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**BACHELOR THESIS**

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# Resolutions of singularities using blow-ups

Department of Algebra

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Abstract: The thesis is focused on the resolution of singularities via method of blow-ups. We simplified an approach from the literature and showed usage of this method on an example.

Keywords: blow-up algebraic geometry singularity resolution

# Contents

# Introduction

In this thesis we introduced the theory of affine and projective varieties, and we introduced concepts that help us with resolution of their singularities.

Singularities are a very common problem not only in geometry but also in different disciplines of mathematics. They are often the reason why we cannot solve problems generally and have to give up on our obtained insight into these special places.

This is not only a problem in pure math but also in real-life applications. Equations defined by polynomials are used almost everywhere where you have computer graphics, robotics and also numerical calculations. There is still ongoing research in this area, which is a sign of the importance of this topic.

In our thesis, we use the method of blow-ups which is very powerful but quite hard to use everywhere, and it needs to solve a lot of special cases if you do not have deep enough insight into this theory.

We first introduce the basics of algebraic geometry, and then we build upon them and introduce blow-ups, which will allow us to find equivalent variety without singularities.

# Chapter 1

# Affine varieties

Unless stated otherwise, $K$ will denote a field and $\overline{K}$ will denote its algebraic closure. The set of all polynomials in variables $x_1, \ldots, x_n$ will be denoted by $K[x_1, \ldots, x_n]$. We will also assume that $n \in \mathbb{N}$. This chapter is mainly from [7].

**Definition 1** (Affine space)**.** Affine space *of dimension* $n \in \mathbb{N}$ *over field* $K$ *is simply the Cartesian product*

$$\mathbb{A}_K^n = \underbrace{K \times \cdots \times K}_{n \text{ times}}.$$

*An element of* $\mathbb{A}_K^n$ *or simply* $\mathbb{A}^n$ *will be called* point, *and if* $P = (a_1, \ldots, a_n), a_i \in K$*, then the* $a_i$ *will be called the* coordinates *of* $P$*.*

**Definition 2** (Affine algebraic set)**.** *A set* $A$ *is an* affine algebraic set *if there exist* $I \subseteq K[x_1, \ldots, x_n]$ *such that*$A$ *is the* zero set *of* $I$*.*

*That is* $A = V(I) = \{a \in K^n \mid f(a) = 0\, \forall f \in I\} \subseteq \mathbb{A}_K^n$*. Note the set* $I$ *can be arbitrary, not necessarily finite or countable.*

*Note.* Given a set of polynomials $f_1, \ldots, f_m \in K[x_1, \ldots x_n]\}, \ m \in \mathbb{N}$ we use a slightly incorrect notation $V(f_1, \ldots, f_m)$ instead of the formally correct $V(\{f_1, \ldots, f_n\})$.

**Lemma 1** (Closedness of algebraic sets)**.** *The following hold:*

1. $\emptyset$ *and* $\mathbb{A}_K^n$ *are affine algebraic sets.*

2. *Union of two algebraic sets is algebraic.*

3. *Arbitrary intersection of algebraic subsets of* $\mathbb{A}_K^n$ *is again an algebraic set.*

*Proof*.

1. We have $\emptyset = V(1)$ and $V(0) = \mathbb{A}^n_K$.
2. It holds that $V(I_1) \cup V(I_2) = V(\{f_1 f_2 \mid f_1 \in I_1, f_2 \in I_2\})$, which is algebraic.
3. Let $\{V(I_i)\}_{i\in\alpha}$ be a set of affine algebraic varieties. Then $\bigcap_{i\in\alpha} V(I_i) = V(\bigcup_{i\in\alpha} I_i)$.

$\square$

**Definition 3** (Topology)**.** *A* topological space *is a pair $(X, \tau)$ consisting of a set $X$ and a faimily of subsets of $X$ satisfying the following conditions:*

1. *$\emptyset \in \tau$ and $X \in \tau$.*
2. *If $U_1 \in \tau$ and $U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.*
3. *If $\mathcal{A} \subseteq \tau$, then $\bigcup \mathcal{A} \in \tau$.*

*Elements of $\tau$ are called* open sets*, and their complements in $X$ are called* closed sets.

**Definition 4** (Continuous map)**.** *Let $(X, \tau)$ and $(Y, \tau_1)$ be topological spaces. A mapping $f : X \to Y$ is called continous if $f^{-1}(U) \in \tau$ for any $U \in \tau_1$.*

**Definition 5** (Induced topology)**.** *Let $(X, \tau)$ be a topological space. For $A \subseteq X$ we define topology* induced *by $\tau$ on $A$ as $\tau_A = \{B \cap A \mid B \in \tau\}$.*

**Definition 6** (Closure in topology)**.** *Let $(X, \tau)$ be a topological space. For any $A \subseteq X$ consider the family $\mathcal{C}_A$ of all closed sets containing $A$. We define $\overline{A} = \cap \mathcal{C}_A$. Obviously $\overline{A}$ is the smallest closed set containing $A$.*

**Definition 7** (Homeomorphism)**.** *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. A mapping $f : X \to Y$ is said to be a* homeomorphism *if it is a continous bijection with a continous inverse (it is* open *mapping).*

**Definition 8** (Dense set)**.** *Let $(X, \tau_X)$ be a topological space. A set $A \subseteq X$ is called* dense set *in $X$ if $\overline{A} = X$. Or equivalently, for each open non-empty $G \subseteq X$, $A \cap G \neq \emptyset$.*

**Definition 9** (Zariski open set)**.** *Let $Z$ be the set of all complements of affine algebraic varieties on $\mathbb{A}^n_K$*

$$Z = \{X_S \mid X_S = \mathbb{A}^n_K \setminus V(S),\ S \subseteq K[x_1, \ldots, x_n]\}.$$

*This defines a topology on $\mathbb{A}^n_K$ called* Zariski topology.

Correctness of the definition can be infered from Lemma 1.

**Definition 10** (Noetherian ring)**.** *A commutative ring $R$ is called* noetherian *if there is no infinite strictly increasing chain of ideals $I_1 \subseteq eqI_2 \subsetneq I_3 \subsetneq \dots$. Equivalently, $R$ is noetherian if each ideal $I \subseteq R$ is finitely generated.*

*Note.* Equivalence of both conditions is proven in [3].

**Theorem 2** (Hilbert Basis Theorem)**.** *Commutative ring $R$ is a noetherian if and only if $R[x]$ is noetherian. In particular, $K[x_1, \dots, x_n]$ is noetherian for each field $K$ and natural number $n \geq 1$.*

*Proof.* Proof can be found for instance in [3].

□

*Corollary* 1. For each algebraic set $X \subseteq \mathbb{A}^n_K$ there exist an ideal $I \subseteq K[x_1, \dots, x_n]$ so that $X = V(I)$. By Theorem 2 there exists $r \in \mathbb{N}$ and polynomials $f_1, \dots, f_r$ so that $X = V(f_1, \dots, f_r)$.

**Definition 11** (Basis of topology)**.** *A family $\mathcal{B} \subseteq \tau$ is a called a* basis *of a topological space $(X, \tau)$ if every open subset of $X$ can be represented as the union of a subfamily of $\mathcal{B}$. [1]*

Note that empty set can be obtained as $\emptyset = \cup\emptyset$.

**Lemma 3** (Characterization of basis)**.** *A familly of subsets $\mathcal{B} \subseteq \tau$ which satisfies conditions (B1) and (B2) below if and only if it is a basis of topological space $(X, \tau)$, where $\tau = \{\bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B}\}$.*

*(B1) For any $U_1,\ U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U \in \mathcal{B}$ such that $x \in U \subseteq U_1 \cap U_2$.*

*(B2) For every $x \in X$ there exists a $U \in \mathcal{B}$ such that $x \in U$. [1]*

*Proof.* "$\Rightarrow$"
We have $X = \bigcup_{x \in X} U_x$, where $x \in U_x$ from (B2) and $\emptyset$ as the trivial union. Fix $U$ an open set $U \in \tau$. Then $U = \bigcup_{x \in U} V_x$ where $x \in V_x \subseteq U \cap U = U$ so $\mathcal{B}$ is a basis of $(X, \tau)$. If $U_1,\ U_2 \in \mathcal{B}$ then $U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} V_x$ where $V_x \subseteq U_1 \cap U_2$ from (B1).
"$\Leftarrow$"
Trivial.

□

**Definition 12** (Irreducible set). *Let $X$ be a topological space and let $Y$ be its nonempty subset. $Y$ is* irreducible *if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets $Y_1, Y_2 \neq \emptyset$, closed in $Y$.*

**Lemma 4** (Characterization of irreducibility). *A nonempty topological space $X$ is irreducible if and only if each non-empty open subset of $X$ is dense in $X$.*

*Proof.* If $X$ is reducible, then there are $A, B$ closed proper nonempty subsets of $X$ such that $X = A \cup B$. $X \setminus B$ and $X \setminus A$ are both open and nonempty, and but their intersection is equal to $X \setminus (A \cup B) = \emptyset$, thus neither of them is dense in $X$.

In order to prove the converse, let us assume that there is a non-empty open subset $U \subseteq X$ which is not dense in $X$. Then there is open subset $G \subseteq X$ such that $G \cap U = \emptyset$. But $X = X \setminus U \cup \overline{U}$. Both of these sets are closed, non-empty and proper, since $\overline{U} \neq X$ and hence $X$ is reducible.

□

**Definition 13** (Quasi-affine variety). *An open subset of an affine variety is called a* quasi-affine variety. *Equivalently, a quasi-affine variety is the intersection of an open set and a closed irreducible set in the Zariski topology.*

**Definition 14** (Ideal). *The ideal of a set $X \subseteq \mathbb{A}^n_K$ is defined as*

$$I(X) = \{f \in K[x_1, \ldots, x_n] \mid f(P) = 0, \forall P \in X\}.$$

The terminology comes from the fact that $I(X) \subseteq K[x_1, \ldots, x_n]$ is an ideal of the polynomial ring. This is easy to see beacause if $f, g \in I(x)$ and $h \in K[x_1, \ldots, x_n]$ then $(f+g)(x) = f(x)+g(x)$ and $(h \cdot f)(x) = h(x) \cdot f(x) = h(x) \cdot 0 = 0$ thus $f+g, h \cdot f \in I(X)$ and this implies $I(X)$ is an ideal of $K[x_1, \ldots, x_n]$. Now we have a function $V : S \mapsto V(S)$ which maps subsets of $K[x_1, \ldots, x_n]$ to algebraic sets, and a function $I : X \mapsto I(X)$ which maps subsets of $\mathbb{A}^n_K$ to ideals. Some of their basic properties are summarized in following lemma.

**Lemma 5** (Properties of V and I). *Let $n \in \mathbb{N}, X, X_1, X_2 \subseteq \mathbb{A}^n_K$ and $S, S_1, S_2 \subseteq K[x_1, \ldots, x_n]$.*

1. *If $X_1 \subseteq X_2$, then $I(X_1) \supseteq I(X_2)$.*
2. *If $S_1 \subseteq S_2$, then $V(S_1) \supseteq V(S_2)$.*
3. *$I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.*
4. *$I(V(S)) \supseteq S$ and $V(I(X)) \supseteq X$.*

5. $V(I(X)) = \overline{X}$.

6. $I(\emptyset) = K[x_1, \dots, x_n]$ *and, if the field* $K$ *is infinite it also holds that* $I(\mathbb{A}^n_K) = \{0\}$.

7. $I(V(I(X))) = I(X)$ *and* $V(I(V(S))) = V(S)$.

*Proof.* (1), (2), (6) and (3) can be proven straight from definitions.
(4): Fix $A \in X$. For $f \in I(X)$ we have $f(A) = 0$ and this implies $A \in V(I(X))$, thus $X \subseteq V(I(X))$. Similarly, if $f \in I(X)$ then $f(A) = 0$, $\forall A \in V(S)$ and this implies $f \in I(V(S))$.
To prove the statement of ( 7), we need to prove two inclusions. First is by ( 4) we have $I(V(I(X))) \supseteq I(X)(4)$. The other inclusion:

$$V(I(X)) \supseteq X \qquad (4)$$
$$I(V(I(X))) \subseteq I(X) \qquad (1)$$

The proof of $V(I(V(S))) = V(S)$ would be similar.
To prove (5) we prove two inclusions. By definition of the Zariski topology we see that $V(I(X))$ is closed. From ( 4) we have $V(I(X)) \supseteq X$. Next we need to prove that if $Y \supseteq X$ is closed then $V(I(X)) \subseteq Y$. Since Y is closed we have $Y = V(J)$ for some ideal $J \subseteq K[x_1, \dots, x_n]$. Now by applying ( 1) and ( 2), we get that $V(I(V(J))) \supseteq V(I(X))$. The left-hand side can be simplified with ( 7) and we get $Y = V(J) \supseteq V(I(X))$, which is what we wanted.

□

**Definition 15** (Radical ideal). *If* $R$ *is a commutative ring and* $I \subseteq R$ *is and ideal then the* radical *of* $I$*, denoted by* $\sqrt{I}$*, is defined as:*

$$\sqrt{I} = \{f \in R \mid \exists s \in \mathbb{N} \text{ such that } f^s \in I\}$$

**Theorem 6** (Weak Nullstellensaztz). *Let* $K$ *be an algebraically closed field,* $n \in \mathbb{N}$ *and* $I \subseteq K[x_1, \dots, x_n]$ *be a proper ideal. Then* $V(I)$ *is non-empty.*

**Theorem 7** (Hilbert's Nullstellensatz). *Let* $K$ *be algebraically closed field,* $n \in \mathbb{N}$ *and* $J \subseteq K[x_1, \dots, x_n]$ *be an ideal. Then* $I(V(J)) = \sqrt{J}$.

*Proof*. Both proofs can be found in [7].

□

**Lemma 8** (Characterization of irreducible algebraic sets). *An algebraic set* $X \subseteq \mathbb{A}^n_K$ *is irreducible if and only if its ideal* $I(X)$ *is a prime ideal.*

*Proof* . We first show that irreducibility of $X$ implies $I(X)$ is prime ideal. Toward a contradiction, suppose that $A, B \subseteq K[x_1, \ldots, x_n]$ are ideals such that $AB \subseteq I(X)$ and both $A, B \nsubseteq I(X)$. This implies that there exists $f \in A$ and $g \in B$ such that $f(x) \neq 0 \neq g(y)$ for some $x, y \in X$. We work over integral domain $K$ so $f(x)g(x) \neq 0$ but $fg \in AB \subseteq I(X)$ and that is contradiction with our assumption.
For the other implication let $I(X)$ be prime ideal and $X = X_1 \cup X_2$. Thus we have $I(X) = I(X_1) \cap I(X_2)$, but $I(X)$ is a prime ideal and that means WLOG $I(X) = I(X_1)$. Hence $X = X_1$ and $X$ is irreducible.

□

*Corollary* 2. Let $\mathcal{M}$ be a maximal ideal of $A = K[x_1, \ldots, x_n]$. Then $\mathcal{M}$ corresponds to minimal irreducible closed subset of $\mathbb{A}^n_K$, that is, a point.

**Definition 16** (Polynomial map). *Let $n, l \in \mathbb{N}$, $X \subseteq \mathbb{A}^n_K$ and $Y \subseteq \mathbb{A}^\ell_K$ be algebraic sets. A map $f : X \to Y$ is a* polynomial map *if there exist polynomials $f_1, , \ldots, f_\ell \in K[x_1, \ldots, x_n]$ such that for each $P = (a_1, \ldots, a_n) \in X$ we have*

$$f(P) = (f_1(P), \ldots, f_\ell(P)).$$

**Lemma 9** (Polynomial maps are continous). *Polynomial maps $f : X \to Y$ are continous with respect to the Zariski topologies on X and Y.*

*Proof.* Suppose that $f : X \to Y$, $X \subseteq \mathbb{A}^n_K, Y \subseteq \mathbb{A}^\ell_K$ is a polynomial map such that $f(P) = (f_1(P), \ldots, f_\ell(P))$ for $f_i \in K[x_1, \ldots, x_n]$. It is sufficient to show that the preimage of a closed set in $f$ is again a closed set. Fix a closed set $C \subseteq Y$. By corollary of the Hilbert basis theorem 2 we can find polynomials $g_1, \ldots, g_r$, so that $C = V(g_1, \ldots, g_r)$. This can also be understood as $C = g^{-1}((0, \ldots, 0))$, $(0, \ldots, 0) \in \mathbb{A}^r_K$, where

$$\begin{aligned} g : Y &\to \mathbb{A}^r_K \\ P &\mapsto (g_1(P), \ldots, g_r(P)). \end{aligned}$$

By the following computation we see that

$$f^{-1}(C) = f^{-1}\left(g^{-1}((0, \ldots, 0))\right) = (g \circ f)^{-1}((0, \ldots, 0)) = V(h_1, \ldots, h_r).$$

Here $h_i(P) = g_i(f_1(P), \ldots, f_\ell(P))$, for $P \in X$. The preimage of C with respect to $f$ is clearly closed and that is what we wanted to show. □

**Definition 17** (Affine coordinate ring). *The set $\{f : X \to \mathbb{A}^1_K \mid f$ is a polynomial map $\}$ is called the* coordinate ring *of $X$ and is denoted by $K[X]$.*

# Chapter 2

# Projective Space

In this chapter, we will present projective space and some constructions analogous to constructions from Chapter 1. We will use a natural compactification of $\mathbb{A}^n_K$ obtained by adding an infinitely distant point in every direction called protective space. The concept of projective space is very important in understanding sets of solutions of polynomial equations. Instead of thinking about a point in space, we will be thinking about lines through origin. Each line is represented by its direction, hence we remove $0$ because it does not generate a line. This approach, originally created by Desargues to study the conics, offers us tremendous advantages in generalizing special cases of affine geometry, into all-inclusive statements. [4]

**Definition 18** (Projective space)**.** Projective space *is the set of all one dimensional subspaces of the vector space $K^{n+1}$ that is set of all lines through the origin in $K^{n+1}$. Projective space is denoted by $\mathbb{P}^n_K$. Formally we will define it as*

$$\mathbb{P}^n_K = \frac{\mathbb{A}^{n+1}_K \setminus \{0\}}{\sim}$$

*where $\sim$ denotes the equivalence relation of points lying on the same line through the origin: $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if there exists $\lambda \in K, \lambda \neq 0$ such that $(x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n)$. We will denote $[x_0 : \cdots : x_n] = [(x_0, \ldots, x_n)]_\sim$. We call this* homogenous coordinates *of the point in the projective space.*

This notation emphasizes that the homogeneous coordinates are defined only up to their nonzero scalar multiple. [6]

The value of a polynomial at a point $x \in \mathbb{P}^n$ is generally not well-defined. Thus, the naive way of defining vanishing set of a polynomial will fail.

*Observation.* For polynomial $f = x^2 - y \in K[x, y]$ and $[1 : 1] = [2 : 2] \in \mathbb{P}^1$, but we have

$$f(1,1) = 1^2 - 1 = 0$$
$$f(2,2) = 2^2 - 2 = 2$$

However if every term of the polynomial would be of the same degree, we could factor out the scaling constant. This motivates the definition of *homogenous polynomial.*

**Definition 19** (Homogenous polynomial)**.** *A polynomial $f \in K[x_1, \ldots, x_n]$ is called* homogenous *(or a* form*) if all of its terms have the same degree.*

Note that every polynomial $f \in K[x_1, \ldots, x_n]$ of degree $d$ has a unique expression $f = f_0 + f_1 + \cdots + f_d$, where $f_i$ is a form of degree $i$.

*Observation.* Let $m, n \in \mathbb{N}$ and $d \in \mathbb{N}, d \geq 0$ and let $f \in K[x_1, \ldots, x_n]$ be homogenous of degree $d \geq 0$. Let $\lambda \in K,\ \lambda \neq 0$ and $[(a_1, \ldots, a_n)]_\sim \in \mathbb{P}^{n-1}$. Since $f$ is homogenous of degree $d$, we can see that:

$$f(\lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_1, \ldots, a_n).$$

So if $f(a_1, \ldots, a_n) = 0$, then $f(\lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_1, \ldots, a_n) = \lambda^d \cdot 0 = 0$. Hence $f$ is a zero for all representatives $(a_1, \ldots, a_n)$ of $[(a_1, \ldots, a_n)]_\sim$ and the statement "$P \in \mathbb{P}$ is a zero of $f$" is not unreasonable.

This allows us to define the vanishing set of a polynomial in a projective space as well as other constructions from Chapter 1. But before we do that let us introduce a couple of definitions which will help us describe homogenous polynomials better.

**Definition 20** (Homogenous ideal)**.** *An ideal $I$ is* homogenous *if it can be expressed as $I = (f_1, \ldots, f_m)$ where all $f_i$ are homogenous polynomials.*

**Definition 21** (Projectvie algebraic set)**.** *We say that $[a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$ is* zero *of a homogenous polynomial $f \in K[x_1, \ldots, x_{n+1}]$ or that $f$* vanishes *on $[a_1 : \cdots : a_{n+1}]$, if $f(a_1, \ldots, a_{n+1}) = 0$. For a set $S$ of homogeneous polynomials of $K[x_1, \ldots, x_{n+1}]$ we will denote* vanishing set *$\mathcal{V}(S) = \{P \in \mathbb{P} \mid P \text{ is a zero of } f, \forall f \in S\}$. A set $X \subseteq \mathbb{P}^n$ is a* projective algebraic set *if $X = \mathcal{V}(T)$ for a set $T$ of homogenous polynomials.*

The properties of projective algebraic sets are very similar to properties of affine algebraic sets as we can see in following lemma.

**Lemma 10** (Projective sets are closed)**.** *The following hold:*

1. *$\emptyset$ and $\mathbb{A}^n_K$ are projective algebraic sets.*

2. *Union of two projective algebraic sets is algebraic.*

3. *Arbitrary intersection of projective algebraic subsets of $\mathbb{A}_K^n$ is again an algebraic set.*

*Proof* . This proof is basically the same as the proof of Lemma 1.

□

**Definition 22** (Zariski topology on $\mathbb{P}^n$). *Similarly to the affine case, we define* Zariski topology *on $\mathbb{P}^n$ as*

$$Z = \{X_S \mid,\ X_S = \mathbb{P}^n \setminus \mathcal{V}(S),\ S \subseteq K[x_1, \dots, x_{n+1}]\}.$$

**Definition 23** (Projective variety). A Projective variety *is an irreducible algebraic set in $\mathbb{P}^n$. Equivalently, it is a closed irreducible set in $(\mathbb{P}^n, Z)$. Intersection of a projective variety and a Zariski open set is called* quasi-projective variety.

## 2.1 Open cover of projective space

In this section, we will decompose projective space into simpler parts and establish a connection between these parts and the whole space. In fact, $\mathbb{P}^n$ admits an open cover consisting of open sets, each of which is homeomorphic to the affine space $\mathbb{A}_K^n$. We saw in the previous section, that projective space consists of equivalence classes of affine points, which correspond to lines through the origin. We would like to better understand the connection between the affine word and the projective word.

The significance of this approach becomes evident in the subsequent development of the text, where it is utilized to solve a problem in projective space by solving it "locally" and then reconstructing the solution. Our first step will be to look at the intersections of "lines" with the hyperplane not passing through the origin. To put this formally, we first define *affine charts*, denoted by $U_i$, as complements of $\mathcal{V}(x_i)$ in the projective space $\mathbb{P}^n$. Equivalently we could say that $U_i = \{[(a_1 : \dots : a_{n+1})] \in \mathbb{P}^n \mid a_i = 1\}$. We now claim that every affine chart is homeomorphic to $\mathbb{A}_K^n$.

**Theorem 11** (Homeomorphism of affine charts). *Let $i \in \{1, \dots, n+1\}$. The affine chart $U_i \subseteq \mathbb{P}^n$ with its induced topology, is homeomorphic to $\mathbb{A}_K^n$ with its Zariski topology via homeomorphism $\varphi_i$ defined as:*

$$\varphi_i : U_i \to \mathbb{A}_K^n$$
$$(a_1 : \dots : a_i : \dots : a_{n+1}) \mapsto (\frac{a_1}{a_i}, \dots \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i} \dots, \frac{a_{n+1}}{a_1})$$

*Proof* . Without loss of generality we will prove this for $i = 1$. Denote $U = U_1$ and $\varphi = \varphi_1$ and also denote $S = K[y_1, \ldots, y_{n+1}]$ and the set of homogeneous elements of $K[x_1, \ldots, x_n]$ by $S^h$. The mapping $\varphi_1$ is clearly a bijection. We will prove that both $\varphi$ and $\varphi^{-1}$ map closed sets to closed sets. For this proof we will introduce two maps:

$$\begin{tikzcd} K[x_1,\ldots,x_{n+1}] & & K[y_1,\ldots,y_n] \\ \cup\!\mid & & \parallel \\ S^h \arrow[rr, bend left, "\alpha"] & & S \arrow[ll, bend left, "\beta"] \end{tikzcd}$$

Map $\alpha : f \mapsto f(1, y_1, \ldots, y_n)$ for $f \in S^h$ and $\beta$ is defined as the homogeneous polynomial $\beta(g) = x_1^d g(x_2/x_1, \ldots, x_{n+1}/x_1)$ for $g \in K[x_1, \ldots, x_n]$, $\deg(g) = d$. First we will fix $T \subseteq U$ to be a closed subset. Let $\overline{Y}$ be the closure of $Y$ in $\mathbb{P}^n$, thus $\overline{Y} = \mathcal{V}(T),\ T \subseteq S^h$. Let $T' = \alpha(T)$. We have $(b_1, \ldots, b_n) \in \varphi(Y) \Leftrightarrow [1 : b_1 : \cdots : b_n] \Rightarrow f(1, b_1, \ldots, b_n) = 0,\ \forall f \in T \Leftrightarrow g(b_1, \ldots, b_n) = 0,\ \forall g \in T' = \alpha(T)$ $\Leftrightarrow (b_1, \ldots, b_n) \in V(T')$. We get the other implication by $f(1, b_1, \ldots, b_n) = 0,\ \forall f \in T \Rightarrow [1 : b_1 : \cdots : b_n] \in \overline{Y}$. Since $Y \subseteq U$ is closed in $U$ we have $\overline{Y} \cap U = Y$ and clearly $[1 : b_1 : \cdots : b_n] \in U$ thus $[1 : b_1 : \cdots : b_n] \in Y$.
Conversely, let $W$ be a closed subset of $\mathbb{A}^n_K$. Then $W = V(T')$ for some subset $T' \subseteq K[x_1, \ldots, x_n]$.

$$\varphi^{-1}(W) = \{[1 : a_1 : \cdots : a_n] \mid f(a_1, \ldots, a_n) = 0,\ \forall f \in T'\} = \mathcal{V}(\beta(T')) \cap U.$$

So $\varphi$ is homeomorphism.

□

The following lemma then allows us to compute the closure in projective space by computing closures of each part separately.

**Lemma 12** (Homeomorphism preserve closure)**.** *Let $f : X \to Y$ be a homeomorphism between topological spaces. Let $A \subseteq X$. Then $f(\overline{A}) = \overline{f(A)}$.*

*Proof* . From continuity of $f$ we have that $f^{-1}(f(\overline{A})) \supset \overline{A}$ and by aplying $f$ to both sides we get $f(\overline{A}) \supseteq f(\overline{A})$
We know that $f^{-1}$ is continuous hence $f(\overline{A})$ is closed. We have $f(A) \subseteq f(\overline{A})$ and this implies $\overline{f(A)} \subseteq f(\overline{A})$. Now we can apply $f^{-1}$ to both sides and we get

$$\overline{A} = f^{-1}(\overline{f(A)}) \subseteq f^{-1}(f(\overline{A})) = \overline{A}$$

□

## 2.2 Product of varieties

### 2.2.1 The Segre Embedding

The Segre embedding is a mathematical construction that embeds the product of two projective spaces into a higher-dimensional projective space. We define a map:

$$\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$$
$$[a_1 : \dots : a_{r+1}] \times [b_1 : \dots b_{s+1}] \mapsto [a_1b_1 : \dots : a_ib_j : \dots a_{r+1}b_{s+1}]$$

where $N = (r+1)(s+1)-1 = rs+r+s$. We can think about this in a language of matrices. The space $\mathbb{P}^N$ is the projective space of $(r+1) \times (s+1)$ matrices. Vectors $[a_1 : \dots : a_{n+1}]$ and $[b_1 : \dots : b_{n+1}]$ get mapped to their matrix product:

$$[a_1 : \dots : a_{n+1}]^T [b_1 : \dots : b_{n+1}] = [(a_ib_j)_{i,j}],$$

where by $[(a_ib_j)_{i,j}]$ we mean all elements of the product understood as elements of $\mathbb{P}^N$ in obvious left to right order. Clearly, if we multiply any of the input vectors by a nonzero scalar, the change will propagate into the projective matrix but we will get the same element under equivalence. Also, we have at least one pair of $i, j$ that $a_i$ and $b_j$ is nonzero, thus $(a_ib_j)_{i,j}$ is an element of $\mathbb{P}^N$ and it is a well-defined mapping. The image of this map is clearly the set of all rank 1 matrices. They can be characterized by having all $2 \times 2$ minors equal to zero while not being a zero matrix:

$$\begin{vmatrix} a_ib_j & a_ib_\ell \\ a_kb_j & a_kb_\ell \end{vmatrix} = \begin{vmatrix} z_{ij} & z_{i\ell} \\ z_{kj} & z_{k\ell} \end{vmatrix} = 0$$

Thus it can be expressed as projective variety in $\mathbb{P}^N$:

$$\mathcal{V}(\{z_{ij}z_{k\ell} - z_{i\ell}z_{kj} \mid 1 \leq i < j \leq r+1,\ 1 \leq k < \ell \leq s+1\})$$

### 2.2.2 Products of Affine Varieries

**Theorem 13** (Product of affine varieties)**.** *Let $X \subseteq \mathbb{A}^n_K$ and $Y \subset \mathbb{A}^m_K$ affine varieties. Then $X \times Y \in \mathbb{A}^{n+m}_K$ is affine algebraic variety. (Solution of Exercise 3.15[2].)*

*Proof* . From Corollary 1 we know that we can write varieties as $X = V(f_1, \dots, f_k)$ and $Y = V(g_1, \dots, g_\ell)$. We need to prove irreducibility of $X \times Y$.

If $X \times Y = Z_1 \cup Z_2$ for $\emptyset \neq Z_1, Z_2 \subseteq eqX \times Y$ are closed nontrivial sets, we can find two distinct $(x_1, y_1) \in Z_1 \setminus Z_2, (x_2, y_2) \in Z_2 \setminus Z_1$ such that either $x_1 = x_2$ or $y_1 = y_2$. If this would not be possible then $\forall (x_1, y_1) \in Z_1 : \{x_1\} \times Y, \subseteq Z_1 X \times \{y_1\} \subseteq Z_1$ and $\forall (x_2, y_2) \in Z_2 : \{x_2\} \times Y, X \times \{y_2\} \subseteq Z_2$ but that is contradiction with nontriviality of $Z_1, Z_2$. WLOG $x_1 = x_2$. We can define function $f : Y \to X \times Y$, $f(y) = (x_1, y)$. Then $f$ is a polynomial function hence continuous, but $Y = f^{-1}(\{(x_1, y) \in Z_1\}) \cup f^{-1}(\{(x_1, y) \in Z_2\}$ which are nontrivial from the choice of $(x_1, y_1), (x_2, y_2)$.

$\square$

# Chapter 3

# Blowups

## 3.1 Morphism of varietes

**Definition 24** (Regular function)**.** *Let $X \subseteq \mathbb{A}^n_K$ be a quasi-affine variety. A functions $f : X \to \mathbb{A}^1_K$ is* regular *at $P$ if there exist $g, h \in K[X]$ and open neighbourhood $U$ with $P \in U \subseteq X$ such that $h \neq 0$ on $U$, and $f = \frac{g}{h}$ on $U$. We say that $f$ is regular on $X$ if it is regular at every $P \in X$.*

**Lemma 14** (Continuity of regular function on affine variety)**.** *Lemma 3.1 [2] A regular function is continuous when $K$ is identified with $\mathbb{A}^1_K$ in its Zariski topology.*

*Proof.* Proof similar to the proof for quasi-projective varieties stated below. □

**Lemma 15** (Equivalent condition for a closed subset)**.** *A subset $Z$ of topological space $Y$ is closed in $Y$ if and only if there exists open cover $\mathcal{B}$ of $Y$ such that $Z \cap U$ is closed in $U$ for every $U \in \mathcal{B}$.*

*Proof.* "$\Rightarrow$"

Trivial from the definition of subset topology. "$\Leftarrow$"
We have that $Z \cap U_i$ is closed in $U_i$ for all $U_i \in \mathcal{B}$. We want to prove that $Z$ is closed or equivalently that $X \setminus Z$ is open. We have:

$$X \setminus Z = \left( \bigcup_{U_i \in \mathcal{B}} U_i \right) \setminus Z = \bigcup_{U_i \in \mathcal{B}} (U_i \setminus Z)$$

Since $U_i \setminus Z$ is open in $U_i$, which is open, thus $U_i \setminus Z$ is open, and this imply that $Z$ is closed in $Y$. □

**Lemma 16** (Continuity of regular function on quasi-affine variety)**.** *Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety and $f : Y \to \mathbb{A}^1_K$ be regular function. Then $f$ is continuous with respect to Zariski topology.*

*Proof.* It is enough to show that $f^{-1}(a)$ for $a \in \mathbb{A}^1_K$ is a closed set, because only closed sets of $\mathbb{A}^1_K$ are a finite union of such points or the whole set $\mathbb{A}^1_K$, which preimage is clearly closed. For $f^{-1}(a) = \{P \in Y \mid f(P) = a\}$, $a \in \mathbb{A}^n_K$ to be closed means that it can be covered by open subsets $U \subseteq Y$, such that $U \cap f^{-1}(a)$ is closed in $U$. Let $U \subseteq Y$ be an open subset, such that $f = \frac{g}{h}, g, h \in K[x_1, \ldots, x_n]$, homogenous of the same degree and $h \neq 0$ on $U$. Then $f^{-1}(a) \cap U = \{P \in Y \mid f(P) = \frac{g(P)}{h(P)} = a\} = V(g(P) - ah(P)) \cap U$ is closed in $U$ by definition of closed sets in Zariski topology. Hence $f^{-1}(a)$ is closed in $Y$. $\square$

*Corollary* 3. Let $f$ and $g$ be regular functions on a variety $X$. If $f = g$ on some nonempty open subset $U \subseteq X$, then $f = g$ everywhere.

*Proof.* The set of points $V(f - g)$ is clearly closed. But by assumption, it is at least open nonempty, hence dense by Lemma 4 and thus equal to X. $\square$

## 3.2 Rational maps

**Definition 25** (Morphism of varieties)**.** *Let $K$ be an algebraically closed field. We define* variety *over $K$ as any affine, quasi-affine, projective or quasi-projective variety. If $X, Y$ are two varieties, a* morphism *$\varphi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $f : V \to K$, the function $f \circ \varphi : \varphi^{-1}(V) \to K$ is regular.*

**Definition 26** (Rational map)**.** *Let $X, Y$ be varieties. A* rational map *$\varphi : X \to Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where $U$ is a nonempty open subset $U \subseteq X$, $\varphi_U$ is a morphism of $U$ to $Y$, and where $\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle$ are equivalent if $\varphi_U = \varphi_V$ on $U \cap V$. If the image of $\varphi_U$ is dense in $Y$ for any $U$ we say that $\varphi$ is* dominant.

*Note.* It does not matter which $U$ we will pick in definition of dominant map: Fix $U \subseteq X$ open nonempty set with dense image in $Y$. Then fix any other open nonempty set $U_1 \subseteq X$. Assume, for the sake of contradiction, that there is an open and nonempty set $G \subseteq Y$ such that $G \cap \varphi_{U_1}(U_1) = \emptyset$. From the equivalence, we have that $\varphi_{U_1}(U \cap U_1) = \varphi_U(U \cap U_1)$. It follows that

$$\emptyset = \varphi_{U_1}(U_1) \cap G \supseteq \varphi_U(U \cap U_1) \cap G = \varphi(U \cap U_1 \cap \varphi^{-1}(G))$$

So the set $U \cap U_1 \cap \varphi^{-1}(G)$ is empty. But this set is also intersection of nonempty open sets of irreducible algebraic set, thus dense, hence it is nonempty and that is contradiction.

*Definition* 27 (Birational map). *A* birational map $\varphi : X \to Y$ *is a rational map for which there exists a rational map* $\psi : Y \to X$ *such that* $\psi \circ \varphi = \mathrm{id}_X$ *and* $\varphi \circ \psi = \mathrm{id}_Y$*. If there is birational map from* $X$ *to* $Y$*, we say that* $X$ *and* $Y$ *are* birationally equivalent.

## 3.3 Examples

In this section, we will denote the origin of an affine space, $(0, \ldots, 0)$, by $O$.

We will work with $\mathbb{A}^n_K \times \mathbb{P}^{n-1}$, which is a quasi-projective variety and can be thought of as affine space to which we add all the lines through $O$. We will define *blowing-up* of $\mathbb{A}^n$ at the point $O$ to be the closed subset $X$ of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = y_i x_j \mid i, j \in \{1, \ldots, n\}\}$.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{A}^n_K \times \mathbb{P}^{n-1} \arrow[d] \\
& \mathbb{A}^n_K
\end{tikzcd}$$

The morphism $\varphi : X \to \mathbb{A}^n$ is obtained by restricting the projection map of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ onto the first coordinate of $\mathbb{A}^n$.

Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n \setminus \{O\}$. WLOG $a_1 \neq 0$. Now if $P, (y_1, \ldots, y_n) \in \varphi^{-1}(P)$, then for each $j, y_j = \frac{a_j}{a_1} y_1$. Thus $\varphi^{-1}(P)$ consists of a single point. Furthermore, we can define $\psi(P) = (P, [a_1 : \cdots : a_n])$ defines an inverse morphism to $\varphi$, showing that $X \setminus \varphi^{-1}(O)$ is isomorphic to $\mathbb{A}^n$.

The set $\varphi^{-1}(O)$ consist of all points $(O, Q),\ Q \in \mathbb{P}^n$, because the condition on $X$ is $0 = 0$ thus always valid.

*Definition* 28 (Blow-up). *If* $Y$ *is a closed subvariety of* $\mathbb{A}^n_K$*,* $O \in Y$*, we define* blow-up *of* $Y$ *at the point* $O$ *to be* $\tilde{Y} = \overline{(\varphi^{-1}(Y - O))}$*, where* $\varphi : X \to \mathbb{A}^n_K$*, is as above. To blow up other point than* $O$ *we make a linear change of coordinates sending it to* $O$

*Definition* 29 (Singularity). *Let* $Y \subseteq \mathbb{A}^n$ *be an affine variety, and let* $g_1, \ldots, g_\ell \in K[x_1, \ldots, x_n]$ *is set of generators for the ideal of* $Y$. $Y$ *is* nonsingular at a point $P \in Y$ *if the rank of the matrix* $\|\frac{\partial g_i}{\partial x_j}(P)\|$ *is equal to* $n - r$*, where* $r$ *is the dimension of* $Y$. $Y$ *is nonsingular if it is nonsingular at every point. Definition of dimension [2]*

*Example* 1. *Let* $X$ *be a curve* $V(y^2 - x^3 - x^2)$*. We can easily see that* $O$ *is a singular point on* $X$*. If we substitute* $(t, mt)$ *for* $(x, y)$ *in* $F(x, y) = y^2 - x^3 - x^2$*,* *then will get $F(t, mt) = (-1+m^2)t^2 - t^3$. Let us think about the projection map $\varphi : \mathbb{A}^2_K \times \mathbb{P}^1 \to \mathbb{A}^2_K$ defined by $\varphi((x,y),[x:y]) = (x,y)$. We can define the blow-up of $\mathbb{A}^n_K = (\{\{(0,0)\} \times \mathbb{P}^1\}) \cup \{\{((x,y),[x:y]) \in \mathbb{A}^2_K \times \mathbb{P}^1 \mid (x,y) \neq (0,0)\}\}$. The preimage of $X \setminus \{0\}$ is determined by:*

$$\phi(X \setminus \{0\}) = \{((x,y),[x:y]) \in \mathbb{A}^2_K \times \mathbb{P}^1 \mid (x,y) \in X \setminus \{0\}\}.$$

*As we saw in Section 2.1, $\mathbb{P}^1$ can be thought of as an union of two affine spaces, namely $\mathbb{P}^1 = U_0 \cup U_1$ where $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$, we can also think of $\varphi^{-1}(X\{0\})$ as a union of $\varphi^{-1}(X\setminus\{0\}) \cap (\mathbb{A}^2_K \times U_0)$ and $\varphi^{-1}(X\setminus\{0\}) \cap (\mathbb{A}^2_K \times U_1)$. Using this we have:*

$$\begin{aligned}
\varphi^{-1}(X \setminus \{0\}) \cap (\mathbb{A}^2_K \times U_0) = \\
&= \{((x,y),[x:y]) \in \mathbb{A}^2_K \times \mathbb{P}^1 \mid (x,y) \in X,\ x \neq 0\} \\
&= \{((x,y),[x:y]) \in \mathbb{A}^2_K \times \mathbb{P}^1 \mid y^2 = x^3 + x^2,\ x \neq 0\} \\
&= \{((x,mx),[1:m]) \in \mathbb{A}^2_K \times \mathbb{P}^1 \mid m^2 = x+1,\ x \neq 0, m \in K\} \\
&\cong \{(x,m) \in \mathbb{A}^2_K \mid m^2 = x+1,\ x \neq 0, m \in K\}
\end{aligned}$$

*Since the Zariski closure of $\{(x,m) \in \mathbb{A}^2_K \mid m^2 = x+1, x \neq 0, m \in K\}$ in $\mathbb{A}^2_K$ is $V(m^2 - x - 1)$, if we think about the corresponding set in $\mathbb{A}^n_K \times U_0$, then we can conclude that Zariski closure of $\varphi^{-1}(X \setminus \{0\}) \cap (\mathbb{A}^2_K \times U_0)$ in $(\mathbb{A}^2_K \times U_0)$ is*

$$(\varphi^{-1}(X \setminus \{0\})) \cap (\mathbb{A}^2_K \times U_0)) \cup \Big\{((0,0),[1:1]),((0,0),[1,-1])\Big\},$$

*similarly we can also find that the Zariski closure of $\varphi^{-1}(X \setminus \{0\}) \cap (\mathbb{A}^2_K \times U_1)$ in $(\mathbb{A}^2_K \times U_1)$ is*

$$(\varphi^{-1}(X \setminus \{0\})) \cap (\mathbb{A}^2_K \times U_1)) \cup \Big\{((0,0),[1:1]),((0,0),[1,-1])\Big\},$$

*As a result, the blow-up of $X$ at point 0 is*

$$\begin{aligned}
B_p(X) &= \overline{(\varphi^{-1}(X \setminus \{0\}))} \\
&= \Big(\overline{(\varphi^{-1}(X \setminus \{0\})} \cap (\mathbb{A}^2_K \times U_0)\Big) \cup \Big(\overline{(\varphi^{-1}(X \setminus \{0\})} \cap (\mathbb{A}^2_K \times U_1)\Big) \\
&= (\varphi^{-1}(X \setminus \{0\})) \cup \Big\{((0,0),[1:1]),((0,0),[1:-1])\Big\}
\end{aligned}$$

*This computation was from [5].*

*Example 2. Let $Y$ be given by the equation $Y = V(xy - z^3)$. We see that $Y$ has a singularity in $O$. We will blow up $Y$ at $O$. Let $u, v, w$ be homogeneous coordinates for $\mathbb{P}^2$. Then $X$, the blow-up of $\mathbb{A}^n_K$ at $O$, is defined by the equations $\{yw = zv, xv = yu, zu = xw\}$ inside $\mathbb{A}^3_K \times \mathbb{P}^2$.*

*The preimage of $O$ in $\varphi$ is the set of points in $X$ such that $(x,y,z) = (0,0,0)$ and from equations defining $X$ we can easily see that preimage of $O$ is the whole*

$\{(0,0,0)\} \times \mathbb{P}^2 \cong \mathbb{P}^2$. *As we saw in Section 2.1 projective space has open cover by sets* $U_\omega,\ \omega \in \{u, v, w\}$. *To compute the blowup, we will first compute its "parts". Let us denote the part of the preimage,* $\varphi^{-1}(Y \setminus \{0\}) \cap (\mathbb{A}^3_K \times U_\omega)$, *by* $W_\omega$ *for* $\omega \in \{u, v, w\}$.

$$\begin{aligned} W_u &= \left(\varphi^{-1}(Y \setminus O) \cap (\mathbb{A}^3_K \times U_u)\right) = \\ &= \Big\{(x, y, z) : [u : v : w] \in \mathbb{A}^3_K \times \mathbb{P}^2 \mid (x, y, z) \neq (0, 0, 0), u \neq 0, \\ &\qquad yw = zv, xv = yu, zu = xw, xy - z^3 = 0\Big\} \end{aligned}$$

*We can assume that* $u = 1$ *and since it holds that* $[x : y : z] = [u : v : w]$, *we also have* $x \neq 0$. *From the blowup equations for* $u = 1$ *we see that* $xv = y$ *and* $wx = z$. *With this we can rewrite* $xy - z^3$ *as* $xvx - w^3x^3 = x^2(v - w^3x)$. *But since* $x \neq 0$, *the zero set of* $x^2(v - w^3x)$ *is equal to the zero set of* $v - w^3x$. *From that we have* $W_u = \{(x, vx, wx), [1 : v : w] \mid v = w^3x, x \neq 0\} = \{(x, w^3x^2, wx), [1 : w^3x : w] \mid x \neq 0\} \cong A^2 \setminus V(x)$. *Mapping* $\psi : W_u \to \mathbb{A}^2,\ ((a, b, c), [d : e : f]) \mapsto (a, \frac{c}{a})$ *with inverse* $\psi^{-1} : (a, b) \mapsto ((a, b^3a^3, ab), [1, b^3a, b])$ *which is birational equivalence. The set* $\mathbb{A}^2 \setminus V(x)$ *is dense by lemma Lemma 4. The closure of* $W_u$ *can be computed as follows: We have* $\overline{W_U} = \overline{\psi^{-1}(A^2 \setminus V(X))} = \psi^{-1}(A^2) = \{(x, w^3x^2, wx), [1 : w^3x : w] \in\}$. *We saw before that this set is defined by polynomials* $yw - zv, xv - yu, zu - xw, v - w^3x$. *If we homogenize the polynomials we get quasi-projective variety* $\mathcal{V}(yw - zv, xv - yu, zu - xw, u^2v - w^3x) \subseteq \mathbb{A}^3 \times \mathbb{P}^2$.

*In* $\mathbb{A}^3 \times \mathbb{P}^2$ *it holds that* $\mathcal{V}(yw - zv, xv - yu, zu - xw, u^2v - w^3x)$ *is Zariski closed set. It contains the set* $\varphi^{-1}(Y \setminus O)$, *which is irreducible, because it is isomorphic to* $Y \setminus O$, *which is irreducible, because it is isomorphic to* $Y \setminus O$ *which is irreducible. From its irreducibility we have that* $\varphi^{-1}(Y \setminus O) \cap U_u$ *is dense in* $\varphi^{-1}(Y \setminus O)$. *That means that every closed set in* $\mathbb{A}^3 \times \mathbb{P}^2$ *containing* $\varphi^{-1}(Y \setminus O) \cap U_u$ *also contains* $\varphi^{-1}(Y \setminus O)$. *And this implies that implies*

*Since the original polynomial is symmetric in* $x, y$ *we can repeat almost same process for* $W_v$ *and we will get that* $\overline{W_v} = \mathcal{V}(yw - zv, xv - yu, zu - xw, v^2u - w^3y)$. *The set* $W_w = \{(zu, zv, z), [u : v : 1] \in \mathbb{A}^3 \times \mathbb{P}^2\} = \mathcal{V}(yw - zv, xv - yu, zu - xw, uv - w^2z)$ *The same thing as for* $W_u$ *holds for the other two sets, and hence we have:*

$$\overline{\varphi(Y \setminus O)} \subset \mathcal{V}(yw - zv, xv - yu, zu - xw, u^2v - w^3x, uv^2 - w^3y, uv - w^2z) =: V$$

*Together we have that*

$$\begin{aligned} \overline{\varphi^{-1}(Y \setminus O)} &= \mathcal{V}(yw - zv, xv - yu, zu - xw, u^2v - w^3x) \cap U_u \\ &\cup \mathcal{V}(yw - zv, xv - yu, zu - xw, uv^2 - w^3y) \cap U_v \\ &\cup \mathcal{V}(yw - zv, xv - yu, zu - xw, uv - w^2z) \cap U_w \end{aligned}$$

*Which is same as* $\overline{\varphi^{-1}(Y \setminus O)} = (V \cap u_u) \cup (V \cap U_v) \cup (V \cap U_w)$

# Conclusion

The aim of this thesis was to introduce a simple and comprehensible way to understand the method of blow-ups and how to resolve singularities using it in special cases.

We first introduced affine and projective space and proved some of its properties. We then used a more basic approach to blow-ups to introduce them to a wider audience. We also tried to fill in gaps in the literature on this topic.

We showed how to use the method of blow-ups on exercises and resolved certain singularities.

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