FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

HABILITATION THESIS

EXTREMAL PROBLEMS IN DISCRETE GEOMETRY

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2021

ACKNOWLEDGMENTS

First, I would like to thank all my collaborators with whom I had the pleasure and honor to work. I also thank to members of KAM and IÚUK for a very friendly and encouraging environment. In addition, I thank to all my excellent students whose Bachelor or Master theses and software projects helped to investigate open research problems and discover new promising ideas. Last but not least, I would like to thank to my family for all the support.

The research in this thesis was supported by the following grants and projects: ERC Advanced Research Grant No. 267165 (DISCONV), European Union's Horizon 2020 research and innovation programme under grant agreements No. 678765 and 810115, Czech Science Foundation Grants 14-14179S, 18-13685Y, 18-19158S, and CE-ITI P202/12/G061, grant 1452/15 from Israel Science Foundation, Charles University grants GAUK 690214, GAUK 1262213, and PRIMUS/17/SCI/3, the bilateral research project 7AMB15A T023 between Graz and Prague supported by the Ministry of Education of the Czech Republic, and the project UNCE/SCI/004 (Center of Modern Computer Science) of Charles University.

PREFACE

This thesis is based on the following ten research articles (published or accepted for publication) and two yet unpublished manuscripts listed below. A copy of each article is included in the thesis.

All these works are motivated by extremal problems in discrete geometry and all the main results are listed and described in the Introduction. In these publications from years between 2017 and 2021, we solve several open problems posed by various researchers and we significantly improve previously known results.

[Aic+20] O. Aichholzer, M. Balko, T. Hackl, J. Kynčl, I. Parada, M. Scheucher, P. Valtr, and B. Vogtenhuber. "A superlinear lower bound on the number of 5-holes." In: *J. Combin. Theory Ser. A* 173 (2020), pp. 105236, 31. DOI: 10.1016/j.jcta.2020.105236.

[Bal19] M. Balko. "Ramsey numbers and monotone colorings." In: *J. Combin. Theory Ser. A* 163 (2019), pp. 34–58. DOI: 10.1016/j.jcta.2018.11.013.

[Bal+20] M. Balko, J. Cibulka, K. Král, and J. Kynčl. "Ramsey numbers of ordered graphs." In: *Electron. J. Combin.* 27 (1 2020), P1.16.

[BCV18] M. Balko, J. Cibulka, and P. Valtr. "Drawing graphs using a small number of obstacles." In: *Discrete Comput. Geom.* 59.1 (2018), pp. 143–164. DOI: 10.1007/s00454-017-9919-2.

[BCV19] M. Balko, J. Cibulka, and P. Valtr. "Covering lattice points by subspaces and counting point-hyperplane incidences." In: *Discrete Comput. Geom.* 61.2 (2019), pp. 325–354. DOI: 10.1007/s00454-018-9970-7.

[BJV19] M. Balko, V. Jelínek, and P. Valtr. "On ordered Ramsey numbers of bounded-degree graphs." In: *J. Combin. Theory Ser. B* 134 (2019), pp. 179–202. DOI: 10.1016/j.jctb.2018.06.002.

[Bal+17] M. Balko, V. Jelínek, P. Valtr, and B. Walczak. "On the Beer index of convexity and its variants." In: *Discrete Comput. Geom.* 57.1 (2017), pp. 179–214. DOI: 10.1007/s00454-016-9821-3.

[BSV21a] M. Balko, M. Scheucher, and P. Valtr. *Holes and islands in random point sets*. To appear in *Random Structures & Algorithms.* Extended abstract in: *36th International Symposium on Computational Geometry (SoCG 2020)*. Vol. 164. LIPIcs, 2020, Art.14–16. 2021.

[BSV21b] M. Balko, M. Scheucher, and P. Valtr. *Tight bounds on the expected number of holes in random point sets*. Submitted. 2021.

[BV17] M. Balko and P. Valtr. "A SAT attack on the Erdős–Szekeres conjecture." In: *European J. Combin.* 66 (2017), pp. 13–23. DOI: 10.1016/j.ejc.2017.06.010.

[BV20] M. Balko and M. Vizer. "Edge-ordered Ramsey numbers." In: *European J. Combin.* 87 (2020), pp. 103100, 11. DOI: 10.1016/j.ejc.2020.103100.

[BV21] M. Balko and M. Vizer. *On ordered Ramsey numbers of tripartite 3-uniform hypergraphs*. Submitted. 2021.

While not included directly in this thesis, one additional manuscript [BP21] I co-authored is discussed in the Introduction too.

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1 INTRODUCTION

In this chapter, we survey all the main results that are contained in this thesis. First, we state some necessary preliminaries and give a brief introduction to Ramsey theory and discrete geometry. The content of the remaining sections of this chapter is then described at the end of Section 1.1.

1.1 PRELIMINARIES

We assume that the reader is familiar with the basics of graph theory to the extent covered, for example, by [MN09]. Throughout the whole thesis, we consider only finite simple graphs with no loops nor multiple edges. We also assume familiarity with the basics of geometry and linear algebra to the extend covered, for example, by [Mat02].

For a positive integer n, we use $[n]$ to denote the set $\{1, \dots, n\}$. We omit floor and ceiling signs whenever they are not crucial and we use log and ln to denote base 2 logarithm and the natural logarithm, respectively.

For a positive integer r, an *r-coloring* of a hypergraph H is any function that assigns one of r *colors* to each edge of H. Unless stated otherwise, we assume that the colors of an r-coloring form the set $[r]$.

We let $\lambda_d(K)$ be the d-dimensional Lebesgue measure of a Lebesgue measurable subset K of $\mathbb{R}^d$. We say that $\lambda_d(K)$ is the *volume of* K. The closed d-dimensional ball with the radius $r \in \mathbb{R}$, $r \geq 0$, centered in the origin is denoted by $B^d(r)$. We simply write B^d instead of $B^d(1)$.

For functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we write $f(n) \leq O(g(n))$ if there is a constant c_1 such that $f(n) \leq c_1 \cdot g(n)$ for all $n \in \mathbb{N}$. Similarly, we write $f(n) \geq \Omega(g(n))$ if there is a constant $c_2 > 0$ such that $f(n) \geq c_2 \cdot g(n)$ for all $n \in \mathbb{N}$. If the constants c_1 and c_2 depend on some parameters $a_1, \dots, a_t$, then we emphasize this by writing $f(n) \leq O_{a_1,\dots,a_t}(g(n))$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(g(n))$, respectively. If $f(n) \leq O_{a_1,\dots,a_t}(n)$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(n)$, then we write $f(n) = \Theta_{a_1,\dots,a_t}(n)$.

1.1.1 *Ramsey theory*

In broad sense, *Ramsey theory* refers to any result whose underlying philosophy can be captured by the statement "Every structure of a given kind contains a large well-organized substructure". This part of discrete mathematics has developed enormously in the last few decades, emerging into a field with important statements in many areas, including combinatorics, geometry, logics, and number theory.

A classical example of a Ramsey-type statement, and one of the oldest such results, is the *Erdős–Szekeres lemma* on monotone subsequences proved by Erdős and Szekeres [ES35] in 1935.

Theorem 1.1.1 (The Erdős–Szekeres lemma [ES35])**.** *For every positive integer n, every sequence of at least $(n-1)^2+1$ distinct real numbers contains an increasing or a decreasing subsequence of length n. Moreover, this bound is tight.*

Here, the structure that we study are sequences of distinct real numbers and the well-organized substructures that we are looking for are increasing and decreasing subsequences of length n. In this case, we know the exact minimum size of the structure that guarantees the existence of the well-organized substructures, it equals $(n-1)^2+1$.

One of the cornerstones of Ramsey theory, from which Ramsey theory derives its name, is *Ramsey's theorem* [Ram29].

Theorem 1.1.2 (Ramsey's theorem [Ram29])**.** *For all positive integers r, k and all k-uniform hypergraphs $H_1,\dots,H_r$, there is a positive integer N such that in every r-coloring χ of $K_N^{(k)}$ there is $i \in [r]$ and a subhypergraph of $K_N^{(k)}$ isomorphic to H_i with all edges of color i in χ.*

If we have only two colors ($r=2$), we use $R(H_1,H_2)$ to denote the smallest such integer N and we call it the *Ramsey number* of H_1 and H_2. If the hypergraphs H_1 and H_2 are isomorphic, we simply write $R(H_1)$ instead of $R(H_1,H_2)$. This is called the *diagonal case*. Otherwise we call it the *non-diagonal case*.

By Ramsey's theorem (Theorem 1.1.2), Ramsey numbers are always finite and thus well-defined. Estimating the growth rate of Ramsey numbers is a notoriously difficult problem in general. Despite several attempts over the last 70 years, Ramsey numbers are not fully understood even for complete graphs. Although there have been smaller term improvements [Con09, Ash20, Spe75], the best known bounds on $R(K_n)$ essentially are

$$2^{n/2} \le R(K_n) \le 2^{2n}. \tag{1.1}$$

The lower bound is due to Erdős [Erd47] while the upper bound was proved by Erdős and Szekeres [ES35] in their seminal paper from 1935. In this paper, Erdős and Szekeres independently rediscovered Ramsey's theorem and their work is thus one of the starting points of Ramsey theory.

For sparser graphs, Ramsey numbers grow at a significantly slower rate. For example, the following classical result by Chvátal, Rödl, Szemerédi, and Trotter [Chv+83] says that Ramsey numbers of bounded-degree graphs are only linear in the number of vertices.

Theorem 1.1.3 ([Chv+83])**.** *For every positive integer Δ, there is a positive integer $C=C(\Delta)$ such that every n-vertex graph G with maximum degree Δ satisfies*

$$R(G) \le C \cdot n.$$

The Ramsey numbers $R(K_n^{(k)})$ are even less understood for complete k-uniform hypergraphs with $k \geq 3$. For example, it is only known that

$$2^{\Omega(n^2)} \leq R(K_n^{(3)}) \leq 2^{2^{O(n)}}, \tag{1.2}$$

as shown by Erdős, Hajnal, and Rado [EHR65]. A famous conjecture of Erdős, for whose proof Erdős offered \$500 reward, states that there is a constant $c > 0$ such that $R(K_n^{(3)}) \geq 2^{2^{cn}}$. The case $k = 3$ is of particular importance, as if one determines the growth rate of $R(K_n^{(3)})$ precisely, then the so-called *Stepping-up lemma* by Erdős and Rado [ER52] would determine the growth rate of $R(K_n^{(k)})$ for every $k \geq 4$. The Stepping-up lemma is quantitatively stated as follows.

Theorem 1.1.4 (The Stepping-up lemma [ER52]). *For all integers $k \geq 3$, $s \geq k$, and $t \geq k$, we have*

$$R(K_s^{(k)}, K_t^{(k)}) \geq 2^{\binom{R\left(K_{s-1}^{(k-1)}, K_{t-1}^{(k-1)}\right)}{k-1}} + k - 2.$$

The *tower function* $t_k(x)$ of *height* $k-1$ is defined by the recursive formula $t_1(x) = x$ and $t_k(x) = 2^{t_{k-1}(x)}$ for every $k \geq 2$. Using (1.2) together with the Stepping-up lemma (Theorem 1.1.4), one can derive lower bounds on $R(K_n^{(k)})$ for $k \geq 4$. However, there is a difference of one exponential between known upper and lower bounds. More precisely, we have

$$t_{k-1}(\Omega(2^{-k}n^2)) \leq R(K_n^{(k)}) \leq t_k(O(n)) \tag{1.3}$$

for every $k \geq 3$.

Perhaps surprisingly, the Ramsey number $R(H)$ of every k-uniform hypergraph H with bounded k and with bounded maximum degree is at most linear in the number of vertices of H [Chv+83, CFS09, Coo+08, Coo+09, Ish07, Nag+08].

1.1.2 *Discrete geometry*

In their celebrated paper from 1935, Erdős and Szekeres [ES35] not only established the foundations of Ramsey theory, but they also initiated the study of *discrete geometry*, one of the fields of combinatorics with a wealth of interesting and difficult problems. In discrete geometry, the emphasis is on combinatorial properties of simple geometric objects such as finite sets of points, lines, hyperplanes, circles, and so on. Many questions in discrete geometry are very natural and easy to state, yet their solutions might be very difficult or even out of reach by current methods.

The connection between Ramsey theory and discrete geometry is almost as old as Ramsey theory itself, since one of the earliest and most popular applications of Ramsey's theorem is the *Erdős–Szekeres theorem*, a foundational result in discrete geometry. To state it, we first need to state some definitions.

A finite set P of points in the plane is in *general position* if no three points from P lie on a common line. A finite set of points is in *convex position* if its points form vertices of a convex polygon.

Theorem 1.1.5 (The Erdős–Szekeres theorem [ES35]). *For every positive integer n, there is a positive integer $N(n)$ such that every set of at least $N(n)$ points in general position in the plane contains n points in convex position.*

We use $ES(n)$ to denote the smallest such integer $N(n)$. The statement of the Erdős–Szekeres theorem is a generalization of Esther Klein's problem, which was named the *Happy Ending Problem* by Paul Erdős, as it eventually led to the marriage of George Szekeres and Esther Klein. Erdős and Szekeres provided two proofs of this famous result. One is an application of Ramsey's theorem (Theorem 1.1.2) and yields a rather poor upper bound on the function $ES(n)$. The other proof uses more geometry and gives the estimate

$$ES(n) \leq \binom{2n-4}{n-2} + 1 \tag{1.4}$$

for every $n \geq 2$. Already in 1935, Erdős and Szekeres believed that this bound can be significantly improved. Based on their results for $n = 2, 3, 4$, they posed the famous and still open *Erdős–Szekeres conjecture*, for whose proof Erdős offered $500 reward.

Conjecture 1.1.6 (The Erdős–Szekeres conjecture [ES35]). *For every integer $n \geq 2$, we have*

$$ES(n) = 2^{n-2} + 1.$$

In the 1960s, Erdős and Szekeres [ES60] supported Conjecture 1.1.6 with the lower bound

$$ES(n) \geq 2^{n-2} + 1. \tag{1.5}$$

Despite several attempts over the years [CG98, KP98, TV98, MV16, NY16, SP06], it is still open to decide whether the upper bound $ES(n) \leq 2^{n-2} + 1$ holds. It is only known that the conjecture is true for $n \leq 6$ [SP06]. However, there has been a recent breakthrough by Suk [Suk17], who proved a very close estimate $ES(n) \leq 2^{n+o(n)}$.

Of course, the field of discrete geometry offers much more than Erdős–Szekeres-type questions. There are, for example, beautiful problems about numbers of incidences, unit distances in finite point sets, visibility problems, estimating numbers of faces of polytopes, and many more. Some of these problems are explored later in this thesis.

1.1.3 *Synopsis of the thesis*

We study extremal problems with motivation coming from the field of discrete geometry. In particular, most of the problems are motivated by the Erdős–Szekeres theorem (Theorem 1.1.5).

Quite recently, several authors noticed that the Erdős–Szekeres lemma (Theorem 1.1.1) and the upper bound (1.4) can be derived from results about Ramsey numbers of 2- and 3-uniform paths with a particular ordering of their vertices [CP02, EM13, Fox+12, MSW15, MS14]. This initiated a study of so-called *ordered Ramsey numbers* [Bal+20, Con+17], which are an analogue of Ramsey numbers for hypergraphs with a fixed ordering of their vertex sets.

In Section 1.2, we survey known results about ordered Ramsey numbers of ordered graphs. We state the necessary definitions and then we cover known estimates on ordered Ramsey numbers for general classes of ordered graphs as well as for classes of specific graph orderings. In particular, we solve two open problems posed by Conlon, Fox, Lee, and Sudakov [Con+17] and refute a conjecture of Rohatgi [Roh19]. We conclude this section with a rich list of open problems in this relatively new part of Ramsey theory.

We continue our study of ordered Ramsey numbers in Section 1.3, where we focus on ordered hypergraphs. We list known results about ordered Ramsey numbers of ordered k-uniform hypergraphs with $k \geq 3$ and we mention a connection between the Erdős–Szekeres theorem and ordered Ramsey numbers of monotone 3-uniform paths. We refute a conjecture of Peters and Szekeres [SP06] about an abstract combinatorial strengthening of the Erdős–Szekeres conjecture (Conjecture 1.1.6). When investigating the ordered Ramsey numbers of monotone paths with restricted colorings, we solve an open problem posed by Eliáš and Matoušek [EM13] and also by Moshkovitz and Shapira [MS14]. We also introduce a variant of ordered Ramsey numbers for graphs with ordered edges instead of vertices. Again, this section is concluded with a list of several open problems.

In Section 1.4, we focus on a strengthening of the Erdős–Szekeres theorem to so-called *holes* proposed by Erdős [Erd78]. We survey known bounds for this problem with a particular emphasis on the minimum number of 5-holes, where we solve a folklore problem that has been open since the 1980s and that is mentioned, for example, by Brass, Moser, and Pach [BMP05]. We also explore the problem of estimating the minimum number of holes in random point sets, obtaining several asymptotically tight estimates.

Section 1.5 is devoted to *visibility problems*, a classical topic in discrete geometry. We study so-called *obstacle numbers* of graphs, where we refute a conjecture of Mukkamala, Pach, and Pálvölgyi [MPP12]. We also explore the *index of convexity* of measurable subsets of $\mathbb{R}^d$, giving an affirmative answer to a conjecture of Cabello et al. [Cab+17].

Finally, we explore *covering* and *incidence problems* in Section 1.6. This part falls within the intersection of discrete geometry and geometry of numbers. We prove new results about covering lattice points by linear subspaces, nearly settling a problem mentioned in the book by Brass, Moser, and Pach [BMP05]. We also use these results to obtain the currently strongest bounds on the number of incidences between points and hyperplanes from $\mathbb{R}^d$.

1.2 GRAPH ORDERED RAMSEY NUMBERS

An *ordered graph* is a pair $\mathcal{G} = (G, \prec)$ where $\prec$ is a linear ordering of the vertex set of G. We call $\mathcal{G}$ an *ordering* of G. Two ordered graphs $\mathcal{G} = (G, \prec_1)$ and $\mathcal{H} = (H, \prec_2)$ are *isomorphic* if the graphs G and H are isomorphic via a one-to-one correspondence $g\colon V(G) \to V(H)$ that preserves the orderings. That is, we have $u \prec_1 v$ if and only if $g(u) \prec_2 g(v)$ for all $u, v \in V(G)$. Note that, for every positive integer n, there is only one ordered complete graph on n vertices up to isomorphism. We use $\mathcal{K}_n$ to denote such ordered complete graph.

An ordered graph $\mathcal{G} = (G, \prec_1)$ is an *ordered subgraph* of an ordered graph $\mathcal{H} = (H, \prec_2)$ if G is a subgraph of H and $\prec_1$ is a suborder of $\prec_2$. If an ordered graph $\mathcal{H}$ contains an ordered subgraph isomorphic to an ordered graph $\mathcal{G}$, then we say that $\mathcal{H}$ contains a *copy* of $\mathcal{G}$.

We can now define an analogue of Ramsey numbers for ordered graphs. The *ordered Ramsey number* $\overline{R}(\mathcal{G}, \mathcal{H})$ of two ordered graphs $\mathcal{G}$ and $\mathcal{H}$ is the minimum positive integer N such that every red-blue coloring of the edges of $\mathcal{K}_N$ contains a red copy of $\mathcal{G}$ or a blue copy of $\mathcal{H}$. If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic, then we simply write $\overline{R}(\mathcal{G})$ instead of $\overline{R}(\mathcal{G}, \mathcal{H})$ and we call it the *diagonal case*. We refer to $\overline{R}(\mathcal{G}, \mathcal{H})$ with non-isomorphic $\mathcal{G}$ and $\mathcal{H}$ as the *non-diagonal case*.

Note that we have $R(G) \leq \overline{R}(\mathcal{G})$ for every graph G and each its ordering $\mathcal{G}$. Moreover, $R(K_n) = \overline{R}(\mathcal{K}_n)$ for every positive integer n. Since every ordered graph $\mathcal{G} = (G, \prec)$ on n vertices is an ordered subgraph of $\mathcal{K}_n$, we thus obtain the bounds

$$R(G) \leq \overline{R}(\mathcal{G}) \leq R(K_n).$$

It follows that the number $\overline{R}(\mathcal{G})$ is finite for every ordered graph $\mathcal{G}$. Similarly, $\overline{R}(\mathcal{G}, \mathcal{H})$ is also finite for any ordered graphs $\mathcal{G}$ and $\mathcal{H}$ and thus ordered Ramsey numbers are well-defined.

From some point of view, the study of ordered Ramsey numbers is as old as Ramsey theory itself. For example, the Erdős–Szekeres lemma (Theorem 1.1.1) is a special case of a Ramsey-type result for ordered graphs. To see this, consider the following ordering $\mathcal{P}_n = (P_n, \prec)$, called the *monotone path*, of the path P_n on n vertices. If $v_1 \prec \cdots \prec v_n$ are the vertices of $\mathcal{P}_n$, then the edges of $\mathcal{P}_n$ are the pairs $\{v_i, v_{i+1}\}$ for every $i = 1, \dots, n-1$; see Figure 1.1. Given a sequence $S = (s_1, \dots, s_N)$ of distinct real numbers, we construct an ordered graph $(K_N, \prec)$ with vertex set S and the ordering of the vertices given by their positions in S. That is, for $s_i, s_j \in S$, we have $s_i \prec s_j$ if $i < j$. Then we color an edge $\{s_i, s_j\}$ with $i < j$ red if $s_i < s_j$ and blue otherwise. Afterwards, red monotone paths on n vertices correspond to increasing subsequences of S of length n and blue monotone paths on n vertices to decreasing subsequences of S of length n. The Erdős–Szekeres lemma now follows from the fact

$$\overline{R}(\mathcal{P}_n) = (n-1)^2 + 1 \tag{1.6}$$

proved, for example, by Choudum and Ponnusamy [CP02] or by Milans, Stolee, and West [MSW15].

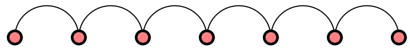

Figure 1.1: The monotone path $\mathcal{P}_n$ for $n = 7$. In all figures of ordered graphs in this thesis, the vertices are ordered from left to right.

Similarly, the Erdős–Szekeres theorem (Theorem 1.1.5) can be derived from estimates on ordered Ramsey numbers of 3-uniform monotone hyperpaths; see Subsection 1.3.2. The ordered Ramsey numbers are also closely connected to the extremal theory of $\{0, 1\}$-matrices and to variants of Ramsey numbers from discrete geometry, for example, to so-called *geometric Ramsey numbers* [KPT97, Kár+98]. Given these results and connections, there is a strong motivation to study ordered Ramsey numbers and their variants.

While there had been a lot of results about ordered Ramsey numbers of monotone hyperpaths [CP02, EM13, Fox+12, MSW15, MS14], there was a surprisingly little work on ordered Ramsey numbers of more general ordered graphs and hypergraphs. The first systematic study of ordered Ramsey numbers was conducted by Balko, Cibulka, Král, and Kynčl [Bal+20] and independently by Conlon, Fox, Lee, and Sudakov [Con+17]. Since then there has been much progress on understanding the ordered Ramsey numbers.

In this section, we survey the recent developments about ordered Ramsey numbers with focus on the graph Ramsey theory. We start by mentioning general bounds on ordered Ramsey numbers. Then, we will focus on specific classes of ordered graphs such as ordered matchings, paths, or cycles, where we can prove much more precise estimates and, in many cases, even the exact formulas for the ordered Ramsey numbers. We conclude this section by mentioning several open problems about ordered Ramsey numbers and directions for future research in this relatively new field.

1.2.1 *General bounds*

Since $\overline{R}(\mathcal{G}) \leq R(K_n)$ for every ordered graph $\mathcal{G}$ on n vertices, we see from (1.1) that $\overline{R}(\mathcal{G})$ grows at most exponentially in n. For dense ordered graphs, a standard probabilistic argument shows that this is asymptotically tight. However, the ordered Ramsey numbers may differ substantially from the usual Ramsey numbers for sparser graphs. This result was proved independently by Balko, Cibulka, Král, and Kynčl [Bal+20] and by Conlon, Fox, Lee, and Sudakov [Con+17] who showed that there are ordered matchings with superpolynomial ordered Ramsey numbers. Here, a *matching* is a graph with maximum degree 1.

Theorem 1.2.1 ([Bal+20, Con+17]). *There is a constant $C > 0$ such that for every even $n \geq 2$, there is an ordered matching $\mathcal{M}_n$ on n vertices satisfying*

$$\overline{R}(\mathcal{M}_n) \geq n^{C \log n / \log\log n}.$$

Note that this result is in sharp contrast with Theorem 1.1.3. For ordered matchings this is a rather typical behavior, as Conlon, Fox, Lee, and Sudakov [Con+17] proved that this superpolynomial lower bound holds for almost all ordered matchings.

Given the lower bound from Theorem 1.2.1, it is natural to ask how fast can ordered Ramsey numbers grow for ordered graphs with bounded maximum degree. Balko, Cibulka, Král, and Kynčl [Bal+20] showed that if we additionally bound the following analogue of the chromatic number, then we obtain polynomial upper bounds on ordered Ramsey numbers.

A subset I of vertices of an ordered graph $\mathcal{G} = (G, \prec)$ is an *interval* if for every pair u, v of vertices of I with $u \prec v$, every vertex w of G satisfying $u \prec w \prec v$ is contained in I. The *interval chromatic number* of $\mathcal{G}$ is the minimum number of intervals the vertex set of $\mathcal{G}$ can be partitioned into so that there is no edge between vertices of the same interval. We note that there is a variant of the *Erdős–Stone–Simonovits theorem* for ordered graphs proved by Pach and Tardos [PT06], which is expressed in terms of the interval chromatic number. For a positive integer d, a graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d.

Theorem 1.2.2 ([Bal+20]). *There is a constant $C > 0$ such that every ordered d-degenerate graph $\mathcal{G}$ on n vertices with interval chromatic number χ satisfies*

$$\overline{R}(\mathcal{G}) \leq n^{Cd^{\lceil \log \chi \rceil}}.$$

Conlon, Fox, Lee, and Sudakov [Con+17] independently proved the following stronger upper bound. For positive integers $t, n_1, \ldots, n_t$, let $\mathcal{K}_{n_1,\ldots,n_t}$ be an ordering of the complete t-partite graph $K_{n_1,\ldots,n_t}$ in which the vertices of the color class of size n_i form the ith interval. If $n_1 = \cdots = n_t = n$, we simply write $\mathcal{K}_t(n)$ instead of $\mathcal{K}_{n_1,\ldots,n_t}$.

Theorem 1.2.3 ([Con+17]). *Let $\mathcal{G}$ be an ordered d-degenerate graph on n vertices with maximum degree Δ. For positive integers n' and χ, let $s = \lceil \log \chi \rceil$ and $D = 8\chi^2 n'$. Then*

$$\overline{R}(\mathcal{G}, \mathcal{K}_\chi(n')) \leq 2^{s^2 d + s} \Delta^s n^s D^{ds+1}.$$

In particular, if $\mathcal{G}$ is an ordered d-degenerate graph with n vertices and with interval chromatic number χ, then Theorem 1.2.3 implies

$$\overline{R}(\mathcal{G}) \leq n^{32d \log \chi}, \tag{1.7}$$

which is a stronger upper bound than the one from Theorem 1.2.2. Also note that for $d = 1$ this bound almost matches the lower bound from Theorem 1.2.1.

Using a result of Erdős and Szemerédi [ES72], Conlon, Fox, Lee, and Sudakov [Con+17] derived the following result from the proof of Theorem 1.2.2, which shows that the ordered Ramsey numbers behave more like the usual Ramsey number for denser ordered graphs.

Theorem 1.2.4 ([Con+17])**.** *There is a constant $C > 0$ such that every ordered d-degenerate graph $\mathcal{G}$ on n vertices satisfies*

$$\overline{R}(\mathcal{G}) \leq 2^{Cd\log^2(2n/d)}.$$

This result is close to sharp for very small d by Theorem 1.2.1 and for very large d by (1.1).

Besides the interval chromatic number, another natural parameter for ordered graphs is their bandwidth. For an ordered graph $\mathcal{G} = (G, \prec)$, the *bandwidth* of $\mathcal{G}$ is the length of the longest edge in $\mathcal{G}$. That is, it is the maximum from $|i - j|$ taken over all edges $\{u, v\}$ of $\mathcal{G}$, where i is the position of u and j is the position of v in $\prec$. We call the number $|i - j|$ the *length* of the edge $\{u, v\}$.

Conlon, Fox, Lee, and Sudakov [Con+17] proved that, for every positive integer k, every ordered matching $\mathcal{M}$ on n vertices with bandwidth at most k satisfies $\overline{R}(\mathcal{M}) \leq n^{\lceil \log k \rceil + 2}$. They also asked whether this result can be extended by proving a polynomial upper bound on ordered Ramsey numbers of all ordered graphs with bounded bandwidth. This problem was solved by Balko, Cibulka, Král, and Kynčl [Bal+20] who proved the following result.

Theorem 1.2.5 ([Bal+20])**.** *For every positive integer k, there is a constant $C = C(k)$ such that every n-vertex ordered graph $\mathcal{G}$ with bandwidth k satisfies*

$$\overline{R}(\mathcal{G}) \leq C \cdot n^{128k}.$$

Observe that every n-vertex ordered graph $\mathcal{G}$ with bandwidth at most k is an ordered subgraph of the n-vertex ordered graph $\mathcal{P}_n^k$ that contains all edges of length at most k. In particular, $\overline{R}(\mathcal{G}) \leq \overline{R}(\mathcal{P}_n^k)$. Note that $\mathcal{P}_n^1 = \mathcal{P}_n$ and thus $\overline{R}(\mathcal{P}_n^1) = (n-1)^2 + 1$ by (1.6). Mubayi [Mub17] improved the upper bound on $\overline{R}(\mathcal{P}_n^k)$ in the case $k = 2$ by showing

$$\overline{R}(\mathcal{P}_n^2) \leq O(n^{19.487})$$

for every $n \geq 2$. He also used this result to determine the correct tower growth rate of the k-uniform hypergraph Ramsey number of a $(k+1)$-clique versus a monotone k-uniform path; see Section 1.3 for the definitions.

1.2.2 *Ordered matchings*

For every ordered matching $\mathcal{M}$ on n vertices, Conlon, Fox, Lee, and Sudakov [Con+17] proved the upper bound $n^{\lceil \log n \rceil}$. A simple argument shows that if an ordered matching $\mathcal{M}$ has the interval chromatic number 2, then this bound can be significantly improved to $O(n^2)$. This is quite close to the truth, as Conlon, Fox, Lee, and Sudakov [Con+17] constructed an ordered matching $\mathcal{M}$ on n vertices with interval chromatic number 2 such that $\overline{R}(\mathcal{M}) \geq \frac{cn^2}{\log^2(n)\log\log n}$ for some constant $c > 0$. This was improved by Balko, Jelínek, and Valtr [BJV19], who proved a stronger bound, which additionally holds for almost all ordered matchings with interval chromatic number 2. To state this result, we need to introduce some definitions first.

For a positive integer n, the *random n-permutation* is a permutation of the set $[n]$ chosen independently uniformly at random from the set of all $n!$ permutations of the set $[n]$. For a positive integer n and the random n-permutation π, the *random ordered n-matching* $\mathcal{M}(\pi)$ is the ordered matching with the vertex set $[2n]$ and with edges $\{i, n+\pi(i)\}$ for every $i \in [n]$. Note that the interval chromatic number of every random ordered n-matching is 2. The random ordered n-matching satisfies an event A *asymptotically almost surely* if the probability that A holds tends to 1 as n goes to infinity.

Theorem 1.2.6 ([BJV19]). *There is a constant $C > 0$ such that the random ordered n-matching $\mathcal{M}(\pi)$ asymptotically almost surely satisfies*

$$\overline{R}(\mathcal{M}(\pi)) \geq C \cdot \left(\frac{n}{\log n}\right)^2.$$

For the non-diagonal ordered Ramsey numbers, Conlon, Fox, Lee, and Sudakov [Con+17] investigated the ordered Ramsey numbers $\overline{R}(\mathcal{M}, \mathcal{K}_3)$, where $\mathcal{M}$ is an ordered matching on n vertices. It follows from the well-known bound $R(K_n, K_3) \leq O(n^2/\log n)$ by Ajtai, Komlós, and Szemerédi [AKS80] that

$$\overline{R}(\mathcal{M}, \mathcal{K}_3) \leq O\left(\frac{n^2}{\log n}\right).$$

For $R(K_n, K_3)$, this bound is tight as shown by Kim [Kim95], but Conlon, Fox, Lee, and Sudakov [Con+17] expect that this upper bound is far from optimal for $\overline{R}(\mathcal{M}, \mathcal{K}_3)$. They constructed an ordered matching $\mathcal{M}$ on n vertices satisfying

$$\overline{R}(\mathcal{M}, \mathcal{K}_3) \geq O\left(\left(\frac{n}{\log n}\right)^{4/3}\right)$$

and posed the following problem.

Problem 1.2.7 ([Con+17]). *Does there exist an $\varepsilon > 0$ such that every ordered matching $\mathcal{M}$ on n vertices satisfies $\overline{R}(\mathcal{M}, \mathcal{K}_3) \leq O(n^{2-\varepsilon})$?*

Problem 1.2.7 is still open and seems to be difficult, but there has been some partial progress. Rohatgi [Roh19] proved that almost every ordered matching $\mathcal{M}$ on n vertices with interval chromatic number 2 satisfies $\overline{R}(\mathcal{M},\mathcal{K}_3) \leq O(n^{24/13})$. He also showed that if $\mathcal{M}$ is a non-crossing ordered matching, then, for every $\varepsilon > 0$, we have $\overline{R}(\mathcal{M},\mathcal{K}_3) \leq O_\varepsilon(n^{1+\varepsilon})$. Here, an ordered graph $\mathcal{G} = (G, \prec)$ is *non-crossing* if it does not contain two edges $\{u, v\}$ and $\{x, y\}$ with $u \prec x \prec v \prec y$.

A basic building block in the proof of the latter result by Rohatgi [Roh19] is based on so-called nested matchings. An ordered matching $\mathcal{M} = (M, \prec)$ on vertices $v_1 \prec \cdots \prec v_n$ is *nested* if n is even and $\mathcal{M}$ has edges $\{v_i, v_{n-i+1}\}$ for every $i = 1, \ldots, n/2$. We use $\mathcal{NM}_k$ to denote the nested matching on $2k$ vertices.

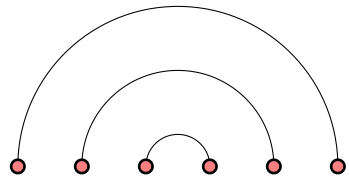

Figure 1.2: The nested matching $\mathcal{NM}_k$ for $k = 3$.

Rohatgi [Roh19] proved

$$4k - 1 \leq \overline{R}(\mathcal{NM}_k, \mathcal{K}_3) \leq 6k.$$

for every positive integer k. He believed that the lower bound is tight and posed the following conjecture.

Conjecture 1.2.8 ([Roh19]). *For every positive integer k, we have*

$$\overline{R}(\mathcal{NM}_k, \mathcal{K}_3) = 4k - 1.$$

Conjecture 1.2.8 is true for $k \leq 3$, but Balko and Poljak [BP21] disproved it for any $k \geq 4$ by showing the following bounds.

Theorem 1.2.9 ([BP21]). *For every positive integer k, we have*

$$\overline{R}(\mathcal{NM}_k, \mathcal{K}_3) \leq (3 + \sqrt{5})k < 5.3k.$$

If $k \geq 6$, we have

$$\overline{R}(\mathcal{NM}_k, \mathcal{K}_3) \geq 4k + 1.$$

Moreover, $\overline{R}(\mathcal{NM}_4, \mathcal{K}_3) = 16$ and $\overline{R}(\mathcal{NM}_5, \mathcal{K}_3) = 20$.

Using the lower bounds from Theorem 1.2.9, Balko and Poljak improved the best-known bounds on the maximum chromatic number of so-called *k-queue graphs*, which addresses a problem posed by Dujmovič and Wood [DW04].

1.2.3 *Ordered stars*

For a positive integer n, a *star* with n vertices is the complete bipartite graph $K_{1,n-1}$. Ramsey numbers of unordered stars are known exactly [BR73] and they are given by

$$R(K_{1,n-1};c) = \begin{cases} c(n-2)+1 & \text{if } c \equiv n-1 \equiv 0 \pmod 2, \\ c(n-2)+2 & \text{otherwise.} \end{cases}$$

The position of the central vertex of an ordered star determines the ordering of a star uniquely up to isomorphism. Thus, we use $\mathcal{S}_{r,s}$ to denote the ordered star with $r-1$ vertices to the left and $s-1$ vertices to the right of the central vertex; see Figure 1.3.

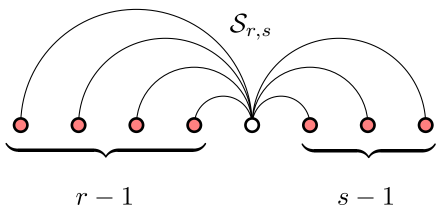

Figure 1.3: The ordered star $\mathcal{S}_{r,s}$ with $r=5$ and $s=4$.

Ordered stars are one of the very few classes of ordered graphs for which we know ordered Ramsey numbers exactly. Choudum and Ponnusamy [CP02] determined the ordered Ramsey numbers of all pairs of ordered stars by the following recursive formulas.

Theorem 1.2.10 ([CP02])**.** *For all integers $r_1, r_2 > 2$, we have*

$$\overline{R}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) = \left\lfloor \frac{-1+\sqrt{1+8(r_1-2)(r_2-2)}}{2} \right\rfloor + r_1 + r_2 - 2.$$

Moreover, for all integers $r_1, r_2, s_1, s_2 \geq 2$, we have

$$\overline{R}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,s_2}) = \overline{R}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) + r_1 + s_2 - 3$$

and

$$\overline{R}(\mathcal{S}_{r_1,s_1}, \mathcal{S}_{r_2,s_2}) = \overline{R}(\mathcal{S}_{r_1,1}, \mathcal{S}_{r_2,s_2}) + \overline{R}(\mathcal{S}_{1,s_1}, \mathcal{S}_{r_2,s_2}) - 1.$$

When we extend the definition of ordered Ramsey numbers to more than two colors, Balko, Cibulka, Král, and Kynčl [Bal+20] showed that ordered Ramsey numbers of all ordered stars are linear with respect to the number of vertices and at most exponential with respect to the number of colors.

1.2.4 *Ordered paths*

The Ramsey numbers of unordered paths are known exactly for a long time. The exact values were determined by Gerencsér and Gyárfás [GG67] who proved that, for $2 \leq r \leq s$, we have

$$R(P_r, P_s) = s + \left\lfloor \frac{r}{2} \right\rfloor - 1.$$

For general ordered paths, Cibulka et al. [Cib+15] showed that, for every ordered path $\mathcal{P}_r$ and every s, we have

$$\overline{R}(\mathcal{P}_r, \mathcal{K}_s) \leq 2^{\lceil \log s \rceil (\lceil \log r \rceil + 1)}.$$

In particular, every ordered path $\mathcal{P}_n$ satisfies $\overline{R}(\mathcal{P}_n) \leq n^{O(\log n)}$. This bound also follows from (1.7). It follows the remark after Theorem 1.2.1 that the ordered Ramsey numbers of ordered paths are typically superpolynomial.

The exact values of ordered Ramsey numbers of general ordered paths are not known, but there are some specific orderings for which we known the ordered Ramsey numbers exactly or we at least have very close bounds.

The monotone paths are perhaps the most natural due to their connection to the Erdős–Szekeres lemma (Theorem 1.1.1). Recall that the *monotone path* $\mathcal{P}_n$ on n vertices is an ordering of the path P_n where edges connect consecutive vertices in the vertex order. Choudum and Ponnusamy [CP02] and independently Milans, Stolee, and West [MSW15] proved the formula

$$\overline{R}(\mathcal{P}_{n_1}, \mathcal{P}_{n_2}) = (n_1 - 1)(n_2 - 1) + 1$$

and even extended this result to an arbitrary number of colors. In particular, it follows that the ordered Ramsey number $\overline{R}(\mathcal{P}_n)$ is quadratic in n.

On the other hand, there are some orderings of the path P_n for which the diagonal ordered Ramsey numbers are only linear in n. The *alternating path* $\mathcal{P}_n^{alt} = (P_n, \prec)$ is the ordering of P_n on vertices $v_1 \prec \cdots \prec v_n$ with edges $\{v_i, v_j\}$ for $i + j \in \{n+1, n+2\}$; see Figure 1.4. Note that the alternating path $\mathcal{P}_n^{alt}$ is an ordered subgraph of $\mathcal{K}_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$, and so it has interval chromatic number 2.

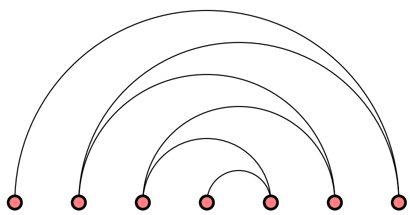

Figure 1.4: The alternating path $\mathcal{P}_n^{alt}$ for $n = 7$.

Using a result from the extremal theory of $\{0,1\}$-matrices, Balko, Cibulka, Král, and Kynčl [Bal+20] proved the following linear bounds on $\overline{R}(\mathcal{P}_n^{alt})$.

Proposition 1.2.11 ([Bal+20]). *For every integer $n > 2$, we have*

$$5\lfloor n/2 \rfloor - 4 \leq \overline{R}(\mathcal{P}_n^{alt}) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}.$$

The proof of the lower bound was later extended by Neidinger and West [NW19] to obtain various linear lower bounds for various classes of ordered graphs with interval chromatic number 2.

The precise multiplicative factor in $\overline{R}(\mathcal{P}_n^{alt})$ is unknown, but the computer experiments by Balko, Cibulka, Král, and Kynčl [Bal+20] indicate that $\overline{R}(\mathcal{P}_n^{alt})$ could be equal to $\lfloor (n-2)\frac{1+\sqrt{5}}{2} \rfloor + n$; see Table 1.1.

n	2	3	4	5	6	7	8	9	10	11	12
$\overline{R}(n)$	2	4	7	9	12	15	17	≥ 20	≥ 22	≥ 25	≥ 28

Table 1.1: Estimates and precise values of the ordered Ramsey numbers $\overline{R}(n) = \overline{R}(\mathcal{P}_n^{alt})$ for $n \leq 12$.

Balko, Jelínek, and Valtr [BJV19] also proved the following Turán-type result for alternating paths.

Proposition 1.2.12 ([BJV19]). *Let $\varepsilon > 0$ be a real constant. Then, for every integer n, every ordered graph on $N \geq n/\varepsilon$ vertices with at least εN^2 edges contains $\mathcal{P}_n^{alt}$ as an ordered subgraph.*

The alternating paths form an interesting class of path orderings as their diagonal ordered Ramsey numbers seem to be minimal among all path orderings; see Problem 1.2.25 in Subsection 1.2.7.

1.2.5 *Ordered cycles*

As the last class of specific ordered graphs, we mention ordered cycles. Again, except of some specific orderings, we do not know any precise formulas for ordered Ramsey numbers of ordered cycles. However, we can determine the ordered Ramsey numbers for a particularly natural ordering of the cycle C_n, called the *monotone cycle* $\mathcal{C}_n$, which is obtained from the monotone path $\mathcal{P}_n$ by adding an edge between the first and the last vertex in the vertex ordering; see Figure 1.5.

The precise formula for ordered cycles covers even in the non-diagonal case and is given by the following result by Balko, Cibulka, Král, and Kynčl [Bal+20].

Theorem 1.2.13 ([Bal+20]). *For all integers $r \geq 2$ and $s \geq 2$, we have*

$$\overline{R}(\mathcal{C}_r, \mathcal{C}_s) = 2rs - 3r - 3s + 6.$$

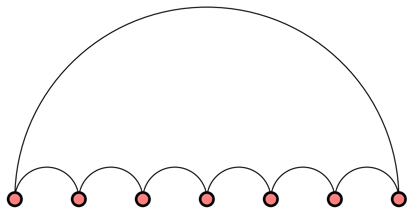

Figure 1.5: The monotone cycle $\mathcal{C}_n$ for $n = 7$.

Theorem 1.2.13 has an application in discrete geometry. The *geometric Ramsey numbers* [Cib+15, KPT97, Kár+98] are natural analogues of ordered Ramsey numbers. For a finite set of points $P \subset \mathbb{R}^2$ in general position, let K_P be the *complete geometric graph on* P, which is a complete graph drawn in the plane so that its vertices are represented by the points in P and the edges are drawn as straight-line segments between the pairs of points in P. The graph K_P is *convex* if P is in convex position. The *geometric Ramsey number* of a graph G, denoted by $Rg(G)$, is the smallest N such that every complete geometric graph K_P on N vertices with edges colored by two colors contains a non-crossing monochromatic drawing of G. If we consider only convex complete geometric graphs K_P in the definition, then we get so-called *convex geometric Ramsey number* $Rc(G)$. Note that these numbers are finite only if G is outerplanar and that $Rc(G) \leq Rg(G)$ for every outerplanar graph G.

Balko, Cibulka, Král, and Kynčl [Bal+20] observed that the geometric and convex geometric Ramsey numbers of cycles are equal to the ordered Ramsey numbers of monotone cycles.

Corollary 1.2.14 ([Bal+20]). *For every integer $n \geq 3$, we have $Rc(C_n) = Rg(C_n) = 2n^2 - 6n + 6$.*

We also note that Ramsey numbers of unordered cycles are known exactly by results of Rosta [Ros73] and Faudree and Schelp [FS74] who extended earlier works by Chartrand and Chuster [CS71] and by Bondy and Erdős [BE73]. Together, these results give

$$R(C_r, C_s) = \begin{cases} 2r - 1 & \text{if } (r,s) \neq (3,3), r \geq s \geq 3, \\ & s \text{ is odd}, \\ r + s/2 - 1 & \text{if } (r,s) \neq (4,4), r \geq s \geq 4, \\ & r, s \text{ are even}, \\ \max\{r + s/2, 2s\} - 1 & \text{if } r > s \geq 4, \\ & s \text{ is even and } r \text{ is odd}. \end{cases}$$

1.2.6 *Minimum ordered Ramsey numbers*

Conlon, Fox, Lee, and Sudakov [Con+17] characterized graphs G for which the ordered Ramsey number $\overline{R}(\mathcal{G})$ is linear in the number of vertices of G for every ordering $\mathcal{G}$ of G. These are precisely graphs G whose edges can be covered by a constant number vertices. A similar problem is to determine graphs that admit an ordering for which the corresponding ordered Ramsey number is linear. This motivates the following definition.

For an unordered graph G, the *minimum ordered Ramsey number of G* is defined as

$$\min \overline{R}(G) = \min\{\overline{R}(\mathcal{G}) : \mathcal{G} \text{ is an ordering of } G\}.$$

Since Ramsey numbers of bounded-degree graphs are linear in the number of vertices by Theorem 1.1.3, it is natural to ask whether the minimum ordered Ramsey numbers of bounded-degree graphs are always at most linear. Conlon, Fox, Lee, and Sudakov [Con+17] considered this unlikely for random regular graphs and posed the following problem.

Problem 1.2.15 ([Con+17])**.** *Do random 3-regular graphs have superlinear ordered Ramsey numbers for all orderings?*

Balko, Jelínek, and Valtr [BJV19] gave an affirmative answer to Problem 1.2.15. In fact, they solved the problem in a slightly more general setting, by extending the concept of d-regular graphs to non-integral values of d.

For a real number $\rho > 0$ and a positive integer n with $\lceil \rho n \rceil$ even, a graph G on n vertices is *ρ-regular*, if every vertex of G has degree $\lfloor \rho \rfloor$ or $\lceil \rho \rceil$ and the total number of edges of G is $\lceil \rho n \rceil /2$. Note that this definition coincides exactly with the standard notion of ρ-regular graphs when ρ is an integer. We let $G(\rho, n)$ denote the random ρ-regular graph on n vertices drawn uniformly and independently from the set of all ρ-regular graphs on the vertex set $[n]$.

Theorem 1.2.16 ([BJV19])**.** *The following two statements are true.*

(a) For every fixed real number $\rho > 2$, asymptotically almost surely

$$\min \overline{R}(G(\rho, n)) \geq \frac{n^{3/2-1/\rho}}{4 \log n \log \log n}.$$

In particular, almost every 3-regular graph G on n vertices satisfies

$$\min \overline{R}(G) \geq n^{7/6}/(4 \log n \log \log n),$$

(b) Asymptotically almost surely,

$$\min \overline{R}\left(G\left(2 + \frac{9 \log \log n}{\log n}, n\right)\right) \geq \frac{n \log n}{2 \log \log n}.$$

Part (a) of Theorem 1.2.16 shows that random ρ-regular graphs have superlinear minimum ordered numbers for any fixed real number $\rho > 2$. Part (b) shows that there are actually "almost 2-regular" graphs with superlinear minimum ordered Ramsey numbers.

Note that the minimum ordered Ramsey numbers of ordered matchings are linear. This can be seen by considering the nested matchings $\mathcal{NM}_k$ and by applying a simple pigeonhole-type argument to obtain $\overline{R}(\mathcal{NM}_k) \leq 4k - 2$. Thus, it remained to decide whether 2-regular graphs always admit an orderings with linear ordered Ramsey numbers. Balko, Jelínek, and Valtr [BJV19] showed that this is indeed the case by proving the following result.

Theorem 1.2.17 ([BJV19])**.** *There is a constant C such that for every graph G on n vertices with maximum degree 2, we have*

$$\min \overline{R}(G) \leq Cn.$$

In fact, Balko, Jelínek, and Valtr [BJV19] proved the following stronger Turán-type statement for bipartite graphs.

Theorem 1.2.18 ([BJV19])**.** *For every real $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that, for every integer n, every bipartite graph G on n vertices with maximum degree 2 admits an ordering $\mathcal{G}$ of G that is contained in every ordered graph with $N = C(\varepsilon)n$ vertices and with at least εN^2 edges.*

Note that no such Turán-type statement is true for a 2-regular graph G that is not bipartite, since then G contains an odd cycle and thus no ordering of such a graph is contained in any ordering of the complete bipartite graph $K_{N/2,N/2}$ with $N^2/4$ edges.

For the upper bounds in the case of a larger maximum degree, a simple corollary of Theorem 1.2.3 states that every graph G on n vertices with constant maximum degree Δ admits an ordering $\mathcal{G}$ with $\overline{R}(\mathcal{G})$ polynomial in n. More precisely, every graph G with n vertices and with maximum degree Δ satisfies

$$\min \overline{R}(G) \leq O(n^{(\Delta+1)\lceil \log(\Delta+1) \rceil + 1}). \tag{1.8}$$

1.2.7 *Open problems*

Since the study of ordered Ramsey numbers is a relatively new topic in Ramsey theory, there are many interesting and difficult new open problems. Here, we would like to draw attention to some of them.

By Theorem 1.2.1, there are ordered matchings $\mathcal{M}$ on n vertices such that $\overline{R}(\mathcal{M}) \geq n^{\Omega(\log n / \log \log n)}$. On the other hand, it follows from (1.7) that every n-vertex ordered matching $\mathcal{M}$ satisfies $\overline{R}(\mathcal{M}) \geq n^{O(\log n)}$. Although the bounds are quite close, there is still a small gap in the exponent. Conlon, Fox, Lee, and Sudakov [Con+17] thus posed the following problem.

Problem 1.2.19 ([Con+17]). *Close the gap between the upper and lower bounds for ordered Ramsey numbers of matchings.*

They also posed a similar problem for ordered matchings with interval chromatic number 2, where the best-known bounds are between $\Omega((n/logn)^2)$ and $O(n^2)$.

Problem 1.2.20 ([Con+17]). *Close the gap between the upper and lower bounds for ordered Ramsey numbers of matchings with interval chromatic number 2.*

Geneson et al. [Gen+19] posed a similar problem for ordered paths with interval chromatic number 2. Here, it follows from a modified proof of Theorem 1.2.2 that the ordered Ramsey number $\overline{R}(\mathcal{P})$ of any ordered path $\mathcal{P}$ on n vertices with interval chromatic number 2 is at most $O(n^3)$ while the best known lower bound $\Omega((n/logn)^2)$ follows from Theorem 1.2.6.

Problem 1.2.21 ([Gen+19]). *Is it true that $\overline{R}(\mathcal{P}) \leq O(n^2)$ for every ordering $\mathcal{P}$ of the path on n vertices with interval chromatic 2?*

It follows from (1.7) (or from Theorem 1.2.2) that bounded-degree ordered graphs with bounded interval chromatic number have polynomial ordered Ramsey numbers. However, we have no non-trivial lower bounds for this case and thus Balko, Cibulka, Král, and Kynčl [Bal+20] stated the following problem.

Problem 1.2.22 ([Bal+20]). *Is there a constant $c > 0$ such that for every fixed Δ there is a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of ordered Δ-regular graphs $\mathcal{G}_n$ with n vertices and interval chromatic number 2 such that $\overline{R}(\mathcal{G}_n) \geq n^{c\Delta}$?*

By Theorem 1.2.5, ordered graphs of bounded bandwidth have polynomial ordered Ramsey numbers. Again, we do not have any nontrivial lower bounds in this case. Since every such ordered graph on n vertices is an ordered subgraph of the ordered graph $\mathcal{P}_n^k$, one might consider only the ordered graphs $\mathcal{P}_n^k$ for the lower bounds. The currently best lower bound is only quadratic and follows from (1.6), which is the case of $\mathcal{P}_n^k$ for $k = 1$.

Problem 1.2.23 ([Bal+20]). *For an integer $k \geq 2$, what is the growth rate of $\overline{R}(\mathcal{P}_n^k)$ with respect to n?*

When addressing Problem 1.2.7, Rohatgi [Roh19] considered a variant of this problem for bounded interval chromatic number and posed the following interesting conjecture.

Conjecture 1.2.24 ([Roh19]). *For a positive integer χ, there is a constant $\varepsilon(\chi) >$ such that*

$$\overline{R}(\mathcal{M}, \mathcal{K}_3) \leq O(n^{2-\varepsilon(\chi)})$$

for every ordered matching $\mathcal{M}$ on n vertices with interval chromatic number χ.

Balko, Cibulka, Král, and Kynčl [Bal+20] proved the ordered Ramsey number of the alternating paths are linear with respect to the number of vertices. They also asked whether these orderings minimize ordered Ramsey numbers of ordered paths.

Problem 1.2.25 ([Bal+20])**.** *For some positive integer n, is there an ordering $\mathcal{P}$ of the path P_n on n vertices such that $\overline{R}(\mathcal{P}) < \overline{R}(\mathcal{P}_n^{alt})$?*

The computer experiments performed by Balko, Cibulka, Král, and Kynčl [Bal+20] also suggested a possible formula for $\overline{R}(\mathcal{P}_n^{alt})$. Here, we state this as an open problem.

Problem 1.2.26 ([Bal+20])**.** *For every integer $n \geq 2$, is it true that*

$$\overline{R}(\mathcal{P}_n^{alt}) = \left\lfloor (n-2)\frac{1+\sqrt{5}}{2} \right\rfloor + n?$$

Finally, there is the question about determining the minimum ordered Ramsey numbers of bounded-degree graphs. The following problem is mentioned by Balko, Jelínek, and Valtr [BJV19].

Problem 1.2.27 ([BJV19])**.** *Close the gap between the upper and lower bounds for minimum ordered Ramsey numbers of 3-regular graphs.*

The currently best known bounds on minimum ordered Ramsey numbers of 3-regular graphs are of order $\Omega(n^{7/6}/(\log n \log\log n))$ and $O(n^{4\lceil \log(4) \rceil + 1}) = O(n^9)$ by Theorem 1.2.16 and by (1.8), respectively. Note that the gap is rather large.

1.3 GENERALIZED ORDERED RAMSEY NUMBERS

This section is devoted to generalizations and different variants of the graph ordered Ramsey numbers. First, we extend the notion of ordered Ramsey numbers to k-uniform hypergraphs and we survey some of the known results. The hypergraph ordered Ramsey numbers were mostly studied for so-called monotone paths because of their connections to the Erdős–Szekeres theorem (Theorem 1.1.5). We explain these connections and we discuss some results about the ordered Ramsey numbers of monotone paths for restricted colorings and their applications in discrete geometry. Finally, we survey the current state of knowledge about the newly introduced variant of Ramsey numbers for graphs with ordered edge sets. At the end of this section, we again list some open problems.

1.3.1 *k-uniform hypergraphs*

We use $K_n^{(k)}$ to denote the complete k-uniform hypergraph on n vertices, that is, the k-uniform hypergraph with $|V| = n$ and $E = \binom{V}{k}$. For every integer $k \geq 2$, the ordered Ramsey numbers have their natural analogue for k-uniform hypergraphs. The *ordered k-uniform hypergraph* is a pair $\mathcal{H} = (H, \prec)$ consisting of a k-uniform hypergraph H and a linear ordering $\prec$ of its vertex set. The notions of an *ordered subhypergraph* and *isomorphism* of ordered k-uniform hypergraphs are analogous to their graph counterparts. Again, there is a unique ordered complete k-uniform hypergraph on n vertices up to isomorphism and we denote it by $\mathcal{K}_n^{(k)}$.

The *ordered Ramsey number* $\overline{R}(\mathcal{H}, \mathcal{G})$ of two ordered k-uniform hypergraphs $\mathcal{H}$ and $\mathcal{G}$ is the smallest $N \in \mathbb{N}$ such that every red-blue coloring of the hyperedges of $\mathcal{K}_N^{(k)}$ contains a blue ordered subhypergraph isomorphic to $\mathcal{H}$ or a red ordered subhypergraph isomorphic to $\mathcal{G}$. In the *diagonal case* $\mathcal{H} = \mathcal{G}$, we just write $\overline{R}(\mathcal{H})$ instead of $\overline{R}(\mathcal{H}, \mathcal{H})$.

Similarly as for ordered graphs, the ordered Ramsey number of every n-vertex k-uniform hypergraph is bounded from above by $R(K_n^{(k)})$. In particular, these numbers are always finite. It is also easy to see that the ordered Ramsey numbers of k-uniform hypergraphs grow at least as fast as the standard Ramsey numbers.

Actually, very little is known about ordered Ramsey numbers of ordered k-uniform hypergraphs with $k \geq 3$ as ordered Ramsey numbers have been studied mostly for ordered graphs only. Balko and Vizer [BV21] studied ordered Ramsey numbers of 3-uniform hypergraphs and obtained some of the first nontrivial estimates, so we primarily focus on the case $k = 3$.

A simple probabilistic argument provides the lower bound $\overline{R}(\mathcal{H}) \geq 2^{\Omega(n^2)}$ for every ordered 3-uniform hypergraph with n vertices and

$\Omega(n^3)$ hyperedges, which is of the same asymptotic growth rate as we have for $\overline{R}(\mathcal{K}_n^{(3)})$ by (1.2). Thus, we consider mostly sparse 3-uniform hypergraphs.

The *degree* of a vertex v in a hypergraph H is the number of hyperedges of H that contain v. It follows from a result by Moshkovitz and Shapira [MS14] that there are ordered 3-uniform hypergraphs $\mathcal{H}$ on n vertices with maximum degree 3 such that $\overline{R}(\mathcal{H}) \geq 2^{\Omega(n)}$.

Therefore, in order to obtain smaller upper bounds on the ordered Ramsey numbers, it is necessary to bound other parameter besides the maximum degree. A natural choice is the interval chromatic number, which is defined analogously as for ordered graphs; see Subsection 1.2.1. Recall that for ordered graphs, bounding both parameters indeed helps, as the ordered Ramsey number $\overline{R}(\mathcal{G})$ of every n-vertex ordered graph $\mathcal{G}$ with bounded maximum degree d and bounded interval chromatic number χ is at most polynomial in the number of vertices by Theorem 1.2.3, which actually gives the stronger estimate

$$\overline{R}(\mathcal{G}, \mathcal{K}_\chi(n)) \leq n^{32d \log \chi}. \tag{1.9}$$

A natural question is whether we can get similar bounds for ordered k-uniform hypergraphs with $k \geq 3$. For integers $k \geq 2$ and $\chi \geq k$, we use $K_\chi^{(k)}(n)$ to denote the *complete k-uniform χ-partite hypergraph*, that is, the vertex set of $K_\chi^{(k)}(n)$ is partitioned into χ sets of size n and every k-tuple with at most one vertex in each of these parts forms a hyperedge. Let $\mathcal{K}_\chi^{(k)}(n)$ be the ordering of $K_\chi^{(k)}(n)$ in which the color classes form consecutive intervals. Conlon, Fox, and Sudakov [CFS11] showed that, for all positive integers $\chi \geq 3$ and n,

$$R(K_\chi^{(3)}(n)) \leq 2^{2^{2R}n^2},$$

where $R = R(K_{\chi-1})$. Since every ordering of $K_\chi^{(3)}(\chi n)$ contains an ordered subhypergraph isomorphic to $\mathcal{K}_\chi^{(3)}(n)$ and every ordered 3-uniform hypergraph on n vertices with interval chromatic number χ is an ordered subhypergraph of $\mathcal{K}_\chi^{(3)}(n)$, we obtain the following bound.

Corollary 1.3.1 ([CFS11, BV21]). *For all positive integers $\chi \geq 3$ and n, every ordered 3-uniform hypergraph $\mathcal{H}$ on n vertices with interval chromatic number χ satisfies*

$$\overline{R}(\mathcal{H}) \leq 2^{2^{2R}\chi^2 n^2},$$

where $R = R(K_{\chi-1})$. In particular, if the interval chromatic number χ of $\mathcal{H}$ is fixed, we have

$$\overline{R}(\mathcal{H}) \leq 2^{O(n^2)}.$$

Note that the last bound is asymptotically tight for dense ordered hypergraphs with bounded interval chromatic number. Thus, Balko and Vizer [BV21] considered the sparse case in which we additionally bound the maximum degree. Since the situation for ordered hypergraphs seems to be more difficult than for ordered graphs, Balko and Vizer [BV21] focused on the first nontrivial case, which is for ordered 3-uniform hypergraphs with interval chromatic number 3. Then they obtained a better upper bound on $\overline{R}(\mathcal{H})$ than $2^{O(n^2)}$ by proving an estimate with a subquadratic exponent.

Theorem 1.3.2 ([BV21]). *Let $\mathcal{H}$ be an ordered 3-uniform hypergraph on t vertices with maximum degree d and let s be a positive integer. Then there are constants $C = C(d)$ and $c > 0$ such that*

$$\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(s)) \leq t \cdot 2^{C(s^{2-1/(1+cd^2)})}.$$

In particular, for $s = t = n$ and bounded d, we get the estimate

$$\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n)) \leq 2^{O(n^{2-1/(1+cd^2)})}. \tag{1.10}$$

The main idea of the proof of Theorem 1.3.2 is based on an embedding lemma from [CFS12], where the authors study Erdős–Hajnal-type theorems for 3-uniform tripartite hypergraphs. Theorem 1.3.2 immediately gives the following corollary.

Corollary 1.3.3 ([BV21]). *Let $\mathcal{H}$ be an ordered 3-uniform hypergraph on n vertices with maximum degree d and with interval chromatic number 3. Then there exists an $\varepsilon = \varepsilon(d) > 0$ such that*

$$\overline{R}(\mathcal{H}) \leq 2^{O(n^{2-\varepsilon})}.$$

The upper bound (1.10) is quite close to the truth, as even when $\mathcal{H}$ is fixed we get a superexponential lower bound on $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n))$, as shown by Fox and He [FH19] and independently by Balko and Vizer [BV21].

Theorem 1.3.4 ([FH19, BV21]). *For every $t \geq 3$ and every positive integer n, we have*

$$\overline{R}(\mathcal{K}_{t+1}^{(3)}, \mathcal{K}_3^{(3)}(n)) \geq 2^{\Omega(n \log n)}.$$

For ordered hypergraphs of uniformity $k > 3$, we recall that it follows from (1.3) that their ordered Ramsey numbers can grow as a tower of height $k-1$. By modifying a result of Conlon, Fox, and Sudakov [CFS10], Balko and Vizer [BV21] showed that we do not have a tower-type growth rate for $\overline{R}(\mathcal{H})$ once the uniformity and the interval chromatic number of $\mathcal{H}$ are bounded.

Proposition 1.3.5. *Let χ, k be integers with $\chi \geq k \geq 2$ and let $\mathcal{H}$ be an ordered k-uniform hypergraph on n vertices with interval chromatic number χ. Then there is a constant c such that*

$$\overline{R}(\mathcal{H}) \leq 2^{R^{\chi(\chi-1)}(c\chi n)^{\chi-1}},$$

where $R = R(K_\chi^{(k)})$. *In particular, if the uniformity* k *and the interval chromatic number* χ *of* $\mathcal{H}$ *are fixed, we have*

$$\overline{R}(\mathcal{H}) \leq 2^{O(n^{\chi-1})}.$$

Finally, we mention a very interesting connection between ordered Ramsey numbers and hypergraph Ramsey numbers observed by Conlon, Fox, Lee, and Sudakov [Con+17]. They showed that for any 3-uniform hypergraph H, there is a family of ordered graphs $\mathcal{S}_H$ such that the Ramsey number of H is bounded in terms of the ordered Ramsey number of the family $\mathcal{S}_H$. Here, the *ordered Ramsey number* $\overline{R}(\mathcal{F})$ *of a family* $\mathcal{F}$ of ordered graphs is the smallest positive integer N such that every 2-coloring of the edges of $\mathcal{K}_N$ contains a monochromatic ordered copy of some ordered graph from $\mathcal{F}$.

For an ordered graph $\mathcal{G}$ with the vertex set $[n]$, let $T(\mathcal{G})$ be a 3-uniform hypergraph on vertex set $[n+1]$ obtained by taking all triples whose first pair is an edge of $\mathcal{G}$. For a 3-uniform hypergraph H on $n+1$ vertices, we let $\mathcal{S}_H$ be the collection of ordered graphs $\mathcal{G}$ on $[n]$ such that H is a subhypergraph of $T(\mathcal{G})$. Conlon, Fox, Lee, and Sudakov [Con+17] then related upper bounds on Ramsey numbers of 3-uniform hypergraphs to ordered Ramsey numbers by proving the following result.

Theorem 1.3.6 ([Con+17]). *Every 3-uniform hypergraph* H *satisfies*

$$R(H) \leq 2^{\binom{\overline{R}(\mathcal{S}_H)}{2}} + 1.$$

Conlon, Fox, Lee, and Sudakov [Con+17] expect that the bound from Theorem 1.3.6 is close to sharp in many cases. For example, the choice $H = K_{n+1}^{(3)}$ satisfies $\mathcal{S}_H = \{\mathcal{K}_n\}$, for which Theorem 1.3.6 produces the double-exponential bound from (1.2), which is believed to be tight. However, Conlon, Fox, Lee, and Sudakov [Con+17] constructed some cases where the bound from Theorem 1.3.6 is far from the truth.

1.3.2 *Monotone paths*

We showed at the beginning of Section 1.2 that the Erdős–Szekeres lemma (Theorem 1.1.1) is a consequence of a stronger Ramsey statement about monotone paths. There is a similar connection between the Erdős–Szekeres theorem (Theorem 1.1.5) and ordered Ramsey numbers of 3-uniform hypergraphs.

For an integer $k \geq 2$, the *monotone k-uniform path* on n vertices, denoted by $\mathcal{P}_n^{(k)} = (P_n^{(k)}, \prec)$, is an ordered k-uniform n-vertex hypergraph with edges formed by k-tuples of consecutive vertices in $\prec$; see Figure 1.6. Note that the monotone path $\mathcal{P}_n$ corresponds to $\mathcal{P}_n^{(2)}$. The monotone paths are sometimes called *tight paths* in the literature.

We can again rather easily show that $ES(n) \leq \overline{R}(\mathcal{P}_n^{(3)})$. Consider a set P of points $p_1, \dots, p_N$ in general position in the plane, ordered

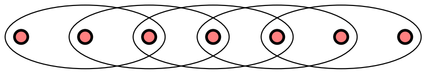

Figure 1.6: The monotone 3-uniform path $\mathcal{P}_n^{(3)}$ for $n = 7$.

according to their increasing x-coordinates. By rotating the plane if necessary, we may assume that no two points from P are on a vertical line. A set of n points from P forms an *n-cup* if its points lie on the graph of a convex function. If the points lie on the graph of a concave function, we call the set an *n-cap*; see Figure 1.7. Note that points of each n-cup are in convex position and the same is true for points of an n-cap. Consider a red-blue-coloring χ_P of the edges of $\mathcal{K}_N^{(3)}$ on $[N]$ where $\chi_P(\{i,j,k\})$ is red if $\{p_i, p_j, p_k\}$ forms a 3-cap and blue otherwise, that is, if $\{p_i, p_j, p_k\}$ forms a 3-cup. Then a sequence of n points from P forms an n-cap or an n-cup if and only if the corresponding vertices of $\mathcal{K}_N^{(3)}$ form a monochromatic copy of $\mathcal{P}_n^{(3)}$ in χ_P. Since n-caps and n-cups are in convex position, we obtain $ES(n) \leq \overline{R}(\mathcal{P}_n^{(3)})$.

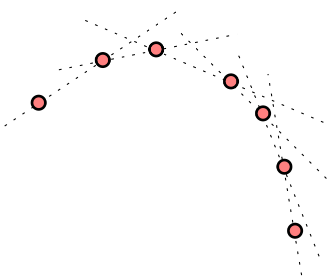

Figure 1.7: An example of (a) an n-cap and (b) an n-cup for $n = 7$.

Recall that Erdős and Szekeres [ES35] proved the bound $ES(n) \leq \binom{2n-4}{n-2} + 1$ for the Erdős–Szekeres theorem; see (1.4). This bound now follows from the fact

$$\overline{R}(\mathcal{P}_n^{(3)}) = \binom{2n-4}{n-2} + 1 \tag{1.11}$$

for every $n \geq 2$, which was proved by Moshkovitz and Shapira [MS14]. Moreover, several other interesting geometric applications of estimates on $\overline{R}(\mathcal{P}_n^{(k)})$ for $k \geq 3$ appeared, for example, variants of the Erdős–Szekeres Theorem for convex bodies [Fox+12] or the higher-order Erdős–Szekeres theorems [EM13].

Given this motivation, the ordered Ramsey numbers $\overline{R}(\mathcal{P}_n^{(k)})$ have been quite intensively studied [CP02, EM13, Fox+12, MSW15, MS14] and their growth rate is nowadays well understood. Moshkovitz and Shapira [MS14] showed that, for all positive integers n and $k \geq 3$,

$$\overline{R}(\mathcal{P}_{n+k-1}^{(k)}) = t_{k-1}((2 - o(1))n),$$

where $t_{k-1}(\cdot)$ is the tower function of height $k-2$; see Subsection 1.1.1 for its definition.

Note that the tower is of height one smaller than the tower-type upper bound on $R(K_n^{(k)})$ from (1.3) obtained with the Stepping-up lemma (Theorem 1.1.4). In fact, Moshkovitz and Shapira [MS14] proved

$$\overline{R}(\mathcal{P}^{(k)}_{n+k-1}) = \rho_k(n) + 1,$$

where $\rho_k(n)$ is the number of *line partitions of n of order k* (see [MS14] for definitions). For $k = 3$, this gives the exact formula $\overline{R}(\mathcal{P}_n^{(3)}) = \binom{2n-4}{n-2} + 1$.

Their coloring χ_P of $\mathcal{K}_N^{(3)} = (K_N^{(3)}, \prec)$ that gives $\overline{R}(\mathcal{P}_n^{(3)}) > \binom{2n-4}{n-2}$ satisfies the following *transitivity property*: if $v_1 \prec v_2 \prec v_3 \prec v_4$ are vertices of $\mathcal{K}_N^{(3)}$ such that $\chi_P(\{v_1, v_2, v_3\}) = \chi_P(\{v_2, v_3, v_4\})$, then all triples from $\binom{\{v_1,v_2,v_3,v_4\}}{3}$ have the same color in χ_P. More generally, for an integer $k \geq 2$, a 2-coloring χ of $\mathcal{K}_N^{(k)} = (K_N^{(k)}, \prec)$ is called *transitive* if for every $(k+1)$-tuple of vertices $\{v_1, \dots, v_{k+1}\}$ that satisfies $v_1 \prec \cdots \prec v_{k+1}$ and $\chi(\{v_1, \dots, v_k\}) = \chi(\{v_2, \dots, v_{k+1}\})$ it holds that all k-tuples from $\binom{\{v_1,\dots,v_{k+1}\}}{k}$ have the same color in χ.

Perhaps surprisingly, the colorings of $\mathcal{K}_N^{(k)}$, which were found by Moshkovitz and Shapira [MS14] and which give $\overline{R}(\mathcal{P}^{(k)}_{n+k-1}) > \rho_k(n)$, are not transitive for $k > 3$. Thus it is natural to ask the following question, which was also considered by Eliáš and Matoušek [EM13].

Problem 1.3.7 ([EM13, MS14])**.** *What is the growth rate of $\overline{R}(\mathcal{P}_n^{(k)})$ when restricted only to transitive colorings?*

Problem 1.3.7 was settled for $k \leq 4$ by a result of by Eliáš and Matoušek [EM13], who showed that the corresponding numbers grow at most as a tower of height $k-2$ and as $t_3(\Theta(n))$ for $k = 4$.

Balko [Bal19] settled Problem 1.3.7 by constructing, for all n and $k \geq 3$, transitive colorings χ_k of $\mathcal{K}_N^{(k)}$ with no monochromatic copy of $\mathcal{P}^{(k)}_{2n+k-1}$, where $N \geq t_{k-1}((1-o(1))n)$. In fact, the colorings χ_k satisfy so-called *monotonicity property*, which is much more restrictive than the transitivity property and which admits several geometric interpretations. Before stating this result, we first introduce the monotonicity property.

Let S be a sequence of n elements from some set. For $i \in [n]$, we use $S^{(i)}$ to denote the subsequence of S obtained by deleting the element from S that is at position i. For $k \geq 2$, a 2-coloring χ of $\mathcal{K}_N^{(k)} = (K_N^{(k)}, \prec)$ is a *k-monotone coloring* of $\mathcal{K}_N^{(k)}$ if it assigns -1 or $+1$ to every edge of $\mathcal{K}_N^{(k)}$ such that the following *monotonicity property* is satisfied: for every sequence S of $k+1$ vertices of $\mathcal{K}_N^{(k)}$ ordered by $\prec$ and all integers a, b, c with $1 \leq a < b < c \leq k+1$, we have $c(S^{(c)}) \leq c(S^{(b)}) \leq c(S^{(a)})$ or $c(S^{(c)}) \geq c(S^{(b)}) \geq c(S^{(a)})$. In other words, the monotonicity condition says that there is at most one change of a sign in the sequence $(c(S^{(k+1)}), \ldots, c(S^{(1)}))$.

Note that every k-monotone coloring of $\mathcal{K}_N^{(k)}$ is a transitive 2-coloring of $\mathcal{K}_N^{(k)}$. For $k = 2$, transitive and 2-monotone colorings coincide. However, for $k \geq 3$, the monotonicity property is much more restrictive.

The notion of monotone colorings has been considered by several researchers [FW01, Miy17, Zie93] under different names. In some sense, monotone colorings can be viewed as more natural than transitive colorings, as they admit various geometric interpretations [Bal19]. This includes *k-intersecting pseudoconfigurations of points* [Miy17], *C_d-arrangements of n pseudohyperplanes* in $\mathbb{R}^d$ [FW01], and extensions of the cyclic arrangement of hyperplanes with a pseudohyperplane [Zie93].

To state the estimate by Balko [Bal19], we introduce ordered Ramsey numbers restricted to k-monotone colorings. For an integer $k \geq 2$, the *monotone Ramsey number* $\overline{R}_{mon}(\mathcal{H})$ of an ordered k-uniform hypergraph $\mathcal{H}$ is the minimum positive integer N such that for every k-monotone coloring χ of $\mathcal{K}_N^{(k)}$ there is an ordered sub-hypergraph of $\mathcal{K}_N^{(k)}$ that is monochromatic in χ and isomorphic to $\mathcal{H}$.

Theorem 1.3.8 ([Bal19]). *For positive integers k and n with $k \geq 3$, we have*

$$\overline{R}_{mon}(\mathcal{P}^{(k)}_{2n+k-1}) \geq t_{k-1}((1-o(1))n).$$

Since every k-monotone coloring is transitive, Theorem 1.3.8 settles Problem 1.3.7. For $k \in \{3, 4\}$, the lower bounds from Theorem 1.3.8 asymptotically match the lower bounds obtained from results of Erdős and Szekeres [ES35] and Eliáš and Matoušek [EM13], respectively.

Despite having several natural geometric interpretations, the k-monotone colorings seem to be quite unexplored. The first non-trivial estimate on the number of k-monotone colorings of $\mathcal{K}_n^{(k)}$ for $k > 3$ was given by Balko [Bal19].

Theorem 1.3.9 ([Bal19]). *For integers $k \geq 3$ and $n \geq k$, the number $S_k(n)$ of k-monotone colorings of $\mathcal{K}_n^{(k)}$ satisfies*

$$2^{n^{k-1}/k^{4k}} \leq S_k(n) \leq 2^{2^{k-2}n^{k-1}/(k-1)!}.$$

Note that the bounds are reasonably close together, even with respect to k. It follows form one of the geometric interpretations of 3-monotone colorings that Theorem 1.3.9 is a generalization of the well-known fact that the number of simple arrangements of n pseudolines is $2^{\Theta(n^2)}$.

Finally, we mention a possible strengthening of the Erdős–Szekeres conjecture (Conjecture 1.1.6) introduced by Peters and Szekeres [SP06]. The proof of the Erdős–Szekeres theorem (Theorem 1.1.5) using caps and cups has a natural abstract combinatorial form, which asks about the value of $\overline{R}(\mathcal{P}_n^{(3)})$. We also know that $\overline{R}(\mathcal{P}_n^{(3)})$ is exactly the minimum size of a planar point set in general position that guarantees the existence of an n-cap or an n-cup. However, there are sets of n points in convex position that are not an n-cap nor an n-cup, thus the problem of estimating $\overline{R}(\mathcal{P}_n^{(3)})$ does not exactly correspond to the Erdős–Szekeres conjecture. Peters and Szekeres [SP06] proposed the following abstract reformulation of convex position using 3-uniform monotone hyperpaths.

If P is a point set in the plane in general position, then every n-tuple of points from P in convex position is a union of an a-cap and a u-cup that share only common endpoints where a and u are some integers satisfying $a + u - 2 = n$. For $n \geq 2$, an ordered 3-uniform hypergraph $\mathcal{H}$ on n vertices is called a *(convex) n-gon* if $\mathcal{H}$ is a union of a red monotone path and a blue monotone path that are vertex disjoint except for the two common end-vertices. In this definition, we allow paths in $\mathcal{H}$ with two vertices and no edges.

Let $PS(n)$ be the maximum number N such that there is a coloring of $\mathcal{K}_N^{(3)}$ with no n-gon.

If P is a set of points in the plane in general position, then n-tuples of points from P in convex position are in one-to-one correspondence with n-gons in the coloring χ_P of $\mathcal{K}_{|P|}^{(3)}$ obtained from P as in the proof $ES(n) \leq \overline{R}(\mathcal{P}_n^{(3)})$ from the beginning of this subsection. Thus we have $2^{n-2} \leq ES(n) - 1 \leq PS(n)$ for every $n \geq 2$ by (1.5). On the other hand, every monochromatic monotone path on n vertices is an n-gon and thus $PS(n) \leq \binom{2n-4}{n-2}$ by (1.11).

Using a computer-assisted proof, Peters and Szekeres [SP06] showed $PS(n) = 2^{n-2}$ for every n with $2 \leq n \leq 5$. Peters and Szekeres also conjectured that this equality is true for every $n \geq 2$.

Conjecture 1.3.10 ([SP06]). *For each $n \geq 2$, $PS(n) = 2^{n-2}$.*

Using a refinement of the Erdős–Szekeres conjecture proved by Erdős, Tuza, and Valtr [ETV96], Balko and Valtr [BV17] refuted Conjecture 1.3.10.

Theorem 1.3.11 ([BV17]). *We have $PS(7) > 32$, $PS(8) > 64$, and $PS(9) > 128$.*

The proof of Theorem 1.3.11 was carried out using a computer-assisted proof based on SAT solvers. For 3-monotone colorings, Balko and Valtr [BV17] did not find any counterexamples and they verified the refined Erdős–Szekeres conjecture in several new cases.

1.3.3 *Edge-ordered graphs*

Besides vertices, we can also order edges of a given graph and then study its extremal properties. An *edge-ordered graph* $\mathfrak{G} = (G, \prec)$ consists of a graph $G = (V, E)$ and a linear ordering $\prec$ of the set of edges E. We sometimes use the term *edge-ordering of G* for the ordering

$\prec$ and also for $\mathfrak{G}$. An edge-ordered graph $(G, \prec_1)$ is an *edge-ordered subgraph* of an edge-ordered graph $(H, \prec_2)$ if G is a subgraph of H and $\prec_1$ is a suborder of $\prec_2$. We say that $(G, \prec_1)$ and $(H, \prec_2)$ are *isomorphic* if there is a graph isomorphism between G and H that also preserves the edge-orderings $\prec_1$ and $\prec_2$.

Motivated by extremal results for edge-graphs obtained by Gerbner et al. [Ger+19], Balko and Vizer [BV20] introduced Ramsey numbers for edge-ordered graphs. The *edge-ordered Ramsey number* $\overline{R}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum positive integer N such that there exists an edge-ordering $\mathfrak{K}_N$ of K_N such that every red-blue coloring of the edges of $\mathfrak{K}_N$ contains a red copy of $\mathfrak{G}$ or a blue copy of $\mathfrak{H}$ as an edge-ordered subgraph of $\mathfrak{K}_N$.

Note that the definition of edge-ordered Ramsey numbers is defined quite differently than ordered Ramsey numbers as the edge-ordering of the complete graph whose edges are being colored depends on the given edge-ordered graphs. This is necessary, as otherwise there might be an edge-ordered graph $\mathfrak{G}$ and an edge-order of $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.

The finiteness of ordered Ramsey numbers was quite easy to show, as it followed from the finiteness of standard Ramsey numbers. This is not the case for edge-ordered Ramsey numbers, where it takes some effort to prove that these numbers are finite for any pair of edge-ordered graphs. This was proved by Balko and Vizer [BV20] who proved the following result.

Theorem 1.3.12 ([BV20]). *For every edge-ordered graph $\mathfrak{G}$, the edge-ordered Ramsey number $\overline{R}_e(\mathfrak{G})$ is finite.*

Theorem 1.3.12 also follows from a recent deep result of Hubička and Nešetřil [HN19, Theorem 4.33] about Ramsey numbers of general relational structures. In comparison, the proof of Theorem 1.3.12 is less general, but it is much simpler and produces better and more explicit bound on $\overline{R}_e(\mathfrak{G})$. The proof of Theorem 1.3.12 yields a stronger induced-type statement where additionally the ordering of the vertex set is fixed and the colorings can use an arbitrary number of colors.

However, the bound on the edge-ordered Ramsey numbers obtained in the proof of Theorem 1.3.12 is enormous, it grows faster than, for example, a tower function of any fixed height. Fox and Li [FL20] improved the bound on edge-ordered Ramsey numbers to single exponential type.

Theorem 1.3.13 ([FL20]). *For each positive integer n, there is an edge-ordered graph $\mathfrak{G}$ on $N = 2^{100n^2 log^2 n}$ vertices such that, for every 2-coloring of the edges of $\mathfrak{G}$, there exists a monochromatic subgraph containing a copy of every n-vertex edge-ordered graph.*

In particular, if $\mathfrak{G}$ is an edge-ordered graph on n vertices, then

$$\overline{R}_e(\mathfrak{G}) \leq 2^{100n^2 log^2 n}.$$

Fox and Li [FL20] also extended their result to more colors and proved the following polynomial upper bound on edge-ordered Ramsey numbers of edge-ordered graphs of bounded degeneracy, improving earlier estimates by Balko and Vizer [BV20].

Theorem 1.3.14 ([FL20]). *If $\mathfrak{G}$ is an edge-ordered d-degenerate graph on n vertices, then*

$$\overline{R}_e(\mathfrak{G}) \leq n^{600d^3 \log (d+1)}.$$

1.3.4 *Open problems*

The ordered Ramsey numbers for ordered k-uniform hypergraphs with $k \geq 3$ and the edge-ordered Ramsey numbers are quite unexplored, so there is a plenty of open problems. In this subsection, we mention only few of them; more comprehensive lists of open problem can be found, for example, in [BV20, BV21, FL20].

Theorems 1.3.2 and 1.3.4 give estimates on the ordered Ramsey numbers $\overline{R}(\mathcal{G}, \mathcal{K}_3^{(3)}(n))$ and although the exponents in the bounds are reasonably close, there is still a gap between them and it would be interesting to close it.

Problem 1.3.15 ([BV21]). *Let d be a fixed positive integer and let $\mathcal{H}$ be an ordered 3-uniform hypergraph on n vertices with maximum degree d. Close the gap between the lower and upper bounds on $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n))$.*

Another interesting problem is to extend the upper bound with subquadratic exponent from Corollary 1.3.3 to ordered 3-uniform hypergraphs with bounded maximum degree and fixed interval chromatic number that is larger than 3.

Problem 1.3.16 ([BV21]). *Let d and χ be fixed positive integers. Is there an $\varepsilon = \varepsilon(d, \chi) > 0$ such that, for every ordered 3-uniform hypergraph $\mathcal{H}$ on n vertices with maximum degree d and with interval chromatic number χ, we have*

$$\overline{R}(\mathcal{H}) \leq 2^{O(n^{2-\varepsilon})}?$$

In general, we are not aware of any nontrivial upper bounds on ordered Ramsey numbers of ordered 3-uniform hypergraphs with bounded maximum degree.

Problem 1.3.17 ([BV21]). *What is the upper bound on ordered Ramsey numbers of ordered 3-uniform hypergraphs with bounded maximum degree?*

For k-monotone colorings, although Theorem 1.3.8 gives asymptotically tight estimates, it is still open to decide whether $\overline{R}(\mathcal{P}_n^{(k)}) = \overline{R}_{mon}(\mathcal{P}_n^{(k)})$ for all k and n. Currently, this equality is known only for $k \leq 3$.

Problem 1.3.18 ([Bal19]). *Is it true that* $\overline{R}_{mon}(\mathcal{P}_n^{(k)}) = \overline{R}(\mathcal{P}_n^{(k)})$ *for all* $k \geq 2$ *and* n*?*

For edge-ordered graphs, a major open question is to find a better estimate for edge-ordered Ramsey numbers of edge-ordered complete graphs on n vertices. In particular, Fox and Li [FL20] asked whether the exponential lower bound is tight for general edge-orderings of K_n. For sparser edge-ordered graphs, Fox and Li [FL20] conjectured that the upper bound from Theorem 1.3.14 can be improved.

Conjecture 1.3.19 ([FL20]). *If* $\mathfrak{H}$ *is an edge-ordered* d*-degenerate graph on* n *vertices, then* $\overline{R}_e(\mathfrak{H}) \leq n^{O(d)}$*.*

Currently, there are no known edge-ordered d-degenerate graphs with superlinear edge-ordered Ramsey numbers. However, Fox and Li [FL20] believe that there are such examples and conjectured that the upper bound from Conjecture 1.3.19 is tight up to the constant in the exponent.

The edge-ordered Ramsey numbers can be naturally extended to edge-ordered k-uniform hypergraphs with any $k \geq 2$. The existence of such numbers follows from a result of Hubička and Nešetřil [HN19], but the resulting bounds are enormous. Fox and Li [FL20] posed a natural problem to give a better estimate on such hypergraph edge-ordered Ramsey numbers.

1.4 COUNTING HOLES IN POINT SETS

Here, we consider a strengthening of the Erdős–Szekeres theorem (Theorem 1.1.5) suggested by Erdős [Erd78] in the 1970s. For an integer $k \geq 3$, a *k-hole* in a finite set P of points in general position in the plane is a k-tuple H of points from P that is in convex position and that contains no point from P in the interior of the convex hull of H; see Figure 1.8. Erdős [Erd78] asked whether, for every positive integer k, every sufficiently large finite point set in general position in the plane contains a k-hole. Thus, when compared to the statement of the Erdős–Szekeres theorem, we want to find a k-tuple of points from P in convex position whose convex hull is additionally empty of other points from P.

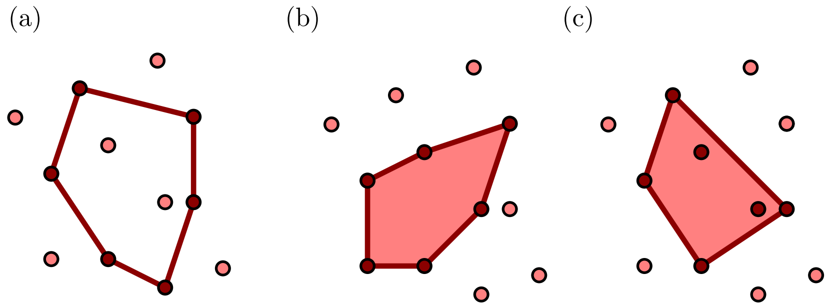

Figure 1.8: An example of a point set P with (a) a 6-tuple of points from P in convex position, (b) a 6-hole in P, and (c) a 6-island in P.

Clearly, every set of at least 3 points in general position in the plane contains a 3-hole. It is also easy to see that every set of at least 5 points in general position in the plane contains a 4-hole and that there are sets of 4 points without a 4-hole. Harborth [Har78] proved that there is a 5-hole in every planar set of 10 points in general position and gave a construction of 9 points in general position with no 5-hole.

After unsuccessful attempts to answer Erdős' question affirmatively for any fixed integer $k \geq 6$, the problem was settled by Horton [Hor83] who constructed, for every positive integer n, a set of n points in general position in the plane with no 7-hole. This result gave a negative answer to the question of Erdős and showed that a strengthening of the Erdős–Szekeres theorem for holes is impossible. The question about the existence of 6-holes remained a longstanding open problem until 2007, when Gerken [Ger08] and Nicolas [Nic07] independently showed that every sufficiently large set of points in general position in the plane contains a 6-hole. Thus, the existence of k-holes in sufficiently large point sets was settled for every positive integer k.

1.4.1 *Counting 5-holes*

Since the existence of k-holes was settled, many researchers started to study the growth rate of the minimum number of k-holes a set of n points in general position in the plane can have. By the result of Horton [Hor83], this problem is nontrivial only for $k \leq 6$, as otherwise the answer is 0.

For positive integers n and $k \geq 3$, let $h_k(n)$ be the minimum number of k-holes in a set of n points in general position in the plane. The growth rate of the functions $h_3(n)$ and $h_4(n)$ is known to be quadratic in n. The best known upper bounds on the minimum numbers of 3-holes and 4-holes were proved by Bárány and Valtr [BV04] who showed

$$h_3(n) \leq 1.6196n^2 + o(n^2) \qquad \text{and} \qquad h_4(n) \leq 1.9397n^2 + o(n^2).$$

Using a result of García [Gar11] and their new estimate on $h_5(n)$, Aichholzer et al. [Aic+20] obtained the following currently strongest lower bounds on $h_3(n)$ and $h_4(n)$, improving earlier estimates by Aichholzer et al. [Aic+14] with linear smaller order terms.

Theorem 1.4.1 ([Aic+20]). *The following two bounds are satisfied for every positive integer n:*

$$h_3(n) \geq n^2 + \Omega(n \log^{2/3} n) \qquad \textit{and} \qquad h_4(n) \geq \frac{n^2}{2} + \Omega(n \log^{3/4} n).$$

No asymptotically matching bounds are known or $h_5(n)$ and $h_6(n)$. The best known upper bounds were again obtained by Bárány and Valtr [BV04] who proved the following quadratic estimates

$$h_5(n) \leq 1.0207n^2 + o(n^2) \qquad \text{and} \qquad h_6(n) \leq 0.2006n^2 + o(n^2).$$

It is widely conjectured that the minimum numbers of 5-holes and 6-holes are both quadratic, however, despite many attempts in the past decades, the best known lower bounds on $h_5(n)$ and $h_6(n)$ were only linear in n. Even the following weaker problem, mentioned for example in the book by Brass, Moser, and Pach [BMP05], was open since the 1980s.

Problem 1.4.2 ([BMP05]). *Is it true that*

$$\lim_{n\to\infty} \frac{h_5(n)}{n} = \infty?$$

For the minimum number of 5-holes, Bárány and Füredi [BF87] noted that Harborth's result [Har78] implies $h_5(n) \geq \lfloor n/10 \rfloor$. This estimate was improved by Bárány and Károlyi [BK01] to $h_5(n) \geq n/6 - O(1)$. In 1987, Dehnhardt [Deh87] proved $h_5(n) \geq 3\lfloor n/12 \rfloor$. Further improved lower bounds included $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ by García [Gar11],

$h_5(n) \geq \lceil\frac{3(n-11)}{7}\rceil$ by Aichholzer, Hackl, and Vogtenhuber [AHV11], $h_5(n) \geq n/2 - O(1)$ by Valtr [Val12], and $h_5(n) \geq 3n/4 - o(n)$ by Aichholzer et al. [Aic+14].

All these estimates improved only the leading constant in the lower bounds on $h_5(n)$. However, Aichholzer et al. [Aic+20] recently proved the following superlinear lower bound on $h_5(n)$ and thus solved Problem 1.4.2 affirmatively.

Theorem 1.4.3 ([Aic+20]). *There is a constant $c > 0$ such that, for every integer $n \geq 10$, we have*

$$h_5(n) \geq cn \log^{4/5} n.$$

Let P be a finite set of points in the plane in general position and let ℓ be a line containing no point of P. We say that P is *ℓ-divided* if there is at least one point of P in each of the two halfplanes determined by ℓ; see Figure 1.9. For an ℓ-divided set P, we use $P = A \cup B$ to denote the fact that ℓ partitions P into the subsets A and B.

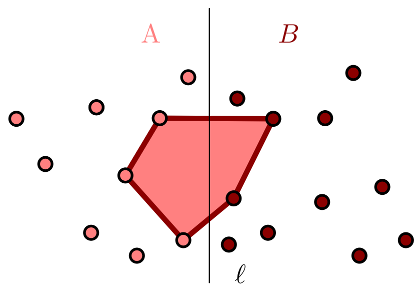

Figure 1.9: An example of an ℓ-divided point set $P = A \cup B$ with an ℓ-divided 5-hole in P.

The following result is a crucial step in the proof of Theorem 1.4.3. It was obtained by Aichholzer et al. [Aic+20] using a computer-assisted proof and it might be of independent interest.

Theorem 1.4.4 ([Aic+20]). *Let $P = A \cup B$ be an ℓ-divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ-divided 5-hole in P.*

We note that the assumption $|A|, |B| \geq 5$ in Theorem 1.4.4 is necessary, as Aichholzer et al. [Aic+20] constructed arbitrarily large ℓ-divided sets $P = A \cup B$ with $|A| = 4$ and with no ℓ-divided 5-holes.

The best known lower bound on the minimum number of 6-holes remains only linear. The strongest lower bound $h_6(n) \geq n/229 - 4$ was proved by Valtr [Val12]. The techniques used to prove the superlinear lower bound on $h_5(n)$ do not seem to be applicable here, since there are too large point sets with no 6-holes [Ove03], which is too demanding for current computers.

1.4.2 *Random point sets*

The quadratic upper bound $h_3(n) \leq O(n^2)$ can be also obtained using random points sets. This was proved by Bárány and Füredi [BF87] who actually extended this upper bound to higher dimensions. Before stating their result, we extend the notion of holes to higher dimensions and we introduce random point sets.

For an integer $d \geq 2$, a set S of points from $\mathbb{R}^d$ is in *general position* if, for every $k = 1, \ldots, d-1$, no $k+2$ points of S lie in an affine k-dimensional subspace of $\mathbb{R}^d$. We say that S is in *convex position* if the points of S are vertices of a convex polytope. For an integer $k \geq d+1$, a set H of k points from S is a *k-hole* in S if H is in convex position and the interior of the convex hull of H does not contain any point from S.

A *convex body* in $\mathbb{R}^d$ is a compact convex set in $\mathbb{R}^d$ with a nonempty interior. We use $\mathcal{K}_d$ to denote the set of all convex bodies in $\mathbb{R}^d$ of volume $\lambda_d(K) = 1$. Let $k \geq d+1$ be an integer and K be a convex body from $\mathcal{K}_d$. We let $EH^K_{d,k}(n)$ be the expected number of k-holes in sets of n points chosen independently and uniformly at random from K.

Bárány and Füredi [BF87] proved the following upper bound on the expected number $EH^K_{d,d+1}(n)$ of $(d+1)$-holes (also called *empty simplices*). They showed

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}$$

for every $K \in \mathcal{K}_d$. Valtr [Val95] improved this bound in the plane by showing $EH^K_{2,3}(n) \leq 4\binom{n}{2}$ for any $K \in \mathcal{K}_2$. Very recently, Reitzner and Temesvari [RT19] showed that Valtr's bound on $EH^K_{2,3}(n)$ is tight for every $K \in \mathcal{K}_2$ up to smaller order terms. This follows from their more general bounds

$$\lim_{n\to\infty} n^{-2} EH^K_{2,3}(n) = 2$$

and

$$\frac{2}{d!} \leq \lim_{n\to\infty} n^{-d} EH^K_{d,d+1}(n) \leq \frac{d}{(d+1)} \frac{\kappa^{d+1}_{d-1}\kappa_{d^2}}{\kappa^{d-1}_{d}\kappa_{(d-1)(d+1)}} \tag{1.12}$$

for $d \geq 2$ and $K \in \mathcal{K}_d$, where $\kappa_d = \pi^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)^{-1}$ is the volume of the d-dimensional Euclidean unit ball. Moreover, the upper bound in (1.12) holds with equality in the case $d = 2$ and if K is a d-dimensional ellipsoid with $d \geq 3$.

For larger holes, that is, for k-holes with $k > d+1$, the question about the growth rate of the expected value $EH^K_{d,k}(n)$ was settled by Balko, Scheucher, and Valtr [BSV21a, BSV21b]. First, they proved the estimate $O(n^d)$ on the expected number of k-holes in a random set of n points in $\mathbb{R}^d$ for any fixed d and k.

Theorem 1.4.5 ([BSV21a]). *Let $d \geq 2$ and $k \geq d+1$ be integers and let K be a convex body from $\mathcal{K}_d$. If $n \geq k$, then*

$$EH^K_{d,k}(n) \leq \frac{2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot n(n-1)\cdots(n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}}.$$

In particular, $EH^K_{d,k}(n) \leq O(n^d)$ for any fixed d and k.

Later, Balko, Scheucher, and Valtr [BSV21b] found an asymptotically matching lower bond on $EH^K_{d,k}(n)$.

Theorem 1.4.6 ([BSV21b]). *For all integers $d \geq 2$ and $k \geq d+1$, there are constants $C = C(d,k) > 0$ and $n_0 = n_0(d,k)$ such that, for every integer $n \geq n_0$ and every convex body $K \in \mathcal{K}_d$, we have*

$$EH^K_{d,k}(n) \geq C \cdot n^d.$$

Theorems 1.4.5 and 1.4.6 show that $EH^K_{d,k}(n) = \Theta(n^d)$ for all fixed integers d and k and every $K \in \mathcal{K}_d$, which determines the asymptotic growth rate of $EH^K_{d,k}(n)$. Thus, it remains to determine the leading constants $\lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$.

For a convex body $K \subseteq \mathbb{R}^d$ (of a not necessarily unit volume), we use p^K_d to denote the probability that the convex hull of $d+2$ points chosen uniformly and independently at random from K is a d-simplex. That is, the probability that one of the $d+2$ points falls in the convex hull of the remaining $d+1$ points. The problem of computing p^K_d is known as the d-dimensional *Sylvester's convex hull problem* for K and it has been studied extensively. Let $p_d = \max_K p^K_d$, where the maximum is taken over all convex bodies $K \subseteq \mathbb{R}^d$.

Balko, Scheucher, and Valtr [BSV21b] improved the lower bound on the expected number $EH^K_{d,d+1}(n)$ of empty simplices in random sets of n points in K from (1.12) by Reitzner and Temesvari [RT19] by a factor of d/p_{d-1}.

Theorem 1.4.7 ([BSV21b]). *For every integer $d \geq 2$ and every convex body $K \in \mathcal{K}_d$, we have*

$$\lim_{n\to\infty} n^{-d} EH^K_{d,d+1}(n) \geq \frac{2}{(d-1)!p_{d-1}}.$$

The leading constant in the estimate from Theorem 1.4.7 is asymptotically tight in the planar case [BSV21b]. We also note that by combining Theorem 1.4.7 with some known results about the expected volume of a random simplex in K, the lower bound on $EH^K_{d,d+1}(n)$ by Reitzner and Temesvari [RT19] can be improved by a factor of $d^{\Omega(d)}$.

Besides empty simplices, the leading constants in the expectation for larger k-holes were also considered in the literature. The expected number $EH^K_{2,4}(n)$ of 4-holes in random planar sets of n points was estimated by Fabila-Monroy, Huemer, and Mitsche [FHM15], who showed

$$EH^K_{2,4}(n) \leq 18\pi D^2 n^2 + o(n^2)$$

for any $K \in \mathcal{K}_2$, where $D = D(K)$ is the diameter of K. It can be shown that the leading constant in their bound is at least 72 for any $K \in \mathcal{K}_2$ [BSV21b]. This estimate was strengthened by Balko, Scheucher, and Valtr [BSV21a] to $EH^K_{2,4}(n) \leq 12n^2 + o(n^2)$ for every $K \in \mathcal{K}_2$. Later, Balko, Scheucher, and Valtr [BSV21b] determined the leading constant in $EH^K_{2,4}(n)$ exactly.

Theorem 1.4.8 ([BSV21b])**.** *For every convex body $K \in \mathcal{K}_2$, we have*

$$\lim_{n\to\infty} n^{-2} EH^K_{2,4}(n) = 10 - \frac{2\pi^2}{3} \approx 3.420.$$

For larger k-holes in the plane, the values $\lim_{n\to\infty} n^{-2} EH^K_{2,k}(n)$ are not determined exactly, but Balko, Scheucher, and Valtr [BSV21b] showed that they exist and do not depend on the convex body K. We recall that this is not true in larger dimensions already for empty simplices.

Finally, we note that Theorem 1.4.5 can be significantly strengthened by considering more general k-tuples of points than k-holes. A set I of k points from a point set $S \subseteq \mathbb{R}^d$ is a *k-island* in S if every point of S in the convex hull of I lies in I; see part (c) of Figure 1.8. Note that k-holes in S are exactly those k-islands in S that are in convex position.

The following result by Balko, Scheucher, and Valtr [BSV21a] shows that the $O(n^d)$ upper bound holds also for the expected number of k-islands, although the leading constant is a bit worse than in Theorem 1.4.5.

Theorem 1.4.9 ([BSV21a])**.** *Let $d \geq 2$ and $k \geq d+1$ be integers and let K be a convex body from $\mathcal{K}_d$. If S is a set of $n \geq k$ points chosen uniformly and independently at random from K, then the expected number of k-islands in S is at most*

$$2^{d-1} \cdot \frac{\left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot (k-d) \cdot n(n-1)\cdots(n-k+2)}{(n-k+1)^{k-d-1}},$$

which is in $O(n^d)$ for any fixed d and k.

1.4.3 *Open problems*

It still remains an open problem to decide whether the minimum numbers of 5-holes and 6-holes grow quadratically in n. For 6-holes, it is not even known whether the number $h_6(n)$ grows superlinearly in n. We note that Pinchasi, Radoičić and Sharir [PRS06] showed that if $h_3(n) \geq (1+\varepsilon)n^2 - o(n^2)$ for some $\varepsilon > 0$, then $h_5(n) = \Omega(n^2)$.

The leading constants $\lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$ are determined exactly for small holes. In particular, we know that these limits exist in such cases. However, it remains an interesting open problem to determine whether the limits $\lim_{n\to\infty} n^{-d} EH^K_{d,k}(n)$ exist for all positive integers d and k with $k \geq d+1$. It follows from results by Reitzner and Temesvari [RT19] and by Balko, Scheucher, and Valtr [BSV21b] that these limits exist if $k = d+1$ or if $k \geq 3$ and $d = 2$.

Balko, Scheucher, and Valtr [BSV21b] also posed the following problem asking for the value of the expected umber of empty simplices in tetrahedron.

Problem 1.4.10 ([BSV21b])**.** *Let K be a 3-dimensional simplex of unit volume. Determine the leading constant* $\lim_{n\to\infty} n^{-d} EH^K_{3,4}(n)$.

1.5 VISIBILITY PROBLEMS

The notion of visibility is one of the classical topics in discrete geometry, involving several interesting results and difficult open problems. We call two points u and v in $\mathbb{R}^d$ *visible* with respect to some set $X \subseteq \mathbb{R}^d$ if there is no point from X on the line segment between u and v. Some form of visibility appears, for example, in the famous *Art-gallery problem* [Mat02], visibility problems for lattice points [BMP05], or in the beautiful *Big-Line-or-Big-Clique* conjecture by Kára, Pór, and Wood [KPW05].

In this section, we focus on two visibility problems. First, we mention some results about so-called *obstacle representations* of graphs. Here, graphs are represented as straight-line drawings surrounded by polygonal obstacles so that edges are only between points that are mutually visible with respect to these obstacles. Second, we discuss results about so-called *index of convexity*, which measures a convexity of a given subset S of $\mathbb{R}^d$ based on the probability that two random points from S are mutually visible.

1.5.1 *Obstacle numbers*

In a *geometric drawing* of a graph G, the vertices of G are represented by distinct points in the plane and each edge of G is represented by the line segment between two points that represent the corresponding end-vertices. As usual, we identify vertices and their images, as well as edges and the line segments representing them.

An *obstacle* is a polygon in the plane. An *obstacle representation* of a graph G is a geometric drawing D of G together with a set $\mathcal{O}$ of obstacles such that two vertices of G are connected by an edge e if and only if the line segment representing e in D is disjoint from all obstacles in $\mathcal{O}$; see Figure 1.10. The *obstacle number* $obs(G)$ of G is the minimum number of obstacles in an obstacle representation of G. The *convex obstacle number* $obs_c(G)$ of a graph G is the minimum number of obstacles in an obstacle representation of G in which all the obstacles are required to be convex. Clearly, $obs(G) \leq obs_c(G)$ for every graph G. For a positive integer n, let $obs(n) = \max_G obs(G)$, where the maximum is taken over all graphs G on n vertices.

The obstacle numbers were introduced by Alpert, Koch, and Laison [AKL10], who proved that they can be arbitrarily large. Later, Pach and Sarıöz [PS11] proved that there are bipartite graphs with arbitrarily large obstacle number.

Mukkamala, Pach, and Sarıöz [MPS10] showed that the number of labeled n-vertex graphs with obstacle number at most h is at most $2^{O(hn\log^2 n)}$ for every fixed positive integer h. It follows that $obs(n) \geq \Omega(n/\log^2 n)$. Later, Mukkamala, Pach, and Pálvölgyi [MPP12] improved the lower bound to $obs(n) \geq \Omega(n/\log n)$. Currently, the

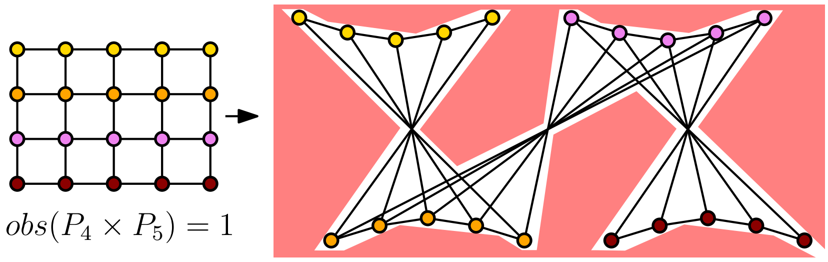

Figure 1.10: An example of an obstacle representation of the grid graph $G = P_4 \times P_5$ showing that $obs(G) = 1$.

strongest lower bound is due to Dujmović and Morin [DM15] who showed $obs(n) \geq \Omega(n/(\log\log n)^2)$.

Clearly, we have the trivial upper bound $obs(G) \leq \binom{n}{2}$ for every graph G on n vertices, as we can place a single-point obstacle on each non-edge of G. Alpert, Koch, and Laison [AKL10] asked whether $obs(n)$ can be bounded from above by a linear function in n. This problem is still open, but Balko, Cibulka, and Valtr [BCV18] proved that this is true for graphs with bounded chromatic number. On the other hand, a modification of the proof of the lower bound by Mukkamala, Pach, and Pálvölgyi [MPP12] gives bipartite graphs G on n vertices with $obs(G) \geq \Omega(n/\log n)$ for every positive integer n.

Theorem 1.5.1 ([BCV18]). *For every positive integer n and every graph G on n vertices, the convex obstacle number of G satisfies*

$$obs_c(G) \leq (n-1)(\lceil \log \chi(G) \rceil + 1),$$

where $\chi(G)$ denotes the chromatic number of G.

Note that Theorem 1.5.1 gives the linear upper even for the convex obstacle number. In fact, Balko, Cibulka, and Valtr [BCV18] proved the linear upper bound on $obs(G)$ even for graphs G with bounded *subchromatic number* in which each color class induces a disjoint union of cliques. This strengthening then gives this upper bound for *split graphs*.

In contrast to the question of Alpert, Koch, and Laison [AKL10], Mukkamala, Pach, and Pálvölgyi [MPP12] conjectured that the maximum obstacle number of n-vertex graphs is around n^2. Theorem 1.5.1 refutes this conjecture. Moreover, its proof can be modified to give the following stronger bound.

Theorem 1.5.2 ([BCV18]). *For every positive integer n and every graph G on n vertices, the convex obstacle number of G satisfies*

$$obs_c(G) \leq n\lceil \log n \rceil - n + 1.$$

For positive integers h and n, let $g(h,n)$ be the number of labeled n-vertex graphs with obstacle number at most h. The lower bounds on $obs(n)$ by Mukkamala, Pach, and Pálvölgyi [MPP12] and by Dujmović and Morin [DM15] are both based on the upper bound $g(h,n) \leq 2^{O(hn \log^2 n)}$. In fact, any improvement on the upper bound for $g(h,n)$ will translate into an improved lower bound on the obstacle number [DM15]. Dujmović and Morin [DM15] conjectured $g(h,n) \leq 2^{f(n) \cdot o(h)}$ where $f(n) \leq O(n \log^2 n)$. Balko, Cibulka, and Valtr [BCV18] proved the following lower bound on $g(h,n)$.

Theorem 1.5.3 ([BCV18])**.** *For every pair of integers n and h satisfying* $0 < h < n$*, we have*

$$g(h,n) \geq 2^{\Omega(hn)}.$$

This lower bound on $g(h,n)$ is not tight in general [BCV18]. We also note that the constructions used in the proofs of the above results can be also applied to obtain new lower bounds on so-called *complexity of faces in arrangements of line segments* [BCV18], which in some cases match upper bounds by Aronov, Edelsbrunner, Guibas, and Sharir [Aro+92].

1.5.2 *Index of convexity*

In this subsection, we discuss two measures of convexity of subsets of $\mathbb{R}^d$ and we investigate the relationship between them.

The first such measure, called the *convexity ratio*, measures the convexity of a Lebesgue measurable set $S \subseteq \mathbb{R}^d$ with respect to the largest convex subset of S. Let $smc(S)$ denote the supremum of the Lebesgue measures of convex subsets of S. Since all convex subsets of $\mathbb{R}^d$ are Lebesgue measurable [Lan86], the value of $smc(S)$ is well defined. Moreover, Goodman's result [Goo81] implies that the supremum is achieved on compact sets S, hence it can be replaced by maximum in this case. If S has finite positive Lebesgue measure, we define the *convexity ratio* $c(S)$ as

$$c(S) = smc(S)/\lambda_d(S).$$

The second convexity measure, called the *Beer index of convexity*, measures convexity of $S \subseteq \mathbb{R}^d$ using the notion of visibility. For a point $A \in S$, let $Vis(A,S)$ be the set of points that are visible from A in S. More generally, for a subset T of S, we use $Vis(T,S)$ to denote the set of points that are visible in S from T. That is, $Vis(T,S)$ is the set of points $A \in S$ for which there is a point $B \in T$ such that the line segment $\overline{AB}$ between A and B is contained in S. Let the *segment set* of S be the set $Seg(S) = \{(A,B) \in S \times S \colon \overline{AB} \subseteq S\} \subseteq (\mathbb{R}^d)^2$. If S has a finite positive Lebesgue measure and $Seg(S)$ is Lebesgue measurable, we define the *Beer index of convexity* $b(S) \in [0,1]$ (or just *Beer index*) as

$$b(S) = \frac{\lambda_{2d}(Seg(S))}{\lambda_d(S)^2}.$$

We leave $b(S)$ undefined for all other sets $S \subseteq \mathbb{R}^d$. Thus, the Beer index can be interpreted as the probability that two points of S chosen uniformly independently at random see each other in S. Note that if $b(S)$ is defined, then $c(S)$ is defined as well.

The Beer index was introduced by Beer [Bee73a, Bee73b, Bee74] in the 1970s under the name 'the index of convexity'. Beer was motivated by studying the continuity properties of $\lambda_d(Vis(A, S))$ as a function of A. For polygonal regions, an equivalent parameter was later independently defined by Stern [Ste89], who called it 'the degree of convexity'. Stern was motivated by the problem of finding a computationally tractable way to quantify how close a given set is to being convex and he approximated the Beer index of a polygon P by a Monte Carlo estimation. Later, Rote [Rot13] showed that for a polygonal region P with n edges the Beer index can be evaluated in polynomial time as a sum of $O(n^9)$ closed-form expressions. Cabello et al. [Cab+17] studied the relationship between the Beer index and the convexity ratio, and applied their results in the analysis of their near-linear-time approximation algorithm for finding the largest convex subset of a polygon.

For general subsets S of $\mathbb{R}^2$ with Lebesgue measurable $Seg(S)$, there is a simple lower bound $b(S) \geq c(S)^2$ that is tight in the some cases. To derive this bound, note that, for every $\varepsilon > 0$, the set S contains a convex subset K of measure at least $(c(S) - \varepsilon)\lambda_2(S)$. Two random points of S both belong to K with probability at least $(c(S) - \varepsilon)^2$, hence $b(S) \geq (c(S) - \varepsilon)^2$. The tightness of this bound is witnessed by a set S which is a disjoint union of a single large convex component and a large number of small components of a negligible size.

It is more challenging to find an upper bound on $b(S)$ in terms of $c(S)$ under additional assumptions on S. As a motivating example, observe that a set S consisting of n disjoint convex components of the same size satisfies $b(S) = c(S) = \frac{1}{n}$. It is easy to modify this example to obtain, for any $\varepsilon > 0$, a simple star-shaped polygon P with $b(P) \geq \frac{1}{n} - \varepsilon$ and $c(P) \leq \frac{1}{n}$; see Figure 1.11. Here, a subset S of $\mathbb{R}^d$ is *star-shaped* if it contains a point that sees every other point of S. Thus, there are simple polygons P with $b(P) \geq \Omega(c(P))$.

Cabello et al. [Cab+17] showed that the above example is essentially optimal for weakly star-shaped polygons, as they proved the following linear upper bound on $b(S)$. A set S is *weakly star-shaped* if there is a line segment $\ell \subseteq S$ with $Vis(\ell, S) = S$.

Theorem 1.5.4 ([Cab+17]). *Every weakly star-shaped simple polygon P satisfies*

$$b(P) \leq 18c(P).$$

For polygons that are not weakly star-shaped, Cabello et al. [Cab+17] gave the following slightly weaker bound.

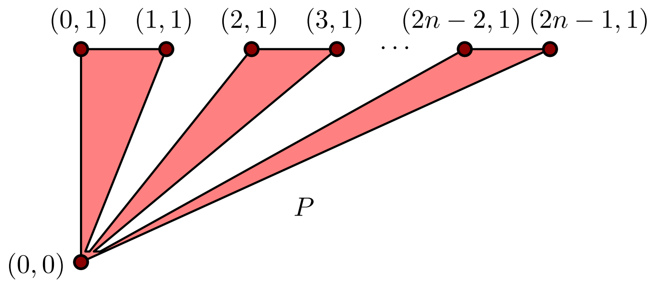

Figure 1.11: A star-shaped polygon P with $b(P) \geq \frac{1}{n} - \varepsilon$ and $c(P) \leq \frac{1}{n}$. The polygon P with $4n-1$ vertices is a union of n triangles $(0,0)(2i,1)(2i+1,1)$, $i = 0,\dots,n-1$, and of a triangle $(0,0)(0,\delta)((2n-1)\delta,\delta)$, where δ is very small.

Theorem 1.5.5 ([Cab+17]). *Every simple polygon P satisfies*

$$b(P) \leq 12c(P)\left(1 + \log_2 \frac{1}{c(P)}\right).$$

Cabello et al. [Cab+17] also conjectured that even for a general simple polygon P, the parameter $b(P)$ can be bounded from above by a linear function of $c(P)$. Balko, Jelínek, Valtr, and Walczak [Bal+17] confirmed this conjecture. In fact, they proved more general statement and to state it in the full generality, we need to introduce some notation.

For a set $X \subseteq \mathbb{R}^2$, the equivalence classes on X of the equivalence relation "A and B can be connected by a polygonal line in X" form the *p-components* of X. A set S is *p-componentwise simply connected* if every p-component of S is simply connected.

Theorem 1.5.6 ([Bal+17]). *Every p-componentwise simply connected set $S \subseteq \mathbb{R}^2$ whose $b(S)$ is defined satisfies*

$$b(S) \leq 180 \cdot c(S).$$

Since every simple polygon satisfies the assumptions of Theorem 1.5.6, we obtain the linear bound $b(P) \leq 180 \cdot c(P)$ also for simple polygons P, which confirms the conjecture of Cabello et al. [Cab+17].

Theorem 1.5.6 fails in higher dimensions as there is a construction of star-shaped sets $S_d \subseteq \mathbb{R}^d$ with $c(S_d) = 0$ and $b(S_d) = 1$ for every $d \geq 3$ [Bal+17]. Despite these examples, Balko, Jelínek, Valtr, and Walczak [Bal+17] showed that there are meaningful higher-order generalizations of the Beer index for which analogues of Theorem 1.5.6 hold in higher dimensions.

For a set $S \subseteq \mathbb{R}^d$, we define the *k-simplex set* of S as

$$Simp_k(S) = \{(A_0,\dots,A_k) \in S^{k+1} \colon conv(\{A_0,\dots,A_k\}) \subseteq S\},$$

where $conv(X)$ denotes the convex hull of a set X. Note that we have $Simp_1(S) = Seg(S)$. For $k \in [d]$ and $S \subseteq \mathbb{R}^d$ with finite positive

Lebesgue measure and with Lebesgue measurable $Simp_k(S)$, we define the *k-index of convexity* of S as

$$b_k(S) = \frac{\lambda_{(k+1)d}(Simp_k(S))}{\lambda_d(S)^{k+1}}.$$

We again leave $b_k(S)$ undefined otherwise. Thus, $b_k(S)$ is the probability that the convex hull of $k+1$ points chosen from S uniformly independently at random is contained in S. Note that $b_1(S) = b(S)$ and $b_1(S) \geq b_2(S) \geq \cdots \geq b_d(S)$, provided all the numbers $b_k(S)$ are defined.

We remark that the sets S_d satisfy $c(S_d) = 0$ and $b_1(S_d) = b_2(S_d) = \cdots = b_{d-1}(S_d) = 1$. Thus, for a general set $S \subseteq \mathbb{R}^d$, only the d-index of convexity can conceivably admit a nontrivial upper bound in terms of $c(S)$. Balko, Jelínek, Valtr, and Walczak [Bal+17] showed that such an upper bound indeed holds.

Theorem 1.5.7 ([Bal+17])**.** *For every integer $d \geq 2$, there is a constant $\beta = \beta(d) > 0$ such that every set $S \subseteq \mathbb{R}^d$ with defined $b_d(S)$ satisfies*

$$b_d(S) \leq \beta c(S).$$

Balko, Jelínek, Valtr, and Walczak [Bal+17] also constructed examples that show that the bound is optimal up to a logarithmic factor.

Theorem 1.5.8 ([Bal+17])**.** *For every integer $d \geq 2$, there is a constant $\gamma = \gamma(d) > 0$ such that for every $\varepsilon \in (0,1)$, there is a set $S \subseteq \mathbb{R}^d$ satisfying*

$$c(S) \leq \varepsilon \qquad \text{and} \qquad b_d(S) \geq \gamma \frac{\varepsilon}{\log 1/\varepsilon}.$$

In particular, we have $b_d(S) \geq \gamma \frac{c(S)}{\log 1/c(S)}$.

1.5.3 *Open problems*

We start by mentioning open problems about the obstacle numbers. First, we recall the open problem by Alpert, Koch, and Laison [AKL10] who asked whether the obstacle numbers are bounded from above by a linear function with respect to the number of vertices.

Problem 1.5.9 ([AKL10])**.** *Is the obstacle number of a graph with n vertices bounded from above by a linear function of n?*

Alpert et al. [AKL10] also asked about the existence of a planar graph with obstacle number greater than one. Berman et al. [Ber+17] proved that the icosahedron has obstacle number 2. However, it is still an open problem to decide whether obstacle numbers of planar graphs can be bounded from above by a constant.

Problem 1.5.10 ([GOV18])**.** *Are there planar graphs graphs with arbitrarily large obstacle number?*

Balko, Cibulka, and Valtr [BCV18] also asked about the behavior of the obstacle numbers when adding an edge.

Problem 1.5.11 ([BCV18]). *If G is a graph and e is a non-edge of G, how much larger can $obs(G+e)$ be when compared to $obs(G)$?*

The same question can be also asked for the convex obstacle number. Note that $obs(G+e) \geq obs(G) - 1$ and $obs_c(G+e) \geq obs_c(G) - 1$ for every graph G and every non-edge e of G.

We also mention several open problems about the Beer index of convexity and its variants. First, we know that Theorem 1.5.6 does not hold for general subsets of $\mathbb{R}^2$. However, Balko, Jelínek, Valtr, and Walczak [Bal+17] conjectured that a large value of $b(S)$ implies the existence of a large convex set whose boundary belongs to S.

Conjecture 1.5.12 ([Bal+17]). *For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^2$ is a set with $b(S) \geq \varepsilon$, then there is a bounded convex set $C \subseteq \mathbb{R}^2$ with $\lambda_2(C) \geq \delta\lambda_2(S)$ and $\partial C \subseteq S$.*

Theorem 1.5.6 shows that Conjecture 1.5.12 holds for p-componentwise simply connected sets, with δ being a constant multiple of ε. It is possible that even in the setting of Conjecture 1.5.12, δ can be taken as a constant multiple of ε.

Balko, Jelínek, Valtr, and Walczak [Bal+17] proposed a stronger version of Conjecture 1.5.12, where the convex set C is required to be a triangle.

Conjecture 1.5.13 ([Bal+17]). *For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^2$ is a set with $b(S) \geq \varepsilon$, then there is a triangle $T \subseteq \mathbb{R}^2$ with $\lambda_2(T) \geq \delta\lambda_2(S)$ and $\partial T \subseteq S$.*

Balko, Jelínek, Valtr, and Walczak [Bal+17] also generalised Conjecture 1.5.13 to higher dimensions and to higher-order indices of convexity. To state this general conjecture, we introduce the following notation: for a set $X \subseteq \mathbb{R}^d$, let the *k-dimensional skeleton* of T be defined as

$$Skel_k(X) = \bigcup_{Y \in \binom{X}{k+1}} conv(Y).$$

Roughly speaking, the following general conjecture states that sets with large k-index of convexity should contain the k-dimensional skeleton of a large simplex.

Conjecture 1.5.14 ([Bal+17]). *For every $k, d \in \mathbb{N}$ such that $1 \leq k \leq d$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^d$ is a set with $b_k(S) \geq \varepsilon$, then there is a simplex T with vertex set X such that $\lambda_d(T) \geq \delta\lambda_d(S)$ and $Skel_k(X) \subseteq S$.*

For every $d \geq 2$, there is a constant $\beta = \beta(d)$ such that every convex set $K \subseteq \mathbb{R}^d$ contains a simplex of measure at least $\beta\lambda_d(K)$ [Las11]. Therefore, Theorem 1.5.7 gives the following result.

Corollary 1.5.15 ([Bal+17]). *For every $d \geq 2$, there is a constant $\alpha = \alpha(d) > 0$ such that every set $S \subseteq \mathbb{R}^d$ whose $b_d(S)$ is defined contains a simplex of measure at least $\alpha b_d(S)\lambda_d(S)$.*

Corollary 1.5.15 thus asserts that Conjecture 1.5.14 holds if $k = d \geq 2$, since $Skel_d(X) = conv(X) = T$.

1.6 INCIDENCES AND COVERING BY SUBSPACES

In this section, we focus on the minimum number of linear subspaces needed to cover points that are contained in the intersection of a given lattice with a given symmetric convex body. We also apply our results to the problem of estimating the maximum number of incidences between a set of points and an arrangement of hyperplanes. First, we state some necessary definitions.

For linearly independent vectors $b_1, \ldots, b_d \in \mathbb{R}^d$, the d-dimensional *lattice* $\Lambda = \Lambda(b_1, \ldots, b_d)$ with *basis* $\{b_1, \ldots, b_d\}$ is the set of all linear combinations of the vectors $b_1, \ldots, b_d$ with integer coefficients. The *determinant of* Λ is $\det(\Lambda) = |\det(B)|$, where B is the $d \times d$ matrix with the vectors $b_1, \ldots, b_d$ as columns. For a positive integer d, we let $\mathcal{L}^d$ be the set of d-dimensional lattices Λ, that is, lattices with $\det(\Lambda) \neq 0$.

A convex body K is *symmetric about the origin* if $K = -K$. We let $\mathcal{K}^d$ be the set of d-dimensional compact convex bodies in $\mathbb{R}^d$ that are symmetric about the origin.

1.6.1 *Covering lattice points by subspaces*

For an integer $d \geq 2$, a collection $\mathcal{S}$ of subsets in $\mathbb{R}^d$ *covers* a set P of points from $\mathbb{R}^d$ if every point from P lies in some set from $\mathcal{S}$. For positive integers k, n, and r with $1 \leq k \leq d-1$, we let $l(d,k,n,r)$ be the maximum size of a set $S \subseteq \mathbb{Z}^d \cap B^d(n)$ such that every k-dimensional *linear* subspace of $\mathbb{R}^d$ contains at most $r-1$ points of S. We also let $g(d,k,n)$ be the minimum number of k-dimensional linear subspaces of $\mathbb{R}^d$ necessary to cover $\mathbb{Z}^d \cap B^d(n)$.

It follows from the definitions that $l(d,k,n,r) \leq (r-1)g(d,k,n)$. For fixed d and $k = d-1$, Bárány, Harcos, Pach, and Tardos [Bár+01] obtained the following asymptotically tight estimates:

$$l(d,d-1,n,d) = \Theta_d(n^{d/(d-1)}) \quad \text{and} \quad g(d,d-1,n) = \Theta_d(n^{d/(d-1)}).$$

In fact, Bárány et al. [Bár+01] proved a stronger result about covering $\Lambda \cap K$ for a given lattice $\Lambda \in \mathcal{L}^d$ and a body $K \in \mathcal{K}^d$. To state this result, we need to introduce some notation.

For a lattice $\Lambda \in \mathcal{L}^d$, a body $K \in \mathcal{K}^d$, and $i \in [d]$, we let $\lambda_i(\Lambda, K)$ be the *ith successive minimum of* Λ *and* K. That is,

$$\lambda_i(\Lambda, K) = \inf\{\lambda \in \mathbb{R} \colon \dim(\Lambda \cap (\lambda \cdot K)) \geq i\},$$

where $\dim(X)$ is the dimension of the affine hull of a set $X \subseteq \mathbb{R}^d$; see Figure 1.12. Since K is compact, it is easy to see that the successive minima are achieved. Note that we have $\lambda_1(\Lambda, K) \leq \cdots \leq \lambda_d(\Lambda, K)$ and $\lambda_1(\mathbb{Z}^d, B^d(n)) = \cdots = \lambda_d(\mathbb{Z}^d, B^d(n)) = 1/n$.

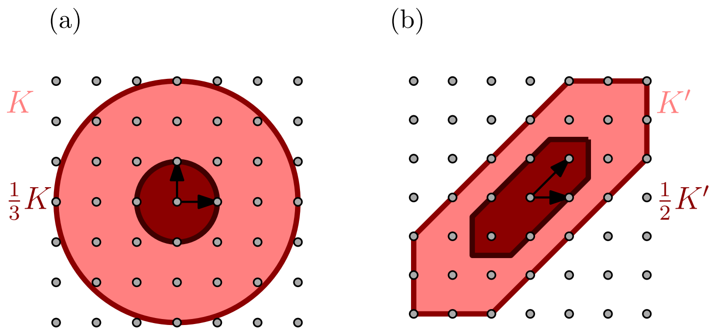

Figure 1.12: Examples of convex bodies K and K' from $\mathcal{K}^2$ such that (a) $\lambda_1(\mathbb{Z}^2, K) = \lambda_2(\mathbb{Z}^2, K) = 1/3$ and (b) $\lambda_1(\mathbb{Z}^2, K') = 1/3$ and $\lambda_2(\mathbb{Z}^2, K') = 1/2$.

Theorem 1.6.1 ([Bár+01]). *For an integer $d \geq 2$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i = \lambda_i(\Lambda, K)$ for every $i \in [d]$. If $\lambda_d \leq 1$, then the set $\Lambda \cap K$ can be covered with at most*

$$c2^d d^2 \log d \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

$(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, where c is some constant.

On the other hand, if $\lambda_d \leq 1$, then there is a subset S of $\Lambda \cap K$ of size

$$\frac{1-\lambda_d}{16d^2} \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

such that no $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ contains d points from S.

For linear subspaces of lower dimension, Brass and Knauer [BK03] conjectured that $l(d, k, n, k+1) = \Theta_{d,k}(n^{d(d-k)/(d-1)})$ for d fixed. This conjecture was refuted by Lefmann [Lef12] who showed that, for all d and k with $1 \leq k \leq d-1$, there is a constant c such that $l(d, k, n, k+1) \leq c \cdot n^{d/\lceil k/2 \rceil}$ for every positive integer n. This bound is asymptotically smaller in n than the growth rate conjectured by Brass and Knauer for sufficiently large d and almost all values of k with $1 \leq k \leq d-1$.

The following problem about covering lattice points by linear subspaces is also posed in the book by Brass, Moser, and Pach [BMP05].

Problem 1.6.2 ([BMP05]). *What is the minimum number of k-dimensional linear subspaces necessary to cover the d-dimensional $n \times \cdots \times n$ lattice cube?*

Balko, Cibulka, and Valtr [BCV19] nearly settled Problem 1.6.2 by proving new bounds on the minimum number of k-dimensional linear subspaces that are necessary to cover points in the intersection of a given lattice with a body from $\mathcal{K}^d$. First, we state their upper bound.

Theorem 1.6.3 ([BCV19]). *For integers d and k with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i = \lambda_i(\Lambda, K)$ for $i = 1, \ldots, d$. If $\lambda_d \leq 1$, then we can cover $\Lambda \cap K$ with $O_{d,k}(\alpha^{d-k})$ k-dimensional linear subspaces of $\mathbb{R}^d$, where*

$$\alpha = \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

Balko, Cibulka, and Valtr [BCV19] also showed the following lower bound.

Theorem 1.6.4 ([BCV19]). *For integers d and k with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i = \lambda_i(\Lambda, K)$ for $i = 1, \ldots, d$. If $\lambda_d \leq 1$, then, for every $\varepsilon \in (0,1)$, there is a positive integer $r = r(d, \varepsilon, k)$ and a set $S \subseteq \Lambda \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, where*

$$\beta = \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)},$$

such that every k-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from S.

Since $\lambda_i(\mathbb{Z}^d, B^d(n)) = 1/n$ for every $i \in [d]$, Theorem 1.6.4 with $\Lambda = \mathbb{Z}^d$ and $K = B^d(n)$ gives the following lower bound on $l(d,k,n,r)$.

Corollary 1.6.5 ([BCV19]). *Let d and k be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, there is an $r = r(d, \varepsilon, k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have*

$$l(d,k,n,r) \geq \Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}).$$

This bound is very close to the bound conjectured by Brass and Knauer [BK03]. Thus it seems that the conjectured growth rate of $l(d,k,n,r)$ is true if we allow r to be (significantly) larger than $k+1$.

Since $l(d,k,n,r) \leq (r-1)g(d,k,n)$ for every $r \in \mathbb{N}$, Theorem 1.6.3 and Corollary 1.6.5 give the following almost tight estimates on $g(d,k,n)$, nearly settling Problem 1.6.2.

Corollary 1.6.6 ([BCV19]). *Let d, k, and n be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, we have*

$$\Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}) \leq g(d,k,n) \leq O_{d,k}(n^{d(d-k)/(d-1)}).$$

1.6.2 *Point-hyperplane incidences*

The problem of determining $l(d,n,k,r)$ is related to bounding the maximum number of point-hyperplane incidences, a classical problem in discrete geometry. For an integer $d \geq 2$, let P be a set of n points in $\mathbb{R}^d$ and let $\mathcal{H}$ be an arrangement of m hyperplanes in $\mathbb{R}^d$. An *incidence between P and $\mathcal{H}$* is a pair (p, H) such that $p \in P$, $H \in \mathcal{H}$, and $p \in H$. The number of incidences between P and $\mathcal{H}$ is denoted by $inc(P, \mathcal{H})$.

In the plane, the famous *Szemerédi–Trotter theorem* [ST83] says that the maximum number of incidences between a set of n points in $\mathbb{R}^2$ and an arrangement of m lines in $\mathbb{R}^2$ is at most $O((mn)^{2/3}+m+n)$. This is known to be asymptotically tight, as a matching lower bound was found earlier by Erdős [Erd46].

For $d \geq 3$, it suffices to consider all points from P lying in an affine subspace that is contained in every hyperplane from an arrangement $\mathcal{H}$. Then the number of incidences is maximum possible, that is $inc(P,\mathcal{H}) = mn$. In order to avoid this degenerate case, we forbid large complete bipartite graphs in the *incidence graph $G(P,\mathcal{H})$ of P and $\mathcal{H}$*, which is the bipartite graph on the vertex set $P \cup \mathcal{H}$ with edges $\{p,H\}$ where (p,H) is an incidence between P and $\mathcal{H}$.

With this restriction, bounding $inc(P,\mathcal{H})$ becomes more difficult. It follows from the works of Chazelle [Cha93], Brass and Knauer [BK03], and Apfelbaum and Sharir [AS07] that the number of incidences between any set P of n points in $\mathbb{R}^d$ and any arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ with $K_{r,r} \not\subseteq G(P,\mathcal{H})$ satisfies

$$inc(P,\mathcal{H}) \leq O_{d,r}\left((mn)^{1-1/(d+1)}+m+n\right). \tag{1.13}$$

The following estimate proved by Balko, Cibulka, and Valtr [BCV19] using Corollary 1.6.5 is the best general lower bound on $inc(P,\mathcal{H})$. It improves an earlier lower bound by Brass and Knauer [BK03].

Theorem 1.6.7 ([BCV19])**.** *For every integer $d \geq 2$ and $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon) \in \mathbb{N}$ such that for all positive integers n and m the following statement is true. There is a set P of n points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and*

$$inc(P,\mathcal{H}) \geq \begin{cases} \Omega_{d,\varepsilon}\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right) & \text{if } d \text{ is odd,} \\ \Omega_{d,\varepsilon}\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) & \text{if } d \text{ is even.} \end{cases}$$

The parameter ε in the exponent can be removed for $d \leq 3$ [BK03, BCV19]. That is, we have the bounds $\Omega((mn)^{2/3})$ for $d = 2$ and $\Omega((mn)^{7/10})$ for $d = 3$. However, the bounds do not match the upper bounds from (1.13) for $d \geq 3$.

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[BJV19] M. Balko, V. Jelínek, and P. Valtr. "On ordered Ramsey numbers of bounded-degree graphs." In: *J. Combin. Theory Ser. B* 134 (2019), pp. 179–202. DOI: 10.1016/j.jctb.2018.06.002.

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GENERALIZED ORDERED RAMSEY NUMBERS

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[BV17] M. Balko and P. Valtr. “A SAT attack on the Erdős–Szekeres conjecture.” In: *European J. Combin.* 66 (2017), pp. 13–23. DOI: 10.1016/j.ejc.2017.06.010.

[BV20] M. Balko and M. Vizer. “Edge-ordered Ramsey numbers.” In: *European J. Combin.* 87 (2020), pp. 103100, 11. DOI: 10.1016/j.ejc.2020.103100.

[BV21] M. Balko and M. Vizer. *On ordered Ramsey numbers of tripartite 3-uniform hypergraphs*. Submitted. 2021.

4

COUNTING HOLES IN POINT SETS

This chapter contains the following publications:

[Aic+20] O. Aichholzer, M. Balko, T. Hackl, J. Kynčl, I. Parada, M. Scheucher, P. Valtr, and B. Vogtenhuber. “A superlinear lower bound on the number of 5-holes.” In: *J. Combin. Theory Ser. A* 173 (2020), pp. 105236, 31. DOI: 10.1016/j.jcta.2020.105236.

[BSV21a] M. Balko, M. Scheucher, and P. Valtr. *Holes and islands in random point sets*. To appear in *Random Structures & Algorithms.* Extended abstract in: *36th International Symposium on Computational Geometry (SoCG 2021)*. Vol. 164. LIPIcs, 2020, Art.14–16. 2021.

[BSV21b] M. Balko, M. Scheucher, and P. Valtr. *Tight bounds on the expected number of holes in random point sets*. Submitted. 2021.

5 VISIBILITY PROBLEMS

This chapter contains the following publications:

[BCV18] M. Balko, J. Cibulka, and P. Valtr. “Drawing graphs using a small number of obstacles.” In: *Discrete Comput. Geom.* 59.1 (2018), pp. 143–164. DOI: 10.1007/s00454-017-9919-2.

[Bal+17] M. Balko, V. Jelínek, P. Valtr, and B. Walczak. “On the Beer index of convexity and its variants.” In: *Discrete Comput. Geom.* 57.1 (2017), pp. 179–214. DOI: 10.1007/s00454-016-9821-3.

6 INCIDENCES AND COVERING BY SUBSPACES

This chapter contains the following publication:

[BCV19] M. Balko, J. Cibulka, and P. Valtr. "Covering lattice points by subspaces and counting point-hyperplane incidences." In: *Discrete Comput. Geom.* 61.2 (2019), pp. 325–354. DOI: 10.1007/s00454-018-9970-7.