Report on “Convergence in Banach spaces”

The thesis consists of 3 papers, one of which is published (in JFA) and two accepted for publication (in Studia Math. and JMAA).

Brief summary of the content of the thesis: The first two papers deal with the notion of weak*-derived sets of higher orders. Recall that given a Banach space X and $A \subset X^*$, the *weak* derived set* $A^{(1)}$ *of* A consists of all weak* limits of bounded convergent nets in A. We put $A^{(0)} = A$, given a successor ordinal α, we put $A^{(\alpha)} = (A^{(\alpha-1)})^{(1)}$ and for a limit ordinal α we put $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)}$. The *order of* A is the least ordinal α such that $A^{(\alpha)} = A^{(\alpha+1)}$.

First paper: It follows from a result by M. Ostrovskii from 2011 that a Banach space is reflexive, if and only if the order of every convex set $A \subset X^*$ is either 0 or 1. The most difficult part was to show that in nonreflexive spaces there exists a convex set in X^* of order greater than 1. The main result of the first paper contained in the thesis (published in JFA) is that in duals of nonreflexive spaces one can find a convex set of order α for any $\alpha \in \mathbb{N} \cup \{\omega + 1\}$. This significantly improves the result of M. Ostrovskii mentioned above. This first paper contained in the thesis was already cited in a work of M. Ostrovskii, where he further develops and improves Silber’s ideas and proves that the same holds for any countable successor ordinal α, that is, in duals of nonreflexive spaces one may find a convex set of order $\alpha + 1$ for any countable ordinal α.

Second paper: In the paper from 2011 by Ostrovskii mentioned above it is also proved that given a Banach space X, the following conditions are equivalent.

- There exists a linear subspace $A \subset X^*$ such that $A^{(1)}$ is a proper norm-dense subset of X^*.
- X is a non-quasi-reflexive and contains infinite-dimensional subspace with separable dual.

In the second paper contained in the thesis (accepted in Studia Math.) the author extends the result to higher ordinals. Namely, he proves that the two conditions above are equivalent to the fact that for each countable ordinal α there is a subspace $A \subset X^*$ such that $A^{(\alpha+1)}$ is a proper norm dense subspace of X^*.

Third paper: In the third paper (accepted in JMAA) the author is dealing with a quantified version of a variant of the ξ-Banach-Saks property and weak ξ-Banach-Saks property for countable ordinals ξ. The author suggests several ways of quantitative versions of those properties and very deeply investigates those. The main results include improvements of some deep results proved by Argyros et al, or negative answer to a Question published in a paper by H. Bendová, O. Kalenda and J. Spurný. In the paper the author also suggests new interesting questions/problems suitable for further investigations.

Mathematical level of the results: What I find common for all the papers contained in the thesis is that all the results are very deep, proofs are very involved both technically and combinatorially. Since the first two papers were already accepted at the time the thesis was written, I did not check all the proofs in a detail.

The last paper included in the thesis was accepted only later, so I did read it in a detail and I can responsibly say that the level of writing is very high - much better when compared e.g. with the cited literature. There are almost no typos. Considering how involved the topic of this paper is, arguments are explained very clearly. There is only one slip which is fully understandable in such a technically involved paper and does not change my opinion on the thesis at all - however, since the paper was accepted already and it seems no corrections are possible in the accepted version, I do include some remarks concerning this paper at the end of the report for the use of experts trying to understand the paper in the future.

Summary: The author has for sure proved he is able to read scientific papers (even the ones which are difficult to read), think about those, push forward our knowledge and inspire other mathematicians to develop the area. For those reasons I believe the author deserves the degree of Doctor of Philosophy.

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Some remarks concerning the paper "Quantification of Banach-Saks properties of higher orders"

All the remarks below were discussed with the author who agrees with those. The most important remark is the one emphasize with the red color.

- page 37, the end of Section 3.2.3: I believe that at this place it would be helpful to mention some more properties of ξ_k^M. I would recommend to start with the following ones

  (P-1) for any $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $i, k \in \mathbb{N}$ we have
  $$\xi_{i+k}^M = \xi_i^{M \setminus \bigcup_{j=1}^k \operatorname{suppt} \xi_j^M}$$
  (the proof is by induction over ξ, quite tedious, but straightforward .. so hopefully can be left as an excercise to the reader)

  (P-2) for any $M, N \in [\mathbb{N}]$, $\xi < \omega_1$ and $k \in \mathbb{N}$ we have that whenever
  $$M \cap [1, \max \operatorname{suppt} \xi_k^M] = N \cap [1, \max \operatorname{suppt} \xi_k^M],$$
  then $\xi_i^M = \xi_i^N$ for every $i \leq k$.
  (the proof is again by induction over ξ, using the above it is quite short and straightforward .. so hopefully can be left as an excercise to the reader as well)

  And then mention that using those two one can inductively prove properties P.3 and P.4 from [3, page 171] (those are used at various places later).
- page 49, line -8: there should be N instead of M (in $ca((x_n)_{n \in M})$)
- page 51, Definition of non-increasing block convex combination: one should assume moreover that the numbers $\alpha(j)$ are positive
- proof of Lemma 3.20: there is a gap in the proof and at this moment it is not clear whether it can be fixed.

  The problematic place is on page 54, lines -8 and -7: by definition we have $\zeta_k^P = [\zeta_{n_k}]_1^{P_k}$ where $n_k = \min P_k$ (and not $[\zeta_k]_1^{P_k}$ as the author claims). Thus, it is not clear whether we have
  $$\zeta_{k+1}^N = [\zeta_{n_0+k}]_1^{N_k}$$
  on the last line at page 54, which is what is frequently used in what follows.

  Thus, it is not clear whether Lemma 3.20 holds. Consequently, it is not clear whether Proposition 3.21 holds (because Lemma 3.20 is used in its proof). Finally, the quantity $\delta_0(A)$ defined on page 57 should be rather defined by the formula
  $$\delta_0(A) := \inf_{\xi < \omega_1} bs_\xi^s(A)$$
  as at this moment it is not clear whether the minimum (used in the paper instead of infimum) exists. As far as I know, once the quantity δ_0 is redefined, all the proofs work just fine, so this does not influence validity of what follows.

- page 58, Lem 3.23: there is a gap and Lemma needs to be corrected. Namely, in the second paragraph of the proof of from $bs^s_\xi(A) > 4c$ we obtain (x_n) in A which weakly converges to some x and $(x_n - x)$ generates $\ell_1^{\xi+1}$-spreading model with constant c (the same inaccuracy occurred at more places above, but here it seems to be really relevant for the presented argument). Thus, I do not see how to obtain that $\{(x_n)_{n\in F} \colon F \in \mathcal{S}_{\xi+1}\} \subset \mathcal{T}$. It seems to me that one needs to define $\mathcal{T}$ in a slightly different way (e.g. as a subset of $(\overline{A - \overline{A}^w})^n$), reformulate Lemma 3.23 (basis of ℓ_1 embeds into $\overline{A - \overline{A}^w}$) and then modify its use in Prop 3.24
- page 59, third example (in c_0): the sequence witnessing $\beta(A) = 2$ should be rather $e_1 + \ldots + e_n - e_{n+1}$ (and not $\ldots - e_{n-1}$)
- page 62, line 2: inequalities which may be strict are the third and fourth (and not second and third)