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Nonlinear Electrodynamics

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Abstract: Nonlinear electrodynamics, introduced in the 1930s to remedy divergences associated with Maxwell's theory, has become a recurring theme in theoretical physics. Recent developments in the area of nonlinear electrodynamics coupled to gravity have prompted the creation of an accessible ground up reformulation of the basic structure. We develop the formalism by building upon classical electromagnetism in Minkowski spacetime, deriving the fundamental equations by the action principle before re-deriving the Lagrangians of two important models from the founding era and describing the corresponding regular static spherically symmetric solutions. The focus is then shifted to the examination of a recently discovered model through which we develop a basic background for the coupling of nonlinear electrodynamics to gravity and AdS black hole thermodynamics.

Keywords: Maxwell's equations, Nonlinear electrodynamics, Exact solutions

Contents

Introduction

Nonlinear electrodynamics (NE) began as an attempt at resolving divergences associated with classical models of a charged particle. These include the problems of infinite self-energy and electric field divergence of the electron. This approach attempts to resolve these divergences within the framework of classical field theory by relaxing the condition of linearity in the field equations. The first and most well known model to use this approach was published in 1934 by M. Born and L. Infeld [1], where the fundamental idea was to impose an upper bound on the field strength in analogy with the consequences a finite speed of light has on the action of a free particle in special relativity. This elegant idea did in fact lead to a spherically symmetric non-trivial electrostatic solution with finite self energy, but did not prove to be the grand new holistic field theory unifying the descriptions of matter and electromagnetic field Born envisioned it to be. The Born-Infeld (BI) model has many exceptional properties among other NE theories, discussed further in chapter 2, making it the most successful and well-known such theory. BI however still suffers from some theoretical issues including a discontinuity of the electric field $\mathbf{E}$ at the origin. This problem was soon pointed out and new NE theories were being developed on these purely theoretical grounds by B. Hoffmann [2] and N. Rosen [3] for example. The former theory by Hoffman is expanded upon further in chapter 3.

NE has been shelved and revived many times throughout its existence but has proven to warrant its presence in the theorist's toolbox by reappearing in a variety of contexts. For example, only a year after the Born-Infeld paper, following Dirac's description of the positron, Heisenberg and Euler found a way to add one loop vacuum polarization corrections to the Maxwell Lagrangian resulting in an effective nonlinear Lagrangian capable of describing the effects of photon-photon scattering in quantum electrodynamics (QED) [4, 5]. Other such effective Lagrangians have been used in condensed matter theory to phenomenologically describe nonlinear effects in Dirac materials for example [6]. In particular the BI Lagrangian has even appeared in string theory as the Dirac-Born-Infeld (DBI) D-brane action [7]. NE thus seems to have a ubiquitous presence in theoretical physics.

Further, NE coupled to gravity has been used to construct various black holes and worm holes [8]. As this is a generalization of the standard Einstein-Maxwell theory, it has also been used to search for generalized exact solutions to various charged black hole spacetimes. One example is the result presented by Ayon-Beato and García in [9], interpreting a so-called regular black hole solution (sometimes also referred to as a Bardeen black hole) as sourced be a NE model with magnetic charge in spherically symmetric static geometry that possesses no spacetime curvature singularities. Bardeen black holes are named after James M. Bardeen, who first presented such a regular solution with a horizon generated by a certain stress-energy tensor at the GR5 conference in 1968 in Tbilisi [10]. The NE reinterpretation of this result by Ayon-Beato and García came much later in 2000. This was achieved with a Lagrangian that doesn't give the Maxwell weak-field limit:

$$\mathcal{L}_{Bar}(F) = \frac{3}{2sg^2}\left(\frac{\sqrt{-2g^2F}}{1+\sqrt{-2g^2F}}\right)^{5/2}, \tag{1}$$

where $s = \frac{|g|}{2m}$, g a parameter associated with the magnetic charge and m a parameter associated with the black hole mass. This result reignited the foundational idea in NE of a gauge theory with an everywhere regular field. However it has since been proven that no theories admitting static spherically symmetric solutions with purely electric sources satisfying the Maxwell weak-field limit exist [11].

To gauge the physical properties of NE models coupled to gravity that distinguish them from the Maxwell case we may study for example their implications for: the existence of exact solutions in certain geometries (i.e. generalizations to some, for example, rotating or non-spherically-symmetric case), the black hole thermodynamics, or systems where extreme field strengths become relevant.

General properties of NE theories were thoroughly studied in the 1960's and 70's. Many of the findings of these studies were covered in lecture notes by Jerzy Plebański [12]. As a result of this endeavour a general formalism of NE exists through which the different models can be communicated. We therefore introduce the basics of this formalism in the first chapter. Then in the following two chapters we find and examine the Lagrangians of two particular famous models of NE in Minkowski spacetime to better understand how they behave and differ on this level, before going into more detail on a new Lagrangian, recently discovered in [13], called 'RegMax' (shortened from 'regularized Maxwell'). For each model we find the spherically symmetric electrostatic solution corresponding to a unit point charge and calculate its self-energy. In the last chapter we first compare analytic solutions to some elementary static configurations in the Maxwell, Born-Infeld and RegMax theories. We then move on to describe the black hole thermodynamics for the RegMax Lagrangian as this was studied as part of a recent review on this model [14].

Chapter 1

Maxwell's Theory and General Characteristics of NE

In order to introduce relevant physical ideas and notation used further in the thesis we begin by deriving Maxwell's equations in a sequence inspired by [15]. Here and we restrict ourselves to the flat Minkowski spacetime equipped with a metric with signature $(-,+,+,+)$ in order to focus solely on the electrodynamics and not worry about coupling NE with GR.

The action of the field with a massive charged particle moving through it can be split into three parts: $\mathcal{S} = \mathcal{S}_m + \mathcal{S}_{mf} + \mathcal{S}_f$ corresponding to the 'geometric' action of the particle in a inertial reference frame, the field-particle interaction calculated along its world line and the field's own action respectively.

The property that quantifies the magnitude and character of a particle's interaction with the electromagnetic field is the electric charge q; its value can be positive, or negative. The electromagnetic field is generally described by a four-vector quantity A_μ called the four-potential. The interaction action is thus dependent on A_μ scaled by the charge integrated along the particle's world line and the field action only on A_μ in some way (this will be discussed later). The first two action functions can be then written (assuming the summation notation) as[1]:

$$\mathcal{S}_m = -m_0 \int_{\tau_1}^{\tau_2} d\tau, \tag{1.1}$$

$$\mathcal{S}_{mf} = q \int_{\gamma(\tau_1,\tau_2)} A_\mu \, dx^\mu = q \int_{\tau_1}^{\tau_2} A_\mu \frac{dx^\mu}{d\tau} \, d\tau, \tag{1.2}$$

[1]natural units, where the speed of light c, the vacuum electric permittivity ε_0, and $\hbar$ are all set to one will be assumed everywhere unless explicitly stated otherwise

where m_0 is the particle's rest mass, τ_1, τ_2 the proper time of spacetime events representing the initial and final positions of the particle, $d\tau = \sqrt{-dx_\mu dx^\mu}$, and $\frac{dx^\mu}{d\tau} = u^\mu$ is the four-velocity.

To find the equations of motion of this particle by the principle of stationary action we can assume the field is given and vary the particle trajectory. The first two expressions 1.1 & 1.2 are therefore sufficient:

$$\begin{aligned}
0 = \delta\mathcal{S} = \delta \int_{\tau_1}^{\tau_2} & (-m_0\, d\tau + qA_\mu u^\mu\, d\tau) + \delta\mathcal{S}_f \\
= \int_{\tau_1}^{\tau_2} & (-m_0 \delta\, d\tau + qA_\mu \delta(u^\mu\, d\tau) + q\delta A_\mu u^\mu\, d\tau) \\
= \int_{\tau_1}^{\tau_2} & (p_\mu\, d\delta x^\mu + qA_\mu\, d\delta x^\mu + q\delta A_\mu u^\mu\, d\tau) \\
= \int_{\tau_1}^{\tau_2} & \left(p_\mu \frac{d\delta\, x^\mu}{d\tau} + qA_\mu \frac{d\delta x^\mu}{d\tau} + q\delta A_\mu u^\mu \right) d\tau \\
= \int_{\tau_1}^{\tau_2} & \left(-\frac{dp_\mu}{d\tau}\delta x^\mu - q\frac{dA_\mu}{d\tau}\delta x^\mu + q\delta A_\mu u^\mu \right) d\tau + \\
& + \left[(p_\mu + qA_\mu)\, \delta x^\mu \right]_{\tau_1}^{\tau_2} \\
= \int_{\tau_1}^{\tau_2} & \left(-\frac{dp_\mu}{d\tau} + q\left(-\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu} \right) u^\nu \right) \delta x^\mu\, d\tau \\
\implies & \frac{dp_\mu}{d\tau} = q\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u^\nu,
\end{aligned} \tag{1.3}$$

we define: $$F_{\mu\nu} := \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right), \tag{1.4}$$

where τ_1, τ_2 are the proper times of the spacetime events between which the particle trajectory lies and $p^\mu = m_0 u^\mu$ is the four-momentum. $F_{\mu\nu}$ is the electromagnetic field strength tensor, an antisymmetric twice covariant tensor, whose entries in three dimensional notation correspond to the electric and magnetic field vectors, **E** & **B**, in the following way:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{1.5}$$

This expression can be used to check that the right side of the three spacial equations in 1.3 are indeed the Lorentz force.

From the definition of $F_{\mu\nu}$ in 1.4 we immediately get:

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0, \tag{1.6}$$

These equations are equivalent with the homogeneous Maxwell equations as can be seen better from the equivalent form:

$$\epsilon^{\mu\nu\lambda\kappa}\frac{\partial F_{\lambda\kappa}}{\partial x^{\nu}} = 0, \tag{1.7}$$

where $\epsilon^{\mu\nu\lambda\kappa}$ is the four dimensional *Levi-Civita* tensor; in this form it is obvious that these are only three independent equations. In space-vector notation these become:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.9}$$

Since $F_{\mu\nu}$ is indeed a tensor , it can be used to create Lorentz-invariant (this easily generalizes to general relativistic invariance when concerned with GR) scalar quantities. It turns out (see [15]) there are only two independent ways of doing this; one of them generates a true scalar, the other a pseudoscalar:

$$\text{the scalar invariant: } F := -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \tag{1.10}$$

$$\text{the pseudoscalar invariant: } G := \frac{1}{8}\epsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}F_{\lambda\kappa} = \mathbf{E}\cdot\mathbf{B}, \tag{1.11}$$

invariant scalar expressions can thus be created from F and G^2.

As stated above, $\mathcal{S}_f$ can only depend on the field, which is characterized by A_μ. Since the fields $\mathbf{E}$ and $\mathbf{B}$ (i.e. the entries of $F_{\mu\nu}$) represent measurable physical quantities and A_μ is not unique (the four-potential A_μ has gauge freedom, see again [15] for details), A_μ cannot explicitly appear in the field equations, nor under the integral in the action. On an experimental basis, we observe that the fields obey superposition, which indicates the field equations are linear differential equations. Thus the expression under the integral in the action should be quadratic in the fields. This leaves only a single option for $\mathcal{S}_f$ up to multiplication by a scalar, which, by convention, we set to $-\frac{1}{4}$:

$$\mathcal{S}_f = \int_{\tau_1}^{\tau_2}\int \mathcal{L}_M \, dV_0 \, d\tau = -\frac{1}{4}\int_{\tau_1}^{\tau_2}\int F_{\mu\nu}F^{\mu\nu} \, dV_0 \, d\tau, \tag{1.12}$$

where we introduce the Maxwell field Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$.

Before deriving the second pair of Maxwell equations the current four-vector. A element of charge may be written as $dq = \rho \, dV$, where $\rho(x^\mu)$ is the charge density. Multiplying this expression by dx^μ we get:

$$dq \, dx^\mu = \rho \, dV \, dx^\mu = \rho \, dV \, dt \, \frac{dx^\mu}{dt}. \tag{1.13}$$

Since the left side is a four-vector, the right side must also be; and $dV\,dt = (dV_0\frac{1}{\gamma})(\gamma d\tau)$ is a scalar. This implies that the four-current defined as:

$$J^\mu := \rho\frac{dx^\mu}{dt} = \rho_0\frac{dx^\mu}{d\tau} \tag{1.14}$$

must be a four-vector.

In order to obtain the remaining field equations by the principle of stationary action we assume the motion of the charge and vary the four-potential A_μ (treating it as the generalized coordinates):

$$\begin{aligned}
0 = \delta\mathcal{S} = \delta\mathcal{S}_m + \delta\int_{\gamma(\tau_1,\tau_2)} qA_\mu dx^\mu - \frac{1}{4}\delta\int_\Omega F_{\mu\nu}F^{\mu\nu}\,d\Omega \\
= \delta\int_{\tau_1}^{\tau_2} qA_\mu u^\mu d\tau - \frac{1}{4}\delta\int_\Omega F^{\mu\nu}F_{\mu\nu}\,d\Omega \\
= \delta\int_\Omega\left(J^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right)d\Omega \\
= \int_\Omega\left(J^\mu\delta A_\mu - \frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu}\right)d\Omega \\
= \int_\Omega\left(J^\mu\delta A_\mu - \frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial x^\mu}\delta A_\nu + \frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu\right)d\Omega \\
= \int_\Omega\left(J^\mu\delta A_\mu + F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu\right)d\Omega \\
= \int_\Omega\left(J^\mu\delta A_\mu - \frac{\partial F^{\mu\nu}}{\partial x^\nu}\right)\delta A_\mu\,d\Omega + \int_{\partial\Omega}F^{\mu\nu}\delta A_\mu\,d\Sigma_\nu \\
= \int_\Omega\left(J^\mu - \frac{\partial F^{\mu\nu}}{\partial x^\nu}\right)\delta A_\mu\,d\Omega, \qquad (1.15) \\
\implies \frac{\partial F^{\mu\nu}}{\partial x^\nu} = J^\mu, \qquad (1.16)
\end{aligned}$$

where Ω is the considered spacetime region between τ_1 and τ_2, $d\Omega = dt\,dV = d\tau\,dV_0$, in the first row $\delta\mathcal{S}_m = 0$, then the definitions of J^μ, $F^{\mu\nu}$, its antisymmetry and integration by parts were all used successively. Lastly, the integral over $\partial\Omega$ vanishes since we assume A_μ fixed on the boundary ($\delta A_\mu = 0$ on $\partial\Omega$). 1.16 are the remaining pair of field equations. These can be rewritten in space-vector notation to the familiar form:

$$\nabla\cdot\mathbf{E} = \rho, \tag{1.17}$$

$$\nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial\mathbf{E}}{\partial t}, \tag{1.18}$$

where $\mathbf{j}$ is the space-vector of the spacial components of J^μ, ie. the standard current density vector.

1.16 may also be used to quickly derive the continuity equation — taking the derivative of both sides with respect to x^μ and using the antisymmetry of $F^{\mu\nu}$ (the operator $\frac{\partial^2}{\partial x^\mu \partial x^\nu}$ is symmetric):

$$\frac{\partial^2 F^{\mu\nu}}{\partial x^\mu \partial x^\nu} = 0 = \frac{\partial J^\mu}{\partial x^\mu}, \tag{1.19}$$

the second equality *is* the continuity equation; the statement of conservation of charge, which, by Noether's theorem, is a result of the $U(1)$ gauge symmetry of A_μ.

The $\mathbf{E}$ field solution for a static unit point charge is the Coulomb field $\mathbf{E} = \frac{e}{4\pi r^2}\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit radial vector. The self-energy of this charge can be calculated as:

$$U = \frac{1}{2}\int |\mathbf{E}|^2 \, dV = \frac{e^2}{8\pi}\int_0^\infty \frac{1}{r^2}\, dr \to \infty. \tag{1.20}$$

The elimination of this divergence and identification of a corresponding finite value with the electronic mass was one of the primary motivations for the development of nonlinear electrodynamics.

General Characteristics of NE Theories

There are similarities between NE theories that are worth covering before studying the specifics of some NE models. Here we examine some features of source-free, relativistic, gauge invariant and non-linear theories.[16, 17]

In general, NE is concerned with altering the Lagrangian density function $\mathcal{L}_{NE}$ in the expression for the field action $\mathcal{S}_f = \int \mathcal{L}_{NE}\, d\Omega$ to a invariant function of the electromagnetic field invariants F and G^2 thus taking the form $\mathcal{L}_{NE} = \mathcal{L}(F, G^2)$. Lagrangians that in the weak-field limit approach the Maxwell Lagrangian $\mathcal{L}_M$ are said to obey the *principle of correspondence.* Although it is often assumed an NE theory obeys this principle, general NE Lagrangians do not, and many such Lagrangians have also been studied [9]; the general theory is independent of this assumption. Since any NE Lagrangian leads to a different set of field equations by the action principle, it would be useful to find a unifying description for the form of these NE field equations.

The electromagnetic field is described by the four-potential A_μ which, in order to preserve gauge invariance, appears in the Lagrangian and any other derived measurable physical quantity, only as elements of the field strength tensor $F_{\mu\nu}$ defined in 1.4. NE makes an analogy with the classical theory describing electrodynamics of continuous media, where electric displacement field $\mathbf{D}_{\text{media}}$ and magnetic field $\mathbf{H}_{\text{media}}$ are introduced as depending on the fields $\mathbf{E}$ and $\mathbf{B}$ as well as properties of the material media through constitutive relations. NE theory introduces a new field described by the four-*anti-potential* $\mathfrak{P}_\mu$, which in order to preserve gauge invariance, appears in $\mathcal{L}_{NE}$ only as a new field strength tensor defined by the antilinear combination $p_{\mu\nu} =: \partial_\mu \mathfrak{P}_\nu - \partial_\nu \mathfrak{P}_\mu$ that obeys equations of the same form as the source-free Maxwell equations . This new tensor field $p_{\mu\nu}$ is bound to the components of $f_{\mu\nu}$ (we reserve $F_{\mu\nu}$ for the Maxwell field and use $f_{\mu\nu}$ for the resulting NE $\mathbf{E}$ and $\mathbf{B}$ field) by the NE *constitutive relations* in Lagrangian formalism:

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B}), \tag{1.21}$$

$$\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B}), \tag{1.22}$$

where $\mathbf{D}$ and $\mathbf{H}$ are space-vectors from the components of the new $p_{\mu\nu}$ tensor (see 1.32). These constitutive relations may be derived from the form of $\mathcal{L}_{NE}$ as will be shown (see 1.32). The first pair of equations that follow immediately from the definition of $f_{\mu\nu}$ (see 1.6, 1.7 and 1.8) remain in completely unaltered form. The second pair of equations (source-free) becomes:

$$\nabla \cdot \mathbf{D} = 0, \tag{1.23}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{1.24}$$

These equations can alternatively be supplemented with constitutive relations in Hamiltonian formalism:

$$\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \tag{1.25}$$

$$\mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B}), \tag{1.26}$$

where $\mathbf{D}$ and $\mathbf{B}$ are chosen to be the independent variables. This leads to a consistent Hamiltonian formulation of NE theory with field equations that resemble *Hamilton's* canonical equations of motion in classical mechanics [16]:

$$\begin{array}{ccc} \dot{q} = \frac{\partial H}{\partial p} & & \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} \\ & vs. & \\ \dot{p} = -\frac{\partial H}{\partial q} & & \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \end{array} \tag{1.27}$$

Next we prove the above assertion on the form of second pair of NE equations in Lagrangian formalism by the variational principle applied to the field action. The Lagrangian density in a general NE theory (obeying the conditions listed above) is any function of the invariants F and G^2 defined in 1.10 and 1.11: $\mathcal{L} = \mathcal{L}(F, G^2)$; to simplify the calculation we work here with the Lagrangian in the form $\mathcal{L} = \mathcal{L}(F, G)$ but to keep the theory covariant, only Lagrangians with G^2 should be considered. The second pair of field equations in the absence of charges, as assumed, can then be found by the following calculation:

$$\begin{aligned} 0 = \delta\mathcal{S}_f &= \delta \int_\Omega \mathcal{L}(F, G)\, d\Omega \\ &= \int_\Omega \frac{\partial \mathcal{L}}{\partial F}\delta F + \frac{\partial \mathcal{L}}{\partial G}\delta G\, d\Omega = \star, \end{aligned}$$

we calculate δF and δG separately:

$$\begin{aligned} \delta F =& -\frac{1}{4}\delta\left(F^{\mu\nu}F_{\mu\nu}\right) = -\frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{2}F^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\delta A_\nu - \frac{\partial}{\partial x^\nu}\delta A_\mu\right) = \\ =& -\frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial x^\mu}\delta A_\nu + \frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu = \\ =& -\frac{1}{2}F^{\nu\mu}\frac{\partial}{\partial x^\nu}\delta A_\mu + \frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu = F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu, \end{aligned} \tag{1.28}$$

$$\begin{aligned} \delta G =& \frac{\partial G}{\partial F_{\mu\nu}}\delta F_{\mu\nu} + \frac{\partial G}{\partial F_{\lambda\kappa}}\delta F_{\lambda\kappa} = \\ =& \frac{1}{8}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}\left(-2\frac{\partial}{\partial x^\mu}\delta A_\nu\right) + \frac{1}{8}\epsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}\left(-2\frac{\partial}{\partial x^\lambda}\delta A_\kappa\right) = \\ =& -\frac{1}{4}\left(\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}\frac{\partial}{\partial x^\mu}\delta A_\nu + \epsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}\frac{\partial}{\partial x^\lambda}\delta A_\kappa\right) = \\ =& -\frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}\frac{\partial}{\partial x^\mu}\delta A_\nu = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}\frac{\partial}{\partial x^\nu}\delta A_\mu =: {}^\star F^{\mu\nu}\frac{\partial}{\partial x^\nu}\delta A_\mu, \end{aligned} \tag{1.29}$$

where we denoted the *Hodge dual* of $F^{\mu\nu}$ as ${}^{\star}F^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$. Then:

$$\begin{aligned}
\star &= \int_{\Omega} \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} \right) \frac{\partial}{\partial x^{\nu}} \delta A_{\mu} \, d\Omega \\
&= - \int_{\Omega} \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} \right) \delta A_{\mu} \, d\Omega + \\
&\quad + \int_{\partial\Omega} \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} \right) \delta A_{\mu} \, d\Sigma_{\nu} \\
&= - \int_{\Omega} \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} \right) \delta A_{\mu} \, d\Omega, \qquad (1.30) \\
&\implies \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} \right) = 0. \qquad (1.31)
\end{aligned}$$

With analogy to Maxwell's theory (source-free) we may now define the new antisymmetric tensor:

$$p^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial G} {}^{\star}F^{\mu\nu} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}. \qquad (1.32)$$

The components of this tensor define vectors $\mathbf{D}$ and $\mathbf{H}$ in NE. The remaining field equations may now be written in the form

$$\frac{\partial p^{\mu\nu}}{\partial x^{\nu}} = 0, \qquad (1.33)$$

which in space-vector notation becomes equations 1.23 & 1.24. With a source the calculation is again analogous to 1.16, where J^{μ} now has the characteristic of a free four-current. Then:

$$\nabla \cdot \mathbf{D} = \rho, \qquad (1.34)$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}. \qquad (1.35)$$

Although the canonical formulation of NE will not be explored further, it is useful to calculate the Hamiltonian density $\mathcal{H}$ proportional to the field energy density u and the 00-th component of the energy-momentum tensor $\mathbf{T}$. We can then calculate the self energy of point charges in various NE models and check if it turns out finite. Born and Infeld in [1] even attempted to explain the electronic mass with their theory by use of the energy-mass relation; setting the self energy $U = m_e c^2$ and obtaining a value for the BI parameter β (this is discussed further in Chapter 2). The Hamiltonian density is obtained by a Legendre transform of the Lagrangian:

$$\mathcal{H} = \mathbf{T}^{00} = 4\pi u = \mathbf{D} \cdot \mathbf{E} - \mathcal{L}. \tag{1.36}$$

For more details on the general characteristics of NE theories, the algebraic and general physical properties or the canonical formulation see [16, 12].

The Field of a Static Point Charge in NE

If we have a Lagrangian given explicitly in terms of F and G, finding the electric field of a static point charge the absence of all magnetic fields ($\mathbf{B} = \mathbf{H} = \mathbf{0}$) and any external electric fields reduces to finding the solutions of the field equations for a spherically symmetric electrostatic field; these have the form:

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{1.37}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.38}$$

Which in spherical coordinates reduces further to:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 D_r\right) = 0. \tag{1.39}$$

With

$$D_r = \frac{\partial \mathcal{L}}{\partial F} E_r \tag{1.40}$$

from 1.32, since $\mathcal{L}$ is now independent of G. The first equation 1.37 gives $E_r = -\phi'(r)$ for some potential function $\phi(r)$. F now also (by definition) becomes $F = \frac{1}{2}E_r^2$. The solution to the spherically symmetric electrostatic field equations is then:

$$D_r = \frac{e}{4\pi r^2}, \tag{1.41}$$

where e is an integration constant with dimensions of charge that we may set equal to the elementary charge and the 4π is added to associate the result with Coulomb's law in the used natural unit system ($\varepsilon_0 = \mu_0 = c = \hbar = 1$). Thus if the NE has a weak-field limit corresponding to the Maxwell Lagrangian, then $E_r^{NE} \xrightarrow{r\to\infty} E_r^{Max} = \frac{e}{4\pi r^2}$. The electric field $\mathbf{E}$ of an electrostatic elementary charge in this situation is now described by valid solutions $E_r(r)$ to the equation

$$\frac{e}{4\pi r^2} = \frac{\partial \mathcal{L}}{\partial F} E_r. \tag{1.42}$$

Chapter 2

Born-Infeld Theory

The foundational argument for motivating their model in [1, 18], M. Born and L. Infeld (this model will further be referred to as BI) claim the 'philosophical superiority' of what they call the *unitarian standpoint* concerning *the relation of matter to the electromagnetic field* in opposition to the *dualistic standpoint* representing the direction fundamental physics was headed in at the time.

The unitarian standpoint asserts the existence of a single physical entity: the electromagnetic field, from which the mass of particles may be derived from their field energy. Infinite self-energy of a point charge and the general characteristics of Maxwell's theory made it exceedingly difficult to explain the existence of the electron; a problem known as the search for a classical model for the electron. Earlier attempts to find unitarian models failed, either by introducing new forces of non-electromagnetic origin as in the case of Heaviside, Searle, Thomson and others, or by abandoning gauge invariance as in the case of G. Mie [19]. For more information on classical electron models see for example [20].

The BI theory is thus an attempt to construct a theory of electrodynamics based on the unitarian standpoint that, unlike previous attempts, obeys the principle of general covariance and gauge invariance by setting a maximum field strength.

Born's Idea

In order to obtain a Lagrangian that sets a finite upper bound on the field strength, M. Born makes an analogy with how the postulate of finite maximal velocity changes the Newtonian action function of a free particle $\frac{1}{2}mv^2$ to the special relativistic expression $m_0c^2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$, where c is the speed of light in vacuum. This leads him to alter the Maxwellian electromagnetic field action $\mathbf{L}_M$ defined in 1.12 to the following form:

$$\mathcal{L}_B := \beta^2\left(1-\sqrt{1-\frac{(\mathbf{E}^2-\mathbf{B}^2)}{\beta^2}}\right) \overset{1.10}{=} \beta^2\left(1-\sqrt{1-2\frac{F}{\beta^2}}\right), \tag{2.1}$$

where β is a constant, referred to as the *absolute field*, with units of electromagnetic field intensity, which, in the electrostatic case represents the maximal field strength, and in general 'the quotient of the field strength expressed in the conventional units divided by the field strength in the natural' [1].

Since the derivation of 2.1 is far from rigorous, Born, now joined by L. Infeld, derived a Lagrangian based on the same idea by demanding its general covariance. In [1] they *wrongly* claim that the discovered Lagrangian is singled out by its covariance through the method they employ to derive it. The fact is, that taking the Lagrangian to be any invariant function of F and G^2 leads to an invariant expression; this certainly nullifies their claim of uniqueness. This method by which the BI Lagrangian was obtained is however quite elegant and will be briefly restated in the following section.

BI Lagrangian Derivation

We begin by assuming the existence of any covariant tensor field $a_{\mu\nu}$. It is easy to prove (see 5) that the spacetime integral of expression $\sqrt{|a_{\mu\nu}|}$ is a general-relativistic scalar invariant; and so the idea is to use it for the field Lagrangian as the integrand in the action $\mathcal{S}_f$. If we were to allow the field to be determined by multiple tensor fields, an invariant scalar may be created by a sum of such terms. We wish to describe both the metrical and electromagnetic fields by the arbitrary $a_{\mu\nu}$. Any tensor may be decomposed into its symmetric and antisymmetric parts; we plan to identify the symmetric part of $a_{\mu\nu}$, defining $g_{\mu\nu}$, with the metric tensor, which implies the antisymmetric part must also be covariant; we thus identify *it*, $f_{\mu\nu}$, with the electromagnetic field. A Lagrangian leading to an invariant action may thus be assumed to take the following form:

$$\mathcal{L} \propto \sqrt{-|a_{\mu\nu}|} + A\sqrt{-|g_{\mu\nu}|} + B\sqrt{|f_{\mu\nu}|} \tag{2.2}$$

(the negative signs are there because the determinant of the metric is negative). Since $f_{\mu\nu}$ is assumed to be of the form 1.4, its spacetime integral vanishes ; and therefore we can set $B = 0$. In order to determine A, we explicitly calculate the value $|a_{\mu\nu}|$ in Cartesian coordinates. We would like the new Lagrangian density to correspond to the Maxwell Lagrangian $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ in the weak field limit (when we can neglect higher powers (>2) of $f_{\mu\nu}$). By direct calculation we get:

$$
\begin{aligned}
-|a_{\mu\nu}| = -|\eta_{\mu\nu} + f_{\mu\nu}| &= - \begin{vmatrix} -1 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 1 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 1 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 1 \end{vmatrix} \\
&\overset{5.37}{=} 1 - f_{12}^2 - f_{13}^2 - f_{14}^2 + f_{23}^2 + f_{24}^2 + f_{34}^2 - |f_{\mu\nu}|, \qquad (2.3)
\end{aligned}
$$

where, in the weak field limit, we neglect the term $|f_{\mu\nu}|$ and any higher order terms that come up in the Taylor expansion of the square-root. $\sqrt{|a_{\mu\nu}|}$ then becomes $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ when we set $A = -1$. Thus after multiplying the right hand side of 2.2 by -1 to get the Maxwell limit we obtain:

$$
\mathcal{L}_{BI} = \sqrt{-|g_{\mu\nu}|} - \sqrt{-|g_{\mu\nu} + f_{\mu\nu}|}, \qquad (2.4)
$$

which in flat space Cartesian coordinates becomes:

$$
\mathcal{L}_{BI} = 1 - \sqrt{1 + \mathtt{F} - \mathtt{G}^2}, \qquad (2.5)
$$

where

$$
\mathtt{F} = \frac{1}{2} f_{\mu\nu} f^{\mu\nu} = -f_{12}^2 - f_{13}^2 - f_{14}^2 + f_{23}^2 + f_{24}^2 + f_{34}^2 \propto -2F, \qquad (2.6)
$$

$$
\mathtt{G} = \sqrt{|f_{\mu\nu}|} \overset{5.38}{=} f_{12}f_{34} + f_{14}f_{23} - f_{24}f_{13} \propto G. \qquad (2.7)
$$

We now introduce the constant β: 'the quotient of the field strength expressed in the conventional units divided by the field strength in the natural units'[1] and write:

$$
\mathtt{F} = -\frac{2}{\beta^2} F = \frac{1}{\beta^2}\left(\mathbf{B}^2 - \mathbf{E}^2\right), \qquad (2.8)
$$

$$
\mathtt{G} = \frac{G}{\beta^2} = \frac{1}{\beta^2}\mathbf{E}\cdot\mathbf{B}, \qquad (2.9)
$$

$$
L_{BI} = \beta^2 \left(1 - \sqrt{1 + \frac{1}{\beta^2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) - \frac{1}{\beta^4}\left(\mathbf{E}\cdot\mathbf{B}\right)^2}\right), \qquad (2.10)
$$

$$
= \beta^2 \left(1 - \sqrt{1 - 2\frac{F}{\beta^2} - \frac{G^2}{\beta^4}}\right). \qquad (2.11)
$$

With this in mind we may now calculate $p^{\mu\nu}$ in flat spacetime BI theory in Cartesian coordinates in terms of $f_{\mu\nu}$ using 1.32 and 2.8:

$$p^{\mu\nu} = \frac{\partial L_{BI}}{\partial F} F^{\mu\nu} + \frac{\partial L_{BI}}{\partial G} {}^\star F^{\mu\nu} = -2\frac{\partial \mathcal{L}_{BI}}{\partial \mathtt{F}} f^{\mu\nu} + \frac{\partial \mathcal{L}_{BI}}{\partial \mathtt{G}} {}^\star f^{\mu\nu}, \tag{2.12}$$

$$\mathbf{D}_{BI} = \frac{\mathbf{E} + \mathtt{G}\mathbf{B}}{\sqrt{1 + \mathtt{F} - \mathtt{G}^2}}, \tag{2.13}$$

$$\mathbf{H}_{BI} = \frac{\mathbf{B} - \mathtt{G}\mathbf{E}}{\sqrt{1 + \mathtt{F} - \mathtt{G}^2}}. \tag{2.14}$$

The Field of a Static Point Charge in BI Theory

With the theory we have developed so far we can solve the Born-Infeld field equations for the simple case of a static, spherically symmetric field with $\mathbf{B} = \mathbf{H} = \mathbf{0}$. From the discussion in the previous chapter, the electric field $\mathbf{E}$ can be obtained as the solution to:

$$\frac{e}{4\pi r^2} = \frac{\partial \mathcal{L}}{\partial F} E_r, \tag{2.15}$$

where $F = \frac{1}{2} E_r^2$ and

$$\frac{\partial \mathcal{L}}{\partial F} = \frac{\partial L_{BI}}{\partial F} = -2\frac{\partial \mathcal{L}_{BI}}{\partial \mathtt{F}}. \tag{2.16}$$

Then from 2.13 and 2.10:

$$D_r = \frac{E_r}{\sqrt{1 - \left(\frac{E_r}{\beta}\right)^2}} = -\frac{\phi'(r)}{\sqrt{1 - \left(\frac{\phi'(r)}{\beta}\right)^2}} = \frac{e}{4\pi r^2}. \tag{2.17}$$

We introduce a new dimensionless quantity representing a characteristic 'radius of an electron':

$$r_0 := \sqrt{\frac{e}{4\pi\beta}}. \tag{2.18}$$

Then from 2.17:

$$\frac{e^2}{(4\pi)^2 r^4}\left(1 - \frac{\phi'(r)^2}{\beta^2}\right) = \phi'(r)^2 \implies \phi'(r)^2\left(1 + \frac{e^2}{(4\pi)^2 r^4 \beta^2}\right) = \frac{e^2}{(4\pi)^2 r^4},$$

$$\implies E_r = -\phi'(r) = \frac{e\beta}{\sqrt{e^2 + (4\pi)^2 r^4 \beta^2}} = \frac{\beta}{\sqrt{1 + \left(\frac{r}{r_0}\right)^4}}. \tag{2.19}$$

Since the field is pointing radially outwards from the origin with non-zero magnitude everywhere (see figure 2.1), we get a discontinuous change in its

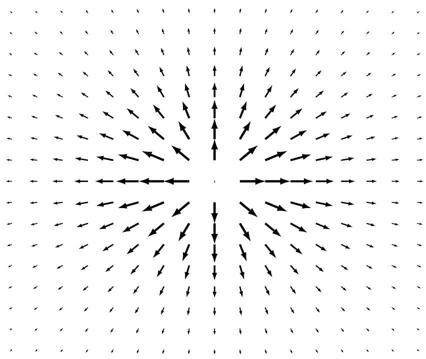

Figure 2.1 $\mathbf{E}$ field of a static point charge in BI theory.

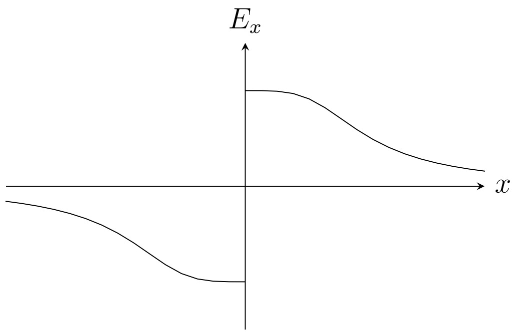

Figure 2.2 Radial component of the $\mathbf{E}$ field of a static point charge in BI theory on an axis going through its origin.

direction when moving along an axis going through the origin between two points very close either side of it. This can be seen explicitly by converting to Cartesian coordinates and in figure 2.2; the x-component of the field becomes:

$$E_x = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{\beta}{\sqrt{1+\frac{(x^2+y^2+z^2)^2}{r_0^4}}}.$$

The energy density of this field is then by 1.36:

$$\begin{aligned} 4\pi u =& \mathbf{D}_{BI} \cdot \mathbf{E} - L_{BI} = \frac{\mathbf{E}^2}{\sqrt{1-\frac{\mathbf{E}^2}{\beta^2}}} - \beta^2 \left(1 - \sqrt{1-\frac{\mathbf{E}^2}{\beta^2}}\right) \\ =& \beta^2 \left(\frac{1}{\sqrt{1-\frac{E_r^2}{\beta^2}}} - 1\right) \overset{2.19}{=} \beta^2 \left(\sqrt{\left(\frac{r_0}{r}\right)^4 + 1} - 1\right). \end{aligned} \tag{2.20}$$

When integrated over space we get the energy of a static charge in BI theory:

$$\begin{aligned} U =& \frac{1}{4\pi} \int \beta^2 \left(\sqrt{\left(\frac{r_0}{r}\right)^4 + 1} - 1\right) = \frac{3\beta^2 r_0^3 \Gamma\left(-\frac{3}{4}\right)^2}{32\sqrt{\pi}} \\ \approx& 1.2361\beta^2 r_0^3 = 1.2361 \frac{e^2}{(4\pi)^2} \frac{1}{r_0}, \end{aligned} \tag{2.21}$$

where Γ is the Gamma function.

Chapter 3

Hoffmann-Born-Infeld

Soon after the publication of the BI paper [1], objections were made concerning critical arguments in the paper [21, 3, 2]. A big point of critique was the erroneous assertion that the BI Lagrangian is uniquely determined by its general covariance, but this was not the only problem: there is also the discontinuity of the field at the origin in Cartesian coordinates discussed at the end of the last chapter and spacetime turns out singular at the centre of a point charge when general relativity is considered among other problems mentioned in [2].

One of the proposals to fix these issues came from Hoffmann and Infeld in [2]. The main idea was to make the E field vanish at the centre of a point-particle, thus eliminating the discontinuity at the origin in Cartesian coordinates, by introducing a logarithmic term to the Lagrangian. This Lagrangian was derived by utilizing the variational principle applied to a 'new action function' of the form $\mathcal{T} = \mathcal{L} + \mathcal{H}$; this approach is based on a paper by Infeld [22]. A brief overview precedes the derivation of the HBI Lagrangian in the next section.

HBI Lagrangian Derivation

Infeld's Method

In this section we use:

$$\mathtt{F} := \frac{1}{2} f_{\mu\nu} f^{\mu\nu}, \;\; \mathtt{P} := \frac{1}{2} {}^{\star}p_{\mu\nu} {}^{\star}p^{\mu\nu}, \;\; \mathtt{R} := \frac{1}{2} f_{\mu\nu} p^{\mu\nu} = -\frac{1}{2} {}^{\star}f_{\mu\nu} {}^{\star}p^{\mu\nu}$$

Before deriving this Lagrangian with B. Hoffmann, L. Infeld came up with a new formalism for deriving generally invariant action functions [22]. This formalism is based in part on observations in Born's original paper [18] involving the action principle in the form $\delta \int \mathcal{H}_B \, d\tau = 0$, where $\mathcal{H}(\mathtt{P}) = \sqrt{1+\mathtt{P}} - 1$, yields an equivalent set of field equations as the traditional form $\delta \int \mathcal{L}_B \, d\tau = 0$, where $\mathcal{L}_B(\mathtt{F}) = \sqrt{1+\mathtt{F}} - 1$; the Lagrangian and Hamiltonian being connected by a Legendre transformation.

In this section we restrict ourselves to an action function dependent solely on the electric field, since we are concerned with the field of an electrostatic point charge and $\mathbf{B} = \mathbf{0}$. Infeld introduced a new form of action function $\mathcal{T} = \mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R})$ by the property:

$$p^{\mu\nu} = \frac{\partial \mathcal{T}}{\partial f_{\mu\nu}} \quad \Longleftrightarrow \quad {}^{\star}f^{\mu\nu} = \frac{\partial \mathcal{T}}{\partial^{\star} p_{\mu\nu}}. \tag{3.1}$$

He then showed in [22] that $\mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R})$ can always be represented as the sum $\mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R}) = \mathcal{L}(\mathtt{F}) + \mathcal{H}(\mathtt{P})$, and in fact also that this representation is equivalent to the equivalence of equations 3.1. The Lagrangian and Hamiltonian may then be recovered from $\mathcal{T}$ by the equations:

$$\mathcal{L}(\mathtt{F}) = \frac{1}{2} \left(\mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R}) + \mathtt{R} \right), \tag{3.2}$$

$$\mathcal{H}(\mathtt{P}) = \frac{1}{2} \left(\mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R}) - \mathtt{R} \right), \tag{3.3}$$

and the relations:

$$\mathtt{R} = \mathcal{T}_{\mathtt{F}} \mathtt{F} - \mathcal{T}_{\mathtt{P}} \mathtt{P}, \tag{3.4}$$

$$\mathtt{R}^2 = - \mathtt{F}\mathtt{P}, \tag{3.5}$$

$$0 = \mathcal{T}_{\mathtt{F}} \mathtt{F} + \mathcal{T}_{\mathtt{P}} \mathtt{P} + \mathcal{T}_{\mathtt{R}} \mathtt{R}, \tag{3.6}$$

which are derived from the properties of $\mathcal{T}$ (here the notation $\mathcal{T}_{\mathtt{F}}$ represents the partial derivative of $\mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R})$ with respect to $\mathtt{F}$). These relations allow us to determine $\mathtt{F} = \mathtt{F}(\mathtt{P})$ and $\mathtt{R} = \mathtt{R}(\mathtt{P})$, or $\mathtt{P} = \mathtt{P}(\mathtt{F})$ and $\mathtt{R} = \mathtt{R}(\mathtt{F})$. It also follows that:

$$\frac{\partial \mathcal{L}(\mathtt{F})}{\partial f_{\mu\nu}} = \frac{\partial \mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R})}{\partial f_{\mu\nu}} = p^{\mu\nu}, \tag{3.7}$$

$$\frac{\partial \mathcal{H}(\mathtt{P})}{\partial^{\star} p_{\mu\nu}} = \frac{\partial \mathcal{T}(\mathtt{F}, \mathtt{P}, \mathtt{R})}{\partial^{\star} p_{\mu\nu}} = {}^{\star}f^{\mu\nu}, \tag{3.8}$$

The action principle now takes the form $\delta \int \mathcal{T} \, d\tau = 0$ and yields two forms of the first pair of field equations (Infeld calls these integrability conditions):

$$\frac{\partial^{\star} f^{\mu\nu}}{\partial x^{\nu}} = 0, \quad \frac{\partial p^{\mu\nu}}{\partial x^{\nu}} = 0, \tag{3.9}$$

and two versions of the second pair (the Euler equations):

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{T}}{\partial f_{\mu\nu}}\right) = 0, \quad \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{T}}{\partial^\star p_{\mu\nu}}\right) = 0, \tag{3.10}$$

these equations reduce to the field original equations obtained by varying the Lagrangian when the properties 3.1 of $\mathcal{T}$ hold.

HBI Lagrangian

We now turn our attention to the derivation of the action function proposed in the HBI paper [2]. We assume the action function in the form $\mathcal{T} = \mathcal{T}(\mathtt{F}, \mathtt{P})$. From the definitions of the tensors $f^{\mu\nu}$ and $p^{\mu\nu}$ as the curls of the potential vector A_μ and anti-potential vector $\mathfrak{P}_\mu$ (the analogous quantity to A_μ for $p_{\mu\nu}$, ie.: $p_{\mu\nu} =: \partial_\mu \mathfrak{P}_\nu - \partial_\nu \mathfrak{P}_\mu$) respectively, the first pair of field equations:

$$\frac{\partial^\star f^{\mu\nu}}{\partial x^\nu} = 0, \tag{3.11}$$

$$\frac{\partial p^{\mu\nu}}{\partial x^\nu} = 0, \tag{3.12}$$

are satisfied automatically. We also expect $f^{\mu\nu}$ and $p^{\mu\nu}$ to be *conjugate with respect to the action function* as in the section on general NE. This translates to the equations:

$$p^{\mu\nu} = \frac{\partial \mathcal{L}_{HBI}(\mathtt{F})}{\partial f_{\mu\nu}} = \frac{\partial \mathcal{T}(\mathtt{F}, \mathtt{P})}{\partial f_{\mu\nu}} = 2\mathcal{T}_{\mathtt{F}} f^{\mu\nu}, \tag{3.13}$$

$$^\star f^{\mu\nu} = \frac{\partial \mathcal{L}_{HBI}(\mathtt{F})}{\partial^\star p_{\mu\nu}} = \frac{\partial \mathcal{T}(\mathtt{F}, \mathtt{P})}{\partial^\star p_{\mu\nu}} = 2\mathcal{T}_{\mathtt{P}} {}^\star p^{\mu\nu}, \tag{3.14}$$

where the second equalities follow from 3.7 and 3.8. These equations are 3.1. Substituting 3.13 (effectively the constitutive relations) into 3.12 we get field equations in the form:

$$\frac{\partial^\star f^{\mu\nu}}{\partial x^\nu} = 0, \tag{3.15}$$

$$\frac{\partial f^{\mu\nu}}{\partial x^\nu} = \rho^\mu, \tag{3.16}$$

$$\text{where } \rho^\mu := -\frac{\partial \log(2\mathcal{T}_F)}{\partial x^\nu} f^{\mu\nu}. \tag{3.17}$$

Multiplying the first equation by $f_{\mu\nu}$ and the second one by ${}^\star p_{\mu\nu}$ we get:

$$\begin{aligned} \mathtt{R} =& 2\mathcal{T}_{\mathtt{F}}\mathtt{F}, & (3.18)\\ -\mathtt{R} =& 2\mathcal{T}_{\mathtt{P}}\mathtt{P}, & (3.19)\\ \implies& -R^2 = 4\mathcal{T}_{\mathtt{F}}\mathcal{T}_{\mathtt{P}}\mathtt{FP}, & (3.20)\\ \text{by 3.5: } 1 =& 4\mathcal{T}_{\mathtt{F}}\mathcal{T}_{\mathtt{P}}. & (3.21)\end{aligned}$$

Multiplying 3.13 and 3.14 directly we get:

$$0 = \mathcal{T}_{\mathtt{F}}\mathtt{F} + \mathcal{T}_{\mathtt{P}}\mathtt{P}, \tag{3.22}$$

which is 3.6 and coincidentally an expression of *Euler's homogeneous function theorem*, from which it follows that $0 = \sum_i^n x_i \frac{\partial f}{\partial x_i}$ for any function f that is homogeneous of degree zero in its arguments (that is: $f(\alpha \mathbf{x}) = \alpha^0 f(\mathbf{x}) = f(\mathbf{x}) := f(x_1, \ldots, x_n)$). This means we can write $\mathcal{T}$ satisfying this last condition in terms of a new function that is homogeneous degree zero in $\mathtt{F}$ and $\mathtt{P}$:

$$\mathcal{T}(\mathtt{F},\mathtt{P}) = \mathcal{T}(\epsilon(\mathtt{F},\mathtt{P})), \text{ where } \epsilon(\mathtt{F},\mathtt{P}) := \sqrt{-\frac{\mathtt{F}}{\mathtt{P}}}. \tag{3.23}$$

From 3.21 we get:

$$\begin{aligned} 2\mathcal{T}_{\mathtt{F}}\epsilon =& 1, & (3.24)\\ 2\mathcal{T}_{\mathtt{P}}\epsilon^{-1} =& 1, & (3.25)\\ &\text{and} & \\ \mathcal{T}_\epsilon = -\mathtt{P},& \ \ \epsilon^2\mathcal{T}_\epsilon = \mathtt{F}. & (3.26)\end{aligned}$$

We demand the field equations be *well-defined* at every point in spacetime by imposing a regularity condition on the solutions $f_{\mu\nu}$ of equations 3.15 and 3.16 that they belong to the class of at least once differentiable functions everywhere. This also implies for spherically symmetric electrostatic solutions that E_r must be zero at the origin in order to prevent a discontinuity in Cartesian coordinates of E. This assumption is not fulfilled in Born's or BI theory. The following arguments then specify the unique form of action function up to the first order in ϵ:

The regularity condition demands the electric field intensity to vanish at the origin, thus having a first order Taylor expansion of r^n for some $n > 0$:

$$E_r \xrightarrow{r\to 0} 0 \sim r^n, \ \ n > 0. \tag{3.27}$$

We know the solution of D_r in NE is 1.41 $\sim r^{-2}$, thus:

$$\epsilon \equiv \frac{E_r}{D_r} \xrightarrow{r\to 0} 0 \sim r^{2+n}, \ \ n > 0. \tag{3.28}$$

To get the Maxwell limit for large r we need:

$$D_r \xrightarrow{r\to\infty} E_r \xrightarrow{r\to\infty} 0 \sim r^{-2}, \tag{3.29}$$
$$\epsilon \xrightarrow{r\to\infty} 1, \ \ \mathcal{T}_\epsilon \xrightarrow{r\to\infty} 0, \ \ \mathcal{T} \xrightarrow{r\to\infty} 0. \tag{3.30}$$

These conditions give:

$$\mathtt{P} \xrightarrow{r\to 0} \infty \sim r^{-4}, \ \ \frac{1}{\epsilon} \xrightarrow{r\to 0} \infty \sim r^{-(2+n)}, \ \ n > 0. \tag{3.31}$$

Then 'if $\mathcal{T}_\epsilon$ can be expanded in a power series in ϵ, the lowest term *must* contain $\frac{1}{\epsilon}$.' So we assume:

$$-\mathtt{P} = \mathcal{T}_\epsilon = -\frac{1}{\epsilon} + \alpha + ..., \ \ \alpha = const. \tag{3.32}$$

Which after integration gives:

$$\mathcal{T} = k + \alpha\epsilon - \log\epsilon + ..., \ \ k = const. \tag{3.33}$$

Taking the limits

$$\mathcal{T} \xrightarrow{r\to\infty} 0 = k + \alpha.1 - 0 + ...,$$
$$\mathcal{T}_\epsilon \xrightarrow{r\to 0} 0 = -\frac{1}{1} + \alpha + ... \implies \alpha = 1, \tag{3.34}$$
$$\implies k = -1, \tag{3.35}$$
$$\implies \mathcal{T}(\epsilon) = \epsilon - \log\epsilon - 1. \tag{3.36}$$

It now follows that:

$$\mathcal{T}_\epsilon = 1 - \frac{1}{\epsilon} = -\mathtt{P}, \tag{3.37}$$
$$\implies 1 + \mathtt{P} = \sqrt{-\frac{\mathtt{P}}{\mathtt{F}}} = \frac{1}{\epsilon}. \tag{3.38}$$

The last equation can be solved for P in terms of F:

$$\mathtt{P} = -1 - \frac{1 \pm \sqrt{1 + 4\mathtt{F}}}{2\mathtt{F}} \tag{3.39}$$
$$\implies \frac{1}{\epsilon(\mathtt{F}, \mathtt{P}(\mathtt{F}))} = \frac{-1 \pm \sqrt{1 + 4\mathtt{F}}}{2\mathtt{F}}, \tag{3.40}$$
$$\implies \epsilon(\mathtt{F}, \mathtt{P}(\mathtt{F})) = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\mathtt{F}}\right). \tag{3.41}$$

We can now explicitly express $p_{\mu\nu}$ in terms of $f_{\mu\nu}$ by calculating $\mathcal{T}_{\mathtt{F}}(\mathtt{F},\mathtt{P})$ and substituting for P:

$$\begin{aligned}
\mathcal{T}_{\mathtt{F}}(\mathtt{F},\mathtt{P}) =& \mathcal{T}_{\epsilon}(\epsilon(\mathtt{F},\mathtt{P}))\epsilon_{\mathtt{F}}(\mathtt{F},\mathtt{P}) = \left(1-\frac{1}{\epsilon(\mathtt{F},\mathtt{P})}\right)\epsilon_{\mathtt{F}}(\mathtt{F},\mathtt{P}) \\
&\overset{3.37}{=} (-\mathtt{P})\frac{1}{2\epsilon(\mathtt{F},\mathtt{P})}\frac{-1}{\mathtt{P}} = \frac{1}{2\epsilon(\mathtt{F},\mathtt{P})} \qquad (3.42) \\
&\Longrightarrow 2\mathcal{T}_{\mathtt{F}}(\mathtt{F},\mathtt{P}(\mathtt{F})) = \frac{-1\pm\sqrt{1+4\mathtt{F}}}{2\mathtt{F}}. \qquad (3.43)
\end{aligned}$$

We can also find the explicit form of the Lagrangian $\mathcal{L}_{HBI}(\mathtt{F})$ using 3.2 and 3.18:

$$\begin{aligned}
\mathcal{L}_{HBI}(\mathtt{F}) =& \frac{1}{2}\left(\mathcal{T}(\mathtt{F},\mathtt{P}(\mathtt{F})) + 2\mathcal{T}_{\mathtt{F}}(\mathtt{F},\mathtt{P}(\mathtt{F}))\mathtt{F}\right) \\
=& \frac{1}{2}\left(\epsilon(\mathtt{F},\mathtt{P}(\mathtt{F})) - \log\epsilon(\mathtt{F},\mathtt{P}(\mathtt{F})) - 1 + 2\frac{1}{2\epsilon(\mathtt{F},\mathtt{P}(\mathtt{F}))}\mathtt{F}\right) \\
=& \frac{1}{2}\left(\frac{1}{2}\left(1\pm\sqrt{1+4\mathtt{F}}\right) - \log\left(\frac{1\pm\sqrt{1+4\mathtt{F}}}{2}\right) - 1 + \frac{-1\pm\sqrt{1+4\mathtt{F}}}{2\mathtt{F}}\mathtt{F}\right) \\
=& -\frac{1}{2}\left(1\mp\sqrt{1+4\mathtt{F}} + \log\left(\frac{1\pm\sqrt{1+4\mathtt{F}}}{2}\right)\right), \qquad (3.44)
\end{aligned}$$

where the upper sign is defined (for a point charge) in the region $r^4 > q^2b^2$ and the lower sign where $r^4 < q^2b^2$; we get equality of the two expressions at $r^4 = q^2b^2$. It is possible to only chose the upper (or lower) sign Lagrangian in all space if for the inside/outside we adopt a Bertotti-Robinson (BR) type spacetime (uniform magnetic field) and outside/inside a Reissner-Nordtröm (RN) type spacetime (static charged black hole), where in the region of (BR) type we set E_r and D_r constant, obeying continuity to avoid double-valuedness of $\mathbf{D}(\mathbf{E})$: from 3.47 we see that $\mathbf{E}$ vanishes both at zero and at infinity [8]. Choosing the upper sign, using $\mathtt{F} = -2\frac{F}{\beta^2}$ and setting a constant factor to get the Maxwell limit, we get:

$$L_{HBI}(F) = \frac{\beta^2}{2}\left(1-\sqrt{1-8\frac{F}{\beta^2}} + \log\left(\frac{1+\sqrt{1-8\frac{F}{\beta^2}}}{2}\right)\right). \qquad (3.45)$$

The Field of a Static Charge in HBI Theory

From 3.45 and 1.42 we can get the electric field analogously to the case with L_{BI}:

$$D_r = \frac{e}{4\pi r^2} = -2\frac{\partial \mathcal{L}_{HBI}(\mathtt{F})}{\partial \mathtt{F}} E_r = \frac{\partial L_{HBI}(F)}{\partial F} E_r =$$
$$= \beta^2 \frac{1-\sqrt{1-4\frac{E_r^2}{\beta^2}}}{2E_r^2} E_r, \tag{3.46}$$

where we used $\mathtt{F} = -\frac{E_r^2}{\beta^2}$ and $F = \frac{E_r^2}{2\beta^2}$. Then using the same definition for the 'radius of the electron' as in BI theory, $r_0 = \sqrt{\frac{e}{4\pi\beta}}$, we get:

$$\left(2E_r \frac{e}{4\pi r^2 \beta^2} - 1\right)^2 = 1 - 4\frac{E_r^2}{\beta^2} \implies E_r\left(\frac{e^2}{(4\pi)^2 r^4 \beta^2}\right) = \frac{e}{4\pi r^2},$$
$$\implies E_r = -\phi'(r) = \frac{4\pi e r^2 \beta^2}{e^2 + (4\pi)^2 r^4 \beta^2} = \beta \frac{\left(\frac{r}{r_0}\right)^2}{1+\left(\frac{r}{r_0}\right)^4}. \tag{3.47}$$

As we can see explicitly from figures 3.1 and 3.2, this model effectively eliminates the discontinuity in $\mathbf{E}$ at the origin.

The energy density of the HBI point-charge field is given by 1.36:

$$\begin{aligned} 4\pi u =& \mathbf{D}_{HBI} \cdot \mathbf{E} - L_{HBI} \\ =& \frac{\beta^2}{2}\left(1-\sqrt{1-4\frac{\mathbf{E}^2}{\beta^2}}\right) - \frac{\beta^2}{2}\left(1-\sqrt{1-4\frac{\mathbf{E}^2}{\beta^2}} + \log\left(\frac{1+\sqrt{1-4\frac{\mathbf{E}^2}{\beta^2}}}{2}\right)\right) \\ =& -\frac{\beta^2}{2}\log\left(\frac{1+\sqrt{1-4\frac{\mathbf{E}^2}{\beta^2}}}{2}\right) = -\frac{\beta^2}{2}\log\left(\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\left(\frac{r}{r_0}\right)^4}{\left(1+\left(\frac{r}{r_0}\right)^4\right)^2}}\right) \\ =& -\frac{\beta^2}{2}\log\left(\frac{1}{2}\left(1+\frac{r^4-r_0^4}{r^4+r_0^4}\right)\right) \end{aligned} \tag{3.48}$$

Integrating over all space we get for a static point-charge in HBI theory the energy:

$$U = \int u\, dV = \frac{1}{9}\left(3\sqrt{2}\pi - 4\right)\beta^2 r_0^3 \approx 1.03652 \frac{e^2}{(4\pi)^2}\frac{1}{r_0}. \tag{3.49}$$

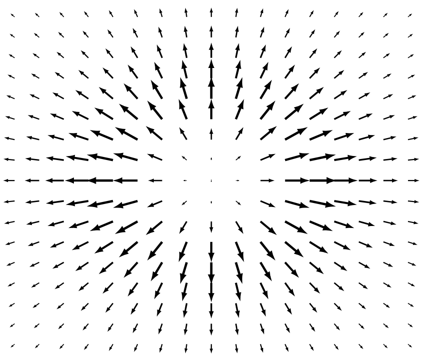

Figure 3.1 $\mathbf{E}$ field of a static point charge in HBI theory.

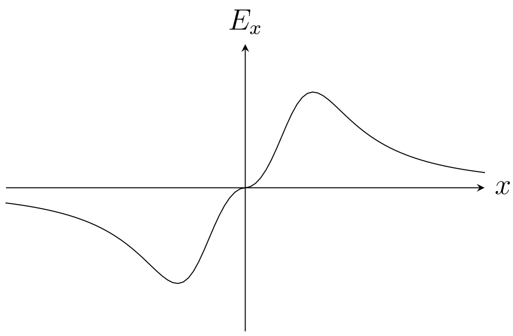

Figure 3.2 Radial component of the $\mathbf{E}$ field of a static point charge in HBI theory on an axis going through its origin.

Chapter 4

RegMax

In this chapter we provide a brief overview of NE coupled to gravity and then explore a particular recently discovered model that arises in the context of exact radiative solutions of black hole spacetimes and slowly rotating black holes [13, 23].

The coupling of classical electrodynamics to gravity is done simply by altering the electromagnetic Lagrangian $\mathcal{L}_{EM}$ in the vacuum Einstein-Hilbert action written for a generic NE theory (in geometric units with $G = 1$):

$$\mathcal{S} = \frac{1}{16\pi} \int \left(R - 2\Lambda + 4\mathcal{L}_{EM}\right) \sqrt{-g}\, d^4x, \tag{4.1}$$

where R is the Ricci scalar, Λ is the cosmological constant and g the metric determinant. This action leads via the variation of the metric to the NE-Einstein field equations with the NE energy momentum tensor on the right hand side:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{4.2}$$

We can find the explicit form of $T_{\mu\nu}$ this way and check the correspondence of its 00-component with the Hamiltonian density in the first chapter:

$$\begin{aligned} \delta S =& \frac{1}{16\pi} \int \sqrt{-g}\delta\left(R - 2\Lambda + 4\mathcal{L}_{EM}\right) + \left(R - 2\Lambda + 4\mathcal{L}_{EM}\right) \delta\sqrt{-g}\, d^4x, \\ =& \frac{1}{16\pi} \int \left(\left(\frac{\delta R}{\delta g^{\mu\nu}} + 4\frac{\delta \mathcal{L}_{EM}}{\delta g^{\mu\nu}} \right) + \right. \\ & \left. \left(\frac{1}{\sqrt{-g}} \left(R - 2\Lambda + 4\mathcal{L}_{EM}\right) \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) \right) \sqrt{-g}\delta g^{\mu\nu}\, d^4x = 0. \end{aligned} \tag{4.3}$$

Terms with R and Λ become after some standard calculations to be found for example in [24], the left hand side of the Einstein field equations. To get the energy-momentum tensor we first calculate the following:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right), \tag{4.4}$$

where the last step involves Jacobi's formula for the derivative of matrix determinants from linear algebra to get $\delta g = \delta \det g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}$, where in the last equality we use the formula for the derivative of an inverse matrix. And also:

$$\begin{aligned}\frac{\delta F^{\lambda\kappa}}{\delta g^{\mu\nu}} =& \frac{\delta\left(g^{\lambda\alpha}g^{\kappa\beta}F_{\alpha\beta}\right)}{\delta g^{\mu\nu}} = F_{\alpha\beta}\left(g^{\lambda\alpha}\delta^{\kappa}_{\mu}\delta^{\beta}_{\nu} + g^{\kappa\beta}\delta^{\lambda}_{\mu}\delta^{\alpha}_{\nu}\right)\\ =& F_{\alpha\nu}g^{\lambda\alpha}\delta^{\kappa}_{\mu} + F_{\nu\beta}g^{\kappa\beta}\delta^{\lambda}_{\mu}.\end{aligned} \tag{4.5}$$

Then:

$$\begin{aligned}\frac{\delta\mathcal{L}(F,G)}{\delta g^{\mu\nu}} =& \frac{\partial\mathcal{L}}{\partial F}\frac{\delta F}{\delta g^{\mu\nu}} + \frac{\partial\mathcal{L}}{\partial G}\frac{\delta G}{\delta g^{\mu\nu}}\\ =& -\frac{1}{4}\frac{\partial\mathcal{L}}{\partial F}\left(2F_{\lambda\kappa}\frac{\delta F^{\lambda\kappa}}{\delta g^{\mu\nu}}\right) + \frac{\partial\mathcal{L}}{\partial G}\left(\frac{\varepsilon_{\lambda\kappa\sigma\rho}}{8\sqrt{-g}}\left(F^{\lambda\kappa}\frac{\delta F^{\sigma\rho}}{\delta g^{\mu\nu}} + F^{\sigma\rho}\frac{\delta F^{\lambda\kappa}}{\delta g^{\mu\nu}}\right)\right.\\ &\left. + \frac{\varepsilon_{\lambda\kappa\sigma\rho}F^{\lambda\kappa}F^{\sigma\rho}}{8}\frac{\delta\frac{1}{\sqrt{-g}}}{\delta g^{\mu\nu}}\right)\\ \overset{4.5,4.4}{=}& -\frac{1}{4}\frac{\partial\mathcal{L}}{\partial F}\left(2F_{\lambda\kappa}\left(F_{\alpha\nu}g^{\lambda\alpha}\delta^{\kappa}_{\mu} + F_{\nu\beta}g^{\kappa\beta}\delta^{\lambda}_{\mu}\right)\right) +\\ & + \frac{\partial\mathcal{L}}{\partial G}\left(\frac{\varepsilon_{\lambda\kappa\sigma\rho}}{8\sqrt{-g}}\left(F^{\lambda\kappa}\left(F_{\alpha\nu}g^{\sigma\alpha}\delta^{\rho}_{\mu} + F_{\nu\beta}g^{\rho\beta}\delta^{\sigma}_{\mu}\right) +\right.\right.\\ & \left.\left. + F^{\sigma\rho}\left(F_{\alpha\nu}g^{\lambda\alpha}\delta^{\kappa}_{\mu} + F_{\nu\beta}g^{\kappa\beta}\delta^{\lambda}_{\mu}\right)\right) + \frac{\varepsilon_{\lambda\kappa\sigma\rho}F^{\lambda\kappa}F^{\sigma\rho}}{8}\frac{1}{2}\frac{1}{\sqrt{-g}}g_{\mu\nu}\right)\\ =& -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial F}\left(F_{\lambda\mu}F^{\lambda}{}_{\nu} + F_{\mu\kappa}F_{\nu}{}^{\kappa}\right) + \frac{\partial\mathcal{L}}{\partial G}\left(\frac{1}{8\sqrt{-g}}\left(\varepsilon_{\lambda\kappa\sigma\mu}F^{\lambda\kappa}F^{\sigma}{}_{\nu} +\right.\right.\\ & \left. + \varepsilon_{\lambda\kappa\mu\rho}F^{\lambda\kappa}F_{\nu}{}^{\rho} + \varepsilon_{\lambda\mu\sigma\rho}F^{\sigma\rho}F^{\lambda}{}_{\nu} + \varepsilon_{\mu\kappa\sigma\rho}F^{\sigma\rho}F_{\nu}{}^{\kappa}\right) +\\ & \left. + \frac{\varepsilon_{\lambda\kappa\sigma\rho}F^{\lambda\kappa}F^{\sigma\rho}}{8}\frac{1}{2}\frac{1}{\sqrt{-g}}g_{\mu\nu}\right)\\ =& \frac{\partial\mathcal{L}}{\partial F}F_{\mu\lambda}F^{\lambda}{}_{\nu} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial G}Gg_{\mu\nu},\end{aligned} \tag{4.6}$$

where the last equality holds due to the anti symmetry of $F_{\mu\nu}$. Thus from

4.3, 4 and 4.6 we get the NE energy-momentum tensor:

$$
\begin{aligned}
8\pi T_{\mu\nu} =& 2\left(2\frac{\delta\mathcal{L}_{EM}}{\delta g^{\mu\nu}} - \mathcal{L}_{EM} g_{\mu\nu}\right), \\
T_{\mu\nu} =& \frac{1}{4\pi}\left(2\frac{\partial\mathcal{L}}{\partial F} F_{\mu\lambda}F^{\lambda}{}_{\nu} + \frac{\partial\mathcal{L}}{\partial G} G g_{\mu\nu} - \mathcal{L}_{EM} g_{\mu\nu}\right).
\end{aligned}
\tag{4.7}
$$

We may now check this is in accordance with 1.36.

By varying the electromagnetic field we get the same NE field equations with $\mathcal{L}_{EM} = \mathcal{L}(F, G^2)$ as in the first chapter. The standard classical coupling is the Einstein-Maxwell theory, where $\mathcal{L}_{EM} = F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. This theory has a divergence in both the electromagnetic field and spacetime geometry at the centre of a charged black hole.

NE can be used to smooth out at least one of these divergences. The BI model for example removes only the field divergence as discussed in the second chapter. The Bardeen model [9], mentioned in the introduction, removes spacetime divergence but not the field divergence and it doesn't obey the principle of correspondence (i.e. the Maxwell weak field limit). Some models of NE coupled to gravity can thus aid with theoretical problems associated with spacetime divergences.

Since it is known the linear electromagnetic theory does not describe the nonlinear effects arising from QED in the presence of very strong fields and that effective classical NE theories describing these effects to some degree of accuracy do exist, NE may play a role in describing processes in these extreme circumstances as a phenomenological model. Such circumstances include neutron stars or the formation of black holes, where such strong fields may be common [25, 12, 5].

For a more detailed introduction to NE coupled with GR see [26, 12].

Recently, in [13], a new Lagrangian has been discovered that admits exact radiative solutions to the NE-Einstein field equations in the Robinson-Trautman class of exact black hole solutions. Later, the same Lagrangian came up when considering slowly rotating black holes [23]. This prompted a further study of this Lagrangian in [14]; some main results of which, mainly regarding electrodynamic properties and black hole thermodynamics, will be developed in the following sections.

RegMax Electrodynamics

In order to be consistent with a recent review [14], in this section we adopt new conventions for the units, where $\frac{1}{4\pi\varepsilon_0} = 1$, and field invariants:

$$\mathcal{S} := -\,2F = \frac{1}{2} f_{\mu\nu} f^{\mu\nu}, \tag{4.8}$$

$$\mathcal{P} := 2G = \frac{1}{2} f_{\mu\nu} \, {}^{\star} f^{\mu\nu}. \tag{4.9}$$

In the following we explore some basic electrodynamic properties of the so-called 'RegMax' Lagrangian:

$$\mathcal{L}_{\text{RegMax}}(\mathcal{S}) = -2\alpha^2 \left(1 - 3\log(1-s) + \frac{s^3 + 3s^2 - 4s - 2}{2(1-s)} \right), \tag{4.10}$$

$$\text{where } s := \left(\frac{-\mathcal{S}}{\alpha^4} \right)^{\frac{1}{4}}. \tag{4.11}$$

This Lagrangian obeys the principle of correspondence (has the Maxwell weak field limit) and belongs to the restricted class of Lagrangians dependent only on $\mathcal{S}$ (a generalization to $\mathcal{L}(\mathcal{S}, \mathcal{P}^2)$ has not been found yet). Thus in all the following we assume $\mathcal{P} = 0$. Note also this form of the Lagrangian is suited for the electric case where $\mathcal{S} < 0$ (although a magnetic extension has been constructed, see [14]) and so here we only deal with the case $\mathbf{B} = \mathbf{0}$.

A full rigorous derivation of it may be found in the paper it was discovered in [13]. An intuitive way of finding it can be achieved by assuming a simple form of regularization to the Maxwell field of a point charge. We may demand, as is true in the Born-Infeld theory, to have a field maximum at the origin independent of the charge given by a fixed parameter of the theory that we denote β:

$$|E_r| \xrightarrow{r \to 0} \beta. \tag{4.12}$$

There are many functions fulfilling such a requirement including the BI point charge, or

$$\frac{Q}{r^2 + \frac{|Q|}{\beta}} \tag{4.13}$$

for example. From a theoretical point of view it is however satisfying and intuitive for the inverse-square law to stay valid for all r. This requirement can be almost satisfied (depending on the interpretation) if we imagine a small bit of space being 'eaten up' at the origin in order to create the proportional charge source i.e. the inverse square law holds if to the distance from the origin we just add the inaccessible radius $r = \frac{|Q|}{\alpha}$. This reduces the form of the electric field further to:

$$E_r = \frac{Q}{\left(r + \frac{\sqrt{|Q|}}{\alpha}\right)^2}, \tag{4.14}$$

where $\alpha^2 = \beta$. We have effectively just 'cut out' and stitched together the boundary of a ball of radius $\frac{\sqrt{|Q|}}{\alpha}$ from the standard Maxwell point charge field.

The general NE equation 1.42 in the new convention has the form (here, $Q = \frac{e}{4\pi\varepsilon_0} = e$):

$$D_r = \frac{Q}{r^2} = 2\mathcal{L}_{\mathcal{S}} E_r, \tag{4.15}$$

where $\mathcal{L}_{\mathcal{S}} = \frac{\partial \mathcal{L}}{\partial \mathcal{S}}$. Plugging E_r into this using $\mathcal{S} = -E_r^2$ we get:

$$\mathcal{L}_{\mathcal{S}} = \frac{-1}{2\sqrt{-\mathcal{S}}} \frac{Q}{\left(\sqrt{\frac{|Q|}{\sqrt{-\mathcal{S}}}} - \frac{\sqrt{|Q|}}{\alpha}\right)^2} \tag{4.16}$$

$$= \frac{-1}{2\left(1 - \left(\frac{-\mathcal{S}}{\alpha^4}\right)^{\frac{1}{4}}\right)^2}. \tag{4.17}$$

Integrating this w.r.t. to $\mathcal{S}$ yields the RegMax Lagrangian 4.10.

RegMax properties

Here we present some basic known properties of the RegMax theory developed in [13, 23, 14] that categorize it among other NE theories, that make it unique and that motivate its further study.

RegMax is a recently proposed model of NE that follows from the most straightforward regularization to the Coulomb point charge electric field resulting in the complicated looking Lagrangian 4.10. It belongs to the restricted class of NE theories dependent only on the field invariant $\mathcal{S}$, i.e. $\mathcal{L} = \mathcal{L}(\mathcal{S})$, and obeys the principle of correspondence with the linear Maxwell theory in the weak field limit. It is characterized by a dimensionful parameter α dimensionally related by $\beta \propto \alpha^2$ to the BI parameter β with $[\beta] = (\text{length})^{-1}$. The theory is not conformal and does not possess electromagnetic duality like BI or ModMax [27].

RegMax implies vacuum birefringence (the two degrees of freedom (modes) propagated by NE electromagnetic waves do not propagate with the same speed), as does any generic NE theory apart from BI (as proved by Plebański [12]). A problem that may occur with some restricted NE models is that one of the modes may not be causal. It has been proven in [14] however that the propagation of RegMax modes is causal.

At $r = 0$ the gravitating spherically symmetric AdS RegMax solution possesses a singularity and, depending on the parameters in the resulting metric function f_0 5.16, can have zero, one or two horizons.

The theory provides radiative Robinson-Trautman spacetime solutions with 'Maxwell-like' properties [13], solutions to slowly rotating black holes [23], accelerated (*C-metric*) black holes and other important gravitating solutions [14]. The fact all these solutions can be found analytically in a form very close to Maxwell solutions makes the RegMax model unique.

The Field of a Static Charge for the RegMax Lagrangian

Reversing the heuristic derivation of Lagrangian 4.10 and taking the derivative of the Lagrangian with respect to $\mathcal{S}$ we get:

$$\mathcal{L}_{\mathcal{S}}(\mathcal{S}) = -\frac{1}{2\,(s-1)^2}. \tag{4.18}$$

Then the static, spherically symmetric NE equation for a charge Q takes the form:

$$D_r = \frac{Q}{r^2} = -2\mathcal{L}_{\mathcal{S}} E_r = \frac{1}{(s-1)^2} E_r, \tag{4.19}$$

which has a non-diverging solution for the electric field:

$$E_r = \frac{\alpha^2 Q}{\left(\alpha r + \sqrt{|Q|}\right)^2}. \tag{4.20}$$

This may be integrated to obtain the vector potential:

$$\mathbf{A} = \psi_0 dt, \quad \psi_0 = -\frac{\alpha Q}{\alpha r + \sqrt{|Q|}}. \tag{4.21}$$

We can also specify the electromagnetic invariant $\mathcal{S}$:

$$\mathcal{S} = -E_r^2 = -\frac{\alpha^4 Q^2}{\left(\alpha r + \sqrt{|Q|}\right)^4} \tag{4.22}$$

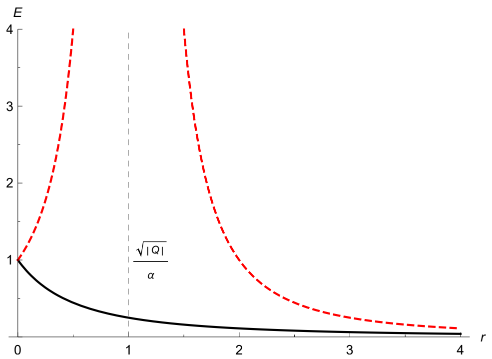

Figure 4.1 The diverging $\tilde{E}_r$ field 4.23 (red dashed) and solution 4.20 (solid black) with $Q = \alpha = 1$.

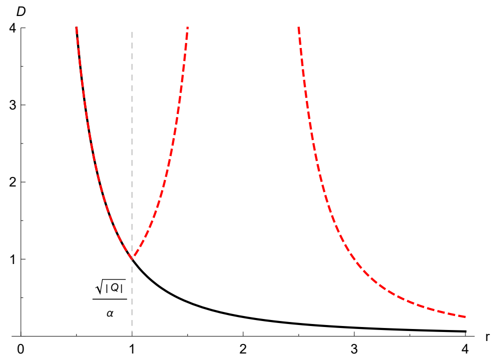

Figure 4.2 The $\tilde{D}_r$ field generated by substituting the diverging solution 4.23 into 4.19 (red dashed) and the unique NE field equations solution $D_r = \frac{Q}{r^2}$ (solid black) with $Q = \alpha = 1$.

There is also a diverging solution with a sign change in the expression in the denominator that solves the algebraic equation for $r < \frac{\sqrt{|Q|}}{\alpha}$:

$$\tilde{E}_r = \frac{\alpha^2 Q}{\left(\alpha r - \sqrt{|Q|}\right)^2}. \tag{4.23}$$

A comparison of E_r and the diverging $\tilde{E}_r$ is shown in figure 4.1, where. When we check if $\tilde{E}_r$ truly solves the NE equation by substituting into 4.19, we get that the left hand side $D_r = \frac{Q}{r^2}$ only equals the right hand side in the region $r < \frac{\sqrt{|Q|}}{\alpha}$. How this solution fails outside the region can be seen in figure 4.2: for $r < \frac{\sqrt{|Q|}}{\alpha}$ the dashed red line and solid black line overlap indicating $\tilde{D}_r = D_r$, but outside this interval the two lines separate, indicating $\tilde{E}_r$ is no longer even a solution outside the region. But even in the region $r < \frac{\sqrt{|Q|}}{\alpha}$ this solution does not correspond to the field generated by a charge Q, as can be checked by integrating the generated charge density over the valid region and finding the result diverges. This calculation is shown in the appendix 5.

We can also find the energy density of the static point-charge by 1.36; the expression simplifies a lot if we assume $Q > 0$. Then:

$$\begin{aligned}u =& D_r E_r - \mathcal{L}_{\text{RegMax}}(\mathcal{S}) = \frac{Q}{r^2}\frac{\alpha^2 Q}{\left(\alpha r + \sqrt{|Q|}\right)^2} - \mathcal{L}_{\text{RegMax}}(F) \\ =& \frac{\alpha^2\sqrt{|Q|}\left(-3\alpha\sqrt{|Q|}r + Q - 6\alpha^2 r^2\right)}{r^2\left(\sqrt{|Q|} + \alpha r\right)} - 6\alpha^4 \log\left(\frac{\alpha r}{\sqrt{|Q|} + \alpha r}\right).\end{aligned} \tag{4.24}$$

This can be integrated over all space to obtain the energy of a static point charge for this Lagrangian:

$$U = \frac{1}{4\pi}\int u\, dV = \frac{8}{3}\pi\alpha Q^{3/2}. \tag{4.25}$$

Chapter 5

Elementary NE Solutions and Black Hole Thermodynamics

Elementary NE Solutions

In this section we compare Maxwell, Born-Infeld and RegMax electrodynamics by observing the solutions of symmetric configurations: point charge, homogeneously charged sphere (electrostatic) and an infinite wire with current (magnetostatic). The calculations are inspired by the work in [28], where only the Born-Infeld case is considered.[29, 30]

The static, spherically symmetric field of a unit point charge in each of the models (using the solutions) is compared in figure 5.1.

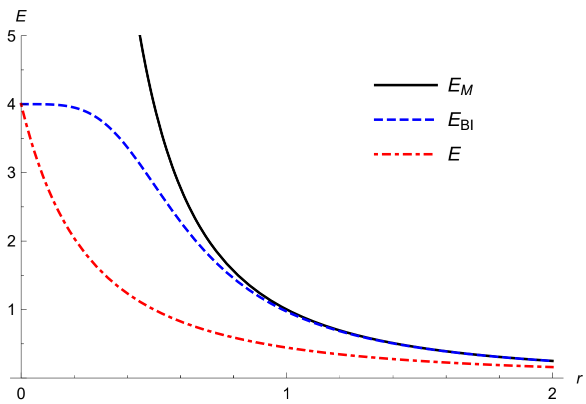

Figure 5.1 A comparison of the electrostatic fields generated by a point charge with $Q = 1$ for the Maxwell theory (solid black line), Born-Infeld with $\beta = 4$ (blue dashed) and RegMax with $\alpha = 2$ (red dot-dashed) [14].

If we assume a static ball of radius R with homogeneous charge density (homogeneous in the sources ρ of $\mathbf{D}$):

$$\begin{cases} \rho & r \leq R \\ 0 & r > R \end{cases}, \tag{5.1}$$

we can use Gauss's law (due to spherical symmetry) to reduce the NE equations in the same fashion as was done for a point charge in the first chapter. Thus:

$$D_r = \begin{cases} \frac{\rho}{3} r & \text{inside} \\ \frac{\rho R^3}{3r^2} & \text{outside} \end{cases}. \tag{5.2}$$

Plugging this into the static, spherically symmetric NE equation we get the radial electric fields (with the convention $\frac{1}{4\pi\varepsilon} = 1$):

$$E_{Max} = \begin{cases} \frac{Q}{R^3} r & \text{inside} \\ \frac{Q}{r^2} & \text{outside} \end{cases}, \tag{5.3}$$

$$E_{BI} = \begin{cases} \frac{\beta r Q}{\sqrt{R^6 \beta^2 + Q^2 r^2}} & \text{inside} \\ \frac{\beta Q}{\sqrt{Q^2 + \beta^2 r^4}} & \text{outside} \end{cases}, \tag{5.4}$$

$$E = \begin{cases} \frac{\alpha^2 Q r}{Qr + \alpha R(\alpha R^2 + 2\sqrt{|Q| r R})} & \text{inside} \\ \frac{\alpha^2 Q}{(\sqrt{|Q|} + \alpha r)^2} & \text{outside} \end{cases}. \tag{5.5}$$

These solutions are compared in figure 5.2

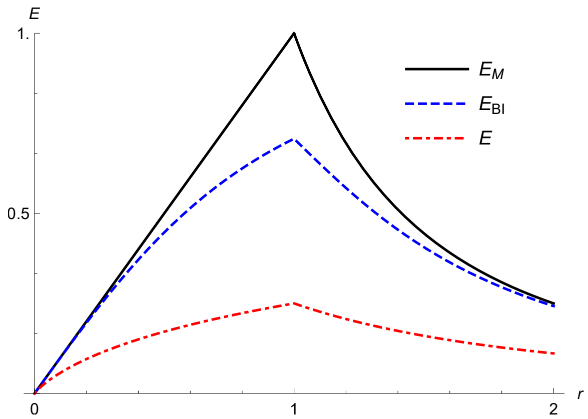

Figure 5.2 A comparison of the electrostatic fields generated by a homogeneously charged ball with $Q = 1$ for the Maxwell theory (solid black line), Born-Infeld with $\beta = 1$ (blue dashed) and RegMax with $\alpha = 1$ (red dot-dashed) [14].

Similarly, in the stationary (magnetostatic) case, we may calculate the magnetic field of generated by a infinite straight wire carrying NE field source 'current' by applying the analogue Ampere's law. The relevant NE field equation and constitutive relation in this case read:

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \tag{5.6}$$
$$\mathbf{H} = -2\frac{\partial \mathcal{L}}{\partial \mathcal{S}}\mathbf{B}, \tag{5.7}$$

where the 4π comes from the convention $\frac{1}{4\pi\varepsilon} = 1 \implies \mu = 4\pi$ by the relation $\mu\varepsilon = \frac{1}{c^2} = 1$. The infinite wire directed in the positive z direction with current I gives the only non-trivial component:

$$H_\varphi = 2\frac{I}{r}. \tag{5.8}$$

Solving the constitutive relation then gives the φ-component of the magnetic field $\mathbf{B}$:

$$B_{Max} = 2\frac{I}{r}, \tag{5.9}$$
$$B_{BI} = \frac{2\beta I}{\sqrt{\beta^2 r^2 + 4I^2}}, \tag{5.10}$$
$$B = \frac{2\alpha^2 I^2}{\alpha^2 I r + 2\sqrt{2}\alpha\sqrt{I^3 r} + 2I^2}. \tag{5.11}$$

These solutions are again compared in figure 5.3. Note there also exists a unphysical diverging solution to the RegMax constitutive relation in this case for $r < 2\frac{I}{\alpha^2}$.

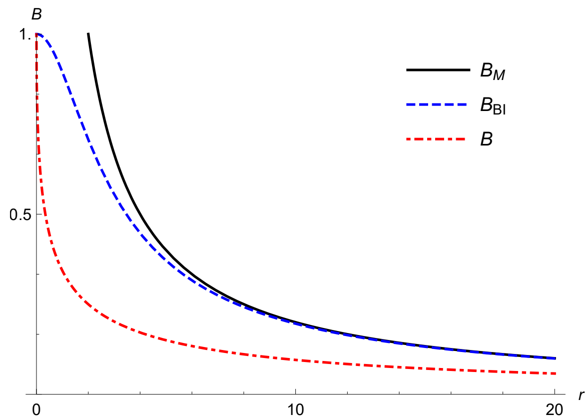

Figure 5.3 A comparison of the magnetic fields generated by a infinite wire with current $I = 1$ for the Maxwell theory (solid black line), Born-Infeld with $\beta = 1$ (blue dashed) and RegMax with $\alpha = 1$ (red dot-dashed).

Black Hole Thermodynamics

In this section we briefly introduce the study of black hole (BH) thermodynamics in Anti de-Sitter (AdS) spacetime coupled to NE (de Sitter space BH thermodynamics turns out to be a lot more complicated due to multiple horizons and is far from being well established). Specifically a sub-branch of the field concerned with involving the cosmological constant Λ into the thermodynamic framework as a pressure term, sometimes called 'black hole chemistry', see [31] for a brief introduction and [32] for a comprehensive review.

In [33], Hawking showed that the surface area of a black hole event horizon (under some 'reasonable assumptions') can never decrease; a statement that resembles the second law of thermodynamics. Bekenstein noticed this similarity and proposed in [34] that black holes can be assigned an entropy proportional to the surface area by $S \propto \frac{Ac^3}{\hbar G}$. Hawking then confirmed this and found the proportionality constant to be $\frac{1}{4}$ in [37]; so we get $S = \frac{Ac^3}{4\hbar G}$. A similar analogy was made between the black hole surface gravity $\frac{\kappa}{8\pi G}$ and temperature. By extending this analogy further, Bardeen, Carter and Hawking formulated *The four laws of black hole mechanics* in [35] that reflect the second, first, zeroth and third laws of thermodynamics. The laws follow (in the order from [33]):

`The Second Law`

The area A of the event horizon of each black hole does not decrease with time, i.e.

$$\delta A \geq 0. \tag{5.12}$$

If two black holes coalesce, the area of the final event horizon is greater than the sum of the initial horizons, i.e.

$$A_3 > A_1 + A_2. \tag{5.13}$$

`The First Law`

For a rotating charged black hole with mass M, angular momentum J and charge Q:

$$\begin{aligned} \delta M =& \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q, \\ =& T\delta S + \Omega\delta J + \Phi\delta Q \quad \text{with } G = c = k_B = 1, \end{aligned} \tag{5.14}$$

where κ is the surface gravity, Ω the angular velocity and Φ the electric potential. T and S are the thermodynamic temperature and entropy defined above.

The Zeroth Law

The surface gravity, κ of a stationary black hole is constant over the event horizon.

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

In the paper [35] they are quick to point out that, despite the analogy, $\frac{\kappa}{8\pi G}$ and A are distinct from the classical temperature and entropy of a black hole; the classical thermodynamic temperature of a black hole is in fact absolute zero, since there is no way a black hole could be in equilibrium with black body radiation at any non-zero temperature. The proposed non-zero thermodynamic temperature arises when the quantum effects that result in Hawking radiation are taken into account.

A question that arises when considering NE black hole spacetimes is consistency with this theory of black hole thermodynamics first developed by Bekenstein and Hawking [36, 37]. A nice property of all theories of NE coupled to gravity is that the validity of the zeroth and first laws of black hole thermodynamics are always given [38].

In the Maxwell case, due to homogeneity of the first law, one can get, by Euler's homogeneous function theorem, a formula for the mass – the so-called Smarr formula [39]. This formula is the BH thermodynamics version of the Gibbs-Duhem relation in classical thermodynamics (the mass is interpreted as the internal energy, or enthalpy if we allow Λ [40]) and greatly simplifies BH thermodynamic calculation. When it is found (which is done by varying all dimensionful coupling constants and applying dimensional analysis as shown in [32]) standard thermodynamic formalism may be applied to get parametric relationships between the thermodynamic potentials and derived thermodynamic variables (for example the canonical free energy F on the absolute thermodynamic temperature T).

Some NE models can then be shown to exhibit nontrivial thermodynamic and *phase transition* behaviour. A famous example of such black hole phase transitions is the so-called Hawking-Page phase transition (first order) between thermal AdS and a black hole, discovered in [41] and the small-black-hole/large-black-hole phase transition of charged black holes [41, 42, 43]. The mechanisms or interpretation of such phase transitions is not well understood; their study mostly relies on the general formalism of thermodynamics and the stability analysis it provides to determine when they occur. In particular, when incorporating the cosmological constant in the thermodynamics as a ‘pressure’ term (so-called black hole chemistry [31]), black hole phase transitions can be modelled by an analogy with Van der Waals fluid [44]. Explicit heat capacities or the speed of sound can be calculated; free energy plots and phase diagrams can be constructed etc.

RegMax Black Hole Thermodynamics

In order to study the black hole thermodynamics in gravity coupled to RegMax, we make use of the full self-gravitating solution generalized from the static, spherically symmetric field 4.20, presented in [14]. If we write the resulting line element in the form:

$$ds^2 = -f_0 dt^2 + \frac{dr^2}{f_0} + r^2 d\Omega^2, \tag{5.15}$$

then the solution is characterized by the metric function f_0 given by:

$$\begin{aligned} f_0 =& 1 - 2\alpha^2|Q| + \frac{4\alpha|Q|^{3/2} - 6m}{3r} + 4r\alpha^3\sqrt{|Q|} - \\ & - 4\alpha^4 r^2 \log\left(1 + \frac{\sqrt{|Q|}}{r\alpha}\right) + \frac{r^2}{\ell^2}, \end{aligned} \tag{5.16}$$

where α is the RegMax parameter, m is the invariant mass, Q the charge, ℓ the angular momentum and $d\Omega$ in the previous expression the solid angle element: $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. This metric function is plotted for various parameter values in 5.4.

The horizon of an AdS black hole is located at the largest root r_+ of this metric function $f_0(r_+) = 0$; it corresponds to the Killing horizon generated by the field $\xi = \partial_t$. We can now determine the thermodynamic variables of the

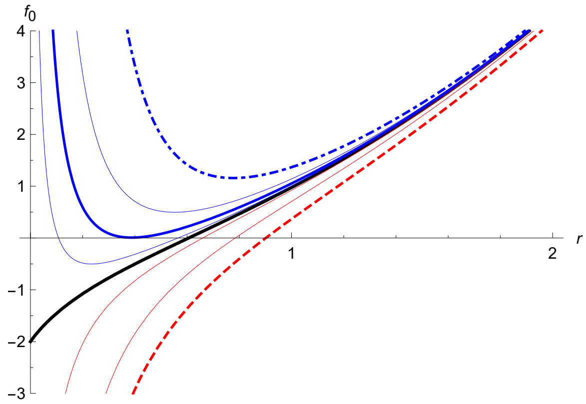

Figure 5.4 The metric function f_0 plotted in various cases (RegMax AdS black holes). Depending on the mass m, RegMax black holes are characterized by two regimes based on the behaviour near $r = 0$: S-type (red lines), and RN-type (blue lines). A BH horizon is characterized by the roots of f_0; thus we see characteristic Schwarzchild behaviour in the S-type, converging precisely to that case for $m \to \infty$ (red dashed). Characteristic Reissner-NordStröm behaviour with 2, 1 and no horizons is seen in the RN regime with the extremal $m \to 0$ case shown in blue, dot-dashed. There exists a marginal mass that splits the two regimes and for which $\lim_{r\to 0} f_0$ is finite (solid black line). This mass is given by $M_m = \frac{2\alpha}{3}|Q|^{3/2}$. The plot is for values $Q = \ell = \alpha = 1$ [14].

RegMax AdS black holes using standard formulae that can be found in [32]:

$$\begin{aligned} T =& \frac{f_0'(r_+)}{4\pi}, \qquad (5.17)\\ =& \frac{1}{4\pi r_+}\left(1 - 2\alpha^2|Q| - 12\alpha^4 r_+^2 \log\left(1 + \frac{\sqrt{|Q|}}{\alpha r_+}\right)\right.\\ &\left. + 4\alpha^3 r_+\sqrt{|Q|}\left(2 + \frac{\alpha r_+}{\sqrt{|Q|} + \alpha r_+}\right) + \frac{3r_+^2}{\ell^2}\right), \end{aligned}$$

$$S = \frac{A}{4} = \pi r_+^2, \qquad (5.18)$$

$$\phi = -\xi \cdot \mathbf{A}|_{r=r_+} = \frac{\alpha Q}{\alpha r_+ + \sqrt{|Q|}}, \qquad (5.19)$$

$$\begin{aligned} M =& m \quad (\text{from } f_0(r_+) = 0), \qquad (5.20)\\ =& \frac{1}{2} r_+\left(1 + \frac{r_+^2}{\ell^2} - 4\alpha^4 r_+^2 \log\left(1 + \frac{\sqrt{|Q|}}{\alpha r_+}\right)\right)\\ &+ \frac{2}{3}\alpha|Q|^{3/2} + 2\alpha^3 r_+^2\sqrt{|Q|} - \alpha^2 r_+|Q|, \end{aligned}$$

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}, \tag{5.21}$$

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha} = \frac{4}{3}\pi r_+^3, \tag{5.22}$$

$$\mu_\alpha = \left(\frac{\partial M}{\partial \alpha}\right)_{S,Q,P}, \tag{5.23}$$

where M was calculated by the conformal method [45] and μ_α is the 'α-polarization potential'. The first law thus holds in an extended form:

$$\delta M = T\delta S + \phi\delta Q + V\delta P + \mu_\alpha\delta\alpha. \tag{5.24}$$

We may also check the corresponding Smarr relation

$$M = 2TS + \phi Q - 2VP - \frac{1}{2}\mu_\alpha\alpha \tag{5.25}$$

holds.

The Canonical Ensemble

In the canonical ensemble we are concerned with a system in equilibrium with an infinite (thermal) reservoir characterized by a fixed temperature. In statistical mechanics, the information about the system is then contained by the canonical partition function $Z = Z(\beta, V, N_i)$, where $\beta = \frac{1}{k_B T}$. For example in the discrete case, the probability that the system at given T is in a state with energy E is given by $\rho = \frac{1}{Z}e^{-\beta E}$. In thermodynamics, this corresponds to the fundamental equation $F = F(T, V, N_i)$, where F is the Helmholtz free energy and, by standard calculation, we get $F = -k_B T \log Z$. The analogy in AdS black hole thermodynamics comes from assuming the black hole spacetime is in a state that fixes the value of the black hole thermodynamic temperature T. Thus we wish to formulate the fundamental relation in terms of T by performing the Legendre transformation of M that exchanges S for T to obtain the canonical free energy F:

$$F = M - TS = F(T, Q, P, \alpha). \tag{5.26}$$

Recall that in this context M actually corresponds to the enthalpy, thus the variables of this black hole F are the variables of the Gibbs free energy $G(T, P, N_i)$ in classical thermodynamics and so the naming of the thermodynamic potentials is 'shifted' like this. This is because without Λ, M is treated simply as the internal energy.

Since 5.17 or 5.20 are not in general solvable for r_+, all calculations and plots are performed by treating r_+ as a parameter.

Charged AdS Maxwell black holes in this ensemble give first order phase transitions between 'small black hole' and 'large black hole' phases and in the Born-Infeld case very interesting and complicated phase transition behaviour arises [46, 44, 47]. First order phase transitions usually terminate for some critical values (P_c, V_c, T_c) at a critical point, which is usually characterized by the relations

$$\frac{\partial P}{\partial V} = 0 = \frac{\partial^2 P}{\partial V^2}. \tag{5.27}$$

The reason Born-Infeld electrodynamics permits such unusual phase behaviour is the fact these relations have two independent solutions for the range of parameters $\beta \in (\frac{1}{\sqrt{8}|Q|}, \frac{1}{2|Q|})$. The two critical points are led up to by two swallow tails that may intersect in complicated ways explaining the unusual behaviour. One of the results of [14] is the fact RegMax NE does not permit any such unusual behaviour.

We obtain $P = P(V, T, Q, \alpha)$ by rewriting 5.17 and using 5.21, 5.22, 5.20. Then 5.27 can be solved for V_c and T_c. To aid the calculation we, without loss of generality, assumed $Q > 0$. Then the result is

$$\begin{aligned} V_c(\alpha, Q) =& 2 \left(\frac{i\left(\sqrt{3}+i\right)\alpha^{4/3}Q^{8/3}}{\sqrt[3]{\alpha^2 Q\left(10-\alpha^2 Q\right)+2\sqrt{-\left(2\alpha^2 Q-1\right)^3}+2}} + \right. \\ &+ \frac{4i\left(\sqrt{3}+i\right)Q^{5/3}}{\alpha^{2/3}\sqrt[3]{\alpha^2 Q\left(10-\alpha^2 Q\right)+2\sqrt{-\left(2\alpha^2 Q-1\right)^3}+2}} + 4Q^2 + \\ &\left. + \frac{Q - i\left(\sqrt{3}-i\right)\alpha Q\sqrt[3]{\frac{Q\left(\alpha^2 Q(10-\alpha^2 Q)+2\sqrt{-(2\alpha^2 Q-1)^3}+2\right)}{\alpha}}}{\alpha^2} \right)^{1/2}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} T_c(\alpha, Q; V_c) =& \frac{Q}{\pi}\left(\frac{32Q^{3/2}\alpha^3}{(\alpha^2 V_c^2 - 4Q)^2} + \right. \\ &\left. + \frac{16Q^2 - 32Q^3\alpha^2 - 8Q\alpha^2 V_c^2 - 8Q^2\alpha^4 V_c^2 + \alpha^4 V_c^4}{QV_c(\alpha V_c^2 - 4Q)^2} \right), \end{aligned} \tag{5.29}$$

and $P_c = P_c(\alpha, Q; V_c(\alpha, Q), T_c(\alpha, Q)) = P(V_c, T_c, Q, \alpha)$. This solution is real and unique for physical values of $\alpha > 0$, $Q > 0$, V and T, and the functions V_c and T_c are injective, as can be seen in 5.5, therefore only one critical point and also one phase transition is permitted for any given value of α. A comparison of this solution with the Born-Infeld case is also shown in figure 5.5.

The behaviour of F heavily relies on α. In particular there exists a

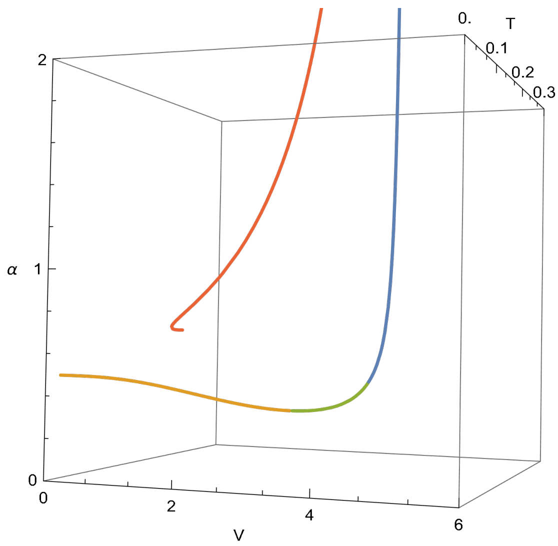

Figure 5.5 Critical points in BI and RegMax: The critical points are plotted parametrically in α for the BI theory (blue-yellow-green line, where the different colours refer to the different solutions parametrized by α) and RegMax (red line). We see a region, where BI dips below the starting point ($V \to 0$) and therefore possesses two critical points for given $\beta = \alpha^2$, whereas the RegMax curve is injective with respect to α. The plot is for $Q = 1$.

critical value

$$\alpha_c = \frac{1}{\sqrt{2}\sqrt{|Q|}}, \tag{5.30}$$

around which the behaviour changes from being similar to the charged-AdS Maxwell case ($\alpha > \alpha_c$, Van der Waals-like) with characteristic swallow-tail behaviour indicating a first degree phase transition between the branches of small and large black holes, see figure 5.6.

By numerically finding the self-intersection points of the swallow tails we can then construct the $P - T$ phase diagram showed in figure 5.7.

We get Schwarzchild-like behaviour for $\alpha < \alpha_c$, see figure 5.8.

At $\alpha = \alpha_c$ we see the behaviour displayed in figure 5.9:

we get the point p at $r_+ = 0$, where the free energy terminates for all ℓ. This point effectively eliminates the possibility for a swallow tail for this case and also all $\alpha < \alpha_c$. We can find the temperature T_p and free energy F_p

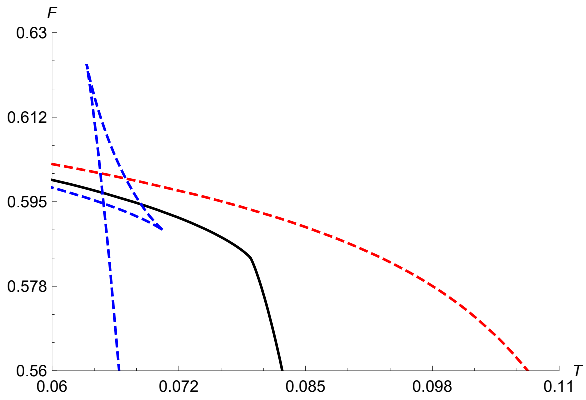

Figure 5.6 F-T diagram: The behaviour of the canonical free energy in the $\alpha > \alpha_c$ regime ($Q = 1$) with a swallow tail for $P < P_c$ (blue dashed). For small T we observe the small black hole phase which, by a first degree phase transition, turns to the large black hole phase. Then a curve with the critical point at $P = P_c$ (solid black) and a smooth curve for $P < P_c$ (red dashed) [14].

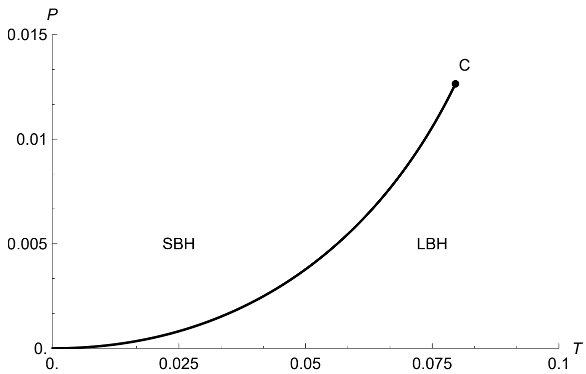

Figure 5.7 P-T diagram: For $\alpha = 1 > \alpha_c$ (and $Q = 1$) there exists a first order phase transition between the small and large black hole phases indicated by (SBH) and (LBH) in the figure. This phase transition terminates at the critical point C [14].

corresponding to this point p by taking the limit $r_+ \to 0$ when $\alpha = \alpha_c$:

$$\begin{aligned} T =& T_p + O(r_+), \quad T_p = \frac{1}{\pi\sqrt{2}|Q|}, \\ F =& \frac{Q\sqrt{2}}{3} + O(R_+^3). \end{aligned} \tag{5.31}$$

By finding the minimum of T, i.e. the cusp edge curve we get the $P - T$ phase diagram for the case $\alpha \leq \alpha_c$; see 5.10 for the case $\alpha = \alpha_c$. For $\alpha < \alpha_c$ the diagram is similar but without the presence of the vertical asymptote at T_p: the (NO BH) region extends to infinity for large enough P.

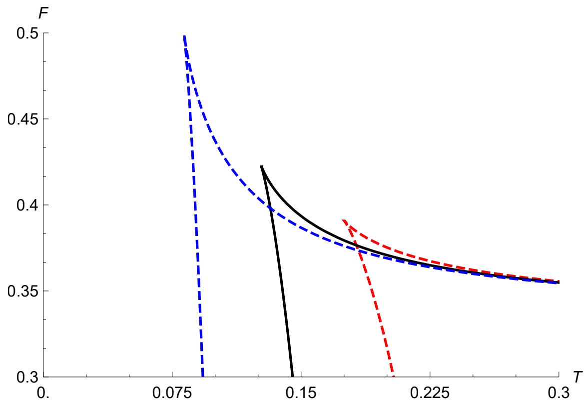

Figure 5.8 F-T diagram: The behaviour of the canonical free energy in the $\alpha = 1 = \alpha_c$ regime ($Q = 1$) for three values of $P = \frac{3}{8\pi\ell^2}$: $\ell = 0.4$ (red dashed), $\ell = 0.7$ (solid black) and $\ell = 0.8$ (blue dashed). We see the point p, from which all these curves emerge ($r_+ \to 0$). The first degree phase transition to small black holes has disappeared and is replaces by a cusp beyond which a region with no black holes (small T) exists and the large black hole phase remains for large enough T [14].

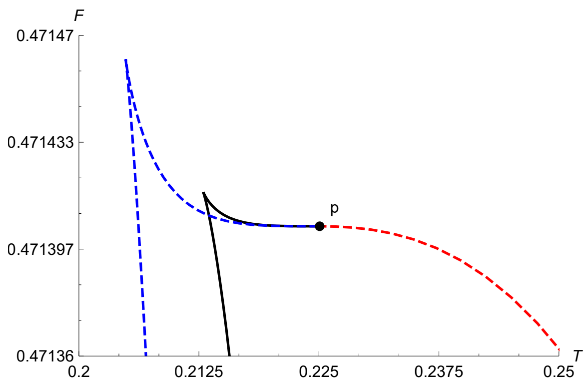

Figure 5.9 F-T diagram: The behaviour of the canonical free energy in the $\alpha = \frac{1}{2} < \alpha_c$ regime ($Q = 1$) for three values of $P = \frac{3}{8\pi\ell^2}$: $\ell = 0.4$ (red dashed), $\ell = 1$ (solid black) and $\ell = 2.2$ (blue dashed). For large enough T we get a large black hole phase, otherwise, beyond the cusp, no black holes exist [14].

A summary of the effect that α has on the behaviour of the free energy F for fixed P and Q is shown in figure 5.11.

The Grand Canonical Ensemble

In the grand canonical ensemble the system is at equilibrium with an infinite thermal reservoir further characterized by a fixed value of the electric potential ϕ. This ensemble allows for the description of neutral thermal

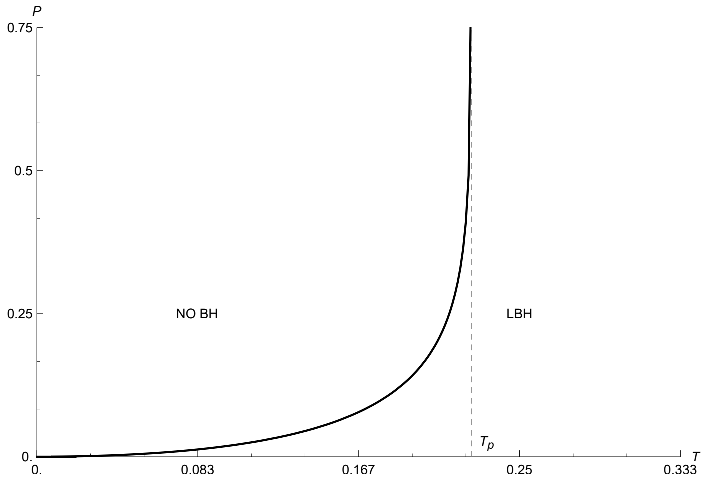

Figure 5.10 P-T diagram: For $\alpha = \alpha_c$ (and $Q = 1$) there exists a region with no black holes, indicated by (NO BH) in the plot, and a region with large black holes (LBH). In this case the (NO BH) region is bounded by the vertical asymptote characterized by the temperature $T_p = \frac{1}{\pi\sqrt{2}|Q|}$.

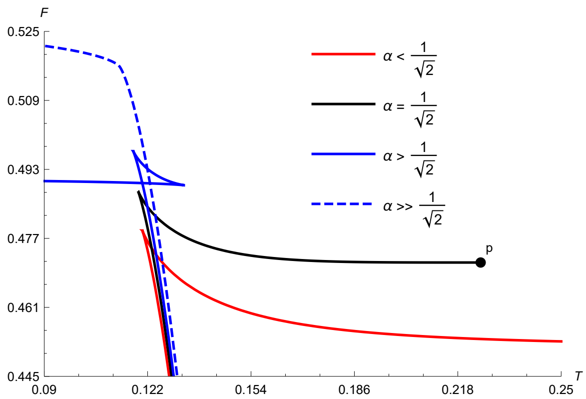

Figure 5.11 F-T diagram: For fixed P ($\ell = 2$ and $Q = 1$) we have plotted the canonical free energy F at the values displayed in the legend. We see the Schwarzschild-like behaviour with a cusp for $\alpha \leq \alpha_c = \frac{1}{\sqrt{2}}$ and VdW-like behaviour in the complementary case [14].

radiation as a thermodynamic phase that may collapse into a black hole. In the Einstein-Maxwell theory this collapse is described by a first order phase transition as shown by Hawking and Page in [41]. The thermodynamic potential in this case exchanges S for T and Q for ϕ as thermodynamic variables, resulting in the grand canonical free energy W:

$$W = M - TS - \phi Q = W(T, \phi, P, \alpha). \tag{5.32}$$

The neutral thermal radiation phase is characterized by:

$$W \approx 0. \tag{5.33}$$

The gravitating solution f_0 and all derived thermodynamic quantities are given for Q; thus we need to express Q in terms of the grand canonical thermodynamic variables. We can find $Q = Q(\phi, \alpha)$ by inverting the relation 5.19:

$$Q = \frac{\phi}{2\alpha^2}\left(2\alpha^2 r_+ + |\phi| + \sqrt{\phi^2 + 4|\phi|\alpha^2 r_+}\right). \tag{5.34}$$

In a similar sense to how there exists a critical α_c for which the behaviour of the $F - T$ diagram changes in the canonical ensemble, there exists a critical

$$\phi_c = \frac{1}{\sqrt{2}} \tag{5.35}$$

in the grand canonical ensemble that has an effect on the $W - T$ diagram.

In the following we assume, without loss of generality, $\alpha = 1$. For $\phi > \phi_c$ the $W - T$ diagram permits, for small enough pressures $P < P_c$ a swallow tail indicating a first order phase transition between the small black hole (small T) and large black hole (large T) phases; see 5.12. This phase transition in the grand canonical ensemble is not present in the linear Maxwell case. There is no radiation phase in this case as the thermodynamically stable configurations (minimal W) are all below $W = 0$ for the entire temperature domain $T > 0$ for any P or α.

The phase diagram for this case is shown in 5.13.

For the case $\phi = \phi_c$ we get what can be seen in figure 5.14: at $r_+ = 0$ a termination point q appears, universal for all P (or ℓ), that blocks the possibility of the small black hole phase and in its place the radiation phase is 'revealed' at $W \approx 0$. We can find the temperature T_q and free energy W_q corresponding to this point by taking the limit $r_+ \to 0$:

$$\begin{aligned} T =& T_q + O(r_+), \quad T_q = \frac{\alpha^2}{\pi\sqrt{2}}, \\ W =& -\frac{1}{6\sqrt{2}\alpha^2} + O(r_+^3). \end{aligned} \tag{5.36}$$

A curiosity of the new radiation-large black hole transition is that for a range of large enough P (as we assume $\alpha = 1$ fixed), where the *cusp* (the point of minimal T of the curves in the corresponding $W - T$ plots) is below $W \approx 0$, we effectively see a discontinuous jump in W. This discontinuity in a

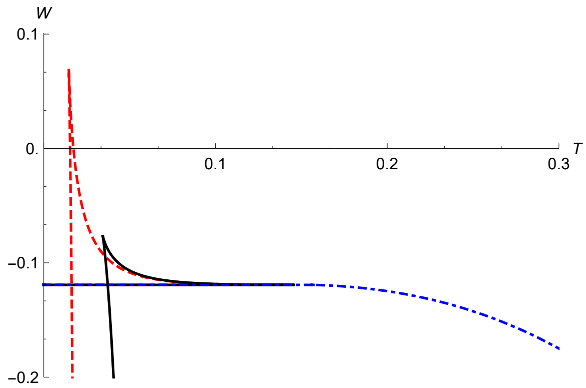

Figure 5.12 $W - T$ diagram: For $\phi > \phi_c$ ($\phi = 0.71$ and $\alpha = 1$) we have plotted the grand canonical free energy W at $P = 0.0008$ (red dashed swallowtail), $P = 0.005$ (solid black swallow tail), $P = 0.25$ (blue dot-dashed smooth decreasing curve). Only the black hole phases exist on the curve with lowest possible W for given T [14].

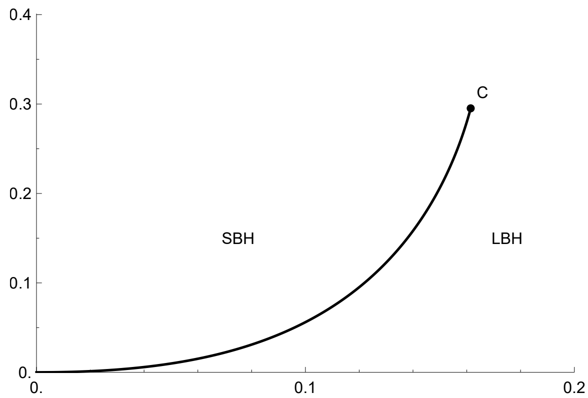

Figure 5.13 $P - T$ diagram: For $\phi > \phi_c$ ($\phi = 0.71$ and $\alpha = 1$) the phase diagram looks similar to the VdW phase transition just as in the canonical case for $\alpha > \alpha_c$. The critical point C can be found by substituting $Q(\phi, \alpha)$ into T_c and P_c [14].

thermodynamic potential has been seen in this context before for example with the complicated 'reentrant phase transitions' behaviour in BI case and has been referred to as a *zeroth degree* phase transition following the Gibbs phase transition categorization naming scheme [47, 48].

When the cusp pierces $W \approx 0$ a first order transition between the large black hole and radiation phases is restored. The presence of the radiation phase also implies there in no 'no black hole' region in the grand canonical ensemble. The $P - T$ phase diagram for this case is shown in fig 5.15. In this $\phi = \phi_c$ case there also exists a vertical asymptote for the zeroth degree phase transitions at T_q in the $P - T$ diagram (not shown in 5.15 due to scale).

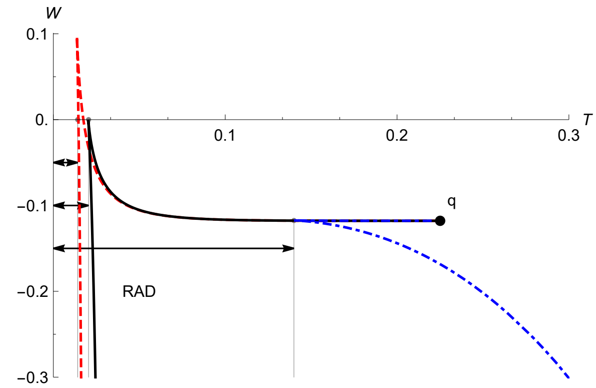

Figure 5.14 $W - T$ diagram: For $\phi = \phi_c$ ($\phi = \frac{1}{\sqrt{2}}$, $\alpha = 1$) we have plotted the grand canonical free energy W at $P = 0.0007$ (red dashed) where the cusp has pierced the $W \approx 0$ thermal radiation level and a first order phase transition exists between them, $P = 0.0016$ (solid black) where the cusp is only touching zero: $W_{cusp} \approx 0$ (maximal P for a first order transition to exist) and $P = 0.153$ (blue dot-dashed) where there is a discontinuous jump in W between the thermal radiation and the large black hole phase. All the curves terminate at the point q when $r_+ \to 0$ [14].

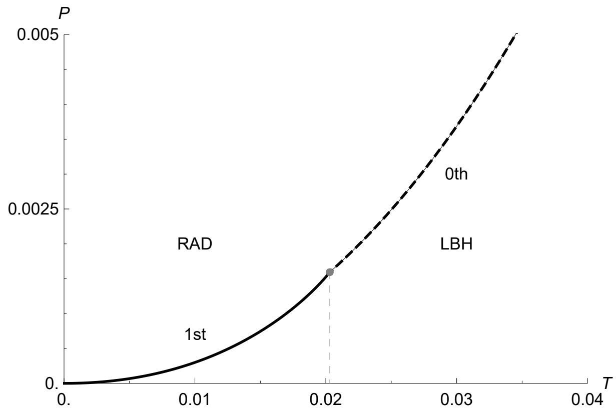

Figure 5.15 $P - T$ diagram: For $\phi = \phi_c$ ($\alpha = 1$) we plot the phase diagram between the radiation (RAD) and large black hole (LBH) phases. For small enough T there is a first order phase transition which turns to a zeroth order transition when the cusp crosses $W_{cusp} \approx 0$. For $\phi = \phi_c$ there exists a vertical asymptote for the zeroth order coexistence curve at $T = T_q = \frac{\alpha^2}{\pi\sqrt{2}}$; for $\phi < \phi_c$ the curve exists for all $T > 0$ [14].

In the $\phi < \phi_c$ case the behaviour of the previous $\phi = \phi_c$ case continues, only the point q disappears and there is no vertical asymptote in the $P - T$ phase diagram – the zeroth order phase transition exists for all large enough T. The $W - T$ plot of this case is shown in figure 5.16

A summary of the effect that ϕ has on the behaviour of the free energy

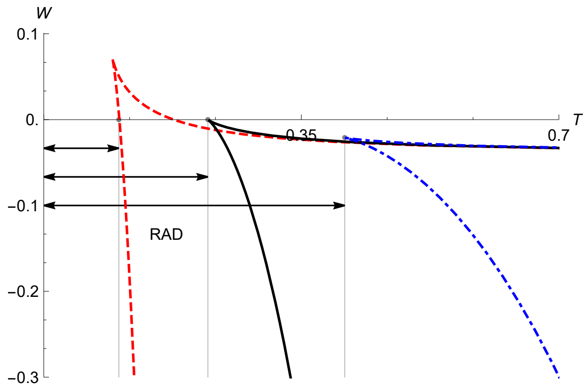

Figure 5.16 $W - T$ diagram: For $\phi < \phi_c$ ($\phi = 0.5$ and $\alpha = 1$) we have plotted the grand canonical free energy W at $P = 0.02$ (red dashed), $P = 0.12$ (solid black) and $P = 0.42$ (blue dot-dashed). We see that for $P \lessapprox 0.12$ (red and black lines) there exists a first order phase transition, while for larger P this becomes a zeroth order phase transition [14].

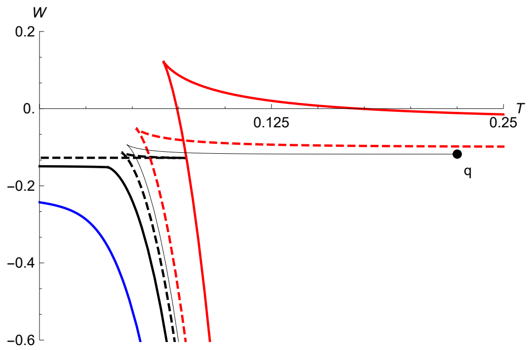

Figure 5.17 $W - T$ diagram (effect of ϕ): For fixed $P = 0.01$ and $\alpha = 1$ we have plotted the grand canonical free energy W at $\phi = 0.5$ (solid red) where a first order RAD-LBH transition exists, $\phi = 0.67$ (red dashed) where a zeroth order RAD-LBH transition exists, $\phi = \phi_c$ (black thin) where the point q and zeroth order RAD-LBH transition exists (whether the cusp is below or above $W \approx 0$ depends on P), $\phi = 0.725$ (black dashed) where a swallow tail and first order SBH-LBH transition exists, $\phi \approx 0.76$ (solid black) with a critical point of the SBH-LBH transition exists and finally $\phi = 0.85$ (solid blue) with no phase transitions [14].

W for fixed $P = 0.01$ and $\alpha = 1$ is shown in figure 5.17.

Finally, putting everything in the grand canonical ensemble together, by we display the $\phi - T$ phase diagram in figure 5.18. We see there are in general four critical points $C1 - 4$. For small enough P the zeroth order transition between $C3$ and $C4$ may vanish due to the cusp piercing the $W \approx 0$ value when $\phi = \phi_c$.

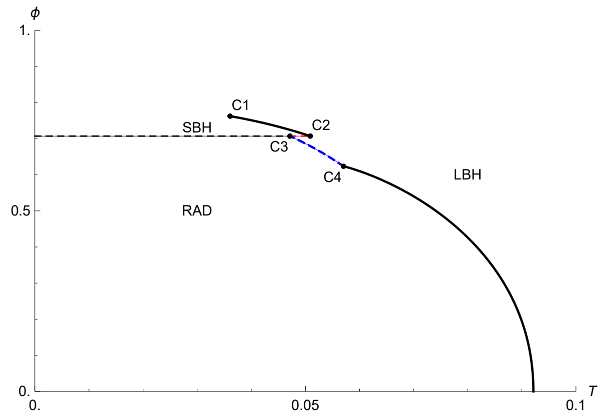

Figure 5.18 $\phi - T$ diagram: For $P = 0.01$ and $\alpha = 1$ we display the phase diagram between the small black hole (SBH), large black hole (LBH) and radiation (RAD) phases. We observe first order phase transitions between the SBH and LBH phases (solid black curve from $C1$ to $C2$) and between the RAD and LBH phases (solid black curve to the right of $C4$). Zeroth order phase transitions exist in general between all the phases and are denoted by dashed curves. The SBH-RAD (black dashed) and SBH-LBH (red dashed between $C2$ and $C3$) zeroth order transitions lie on the $\phi = \phi_c$ line. For small enough pressures P the RAD-LBH (blue dashed from $C3$ to $C4$) zeroth order transition vanishes and $C4$ overlaps $C3$. The diagram is symmetric w.r.t. the $\phi = 0$ axis and the solid black coexistence curve is regular at $\phi = 0$ [14].

To conclude the grand canonical ensemble, we see that the presence of the thermal radiation phase diversifies the phase behaviour of RegMax black holes and a critical value of $\phi = \phi_c$ takes on the role of the critical value $\alpha = \alpha_c$ in the canonical ensemble.

Conclusion

Starting from the linear Maxwell theory we have developed the formalism of nonlinear electrodynamics. Motivated by resolving the self-energy divergence of the electron we assumed the electromagnetic interaction is governed by a new set of field equations that come from applying the action principle to a generalized Lagrangian density of the form $\mathcal{L}_{NE} = \mathcal{L}(F, G^2)$. The resulting NE equations are given the same formal structure as Maxwell's equations by expressing them in terms of of a new anti symmetric second rank tensor $p^{\mu\nu}$ bound to the physical $F^{\mu\nu}$ fields by the NE constitutive relations. In the static spherically symmetric case of a unit charge we obtain a formal general solution that reduces the search for the corresponding electric field of any NE theory to an algebraic equation. We use this result to compare the further discussed NE theories between each other.

The founding Born-Infeld model is introduced and the Lagrangian density is re-derived. We demonstrate one of its perceived drawbacks, the discontinuity of $\mathbf{E}$ at the origin, and find the value of the finite self-energy of a unit 'point' charge $U_{BI} \approx 1.2361 \frac{e^2}{(4\pi)^2} \frac{1}{r_0}$. We follow this up by the Hoffmann-Born-Infeld model, one of the models that was supposed to fix the issues with BI. For this we summarize Infeld's interesting approach of devising an action function with the desired properties. This is done also to aid the derivation of the HBI Lagrangian for which we also calculate the unit charge $\mathbf{E}$ field and self-energy $U_{HBI} \approx 1.03652 \frac{e^2}{(4\pi)^2} \frac{1}{r_0}$.

In the remaining part of the thesis we examine the RegMax NE model and its contemporary context. NE coupled to gravity is introduced through assuming the validity of the Einstein field equations and deriving the form of of the NE energy-momentum tensor by varying the NE-Einstein-Hilbert action w.r.t. the metric $g_{\mu\nu}$. Next a heuristic derivation of the RegMax Lagrangian is given based on regularizing the static Coulomb field in the most straight forward was possible while preserving a skewed re-interpretation of the inverse square law. We also find the unit charge $\mathbf{E}$ field and self-energy $U = \frac{8}{3}\pi\alpha Q^{3/2}$. An extension of the static spherically symmetric case is made to obtain the fields of a homogeneously charged ball with $Q = 1$

and an infinite wire with current $I = 1$ and the RegMax, BI and Maxwell solutions are compared. Then the basics of black hole thermodynamics are introduced before applying them to an analysis of RegMax AdS black hole thermodynamics in the canonical and grand canonical ensembles. We find an explicit analytic expression for all the critical points in the canonical ensemble 5.28, 5.29. We also discover critical values of $\alpha = \alpha_c = \frac{1}{\sqrt{2}\sqrt{|Q|}}$ and $\phi = \phi_c = \frac{1}{\sqrt{2}}$ in the canonical and grand canonical ensembles respectively, for which the phase behaviour fundamentally changes. A series of figures (figures 5.6 – 5.18) is produced to illustrate this behaviour with a key result being the elaborate $\phi - T$ phase diagram of 5.18; these are precisely the findings presented more thoroughly in [14].

Appendix

Invariance of $\int \sqrt{|a_{\mu\nu}|}\, d^4x$

A coordinate transformation $x^\mu \mapsto x'^\mu$ has the Jacobian $J = \frac{\partial x'^\mu}{\partial x^\mu}$. By this transformation the spacetime volume element d^4x becomes $d^4x' = Jd^4x$ since dx^μ are contravariant. $|a_{\mu\nu}|$ becomes $\left|a'_{\mu\nu}\right| = J^{-2}|a_{\mu\nu}|$ since $a_{\mu\nu}$ is twice covariant. Thus taking the square root of $|a_{\mu\nu}|$ makes the Jacobians cancel out and the expression $\int \sqrt{|a_{\mu\nu}|}\, d^4x$ is an invariant.

Determinant calculation in BI

$$
\begin{aligned}
-|a_{\mu\nu}| = -\begin{vmatrix} -1 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 1 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 1 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 1 \end{vmatrix} = -\Bigg(-(-1)^{1+1} \begin{vmatrix} 1 & f_{23} & f_{24} \\ -f_{23} & 1 & f_{34} \\ -f_{24} & -f_{34} & 1 \end{vmatrix} - \\
- f_{12}\,(-1)^{1+2} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ -f_{23} & 1 & f_{34} \\ -f_{24} & -f_{34} & 1 \end{vmatrix} - f_{13}\,(-1)^{1+3} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ 1 & f_{23} & f_{24} \\ -f_{24} & -f_{34} & 1 \end{vmatrix} - \\
- f_{14}\,(-1)^{1+4} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ 1 & f_{23} & f_{24} \\ -f_{23} & 1 & f_{34} \end{vmatrix} \Bigg) = \\
= \left(1 + f_{23}f_{24}f_{34} - f_{23}f_{24}f_{34} + f_{24}^2 + f_{34}^2 + f_{23}^2\right) - \\
- f_{12}\left(f_{12} + f_{14}f_{23}f_{34} - f_{13}f_{24}f_{34} + f_{14}f_{24} + f_{12}f_{34}^2 + f_{13}f_{23}\right) + \\
+ f_{13}\left(f_{12}f_{23} - f_{14}f_{34} - f_{13}f_{24}^2 + f_{14}f_{23}^2 + f_{12}f_{24}f_{34} - f_{13}\right) - \\
f_{14}\left(f_{12}f_{24}f_{34} + f_{14} - f_{13}f_{23}f_{24} + f_{14}f_{23}^2 - f_{12}f_{24} - f_{13}f_{34}\right) = \\
= 1 - f_{12}^2 - f_{13}^2 - f_{14}^2 + f_{23}^2 + f_{24}^2 + f_{34}^2 - |f_{\mu\nu}|, \qquad (5.37)
\end{aligned}
$$

where

$$|f_{\mu\nu}| = \begin{vmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{vmatrix} = -f_{12}(-1)^{1+2} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ -f_{23} & 0 & f_{34} \\ -f_{24} & -f_{34} & 0 \end{vmatrix} -$$

$$- f_{13}\,(-1)^{1+3} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ 0 & f_{23} & f_{24} \\ -f_{24} & -f_{34} & 0 \end{vmatrix} - f_{14}\,(-1)^{1+4} \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ 0 & f_{23} & f_{24} \\ -f_{23} & 0 & f_{34} \end{vmatrix} =$$

$$\begin{aligned} =&f_{12}\left(f_{14}f_{23}f_{34} - f_{13}f_{24}f_{34} + f_{12}f_{34}^2\right) - f_{13}\left(-f_{13}f_{24}^2 + f_{14}f_{23}f_{24} + \right. \\ &\left. + f_{12}f_{24}f_{34}\right) + f_{14}\left(f_{12}f_{23}f_{34} - f_{13}f_{23}f_{24} + f_{14}f_{23}^2\right) = \\ =&\left(f_{12}f_{34} + f_{14}f_{23} - f_{24}f_{13}\right)^2 \end{aligned} \tag{5.38}$$

Divergence of the charge of the second solution

The 'free' charge density of the solution

$$\tilde{E_r} = \frac{\alpha^2 Q}{\left(\alpha r - \sqrt{|Q|}\right)^2} \tag{5.39}$$

can be defined as the divergence of **E**:

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\mathrm{d}(r^2 E_r)}{\mathrm{d}r} = \rho_{\text{free}}. \tag{5.40}$$

This gives:

$$\rho_{\text{free}} = -\frac{2\alpha^2 Q\sqrt{|Q|}}{r\left(\alpha r - \sqrt{|Q|}\right)^3}. \tag{5.41}$$

This function can be integrated in spherical coordinates:

$$\int 4\pi\rho_{\text{free}} r^2\, dr = -\frac{4\pi Q\sqrt{|Q|}\left(\sqrt{|Q|} - 2\alpha r\right)}{\left(\sqrt{|Q|} - \alpha r\right)^2}. \tag{5.42}$$

In the region $r < \frac{\sqrt{Q}}{\alpha}$ (where $\tilde{E_r}$ is a solution) we get:

$$\tilde{Q} = \lim_{r \to \frac{\sqrt{Q}}{\alpha}} \int \rho_{\text{free}} - \lim_{r \to 0} \int \rho_{\text{free}} = \infty - (-4\pi Q) = \infty \tag{5.43}$$

With all other valid solutions of the NE field equations we have an equality between the integrated divergence of **D** (equal to Q) and the integrated divergence of **E** calculated here. Thus this solution is unphysical.

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