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To whom it may concern,

Re: Martin Balko's Habilitation Thesis

In this thesis, the author explores a range of problems at the interface of Ramsey theory and discrete geometry. Continuing earlier work from their PhD thesis, they solve a number of interesting problems related to ordered Ramsey numbers, study a range of themes around the famous Erdős–Szekeres convex sets problem and much more besides. I will give a brief summary of the main results below, but I find the thesis to be an excellent one and warmly support the award of an habilitation.

The thesis splits into five main parts and I will describe the work in each part in turn:

- The first part explores ordered Ramsey numbers, where, given an ordered graph H, its ordered Ramsey number $\overline{R}(H)$ is defined to be the smallest N such that any 2-colouring of the edges of the complete ordered graph with vertex set $\{1, 2, \ldots, N\}$ contains a monochromatic ordered copy of H. This interesting concept was introduced independently by the author of this thesis, together with Cibulka, Král and Kynčl, and also by this reviewer, together with Fox, Lee and Sudakov. The paper by Balko et al. is included here and contains a number of foundational results, including the surprising result, discovered by both groups, that there are ordered matchings that have super-polynomial ordered Ramsey numbers.

  The initial papers on this topic left open a number of promising questions, many of which were subsequently solved by the author and his collaborators. Again the relevant papers are included here. For instance, solving a problem of this reviewer and his collaborators, Balko, Jelínek and Valtr show that there are 3-regular graphs (in fact, almost any 3-regular graph will work) for which the ordered Ramsey number is superlinear in every ordering. Another paper, placed in the next section of the thesis, but to my mind more properly placed in this section, concerns extensions of these results to hypergraphs. Here, Balko and Vizer prove a number of interesting results and state several interesting questions that will doubtless invite further work.

- The second section of the thesis concerns several variations on the ordered Ramsey theme. For instance, in the single-author paper 'Ramsey numbers and monotone colorings', the author defines a notion of monotone colourings for hypergraphs and then a dependent notion of monotone Ramsey numbers. Like ordered Ramsey numbers, this notion is inspired by applications in discrete geometry, particularly to results of Erdős–Szekeres-type that we will discuss under the next section. The main result then gives an essentially tight lower bound for these monotone Ramsey numbers, answering problems raised by Eliáš and Matoušek and by Moshkovitz and Shapira.

  Also in this section is a paper with Vizer on edge-ordered Ramsey numbers. The notion of ordered Ramsey numbers described above concerns vertex orderings, but it is entirely possible to define a similar notion for edge orderings. The paper with Vizer is the foundational paper on this theme and proves a number of basic results, not least that this notion of edge-ordered Ramsey numbers is well-defined, something which is far less obvious here than it is for the vertex-ordered case.

- The Erdős–Szekeres theorem is the statement that any sufficiently large set of points in the plane with no three on a line contains n points that form a convex set. An old question of Erdős asked whether one can in fact find an n-point convex set with no other points in the interior of this set. A remarkable construction of Horton shows that this is not true, even for $n = 7$. However, several questions around these holes remain and that is what the results of this section explore.

On the one hand, there are two papers, both with Scheucher and Valtr, looking at the number of such holes in a randomly distributed point set. More precisely, given a compact convex set K in $\mathbb{R}^d$ and any $k > d$, they show that a random n-point subset of K contains $\Theta(n^d)$ holes with k vertices.

In a different direction, together with Aichholzer, Hackl, Kynčl, Parada, Scheucher, Valtr and Vogtenhuber, the author solves an old problem in the area by showing that any collection of n points in the plane with no three on a line contains at least $n \log^{4/5} n$ holes with 5 vertices. It has been known for some time that there are $\Theta(n^2)$ holes with 3 or 4 vertices and, though the analogous estimate is conjectured for 5-holes, it has remained wide open and even improving on the trivial linear bound resisted attack. The approach taken by Balko and his collaborators is extremely interesting, in that it first relies on a computer proof to establish a key auxiliary result from which the main theorem then follows.

There is also another paper, placed in the second section above, that relates to the Erdős–Szekeres theorem. Indeed, there is an old conjecture of Erdős and Szekeres that states that any set of $2^{n-2} + 1$ points in the plane with no three on a line contains an n-point convex subset, which would be tight. An approach to this conjecture was suggested by Peters and Szekeres, who formulated a Ramsey-theoretic statement, which, if true, would have implied the Erdős–Szekeres conjecture. However, using SAT solvers, Balko and Valtr were able to disprove this conjecture.

- The fourth section concerns visibility problems. We say that two points u and v in $\mathbb{R}^d$ are visible with respect to $X \subset \mathbb{R}^d$ if there is no point of X on the line segment connecting u and v. The thesis contains two papers concerning visibility. The first concerns the obstacle number of a graph H, the minimum number of polygonal obstacles one needs to place in the plane so that there is a set of points such that the set of visible pairs are exactly the edges of H. It is an open question to show that the obstacle number of any graph is at most linear in the number of vertices. Balko's contribution, with Cibulka and Valtr, was to prove this for graphs of bounded chromatic number. Moreover, in general, they give a $O(n \log n)$ upper bound.

  The second paper in this direction concerns the relationship between two different measures of convexity for a general Lebesgue measurable subset of $\mathbb{R}^d$, one of which, the Beer index, is defined in terms of visibility. I will not define the notions here, but suffice to say that there was a conjecture of Cabello, Cibulka, Kynčl, Saumell and Valtr saying that for simple polygons, the Beer index is at most a linear function of the other notion, the convexity ratio. This, along with some more general results, was proved by Balko, Jelínek, Valtr and Walczak.

- The final section contains a single paper, but a nice one, written with Cibulka and Valtr, about covering lattice points by subspaces. Indeed, for any d-dimensional lattice Δ and any symmetric compact convex body K in $\mathbb{R}^d$, they give very accurate estimates for the minimum number of linear subspaces (or affine subspaces) needed to cover all the points in $\Delta \cap K$. As an application, they then improve the lower bounds for the number of incidences between a set of points and a set of hyperplanes in $\mathbb{R}^d$, itself a well-known classical problem.

Overall, I find this to be an extremely interesting and engaging thesis, covering a broad range of themes in discrete geometry and its adjacent areas. The author seems to have excellent taste in terms of his directions of research and writes very well, both of which make the thesis a pleasure to read. I was explicitly asked to comment on the originality of the thesis and can confirm that all of the research is, to the very best of my knowledge, new and original, indeed at the cutting edge of research. I am very happy to recommend that the author be awarded his habilitation on the basis of this fine work.

Yours sincerely,

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