\mu\nu}$ is inverse to $g^{\mu\nu}$, so $g_{\mu\nu}g^{\mu\nu} = 4$ and a fact, that we can change our index notation $\nu \to \delta$ and $\beta \to \gamma$.

However this identity only holds in four dimensions. In d dimensions, we have the following equation

$$T_{\text{elmag}}{}^{\mu}{}_{\mu} = F^{\mu\nu}F_{\mu\nu}\left(\frac{4-d}{4}\right). \tag{2.3}$$

2.2 Perfect fluid stress-energy tensor

Another very important form of stress-energy tensor for us is the one for a perfect fluid, since we would like to merge these two sources.

Definition of stress energy tensor for a perfect fluid is of a form

$$T^{\alpha\beta} = (\rho + p)\,u^{\alpha}u^{\beta} + pg^{\alpha\beta}, \tag{2.4}$$

where ρ is energy density, p is pressure, u^{α} is four-velocity and $g^{\alpha\beta}$ is metric tensor.

In its rest frame, the stress-energy tensor of a perfect fluid has the form

$$T^{\alpha}{}_{\beta} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{2.5}$$

since in its rest frame, we find $u^{\alpha} = [1, 0, 0, 0]$, $u_{\alpha} = [-1, 0, 0, 0]$ and $g^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} = diag\,(1, 1, 1, 1)$.

2.3 Energy conditions

If we are talking about perfect fluid solutions of Einstein's equations, we must not forget to mention energy conditions [Hawking and Ellis [1975]], which are going to play a significant role in our subsequent work.

Einstein's theory of general relativity does not place any requirements on the stress-energy tensor (apart from symmetry in the first and second index), so it can quite easily happen that the resulting solutions are for a substance with non-physical meaning. To avoid such solutions, it is reasonable for the stress-energy tensor to satisfy the following four conditions

(1) Weak energy condition states, that every (physical) observer has to detect non-negative energy density (X^{α} is tangent to a future directed, time-like worldline):

$$\rho = T_{\alpha\beta} X^{\alpha} X^{\beta} \geq 0. \tag{2.6}$$

(2) Null energy condition states, that even in a limiting case of speed of light, the **weak condition** should hold

$$\nu = T_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0. \tag{2.7}$$

(3) Dominant energy condition requires that the weak energy condition hold true and, additionally, that the momentum flux measured by any observer (the stress-energy tensor projected on their tangent vector field or, as a limiting case, on the tangent vector field to any null trajectory) $-T_{\alpha\beta} X^{\beta}$ be future directed. This means that mass-energy can never be observed moving faster than the speed of light.

(4) Strong energy condition restricts negative pressure

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) X^{\alpha} X^{\beta} \geq 0, \tag{2.8}$$

where $T_{\alpha\beta}$ is stress-energy tensor and $T = T^{\mu}_{\mu}$.

Energy conditions can be reformulated in terms of eigenvalues of perfect fluid stress-energy tensor in its rest-frame into a form

(1) Null energy condition $\rho + p \geq 0$

(2) Weak energy condition $\rho \geq 0$, $\rho + p \geq 0$

(3) Dominant energy condition $\rho \geq |p|$

(4) Strong energy condition $\rho + p \geq 0$, $\rho + 3p \geq 0$,

As we can see above, trying to find a solution satisfying the energy conditions in general relativity means solving a system of nonlinear differential inequalities, which is very difficult, if not completely impossible. There is not even much literature on the subject of differential inequalities (e.g., [Szarski [1965]]), so we have no choice but to try to "blindly" search for solutions and try to see if they satisfy the aforementioned system of differential inequalities.

3. Kantowski-Sachs seed

As we have written in the introduction, we are going to assume a metric of the Kantowski-Sachs-like form. Original Kantowski-Sachs metric has been analysed independently by Kompaneets and Chernov, and Kantowski and Sachs. It is of a form [Griffiths and Podolský [2009]]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + X(t)^2\mathrm{d}\chi^2 + Y(t)^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2\right). \tag{3.1}$$

This metric was chosen primarily because it is a cylindrical spacetime and it is relatively "easy" to exactly solve cylindrical spacetimes. Also, there exist exact solutions for it containing both an electromagnetic field and an ideal fluid, as well as solutions containing cosmological constant (e.g., [Lim [2018]], [Astorino [2012]], [Veselý and Žofka [2019]], [Veselý and Žofka [2021]]). In particular, solutions with the cosmological constant are of interest to us, since experimental data consistently confirm its non-zeroity.

The exact form of the deformed metric we will use in our calculations below is

$$\mathrm{d}s^2 = -g(t)\mathrm{d}t^2 + a(t)\mathrm{d}z^2 + b(t)\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left[1 + \frac{\Lambda}{2}r^2\right]^2}, \tag{3.2}$$

where the radial cylindrical coordinate is $r^2 := x^2 + y^2$.

The Maxwell tensor is further assumed to correspond to a magnetic field parallel to the z-axis, which reads

$$F_{xy} = -F_{yx} = B(r^2, t). \tag{3.3}$$

A cylindrically symmetric spacetime does not necessarily imply a cylindrically symmetric source, that is, a magnetic field. This is merely our choice in a hope that it might lead to a simplification of the resulting equations.

3.1 Right side of Einstein's equations for magnetic field

At first we write down and compute Maxwell equations and find their general solution with no electric charges and currents. We start with potential equations

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0. \tag{3.4}$$

Due to the vanishing of most of the terms of Maxwell's tensor, many equations are going to be trivial. From equations

$$F_{tx,\lambda} + F_{x\lambda,t} + F_{\lambda t,x} = F_{xy,t} = -F_{yx,t} = 0, \tag{3.5}$$

$$F_{xz,\lambda} + F_{z\lambda,x} + F_{\lambda x,z} = F_{yx,z} = -F_{xy,z} = 0, \tag{3.6}$$

we can easily observe, that the function $B(r^2, t)$ is actually time independent. From Maxwell's field equations

$$F^{\mu\nu}{}_{;\mu} = 0, \tag{3.7}$$

we obtain a system of two partial differential equations for unknown function $B(r^2)$

$$2\Lambda x B(r^2) + \left(1 + \frac{\Lambda}{2} r^2\right) \partial_x B(r^2) = 0, \tag{3.8}$$

$$2\Lambda y B(r^2) + \left(1 + \frac{\Lambda}{2} r^2\right) \partial_y B(r^2) = 0. \tag{3.9}$$

General solution of this system is

$$B(r^2) = \frac{c_1}{(1 + \frac{\Lambda}{2} r^2)^2}, \tag{3.10}$$

where c_1 is integration constant. Its dimension is the same as that of the magnetic field so in the following we set $c_1 = B$. Now we can easily calculate corresponding stress-energy tensor $T^{\mu\nu}_{mag}$ according to equation (2.2) and then lower its indices according to equation

$$T^{mag}_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} T^{\mu\nu}_{mag}. \tag{3.11}$$

We gain

$$T^{mag}_{tt} = \frac{g(t) B^2}{8\pi b(t)^2}, \tag{3.12}$$

$$T^{mag}_{xx} = T^{mag}_{yy} = \frac{B^2}{8\pi b(t)(1 + \frac{\Lambda}{2} r^2)^2}, \tag{3.13}$$

$$T^{mag}_{zz} = -\frac{a(t) B^2}{8\pi b(t)^2}. \tag{3.14}$$

3.2 Left side of Einstein's equations

Next step is computing left side of Einstein's equations, namely Einstein tensor and term Λg.

At first, we compute affine connection components using

$$\Gamma^{\mu}{}_{\nu\kappa} = g^{\mu\alpha} \Gamma_{\alpha\nu\kappa} = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\kappa} + g_{\alpha\kappa,\nu} - g_{\nu\kappa,\alpha}), \tag{3.15}$$

from where we gain

$$\Gamma^{t}{}_{tt} = \frac{g'(t)}{2g(t)}, \tag{3.16}$$

$$\Gamma^{t}{}_{xx} = \Gamma^{t}{}_{yy} = \frac{b'(t)}{2g(t)(1 + \frac{\Lambda}{2} r^2)^2}, \tag{3.17}$$

$$\Gamma^{x}{}_{tx} = \Gamma^{x}{}_{xt} = \frac{b'(t)}{2b(t)}, \tag{3.18}$$

$$\Gamma^{y}{}_{ty} = \Gamma^{y}{}_{yt} = \frac{b'(t)}{2b(t)}, \tag{3.19}$$

$$\Gamma^{t}{}_{zz} = \frac{a'(t)}{2g(t)}, \tag{3.20}$$

$$\Gamma^{z}{}_{tz} = \Gamma^{z}{}_{zt} = \frac{a'(t)}{2a(t)}, \tag{3.21}$$

$$\Gamma^{x}_{\ xx} = -\frac{\Lambda x}{1+\frac{\Lambda}{2}r^2}, \tag{3.22}$$

$$\Gamma^{x}_{\ xy} = \Gamma^{x}_{\ yx} = -\Gamma^{y}_{\ xx} = -\frac{\Lambda y}{1+\frac{\Lambda}{2}r^2}, \tag{3.23}$$

$$\Gamma^{y}_{\ xy} = \Gamma^{y}_{\ yx} = -\Gamma^{x}_{\ yy} = -\frac{\Lambda x}{1+\frac{\Lambda}{2}r^2}, \tag{3.24}$$

$$\Gamma^{y}_{\ yy} = -\frac{\Lambda y}{1+\frac{\Lambda}{2}r^2}. \tag{3.25}$$

Any other combination of indices is trivial.

From here we can calculate Ricci tensor and Ricci scalar using the following identities

$$R_{\mu\nu} = R^{\kappa}_{\ \mu\kappa\nu} = \Gamma^{\alpha}_{\ \mu\nu,\alpha} - \Gamma^{\alpha}_{\ \alpha\nu,\mu} + \Gamma^{\alpha}_{\ \alpha\beta}\Gamma^{\beta}_{\ \mu\nu} - \Gamma^{\alpha}_{\ \mu\beta}\Gamma^{\beta}_{\ \alpha\nu}, \tag{3.26}$$

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}_{\ \mu}. \tag{3.27}$$

There we gain

$$\begin{aligned} R_{tt} \ = \ & \frac{1}{4a(t)^2 g(t) b(t)^2}\Big[-2g(t)b(t)^2 a''(t) - 4g(t)b''(t)b(t)a(t)^2 \\ & + g(t)b(t)^2(a'(t))^2 + g'(t)b(t)^2 a(t)a'(t) \\ & + 2a(t)^2 b'(t)[g'(t)b(t) + g(t)b'(t)]\Big], \end{aligned} \tag{3.28}$$

$$R_{xx} = R_{yy} = \frac{2g(t)b''(t)a(t) + [a'(t)g(t) - g'(t)a(t)]b'(t) + 8\Lambda g(t)^2 a(t)}{4a(t)g(t)^2(1+\frac{\Lambda}{2}r^2)}, \tag{3.29}$$

$$\begin{aligned} R_{zz} = & \frac{1}{4b(t)g(t)^2 a(t)}\big[2b'(t)a'(t)g(t)a(t) + 2g(t)a(t)b(t)a''(t) \\ & - g(t)b(t)(a'(t))^2 - g'(t)b(t)a(t)a'(t)\Big], \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} R = & \frac{1}{2a(t)^2 g(t)^2 b(t)^2}\Big[8\Lambda g(t)^2 b(t)a(t)^2 - g(t)(b'(t))^2 a(t)^2 \\ & + 2g(t)b'(t)b(t)a(t)a'(t) + 4g(t)b''(t)b(t)a(t)^2 + 2g(t)b(t)^2 a(t)a''(t) \\ & - g(t)b(t)^2(a'(t))^2 - 2b'(t)g'(t)b(t)a(t)^2 - g'(t)b(t)^2 a(t)a'(t)\Big]. \end{aligned} \tag{3.31}$$

Finally we can get Einstein tensor from formula

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{3.32}$$

We obtain

$$G_{tt} = \frac{8\Lambda g(t)b(t)a(t) + (b'(t))^2 a(t) + 2b'(t)b(t)a'(t)}{4a(t)b(t)^2}, \tag{3.33}$$

$$\begin{aligned} G_{xx} = G_{yy} = & \frac{1}{4g(t)^2 a(t)^2 b(t)(1+\frac{1}{\Lambda}r^2)}\Big[-2g(t)b(t)^2 a(t)a''(t) - \\ & 2g(t)b''(t)b(t)a(t)^2 + g(t)b(t)^2(a'(t))^2 + \\ & b(t)a(t)[g'(t)b(t) - g(t)b'(t)]a'(t) \\ & + a(t)^2 b'(t)[g'(t)b(t) + g(t)b'(t)]\Big], \end{aligned} \tag{3.34}$$

$$G_{zz} = -\frac{a(t)[8\Lambda g(t)^2 b(t) - (b'(t))^2 g(t) + 4g(t)b''(t)b(t) - 2b'(t)g'(t)b(t)]}{4g(t)^2 b(t)^2}. \tag{3.35}$$

3.3 Searching for a perfect fluid

Now, according to (1.3) we add $\Lambda g_{\mu\nu}$ term, subtract $8\pi T^{mag}_{\mu\nu}$ term from Einstein tensor and divide everything with 8π, so we obtain $T^{\text{leftover}}{}_{\mu\nu}$ tensor on the left side of Einstein's equations.

$$\begin{aligned} T^{\text{leftover}}{}_{tt} = & \frac{1}{32\pi a(t)b(t)^2}\Big[a(t)(b'(t))^2 + 2b(t)b'(t)a'(t) \\ & -4g(t)a(t)[B^2 + \Lambda b(t)^2 - 2\Lambda b(t)\Big], \end{aligned} \tag{3.36}$$

$$\begin{aligned} T^{\text{leftover}}{}_{xx} = T^{\text{leftover}}{}_{yy} = & \frac{1}{32\pi a(t)^2 g(t)^2 b(t)[1+\frac{1}{\Lambda}r^2]}\Big[-2g(t)b(t)^2 a(t)a''(t) \\ & - 2g(t)b''(t)b(t)a(t)^2 + g(t)b(t)^2(a'(t))^2 \\ & - a(t)b(t)[g(t)b'(t) - g'(t)b(t)]a'(t) \\ & + a(t)^2\Big[(b'(t))^2 g(t) + b'(t)g'(t)b(t) \\ & -4g(t)^2[B^2 - \Lambda b(t)^2]\Big]\Big], \end{aligned} \tag{3.37}$$

$$\begin{aligned} T^{\text{leftover}}{}_{zz} = & \frac{1}{32\pi g(t)^2 b(t)^2}a(t)\Big[4\Lambda g(t)^2 b(t)^2 + 4B^2 g(t)^2 \\ & - 8\Lambda g(t)^2 b(t) + (b'(t))^2 g(t) - 4g(t)b''(t)b(t) \\ & +2b'(t)g'(t)b(t)]. \end{aligned} \tag{3.38}$$

Now we lift the first index and we choose the rest frame to work in, where ${T_{\text{fluid}}}^{\alpha}{}_{\beta} = \text{diag}(-\rho, p, p, p)$ which corresponds to a fact, that the perfect fluid we are looking for is at rest in the coordinate system connected wit our coordinates (generally, the fluid can be moving according to this system, which would give us different stress-energy tensor). This gives us following equations

$$\begin{aligned} {T_{\text{fluid}}}^{t}{}_{t} = & -\frac{1}{32\pi a(t)b(t)^2 g(t)}\Big[(b'(t))^2 a(t) + 2b'(t)b(t)a'(t) \\ & -2g(t)a(t)[2B^2 + 2\Lambda b(t)^2 - 4\Lambda b(t)]\Big] = -\rho, \end{aligned} \tag{3.39}$$

$$\begin{aligned} {T_{\text{fluid}}}^{x}{}_{x} = {T_{\text{fluid}}}^{y}{}_{y} = & \frac{1}{32\pi b(t)^2 a(t)^2 g(t)^2}\Big[-2g(t)b(t)^2 a(t)a''(t) \\ & - 2g(t)b''(t)b(t)a(t)^2 + g(t)b(t)^2(a'(t))^2 \\ & + b(t)a(t)[g'(t)b(t) - g(t)b'(t)]a'(t) \\ & + a(t)^2\Big[(b'(t))^2 g(t) + b'(t)g'(t)b(t) \\ & -2g(t)^2[2B^2 - 2\Lambda b(t)^2]\Big]\Big] = p, \end{aligned} \tag{3.40}$$

$$\begin{aligned} {T_{\text{fluid}}}^{z}{}_{z} = & \frac{1}{32\pi b(t)^2 g(t)^2}\Big[-4g(t)b''(t)b(t) + (b'(t))^2 g(t) \\ & + 2b'(t)g'(t)b(t) + 2g(t)^2\Big[2B^2 + 2\Lambda b(t)^2 \\ & -4\Lambda b(t)]] = p. \end{aligned} \tag{3.41}$$

Now we have one condition on the metric functions $a(t), b(t)$ and $g(t)$, which comes from the equality of partial pressures (2.5), namely ${T_{\text{fluid}}}^{x}{}_{x} = {T_{\text{fluid}}}^{z}{}_{z}$.

We also have energy conditions so that the stress-energy tensor T_{fluid} has reasonable physical interpretation. We namely want to choose such functions $a(t), b(t)$ and $g(t)$ that $\rho \geq 0$, $\rho \geq |p|$ and $p \geq 0$.

Solving this set of differential inequalities and one differential equation isn't really easy, so we are going to use several assumptions to simplify our problem.

3.3.1 Purely magnetic field ($\rho = p = 0$)

At first, we assume the limiting case of $p = \rho = 0$. This solution correspond to a purely magnetic stress-energy tensor on the right side of Einstein's equations. From this assumption we get three homogeneous equations for three functions $a(t), b(t)$ and $g(t)$

$$T_{\text{fluid}}{}^{t}{}_{t} = 0, \tag{3.42}$$

$$T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{y}{}_{y} = 0, \tag{3.43}$$

$$T_{\text{fluid}}{}^{z}{}_{z} = 0. \tag{3.44}$$

For these equations, there exist two different general solutions for functions $a(t), b(t)$ and $g(t)$.

1st solution

$$b(t) = 1 \pm \frac{\sqrt{-\Lambda B^2 + \Lambda^2}}{4\Lambda} = \text{const.}, \tag{3.45}$$

while $g(t)$ is arbitrary (which suits our gauge freedom) and $a(t)$ is a complicated functional of functions $b(t)$ and $g(t)$. Since we can always rescale the z coordinate, we can set $b(t) = 1$ so that these 2 solutions are, in fact, equivalent. To simplify the solution further, we use the gauge freedom and reparametrize the time so that $g(t) = 1$, which leads to $a(t)$ of the form

$$a(t) = \frac{1}{4}\left(C_1 t + C_2\right)^2 \to \alpha t^2, \tag{3.46}$$

where C_1 and C_2 are integration constants and $\alpha = C_1^2/4$ is obviously positive number, produced by changing the time parametrization. Nullity of terms $T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{y}{}_{y}$ also gives us condition $\Lambda = B^2$.

From here we get a spacetime given by metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^2 \mathrm{d}z^2 + \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left[1 + \frac{\Lambda}{2} r^2\right]^2}, \tag{3.47}$$

where we got rid of α by rescaling the coordinate z.

We can notice that this solution is very similar to the Plebanski-Hacyan solution [Žofka [2019]].

Now we calculate Kretschmann, Ricci, and Maxwell scalars (to examine the limiting cases of our solution and its singularities) according to equation (3.27) respectively according to

$$K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \tag{3.48}$$

and

$$F^2 = F_{\alpha\beta} F^{\alpha\beta}. \tag{3.49}$$

Ricci scalar should come out of calculation to value of 4Λ, as it has been showed in (1.5). So we have an expected value for Ricci scalar and computation with our specific metric tensor should prove it. It should also work as a verification of our previous calculations.

Thus from computations with our metric and Maxwell tensor, solutions are

$$
\begin{aligned}
K =& \frac{(a'(t))^2(b'(t))^2}{2b(t)^2a(t)^2g(t)^2} + \frac{[8\Lambda g(t)b(t) + (b'(t))^2]^2}{4b(t)^4g(t)^2} \\
&+ \frac{[-2a(t)g(t)a''(t) + a'(t)g'(t)a(t) + (a'(t))^2g(t)]^2}{4g(t)^4a(t)^4} \\
&+ \frac{[(b'(t))^2g(t) + b'(t)g'(t)b(t) - 2b''(t)g(t)b(t)]^2}{2g(t)^4b(t)^4},
\end{aligned} \tag{3.50}
$$

and

$$F^2 = \frac{2B^2}{b(t)^2}. \tag{3.51}$$

Ricci scalar is evaluated in equation (3.31).

Now, if we insert our new metric functions from metric (3.47), we get the following expressions for the Kretschmann, Ricci and Maxwell scalars

$$K = 16\Lambda^2, \tag{3.52}$$

$$R = 4\Lambda, \tag{3.53}$$

$$F^2 = 2B^2. \tag{3.54}$$

We can see, that all scalars are constant, although metric function $a(t)$ is time dependent.

This is something we wouldn't have expected and is certainly worth further thought and consideration, or even further work to explore the nature of this spacetime.

2nd solution

Now we want to take a look at the second general solution given by following equations

$$g(t) = \frac{12(b'(t))^2}{16\Lambda b(t)^2 - 96\Lambda b(t) - 3B^2 + 12C_2\sqrt{b(t)}}, \tag{3.55}$$

$$a(t) = C_1 \exp\left(\int \frac{16\Lambda g(t)b(t)^2 + g(t)B^2 - 32\Lambda g(t)b(t) - 4(b'(t))^2}{8b'(t)b(t)} dt\right), \tag{3.56}$$

and function $b(t)$ is arbitrary.

If we substitute equation (3.55) into equation (3.56), integral in equation (3.56) changes into form

$$
\begin{aligned}
&\int \frac{16\Lambda g(t)b(t)^2 + g(t)B^2 - 32\Lambda g(t)b(t) - 4(b'(t))^2}{8b'(t)b(t)} dt = \\
&\int \left(\frac{48\Lambda b(t)^2 - 96\Lambda b(t) + 3B^2}{4b(t)\left[16\Lambda b(t)^2 - 96\Lambda b(t) - 3B^2 + 12C_2\sqrt{b(t)}\right]} - \frac{1}{2b(t)} \right) b'(t) dt,
\end{aligned} \tag{3.57}
$$

where C_2 is integration constant.

Now we can change integration variable and edit the term in brackets into a term with common denominator, so that the integral goes into form

$$\begin{aligned}&\int\left(\frac{48\Lambda b(t)^2-96\Lambda b(t)+3B^2}{4b(t)\left[16\Lambda b(t)^2-96\Lambda b(t)-3B^2+12C_2\sqrt{b(t)}\right]}-\frac{1}{2b(t)}\right)b'(t)dt=\\&\int\left(\frac{16\Lambda b(t)^2-6\sqrt{b(t)}C_2+3B^2}{b(t)\left[16\Lambda b(t)^2-96\Lambda b(t)-3B^2+12C_2\sqrt{b(t)}\right]}\right)b'(t)dt,\end{aligned}\tag{3.58}$$

This can also be written in a form

$$\int\frac{Ax^2+B\sqrt{x}+C}{x\left(Ax^2+Dx-2B\sqrt{x}-C\right)}dx,\tag{3.59}$$

which is general form of an integral we can not solve. We tried to factorize it, but without success.

3.3.2 Zero partial pressures ($\rho \neq 0$, $p = 0$)

Now we would like to take a look at solution with nonzero ρ in ${T_{\text{fluid}}}^{\alpha}{}_{\beta}$, but with zero partial pressures, which can be described by following set of equations

$${T_{\text{fluid}}}^{t}{}_{t}=-\rho,\tag{3.60}$$

$${T_{\text{fluid}}}^{x}{}_{x}={T_{\text{fluid}}}^{y}{}_{y}=0,\tag{3.61}$$

$${T_{\text{fluid}}}^{z}{}_{z}=0.\tag{3.62}$$

This type of stress-energy tensor physically corresponds to incoherent dust.

To be able to move forward, we are going to need time parametrization again, so we use $g(t) = 1$ to simplify equations above. Otherwise equations are too hard to solve and only solutions we have found were those for zero ρ, which we have discussed earlier. The equations above after parametrization go into form

$$\begin{aligned}{T_{\text{fluid}}}^{t}{}_{t}=&-\frac{1}{32\pi a(t)b(t)^2}\Big[(b'(t))^2a(t)+2b'(t)b(t)a'(t)\\&-4a(t)[B^2+\Lambda b(t)^2-2\Lambda b(t)\Big]=-\rho(t),\end{aligned}\tag{3.63}$$

$$\begin{aligned}{T_{\text{fluid}}}^{x}{}_{x}={T_{\text{fluid}}}^{y}{}_{y}=&\frac{1}{32\pi b(t)^2a(t)^2}\Big[-2b(t)^2a(t)a''(t)-2b''(t)b(t)a(t)^2\\&+b(t)^2(a'(t))^2-b(t)a(t)b'(t)a'(t)\\&+a(t)^2\Big[(b'(t))^2-4[B^2-\Lambda b(t)^2]\Big]\Big]=0,\end{aligned}\tag{3.64}$$

$$\begin{aligned}{T_{\text{fluid}}}^{z}{}_{z}=&\frac{1}{32\pi b(t)^2}\Big[-4b''(t)b(t)+(b'(t))^2\\&+4\Big[B^2+\Lambda b(t)^2-2\Lambda b(t)\Big]\Big]=0.\end{aligned}\tag{3.65}$$

After this simplification, we have two homogeneous nonlinear differential equations for functions $b(t)$ and $a(t)$ and one non-homogeneous differential equation described by equation (3.63). We would like to find metric functions $a(t)$ and $b(t)$ from equations (3.64) and (3.65) and when we have them, we would like to insert them into equation (3.63) and find a function $\rho(t)$.

We can see, that equation (3.65) is differential equation only in function $b(t)$. This means, that the solution of the whole set should be findable through solving equation (3.65) and finding function $b(t)$, inserting the solution into equation (3.64) and solving it for function $a(t)$ and then inserting both functions $a(t)$ and $b(t)$ into equation (3.63) and finding function $\rho(t)$.

Unfortunately, the first step is a problem. Though the equation (3.65) is solvable, it does not lead to any analytical function. The only thing we came up with is using another simplification. We assume that $b(t) = 1$ (any different analytical function assumptions lead to solutions for only few certain times or have no solution at all, which is in direct conflict with physical reality). This leads to a system of one differential equation for function $a(t)$, one equation restricting values of constants and one equation for $\rho(t)$

$$T_{\text{fluid}}{}^{t}{}_{t} = \frac{1}{8\pi}\left(B^2 - \Lambda\right) = -\rho, \tag{3.66}$$

$$T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{y}{}_{y} = \frac{(a'(t))^2 - 4a(t)a''(t) - 2a(t)^2\,(B^2 - \Lambda)}{32\pi a(t)} = 0, \tag{3.67}$$

$$T_{\text{fluid}}{}^{z}{}_{z} = \frac{1}{8\pi}\left(B^2 - \Lambda\right) = 0. \tag{3.68}$$

We can easily obtain, that for this solution $\rho(t) = 0$. This leads to the solution we discussed in previous subsection.

3.3.3 General case $(\rho, p \neq 0)$

Finally we are going to try to solve general case, assuming only equality between partial pressures. This situation can be described with following set of equation

$$T_{\text{fluid}}{}^{t}{}_{t} = -\rho, \tag{3.69}$$

$$T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{y}{}_{y} = T_{\text{fluid}}{}^{z}{}_{z} = p. \tag{3.70}$$

The idea of a solution is to look at equation (3.70) and try to find functions $a(t)$, $b(t)$ and $g(t)$ that meet the condition given by partial pressure equality at first. After that, we insert these functions into $-T_{\text{fluid}}{}^{t}{}_{t}$ and into $T_{\text{fluid}}{}^{x}{}_{x}$ and find corresponding mass-energy density ρ and pressure p.

We have found two following solutions for equation (3.70)

1st solution

$$a(t) = C_1, \tag{3.71}$$

$$b(t) = \frac{B^2}{\Lambda}, \tag{3.72}$$

and function $g(t)$ is arbitrary. If we use time parametrization $g(t) = 1$ and insert functions $a(t)$, $b(t)$ and $g(t) = 1$ into equation (3.69) and (3.70), we obtain following mass-energy density and pressure

$$\rho = \frac{\Lambda^2 - B^2\Lambda}{8\pi B^2}, \tag{3.73}$$

$$p = -\frac{\Lambda^2 - B^2\Lambda}{8\pi B^2}. \tag{3.74}$$

From the resulting form of ρ and p we can easily see equation of state for our ideal fluid

$$\rho = -p. \tag{3.75}$$

If we compare this solution with our energy conditions (namely $\rho \geq 0$, $\rho \geq |p|$ and $p \geq 0$), it is not hard to see that we need

$$\rho = p = 0 \tag{3.76}$$

to satisfy the energy conditions. This means, that if we want to comply with the energy conditions for our perfect fluid, this solution can not be a one of an perfect fluid, but (again) only for a strictly magnetic field.

2nd solution

$$g(t) = \frac{b(t)^2(a'(t))^2 - 2b'(t)b(t)a(t)a'(t) + a(t)^2(b'(t))^2}{a(t)\left[\int -\frac{8(-B^2+\Lambda b(t))(b'(t)a(t)-b(t)a'(t))}{b(t)}dt + C_1\right]}, \tag{3.77}$$

and functions $a(t)$ and $b(t)$ are arbitrary while C_1 is an integration constant. We now try and insert some simple analytical functions $a(t)$ and $b(t)$ and compute the remaining function $g(t)$ and then take a look at how our ρ and p are evaluated. We have tried several combinations for the most basic functions, namely t, t^2, $\exp(t)$, $\sin(t)$ and $\cos(t)$.

Whenever the function $b(t)$ is constant in time, we can lay it equal to one without loss of generality by a simple reparametrization of x and y coordinates. We can also always choose gauge $g(t) = 1$ without any loss of generality. For such $b(t)$ and $g(t)$ are ρ and p always in the same form

$$\rho = \frac{\Lambda - B^2}{8\pi}, \tag{3.78}$$

$$p = -\rho = -\frac{\Lambda - B^2}{8\pi}, \tag{3.79}$$

what can be easily seen from equation (3.63) and (3.65). These solutions (again) lead to $\rho = p = 0$ if we want to satisfy energy conditions and are again the same case of solution for (electro-)magnetic field. In addition, the energy conditions again give us a condition $\Lambda = B^2$ and formula for function $a(t)$ is given by equation (3.46).

Now, if we substitute non-constant $b(t)$, we gain solutions, that satisfy energy conditions for some non-trivial time intervals (if we choose suitable values for constants Λ, B and C_1). As we have written earlier, we have tried several basic functions for $a(t)$ and $b(t)$, from where function $g(t)$, $\rho(t)$ and $p(t)$ were computed and the results are following

(1) $a(t) = 1$, $b(t) = t$

$$g(t) = -\frac{1}{-2B^2 ln(t) + 8t\Lambda - C_1}, \tag{3.80}$$

$$\rho(t) = \frac{8B^2 ln(t) - 4\Lambda t^2 - 4B^2 + C_1}{32\pi t^2}, \tag{3.81}$$

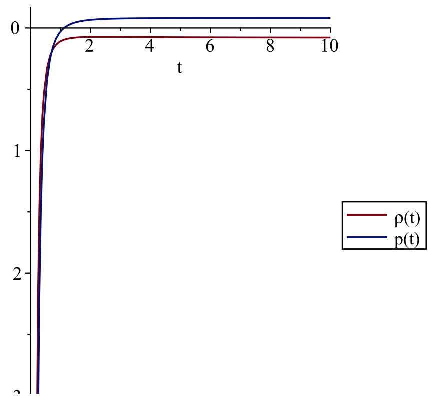


Figure 3.1: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

$$p(t) = \frac{8B^2 ln(t) + 4\Lambda t^2 - 12B^2 + C_1}{32\pi t^2}, \tag{3.82}$$

(2) $a(t) = 1$, $b(t) = t^2$

$$g(t) = -\frac{t^2}{-4B^2 ln(t) + 2\Lambda t^2 - C_1}, \tag{3.83}$$

$$\rho(t) = \frac{-\Lambda t^4 + 4B^2 ln(t) - B^2 + C_1}{8\pi t^4}, \tag{3.84}$$

$$p(t) = \frac{\Lambda t^4 + 4B^2 ln(t) - 3B^2 + C_1}{8\pi t^4}, \tag{3.85}$$

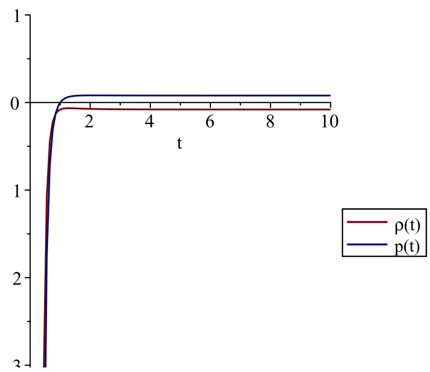


Figure 3.2: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(3) $a(t) = t$, $b(t) = t^2$

$$g(t) = -\frac{3t^3}{8\Lambda t^3 - 24tB^2 - 3C_1}, \tag{3.86}$$

$$\rho(t) = \frac{-3\Lambda t^5 - 10\Lambda t^3 + 45tB^2 + 6C_1}{24\pi t^5}, \tag{3.87}$$

$$p(t) = \frac{3\Lambda t^5 + 2\Lambda t^3 + 27tB^2 + 6C_1}{24\pi t^5}, \tag{3.88}$$

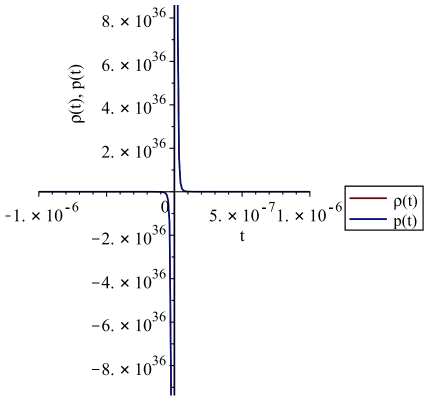


Figure 3.3: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(4) $a(t) = t^2$, $b(t) = t$

$$g(t) = \frac{3t^2}{-12B^2t^2 + 8\Lambda t^3 + 3C_1}, \tag{3.89}$$

$$\rho(t) = \frac{-12\Lambda t^4 - 72B^2t^2 + 64\Lambda t^3 + 15C_1}{96\pi t^4}, \tag{3.90}$$

$$p(t) = \frac{12\Lambda t^4 - 32\Lambda t^3 + 15C_1}{96\pi t^4}, \tag{3.91}$$

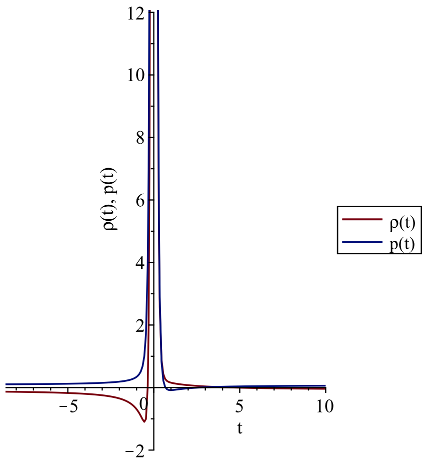


Figure 3.4: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(5) $a(t) = 1$, $b(t) = e^t$

$$g(t) = -\frac{e^{2t}}{-8tB^2 + 8\Lambda e^t - C_1}, \tag{3.92}$$

$$\rho(t) = \left[-4\Lambda e^{2t} + B^2(8t - 4) + C_1\right]\frac{1}{32\pi e^{2t}}, \tag{3.93}$$

$$p(t) = \left[4\Lambda e^{2t} + B^2(8t - 12) + C_1\right]\frac{1}{32\pi e^{2t}}, \tag{3.94}$$

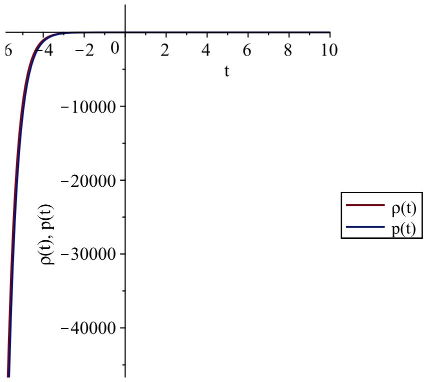


Figure 3.5: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(6) $a(t) = t$, $b(t) = e^t$

$$g(t) = -\frac{\left(e^t\right)^2 (t-1)^2}{t\left(-4B^2\ t^2 + 8tB^2 + 8\Lambda e^t t - 16\Lambda e^t - C_1\right)}, \tag{3.95}$$

$$\begin{aligned}\rho(t) =& \frac{e^{-2t}}{16\pi (t-1)^2}\Big[-2\Lambda (t-1)^2 e^{2t} + -4\Lambda (2t-5) e^t \\ & +2B^2t^3 - 2B^2t^2 + \left(-4B^2 + \frac{C_1}{2}\right) t - 2B^2 + C_1\Big],\end{aligned} \tag{3.96}$$

$$\begin{aligned}p(t) =& \frac{2e^{-2t}}{8\pi (t-1)^3}\Big[2\Lambda (t-1)^3 e^{2t} - 4\Lambda (t-5) e^t + 2B^2t^4 \\ & - 8B^2t^3 + \left(10B^2 + \frac{C_1}{2}\right) t^2 + \left(-10B^2 + \frac{C_1}{2}\right) t \\ & -2B^2 + C_1\Big],\end{aligned} \tag{3.97}$$

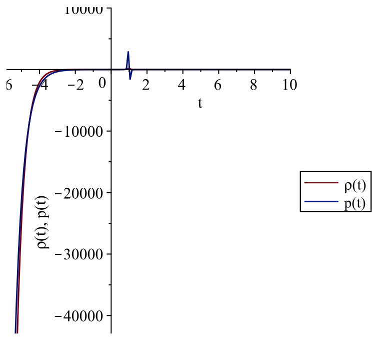


Figure 3.6: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(7) $a(t) = t^2$, $b(t) = e^t$

$$g(t) = -\frac{3\left(e^t\right)^2 (t-2)^2}{-8B^2t^3 + 24B^2t^2 + 24\Lambda e^t t^2 - 96\Lambda e^t t + 96\Lambda e^t - 3C_1}, \tag{3.98}$$

$$\begin{aligned}\rho(t) =& \frac{e^{-2t}}{24\pi t (t-2)^2}\Big[-3t\Lambda (t-2)^2 e^{2t} - 24\Lambda (t-2)^2 e^t \\ & +2B^2t^4 - B^2t^3 - 12B^2t^2 + \left(-12B^2 + \frac{3C_1}{4}\right) t + 3C_1\Big],\end{aligned} \tag{3.99}$$

$$p(t) = \frac{4e^{-2t}}{24\pi (t-2)^3} \left[3\Lambda (t-2)^3 e^{2t} + 2B^2t^4 - 11B^2t^3 + 18B^2t^2 + \left(-12B^2 + \frac{3C_1}{4} \right) t - 24B^2 + \frac{3C_1}{4} \right], \tag{3.100}$$

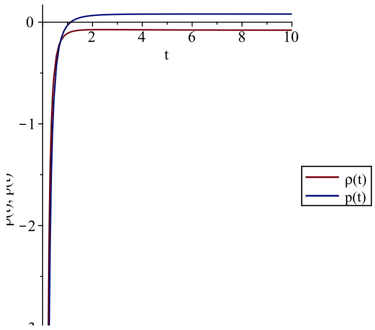


Figure 3.7: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(8) $a(t) = 1$, $b(t) = \sin(t)$

$$g(t) = -\frac{\cos(t)^2}{-8B^2 \ln(\sin(t)) + 8\Lambda \sin(t) - C_1}, \tag{3.101}$$

$$\rho(t) = \frac{8B^2 \ln(\sin(t)) - 4\Lambda \sin(t)^2 - 4B^2 + C_1}{32\pi \sin(t)^2}, \tag{3.102}$$

$$p(t) = \frac{8B^2 \ln(\sin(t)) + 4\Lambda \sin(t)^2 - 12B^2 + C_1}{32\pi \sin(t)^2}, \tag{3.103}$$

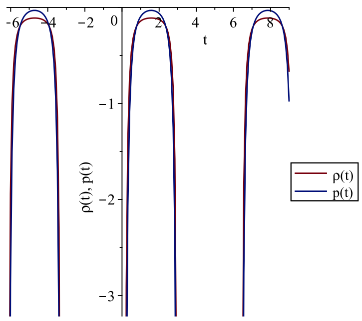


Figure 3.8: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(9) $a(t) = 1$, $b(t) = \cos(t)$

$$g(t) = -\frac{\sin(t)^2}{-8B^2 \ln(\cos(t)) + 8\Lambda \cos(t) - C_1}, \tag{3.104}$$

$$\rho(t) = \frac{8B^2 \ln(\cos(t)) - 4\Lambda \cos(t)^2 - 4B^2 + C_1}{32\pi \cos(t)^2}, \tag{3.105}$$

$$p(t) = \frac{8B^2 \ln(\cos(t)) + 4\Lambda \cos(t)^2 - 12B^2 + C_1}{32\pi \cos(t)^2}, \tag{3.106}$$

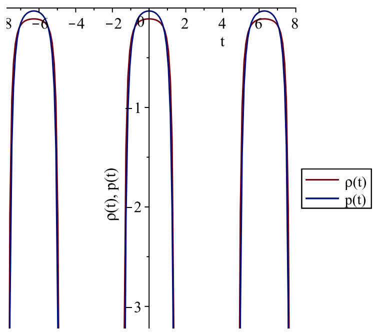


Figure 3.9: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(10) $a(t) = \cos(t)$, $b(t) = \sin(t)$

$$g(t) = -\frac{1}{\cos(t)\left[-8B^2 \ln\left(\frac{1}{\sin(t)} - \cot(t)\right) + 8\Lambda t - C_1\right]}, \tag{3.107}$$

$$\begin{aligned}\rho(t) =& \frac{1}{32\pi\sin(t)^2}\left[24B^2\cos(t)\left(\cos(t)^2 - \frac{2}{3}\right)\ln\left(\frac{1-\cos(t)}{\sin(t)}\right)\right.\\ &+ (-24\Lambda t + 3C_1)\cos(t)^3 + 4\Lambda\cos(t)^2\\ &\left.+ (16\Lambda t - 2C_1)\cos(t) - 4B^2 + 8\Lambda\sin(t) - 4\Lambda\right],\end{aligned} \tag{3.108}$$

$$\begin{aligned}p(t) =& \frac{1}{32\pi\sin(t)^2}\left[-40B^2\cos(t)\left(\cos(t)^2 - \frac{6}{5}\right)\ln\left(\frac{1-\cos(t)}{\sin(t)}\right)\right.\\ &+ (40\Lambda t - 5C_1)\cos(t)^3 + (-48\Lambda t + 6C_1)\cos(t)\\ &+ \left(-16B^2 + 16\Lambda sin(t) - 4\Lambda\right)\cos(t)^2 + 4B^2\\ &-8\Lambda\sin(t) + 4\Lambda],\end{aligned} \tag{3.109}$$

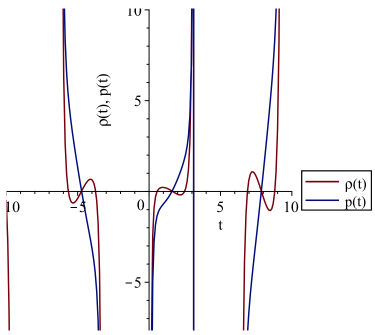


Figure 3.10: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(11) $a(t) = \sin(t)$, $b(t) = \cos(t)$

$$g(t) = \frac{1}{\sin(t)\left[-8B^2 \ln\left(\frac{1}{\cos(t)} + \tan(t)\right) + 8\Lambda t + C_1\right]}, \tag{3.110}$$

$$\begin{aligned}\rho(t) =& \frac{1}{32\pi\cos(t)^2}\left[24B^2\sin(t)\left(\cos(t)^2 - \frac{1}{3}\right)\ln\left(\frac{1+\sin(t)}{\cos(t)}\right)\right.\\ &+ ((-24\Lambda t - 3C_1)\sin(t) - 4\Lambda)\cos(t)^2 + 8\Lambda\cos(t)\\ &\left.(8\Lambda t + C_1)\sin(t) - 4B^2\right],\end{aligned} \tag{3.111}$$

$$
\begin{aligned}
p(t) =& \frac{1}{32\pi\cos(t)^2}\left[-40B^2\sin(t)\left(\cos(t)^2+\frac{1}{5}\right)\ln\left(\frac{1+\sin(t)}{\cos(t)}\right)\right. \\
&- 16\Lambda\cos(t)^3 + \Big((40\Lambda t + 5C_1)\sin(t) + 16B^2 + 4\Lambda\Big)\cos(t)^2 \\
&\left. +8\Lambda\cos(t) + (8\Lambda t + C_1)\sin(t) - 12B^2\right],
\end{aligned} \tag{3.112}
$$

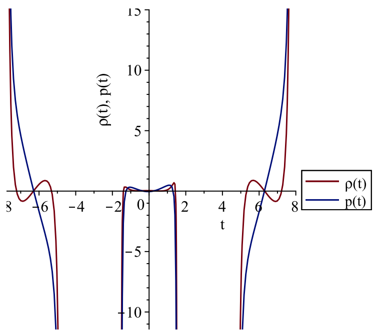


Figure 3.11: Plotted functions $\rho(t)$ and $p(t)$ for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

where C_1 is integration constant.

Unfortunately, it was not possible to find a solution that would satisfy all of the energy conditions on the whole interval $t \in (-\infty; \infty)$ (or in the domain of functions $\rho(t)$ and $p(t)$). This insight can be gained from the asymptotic behaviour of functions $\rho(t)$ and $p(t)$ for each solution and also from graphs plotted for concrete values of constants.

It is important to say, that the functions appearing in our metric are not squared. This could lead to change of signature of our metric tensor. Solutions, which would satisfy energy conditions but assumed negative metric function on the interval where they satisfied the conditions, would be perfect fluids, but the spacetime they would be in wouldn't be one from our universe. Thus they would not have any physical meaning.

Because there have not been found any solution which would satisfy energy conditions, we feel no need to plot or study metric functions, that were inserted into functions $\rho(t)$ and $p(t)$ in points **(1)-(11)**, but they will play significant role in the next chapter.

4. Minkowski seed

Instead, we now assume the metric to be of the form

$$\mathrm{d}s^2 = -g(t)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{b(t)^2} + b(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{4.1}$$

where the radial cylindrical coordinate is $r^2 := x^2 + y^2$. We keep $g(t)$ in the metric although it is merely a gauge—choosing a suitable function here can be advantageous in subsequent calculations.

Again, choosing Minkowski-type of a metric has its reason due to that there are solutions involving electromagnetic field and cosmological term, as was written in introduction, see, e.g., [Thorne [1967]].

The Maxwell tensor corresponding to a (static) magnetic field parallel to the z-axis again reads

$$F_{xy} = -F_{yx} = B(r^2, t). \tag{4.2}$$

Now we are going to proceed as per the previous section.

4.1 Right side of Einstein's equations for magnetic field

Maxwell's potential equations are going to look exactly the same way as the have in previous chapter. From them we gain (again), that function $B(r^2, t)$ is actually time independent. Equations that change beacuase of difference in metric are field ones. From equation (3.7) we obtain system of two partial differential equations for unknown function $B(r^2)$.

$$\partial_x B(r^2) = 0, \tag{4.3}$$

$$\partial_y B(r^2) = 0. \tag{4.4}$$

Solution of this system is quite simple and it is

$$B(r^2) = const. \tag{4.5}$$

Again, we set this constant equal to the magnetic field B. According to equation (2.2), we can now calculate corresponding stress-energy tensor $T^{\mu\nu}_{mag}$ and lower its indices according to equation (3.11). We gain

$$T^{mag}_{tt} = \frac{g(t)B^2}{8\pi b(t)^4}, \tag{4.6}$$

$$T^{mag}_{xx} = T^{mag}_{yy} = \frac{B^2}{8\pi b(t)^2}, \tag{4.7}$$

$$T^{mag}_{zz} = -\frac{B^2}{8\pi b(t)^6}. \tag{4.8}$$

4.2 Left side of Einstein's equations

Now we compute Einstein tensor and Λg term, so we gain left side of Einstein's equations.
We start with computing components of affine connection again, using equation (3.15), from where we gain

$$\Gamma^t{}_{tt} = \frac{g'(t)}{2g(t)}, \tag{4.9}$$

$$\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \frac{b(t)b'(t)}{g(t)}, \tag{4.10}$$

$$\Gamma^x{}_{tx} = \Gamma^x{}_{xt} = \frac{b'(t)}{b(t)}, \tag{4.11}$$

$$\Gamma^y{}_{ty} = \Gamma^y{}_{yt} = \frac{b'(t)}{b(t)}, \tag{4.12}$$

$$\Gamma^t{}_{zz} = -\frac{b'(t)}{g(t)b(t)^3}, \tag{4.13}$$

$$\Gamma^z{}_{tz} = \Gamma^z{}_{zt} = -\frac{b'(t)}{b(t)}. \tag{4.14}$$

Any other combinations of indices is trivial.

From here, we calculate Ricci tensor and Ricci scalar using identities given by equations (3.26) respectively (3.27). We gain

$$R_{tt} = \frac{-2b''(t)g(t)b(t) + b'(t)g'(t)b(t) - 4(b'(t))^2 g(t)}{2g(t)b(t)^2}, \tag{4.15}$$

$$R_{xx} = R_{yy} = \frac{b(t)\left(2b''(t)g(t) - b'(t)g'(t)\right)}{2g(t)^2}, \tag{4.16}$$

$$R_{zz} = \frac{-2b''(t)g(t) + b'(t)g'(t)}{2b(t)^3 g(t)^2}, \tag{4.17}$$

and

$$R = \frac{2b''(t)g(t)b(t) - b'(t)g'(t)b(t) + 2(b'(t))^2 g(t)}{g(t)^2 b(t)^2}. \tag{4.18}$$

At last we get Einstein tensor from equation (3.32)

$$G_{tt} = -\frac{(b'(t))^2}{b(t)^2}, \tag{4.19}$$

$$G_{xx} = G_{yy} = -\frac{(b'(t))^2}{g(t)}, \tag{4.20}$$

$$G_{zz} = \frac{-2b''(t)g(t)b(t) + b'(t)g'(t)b(t) - (b'(t))^2 g(t)}{b(t)^4 g(t)^2}. \tag{4.21}$$

4.3 Searching for a perfect fluid

As we have done it in previous chapter, now we add Λg term to the left side of Einstein's equations and subtract $8\pi T^{mag}_{\mu\nu}$ term from it and than divide everything with 8π. We obtain $T^{\text{leftover}}{}_{\mu\nu}$ on the left side of Einstein's equations, this time for Minkowski-like spacetime.

$$T^{\text{leftover}}{}_{tt} = -\frac{1}{8\pi}\left(\frac{g(t)B^2}{b(t)^4} + \frac{(b'(t))^2}{b(t)^2} + \Lambda g(t)\right), \tag{4.22}$$

$$T^{\text{leftover}}{}_{xx} = T^{\text{leftover}}{}_{yy} = -\frac{1}{8\pi}\left(\frac{B^2}{b(t)^2} + \frac{(b'(t))^2}{g(t)} - \Lambda b(t)^2\right), \tag{4.23}$$

$$T^{\text{leftover}}{}_{zz} = \frac{1}{8\pi}\left(\frac{B^2}{b(t)^6} + \frac{-2b''(t)g(t)b(t) + b'(t)g'(t)b(t) - (b'(t))^2 g(t)}{b(t)^4 g(t)^2} + \frac{\Lambda}{b(t)^2}\right). \tag{4.24}$$

Now we lift the first index and choose a rest frame to work in again, so we get $T_{\text{fluid}}{}^{\alpha}{}_{\beta} = \text{diag}(-\rho, p, p, p)$. This gives us following equations

$$T_{\text{fluid}}{}^{t}{}_{t} = \frac{1}{8\pi}\left(\frac{B^2}{b(t)^4} + \frac{(b'(t))^2}{g(t)b(t)^2} + \Lambda\right) = -\rho, \tag{4.25}$$

$$T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{y}{}_{y} = -\frac{1}{8\pi}\left(\frac{B^2}{b(t)^4} + \frac{(b'(t))^2}{g(t)b(t)^2} - \Lambda\right) = p, \tag{4.26}$$

$$T_{\text{fluid}}{}^{z}{}_{z} = \frac{1}{8\pi}\left(\frac{B^2}{b(t)^4} + \frac{-2b''(t)g(t)b(t) + b'(t)g'(t)b(t) - (b'(t))^2 g(t)}{b(t)^2 g(t)^2} + \Lambda\right) = p. \tag{4.27}$$

Now we have again one condition coming out of equality of partial pressures ($T_{\text{fluid}}{}^{x}{}_{x} = T_{\text{fluid}}{}^{z}{}_{z}$), for metric functions $g(t)$.and $b(t)$.

We also have energy conditions such that $\rho \geq 0$, $\rho \geq |p|$ and $p \geq 0$.

We are going to try to find such functions $g(t)$ and $b(t)$, that everything written above states true.

We have tried all assumptions we have tried in chapter 3 for Kantowski-Sachs metric, but unlike in chapter 3, we have found only solutions for general case, where our only assumption was partial pressures equality. Even though we were not able to find a solution for these assumptions, they do exist for a vanishing cosmological constant [Thorne [1967]].

If we look for example on the assumption where $\rho = p = 0$, we can with simple observation see, that in equations (4.25) and (4.26) term $\frac{(b'(t))^2}{g(t)b(t)^2}$ once equals $-\Lambda - \frac{B^2}{b(t)^4}$ and once $\Lambda - \frac{B^2}{b(t)^4}$, which necessarily means a vanishing cosmological constant.

As we have written a little bit earlier, we have found one general solution for general case, where $\rho, p \neq 0$ and we assume only partial pressures equality, described by equations (3.69) and (3.70). The solution is in a following form

$$g(t) = -\frac{b(t)^2(b'(t))^2}{B^2 - b(t)^2 C_1}, \tag{4.28}$$

where C_1 is integration constant and function $b(t)$ is arbitrary.

4.3.1 Solving for $b(t)$ in various analytical gauges

From here, we are going to proceed as per the previous section. We are going to try to substitute different combinations of basic analytical functions for $b(t)$ and look for a possible perfect fluid, that would satisfy our energy conditions ($\rho \geq 0$, $\rho \geq |p|$ and $p \geq 0$).

Unlike for function $g(t)$ in Kantowski-Sachcs general solution given by equation (3.77), for function $g(t)$ states quite "easy" differential equation (4.28) here, which is solvable. That means, that we can now insert analytical functions for $g(t)$ and get dependent functions $b(t)$ and vice versa.

Namely, we have tried to insert for $b(t)$ and $g(t)$ combinations of functions t, t^2, $\exp(t)$, $\sin(t)$ and $\cos(t)$ again. We approach the problem in this manner since, with a suitable choice of gauge, the resulting expressions might be easier to analyze. The results follow below.

(1) $g(t) = 1$ (time parametrization)

$$b(t) = -\frac{\sqrt{\left[(t - C_2)^2 C_1^2 + B^2\right] C_1}}{C_1}, \tag{4.29}$$

$$\rho(t) = \frac{(-1 - (t - C_2)^2 \Lambda) C_1^2 - B^2 \Lambda}{8\pi \left[(t - C_2)^2 C_1^2 + B^2\right]}, \tag{4.30}$$

$$p(t) = \frac{(-1 + (t - C_2)^2 \Lambda) C_1^2 + B^2 \Lambda}{8\pi \left[(t - C_2)^2 C_1^2 + B^2\right]}, \tag{4.31}$$

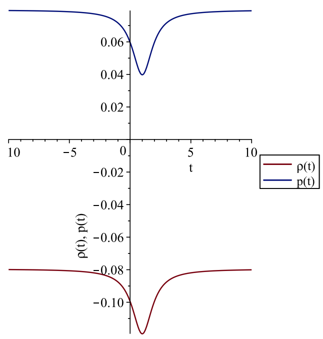


Figure 4.1: Plotted functions $\rho(t)$ and $p(t)$ from **(1)** for $\Lambda = 2$, $B = 1$, $C_1 = 1$ and $C_2 = 1$.

(2) $g(t) = t$

$$b(t) = -\frac{\sqrt{-12\sqrt{-t} C_1^3 C_2 t + (4t^3 - 9C_2^2) C_1^3 + 9B^2 C_1}}{3C_1}, \tag{4.32}$$

$$\rho(t) = -\frac{81}{8\pi\sqrt{-t}\left(12(-t)^{\frac{3}{2}}C_1^2C_2 + 4C_1^2t^3 - 9C_1^2C_2^2 + 9B^2\right)^2}$$
$$\left[-\frac{8\left(\left(-3\Lambda C_2^2 + \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)C_1^2\,(-t)^{\frac{7}{2}}}{9} + \frac{16\Lambda(-t)^{\frac{13}{2}}C_1^4}{81}\right. \tag{4.33}$$
$$+\left(\left(-\Lambda C_2^2 + 1\right)C_1^2 + B^2\Lambda\right)\left(-C_1^2C_2^2 + B^2\right)\sqrt{-t}$$
$$\left.+\frac{8C_2\left(\left(\frac{4}{9}\Lambda t^3 - \Lambda C_2^2 + \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)C_1^2t^2}{3}\right],$$

$$p(t) = -\frac{81}{8\pi\sqrt{-t}\left(12(-t)^{\frac{3}{2}}C_1^2C_2 + 4C_1^2t^3 - 9C_1^2C_2^2 + 9B^2\right)^2}$$
$$\left[-\frac{8\left(\left(-3\Lambda C_2^2 - \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)C_1^2\,(-t)^{\frac{7}{2}}}{9} + \frac{16\Lambda(-t)^{\frac{13}{2}}C_1^4}{81}\right. \tag{4.34}$$
$$+\left(\left(-\Lambda C_2^2 - 1\right)C_1^2 + B^2\Lambda\right)\left(-C_1^2C_2^2 + B^2\right)\sqrt{-t}$$
$$\left.+\frac{8C_2\left(\left(\frac{4}{9}\Lambda t^3 - \Lambda C_2^2 - \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)C_1^2t^2}{3}\right],$$

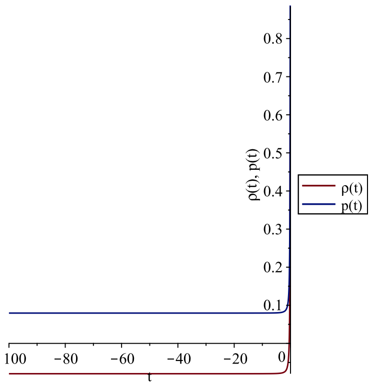


Figure 4.2: Plotted functions $\rho(t)$ and $p(t)$ from **(2)** for $\Lambda = 2$, $B = 1$, $C_1 = 1$ and $C_2 = 1$.

(3) $g(t) = t^2$

$$b(t) = -\frac{\sqrt{C_1^3t^4 - 4C_1^4t^2 + 4C_1^5 + 4B^2C_1}}{2C_1}, \tag{4.35}$$
$$\rho(t) = \frac{-4\Lambda C_1^4 + 4\Lambda C_1^3t^2 + \left(-\Lambda t^4 - 4\right)C_1^2 - 4B^2\Lambda}{8\pi\left(C_1^2t^4 - 4C_1^3t^2 + 4C_1^4 + 4B^2\right)}, \tag{4.36}$$
$$p(t) = \frac{4\Lambda C_1^4 - 4\Lambda C_1^3t^2 + \left(\Lambda t^4 - 4\right)C_1^2 + 4B^2\Lambda}{8\pi\left(C_1^2t^4 - 4C_1^3t^2 + 4C_1^4 + 4B^2\right)}, \tag{4.37}$$

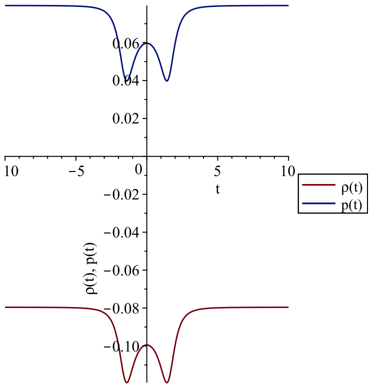


Figure 4.3: Plotted functions $\rho(t)$ and $p(t)$ from **(3)** for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(4) $g(t) = exp(t)$

$$b(t) = \frac{\sqrt{C_1\left(-4\sqrt{-e^t}C_1^2C_2 - C_1^2C_2^2 + B^2 + 4C_1^2e^t\right)}}{C_1}, \tag{4.38}$$

$$\begin{aligned}\rho(t) =& \frac{1}{8\pi\sqrt{-e^t}\left(-4\sqrt{-e^t}C_1^2C_2 - C_1^2C_2^2 + B^2 + 4C_1^2e^t\right)^2} \\ & \left[\left(16\Lambda e^{2t}C_1^4 + \left(\left(-24\Lambda C_2^2 + 4\right)C_1^4 + 8B^2\Lambda C_1^2\right)e^t\right.\right. \\ & \left. + \left(\left(-\Lambda C_2^2 + 1\right)C_1^2 + B^2\Lambda\right)\left(-C_1^2C_2^2 + B^2\right)\right)\sqrt{-e^t} \\ & \left. +8C_2C_1^2\left(4C_1^2\Lambda e^{2t} + e^t\left(\left(-\Lambda C_2^2 + \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)\right)\right],\end{aligned} \tag{4.39}$$

$$\begin{aligned}p(t) =& \frac{1}{8\pi\sqrt{-e^t}\left(-4\sqrt{-e^t}C_1^2C_2 - C_1^2C_2^2 + B^2 + 4C_1^2e^t\right)^2} \\ & \left[\left(16\Lambda e^{2t}C_1^4 + \left(\left(-24\Lambda C_2^2 - 4\right)C_1^4 + 8B^2\Lambda C_1^2\right)e^t\right.\right. \\ & \left. + \left(\left(-\Lambda C_2^2 - 1\right)C_1^2 + B^2\Lambda\right)\left(-C_1^2C_2^2 + B^2\right)\right)\sqrt{-e^t} \\ & \left. +8C_2C_1^2\left(4C_1^2\Lambda e^{2t} + e^t\left(\left(-\Lambda C_2^2 - \frac{1}{2}\right)C_1^2 + B^2\Lambda\right)\right)\right],\end{aligned} \tag{4.40}$$

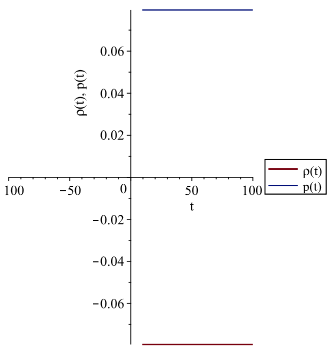


Figure 4.4: Plotted functions $\rho(t)$ and $p(t)$ from **(4)** for $\Lambda = 2$, $B = 1$, $C_1 = 1$ and $C_2 = 1$.

(5) $b(t) = t$

$$g(t) = -\frac{t^2}{B^2 - t^2 C_1}, \tag{4.41}$$

$$\rho(t) = -\frac{\Lambda t^2 + C_1}{8\pi t^2}, \tag{4.42}$$

$$p(t) = \frac{\Lambda t^2 - C_1}{8\pi t^2}, \tag{4.43}$$

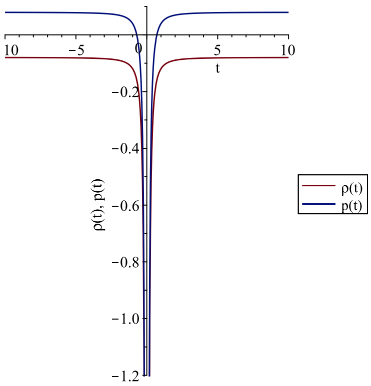


Figure 4.5: Plotted functions $\rho(t)$ and $p(t)$ from **(5)** for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(6) $b(t) = t^2$

$$g(t) = -\frac{4t^6}{-t^4 C_1 + B^2}, \tag{4.44}$$

$$\rho(t) = -\frac{\Lambda t^4 + C_1}{8\pi t^4}, \tag{4.45}$$

$$p(t) = \frac{\Lambda t^4 - C_1}{8\pi t^4}, \tag{4.46}$$

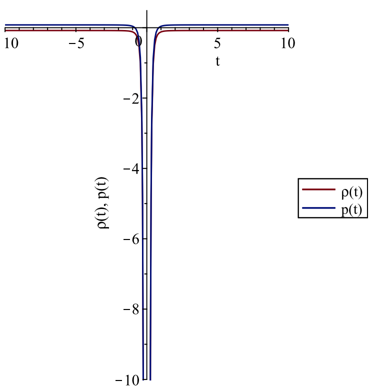


Figure 4.6: Plotted functions $\rho(t)$ and $p(t)$ from **(6)** for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(7) $b(t) = e^t$

$$g(t) = -\frac{e^{4t}}{B^2 - e^{2t}C_1}, \tag{4.47}$$

$$\rho(t) = -\frac{(\Lambda e^{2t} + C_1)\, e^{-2t}}{8\pi}, \tag{4.48}$$

$$p(t) = \frac{-e^{-2t}C_1 + \Lambda}{8\pi}, \tag{4.49}$$

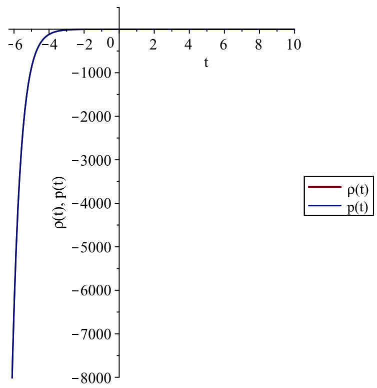


Figure 4.7: Plotted functions $\rho(t)$ and $p(t)$ from **(7)** for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(8) $b(t) = \sin(t)$

$$g(t) = -\frac{\sin(t)^2 \cos(t)^2}{B^2 + C_1 \cos(t)^2 - C_1}, \tag{4.50}$$

$$\rho(t) = \frac{\Lambda \cos(t)^2 - \Lambda - C_1}{8\pi \sin(t)^2}, \tag{4.51}$$

$$p(t) = \frac{-\Lambda \cos(t)^2 + \Lambda - C_1}{8\pi \sin(t)^2}, \tag{4.52}$$

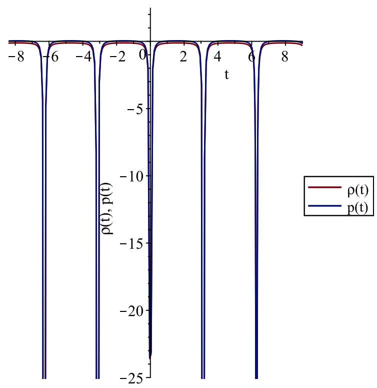


Figure 4.8: Plotted functions $\rho(t)$ and $p(t)$ from **(8)** for $\Lambda = 2$, $B = 1$ and $C_1 = 1$.

(9) $b(t) = \cos(t)$

$$g(t) = -\frac{\cos(t)^2 \sin(t)^2}{B^2 - C_1 \cos(t)^2}, \tag{4.53}$$

$$\rho(t) = \frac{-\Lambda \cos(t)^2 - C_1}{8\pi \cos(t)^2}, \tag{4.54}$$

$$p(t) = \frac{\Lambda \cos(t)^2 - C_1}{8\pi \cos(t)^2}, \tag{4.55}$$

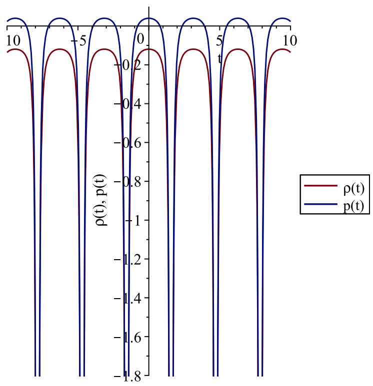


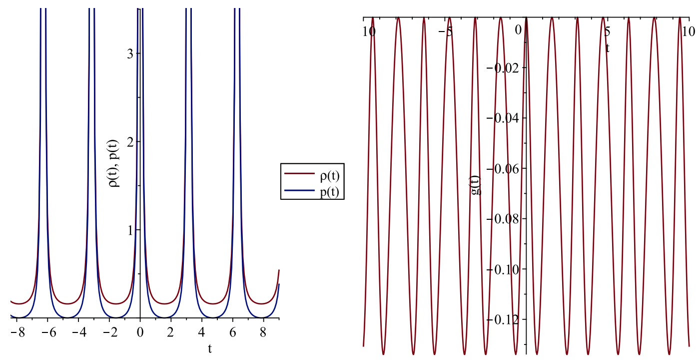
Figure 4.9: Plotted functions $\rho(t)$ and $p(t)$ from **(9)** for $\Lambda = 2$, $B = 1$, $\kappa = 8\pi$ and $C_1 = 1$.

where C_1 and C_2 are integration constants.

For $\rho(t)$ and $p(t)$ in solutions **(8)** and **(9)**, there was possible to find solutions simply by altering the constants values. These solutions have its singularities, but the singularities correspond to singularities in Kretschmann, Ricci and Maxwell scalar (equations (3.49), (3.27) and (3.48)). Unfortunately, the solutions were physical in the sense of ideal fluid and energy conditions imposed on it, but the universe they were found in was not ours due to the sign of metric function $g(t)$ described by equation (4.50), respectively (4.53), which is negative. This gives us metric tensor with signature $(+, +, +, +)$.

Plotted functions $\rho(t)$ and $p(t)$ from solutions **(8)** and **(9)** that satisfy energy conditions (but in a different universe than the one we live in) and corresponding tensor functions $g(t)$ are following

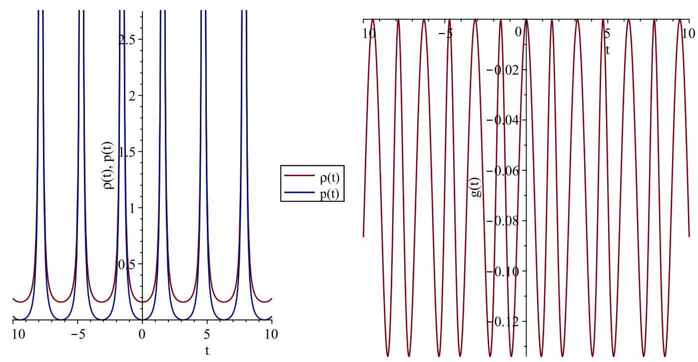
(8)



(a) (b)

Figure 4.10: **(a)** Plotted functions $\rho(t)$ and $p(t)$ from for $\Lambda = -2$, $B = 1$ and $C_1 = -2$ **(b)** Plotted function $g(t)$ for $\Lambda = -2$, $B = 1$ and $C_1 = -2$

(9)



(a) (b)

Figure 4.11: **(a)** Plotted functions $\rho(t)$ and $p(t)$ from for $\Lambda = -2$, $B = 1$ and $C_1 = -2$ **(b)** Plotted function $g(t)$ for $\Lambda = -2$, $B = 1$ and $C_1 = -2$

For the functions $b(t) = \cos(t)$ and $g(t)$ given by equation (4.53), we tried the Wick's rotation, which led to a perfect fluid satisfying the energy conditions, but unfortunately the metric function g was still negative.

No other solution for $t \in (-\infty, \infty)$ (or in the domain of functions $\rho(t)$ and $p(t)$) was found, even for different values of constants. This insight can be gained again from asymptotic behaviour of functions $\rho(t)$ and $p(t)$ for solutions above and also from graphs plotted for concrete values of constants.

4.3.2 General solution for $b(t)$

Now we are going to take a more systematic look at equation (4.28). Solving this differential equation for the metric function $b(t)$, we get the following integral equation

$$-\frac{\sqrt{B^2-b(t)^2C_1}}{C_1}\pm\int_0^t\frac{\sqrt{-\left(B^2-b(t)^2C_1\right)g(x)}}{\sqrt{B^2-b(t)^2C_1}}\,dx+C_2=0, \tag{4.56}$$

where C_1 and C_2 are integration constants.

Integral on the left side of the equation can be simplified into a form

$$\int_0^t\frac{\sqrt{-\left(B^2-b(t)^2C_1\right)g(x)}}{\sqrt{B^2-b(t)^2C_1}}\,dx=i\int_0^t\sqrt{g(x)}\,dx, \tag{4.57}$$

which is solvable depending on $g(x)$.

If we define function $f(t)$ by equation

$$f(t)=\int_0^t\sqrt{g(x)}\,dx, \tag{4.58}$$

equation (4.56) goes into a form

$$-\frac{\sqrt{B^2-b(t)^2C_1}}{C_1}\pm if(t)+C_2=0. \tag{4.59}$$

There exist two solutions of this equation for function $b(t)$

$$b(t)=\pm\frac{\sqrt{C_1\left(f(t)^2C_1^2+2if(t)C_1^2C_2-C_1^2C_2^2+B^2\right)}}{C_1}. \tag{4.60}$$

We can clearly see, that constant C_2 must be equal to zero, because we only want real solutions, thus

$$b(t)=\pm\frac{\sqrt{C_1\left(f(t)^2C_1^2+B^2\right)}}{C_1}. \tag{4.61}$$

Here, we should notice, that if we do not want to violate metric signature, constant C_1 has to be positive and nonzero, due to that $b(t)^2\leq 0$ for $C_1<0$ and diverges for $C_1=0$.

If we now insert function $g(t)$ given by equation (4.28) into equations (4.25) for $\rho(t)$ and (4.26) for $p(t)$, we get

$$\rho(t)=\frac{-\Lambda b(t)^2-C_1}{8\pi b(t)^2}, \tag{4.62}$$

$$p(t)=\frac{\Lambda b(t)^2-C_1}{8\pi b(t)^2}. \tag{4.63}$$

From here, we can easily see, that function $b(t)$ is squared in both equations, thus the sign on the right side of the equation (4.61) has no further meaning in following calculations and we are going to use the plus sign without prejudice to the generality.

If we now insert equation (4.61) into equations (4.62) and (4.63), we obtain that

$$\rho(t)=-\frac{C_1^2}{8\pi\left(f(t)^2C_1^2+B^2\right)}-\frac{\Lambda}{8\pi}, \tag{4.64}$$

$$p(t) = -\frac{C_1^2}{8\pi\left(f(t)^2 C_1^2 + B^2\right)} + \frac{\Lambda}{8\pi}. \tag{4.65}$$

Now, at first sight, we can see, that for $\Lambda = C_1 = 0$, we get strictly (electro)magnetic solution without perfect fluid, but as we have already said, C_1 must not be equal to 0, so there exists no strictly electromagnetic field for the function $b(t)$ of the form (4.61).

We can also quite easily see the equation of state, which is

$$p(t) = \rho(t) + \frac{\Lambda}{4\pi}. \tag{4.66}$$

We can also see, that the term

$$\frac{C_1^2}{8\pi\left(f(t)^2 C_1^2 + B^2\right)} > 0 \tag{4.67}$$

for $C_1 \neq 0$, which we ruled out above.

This means, that at least one of the functions $\rho(t)$ or $p(t)$ has to be negative, depending on sign of Λ, for $t \in \mathbf{R}$. Thus energy conditions can never be met.

Conclusion

In an effort to find solutions to the equations of general relativity for multiple sources at once, we have summarized in the first two chapters both the general and the necessary knowledge for the calculations that take place primarily in Chapters 3 and 4.

In the first chapter, we summarized Einstein's gravitational field equations directly and derived the identity, which we further used in the calculations in Chapter 3.

In the second chapter, we dealt mainly with the right side of Einstein's equations, the stress-energy tensor, and in particular with the two variants of it, that we were interested in in this thesis: the electromagnetic tensor and the tensor corresponding to the perfect fluid. We have also presented the energy conditions on such a tensor, which played a crucial role in our work.

In Chapter 3 we have already dealt directly with the calculations in the spacetime given by the Kantowski-Sachs-like metric with cylindrical symmetry.

We first considered the curvature field generated purely by the electromagnetic stress-energy tensor.

We managed to find one exact solutions of Einstein's equations just for the case of the electromagnetic field, which gave us the expressions for metric functions, namely

$$g(t) = 1,$$
$$b(t) = 1,$$
$$a(t) = t^2,$$

This is a spacetime which is very interesting and which we would like to study in the future, if only because it has constant scalar curvatures but non-constant metric functions.

Subsequently, we were also able to show, that if the metric function $b(t)$ is an arbitrary constant we can lay it equal to one without loss of generality by a simple reparametrization of x and y coordinates. we can also without any loss of generality choose gauge $g(t) = 1$. For such $b(t)$ and $g(t)$ tensor ${T_{\text{fluid}}}^{\mu}{}_{\nu}$ will always eventually vanish and thus will only be a curvature generated by the electromagnetic field and not by a perfect fluid. Energy conditions then also place a condition on the value of the cosmological constant, namely

$$\Lambda = B^2,$$

and function $a(t) = t^2$.

Next, we tried to find a solution for zero partial pressures in ${T_{\text{fluid}}}^{\mu}{}_{\nu}$, which corresponds to incoherent dust, and also for the general case where we assumed only the equality between partial pressures for the tensor ${T_{\text{fluid}}}^{\mu}{}_{\nu}$. Unfortunately, no solution satisfied the energy conditions on the whole time interval (or in the domain of functions $\rho(t)$ and $p(t)$). The only solutions that satisfied them were those for ${T_{\text{fluid}}}^{\mu}{}_{\nu} = 0$, which again leads to a field generated by a purely electromagnetic field.

In Section 4 we tried to use for the same electromagnetic field a different metric describing the curvature of spacetime, namely the Minkowski-like form of metric.

In this chapter, unlike in the chapter 3, we have not been able to find such metric functions $g(t)$ and $b(t)$ corresponding to the field generated by purely electromagnetic field neither by incoherent dust or fluid.

Of course, the work could be continued. It is always possible to choose different metric tensors with different symmetries, and a similar statement holds for the Maxwell tensor, its symmetries and the field it produces. There is an immeasurable number of combinations that could be tried for different metrics and different Maxwell tensors with different symmetries.

It might also be worth considering a field generated by a fluid that is not static in our coordinate system, but is moving. There are indeed many possibilities.

It is also worth noting that a large number of perfect fluids were found in this thesis, unfortunately they just did not meet the conditions we set for them.

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