FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Quantum vacua, curved spacetime and singularities

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Study programme: Physics
Study branch: Theoretical physics

Prague 2021

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I would like to express my gratitude to my supervisor Alfredo Iorio for giving me an interesting topic for my master thesis, for numerous consultations and his great patience. Thanks to his advice and help, the work has improved significantly in all respects.

I would like to thank my friend and schoolmate Zuzka for occasional discussions and help with English.

Finally, I would also like to thank my family for their support during my studies.

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Abstract: In this work we investigate the Weyl anomaly from a new perspective. Our goal is to identify a set-up for which the classical Weyl symmetry is not broken, at the quantum level by the usual arguments related to the Euler invariants, but rather by the impact of other geometrical obstructions. Therefore, we work, mostly, in three spatiotemporal dimensions, where general arguments guarantee the absence of trace anomalies. In particular, our interest here is on whether various types of singularities, emerging in the description of the differential geometry of surfaces, could induce some form of quantum inequivalence, even though the classical symmetry is at work. To this end, we work with a very special three-dimensional metric, whose nontriviality is fully in its spatial two-dimensional part. The last ingredient we use, to clean-up the way from other complications, is to work with physical systems where no Weyl gauge field is necessary, to have the classical invariance. The system we focus on is then the massless Dirac field theory (that, as well known, enjoys local Weyl symmetry) in three-dimensional conformally flat spacetimes.

With these premises, the research programme consists of three steps. The first step is to find the coordinate transformations that link the conformal factor identifying the surface to the spatiotemporal conformal factor. This task is highly nontrivial. In this thesis we have fully solved the issue for the cases in point, filling gaps of the relevant research literature. The second step is to identify the quantum operator that, when applied to the the Dirac field, produces the correct Weyl transformation. Here we have found a possible candidate for such an operator, and the analysis of the impact of singularities of the conformal factor on its regularity are particularly simple. Nonetheless, this operator needs further studies to be fully put into contact with the third, and final point of the programme, that is the study of the effects of the above on the quantum Weyl symmetry of the Dirac system.

Keywords: Weyl symmetry, Weyl anomaly, Singularities, Geometry and Topology, Canonical and Bogoliubov transformation, Quantum inequivalence

Contents

List of Abbreviations

QM	Quantum mechanics
QFT	Quantum field theory
STR	Special theory of relativity
VEV	Vacuum expectation value
CT	Canonical transformation
BT	Bogoliubov transformation
WT	Weyl transformation
QWT	Quantum Weyl transformation
CCR	Canonical commutation relation
CAR	Canonical anticommutation relation

List of Figures

Introduction

The grand goal of the presented thesis is to investigate the phenomenon of the quantum anomaly from a new perspective, the focus being on the *Weyl anomaly*. The latter is often referred to, in the literature, as the *trace, scale* or *conformal* anomaly, that are all related but different concepts. In all cases, as customary, by quantum anomaly we shall mean the quantum mechanical obstruction to a classical symmetry.

We consider systems[1] related by the Weyl classical symmetry. After the quantization of the fields, we want to search for an inequivalence of related Hilbert spaces, stemming from the *singularities of classical nature*: the spacetime singularities such as cusps, boundaries and other. For illustration, see the fig. 1.

Let us add that the singularities play a crucial role when it comes to the inequivalence of Hilbert spaces in the quantum field theory (QFT). We shall see this when we discuss the Bogoliubov transformation (BT) and the inequivalent representations of the canonical commutation relations (CCRs)[2] in chapter 5. This is our motivation to investigate whether the classical singularities cause the inequivalence of Hilbert spaces.

In the following paragraphs we shall introduce a model where the trace anomaly is not present[3], so the quantum inequivalence we are in search of has the potential to be transparent.

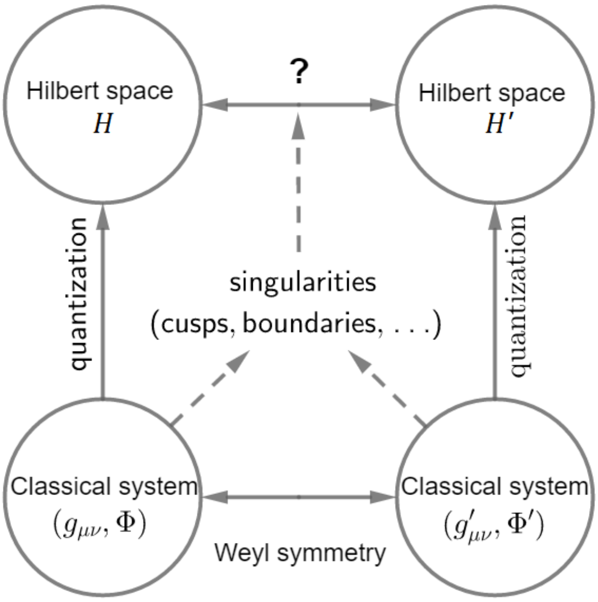


Figure 1: Concept map of the grand goal of this thesis

In order to describe our model, we introduce the basic terminology. The *Weyl transformation* (WT) is a simultaneous rescaling of the metric tensor $g_{\mu\nu}$ and the

[1] When we talk about a "system", we mean some field Φ and some spacetime on which the field "lives". The spacetime is described by its metric tensor $g_{\mu\nu}$. We shall usually denote the system as a pair $(g_{\mu\nu}, \Phi)$.

[2] Similarly, we shall donote the canonical anticommutation relations as "CARs".

[3] In the literature, when it comes to the Weyl anomaly, it is mostly understood as the trace anomaly. In this work, we want to avoid this viewpoint.

field[4] Φ such that [1]:

$$(g_{\mu\nu}, \Phi) \to \left(g'_{\mu\nu}, \Phi'\right), \tag{1}$$

with

$$g'_{\mu\nu}(Q) = e^{2\Sigma(Q)} g_{\mu\nu}(Q), \quad \Phi'(Q) = e^{d_\Phi \Sigma(Q)} \Phi(Q), \tag{2}$$

where $e^{\Sigma(Q)}$ is the *(spacetime) conformal factor*, a function of spacetime coordinates Q^μ, and d_Φ is the scale dimension. It is worth emphasizing that the WT is not a coordinate transformation, but an actual transformation of the spacetime metric/geometry (proper distances are changed).

The action $A[g_{\mu\nu}, \Phi]$ is said to be *Weyl invariant*, if and only if it is invariant under the WT:

$$A[e^{2\Sigma} g_{\mu\nu}, e^{d_\Phi \Sigma}\Phi] = A[g_{\mu\nu}, \Phi]. \tag{3}$$

The actions invariant under the WT usually do not contain any preferable unit of a scale, like mass, which may otherwise break the symmetry.

Moreover, it turns out that the trace of the energy-momentum tensor:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta A}{\delta g_{\mu\nu}}, \tag{4}$$

vanishes:

$$T^\mu_\mu = 0, \tag{5}$$

where g is the determinat of the metric tensor $g_{\mu\nu}$ and $\delta/\delta g_{\mu\nu}$ is the standard functional derivative [2].

Nevertheless, once we quantize[5] the field Φ, the situation changes dramatically:

$$g^{\mu\nu} < T_{\mu\nu} > \sim (\text{Full contractions of the Riemann tensor})^{n/2}, \tag{6}$$

where $< . >$ denotes the vacuum expectation value (VEV) and $n = 2m$, m being a natural number, is the dimension of the spacetime. From (6) it becomes clear why the Weyl anomaly is referred to as the trace anomaly [3].

However, in the odd dimensions $n = 2m + 1$ the situation is dramatically different from (6):

$$g^{\mu\nu} < T_{\mu\nu} >= 0. \tag{7}$$

This is because there are no *Euler invariants* in the odd dimensions [4]. So, if a quantum inequivalence is present, it is not because of the trace anomaly.

We want to identify the cleanest tractable situation to have our problem under scrutiny. The following setup seems to be the most convenient:

1. We focus on the spacetimes of the dimension $n = 3$, because *there are no Euler invariants (so no trace anomaly).* Moreover, we choose a special subset of these spacetimes such that:

[4]With general inner structure, i.e. scalar, spinor or other fields.

[5]In this work we only quantize the fields, but the spacetimes remain classical. This is the so called semiclassical quantization.

1.1. They are *conformally flat*. It means that there exists a coordinate system $Q^\mu \equiv (T, X, Y)$ such that:

$$g_{\mu\nu} = e^{2\Sigma}\eta_{\mu\nu}. \tag{8}$$

Then the dynamics of the field $e^{d_\Phi \Sigma}\Phi$ referes to the flat case:

$$A[e^{2\Sigma}\eta_{\mu\nu}, e^{d_\Phi \Sigma}\Phi] = A[\eta_{\mu\nu}, \Phi]. \tag{9}$$

We shall be interested in how the Hilbert space gets changed when we move from a flat spacetime to a curved (conformally flat) spacetime, which contains singularities (e.g. cusps). Because only the curved spacetime does possess singularities, it will make our investigation, how the singularities affect the quantization, more transparent.

1.2. They are constructed as a *Cartesian product of a flat time and two-dimensional surfaces*. We shall see later that the surfaces cannot be arbitrary, but due to the condition 1.1. they are of constant Ricci (and Gaussian) curvature. These surfaces have been widely studied in the literature and we shall be able to use this knowledge to our advantage. We shall denote such spacetimes as $\mathbb{R} \times \mathcal{M}^2$ and their metric tensor is:

$$g_{\mu\nu}(q) = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ij}(\vec{q}) \end{pmatrix}, \tag{10}$$

where $q^\mu \equiv (t, x, y)$ and $\vec{q} \equiv (x, y)$ denotes unspecified abstract coordinates covering the surface. We shall understand t as the laboratory time and the spatial part of the metric tensor g_{ij} as the metric of a surface.

2. We choose to work with the *Dirac massless field theory*. Its action is invariant under the WT. If we denote the Dirac field, living on the Minkowski spacetime with $\eta_{\mu\nu}$, as ψ_α then the Weyl transformed system is:

$$g_{\mu\nu} = e^{2\Sigma}\eta_{\mu\nu}, \quad \psi_{\Sigma\alpha} = e^{-\frac{n-1}{2}\Sigma}\psi_\alpha, \tag{11}$$

where Σ is a function of spacetime coordinates and we consider the general dimension of spacetime n. Although we are interested in $n = 3$, most of our calculations will be done for a general dimension n.

The classical dynamics of the field ψ_α is preserved under the transformation:

$$A[g_{\mu\nu}, \psi_{\Sigma\alpha}] = A[\eta_{\mu\nu}, \psi_\alpha]. \tag{12}$$

We shall discuss in chapter 1 that the Dirac massless action is invariant under the WT as it is, i.e. we do not need to introduce any additional gauge (Weyl) field, as it is the case for e.g. the scalar field [1].

Once we are in this setup, we quantize both Dirac fields ψ_α and $\psi_{\Sigma\alpha}$ and consider the following *quantum Weyl transformation* (QWT):

$$\begin{aligned} \psi_{\Sigma a} &= W\psi_\alpha W^{-1} \\ &= e^{-\frac{n-1}{2}\Sigma}\psi_\alpha, \end{aligned} \tag{13}$$

where the operator W realizes the quantum transformation. We shall study the QWT in a general dimension n and focus on $n = 3$ when we need to apply considerations related to the geometry of the surfaces of constant Ricci (Gaussian) curvature, as we discussed above. It is natural to expect such operator W to be unitary, since it implements a quantum symmetry. We want to understand how the geometrical obstruction can make this operator irregular, which would break the symmetry (which was thus defined only formally).

We shall discuss in chapter 6 that the QWT is an example of a *canonical transformation* (CT). In QFT, CT is a tranformation which preserves the structure of the CCRs (or CARs), but still leads to different Hilbert spaces. We shall explain the concept of the canonical transformation and related *inequivalent representations* in chapter 5.

Our research can be formulated as a three-step programme:

1. Since we want to understand the impact of the geometrical obstruction on the quantum inequivalence (symmetry breaking), we shall

   1.1. focus on the classical Weyl symmetry, where the essential role is played by the function Σ, giving the spacetime conformal factor $e^{2\Sigma}$;

   1.2. study the conformally flat spacetimes $\mathbb{R} \times \mathcal{M}^2$ of constant Ricci curvature and its spatial parts $\mathcal{M}^2$, the surfaces of constant Ricci curvature, where all of the non-triviality of the spacetimes is present.

   At the end of this step we know the function $\Sigma(T, X, Y)$ including its singularities and we identify the coordinates $Q^\mu \equiv (T, X, Y)$, in which the metric is explicitly conformally flat. This is a crucial step.

   This is the content of the chapters 1, 2, 3, 4 and appendix A.

2. We need to find the quantum operator W which implements the Weyl transformation at the quantum level. This operator is supposed to be unitary because it implements a quantum symmetry [5].

   However, we expect that W depends on Σ and this can be a source of singularities, which would make W irregular. If this happens, the quantum symmetry is broken, since W is not well defined.

   We shall introduce the operator W, which depends on Σ, but we shall also find out that it is quite peculiar. We shall show in chapter 6 that there is no, at least no simple, way how to find such a unitary operator W. We shall define it formally, but we shall find out that it leads to a discrepancy.

   However, it turns out that this discrepancy can be avoided, technically, when we consider that the operator W is not unitary, but hermitian. The link between this operator and the symmetry of the system is unobvious at this moment.

   On the other hand, relaxing the requirement of unitarity, we shall study the operator W and investigate when it becomes singular, in correspondence to the geometrical singularities of the spacetimes, enconded into Σ. The task will turn out to be very demanding and we arrived at a partial solution.

We shall discuss in chapter 5 the general case of irregular quantum transformations among different Hilbert spaces. In the next chapter 6 and appendices B and C we treat the operator W as a hermitian operator and explore under what circumstances it becomes singular.

3. Provided that the point 2. would have been finished completely, the next step is to understand the connection between our W and the quantum Weyl symmetry.

To fulfill the general goal of this project it would be necessary to solve all the three points discussed above. We fully solved step 1. and set-up the stage for the solution of step 2., while step 3. seemed too demanding and further work, based on the results we found here, is necessary. We shall discuss the list of what must be done in Conclusions 7.

Finally, let us briefly comment on the content of each chapter and appendix:

- Chapter 1 is devoted to the introduction of the Weyl symmetry of the Dirac massless action.
- In chapter 2 we discuss on the general ground the spacetimes $\mathbb{R} \times \mathcal{M}^2$ in more details. We show how the condition that the spacetimes under consideration are conformally flat implies that the surfaces must have constant Ricci (Gaussian) curvature.
- Chapter 3 is devoted to the study of the surfaces of constant Ricci (Gaussian) curvature: their local geometry as well as global properties (singularities).
- Chapter 4 is dedicated to the further study of the spacetimes $\mathbb{R} \times \mathcal{M}^2$: their local geometry as well as the global description.
- In chapter 5 we introduce the Bogoliubov transformation and discuss inequivalence between Hilbert spaces on the general ground.
- In chapter 6 we apply the knowledge gained in chapter 5 and discuss the quantum Weyl transformation.
- In chapter 7 we summarize our results and suggest other ways in which the work can be developed.
- In appendix A we explore how the concept of curvature can be generalized to manifolds which are not smooth everywhere, focusing on the conical singularity.
- In appendices B and C we present some technical calculations which, supplementing chapter 6, did not fit directly into the main body of this thesis

1. Weyl symmetry

This chapter is dedicated to the Dirac massless field theory with emphasis on its Weyl symmetry. Its content is as follows:

We start by recalling the action for the Dirac field on flat spacetime. Assuming its mass term disappears ($m = 0$), the action becomes scale invariant[1].

In order to investigate symmetries in curved spaces, we rewrite the action in a diffeomorphic invariant form. The action remains scale invariant, but it is the metric tensor rather than the coordinates that must be transformed. This scale transformation is also called *rigid Weyl transformation*, because its scale factors (for the metric tensor and the field) depend on powers of e^{Σ}, where Σ is independent on the coordinates.

Then we take one more step further and promote Σ into a function $\Sigma(Q)$ of spacetime coordinates Q^{μ}. The transformation is named *local Weyl transformation*. It is not granted that the action remains invariant under this transformation. When the action ceases to be invariant it is necessary to introduce a *gauge field*, which compensates the extra terms in the transformed action and saves the symmetry. However, for the Dirac massless field theory this is not the case. Its action is invariant under the local Weyl transformation without introducing the Weyl field W_{μ}, playing the role of the gauge field. Nevertheless, the Weyl field can be formally introduced, but no physical meaning can be abscribed to it.

The main reference for this chapter is [1], but we also refer extensively to [6] and [7]. Let us just mention that in the last two articles, the Weyl symmetry of the massless Dirac action is put into the interesting context of graphene. We shall also revisit these papers when we focus on geometry of spacetimes, see chapter 2.

1.1 Lorentz invariance and diffeomorphism

We begin by recalling the Dirac equation [8],[9]:

$$i\gamma^a \partial_a \psi_\alpha - m\psi_\alpha = 0, \tag{1.1}$$

where ψ_α is the Dirac spinor with α being a spinor index, γ^b is a standard gamma matrix, satisfying the Clifford algebra:

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}, \tag{1.2}$$

where η_{ab} is the Minkowski metric, a and b are both Lorentz (flat) indices, $\partial_a \equiv \partial/\partial x^a$, and m is mass. Let us add that we work in units where the reduced Planck constant $\hbar$ and the speed of light c are as follows: $\hbar = 1 = c$.

The mass term breaks the symmetry we are looking for, thus we can set it to zero it from now on: $m = 0$.

The Dirac equation is invariant under the Lorentz transformation:

$$\psi'(x') = S\psi(x), \tag{1.3}$$

[1]Here, the coordinates and fields are transformed simultaneously.

where $S = \exp\left(-\frac{1}{2}\epsilon^{ab}J_{ab}\right)$ is a matrix with generators:

$$J_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right] \equiv \frac{1}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a) \tag{1.4}$$

satisfing the Lorentz algebra[2] [8]. The coefficients ϵ^{ab} are constants and specify the Lorentz transformation.

The corresponding action:

$$A[\eta_{ab}, \psi, \partial_a\psi] \equiv i\int d^n x\bar{\psi}(x)\gamma^a\partial_a\psi(x). \tag{1.5}$$

It is invariant under the Lorentz transformation, because the Jacobian of the transformation is: $\det\left(\frac{\partial x'}{\partial x}\right) = +1$.

Next we want to find the generalization of the Dirac action to curved spacetimes. The action must be invariant under:

i) the local Lorentz transformation: locally, the spacetime is flat, therefore, the laws of special theory of relativity (STR) apply;

ii) diffeomorphisms: all well-defined coordinates describing the same spacetime are suitable for its description.

With this in mind, the action must become:

$$A[g_{\mu\nu}, \psi, \nabla_\mu\psi] \equiv i\int d^n x\sqrt{-g}\bar{\psi}(x)\gamma^a e^\mu_{\underline{a}}\nabla_\mu\psi(x), \tag{1.6}$$

where $\nabla_\mu \equiv \partial_\mu + \Omega_\mu$ is the diffeomorphic covariant derivative, μ is the Einstein (curved) index, $e^\mu_{\underline{a}}$ is the inverse Vielbein[3], $\Omega_\mu \equiv \frac{1}{2}\omega^{ab}_\mu J_{ab}$ with ω^{ab}_μ being the spin connection. Ω_μ is necesarry if the action must stay invariant under the local Lorentz transformation:

$$\omega^{ab}_\mu \to \omega^{ab}_\mu + \partial_\mu\epsilon^{ab}, \quad \psi \to \exp\left(-\frac{1}{2}\epsilon^{ab}J_{ab}\right)\psi, \tag{1.7}$$

where ϵ is not a constant anymore, but a function of coordinates x, i.e. $\epsilon(x)$.

Notice that although γ^a was replaced by $\gamma^\mu \equiv \gamma^a e^\mu_{\underline{a}}$, the Dirac adjoint has remained unchanged: $\bar{\psi} \equiv \psi^\dagger\gamma^{\underline{0}}$, where 0 is a flat rather than curved index (which we denote by underlying, e.g.: $\underline{0}$), see [6].

In what follows we shall refer to (1.5) as *flat Dirac action* and (1.6) as *curved Dirac action.*

1.2 Scale and Weyl invariance

It is simple to check that the flat Dirac action (1.5) is invariant under the following transformation:

$$x^a \to e^\Sigma x^a, \quad \psi \to e^{-\frac{n-1}{2}\Sigma}\psi, \tag{1.8}$$

where Σ is a constant. This is an example of scale transformations.

[2] It is the spinor representation of the Lorentz transformation.

[3] A mapping from the local Cartesian system to the global coordinate system.

For curved spaces, we can find an analogue of (1.8):

$$e^a_\mu \to e^{\Sigma} e^a_\mu, \quad \psi \to e^{-\frac{n-1}{2}\Sigma}\psi, \tag{1.9}$$

where $e^a_\mu \equiv \partial x^a/\partial x^\mu$ is the Vielbein. Let us stress that (1.9) is not a coordinate transformation, so $x^\mu \to x^\mu$. The metric tensor $g_{\mu\nu}$ can be expressed using the Vielbeins as:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \tag{1.10}$$

so the transformation implies:

$$g_{\mu\nu} \to e^{2\Sigma} g_{\mu\nu}, \quad \sqrt{-g} \to e^{n\Sigma}\sqrt{-g}. \tag{1.11}$$

This transformation does not change ω^{ab}_μ, because Σ is still a constant. This transformation, (1.9), is also known as the rigid Weyl transformation.

The further step is to promote Σ into a function of spacetime coordinatess:

$$\Sigma \to \Sigma(x). \tag{1.12}$$

Considering (1.12) in (1.9) we get the local Weyl transformation, with non-trivial transformation of the spin connection [6]:

$$\omega^{ab}_\mu \to \omega^{ab}_\mu + \left(e^a_\mu e^{\nu b} - e^b_\mu e^{\nu a}\right)\partial_\nu \Sigma. \tag{1.13}$$

Following the standard procedure of *gauging*, this non-triviality is the reason to introduce a new field, here called *Weyl field* W_μ, which transforms as:

$$W_\mu \to W_\mu + \partial_\mu \Sigma. \tag{1.14}$$

The Weyl covariant derivative is then:

$$D_\mu = \nabla_\mu + \Lambda^\nu_\mu W_\nu, \tag{1.15}$$

where Λ^v_μ is called Virial [10] and is yet unspecified function. It is straightforward to check that:

$$\Lambda^\nu_\mu = \frac{n-1}{2}\delta^\nu_\mu - e^a_\mu e^{b\nu} J_{ab}, \tag{1.16}$$

is the correct choice for the action $A[g_{\mu\nu}, \psi, D_\mu\psi]$ to be invariant under the local Weyl transformations.

This general procedure, though, in the case of the Dirac field leads to:

$$\begin{aligned} A[g_{\mu\nu}, \psi, D_\mu\psi] = i\int d^n x\sqrt{-g}\bar{\psi}\gamma^a e^\mu_a \nabla_\mu \psi \\ +i\int d^n x\sqrt{-g}\bar{\psi}\gamma^a e^\mu_a \Lambda^\nu_\mu \psi = A[g_{\mu\nu}, \psi, \nabla_\mu\psi], \end{aligned} \tag{1.17}$$

because of the following identity:

$$\gamma^a e^\mu_a \Lambda^\nu_\mu = 0. \tag{1.18}$$

The derivation of this identity is straightforward and short and it just needs the use of the Clifford algebra: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

This tells us that $A[g_{\mu\nu}, \psi, \nabla_\mu \psi]$ is already local Weyl invariant as it stands and W_μ is not necessary. Let us check this directly. The extra term in the action coming from the Weyl transformation is following:

$$\begin{aligned} \delta A \equiv A[g'_{\mu\nu}, \psi', \nabla'_\mu \psi] - A[g_{\mu\nu}, \psi, \nabla_\mu \psi] = \\ = i \int d^n x \sqrt{-g} \bar{\psi} \left[-\frac{n-1}{2} \gamma^\mu \partial_\mu \Sigma + \frac{1}{2} \left(e^b_\mu e^{\nu c} - e^{\nu b} e^c_\mu \right) \partial_\nu \Sigma J_{bc} \right] \psi, \end{aligned} \tag{1.19}$$

where $\psi' \equiv e^{-\frac{n-1}{2}\Sigma}\psi$ and $g'_{\mu\nu} \equiv e^{2\Sigma} g_{\mu\nu}$.

Applying $J_{\mu\nu} \equiv \frac{1}{4}[\gamma_\mu, \gamma_\nu] \equiv e^a_\mu e^b_\nu J_{ab}$ and the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, we can simply obtain:

$$\frac{1}{2} \left(e^b_\mu e^{\nu c} - e^{\nu b} e^c_\mu \right) \partial_\nu \Sigma J_{bc} = \frac{n-1}{2} \gamma^\mu \partial_\mu \Sigma. \tag{1.20}$$

From this follows that $\delta A = 0$ and the Dirac action (1.6) is invariant under the local Weyl transformations.

Of course, we can choose to work with fields of different sorts, for intance a scalar field. The latter example has a simpler algebraic structure than the Dirac field, but the action for the free scalar field of $m = 0$ is not invariant under the local Weyl transformation, until we define a gauge field (see [1]). Because we are looking for the clearest situation, we want to avoid the need to define additional fields. Therefore, the Dirac massless field is an ideal candidate to work with.

1.3 Flat and conformally flat regimes

In this thesis we shall focus on that case when the curved metric tensor is conformally flat:

$$g_{\mu\nu} = e^{2\Sigma} \eta_{\mu\nu}. \tag{1.21}$$

from which follows:

$$e^a_\mu = e^\Sigma \delta^a_\mu, \quad \sqrt{g} = e^{n\Sigma}. \tag{1.22}$$

The coordinates, in which the conformal structure of spacetime becomes obvious, will be denoted as $Q^\mu \equiv (T, X, Y)$:

$$g_{\mu\nu}(Q) = e^{2\Sigma(Q)} \eta_{\mu\nu}(Q). \tag{1.23}$$

Because of our choice of metric tensors (1.21), the classical dynamics of the massless Dirac field on conformally flat spacetime is the same as on flat counterpart, as one can see from:

$$A[g_{\mu\nu}, \psi_\Sigma, \nabla_\mu \psi_\Sigma] = A[\eta_{ab}, \psi, \partial_a \psi], \tag{1.24}$$

where

$$\psi_\Sigma \equiv e^{-\frac{n-1}{2}\Sigma} \psi. \tag{1.25}$$

This setup, when the action is invariant under the Weyl transformation and spacetimes are conformally flat, is known as *conformal triviality* [4].

2. Conformal flatness

In chapters 3 and 4 we study two-dimensional surfaces of constant Gaussian (or Ricci) curvature and (2+1) dimensional spacetimes, obtained as a Cartesian product of flat time and these surfaces. We denote such spacetimes as $\mathbb{R} \times \mathcal{M}^2$.

In this chapter, we show that the spacetimes $\mathbb{R} \times \mathcal{M}^2$ are conformally flat. We start with the metric tensor (10), add a condition for conformal flatness and obtain that the surfaces, described by g_{ij} in (10), must have constant Gaussian curvature.

For beginning let us recall the metric tensor (10), expressed in the coordinate frame $q^\mu \equiv (t, x, y)$:

$$g_{\mu\nu}(q) = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ij}(\vec{q}) \end{pmatrix}. \tag{2.1}$$

We take advantage of the fact that any surface is locally conformally flat (see e.g. [11]). It means that there exists so called *isothermal coordinates* $(\tilde{x}, \tilde{y})$, covering the surface, that its line element becomes:

$$dl^2 = e^{2\sigma(\tilde{x},\tilde{y})} \left(d\tilde{x}^2 + d\tilde{y}^2 \right), \tag{2.2}$$

where $e^{\sigma(\tilde{x},\tilde{y})}$ is called *(spatial) conformal factor*, a function describing the surface's local geometry. In these coordinates, the full spacetime metric becomes quite simple:

$$g_{\mu\nu}(\tilde{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^{2\sigma(\tilde{x},\tilde{y})} & 0 \\ 0 & 0 & -e^{2\sigma(\tilde{x},\tilde{y})} \end{pmatrix}, \tag{2.3}$$

where $\tilde{q}^\mu = (t, \tilde{x}, \tilde{y})$. Obviously, it is more confident to work within the coordinate frame $(t, \tilde{x}, \tilde{y})$ rather than (t, x, y), because there is only one unspecified function σ (in comparision to the latter case where there are four unknown functions g_{ij}).

Another useful coordinate frame $Q^\mu \equiv (T, X, Y)$ is where the metric tensor is explicitly conformally flat:

$$g_{\mu\nu}(Q) = e^{2\Sigma(Q)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.4}$$

The matrix (2.4) also fixes our signature.

While the metric tensor in the coordinate frame Q^μ (2.4) is conformally flat by definition, conformal flatness of (2.3) is not guaranteed. Therefore, we expect that the function σ in (2.3) cannot be chosen arbitrary.

The necessary and sufficient condition for a spacetime of dimension $n = 3$ to be conformally flat is that its Cotton tensor (see e.g. [6], [12]):

$$C^{\mu\nu} = \epsilon^{\mu\sigma\rho} \nabla_\sigma R_\rho^{\ \nu} + (\mu \leftrightarrow \nu) \tag{2.5}$$

vanishes:

$$C_{\mu\nu} = 0. \tag{2.6}$$

It is worth mentioning that it is the Cotton tensor $C_{\mu\nu}$ rather than the Weyl tensor $C_{\mu\nu\sigma\rho}$ (e.g. [13]):

$$\begin{aligned} C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} &+ \frac{1}{n-2}\left(R_{\mu\rho}g_{\nu\sigma} - R_{\mu\sigma}g_{\nu\rho} + R_{\nu\sigma}g_{\mu\rho} - R_{\nu\rho}g_{\mu\sigma}\right) \\ &+ \frac{1}{(n-1)(n-2)}R\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right), \end{aligned} \tag{2.7}$$

which must vanish. The Weyl tensor, for $n \geq 4$, vanishes if and only if the spacetime is conformally flat. However, it vanishes identically for $n = 3$ and thus loses its practical applicability to test the spacetime on conformal flatness. Then we must use the Cotton tensor instead of the Weyl tensor [12].

We shall see in a moment that the Cotton tensor (2.5) computed for the metric tensor $g_{\mu\nu}(\tilde{q})$ (2.3) vanishes only if a spacetime has constant Ricci curvature $R^{(3)}$. This constraints Σ, because not every conformally flat spacetime must have constant Ricci curvature.

Let us notice that in the metric tensor (2.3) the curvature can only be carried by a surface because time is flat. Therefore, the Ricci scalar $R^{(2)}$ of the surface must be constant. Now, it is simple to see that:

$$R^{(3)} = -R^{(2)}, \tag{2.8}$$

where the additional minus sign follows from the metric's signature. This can be verified by straightforward mechanical calculation, see e.g. [6].

Finally, it will turn out that σ must be a solution of the *Liouville equation* [6], [14], a famous equation of mathematical physics (see below, (2.18)).

Before we compute the Cotton tensor $C_{\mu\nu}$ for (2.3), let us compute the Christoffel symbols, components of the Riemann and Ricci tensors and the Ricci scalar. The only non-trivial independent Christoffel symbols are:

$$\Gamma^1{}_{11} = \Gamma^2{}_{12} = -\Gamma^1{}_{22} = \partial_{\tilde{x}}\sigma, \tag{2.9a}$$

$$\Gamma^2{}_{22} = \Gamma^1{}_{12} = -\Gamma^2{}_{11} = \partial_{\tilde{y}}\sigma. \tag{2.9b}$$

From that it follows that only two non-trivial components of the Riemann and the Ricci curvature tensors are:

$$R_{11} = R^1{}_{212} = -\Delta_{(\tilde{x},\tilde{y})}\sigma = R^1{}_{212} = R_{22}, \tag{2.10}$$

where $\Delta_{(\tilde{x},\tilde{y})} \equiv \partial^2_{\tilde{x}} + \partial^2_{\tilde{y}}$, $\partial_{\tilde{x}} \equiv \partial/\partial\tilde{x}$, $\partial_{\tilde{y}} \equiv \partial/\partial\tilde{y}$.

During the computation, we assumed that:

$$[\partial_{\tilde{x}}, \partial_{\tilde{y}}]\sigma = 0. \tag{2.11}$$

Thanks to that the following components of the Ricci tensor:

$$R_{12} = -R_{21} = [\partial_{\tilde{x}}, \partial_{\tilde{y}}]\sigma \tag{2.12}$$

becomes zero. This is important to stress, because the interchangeability of order of derivatives is not always granted when singularities are taken into considerations, see [15], [16]. It is usual to condider the following *integrability conditions* [15]:

$$[\partial_\mu, \partial_\nu]\, g_{\sigma\rho} = 0, \quad [\partial_\mu, \partial_\nu]\, \Gamma^\lambda{}_{\sigma\rho} = 0. \tag{2.13}$$

Finally, the Ricci scalar reads as:

$$R^{(3)} = 2e^{-2\sigma}\Delta_{(\tilde{x},\tilde{y})}\sigma. \tag{2.14}$$

We are now in the right position to compute the Cotton tensor. It has only two non-trivial independent components:

$$C^{01} = \epsilon^{021}\nabla_2 R^1_1 + \epsilon^{012}\nabla_1 R^1_2, \tag{2.15a}$$
$$C^{02} = \epsilon^{021}\nabla_2 R^2_1 + \epsilon^{012}\nabla_1 R^2_2. \tag{2.15b}$$

As we already know, the Cotton tensor has to vanish whether spacetime should be conformally flat. From (2.6) and (2.15) follows:

$$\partial_{\tilde{x}}\Delta_{(\tilde{x},\tilde{y})}\sigma - 2\partial_{\tilde{x}}\sigma\Delta_{(\tilde{x},\tilde{y})}\sigma = 0, \tag{2.16a}$$
$$\partial_{\tilde{y}}\Delta_{(\tilde{x},\tilde{y})}\sigma - 2\partial_{\tilde{y}}\sigma\Delta_{(\tilde{x},\tilde{y})}\sigma = 0. \tag{2.16b}$$

The direct consequence of this is that the scalar curvature (2.14) must be constant:

$$\partial_{\tilde{x}} R^{(3)} = 2e^{-2\sigma}\left[\partial_{\tilde{x}}\Delta_{(\tilde{x},\tilde{y})}\sigma - 2\partial_{\tilde{x}}\sigma\Delta_{(\tilde{x},\tilde{y})}\sigma\right] = 0, \tag{2.17a}$$
$$\partial_{\tilde{y}} R^{(3)} = 2e^{-2\sigma}\left[\partial_{\tilde{y}}\Delta_{(\tilde{x},\tilde{y})}\sigma - 2\partial_{\tilde{y}}\sigma\Delta_{(\tilde{x},\tilde{y})}\sigma\right] = 0. \tag{2.17b}$$

Since $R^{(3)} = -R^{(2)}$, then $R^{(2)}$ is constant too, and so is $K \equiv R^{(2)}/2$, the Gaussian curvature. (2.14) can be rewritten as:

$$\Delta_{(\tilde{x},\tilde{y})}\sigma = -Ke^{2\sigma}, \tag{2.18}$$

that is the celebrated *Liouville equation* for $\sigma(\tilde{x},\tilde{y})$ [14].

It is also usual to find the Liouville equation in the following form:

$$\Delta_{(\tilde{x},\tilde{y})} \ln\phi(\tilde{x},\tilde{y}) = -K\phi^2(\tilde{x},\tilde{y}), \tag{2.19}$$

where we defined $\phi \equiv e^{\sigma}$.

Solving the Liouville equation (2.18) or (2.19), we obtain spatial conformal factors such that the associated spacetimes $\mathbb{R} \times \mathcal{M}^2$ are conformally flat.

For $K = 0$, the Liouville equation reduces to the Laplace equation:

$$\Delta_{(\tilde{x},\tilde{y})}\sigma = 0, \tag{2.20}$$

whose solutions are the *harmonic functions* [17]. They represent a local geometry of surfaces with zero Gaussian curvature.

For $K \neq 0$, Liouville found the general solution of his equation (2.19) [14]:

$$\phi(z,\bar{z}) = \frac{2}{\sqrt{|K|}}\frac{f'(z)}{1 \pm |f(z)|^2}, \tag{2.21}$$

where $z \equiv \tilde{x}+i\tilde{y}$, $f(z)$ is a meromorhic function, with at most simple poles, which satisfies $f' \equiv df/dz \neq 0$ for all zs in a simply connected domain. The function f represents a "degree of freedom" - it specifies the solution of the Liouville equation. If $K > 0$, then one chooses '+' and the solution is a conformal factor describing local geometry of a surface with positive constant Gaussian curvature, and vice versa for $K < 0$.

Although we shall deal with specific σs in chapter 3, dedicated to surfaces of constant K, we shall not need to use this result. Therefore, we shall not derive it in this thesis and we just recommend the reference where the proof can be found, see the Liouville's original paper [14].

Before we start studying the surfaces of constant K, let us come back to the function Σ and to the metric tensor (2.4). Since the Gaussian curvature is constant, Σ cannot be arbitrary, but fulfills an analogous equation to (2.18). Let us rederive this constrain [18]. We start by computing the Ricci scalar $R^{(3)}$ in terms of Σ, in the coordinates (T, X, Y). Denoting the partial derivatives as $\partial_0 \equiv \partial_T \equiv \partial/\partial T$, $\partial_1 \equiv \partial_X \equiv \partial/\partial X$, $\partial_2 \equiv \partial_Y \equiv \partial/\partial Y$, the only independent Christoffel symbols are:

$$\Gamma^0{}_{00} = \Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^1{}_{01} = \Gamma^2{}_{02} = \partial_0\Sigma, \tag{2.22a}$$
$$\Gamma^0{}_{01} = \Gamma^1{}_{00} = \Gamma^1{}_{11} = -\Gamma^1{}_{22} = \Gamma^2{}_{12} = \partial_1\Sigma, \tag{2.22b}$$
$$\Gamma^0{}_{02} = \Gamma^1{}_{12} = \Gamma^2{}_{00} = -\Gamma^2{}_{11} = \Gamma^2{}_{22} = \partial_2\Sigma. \tag{2.22c}$$

The relevant components of the Ricci tensor are:

$$R_{00} = \left(-2\partial_0^2 + \partial_1^2 + \partial_2^2\right)\Sigma + (\partial_1\Sigma)^2 + (\partial_2\Sigma)^2, \tag{2.23a}$$
$$R_{11} = \left(\partial_0^2 - 2\partial_1^2 - \partial_2^2\right)\Sigma + (\partial_0\Sigma)^2 - (\partial_2\Sigma)^2, \tag{2.23b}$$
$$R_{22} = \left(\partial_0^2 - \partial_1^2 - 2\partial_2^2\right)\Sigma + (\partial_0\Sigma)^2 - (\partial_1\Sigma)^2. \tag{2.23c}$$

The Ricci scalar is:

$$R^{(3)} = e^{-2\Sigma}\left(R_{00} - R_{11} - R_{22}\right). \tag{2.24}$$

From (2.23) and (2.24) along with $R^{(3)} = -R^{(2)} = -2K$, it follows:

$$\Box\Sigma = -\frac{1}{2}\partial_a\Sigma\partial^a\Sigma + \frac{1}{2}Ke^{2\Sigma}, \tag{2.25}$$

where $\Box \equiv \partial_0^2 - \partial_1^2 - \partial_2^2$ and $\partial_a\Sigma\partial^a\Sigma \equiv (\partial_0\Sigma)^2 - (\partial_1\Sigma)^2 - (\partial_2\Sigma)^2$. We shall refer to the equation (2.25) as the *modified Liouville equation of the first form.*

In [18] the authors found the following solutions of (2.25):

$$\Sigma_{K<0}(T, X, Y) = -\frac{1}{2}\ln\frac{T^2 - X^2 - Y^2}{r^2}, \tag{2.26a}$$
$$\Sigma_{K=0}(T, X, Y) = -\ln\frac{T^2 - X^2 - Y^2}{c^2}, \tag{2.26b}$$

where the subscript of Σ indicates that the solution holds for constant negative/zero K, $r \equiv \sqrt{-K}$ is the constant radius of curvature and c is a constant, required for dimensional reasons.

Since the authors found more suitable to work within the light cone, the case $K > 0$ was not considered, so the solution:

$$\Sigma_{K>0}(T, X, Y) = -\frac{1}{2}\ln\frac{X^2 + Y^2 - T^2}{r^2}, \tag{2.27}$$

where $r \equiv \sqrt{K}$. We shall see that we have a good reason to work in outer region of the light cone. For instance, this is the case of the Rindler spacetime (a wedge of the Minkowski spacetime; for a short review, see e.g. [19]).

The relation between Σ and σ or the coordinate transformations $T(t, \tilde{x}, \tilde{y})$, $X(t, \tilde{x}, \tilde{y})$ and $Y(t, \tilde{x}, \tilde{y})$ were unfounded in the mentioned literature, too. This we shall face in chapter 4.

We shall show that there are many more solutions than (2.26). We shall also discuss the solutions for $K > 0$.

Let us just state that all of this is potentially interesting for graphene physics, see [18],[7] or a short review [20].

3. Surfaces of constant K

In this chapter, we introduce surfaces of constant K and put emphasis on description of their geometry and singularities (cusps, boundaries). We start with the surfaces of $K = 0$ and emphasize case of the *cone*, because it is locally flat everywhere except for its tip, where the *curvature is infinite* (*conical singularity*). We then move to the surfaces of constant $K \neq 0$. While there are three types of surfaces of $K > 0$, there is an *infinite number of surfaces of constant $K < 0$, differing by their topologies* ([7], [21]). In the latter case, we focus mainly, but not only, on surfaces of revolution.

We managed to contribute by original results. We found isothermal coordinates for two particular surfaces of constant $K < 0$: the *elliptic pseudosphere* and the *Dini surface*. This task is non-trivial considering the fact that any coordinate transformation leads to a system of non-linear partial differential equations, a problem difficult to solve in general. To the best of our knowledge, this issue has remained unresolved in the dedicated literature, see [7], [21], [22], [23], [24], [25].

3.1 Surfaces of $K = 0$

For $K = 0$, the Liouville equation (2.18) reduces to the Laplace equation:

$$\Delta_{(\tilde{x},\tilde{y})}\sigma = 0. \tag{3.1}$$

Its solutions are the harmonic functions, defined earlier (2.20) [26].

One solution can be guessed quit easily:

$$e^{(\tilde{x}+i\tilde{y})/c}, \tag{3.2}$$

where c is a real constant which makes the exponent dimensionless. Because we are interested in real functions, we take real or imaginary part of the complex exponential function (3.2):

$$\sigma(\tilde{x},\tilde{y}) = e^{\tilde{x}/c}\cos\frac{\tilde{y}}{c}, \quad \sigma(\tilde{x},\tilde{y}) = e^{\tilde{x}/c}\sin\frac{\tilde{y}}{c}. \tag{3.3}$$

Another solution can be found easily when we assume that it is radially symmetric: $\sigma(\tilde{r})$, $\tilde{r} = \sqrt{\tilde{x}^2+\tilde{y}^2}$. The Laplace eq. (3.1) becomes:

$$\left(\frac{\partial^2}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\right)\sigma(\tilde{r}) = 0. \tag{3.4}$$

The solution of (3.4) is then:

$$\sigma(\tilde{x},\tilde{y}) = \frac{a}{2}\ln\frac{\tilde{x}^2+\tilde{y}^2}{c^2}, \tag{3.5}$$

where a, c are real constants. Another example of a harmonic function is:

$$\sigma(\tilde{x},\tilde{y}) = \arctan\frac{\tilde{x}}{\tilde{y}}. \tag{3.6}$$

More harmonic functions can be generated simply by derivatives, e.g.:

$$\Delta_{(\tilde{x},\tilde{y})}\sigma = 0 \longrightarrow \partial_{\tilde{x}}\Delta_{(\tilde{x},\tilde{y})}\sigma = \Delta_{(\tilde{x},\tilde{y})}\partial_{\tilde{x}}\sigma = 0, \tag{3.7}$$

where we assumed $[\partial_{\tilde{x}}, \partial_{\tilde{y}}]\sigma = 0$ (see also (2.11)). So if σ is a harmonic function, so are $\partial_{\tilde{x}}\sigma$, $\partial_{\tilde{y}}\sigma$ etc.

Now we show how σs are related to local geometry of surfaces of $K = 0$. Focusing on (3.5), the corresponding line element is:

$$dl^2 = e^{2\sigma(\tilde{x},\tilde{y})}\left(d\tilde{x}^2 + d\tilde{y}^2\right) = \left(\frac{\tilde{r}^a}{c^a}d\tilde{r}\right)^2 + \frac{\tilde{r}^{2(a+1)}}{c^{2a}}d\tilde{\phi}^2, \tag{3.8}$$

where we applied the substitution from cartesian $(\tilde{x}, \tilde{y})$ to polar $(\tilde{r}, \tilde{\phi})$ isothermal coordinates: $\tilde{x} = \tilde{r}\cos\tilde{\phi}$, $\tilde{y} = \tilde{r}\sin\tilde{\phi}$. Depending on value of a, we obtain different results. We shall require $a \in [-1, 0]$. Then we distinguish the following cases:

1) $a \neq -1$

$$dl^2 = d\tilde{R}^2 + (a+1)^2\tilde{R}^2 d\tilde{\phi}^2, \quad \tilde{R} \equiv \frac{1}{c^a}\frac{\tilde{r}^{a+1}}{a+1}, \tag{3.9}$$

2) $a = -1$

$$dl^2 = d\tilde{R}^2 + c^2 d\tilde{\phi}^2, \quad \tilde{R} \equiv c\ln\frac{r}{b}, \tag{3.10}$$

where we ignored a constant of integration in the case 1), computing $\tilde{R}$, and b is a constant of integration.

Obviously, the line element (3.9) coincides with the line element of a plane if $a = 0$:

$$dl^2 = d\lambda^2 + \lambda^2 d\phi^2, \tag{3.11}$$

or a cone if $a \in (-1, 0)$:

$$dl^2 = d\lambda^2 + (\sin\alpha)^2\lambda^2 d\phi^2, \tag{3.12}$$

where $\lambda \in [0, +\infty)$, $\phi \in [0, 2\pi)$.

In the case of a plane, the coordinates (λ, ϕ) are standard polar coordinates. In the case of a cone, the coordinates are defined by the plot 3.1 (a). The non-zero angle 2α is the apex angle of the cone and for $\alpha = \pi/2$, the cone becomes a plane.

For $a = -1$, the line element (3.10) coincides with the line element of a cylinder:

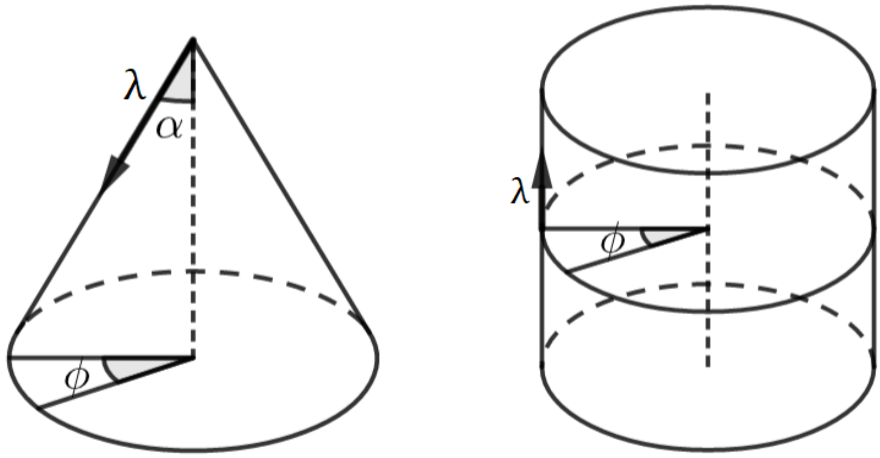
$$dl^2 = d\lambda^2 + \rho^2 d\phi^2, \tag{3.13}$$

where ρ is the radius of the cylinder and $\lambda \in [0, +\infty)$, $\phi \in [0, 2\pi)$. (λ, ϕ) is a cylindrical coordinate system, see the plot 3.1 (b).

The cone is an interesting example because it has zero Gaussian curvature everywhere except for its apex. This singularity is unremovable by a coordinate transformation (on the other hand, the circle boundary can be removed by the assumption of infinite cone). The singularity is called the conical singularity and we describe the method of calculating its scalar curvature in the Appendix A. Here, we present the final result [27]:

$$R = 4\pi(1 - \sin\alpha)\delta^{(2)}(\lambda). \tag{3.14}$$

This result is consistent with our intuition: the curvature is non-zero everywhere, except for a single point, the apex, where the curvature is infinite.



(a) Cone (b) Cylinder

Figure 3.1: Coordinates (λ, ϕ) for the cone and the cylinder

3.2 Surfaces of revolution with $K \neq 0$

A surface of revolution is a surface (in the Euclidean space of dimension three) created by rotating a curve around an axis of rotation. All surfaces (both of constant or nonconstant K) can be parametrized as (e.g. [7], [22]):

$$x(u,v) = R(u)\cos v, \; y(u,v) = R(u)\sin v, \; z(u) = \pm \int^{u} \sqrt{1 - R'^2(\bar{u})} d\bar{u}, \quad (3.15)$$

where x, y, z denotes the Euclidean coordinates, $v \in [0, 2\pi)$ is angular coordinate and u is meridian coordinate, whose range is determined by the condition:

$$1 - R'^2(u) \geq 0, \quad (3.16)$$

where $R'(u) \equiv dR(u)/du$. $R(u)$ is a function specific for particular surface of revolution[1]. The parametirization (3.15) is also refered to as the "*canonical parametrization*" [7].

With respect to the coordinates (u, v), the line element has a simple form:

$$dl^2 = du^2 + R^2(u)dv^2. \quad (3.17)$$

Let us compute the Gaussian curvature in these coordinates. The only non-trivial independent Christoffel symbols are:

$$\Gamma^1{}_{22} = -RR', \quad \Gamma^2{}_{12} = \frac{R'}{R}, \quad (3.18)$$

which gives:

$$R^1{}_{212} = -RR'', \quad R^2{}_{121} = -\frac{R''}{R}. \quad (3.19)$$

From that follows a simple relation for $K = R^{(2)}/2$:

$$\frac{R''}{R} = -K. \quad (3.20)$$

To find solutions of (3.20) we must distinguish (a) $K > 0$ and (b) $K < 0$.

[1]Since we deal with surfaces of constant curvature, hence we shall not have, e.g., $R^{(2)}(u)$, this notation is tenable.

Case (a): $K > 0$

Denoting with r the radius of the constant Gaussian curvature, $K \equiv 1/r^2 > 0$, the solution of (3.20) is:

$$R(u) = c\cos\left(\frac{u}{r} + b\right), \tag{3.21}$$

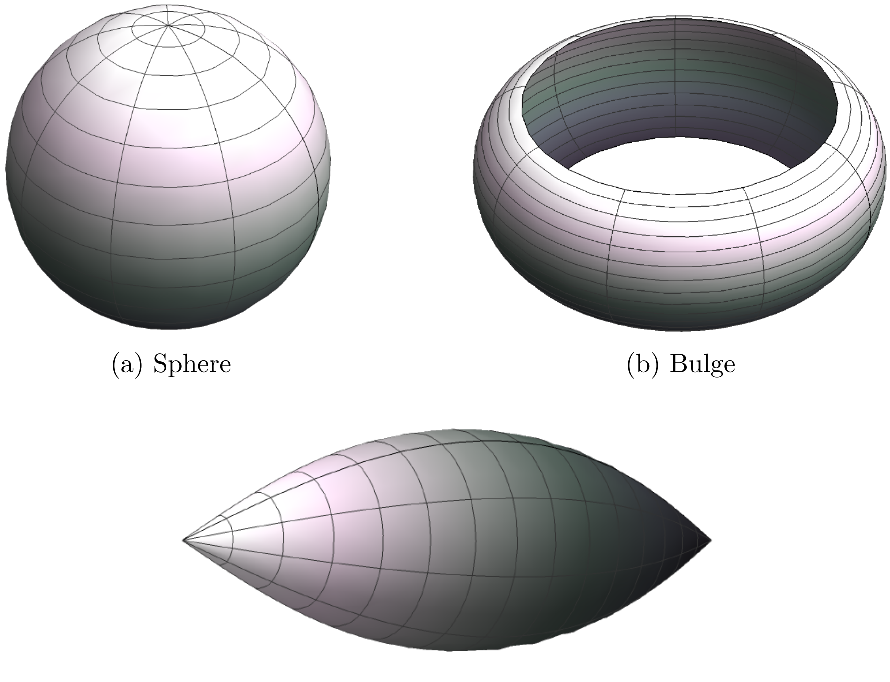
where b, c are constants of integration. Without loss of generality, we can choose $b = 0$ and $c > 0$, so that the maximum value of R corresponds to $u = 0$. Looking at (3.16), it becomes clear that the ratio c/r determinates the range of the u-coordinate, and at the same time the shape of the corresponding surface:

1) When $c = r$, we have the *Sphere*, with $u/r \in [-\pi/2, \pi/2]$.

2) When $c > r$, we have the *Bulge*, with $u/r \in [-\arcsin(r/c), \arcsin(r/c)]$.

3) When $c < r$, we have the *Spindle*, with $u/r \in [-\pi/2, \pi/2]$.

The sphere does not possess any singularity, the Bulge surface has two circles as boundaries, and the Spindle surface has two cusps, see 3.2 (a), (b), (c), respectively.



(a) Sphere

(b) Bulge

(c) Spindle

Figure 3.2: Surfaces of constant $K > 0$; Figures are taken from [28].

Let us note that the last two surfaces can be reduced to the sphere by a simple coordinate redefinition:

$$v \to \bar{v} \equiv \frac{c}{r}v, \tag{3.22}$$

with which the line element (3.17):

$$dl^2 = du^2 + R^2(u)dv^2 = du^2 + \left(R(u)\frac{r}{c}\right)^2 d\bar{v}^2 \equiv du^2 + \bar{R}^2(u)d\bar{v}^2, \tag{3.23}$$

where $\bar{R}(u) \equiv R(u)r/c = r\cos(u/r)$, is the one of the sphere ($c = r$) and we can enjoy the full range of u-coordinates, i.e. the circle boundaries or cusps, discussed before, are absent.

Of course, if $v \in [0, 2\pi)$, then $\bar{v} \in [0, 2\pi c/r)$ and the surface is just a part of the sphere (for $c < r$), or, conversely, is larger in the sense that possesses self-intersections (for $c > r$). These singularities can be avoided when we require $\bar{v} \in [0, 2\pi)$.

Although the singularities of the Bulge and Spindle surfaces can be removed by the coordinate redefinition (obtaining the sphere), they are all different surfaces in laboratory. See, e.g., [28] for possible applications.

We are also interested in finding the spatial conformal factors and isothermal coordinates $(\tilde{x}, \tilde{y})$. It is easy to find (see also [7]):

$$dl^2 = \frac{r^2}{\cosh^2 \tilde{y}} \left(d\tilde{x}^2 + d\tilde{y}^2 \right), \tag{3.24}$$

with

$$\tilde{x} = \frac{cv}{r}, \quad \tilde{y} = \ln\left(1 + \frac{2}{\cot(u/2r) - 1}\right), \quad \sigma(u) = \ln\cos\frac{u}{r}. \tag{3.25}$$

As in the case of the cone, we should bear in mind that σ should be properly defined. It means that $\sigma(u, 0) = \sigma(u, 2\pi)$, where the coordinate points $[u, v = 0]$ and $[u, v = 2\pi]$ correspond to the same point on the manifold. Otherwise, such ambiguity is unpleasent when we wish to make predictions for global geometry. Fortunately, this is not the case for surfaces of $K > 0$ - σ depends only on the u-coordinate, i.e. σ respects rotational symmetry.

Nonetheless, there is subtle point with this σ. Obviously, it diverges for $u \to \pm\pi/2$. This divergence cannot be clearly connected with singular points of the surface (also called the essential singularites). For example, the sphere has no singular points; in the case of the Bulge surface, the values $u = \pm\pi/2$ are even not in the domain of σ.

Case (b): $K < 0$

For $K \equiv -1/r^2 < 0$, the solution of (3.20) is:

$$R(u) = c_1 \sinh\frac{u}{r} + c_2 \cosh\frac{u}{r}, \tag{3.26}$$

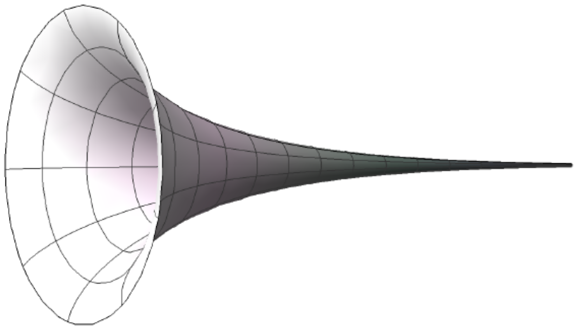
where c_1 and c_2 are constants of integration. The three different pseudospheres are given by:

1) $c \equiv c_1 = c_2$, $u/r \in (-\infty, \ln(r/c)]$ (*Beltrami*);

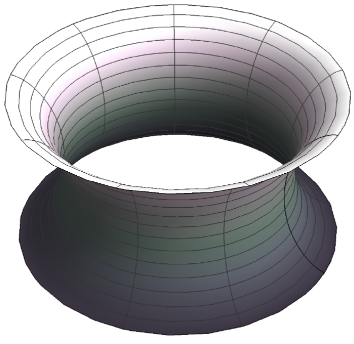
2) $c \equiv c_2$, $c_1 = 0$,
$u/r \in [-\operatorname{arccosh}\left(\sqrt{1 + (r/c)^2}\right), \operatorname{arccosh}\left(\sqrt{1 + (r/c)^2}\right)]$ (*Hyperbolic*);

3) $c \equiv c_1$, $c_2 = 0$, $u/r \in [0, \operatorname{arcsinh}\cot\beta]$, where $c \equiv r\sin\beta$ (*Elliptic*).

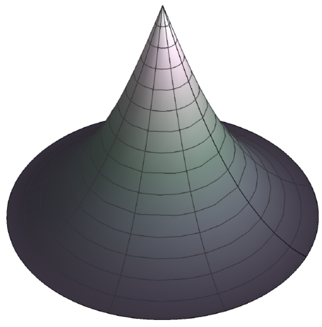
The Beltrami pseudosphere is an infinite surface with a circle of radius r as a boundary at $u_{\max}$, see the fig. 3.3 (a). The Hyperbolic pseudosphere is a finite surface with two circles of radii $\sqrt{c^2 + r^2}$ as its boundaries at $\pm u_{\max}$, see the fig.



(a) Beltrami



(b) Hyperbolic



(c) Elliptic

Figure 3.3: Pseudospheres; Figures are taken from [28].

3.3 (b). The Elliptic pseudosphere is also a finite surface, has an apex at $u_{\min} = 0$ and a circle as a boundary at $u_{\max}$, see the fig. 3.3 (c).

In constrast to the surfaces of $K > 0$, these singularities are unremovable by any coordinate redefinition. This is the consequence of the famous *Hilbert theorem* [23], which establishes:

"There exists no analytical complete surface of constant negative Gaussian curvature in the Euclidean space of dimension three."

A "complete surface" is a surface which does not possess essential singularities (e.g. cusps).

The surfaces with constant $K < 0$ are closely related to hyperbolic (also called "*Lobachevsky*") geometry. Because of that they are also called *Lobachevsky surfaces.* We shall not delve into details about Lobachevsky geometry, rather we shall recommend relevant literature, see [23], [24], [25]. Still, let us say a few words.

Hyperbolic geometry differs from Euclidean geometry for the fifth ("*parallel*") postulate: two different parallels can pass through the same point. All these straight lines (geodesics) then lie in the *Lobachevky plane* (analogue of the "Euclidean plane"). In the Lobachevsky plane, some basic facts, known from Euclidean geometry, do not apply. For instance, the sum of the angles in a triangle is less than π. Another useful concept is the *horocycle*: a curve whose normals all converge asymptotically in the same direction, its center. The area delimited by the horocycle and some of two its normals is called the *horocycle sector* [23]. This sector can be immersed into the Euclidean space; particular example of the horocycle sector and the surface, created from this immersion, is depicted in the fig. 3.4. It is the Beltrami pseudosphere.

We work within the *Poincaré upper half-plane model* $\{(\tilde{x},\tilde{y})|\tilde{y}>0\}$, where $(\tilde{x},\tilde{y})$ are called isothermal coordinates. The straight lines in the hyperbolic plane are represented in this model by circular arcs perpendicular to the $\tilde{x}$-axis (half-circles whose origin is on the $\tilde{x}$-axis) and straight vertical rays perpendicular to the $\tilde{x}$-axis [29], [30].

The Poincaré upper-half-plane model is equipped with the *Poincaré metric tensor* [23], [30]:

$$dl^2 = \frac{r^2}{\tilde{y}^2}\left(d\tilde{x}^2 + d\tilde{y}^2\right). \tag{3.27}$$

The Lobachevsky surfaces are covered by isothermal coordinates $(\tilde{x},\tilde{y})$ and their local geometry is (3.27).

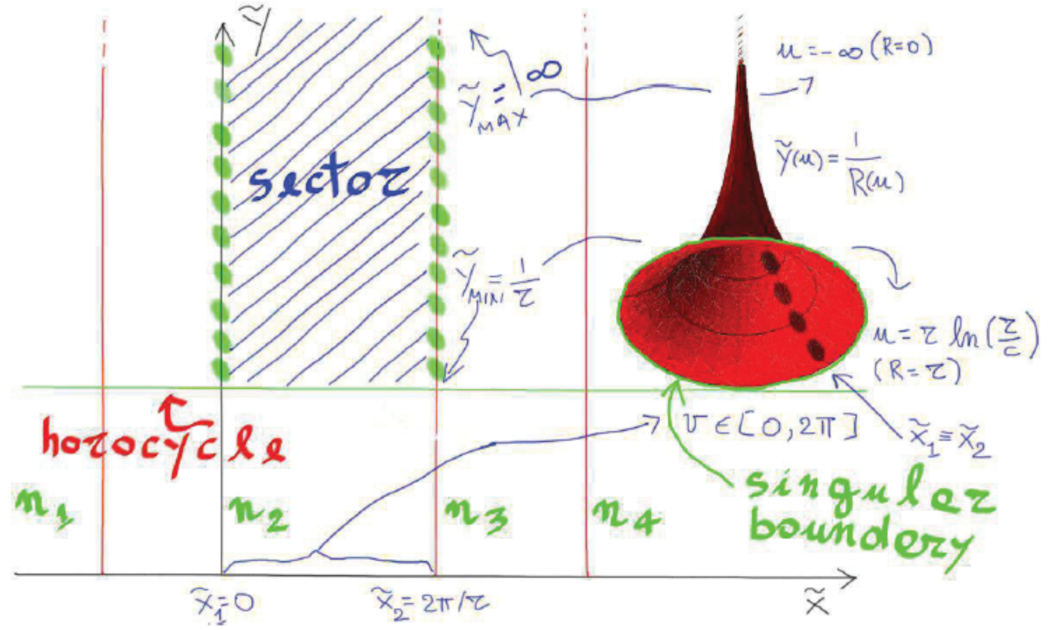


Figure 3.4: Horocycle sector and the Beltrami pseudosphere. ns are normals on the horocycle. Fig. is taken from [7], [31].

Now, we would like to find the coordinate transformation from (u,v), with the line element in the form (3.17) and (3.26), to $(\tilde{x},\tilde{y})$, with line element in the form (3.27), for all pseudospheres.

The Beltrami pseudosphere is obtained with the transformation:

$$\tilde{x} = cv, \quad \tilde{y} = re^{-u/r}. \tag{3.28}$$

For the Hyperbolic pseudosphere, the transformation is not that simple [7]:

$$\tilde{x} = re^{cv/r}\tanh\frac{u}{r}, \quad \tilde{y} = re^{cv/r}\frac{1}{\cosh\frac{u}{r}}. \tag{3.29}$$

To the best of our knowledge, isothermal coordinates for the Elliptic pseudosphere found in the literature do not have an attractive form [7]. Here, we found a clear and attractive form of such coordinates, given by:

$$\tilde{x} = r\frac{\sinh\frac{u}{r}\cos\frac{cv}{r}}{\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}}, \quad \tilde{y} = r\frac{1}{\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}}. \tag{3.30}$$

How we managed to find these coordinates will become clear later, when we shall discuss coordinate transformations in order to find isothermal coordinates (T, X, Y) for the corresponding $\mathbb{R} \times \mathcal{M}^2$ spacetimes, see (4.44).

The spatial conformal factors for the Beltrami, Hyperbolic and Elliptic pseudospheres are following:

$$\sigma_B(u) = \frac{u}{r}, \tag{3.31a}$$

$$\sigma_H(u,v) = -\frac{cv}{r} + \ln\left(\cosh\frac{u}{r}\right), \tag{3.31b}$$

$$\sigma_E(u,v) = \ln\left(\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}\right), \tag{3.31c}$$

respectively. Obviously, σ_B is well defined and becomes singular as $u \to -\infty$ and is finite on its boundary. On the other hand, last two σs are not properly defined:

$$\sigma_{E,H}(u,0) \neq \sigma_{E,H}(u,2\pi). \tag{3.32}$$

For the case of the Hyperbolic pseudosphere it is obvious. For the Elliptic pseudosphere, it is because of $c/r < 1$ and $\sin\frac{cv}{r}$ in (3.31). For this reason, we cannot easily draw conclusions about the global geometry of surfaces.

3.3 Other surfaces of constant $K < 0$

As we said earlier, there is an infinite number of surfaces of constant $K < 0$. Besides the previous three examples, we would like to make a comment on other examples, particularly: the *Dini*, the *Kuen* and the *Breather* surfaces.

In the case of the Dini surface, we found isothermal coordinates with a well-behaved conformal factor. This surface represents a sort of generalization of the Beltrami pseudosphere: it depends on two real parameters a, b, with curvature radius $r \equiv \sqrt{a^2+b^2}$, and for b vanishing it reduces to the Beltrami pseudosphere with $a = r = c$, see (3.28).

In the case of the last two examples, we present their parametrizations and plots.

3.3.1 Dini surface

The Dini surface can be parametrized in the following way [32]:

$$x = a\sin u\cos v, \quad y = a\sin u\sin v, \quad z = a\left[\cos u + \ln\tan\frac{u}{2}\right] + bv, \tag{3.33}$$

where $u \in [0, \pi/2)$, $v \in [0, 2\pi)$ and a, b are real parameters discussed above. These coordinates are also refered to as the "canonical parametrization".

In order to find isothermal coordiates, we assume the following substitution $\rho = \ln\sin u$, which leads to:

$$dl^2 = \left(a^2e^{2\rho} + b^2\right)dv^2 + a^2d\rho^2 + 2ab\sqrt{1-e^{2\rho}}d\rho dv. \tag{3.34}$$

In isothermal coordinates, the line element takes the form:

$$dl^2 = \frac{a^2+b^2}{\tilde{y}^2}\left(d\tilde{x}^2 + d\tilde{y}^2\right). \tag{3.35}$$

Both (3.34) and (3.35) lead to a system of non-linear partial differential equations:

$$a^2 e^{2\rho} + b^2 = \frac{a^2+b^2}{\tilde{y}^2}\left[\left(\frac{\partial \tilde{x}}{\partial v}\right)^2 + \left(\frac{\partial \tilde{y}}{\partial v}\right)^2\right], \tag{3.36}$$

$$a^2 = \frac{a^2+b^2}{\tilde{y}^2}\left[\left(\frac{\partial \tilde{x}}{\partial \rho}\right)^2 + \left(\frac{\partial \tilde{y}}{\partial \rho}\right)^2\right], \tag{3.37}$$

$$ab\sqrt{1-e^{2\rho}} = \frac{a^2+b^2}{\tilde{y}^2}\left[\frac{\partial \tilde{x}}{\partial v}\frac{\partial \tilde{x}}{\partial \rho} + \frac{\partial \tilde{y}}{\partial v}\frac{\partial \tilde{y}}{\partial \rho}\right]. \tag{3.38}$$

We shall solve the (3.36)-(3.38) by the following Ansatz:

$$\tilde{x} = h(v)g_1(\rho) + c, \quad \tilde{y} = h(v)g_2(\rho), \tag{3.39}$$

where c is a constant. We shall omit the detailed, lengthy 'step by step' calculation and present just the result:

$$g_1(\rho) = \frac{a^2}{b^2}\sqrt{1 - \frac{b^4}{a^4}\frac{a^2+b^2}{a^2e^{2\rho}+b^2}}g(\rho), \quad g_2(\rho) = \sqrt{\frac{a^2+b^2}{a^2e^{2\rho}+b^2}}g(\rho), \tag{3.40}$$

$$g(\rho) = \exp\left[\frac{\sqrt{a^2+b^2}b}{a^2}\operatorname{arctanh}\left(\frac{a\sqrt{1-e^{2\rho}}}{\sqrt{a^2+b^2}}\right) - \frac{b}{a}\operatorname{arctanh}\left(\sqrt{1-e^{2\rho}}\right)\right], \tag{3.41}$$

$$h(v) = \exp\left(\frac{b^2}{a^2}v\right), \quad c = -a^2/b^2. \tag{3.42}$$

From (3.40) it follows the condition:

$$e^{2\rho} \geq -\frac{b^2}{a^2} + \frac{b^4}{a^4} + \frac{b^6}{a^6}. \tag{3.43}$$

The right-hand side is negative for $b/a \in \left(0, \sqrt{\frac{\sqrt{5}}{2} - \frac{1}{2}}\right) \approx (0, 0.786)$. For b/a in this range, it holds $\rho \in (-\infty, 0]$, i.e. $u \in [0, \pi/2)$. For b/a larger than $\sqrt{\frac{\sqrt{5}}{2} - \frac{1}{2}}$, $\rho_{\min}$ should be larger than $-\infty$ provided that these isothermal coordinates cover the whole surface. However, $\rho_{\min} = -\infty$, so the coordinates do not.

The limit $b/a \to 0$ reproduces the Beltrami pseudosphere with:

$$\tilde{x} = v, \quad \tilde{y} = e^{-\rho}, \quad a = r = c, \quad v \in [0, 2\pi), \quad \rho \in (-\infty, 0]. \tag{3.44}$$

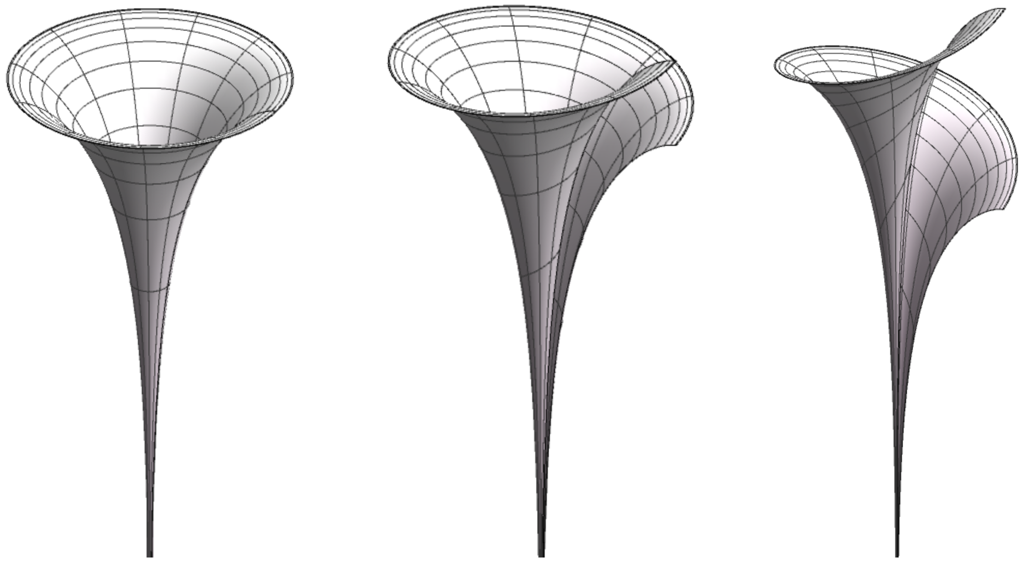
In constrast to (3.28), (3.39) and (3.44) are dimensionless coordinates. Of course, the coordinates can be redefined as:

$$\tilde{x} = \sqrt{a^2+b^2}\left(h(v)g_1(\rho) + c\right), \tilde{y} = \sqrt{a^2+b^2}h(v)g_2(\rho). \tag{3.45}$$

This becomes necessary when we wish to write the conformal factor:

$$\sigma = \frac{1}{2}\frac{a^2e^{2\rho}+b^2}{a^2+b^2} + \frac{b^2}{a^2}\left(v + \frac{\sqrt{a^2+b^2}}{b}\operatorname{arctanh}\frac{a\sqrt{1-e^{2\rho}}}{\sqrt{a^2+b^2}} - \frac{a}{b}\operatorname{arctanh}\sqrt{1-e^{2\rho}}\right). \tag{3.46}$$

Althought the conformal factor (3.46) does depend on v (azimuthal angle), it is still well-defined: no multivalueness is present, because the Dini surface is not rotational symmetric, but it is like a helix, see (3.5).



(a) $a = 1$, $b = 0$ (b) $a = 1$, $b = 0.1$ (c) $a = 1$, $b = 0.3$

Figure 3.5: Dini surface

3.3.2 Kuen and Breather surfaces

The canonical parametrization for the Kuen surface is:

$$x = a\frac{\sin u \cos v}{\frac{1}{2} + \frac{1}{2}v^2 \sin^2 u} + a\frac{\sin u \sin v}{\frac{1}{2} + \frac{1}{2}v^2 \sin^2 u}v, \tag{3.47}$$

$$y = a\frac{\sin u \sin v}{\frac{1}{2} + \frac{1}{2}v^2 \sin^2 u} - a\frac{\sin u \cos v}{\frac{1}{2} + \frac{1}{2}v^2 \sin^2 u}v, \tag{3.48}$$

$$z = a\left[\frac{\cos u}{\frac{1}{2} + \frac{1}{2}v^2 \sin^2 u} + \ln \tan \frac{u}{2}\right], \tag{3.49}$$

where $u \in [0, 2\pi)$, $v \in [0, \pi)$ and a stands for the curvature radius [33], [25].

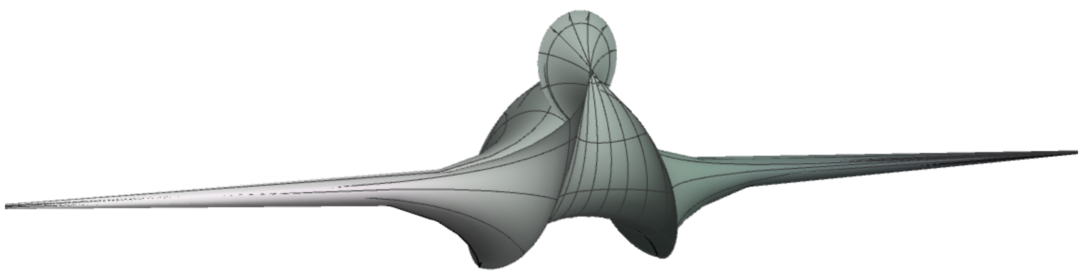


Figure 3.6: Kuen surface

The canonical parametrization for the Breather surface is:

$$x = -\frac{2\sqrt{1-b^2}\cosh bu\left(-\sqrt{1-b^2}\cos v\cos\left(\sqrt{1-b^2}v\right) - \sin v\sin(\sqrt{1-b^2}v)\right)}{b\left[(1-b^2)\cosh^2 bu + b^2\sin^2(\sqrt{1-b^2}v)\right]}, \tag{3.50}$$

$$y = -\frac{2\sqrt{1-b^2}\cosh bu\left(-\sqrt{1-b^2}\sin v\cos\left(\sqrt{1-b^2}v\right) + \cos v\sin(\sqrt{1-b^2}v)\right)}{b\left[(1-b^2)\cosh^2 bu + b^2\sin^2(\sqrt{1-b^2}v)\right]}, \tag{3.51}$$

$$z = u - \frac{2(1-b^2)\cosh bu\sinh bu}{b\left[(1-b^2)\cosh^2 bu + b^2\sin^2(\sqrt{1-b^2}v)\right]}, \tag{3.52}$$

where $u \in (-\infty, +\infty)$, $v \in (-\infty, +\infty)$, $a \in (0, 1)$. The Gaussian curvature is: $K \equiv -1$, so independent on a [23], [25].

Let us add one more observation. Firstly, let us shift the u-coordinate for the Kuen surface as $u \to u + \pi/2$, which symmetrizes its u-range: $u \in [-\pi/2, \pi/2]$. If the u-coordinate is from a vicinity of zero, it holds:

$$x = a\left(\frac{\cos v}{\frac{1}{2} + \frac{1}{2}v^2} + \frac{\sin v}{\frac{1}{2} + \frac{1}{2}v^2}v\right), \tag{3.53}$$

$$y = a\left(\frac{\sin v}{\frac{1}{2} + \frac{1}{2}v^2} - \frac{\cos v}{\frac{1}{2} + \frac{1}{2}v^2}v\right), \tag{3.54}$$

$$z = a\left(u - \frac{u}{\frac{1}{2} + \frac{1}{2}v^2}\right). \tag{3.55}$$

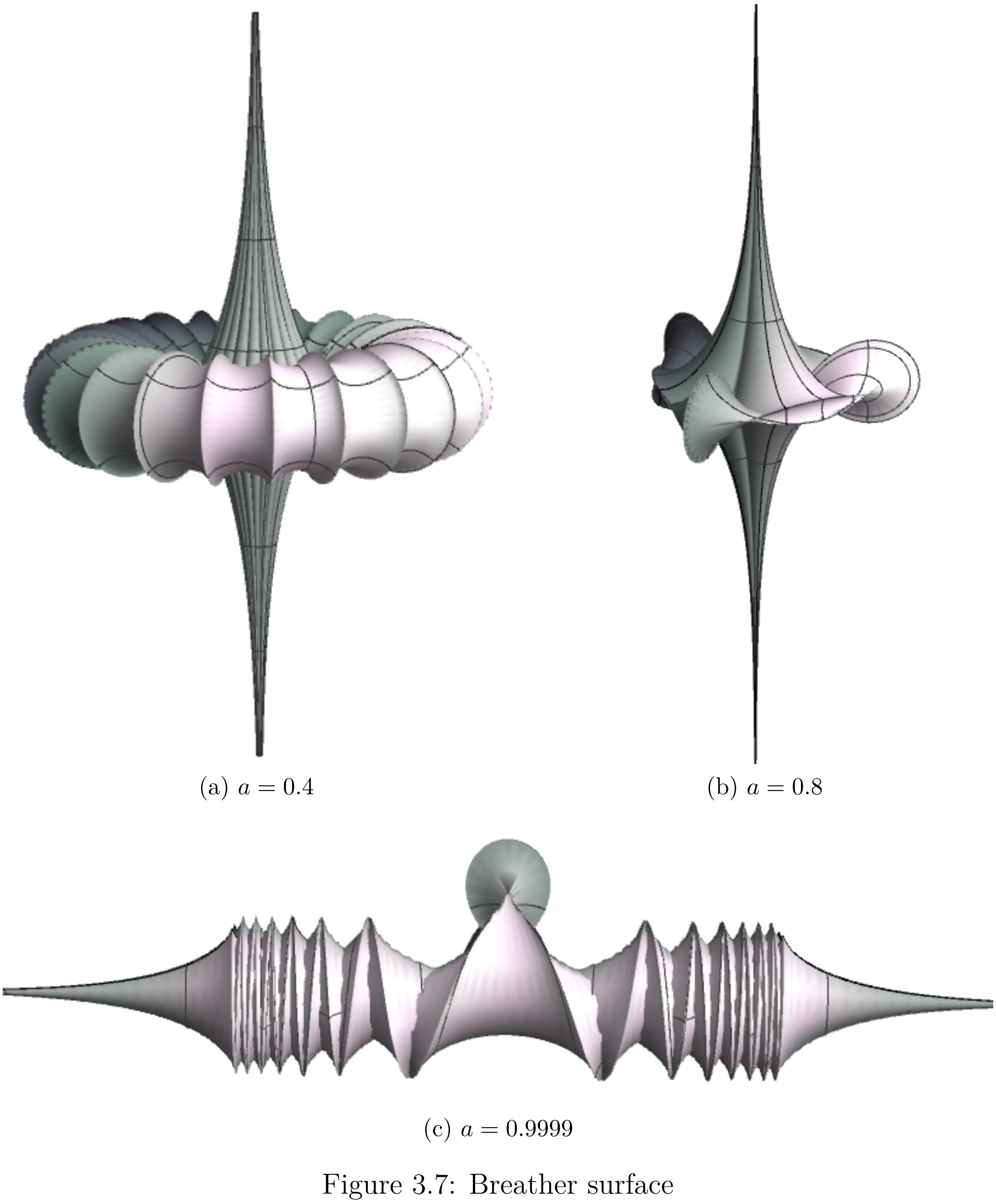
Then, for the breather surface, we define ϵ: $b \equiv 1 - \epsilon$, and make a limit $\epsilon \to 0^+$. Again, if the u-coordinate is from a vicinity of zero, the surface satisfies:

$$x = \frac{\cos v}{\frac{1}{2} + \frac{1}{2}v^2} + \frac{\sin v}{\frac{1}{2} + \frac{1}{2}v^2}v, \tag{3.56}$$

$$y = \frac{\sin v}{\frac{1}{2} + \frac{1}{2}v^2} - \frac{\cos v}{\frac{1}{2} + \frac{1}{2}v^2}v, \tag{3.57}$$

$$z = u - \frac{u}{\frac{1}{2} + \frac{1}{2}v^2}. \tag{3.58}$$

The comparison (3.53)-(3.55) and (3.56)-(3.58) suggests that both surfaces coincide with each other (in prescribed limits).



(a) $a = 0.4$

(b) $a = 0.8$

(c) $a = 0.9999$

Figure 3.7: Breather surface

4. Spacetimes $\mathbb{R} \times \mathcal{M}^2$

In this Chapter, we focus our attention on the spacetimes $\mathbb{R} \times \mathcal{M}^2$, defined earlier in chapter 2. Our goals can be summarized as follows: We shall investigate the modified Liouville equation and search for new solutions, see (2.25) and (2.26). Next, we shall make the effort to find the explicit coordinate transformation from $(t, \tilde{x}, \tilde{y})$ to (T, X, Y):

$$ds^2 = dt^2 - \frac{r^2}{\tilde{y}^2}\left(d\tilde{x}^2 + d\tilde{y}^2\right) = e^{2\Sigma(T,X,Y)}\left(dT^2 - dX^2 - dY^2\right), \tag{4.1}$$

to obtain the functions:

$$T(t, \tilde{x}, \tilde{y}),\ X(t, \tilde{x}, \tilde{y}),\ Y(t, \tilde{x}, \tilde{y}),\ \Sigma(t, \tilde{x}, \tilde{y}). \tag{4.2}$$

We have already found that coordinate transformation can be a difficult task, since it leads to a system of *non-linear partial differential equations* (N-PDE). The task to find (4.2) was faced and partially solved in [7]: the authors found (4.2) for one particular example - the Elliptic pseudosphere:

$$T = re^{t/r}\cosh\frac{u}{r},\quad X = re^{t/r}\sinh\frac{u}{r}\cos\frac{cv}{r},\quad Y = re^{t/r}\sinh\frac{u}{r}\sin\frac{cv}{r}, \tag{4.3a}$$

$$\Sigma_{\mathrm{E}}(T, X, Y) = -\frac{1}{2}\ln\frac{T^2 - X^2 - Y^2}{r^2}, \tag{4.3b}$$

$$\Sigma_{\mathrm{E}}(t) = -\frac{t}{r}. \tag{4.3c}$$

We managed to generalize this result (4.3) to all surfaces of $K = -1/r^2$, with the same conformal factor (4.3c). We have even found another set of functions (4.2) for $K = -1/r^2$, with different $\Sigma(T, X, Y)$, see (4.24). They can be useful for further work in the context of [7] and [18], for instance. We also discuss (4.3) for $K = 0$ and $K > 0$. The latter case was considered unsolved, see [7], but was already found by Roger Penrose himself during the 1960s when he worked on the method of compactification of spacetimes (we recommend [13]). For the case $K = 0$, we recall the results from [18].

4.1 Conformal factors Σs

The modified Liouville equation of the first form (2.25):

$$\Box\Sigma = -\frac{1}{2}\partial_a\Sigma\partial^a\Sigma + \frac{1}{2}Ke^{2\Sigma}, \tag{4.4}$$

can be rewritten as:

$$\Box f = \frac{K}{4}f^5, \tag{4.5}$$

where

$$f \equiv e^{\Sigma/2}. \tag{4.6}$$

We must keep in mind that $f \geq 0$, with $f = 0$ if and only if $\Sigma \to -\infty$. We shall refer to (4.5) as the *modified Liouville equation of the second form.*

Case $K = 0$

For $K = 0$, (4.5) becomes the wave equation:

$$\Box f = 0. \tag{4.7}$$

Considering the following initial conditions:

$$f(0, \vec{Q}) = g_1(\vec{Q}), \quad f_T(0, \vec{Q}) = g_2(\vec{Q}), \tag{4.8}$$

where $\vec{Q} \equiv (X, Y)$, $T \geq 0$ and $f_T(0, \vec{Q}) \equiv \partial_T f(T, \vec{Q})|_{T=0}$, the general solution of the 2-dimensional wave equation (4.7) can be found as [34],[35]:

$$f(T, \vec{Q}) = \frac{1}{2\pi t^2} \int_{B(T,\vec{Q})} \frac{T g_1(\vec{P}) + T^2 g_2(\vec{P}) + T \nabla g_1(\vec{P}) \cdot (\vec{P} - \vec{Q})}{\sqrt{T^2 - |\vec{P} - \vec{Q}|^2}} dP, \tag{4.9}$$

where $\vec{P} \equiv (\tilde{X}, \tilde{Y})$, $dP \equiv d\tilde{X} d\tilde{Y}$ and $B(T, \vec{Q})$ is a disk of radius T about $\vec{Q}$, whose area is $|B(T, \vec{Q})| = \pi T^2$.

Let us introduce a few particular examples of (4.9):

$$f = \sin\left(T - k_X X - k_Y Y\right), \tag{4.10a}$$

$$f = \sin\left(k_X X + k_Y Y - T\right), \tag{4.10b}$$

$$f = \left(\frac{c^2}{T^2 - X^2 - Y^2}\right)^{1/2}, \tag{4.10c}$$

$$f = \left(\frac{c^2}{-T^2 + X^2 + Y^2}\right)^{1/2}, \tag{4.10d}$$

where $k_X^2 + k_Y^2 = 1$ and c is a real constant. Notice that (4.10c) corresponds to:

$$\Sigma = -\ln \frac{T^2 - X^2 - Y^2}{c^2}, \tag{4.11}$$

which was found and presented as the only solution in (2.26).

Once we find the function $f(T, \vec{Q})$, we must focus on the ranges of X, Y such that $f \geq 0$. They are, in general, time dependent.

Let us find another solution: we consider the following Ansatz:

$$f(T, R) = g(T) h(R), \tag{4.12}$$

where $R \equiv \sqrt{X^2 + Y^2}$. Then (4.7) becomes:

$$\partial_T^2 g h - g \partial_R^2 h - \frac{1}{R} g \partial_R h = 0, \tag{4.13}$$

which can be adjusted to:

$$\frac{\partial_T^2 g}{g} = \frac{\partial_R^2 h}{h} + \frac{1}{R} \frac{\partial_R h}{h} \equiv c_1, \tag{4.14}$$

where c_1 is a real constant. We managed to find two equations, each for one function. The solutions of these equations then depend on the sign of c_1. Notice that the equation for $h(R)$ is actually the Bessel equation:

$$R^2 \partial_R^2 h + R \partial_R h - c_1 h R^2 = 0, \tag{4.15}$$

whose solutions are the Bessel functions.

We can have three cases:

$$\text{for } c_1 > 0 : g(T) = e^{\sqrt{c_1}T},\ h(R) = k_1 J_0(i\sqrt{c_1}R) + k_2 Y_0(-i\sqrt{c_1}R), \tag{4.16a}$$

$$\text{for } c_1 < 0 : g(T) = c_2 \sin(\sqrt{|c_1|}T + \phi),\ h(R) = k_1 J_0(\sqrt{|c_1|}R) + k_2 Y_0(\sqrt{|c_1|}R), \tag{4.16b}$$

$$\text{for } c_1 = 0 : g(T) = c_3 T + c_4,\ h(R) = k_1 \ln(k_2 R). \tag{4.16c}$$

where J_0 and Y_0 are examples of the Bessel functions [36]. The Bessel functions of the first and second kind are, respectively:

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\alpha+2k}}{k!\Gamma(\alpha+k+1)},\quad Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}, \tag{4.17}$$

where α is a real parameter. For $\alpha \equiv p$ to be an integer, (4.17) becomes:

$$J_p(x) = J_\alpha(x),\quad Y_p(x) \equiv \lim_{\alpha\to p} Y_\alpha(x). \tag{4.18}$$

Next, the conformal factor can also depend only on one spatial and time coordinate: $f(T, X)$. In this case, (4.7) reduces to the 1-dimensional wave equation. Its general solution is the d'Alembert's formula [35]:

$$f(T, X) = \frac{1}{2}\left[g_1(X-T) + g_1(X+T)\right] + \frac{1}{2}\int_{X-T}^{X+T} g_2(\xi)d\xi. \tag{4.19}$$

We can also assume that f does not depend on time T, so the wave equation reduces to the Laplace equation:

$$\Delta_{(X,Y)} f = 0. \tag{4.20}$$

The solutions of the Laplace equation are the harmonic functions, which we have already discussed, see (3.1).

Case $K \neq 0$

Depending on the sign of K, we found the following solutions:

1) $K \equiv 1/r^2 > 0$

$$f = \left(\frac{r^2}{X^2+Y^2-T^2}\right)^{1/4},\quad \Sigma = -\frac{1}{2}\ln\frac{X^2+Y^2-T^2}{r^2}, \tag{4.21a}$$

$$f = \left(\frac{r^2}{T^2-X^2}\right)^{1/4},\quad \Sigma = -\frac{1}{2}\ln\frac{T^2-X^2}{r^2}, \tag{4.21b}$$

$$f = \left(\frac{r^2}{T^2-Y^2}\right)^{1/4},\quad \Sigma = -\frac{1}{2}\ln\frac{T^2-Y^2}{r^2}. \tag{4.21c}$$

2) $K \equiv -1/r^2 < 0$

$$f = \left(\frac{r^2}{T^2 - X^2 - Y^2}\right)^{1/4}, \quad \Sigma = -\frac{1}{2}\ln\frac{T^2 - X^2 - Y^2}{r^2}, \tag{4.22a}$$

$$f = \left(\frac{r^2}{X^2 - T^2}\right)^{1/4}, \quad \Sigma = -\frac{1}{2}\ln\frac{X^2 - T^2}{r^2}, \tag{4.22b}$$

$$f = \left(\frac{r^2}{Y^2 - T^2}\right)^{1/4}, \quad \Sigma = -\frac{1}{2}\ln\frac{Y^2 - T^2}{r^2}, \tag{4.22c}$$

$$f = \left(\frac{r^2}{X^2 + Y^2}\right)^{1/4}, \quad \Sigma = -\frac{1}{2}\ln\frac{X^2 + Y^2}{r^2}. \tag{4.22d}$$

Let us recall that the solution (4.22a) was found in [18]. Later on, we shall add one more solution for $K > 0$ and the explicit coordinate transformation, which leads to it, see (4.50). This solution was found in [37], but also by Penrose, see [13].

Perhaps, we could find more solutions. However, we showed that there are more solutions of the modified Liouville equation of the first form (4.4) than just found in [18]. Each of them should correspond to particular solution of the coordinate transformation (4.2).

Finally, we must care about the ranges of X, Y, since $f \geq 0$. The next step is to find the explicit coordinate transformation (4.2) and link Σs to particular spacetimes $\mathbb{R} \times \mathcal{M}^2$.

4.2 Coordinate transformations for $K < 0$

We start with $K < 0$ and look for the explicit coordinate transformation from (T, X, Y) to $(t, \tilde{x}, \tilde{y})$.

1st Attempt

In the coordinates $(t, \tilde{x}, \tilde{y})$, the spacetime interval can be rewritten as:

$$\begin{aligned} ds^2 &= \frac{r^2}{\tilde{y}^2}\left(\frac{\tilde{y}^2}{r^2}dt^2 - d\tilde{x}^2 - d\tilde{y}^2\right) \\ &\equiv \frac{r^2}{\tilde{y}^2}ds_R^2, \end{aligned} \tag{4.23}$$

where R stands for "Rindler". It was shown in [7] that ds_R^2 is the line element of the Rindler spacetime (i.e. the right or the left wedge of the Minkowski spacetime, where an accelerated Rindler observer moves). For a review, we recommend e.g. [19]. What is missing in [7] is the explicit coordinate transformation (4.2), which is the following:

$$T = \tilde{y}\sinh\frac{t}{r}, \quad Y = \tilde{y}\cosh\frac{t}{r}, \quad X = \tilde{x}, \tag{4.24a}$$

$$\Sigma_{K<0}(T, Y) = \sigma(\tilde{y}(T, Y)) = -\ln\frac{\tilde{y}(T, Y)}{r} = -\frac{1}{2}\ln\frac{Y^2 - T^2}{r^2}. \tag{4.24b}$$

Both $\Sigma_{K<0}$ and σ are the same function. Let us recall here the local conformal factors of the three pseudospheres (3.31):

$$\begin{aligned} \sigma_B(u) &= \frac{u}{r}, \\ \sigma_H(u,v) &= -\frac{cv}{r} + \ln\left(\cosh\frac{u}{r}\right), \\ \sigma_E(u,v) &= \ln\left(\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}\right). \end{aligned} \tag{4.25}$$

As we have already discussed, the last two factors suffer from the ambiguity (multivalueness):

$$\sigma_{H,E}(u,0) \neq \sigma_{H,E}(u,2\pi). \tag{4.26}$$

This limits our ability to discuss global geometry. Because of this reason, it seems plausible to apply this transformation only on the Beltrami pseudosphere, see the fig. (3.3), and the Dini surface, see the fig. (3.5).

Notice that there is another straighforward way how to obtain the same result (4.24) especially for the spacetime associated to the Beltrami pseudosphere $\mathbb{R} \times \mathcal{M}^2_B$. We start with its spacetime interval:

$$ds_B^2 = dt^2 - du^2 - c^2 e^{2u/r} dv^2 = dz dw - c^2 \frac{1}{e^{w/r} e^{-z/r}} dv^2, \tag{4.27}$$

where we used $z \equiv t + u$, $w \equiv t - u$. Applying $z \equiv -r \ln(p/r)$, $w \equiv r \ln(q/r)$, the line element (4.27) becomes:

$$ds_B^2 = \frac{r^2}{pq}\left[-dpdq - d\left(cv\right)^2\right]. \tag{4.28}$$

Finally, $p \equiv Y - T$, $q \equiv Y + T$, $X \equiv cv$ leads to:

$$ds_B^2 = \frac{r^2}{Y^2 - T^2}\left(dT^2 - dX^2 - dY^2\right), \tag{4.29}$$

which makes the conformal factor evident:

$$\Sigma_{\mathrm{B}}(T,Y) = -\frac{1}{2}\ln\frac{Y^2 - T^2}{r^2}, \tag{4.30a}$$

$$\Sigma_{\mathrm{B}}(u) = \frac{u}{r}. \tag{4.30b}$$

Then, the coordinates transformation from (T, X, Y) to (t, u, v) is:

$$T = re^{-u/r}\sinh\frac{t}{r}, \quad Y = re^{-u/r}\cosh\frac{t}{r}, \quad X = cv. \tag{4.31}$$

Obviously, such Σ differs from the one in (4.3). This will be found by the explicit coordinate transformation in what follows.

2nd Attempt

Here, we would like to find the explicit coordinate transformation from $(t, \tilde{x}, \tilde{y})$ to (T, X, Y), with the global conformal factor $\Sigma_{K<0}$ in (2.26). We shall start with one particular example, the spacetime associated to the Hyperbolic pseudosphere,

$\mathrm{I\!R} \times \mathcal{M}_H^2$, and treat it similarly as we did $\mathrm{I\!R} \times \mathcal{M}_B^2$ above. We shall discover that the same procedure leads to the conformal factor we are looking for. Then we shall find the generalization for all Lobachevsky surfaces.

The spacetime interval of $\mathrm{I\!R} \times \mathcal{M_H}^2$ is:

$$ds_H^2 = dt^2 - du^2 - c^2 \cosh^2 \frac{u}{r} dv^2 = dzdw - \frac{c^2}{4e^{z/r}e^{w/r}} \left(e^{z/r} + e^{w/r}\right)^2 dv^2 \quad (4.32)$$

where $z \equiv t+u$, $w \equiv t-u$. The next substitution is: $z \equiv r\ln(p/r)$, $w = r\ln(q/r)$, from which follows:

$$ds_H^2 = \frac{r^2}{pq}\left[dpdq - \frac{1}{4}(p+q)^2 d\left(\frac{cv}{r}\right)^2\right]. \quad (4.33)$$

Applying $p \equiv \tilde{T} + X_H$, $q \equiv \tilde{T} - X_H$, $\theta \equiv cv/r$, the previous result can be written as:

$$ds_H^2 = \frac{r^2}{\tilde{T}^2 - X_H^2}\left[d\tilde{T}^2 - dX_H^2 - \tilde{T}^2 d\theta^2\right]. \quad (4.34)$$

Finally, we apply $T_H \equiv \tilde{T}\cosh\theta$, $Y_H \equiv \tilde{T}\sinh\theta$ and obtain:

$$ds^2 = \frac{r^2}{T_H^2 - X_H^2 - Y_H^2}\left(dT_H^2 - dX_H^2 - dY_H^2\right) \quad (4.35)$$

which gives for the conformal factor:

$$\Sigma_{\mathrm{H}}(T_H, X_H, Y_H) = -\frac{1}{2}\ln\frac{T_H^2 - X_H^2 - Y_H^2}{r^2}, \quad (4.36a)$$

$$\Sigma_{\mathrm{H}}(t) = -\frac{t}{r}. \quad (4.36b)$$

The final coordinate transformation is:

$$T_H = re^{t/r}\cosh\frac{u}{r}\cosh\frac{cv}{r}, \quad Y_H = re^{t/r}\cosh\frac{u}{r}\sinh\frac{cv}{r}, \quad X_H = re^{t/r}\sinh\frac{u}{r}. \quad (4.37)$$

This can be generalized to all Lobachevsky surfaces in the straighforward way. Let us remind isothermal coordinates corresponding to the Hyperbolic pseudosphere (3.29):

$$\tilde{x}_H = e^{cv/r}\tanh\frac{u}{r}, \quad \tilde{y}_H = e^{cv/r}\frac{1}{\cosh\frac{u}{r}}, \quad (4.38)$$

where isothermal coordinates are dimensionless.

When we replace (u, v), denoting here the coordinates covering the Hyperbolic pseudosphere, by $(\tilde{x}_H, \tilde{y}_H)$ in (4.37), using (4.38), we obtain:

$$T_H = re^{t/r}\frac{\tilde{x}_H^2 + \tilde{y}_H^2 + 1}{2\tilde{y}_H}, \quad Y_H = re^{t/r}\frac{\tilde{x}_H^2 + \tilde{y}_H^2 - 1}{2\tilde{y}_H}, \quad X_H = re^{t/r}\frac{\tilde{x}_H}{\tilde{y}_H}, \quad (4.39)$$

Then the spacetime interval ds_H^2 can be written in the following forms:

$$e^{2\Sigma_{\mathrm{H}}}\left(dT_H^2 - dX_H^2 - dY_H^2\right) = dt^2 - du^2 - c^2\cosh^2\frac{u}{r}dv^2 = dt^2 - \frac{r^2}{\tilde{y}_H^2}\left(d\tilde{x}_H^2 + d\tilde{y}_H^2\right). \quad (4.40)$$

This can be generalized to all Lobachevsky surfaces in very natural way. It is simple to consider (or show directly) that:

$$T = re^{t/r}\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}, \quad Y = re^{t/r}\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}, \quad X = re^{t/r}\frac{\tilde{x}}{\tilde{y}}, \tag{4.41}$$

leads to the following transformation of the spacetime interval:

$$e^{2\Sigma_{K<0}}\left(dT^2 - dX^2 - dY^2\right) = dt^2 - \frac{r^2}{\tilde{y}^2}\left(d\tilde{x}^2 + d\tilde{y}^2\right), \tag{4.42}$$

with the conformal factor same as (4.3) and (4.36):

$$\Sigma_{K<0}(T,X,Y) = -\frac{1}{2}\ln\frac{T^2-X^2-Y^2}{r^2}, \tag{4.43a}$$

$$\Sigma_{K<0}(t) = -\frac{t}{r}. \tag{4.43b}$$

This conformal factor (4.43) is properly defined in the sense that its value does not change replacing $v = 0 \to v = 2\pi$.

Finally, let us state that we can find the global isothermal coordinates for the spacetime associated to the Elliptic pseudosphere, $\mathbb{R}\times\mathcal{M}^2_E$, by a direct calculation, similar to what we did for the Beltrami and the Hyperbolic pseudospheres above, see (4.27) and (4.32). This procedure leads to this parametrization:

$$T = re^{t/r}\cosh\frac{u}{r}, \quad X = re^{t/r}\sinh\frac{u}{r}\cos\frac{cv}{r}, \quad Y = re^{t/r}\sinh\frac{u}{r}\sin\frac{cv}{r} \tag{4.44}$$

and the same conformal factor as (4.43):

$$\Sigma_{\mathrm{E}}(T,X,Y) = -\frac{1}{2}\ln\frac{T^2-X^2-Y^2}{r^2}, \tag{4.45a}$$

$$\Sigma_{\mathrm{E}}(t) = -\frac{t}{r}. \tag{4.45b}$$

Comparing (4.44) and (4.41), we can find the isothermal coordinates for the Elliptic pseudosphere, as shown in (3.30) and below:

$$\tilde{x} = r\frac{\sinh\frac{u}{r}\cos\frac{cv}{r}}{\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}}, \quad \tilde{y} = r\frac{1}{\cosh\frac{u}{r} - \sinh\frac{u}{r}\sin\frac{cv}{r}}. \tag{4.46}$$

4.3 Coordinate transformations for $K > 0$

The problem to find Σ for $K > 0$ and (T, X, Y) as function of (t, u, v), describing the spacetime associated to the sphere, Bulge or Spindle surface, was stated as an open, see e.g. [7]. However, we found that this issue has already been solved by Penrose in the 1960s [13], and again more recently in [37]. To be more precise, this global conformal factor $\Sigma_{K>0}$ was not discussed there in the context of (2+1)-dimensional spacetimes, or even $\mathbb{R}\times\mathcal{M}^2$. The authors were working in higher dimensions and focus on the compactification method or the mathematical cosmology, respectively. But they found the coordinate transformation which leads to this conformal factor.

We start by rewriting the spacetime interval in the following form:

$$ds_S^2 = dt^2 - du^2 - c^2 \sin^2 \frac{u}{r} dv^2 = dt^2 - du^2 - r^2 \sin^2 \frac{u}{r} d\bar{v}^2, \tag{4.47}$$

where $\bar{v} = cv/r$. This looks like the spacetime interval corresponding to the sphere.

Then we apply the following substitution:

$$T = r \tan \frac{t+u}{2r} + r \tan \frac{t-u}{2r}, \quad R = r \tan \frac{t+u}{2r} - r \tan \frac{t-u}{2r}, \tag{4.48}$$

which transformes the spacetime interval as:

$$ds_S^2 = dt^2 - du^2 - r^2 \sin^2 \frac{u}{r} dv^2 = e^{2\Sigma}(dT^2 - dR^2 - R^2 dv^2) \tag{4.49}$$

and the global conformal factor is:

$$e^{\Sigma(t,u)} = \cos \frac{t+u}{2r} \cos \frac{t-u}{2r} = \frac{1}{2}\left(\cos \frac{t}{r} + \cos \frac{u}{r}\right). \tag{4.50}$$

4.4 Coordinate transformations for $K = 0$

Finally, for completness, let us show the explicit coordinate transformation from (t, x, y) to (T, X, Y) [18]. The flat spacetime interval is:

$$ds_{\text{flat}}^2 = e^{2\Sigma_{K=0}}(dT^2 - dX^2 - dY^2) = \frac{c^4}{(T-R)^2(T+R)^2}\left(dT^2 - dR^2 - R^2 d\theta^2\right) \tag{4.51}$$

with $X \equiv R\cos\theta$, $Y \equiv R\sin\theta$. Defining $u \equiv T - R$, $v \equiv T + R$, we obtain:

$$ds_{\text{flat}}^2 = c^2 \frac{du}{u^2} c^2 \frac{dv}{v^2} - \frac{c^4}{u^2 v^2} \frac{(v-u)^2}{4} d\theta^2. \tag{4.52}$$

Applying $z \equiv -c^2/u$, $w \equiv -c^2/v$ we obtain:

$$ds_{\text{flat}}^2 = dz dw - \frac{1}{4}(z-w)^2 d\theta^2. \tag{4.53}$$

From the substitution $t \equiv (z+w)/2$, $r \equiv (z-w)/2$, the flatness of spacetime is obvious:

$$ds_{\text{flat}}^2 = dt^2 - dr^2 - r^2 d\theta^2. \tag{4.54}$$

Finally, we can apply $x \equiv r\cos\theta$ and $y \equiv r\sin\theta$. Then coordinates transformation from (t, x, y) to (T, X, Y) is:

$$T = -c^2 \frac{t}{t^2 - x^2 - y^2}, \quad X = -c^2 \frac{x}{t^2 - x^2 - y^2}, \quad Y = -c^2 \frac{y}{t^2 - x^2 - y^2}. \tag{4.55}$$

5. Canonical transformations

In this chapter, we discuss the profound differences between the systems described by quantum mechanics (QM) and those described by quantum field theory (QFT), mostly referring to the following references: [38], [39] and [40]. The essential difference lies in the *number of degrees of freedom*, which these systems have. For instance, the harmonic oscillator described by one pair of annihilation and creation operators $(a, a^\dagger)$, or equivalently by a pair of canonical coordinates (q, p), has one degree of freedom. On the other hand, an *infinite number of degrees of freedom* is the case of QFT. We can consider a system described by a scalar field ϕ, which is in fact a sum of infinite number of oscillators, hence the system is described by $(a_{\vec{k}}, a^\dagger_{\vec{k}})$, where $\vec{k}$ is the momentum:

$$\phi(x) = \int d\vec{k} N_{\vec{k}} \left(a_{\vec{k}} e^{-ikx} + a^\dagger_{\vec{k}} e^{ikx} \right), \tag{5.1}$$

where N_k is the normalization, $kx \equiv k^0 x^0 - \vec{k} \cdot \vec{x}$ is the scalar product of two spacetime vectors: n-momentum vector $k^\mu = (k^0, \vec{k})$ and n-position vector $x^\mu = (x^0, \vec{x})$, where $k_0 = \sqrt{m^2 + \vec{k}^2}$. We considered the speed of light $c \equiv 1$. Notice that we work in n-dimensional spacetimes, hence $d\vec{k} \equiv d^{n-1}k$.

This difference has far-reaching consequences, which we shall discuss for boson systems, but the generalization for fermions is straightforward, e.g. [41].

In both QM and QFT, the annihilation and creation operators satisfy the canonical commutation relations (CCRs):

$$[a, a] = 0 = [a^\dagger, a^\dagger], \quad [a, a^\dagger] = 1, \tag{5.2a}$$

$$[a_{\vec{k}}, a_{\vec{l}}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{l}}], \quad [a_{\vec{k}}, a^\dagger_{\vec{l}}] = \delta^{(n-1)}(\vec{k} - \vec{l}), \tag{5.2b}$$

where n denotes the dimension of the spacetime and we considered $\hbar \equiv 1$. The defining property of a, as well as a_k, is that it annihilates the vacuum state:

$$a|0\rangle = 0, \tag{5.3a}$$

$$a_{\vec{k}}|0\rangle = 0 \;\; \forall \vec{k}. \tag{5.3b}$$

Let us denote the Fock space, the Hilbert space created from repeated use of the operator $a^\dagger$, as $H[a]$:

$$H[a] = \left\{ \sum_{i=0}^{\infty} c_n |n\rangle; |n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger \right)^n |0\rangle, \sum_{i=0}^{\infty} |c_i|^2 < \infty \right\}. \tag{5.4}$$

The Fock space built from $a^\dagger_k$ $(\forall k)$ will be denoted in the same way $H[a]$.

At this point, the *problem of many vacua* pops up. We can do a transformation of states and operators, through a unitary operator $U(\theta)$, on the Hilbert space[1] $H[a]$:

$$a \to \alpha(\theta) = U(\theta) a U(\theta)^{-1}, \tag{5.5a}$$

$$a_{\vec{k}} \to \alpha_{\vec{k}}(\theta_{\vec{k}}) = U(\theta) a_{\vec{k}} U(\theta)^{-1}, \tag{5.5b}$$

[1] From now on, we shall very often mix the "Fock" and "Hilbert" space, although we shall mean the same object.

which preserves the structure of the CCRs, i.e.:

$$[\alpha(\theta), \alpha^\dagger(\theta)] = 1, \tag{5.6a}$$

$$[\alpha_{\vec{k}}(\theta_{\vec{k}}), \alpha^\dagger_{\vec{l}}(\theta_{\vec{l}})] = \delta^{(n-1)}(\vec{k} - \vec{l}), \tag{5.6b}$$

where θ, as well as $\theta_{\vec{k}}$, is a complex parameter of the transformation. In the case of QFT, $U(\theta)$ depends on $\theta_{\vec{l}}$ for $\forall \vec{l}$. Such a transformation on the Hilbert space, which *preserves the structure of the CCRs*, is called the *canonical transformation* (CT).

One particular example of CT, which we shall study later, is as follows:

$$\alpha(\theta) = a + \theta, \tag{5.7a}$$

$$\alpha_{\vec{k}}(\theta_{\vec{k}}) = a_{\vec{k}} + \theta_{\vec{k}}. \tag{5.7b}$$

It is obvious that $|0\rangle$ *is not the vacuum state for* α*, or* $\alpha_{\vec{k}}(\theta_{\vec{k}})$:

$$\alpha(\theta)|0\rangle = \theta|0\rangle, \tag{5.8a}$$

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0\rangle = \theta_{\vec{k}}|0\rangle. \tag{5.8b}$$

We thus *define a new vacuum state*, denoted same for QM and QFT case, as:

$$\alpha(\theta)|0(\theta)\rangle = 0, \tag{5.9a}$$

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0(\theta)\rangle = 0 \ \ \forall \vec{k}, \tag{5.9b}$$

where the two vacua are connected via the operator $U(\theta)$ as:

$$|0(\theta)\rangle = U(\theta)|0\rangle. \tag{5.10}$$

Repeated application of the operator $\alpha^\dagger(\theta)$, or $\alpha^\dagger_{\vec{k}}(\theta_{\vec{k}})$ on $|0(\theta)\rangle$ creates a new Fock space, which we denote $H[\alpha(\theta)]$ for both QM and QFT. The Fock space with $\theta = 0$ corresponds to $H[\alpha(0)] = H[a]$, i.e. it is the same space.

A CT provides a new representation of the CCRs, which gives a rise to a new vacuum state and in turn to a new Fock space. However, it turns out that there is a *dramatic difference between QM and QFT*.

In QM the *new Fock space* $H[\alpha(\theta)]$ *is equivalent to* $H[a]$*, because the CT is a well-defined, proper unitary transformation, that we are free to perform.* We say that the Fock spaces are *unitarily equivalent.* From this follows that the choice of the representation of the CCRs is purely a matter of convention and after the CT we still work within the same Hilbert (physical) space, just with a different basis. This *unitarly equivalence* between $H[a]$ and $H[\alpha(\theta)]$ is the content of the famous *Stone von Neumann theorem* (see e.g. [42], [43], [44]) and we shall discuss two examples.

On the other hand, the two Fock spaces $H[a]$ and $H[\alpha(\theta)]$ can be *orthogonal to each other in QFT.* If the vacua $|0\rangle$ and $|0(\theta)\rangle$ are such that:

$$\langle 0|0(\theta)\rangle = 0, \tag{5.11}$$

the corresponding Fock spaces, built up from these vacua, are othogonal, too. Then the Fock spaces are said to be *unitarly inequivalent* and the Stone von Neumann theorem no longer applies. The Fock spaces then represent really different physical spaces [39].

In this chapter, we shall not discuss the problem of inequivalent representations of the CCRs in general, but rather discuss two simple examples: the *boson translation* (or the *Bogoliubov translation for coherent states*, which we have already met, see (5.7a)) and the *Bogoliubov transformation for two modes squeezed states.* There we shall become familiar with the concept of the unitarily inequivalence of the Hilbert spaces and the relevant computational techniques, useful for the following.

In what follows of this chapter, we discuss CTs for a classical system of finite degrees of freedom: $(q_i, p_i)_{i=1}^n$. Then we move to QM and QFT.

Althought what we discuss in this chapter is well known, it is useful to recollect those facts here, as this will serve as basis for our when we shall discuss the quantum Weyl transformation, that is, an example of a canonical transformation.

In the following we shall not use special symbols for operators (e.g., $\hat{a}$, etc.), as it will be clear from the context when a quantity is an operator, and when a c-number function.

5.1 CT in classical mechanics

In this section, we briefly recall the concept of CT for a classical system described by n independend coordinates $(q_1, \ldots, q_n)$ and n conjugated momenta $(p_1, \ldots, p_n)$. The key role will be played by the Poisson brackets, invariant under CTs.

Let us begin with Hamilton's equations, describing the dynamics of the classical system:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{5.12}$$

where $i = 1, \ldots, n$ and the dot above q_i, p_i denotes the time derivative. They can be written in more compact way by introducing the new coordinates $(\eta_i)_{i=1}^{2n}$ such that:

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \equiv (\eta_1, \ldots, \eta_n, \eta_{n+1}, \ldots, \eta_{2n}). \tag{5.13}$$

Then Hamilton's equations become:

$$\dot{\eta}_i = J_{ij} \frac{\partial H}{\partial \eta_j}, \tag{5.14}$$

where J_{ij} are the components of the *symplectic matrix* J:

$$J = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n}, \end{pmatrix} \tag{5.15}$$

where $1_{n\times n}$ is the identity matrix with n rows and n columns, $0_{n\times n}$ is the $n \times n$ zero matrix.

Consider a linear transformation from $(\eta_i)_{i=1}^{2n}$ to $(\xi_i)_{i=1}^{2n}$, with the matrix elements $M_{ij} = \partial \xi_i / \partial \eta_j$. Then the form of Hamilton's equations is invariant under a transformation with M satisfying:

$$J = MJM^T. \tag{5.16}$$

Such a transformation is called a *canonical transformation.* Such transformations form a group called the *symplectic group.* For two functions $f(\eta_i)$, $g(\eta_i)$, defined on the phase space, the following expression is invariant under the CT:

$$J_{ij}\frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \eta_j} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \equiv \{f, g\}_{(q,p)}, \tag{5.17}$$

where the Einstein sum convention is applied. The last expression is called the Poisson bracket. Since it is independent from the choice of canonical variables, we can suppress (q, p) and simply denote it by $\{f, g\}$.

It is easy to check that:

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{5.18}$$

The quantization formal procedure consists in:

$$q \to \hat{q}, \quad p \to \hat{p}, \quad \{.,.\} \to i[.,.], \tag{5.19}$$

where $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$, $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively. As announced, we shall omit all hats in what follows, because we shall mainly deal with operators, so there will be no confusion. Notice that we also set $\hbar = 1$.

So the CCRs are:

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\delta_{ij}, \tag{5.20}$$

In straighforward analogy to classical mechanics, the CT for QM is such a transformation on the Hilbert space which does leave the CCRs unchanged. Same statement holds for QFT.

5.2 CT in QM and QFT

We shall now study CTs for two particular examples. In the first example, we shall start with the computations within QM and then we recompute the same problem within QFT. There we shall see the difference between QM and QFT. In the second example, we shall start with the calculations already within QFT.

5.2.1 Boson translation

The system under consideration is described by one pair of $(a, a^\dagger)$:

$$[a, a^\dagger] = 1, \quad [a, a] = 0, \quad [a^\dagger, a^\dagger] = 0, \quad a|0\rangle = 0. \tag{5.21}$$

The Hilbert space $H[a]$ is built by repeated actions of $a^\dagger$ on the vacuum state $|0\rangle$:

$$H[a] = \left\{ \sum_{i=0}^{\infty} c_n|n\rangle; |n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle, \sum_{i=0}^{\infty} |c_i|^2 < \infty \right\}. \tag{5.22}$$

Consider now:

$$a \to \alpha(\theta) = a + \theta, \tag{5.23}$$

where θ is a complex parameter. This is an example of CT:

$$[\alpha(\theta), \alpha^\dagger(\theta)] = 1, \quad [\alpha(\theta), \alpha(\theta)] = 0, \quad [\alpha^\dagger(\theta), \alpha^\dagger(\theta)] = 0. \tag{5.24}$$

This representation of the CCRs has a different vacuum, denoted $|0(\theta)\rangle$:

$$\alpha(\theta)|0(\theta)\rangle = 0, \quad \alpha(\theta)|0\rangle = \theta|0\rangle, \tag{5.25}$$

from which follows:

$$a|0(\theta)\rangle = -\theta|0(\theta)\rangle. \tag{5.26}$$

The Hilbert space generated by repeated action of $\alpha^\dagger(\theta)$ on $|0(\theta)\rangle$ is denoted $H[\alpha(\theta)]$.

We shall assume that the transformation is mediated by the operator U such that:

$$U(\theta) = e^{iG(\theta)}, \tag{5.27a}$$

$$\alpha(\theta) = U(\theta)\alpha U(\theta)^{-1}, \tag{5.27b}$$

where $G(\theta) = G(\theta)^\dagger$ is a hermition operator, which we want to find.

(5.27b) can be rewritten, applying the Baker–Campbell–Hausdorff formula [45]:

$$e^X e^Y = e^Z, \quad Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots, \tag{5.28}$$

as it follows:

$$\begin{aligned} e^{iG(\theta)} a e^{-iG(\theta)} &= a + [iG(\theta), a] + \frac{1}{2!}[iG(\theta), [iG(\theta), a]] + \ldots, \\ &= a + \theta, \end{aligned} \tag{5.29}$$

where the second row follows from (5.23). It is easy to find out:

$$[G(\theta), a] = -i\theta, \tag{5.30}$$

from which follows:

$$G(\theta) = -i\left(\theta^* a - \theta a^\dagger\right). \tag{5.31}$$

The transformed vacuum then looks like this:

$$|0(\theta)\rangle = e^{-\theta a^\dagger + \theta^* a}|0\rangle = e^{-\frac{1}{2}|\theta|^2} e^{-\theta a^\dagger}|0\rangle, \tag{5.32}$$

where we applied, coming from (5.28):

$$e^{-\theta a^\dagger + \theta^* a} = e^{-\frac{1}{2}|\theta|^2} e^{-\theta a^\dagger} e^{\theta^* a}. \tag{5.33}$$

The new ground state $|0(\theta)\rangle$ is a superposition of states with many a-particles, since the scalar product $\langle n|0(\theta)\rangle$ is non-zero for $\forall n$:

$$|0(\theta)\rangle = \sum_n |n\rangle\langle n|0(\theta)\rangle, \tag{5.34}$$

where $\{|n\rangle\}_{n=0}^\infty$ is the base of $H[a]$ and the identity operator, action on $H[a]$, can be written as:

$$\mathbb{1} = \sum_{n=0}^\infty |n\rangle\langle n|. \tag{5.35}$$

This can be done not only for the vacuum state $|0(\theta)\rangle$, but for all states of the Fock space $H[\alpha(\theta)]$: *any vector in* $H[\alpha(\theta)]$ *is a superposition of vectors in* $H[a]$ *and vice versa.* In this sense, *these two Hilbert spaces are the same* ([38]).

Let us denote the number operator of a-particles by:

$$N(a) \equiv a^\dagger a. \tag{5.36}$$

Then it is straighforward to compute the number of a-particles in $|0(\theta)\rangle$:

$$\begin{aligned}\langle 0(\theta)|N(a)|0(\theta)\rangle &= e^{-|\theta|^2}\langle 0|e^{-\theta^* a}N(a)e^{-\theta a^\dagger}|0\rangle \\ &= -\theta e^{-|\theta|^2}\langle 0|e^{-\theta^* a}a^\dagger e^{-\theta a^\dagger}|0\rangle \\ &= |\theta|^2 e^{-|\theta|^2}\langle 0|e^{-\theta^* a}e^{-\theta a^\dagger}|0\rangle \\ &= |\theta|^2,\end{aligned} \tag{5.37}$$

where we used:

$$N(a)e^{-\theta a^\dagger} = e^{-\theta a^\dagger}N(a) - \theta a^\dagger e^{-\theta a^\dagger}, \tag{5.38a}$$

$$e^{-\theta^* a}a^\dagger = a^\dagger e^{-\theta^* a} - \theta^* e^{-\theta^* a}, \tag{5.38b}$$

following from (5.21) and (5.36).

Now, we repeat the calculation for a quantum system with infinite number of degrees of freedom. Now the CCRs include these:

$$[a_{\vec{k}}, a^\dagger_{\vec{q}}] = \delta^{(n-1)}(\vec{k}-\vec{q}), \quad [a_{\vec{k}}, a_{\vec{q}}] = 0, \quad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{q}}] = 0, \tag{5.39}$$

and the vacuum state is $|0\rangle$:

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0\rangle = 0 \quad \forall \vec{k}. \tag{5.40}$$

The transformation for the mode $\vec{k}$ is:

$$a_{\vec{k}} \to \alpha_{\vec{k}}(\theta_{\vec{k}}) = a_{\vec{k}} + \theta_{\vec{k}}, \tag{5.41}$$

which is the CT:

$$[\alpha_{\vec{k}}(\theta_{\vec{k}}), \alpha^\dagger_{\vec{q}}(\theta_{\vec{q}})] = \delta^{(n-1)}(\vec{k}-\vec{q}), \quad [\alpha_{\vec{k}}(\theta_{\vec{k}}), \alpha_{\vec{q}}(\theta_{\vec{q}})] = 0, \quad [\alpha^\dagger_{\vec{k}}(\theta_{\vec{k}}), \alpha^\dagger_{\vec{q}}(\theta_{\vec{q}})] = 0 \tag{5.42}$$

and for all modes $\vec{k}$ we define the new vacuum state:

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0(\theta)\rangle = 0. \tag{5.43}$$

Following the same procedure as before we find this U:

$$U(\theta) = e^{iG(\theta)}, \quad G(\theta) = -i\int d\vec{k}\left(\theta^*_{\vec{k}}a_{\vec{k}} - \theta_{\vec{k}}a^\dagger_{\vec{k}}\right), \tag{5.44}$$

and the following vacuum:

$$|0(\theta)\rangle = e^{-\frac{1}{2}\int d\vec{k}|\theta_{\vec{k}}|^2}e^{-\int d\vec{k}\theta_{\vec{k}}a^\dagger_{\vec{k}}}|0\rangle. \tag{5.45}$$

If we denote the number of particle with a given momentum $\vec{k}$:

$$N_{\vec{k}}(a) = a^{\dagger}_{\vec{k}} a_{\vec{k}}, \tag{5.46}$$

then the vacuum state $|0(\theta)\rangle$ has this number of a-particles:

$$\langle 0(\theta)|N_{\vec{k}}(a)|0(\theta)\rangle = |\theta_{\vec{k}}|^2, \tag{5.47}$$

so the total number of particles is:

$$\mathcal{N}(a) = \int d\vec{k}|\theta_{\vec{k}}|^2. \tag{5.48}$$

The integral in (5.45) and (5.48) depends on the function $\theta_{\vec{k}}$. We restrict our attention to the case in which the vacuum is invariant under the spatial translation, i.e. the function $\theta_{\vec{k}}$ is the Fourier transform of a constant function. Then it holds:

$$\theta_{\vec{k}} = \theta\delta^{(n-1)}(\vec{k}), \tag{5.49}$$

from which follows:

$$\int d\vec{k}|\theta_{\vec{k}}|^2 = \delta^{(n-1)}(0) = +\infty, \tag{5.50}$$

since we assume an infinite volume (otherwise, the vacuum would not be invariant under translation symmetry/homogeneous condensation). Because of that we obtain the following:

$$\mathcal{N}(a) = +\infty, \tag{5.51a}$$

$$|0(\theta)\rangle = e^{-\infty|\theta|^2} e^{-\int d\vec{k}\theta_{\vec{k}} a^{\dagger}_{\vec{k}}}|0\rangle. \tag{5.51b}$$

Of course, the numerical factor in (5.51b) is zero. We shall comment that in what follows, see (5.55).

Let us focus now on (5.51a). The infinite number of particles is actually no surprise, since the vacuum is homogeneous condensation of a-particles and the volume is infinite. Consider for a moment that the volume is finite, and only later we make a limit: $V \to +\infty$. Then we can rewrite $\delta(0)^{(n-1)}$ as:

$$\delta^{(n-1)}(0) = (2\pi)^{-(n-1)} \int d\vec{x} e^{i\vec{k}\cdot\vec{x}}|_{\vec{k}=\vec{0}} = (2\pi)^{-(n-1)}V, \tag{5.52}$$

and define the density of particles as:

$$n(a) = \frac{\mathcal{N}(a)}{V} = (2\pi)^{-(n-1)}|\theta|^2, \tag{5.53}$$

which is finite and well defined for $V \to +\infty$.

Finally, let us discuss (5.51b). Because the numerical factor vanishes: $e^{-\delta^{(n-1)}(0)|\theta|^2} \to 0$, the projection of $|0(\theta)\rangle$ on the basis vectors:

$$|n_k\rangle = \frac{1}{\sqrt{n_k!}} \left(a^{\dagger}_k\right)^n |0\rangle, \tag{5.54}$$

of the Fock space $H[\alpha(\theta)]$ is inevitably zero:

$$\langle n_k|0(\theta)\rangle = 0 \;\; \forall\vec{k}. \tag{5.55}$$

The transformed vacuum $|0(\theta)\rangle$ is orthogonal to the Fock space $H[a]$, so cannot be written as a superposition of vectors from $H[a]$, as it was the case of QM (see (5.34)). This shows us a crucial feature of QFT: *the Hilbert spaces $H[a]$, $H[\alpha(\theta)]$ are not equivalent, because no vector in one space can be written as a superposition of vectors from another space. The spaces are orthogonal as can be seen by computing:*

$$\langle 0|0(\theta)\rangle = e^{-\frac{1}{2}\int d\vec{k}|\theta_{\vec{k}}|^2} = e^{-\delta^{(n-1)}(0)} = 0. \tag{5.56}$$

Let us now show how the CT (5.41) looks at the level of fields. Considering the scalar field:

$$\phi(x) = \int d\vec{k} N_{\vec{k}} \left(a_{\vec{k}} e^{-ikx} + a^\dagger_{\vec{k}} e^{ikx} \right), \tag{5.57}$$

the transformed field is:

$$\phi_\theta(x) = \int d\vec{k} N_{\vec{k}} \left(\alpha(\theta_{\vec{k}}) e^{-ikx} + \alpha^\dagger(\theta_{\vec{k}}) e^{ikx} \right), \tag{5.58}$$

which can be rewritten as:

$$\phi_\theta(x) = \phi(x) + f(x), \tag{5.59}$$

where $f(x)$ is a real function:

$$f(x) \equiv \int d\vec{k} N_{\vec{k}} \left(\theta_{\vec{k}} e^{-ikx} + \theta^*_{\vec{k}} e^{ikx} \right). \tag{5.60}$$

Moreover, the CT (5.59) can be generalized. Consider the following transformation:

$$\phi_\theta(x) = \phi(x) + g(x), \tag{5.61}$$

where $g(x)$ is a function, which does not need to have its Fourier counterpart.

Finally, notice that it may look like the inequivalence of the Hilbert spaces is caused by the infinite volume. This is actually not the right argument. Of course, if we assume that the space is finite, then $\delta^{(n-1)}(0)$ is finite, see (5.52), but that would break the translation symmetry which we required. The inequivalence of Hilbert spaces has its roots in the infinite number of degrees of freedom. [38].

For the discussion of the inequivalence of Hilbert spaces in condesed matter (so finite) systems, or the role of defects (singularities) in the phase space, we recommend the discussion in [38]. We shall open the issue regarding the defects in the next chapter, where we shall discuss the quantum Weyl transformation. Also, an interesting list of references can be found in [40] and in the recent paper [16].

5.2.2 Bogoliubov transformation

We would like to introduce one more example of CT, where we become familiar with a useful technique of computation of $\langle 0|0(\theta)\rangle$, using the functional derivative.

We consider a system described by two sets of ladder operators:

$$[a_{\vec{k}}, a^\dagger_{\vec{l}}] = \delta^{(n-1)}(\vec{k} - \vec{l}) = [b_{\vec{k}}, b^\dagger_{\vec{l}}], \tag{5.62}$$

and the vacuum $|0\rangle$:

$$a_{\vec{k}}|0\rangle = b_{\vec{k}}|0\rangle = 0 \ \ \forall \vec{k}. \tag{5.63}$$

Let us recall that $k^\mu = (k^0, \vec{k})$, $E = \sqrt{m^2 + \vec{k}^2}$, and similarly for $\vec{l}$. The associated Hilbert space will be denoted as $H[a, b]$.

Now, we consider the following transformation:

$$\alpha_{\vec{k}}(\theta_{\vec{k}}) = a_{\vec{k}} \cosh \theta_{\vec{k}} - b^\dagger_{\vec{k}} \sinh \theta_{\vec{k}}, \tag{5.64a}$$

$$\beta_{\vec{k}}(\theta_{\vec{k}}) = b_{\vec{k}} \cosh \theta_{\vec{k}} - a^\dagger_{\vec{k}} \sinh \theta_{\vec{k}}, \tag{5.64b}$$

where $\theta_{\vec{k}}$ is a real parameter. It is simple to show that this transformation is canonical:

$$[\alpha_{\vec{k}}(\theta_{\vec{k}}), \alpha^\dagger_{\vec{l}}(\theta_{\vec{l}})] = \delta^{(n-1)}(\vec{k} - \vec{l}) = [\beta_{\vec{k}}(\theta_{\vec{k}}), \beta^\dagger_{\vec{l}}(\theta_{\vec{l}})]. \tag{5.65}$$

Neither $\alpha_{\vec{k}}(\theta_{\vec{k}})$ or $\beta_{\vec{k}}(\theta_{\vec{k}})$ is an annihilation operator for $|0\rangle$:

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0\rangle = -\sinh \theta_{\vec{k}} b^\dagger_{\vec{k}}|0\rangle, \quad \beta_{\vec{k}}(\theta_{\vec{k}})|0\rangle = -\sinh \theta_{\vec{k}} a^\dagger_{\vec{k}}|0\rangle. \tag{5.66}$$

Therefore, we define a new ground state:

$$\alpha_{\vec{k}}(\theta_{\vec{k}})|0(\theta)\rangle = \beta_{\vec{k}}(\theta_{\vec{k}})|0(\theta)\rangle = 0 \;\; \forall \vec{k}, \tag{5.67}$$

with the corresponding new Hilbert space denoted as $H[\alpha(\theta), \beta(\theta)]$.

We wish to find the unitary operator:

$$U(\theta) = e^{iG(\theta)}, \tag{5.68}$$

so that:

$$\alpha_{\vec{k}}(\theta_{\vec{k}}) = U(\theta) a_{\vec{k}} U(\theta)^{-1}, \tag{5.69a}$$

$$\beta_{\vec{k}}(\theta_{\vec{k}}) = U(\theta) b_{\vec{k}} U(\theta)^{-1} \tag{5.69b}$$

$$\begin{aligned} \alpha_{\vec{k}}(\theta_{\vec{k}}) = e^{iG(\theta)} a_{\vec{k}} e^{-iG(\theta)} &= a_{\vec{k}} + \left[iG(\theta), a_{\vec{k}}\right] + \frac{1}{2!}\left[iG(\theta), \left[iG(\theta), a_{\vec{k}}\right]\right] + \dots \\ &= a_{\vec{k}} - \theta_{\vec{k}} b^\dagger_{\vec{k}} + \frac{1}{2!}\theta^2_{\vec{k}} a_{\vec{k}} - \dots, \end{aligned} \tag{5.70}$$

$$\begin{aligned} \beta_{\vec{k}}(\theta_{\vec{k}}) = e^{iG(\theta)} b_{\vec{k}} e^{-iG(\theta)} &= b_{\vec{k}} + \left[iG(\theta), b_{\vec{k}}\right] + \frac{1}{2!}\left[iG(\theta), \left[iG(\theta), b_{\vec{k}}\right]\right] + \dots \\ &= b_{\vec{k}} - \theta_{\vec{k}} a^\dagger_{\vec{k}} + \frac{1}{2!}\theta^2_{\vec{k}} b_{\vec{k}} - \dots. \end{aligned} \tag{5.71}$$

From the following observation:

$$\left[iG(\theta), a_{\vec{k}}\right] = -\theta_{\vec{k}} b^\dagger_{\vec{k}}, \quad \left[iG(\theta), b_{\vec{k}}\right] = -\theta_{\vec{k}} a^\dagger_{\vec{k}}, \tag{5.72}$$

we simply obtain the generator $G \equiv G(\theta)$:

$$G(\theta) = -i \int d\vec{l} \theta_{\vec{l}} \left(a^\dagger_{\vec{l}} b^\dagger_{\vec{l}} - b_{\vec{l}} a_{\vec{l}}\right). \tag{5.73}$$

The new ground state is then:

$$|0(\theta)\rangle = U(\theta)|0\rangle = e^{iG(\theta)}|0\rangle. \tag{5.74}$$

The computation of (5.74) is more demanding and tricky than (5.45). We start by computing the scalar product $\langle 0|0(\theta)\rangle$:

$$f(\theta) \equiv \langle 0|0(\theta)\rangle = \langle 0|e^{\int d\vec{l}\theta_l\left(a_l^\dagger b_l^\dagger - b_l a_l\right)}|0\rangle. \tag{5.75}$$

Then we apply the functional derivative in the direction $\epsilon\delta^{(n-1)}(\vec{k}-\vec{l})$:

$$\begin{aligned}\frac{\delta}{\delta\theta}f(\theta) \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(f(\theta+\epsilon\delta)-f(\theta)\right) &= \langle 0|U\left(a_k^\dagger b_k^\dagger - b_k a_k\right)|0\rangle = \langle 0|U a_k^\dagger b_k^\dagger|0\rangle \\ &= \langle 0|\left(a_k^\dagger b_k^\dagger - b_k a_k\right)U|0\rangle = -\langle 0|b_k a_k U|0\rangle,\end{aligned} \tag{5.76}$$

where we used a short-hand notation $\epsilon\delta \equiv \epsilon\delta^{(n-1)}(\vec{k}-\vec{l})$. Because of the following identity:

$$U a_k^\dagger b_k^\dagger = \left(a_k^\dagger \cosh\theta_k - b_k \sinh\theta_k\right)\left(b_k^\dagger \cosh\theta_k - a_k \sinh\theta_k\right)U, \tag{5.77}$$

which can be derived simply, when we write:

$$U a_k^\dagger b_k^\dagger = U a_k^\dagger U^{-1} U b_k^\dagger U^{-1} U \tag{5.78}$$

and apply (5.69) and (5.64), we can find from (5.76) the following equation:

$$\frac{\delta}{\delta\theta}f(\theta) = -\delta^{(n-1)}(0)\tanh\theta_k f(\theta), \tag{5.79}$$

whose solution is:

$$f(\theta) = \exp\left(-\delta^{(n-1)}(0)\int d\vec{l}\ln\cosh\theta_l\right). \tag{5.80}$$

At this point, the inequivalence of the Hilbert spaces $H[a,b]$ and $H[\alpha(\theta),\beta(\theta)]$ is showing up: $f(\theta)=0$, since $\delta^{(n-1)}(0)=+\infty$. This is the result we have been looking for.

It remains to find the vacuum $|(\theta)\rangle$. This particular thing will not be of crucial importance for our next discussion, so we shall not show the quite tricky step-by-step procedure here. Instead of that, we recommend e.g. [39] and show the result:

$$|0(\theta)\rangle = \exp\left(-\delta^{(n-1)}(0)\int d\vec{l}\ln\cosh\theta_l\right)\exp\left(\int d\vec{k}\tanh\theta_k a_k^\dagger b_k^\dagger\right)|0\rangle. \tag{5.81}$$

6. Quantum Weyl transformation

In this chapter, we focus our attention on the Weyl transformation (WT) of massless Dirac field (discussed in chapter 1) from a quantum perspective. We shall refer to this as *the quantum Weyl transformation (QWT).* It will turn out that *the QWT is an example of CT*, because the structure of the canonical *anti*commutation relations (CARs) is preserved:

$$\{\psi_\alpha(Q), \pi_\beta(P)\}_{E.T.} = i\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.1a}$$

$$\{\psi_{\Sigma\alpha}(Q), \pi_{\Sigma\beta}(P)\}_{E.T.} = i\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.1b}$$

and the other CARs are zero. We defined the anticommutator $\{A, B\} \equiv AB+BA$ for some operators A, B and $E.T.$ means "equal time".

Let us add that we shall denote the Hilbert space, where ψ, π act, as H. The Hilbert space, where the transformed fields ψ_Σ, π_Σ act, will be denoted as H_Σ.

We want to find the operator W, realizing the QWT:

$$\psi_{\Sigma\alpha} = W\psi_\alpha W^{-1} = e^{-\frac{n-1}{2}\Sigma}\psi_\alpha, \tag{6.2}$$

and search for relation of H with H_Σ. We shall focus on the computation of $\langle 0|W|0\rangle$, as we did in chapter 5.

We shall see that the operator W depends on the function Σ, so the properties of the space-time background are imprinted there. Then it is natural to ask if the space(time) singularities, including the singular behaviour of Σ, can be a source of singularities in W. If so, they might cause the inequivalence of the Hilbert spaces associated to these spacetimes (giving a *new type of Weyl anomaly*).

We open this chapter by proving that QWT is an example of CT. Then we find the operator W, discuss its nature and focus on computation of the scalar product $\langle 0|W|0\rangle$.

6.1 Canonicity of the QWT

Let us write the diffeomorphic invariant action for the Weyl transformed Dirac field (see (1.6), (1.24)):

$$A[g_{\mu\nu}, \psi_\Sigma, \nabla_\mu\psi_\Sigma] \equiv i\int d^n x\sqrt{-g}\bar{\psi}_\Sigma(x)\gamma^\mu\nabla_\mu\psi_\Sigma(x), \tag{6.3}$$

whose Lagrangian is:

$$\mathcal{L}_\Sigma \equiv i\bar{\psi}_\Sigma\gamma^\mu\nabla_\mu\psi_\Sigma. \tag{6.4}$$

The conjugated momentum of the field π_Σ is:

$$\pi_{\Sigma\alpha} \equiv \frac{\delta A}{\delta\psi_{\Sigma\alpha}} = i\sqrt{-g}\bar{\psi}_{\Sigma\beta}\gamma^{\underline{0}}_{\beta\alpha}, \tag{6.5}$$

where α, β are spin indices and the underlined index denotes the Einstein index:

$$\gamma^{\underline{0}} \equiv \gamma^a e^{\underline{0}}_a = \gamma^a e^{-\Sigma}\delta^{\underline{0}}_a = e^{-\Sigma}\gamma^0. \tag{6.6}$$

We write the momentum (6.5) as:

$$\pi_{\Sigma\alpha} = ie^{(n-1)\Sigma}\psi^{\dagger}_{\Sigma\alpha} = ie^{\frac{n-1}{2}\Sigma}\psi^{\dagger}_{\alpha} = e^{\frac{n-1}{2}\Sigma}\pi_{\alpha}. \tag{6.7}$$

After the quantization, the CARs for the non-transformed field and conjugated momentum are:

$$\{\psi_\alpha(Q), \psi_\beta(P)\}_{E.T.} = 0, \tag{6.8a}$$
$$\{\pi_\alpha(Q), \pi_\beta(P)\}_{E.T.} = 0, \tag{6.8b}$$
$$\{\psi_\alpha(Q), \pi_\beta(P)\}_{E.T.} = i\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.8c}$$

where $Q^\mu \equiv (Q^0, \vec{Q})$, $P^\mu \equiv (P^0, \vec{P})$ are flat spacetime coordinates, $E.T.$ means "equal time", so $Q^0 = P^0$. Let us recall that the WT is not a coordinate transformation, so both flat and curved spacetimes are described by same coordinates. It means that $Q^\mu \equiv (Q^0, \vec{Q})$ is a coordinate point, which denotes a point (event) on flat as well as curved spacetime.

The Weyl transformed field and conjugated momentum are:

$$\psi_{\Sigma\alpha} = e^{-\frac{n-1}{2}\Sigma}\psi_\alpha, \tag{6.9a}$$
$$\pi_{\Sigma\alpha} = e^{\frac{n-1}{2}\Sigma}\pi_\alpha, \tag{6.9b}$$

which preserves the structure of the CARs:

$$\{\psi_{\Sigma\alpha}(Q), \psi_{\Sigma\beta}(P)\}_{E.T.} = 0, \tag{6.10a}$$
$$\{\pi_{\Sigma\alpha}(Q), \pi_{\Sigma\beta}(P)\}_{E.T.} = 0, \tag{6.10b}$$
$$\{\psi_{\Sigma\alpha}(Q), \pi_{\Sigma\beta}(P)\}_{E.T.} = i\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.10c}$$

where in (6.10c) we simplified the right hand side as:

$$e^{-\frac{n-1}{2}(\Sigma(Q)-\Sigma(P))}\delta^{(n-1)}(\vec{Q} - \vec{P}) \equiv \delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.11}$$

where we used the properties of δ-function. It means that both sides of (6.11) are zero when $\vec{Q} \neq \vec{P}$ and are the same when $\vec{Q} = \vec{P}$. This confirm that the QWT is CT.

6.2 The operator W

Now we wish to find an operator W such that:

$$\psi_{\Sigma\alpha} = W\psi_\alpha W^{-1}. \tag{6.12}$$

If we assume:

$$W \equiv e^B, \tag{6.13}$$

then the previous equation (6.12) can be rewritten as:

$$e^B\psi_\alpha e^{-B} = e^{-\frac{n-1}{2}\Sigma}\psi_\alpha. \tag{6.14}$$

The left hand side of (6.14) can be expanded as:

$$\psi_\alpha + [B, \psi_\alpha] + \frac{1}{2!}[B, [B, \psi_\alpha]] + \dots, \tag{6.15}$$

while the Taylor series for the right hand side of (6.14) is:

$$\psi_\alpha - \frac{n-1}{2}\Sigma\psi_\alpha + \frac{1}{2!}\left(\frac{n-1}{2}\right)^2 \Sigma^2\psi_\alpha + \dots . \tag{6.16}$$

Comparing both sides, we obtain this simple relation:

$$[B, \psi_\alpha] = -\frac{n-1}{2}\Sigma\psi_\alpha. \tag{6.17}$$

Before we embark to find B, let us explain why it is difficult to define W as a unitary operator.

If W is unitary, then $W^\dagger = W^{-1}$ and

$$\psi_{\Sigma\alpha} = W\psi_\alpha W^{-1}, \tag{6.18a}$$
$$\psi^\dagger_{\Sigma\alpha} = W\psi^\dagger_\alpha W^{-1}. \tag{6.18b}$$

From the following CARs:

$$\{\psi_\alpha(Q), \psi^\dagger_\beta(P)\}_{E.T.} = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.19}$$

and applying (6.18), we obtain:

$$\{\psi_{\Sigma\alpha}(P), \psi^\dagger_{\Sigma\beta}(Q)\}_{E.T.} = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}), \tag{6.20}$$

and similarly for the other CARs.

However, Σ is a real function, so it holds:

$$\psi_{\Sigma\alpha} = e^{-\frac{n-1}{2}\Sigma}\psi_\alpha, \tag{6.21a}$$
$$\psi^\dagger_{\Sigma\alpha} = e^{-\frac{n-1}{2}\Sigma}\psi^\dagger_\alpha. \tag{6.21b}$$

Combining (6.19) and (6.21), we obtain:

$$\begin{aligned}\{\psi_{\Sigma\alpha}(Q), \psi^\dagger_{\Sigma\beta}(P)\}_{E.T.} &= e^{-\frac{n-1}{2}(\Sigma(Q)+\Sigma(P))}\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}) \\ &\equiv e^{-(n-1)\Sigma(P)}\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}),\end{aligned} \tag{6.22}$$

where we used the properties of δ-function in the second row of (6.22). Comparing (6.22) and (6.19), we find that they are in conflict.

In QFT operators which implement a symmetry are either unitary or anti-unitary [5]. Nonetheless, given the real nature of the Weyl transformation (as opposed to the complex nature of a standard gauge transformation $\psi' = e^{i\alpha}\psi$), the simplest solution to the conflict between (6.22) and (6.19) is to relax the request for unitarity in favor of hermiticity. On the one hand, this will make less easy to put such a transformation in direct contact with what customarily considered a *quantum symmetry*. On the other hand, though, it will be a legitimate procedure that will generate the *Weyl transformation* at a quantum level. Furthermore, this will make possible to link singularities of the conformal factor to singular/irregular behaviors of the transformation, which is a crucial point of our work.

For W hermitian, it follows that:

$$\psi^\dagger_{\Sigma\alpha} = W^{-1}\psi^\dagger_\alpha W. \tag{6.23}$$

With this in hands, let us start again with (6.19) and apply W from the left and W^{-1} from the right hand side. Then we can write:

$$\begin{aligned}&\left[W\psi_\alpha(Q)W^{-1}\right]W^2\left[W^{-1}\psi^\dagger_\beta(P)W\right]\left(W^{-1}\right)^2\\&+W^2\left[W^{-1}\psi^\dagger_\beta(P)W\right]\left(W^{-1}\right)^2\left[W\psi_\alpha(Q)W^{-1}\right]=\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q}-\vec{P}).\end{aligned} \tag{6.24}$$

This can be rewritten in a more compact form:

$$\{\psi_{\Sigma\alpha}(Q), W^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2\}_{E.T.} = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q}-\vec{P}). \tag{6.25}$$

We compare (6.25) with (6.22) and find that it should hold:

$$W^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2 = e^{(n-1)\Sigma(P)}\psi^\dagger_{\Sigma\beta}(P). \tag{6.26}$$

This can be easily proved:

$$\begin{aligned}W^2\psi^\dagger_{\Sigma\beta}\left(W^{-1}\right)^2 &= W\psi^\dagger_\beta W^{-1} = e^B\psi^\dagger_\beta e^{-B} = \psi^\dagger_\beta + [B,\psi^\dagger_\beta] + \dots\\&= \psi^\dagger_\beta + \frac{n-1}{2}\Sigma\psi^\dagger_\beta + \dots = e^{\frac{n-1}{2}\Sigma}\psi^\dagger_\beta = e^{(n-1)\Sigma}\psi^\dagger_{\Sigma\beta},\end{aligned} \tag{6.27}$$

where we applied (see also (6.17)):

$$[B,\psi^\dagger_\alpha] = \frac{n-1}{2}\Sigma\psi^\dagger_\alpha. \tag{6.28}$$

Now we multiply (6.26) by the imaginary unit i:

$$\{\psi_{\Sigma\alpha}(Q), iW^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2\}_{E.T.} = i\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q}-\vec{P}), \tag{6.29}$$

and compare it with (6.10c). This leads to the following question:

$$\pi_{\Sigma\beta}(P) \stackrel{?}{=} iW^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2. \tag{6.30}$$

In (6.27) we found:

$$W^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2 = e^{\frac{n-1}{2}\Sigma(P)}\psi^\dagger_\beta(P), \tag{6.31}$$

so our question (6.30) can be formulated as:

$$\pi_{\Sigma\beta}(P) \stackrel{?}{=} iW^2\psi^\dagger_{\Sigma\beta}(P)\left(W^{-1}\right)^2 = e^{\frac{n-1}{2}\Sigma(P)}i\psi^\dagger_\beta(P) = e^{\frac{n-1}{2}\Sigma(P)}\pi_\beta(P), \tag{6.32}$$

which coincides with (6.9b). It seems that W as a hermition operator works well.

Let us add that it becomes clear from our previous computations that the following two relations must hold simultaneously:

$$e^{-B(\Sigma)}\psi^\dagger_\alpha e^{B(\Sigma)} = e^{-\frac{n-1}{2}\Sigma}\psi^\dagger_\alpha, \tag{6.33a}$$

$$e^{B(\Sigma)}\psi^\dagger_\alpha e^{-B(\Sigma)} = e^{\frac{n-1}{2}\Sigma}\psi^\dagger_\alpha, \tag{6.33b}$$

where (6.33a) follows from (6.23) and (6.21b), (6.33b) follows from (6.27). Therefore, it must apply:

$$B(-\Sigma) = -B(\Sigma). \tag{6.34}$$

Let us remind now the conclusions of the previous chapter. We could see for the BT that the operator U is unitary. Its singular behaviour, rooted in an infinite number of degrees of freedom, is the reason for orthogonality of Hilbert spaces. We said about the CCRs that they are *unitarily inequivalent*.

However, the operator W that implements the quantum WT is quite peculiar. In the first place, this operator seems to be hermitian, rather than unitary. Moreover, once we show that the operator W is given by the function Σ, the question arises whether such operator can be singular due to the space-time singularities (including the singularities of the function Σ). If so, the inequivalence of Hilbert spaces might have roots in singularities of classical nature.

Let us now look for an explicit expression of the operator W. Let us recall the equation which the operator B must satisfy (6.17):

$$[B, \psi_\alpha(Q)] = -\frac{n-1}{2}\Sigma(Q)\psi_\alpha(Q). \tag{6.35}$$

We also remind a useful commutation relation which applies for $T \equiv Q_0 = P_0$:

$$[\psi^\dagger_\beta(P)\psi_\beta(P), \psi_\alpha(Q)] = -\delta^{(n-1)}(\vec{P} - \vec{Q})\psi_\alpha(P), \tag{6.36}$$

where we applied the Einstein summation convention on β.

Finally, the following Ansatz comes naturally:

$$B(T, \Sigma) \equiv \frac{n-1}{2}\int d\vec{P}\Sigma(T, \vec{P})\psi^\dagger_\beta(T, \vec{P})\psi_\beta(T, \vec{P}), \tag{6.37}$$

where $\vec{P}$ is symbol for spatial coordinates, e.g. for $n = 3$: $\vec{P} \equiv (X, Y)$. Note that such Ansatz is in good agreement with (6.34). Let us check that this Ansatz works:

$$\begin{aligned}
[B(T,\Sigma), \psi_\alpha(Q)] &\equiv [B(T,\Sigma), \psi_\alpha(T, \vec{Q})] \\
&= \frac{n-1}{2}\int d\vec{P}\Sigma(T, \vec{P})[\psi^\dagger_\beta(T, \vec{P})\psi_\beta(T, \vec{P}), \psi_\alpha(T, \vec{Q})] \\
&= -\frac{n-1}{2}\int d\vec{P}\Sigma(T, \vec{P})\delta^{(n-1)}(\vec{P} - \vec{Q})\psi_\alpha(T, \vec{P}) \\
&= -\frac{n-1}{2}\Sigma(T, \vec{Q})\psi_\alpha(T, \vec{Q}) \\
&\equiv -\frac{n-1}{2}\Sigma(Q)\psi_\alpha(Q),
\end{aligned} \tag{6.38}$$

where we applied (6.36).

Let us notice that the operator W explicitly depends on time: $W(T, \Sigma)$, and the QWT works when all the operators are given at the same time T:

$$\begin{aligned}
\psi_{\Sigma\alpha}(T, \vec{Q}) &= W(T, \Sigma)\psi_\alpha(T, \vec{Q})W(T)^{-1} = e^{B(T,\Sigma)}\psi_\alpha(T, \vec{Q})e^{-B(T,\Sigma)} \\
&= e^{-\frac{n-1}{2}\Sigma(T,\vec{Q})}\psi_\alpha(T, \vec{Q}).
\end{aligned} \tag{6.39}$$

This must be because the commutation relation (6.36) holds at equal time: $T \equiv P_0 = Q_0$. This is an interesting outcome which should be understand, too.

Let us add that we shall usually denote the operator $B(T, \Sigma)$ as:

$$B = \frac{n-1}{2} \int d\vec{P} \Sigma(P) \psi^{\dagger}_{\beta}(P) \psi_{\beta}(P), \tag{6.40}$$

where $P^{\mu} \equiv (T, \vec{P})$. Similarly, we shall usually denote $W(T, \Sigma)$ as W. Also, let us emphasise that we shall always work with operators at the same time T, unless we say otherwise.

6.3 The transformed vacuum $W|0\rangle$

We shall denote the (unnormalized) transformed vacuum as:

$$|0(\Sigma)\rangle \equiv W(T, \Sigma)|0\rangle \tag{6.41}$$

and the projection of $|0(\Sigma)\rangle$ on $|0\rangle$ as:

$$f(\Sigma) \equiv \langle 0|0(\Sigma)\rangle \equiv \langle 0|W(T, \Sigma)|0\rangle. \tag{6.42}$$

Our aim is to explicitly calculate $f(\Sigma) \equiv \langle 0|0(\Sigma)\rangle$, as we did for the BT, see the section (5.2). There we studied the BT on two examples and used two computational techniques to obtain $f(\theta)$:

- 1st approach: expand the operator e^B in a power series, see (5.56);
- 2nd approach: apply the functional derivative, see (5.80).

Before we proceed, we write the Dirac field as a linear combination of the ladder operators:

$$\psi_{\beta}(Q) = \int d\vec{k} N_{\vec{k}} \sum_{s} \left(b(\vec{k}, s) u_{\beta}(\vec{k}, s) e^{-ikQ} + d^{\dagger}(\vec{k}, s) v_{\beta}(\vec{k}, s) e^{ikQ} \right), \tag{6.43}$$

where $\vec{k}$ is momentum, s is spin, $u_{\beta}(\vec{k}, s)$, $v_{\beta}(\vec{k}, s)$ are spinors, $b(\vec{k}, s)$ is the particle annihilation operator, $d(\vec{k}, s)$ is the antiparticle annihilation operator, the hermitian conjugate operators are the creation operators. They satisfy the CARs:

$$\{b(\vec{k}, s), b(\vec{l}, s')^{\dagger}\} = \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'}, \ \{d(\vec{k}, s), d(\vec{l}, s')^{\dagger}\} = \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'} \tag{6.44}$$

and the other CARs are zero.

6.3.1 The 1st approach of computation

Because the operator B has quite complicated structure (6.40), the 1st approach gives the complicated results:

$$f(\Sigma) \equiv \langle 0|W|0\rangle = \langle 0|e^B|0\rangle = 1 + \langle 0|B|0\rangle + \frac{1}{2!}\langle 0|B^2|0\rangle + \frac{1}{3!}\langle 0|B^3|0\rangle + \dots \tag{6.45}$$

Defining the following two matrices:

$$C_{\alpha\beta}(-P+Q) \equiv \int d\vec{k} N_{\vec{k}}^2 \sum_s v_\alpha^\dagger(\vec{k},s) v_\beta(\vec{k},s) e^{ik(-P+Q)} \tag{6.46a}$$

$$= \int d\vec{k} N_{\vec{k}}^2 \sum_s v_\alpha^\dagger(\vec{k},s) v_\beta(\vec{k},s) e^{-i\vec{k}(-\vec{P}+\vec{Q})}, \tag{6.46b}$$

$$\bar{C}_{\alpha\beta}(-P+Q) \equiv \int d\vec{k} N_{\vec{k}}^2 \sum_s u_\alpha^\dagger(\vec{k},s) u_\beta(\vec{k},s) e^{-ik(-P+Q)} \tag{6.46c}$$

$$= \int d\vec{k} N_{\vec{k}}^2 \sum_s u_\alpha^\dagger(\vec{k},s) u_\beta(\vec{k},s) e^{i\vec{k}(-\vec{P}+\vec{Q})}, \tag{6.46d}$$

where $Q^\mu = (T,\vec{Q})$, $P^\mu = (T,\vec{P})$, $kx \equiv k\cdot x \equiv k^0x^0 - \vec{k}\cdot\vec{x}$, we can express the first three orders of B as:

$$\langle 0|B|0\rangle = \frac{n-1}{2}\int d\vec{P}\Sigma(P)\mathrm{Tr}C(0), \tag{6.47a}$$

$$\langle 0|B^2|0\rangle = \left(\frac{n-1}{2}\right)^2 \int d\vec{P} d\vec{L} \Sigma(P)\Sigma(L)\left[(\mathrm{Tr}C(0))^2 + \mathrm{Tr}C(L-P)\bar{C}(P-L)\right], \tag{6.47b}$$

$$\langle 0|B^3|0\rangle = \left(\frac{n-1}{2}\right)^3 \int d\vec{P} d\vec{L} d\vec{K} \Sigma(P)\Sigma(L)\Sigma(K) I(P,L,K), \tag{6.47c}$$

where Tr denotes the trace of a matrix and $I(P,L,K)$ stands for:

$$\begin{aligned} I(P,L,K) = &(\mathrm{Tr}C(0))^3 + \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(P-K)C(K-P) \\ &+ \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(P-L)C(L-P) + \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(K-L)C(L-K) \\ &- \mathrm{Tr}\bar{C}(P-L)C(L-K)C(K-P) \\ &+ \mathrm{Tr}C(-P+L)\bar{C}(-L+K)\bar{C}(-K+P), \end{aligned} \tag{6.48}$$

where $P^\mu = (T,\vec{P})$, $L^\mu = (T,\vec{L})$, $K^\mu = (T,\vec{K})$. The evaluation of higher order terms is a demanding task. Therefore, it seems reasonable to use the 2^{nd} approach.

6.3.2 The 2^{nd} approach of computation

Let us recall $f(\Sigma)$ and define $f(\Sigma+\epsilon\delta)$:

$$f(\Sigma) = \langle 0|\exp\left[\frac{n-1}{2}\int d\vec{P}\Sigma(P)\psi_\beta^\dagger(P)\psi_\beta(P)\right]|0\rangle, \tag{6.49a}$$

$$f(\Sigma+\epsilon\delta) \equiv \langle 0|\exp\left[\frac{n-1}{2}\int d\vec{P}\left(\Sigma(P)+\epsilon\delta^{(n-1)}(\vec{P}-\vec{Q})\right)\psi_\beta^\dagger(P)\psi_\beta(P)\right]|0\rangle. \tag{6.49b}$$

Then the functional derivative of $f(\Sigma)$ is:

$$\frac{\delta}{\delta\Sigma}f(\Sigma) \equiv \lim_{\epsilon\to 0}\frac{f(\Sigma+\epsilon\delta)-f(\Sigma)}{\epsilon} = \frac{n-1}{2}\langle 0|W(T,\Sigma)\psi_\beta^\dagger(Q)\psi_\beta(Q)|0\rangle, \tag{6.50}$$

where $P^\mu \equiv (T,\vec{Q})$, $Q^\mu \equiv (T,\vec{Q})$. Also, we used the Einstein summation convention: repeated indices (here βs) are implicitly summed over.

Strictly speaking, we should write $\frac{\delta}{\delta\Sigma}f(\Sigma)(Q)$ instead of $\frac{\delta}{\delta\Sigma}f(\Sigma)$, but we shall ignore this in what follows.

In what follows, to ease the notation, we shall write, e.g., $\frac{\delta}{\delta\Sigma}f(\Sigma)$ instead of the more explicit expression $\frac{\delta}{\delta\Sigma}f(\Sigma)(Q)$.

It seems useful to change the order of the operators in (6.50):

$$\frac{\delta}{\delta\Sigma}f(\Sigma) = \frac{n-1}{2}e^{\frac{n-1}{2}\Sigma(Q)}\langle 0|\psi^\dagger_\beta(Q)W(T,\Sigma)\psi_\beta(Q)|0\rangle, \tag{6.51}$$

where we used:

$$W(T,\Sigma)\psi^\dagger_\beta(Q) = e^{\frac{n-1}{2}\Sigma(Q)}\psi^\dagger_\beta(Q)W(T,\Sigma), \tag{6.52}$$

because then only the antiparticle operators act nontrivially on the vacuum.

Finally, let us denote the vacuum expectation value (VEV) in (6.51) as the trace of the matrix $A(Q,T,Q)$:

$$\mathrm{Tr}A(Q,T,Q) \equiv A_{\beta\beta}(Q,T,Q) \equiv \langle 0|\psi^\dagger_\beta(Q)W(T)\psi_\beta(Q)|0\rangle. \tag{6.53}$$

We can define the matrix, for general spacetime positions $Q_1^\mu = (Q_1^0, \vec{Q_1})$, $Q_2^\mu = (Q_2^0, \vec{Q_2})$, as:

$$A_{\beta\gamma}(Q_1,T,Q_2) = \langle 0|\psi^\dagger_\beta(Q_1)W(T)\psi_\gamma(Q_2)|0\rangle. \tag{6.54}$$

If we succeed to find the matrix (6.54), we might have a good chance to compute its trace and evaluate $f(\Sigma)$, solving the equation:

$$\frac{\delta}{\delta\Sigma}f(\Sigma) = \frac{n-1}{2}e^{\frac{n-1}{2}\Sigma(Q)}\mathrm{Tr}A(Q,T,Q). \tag{6.55}$$

6.3.3 Computation of $A_{\beta\gamma}(Q,T,Q)$

We expand the Dirac field, see (6.43), and substitute into the equation (6.55). We obtain this result:

$$\begin{aligned} A_{\beta\gamma}(Q_1,T,Q_2) &= f(\Sigma)C_{\beta\gamma}(-Q_1+Q_2) \\ &+ \int d\vec{k}d\vec{l}N_{\vec{k}}N_{\vec{l}}\sum_{s,s'} v^\dagger_\beta(\vec{k},s)v_\gamma(\vec{l},s')e^{-ikQ_1}e^{ilQ_2}\langle 0|d(\vec{k},s)[W,d^\dagger(\vec{l},s')]|0\rangle. \end{aligned} \tag{6.56}$$

Let us recall the matrix $C(-Q_1+Q_2)$:

$$C_{\beta\gamma}(-Q_1+Q_2) = \int d\vec{k}N^2_{\vec{k}}\sum_s v^\dagger_\beta(\vec{k},s)v_\gamma(\vec{k},s)e^{ik(-Q_1+Q_2)}. \tag{6.57}$$

We need to compute the commutator on the right hand side of (6.56).

To achieve this, we use the following well-known identity:

$$e^{-tG}\left[O, e^{tG}\right] = \int_0^t ds e^{-sG}\left[O,G\right]e^{sG}, \tag{6.58}$$

where O, G are some operators, t is a parameter. Notice that the identity (6.58) is trivially satisfied for $t=0$ and, if we differentiate both sides of (6.58) with respect to t, we get the same expression $e^{-tG}[O,G]e^{tG}$ on both sides.

Applying the identity (6.58) and set $t = 1$, we obtain:

$$\left[W, d^\dagger(\vec{l}, s')\right] = W \int_0^1 ds e^{-sB}[B, d^\dagger(\vec{l}, s')]e^{sB}, \tag{6.59}$$

where $W = e^B$. Now, we must compute the commutator of B with $d^\dagger(\vec{l}, s')$. After some calculations, we find:

$$\left[B, d^\dagger(\vec{l}, s')\right] = -\frac{n-1}{2} \int d\vec{P} \Sigma(T, \vec{P}) N_l v_\alpha^\dagger(\vec{l}, s') e^{-ilP} \psi_\alpha(P), \tag{6.60}$$

where $P^\mu \equiv (T, \vec{P})$. With (6.60) in hand, we can finally calculate (6.59). The result of lengthy calculations is:

$$\left[W, d^\dagger(\vec{l}, s')\right] = -N_l v_\alpha^\dagger(\vec{l}, s') \int d\vec{P} \left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right) e^{-ilP} W \psi_\alpha(P). \tag{6.61}$$

After applying (6.61) in (6.56), we obtain a quite involved expression:

$$\begin{aligned} A_{\beta\gamma}(Q_1, T, Q_2) =& f(\Sigma) C_{\beta\gamma}(-Q_1 + Q_2) \\ &- f(\Sigma) \int d\vec{P} \left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right) C_{\beta\alpha}(-Q_1 + P) C_{\alpha\gamma}(-P + Q_2) \\ &+ \int d\vec{k} d\vec{l} N_{\vec{k}} N_{\vec{l}}^2 \sum_{s,s'} v_\beta^\dagger(\vec{k}, s) v_\gamma(\vec{l}, s') v_\alpha^\dagger(\vec{l}, s') \int d\vec{P} \left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right) \\ &\quad \times e^{il(Q_2 - P)} e^{-ikQ_1} \langle 0| \left[W, d(\vec{k}, s)\right] \psi_\alpha(P) |0\rangle. \end{aligned} \tag{6.62}$$

The commutator in (6.62) can be obtained from (6.61). The result of the computation is:

$$\begin{aligned} &A(Q_1, T, Q_2) = f(\Sigma) C(-Q_1 + Q_2) \\ &- f(\Sigma) \int d\vec{P} \left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right) C(-Q_1 + P) C(-P + Q_2) \\ &+ \int d\vec{P} d\vec{L} \left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right) \left(e^{\frac{n-1}{2}\Sigma(Q)} - 1\right) C(-Q_1 + L) A(L, T, P) C(-P + Q_2), \end{aligned} \tag{6.63}$$

where $L^\mu \equiv (T, \vec{L})$, $P^\mu \equiv (T, \vec{P})$.

Notice that the matrix A stands on the left hand side, $A(Q_1, T, Q_2)$, as well as on the right hand side, $A(L, T, P)$, of the equation (6.63). In order to find $A(Q_1, T, Q_2)$, we suggest to apply the procedure analogous to the Dyson series [5], [46]: By repeatedly substituting A on the right side we get:

$$A(Q_1, T, Q_2) = f(\Sigma) \left[C(-Q_1 + Q_2) + E(Q_1, T, Q_2) - F(Q_1, T, Q_2)\right], \tag{6.64}$$

where we defined two new matrices E, F:

$$\begin{aligned} E(Q_1, T, Q_2) = \sum_{k=1}^{\infty} \int d\vec{P}_1 d\vec{P}_2 \dots d\vec{P}_{2k} \left(e^{\frac{n-1}{2}\Sigma(P_1)} - 1\right) \left(e^{\frac{n-1}{2}\Sigma(P_2)} - 1\right) \dots \\ \times \left(e^{\frac{n-1}{2}\Sigma(P_{2k})} - 1\right) C(-Q_1 + P_1) C(-P_1 + P_2) \dots C(-P_{2k} + Q_2), \end{aligned} \tag{6.65}$$

$$F(Q_1,T,Q_2) = \sum_{k=1}^{\infty} \int d\vec{P}_1 d\vec{P}_2 \dots d\vec{P}_{2k-1} \left(e^{\frac{n-1}{2}\Sigma(P_1)} - 1\right)\left(e^{\frac{n-1}{2}\Sigma(P_2)} - 1\right)\dots$$
$$\times \left(e^{\frac{n-1}{2}\Sigma(P_{2k-1})} - 1\right) C(-Q_1+P_1)C(-P_1+P_2)\dots C(-P_{2k-1}+Q_2), \tag{6.66}$$

where $P_i^\mu \equiv (T, \vec{P}_i)$.

The evaluation of (6.53) seems a demanding task. However, we have not yet studied the matrix C in many details. If we found that C actully has a simple form (for instance, like a unit matrix, multiplied by a c-number), it would lead to a tractable result.

Before we get to that point, when we discuss if we can find a more tractable form of the matrices C, $\bar{C}$, let us refer the reader to appendix B, where we show how ladder operators transform under the *"hermitian tranformation"* W, and appendix C, where we show how to derive $\psi_{\Sigma\alpha}$ from ψ_α, using the results of the appendix B.

6.3.4 How the matrices $C(-P+Q), \bar{C}(-P+Q)$ look like

We found in the appendix C that sum of both matrices is:

$$\bar{C}_{\alpha\beta}(-P+Q) + C_{\alpha\beta}(-P+Q) = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q}-\vec{P}). \tag{6.67}$$

Although the equation (6.67) does not determine what each of the matrices is, we may make the following *Ansatz*:

$$\bar{C}_{\alpha\beta}(-P+Q) = C_{\alpha\beta}(-P+Q) = \frac{1}{2}\delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q}-\vec{P}). \tag{6.68}$$

If this Ansatz is right, then (6.65) and (6.66) would be much simplified. Let us discuss this Ansatz in more details.

We recall the well known completness relations:

$$\sum_s u(\vec{k},s)\bar{u}(\vec{k},s) = \not{k} + m, \quad \sum_s v(\vec{k},s)\bar{v}(\vec{k},s) = \not{k} - m. \tag{6.69}$$

If $m = 0$, we can rewrite (6.69) as:

$$\sum_s u_\alpha(\vec{k},s)u_\delta^\dagger(\vec{k},s)\gamma^0_{\delta\beta} = k_a\gamma^a_{\alpha\beta}, \quad \sum_s v_\alpha(\vec{k},s)v_\delta^\dagger(\vec{k},s)\gamma^0_{\delta\beta} = k_a\gamma^a_{\alpha\beta}, \tag{6.70}$$

and substract both equations:

$$\sum_s \left[u_\alpha(\vec{k},s)u_\delta^\dagger(\vec{k},s) - v_\alpha(\vec{k},s)v_\delta^\dagger(\vec{k},s)\right]\gamma^0_{\delta\beta} = 0. \tag{6.71}$$

Considering:

$$\sum_s u_\delta^\dagger(\vec{k},s)u_\alpha(\vec{k},s) = \sum_s v_\delta^\dagger(\vec{k},s)v_\alpha(\vec{k},s), \tag{6.72}$$

we can rewrite the matrix $\bar{C}_{\alpha\beta}(-P+Q)$:

$$\begin{aligned}\bar{C}_{\alpha\beta}(-P+Q) &\equiv \int d\vec{k} N_{\vec{k}}^2 \sum_s u_\alpha^\dagger(\vec{k},s)u_\beta(\vec{k},s)e^{-ik(-P+Q)} \\ &= \int d\vec{k} N_{\vec{k}}^2 \sum_s u_\alpha^\dagger(\vec{k},s)u_\beta(\vec{k},s)e^{i\vec{k}(-\vec{P}+\vec{Q})},\end{aligned} \tag{6.73}$$

in the following form:

$$\bar{C}_{\alpha\beta}(-P+Q) = \int d\vec{k} N_{\vec{k}}^2 \sum_s v_\alpha^\dagger(-\vec{k},s) v_\beta(-\vec{k},s) e^{-i\vec{k}(-\vec{P}+\vec{Q})}. \tag{6.74}$$

We recall the matrix $C_{\alpha\beta}(-P+Q)$:

$$\begin{aligned} C_{\alpha\beta}(-P+Q) &\equiv \int d\vec{k} N_{\vec{k}}^2 \sum_s v_\alpha^\dagger(\vec{k},s) v_\beta(\vec{k},s) e^{ik(-P+Q)} \\ &= \int d\vec{k} N_{\vec{k}}^2 \sum_s v_\alpha^\dagger(\vec{k},s) v_\beta(\vec{k},s) e^{-i\vec{k}(-\vec{P}+\vec{Q})}, \end{aligned} \tag{6.75}$$

and compare (6.75) with (6.74). If the following equation applied:

$$\sum_s v_\delta^\dagger(\vec{k},s) v_\alpha(\vec{k},s) = \sum_s v_\delta^\dagger(-\vec{k},s) v_\alpha(-\vec{k},s), \tag{6.76}$$

it would confirm our Ansatz (6.68). However, we shall introduce one example when (6.76) is not satisfied, which challenges our Ansatz.

We shall consider the following problem: $n = d = 4$, where n is the dimension of the spacetime, d is the dimension of Dirac space. Also, we shall consider $m \neq 0$ for beginning, but then we shall make the limit $m \to 0$. The spinors $v(\vec{k},s)$ in the Dirac representation are:

$$v\left(\vec{k}, s=-\frac{1}{2}\right) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}\vec{k}}{E+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{pmatrix}, \quad v\left(\vec{k}, s=\frac{1}{2}\right) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}\vec{k}}{E+m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}, \tag{6.77}$$

where $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Let us assume that $\vec{k}$ is oriented along the z-axis. Then it holds, for small m (or large $\vec{k}$):

$$\sum_s v_\alpha^\dagger(k_z,s) v_\beta(k_z,s) \propto \begin{pmatrix} \mathbb{1}_{2\times2} & \sigma_z \\ \sigma_z & \mathbb{1}_{2\times2}, \end{pmatrix}, \tag{6.78a}$$

$$\sum_s v_\alpha^\dagger(-k_z,s) v_\beta(-k_z,s) \propto \begin{pmatrix} \mathbb{1}_{2\times2} & -\sigma_z \\ -\sigma_z & \mathbb{1}_{2\times2} \end{pmatrix}, \tag{6.78b}$$

which suggests that (6.76) does not apply, at least not in general. We take this as the argument why we should figure out a new Ansatz, and not settle for (6.68).

We want to modify our first Ansatz (6.68) to include non-diagonal terms. The following observation, based on (6.78), gives us a hint what we should/could take into account. If we denote $\tilde{E}$ as 4×4 matrix such that:

$$\tilde{E} \equiv \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0, \end{pmatrix}, \tag{6.79}$$

then it holds:

$$\mathrm{Tr}\tilde{E} = 0, \quad \tilde{E}^2 = \mathbb{1}_{4\times4}, \quad (\mathbb{1}_{4\times4} + \tilde{E})^2 = 2\left(\mathbb{1}_{4\times4} + \tilde{E}\right). \tag{6.80}$$

We shall build our Ansatz on extrapolation of these properties. We suggest the following *new Ansatz*:

$$\bar{C}_{\alpha\beta}(-P+Q) = \frac{1}{2}\left(\delta_{\alpha\beta} - E_{\alpha\beta}\right)\delta^{(n-1)}(\vec{Q}-\vec{P}), \tag{6.81a}$$

$$C_{\alpha\beta}(-P+Q) = \frac{1}{2}\left(\delta_{\alpha\beta} + E_{\alpha\beta}\right)\delta^{(n-1)}(\vec{Q}-\vec{P}), \tag{6.81b}$$

where the matrix E satisfies $\mathrm{Tr}E = 0$ and $E^2 = \mathbb{1}$. This implies:

$$(\mathbb{1} + E)^n = 2^{n-1} (\mathbb{1} + E) . \tag{6.82}$$

We shall see that this Ansatz gives surprisingly simple outcome.

6.3.5 Calculation of $\langle 0|W|0\rangle$ using the Ansatz for (6.81)

Our goal is to compute the trace of:

$$A(Q,T,Q) = f(\Sigma) \left[C(0) + E(Q,T,Q) - F(Q,T,Q) \right] . \tag{6.83}$$

With (6.81) in hands, we can compute $E(Q,T,Q)$, $F(Q,T,Q)$:

$$\begin{aligned} E(Q,T,Q) &= \frac{\delta^{(n-1)}(0)}{2} \sum_{k=1}^{\infty} \left(e^{\frac{n-1}{2}\Sigma(Q)} - 1 \right)^{2k} (\mathbb{1} + E) , \\ F(Q,T,Q) &= \frac{\delta^{(n-1)}(0)}{2} \sum_{k=1}^{\infty} \left(e^{\frac{n-1}{2}\Sigma(Q)} - 1 \right)^{2k-1} (\mathbb{1} + E) , \end{aligned} \tag{6.84}$$

which leads to this result:

$$A(Q,T,Q) = f(\Sigma) \frac{\delta^{(n-1)}(0)}{2} \left[1 + \left(1 - \frac{1}{y} \right) \sum_{k=1}^{\infty} y^{2k} \right] (\mathbb{1} + E) , \tag{6.85}$$

where we defined y such that:

$$y \equiv e^{\frac{n-1}{2}\Sigma(Q)} - 1. \tag{6.86}$$

If the function Σ is such that:

$$\Sigma(Q) \in \left(-\infty, \frac{2}{n-1} \ln 2 \right) , \tag{6.87}$$

the geometric sum in (6.85) converges and leads to:

$$A(Q,T,Q) = f(\Sigma) \frac{\delta^{(n-1)}(0)}{2} e^{-\frac{n-1}{2}\Sigma(Q)}, \tag{6.88a}$$

$$\mathrm{Tr}A(Q,T,Q) = f(\Sigma) \frac{\delta^{(n-1)}(0)}{2} d e^{-\frac{n-1}{2}\Sigma(Q)}. \tag{6.88b}$$

(6.88c)

Putting (6.88b) into (6.51), we obtain:

$$\frac{\delta}{\delta \Sigma} f(\Sigma) = f(\Sigma) \frac{n-1}{2} d \frac{\delta^{(n-1)}(0)}{2}, \tag{6.89}$$

where d is the dimension of the Dirac space. Solving (6.89), we find $f(\Sigma)$:

$$\begin{aligned} f(\Sigma) &= \exp \left(\frac{n-1}{2} d \frac{\delta^{(n-1)}(0)}{2} \int d\vec{P} \Sigma(T, \vec{P}) \right) \\ &= \exp \left(\frac{n-1}{2} \mathrm{Tr} C(0) \int d\vec{P} \Sigma(T, \vec{P}) \right) . \end{aligned} \tag{6.90}$$

This is the simplest result we could get.

We would also like to compare this result, obtained by the 2nd *approach*, 6.3.2, with the 1st *approach*, 6.3.1, how to compute $f(\Sigma)$. Let us remind the what the 1st approach gives:

$$f(\Sigma) = \langle 0|U|0\rangle = \langle 0|e^B|0\rangle = 1 + \langle 0|B|0\rangle + \frac{1}{2!}\langle 0|B^2|0\rangle + \frac{1}{3!}\langle 0|B^3|0\rangle + \dots . \tag{6.91}$$

As the calculation of higher order terms becomes more and more demanding, we focus only on the first three terms which includes B. Direct calculation gives:

$$\langle 0|B|0\rangle = \frac{n-1}{2}\int d\vec{P}\Sigma(P)\mathrm{Tr}C(0), \tag{6.92a}$$

$$\langle 0|B^2|0\rangle = \left(\frac{n-1}{2}\right)^2 \int d\vec{P}d\vec{L}\Sigma(P)\Sigma(L)\left[(\mathrm{Tr}C(0))^2 + \mathrm{Tr}C(L-P)\bar{C}(P-L)\right], \tag{6.92b}$$

$$\langle 0|B^3|0\rangle = \left(\frac{n-1}{2}\right)^3 \int d\vec{P}d\vec{L}d\vec{K}\Sigma(P)\Sigma(L)\Sigma(K)I(P,L,K), \tag{6.92c}$$

where

$$\begin{aligned} I(P,L,K) = &(\mathrm{Tr}C(0))^3 + \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(P-K)C(K-P) \\ &+ \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(P-L)C(L-P) + \mathrm{Tr}C(0)\mathrm{Tr}\bar{C}(K-L)C(L-K) \\ &- \mathrm{Tr}\bar{C}(P-L)C(L-K)C(K-P) \\ &+ \mathrm{Tr}C(-P+L)\bar{C}(-L+K)\bar{C}(-K+P). \end{aligned} \tag{6.93}$$

Applying (6.81), so $C\bar{C} \propto (\mathbb{1}+E)(\mathbb{1}-E) = (\mathbb{1}-E^2) = 0$, because $E^2 = \mathbb{1}$, we obtain:

$$\langle 0|B|0\rangle = \frac{n-1}{2}d\frac{\delta^{(n-1)}(0)}{2}\int d\vec{P}\Sigma(P), \tag{6.94a}$$

$$\langle 0|B^2|0\rangle = \left(\frac{n-1}{2}d\frac{\delta^{(n-1)}(0)}{2}\int d\vec{P}\Sigma(P)\right)^2 = (\langle 0|B|0\rangle)^2, \tag{6.94b}$$

$$\langle 0|B^3|0\rangle = \left(\frac{n-1}{2}d\frac{\delta^{(n-1)}(0)}{2}\int d\vec{P}\Sigma(P)\right)^3 = (\langle 0|B|0\rangle)^3. \tag{6.94c}$$

As we have shown above, to the third order of the Taylor series both procedures coincide. This is a sign of that we performed our calculations correctly. However, we should not take this as an argument that our Ansatz is right. Both approaches should lead to the same Taylor series anyway.

This is the end of our calculations in this chapter. Let us add some comments about them.

In standard QFT, we work with unitary operators Us, as we could see in (5.2). The fact that the operator W is hermitian and depends explicitly on time T is novelty and it would be interesting to examine the consequences thereof. Some inspiration for further study can be found in this article [6], where the authors had the first remarks on the QWT.

Next, we proposed the Ansatz (6.81) for $C, \bar{C}$, which significantly simplified the calculation of (6.83). This was our main intention. We obtained (6.85) which is simple enough to compute $\mathrm{Tr}A(Q,T,Q)$. However, it contains a geometric series which can diverge if Σ is not from a suitable range, see (6.87):

$$\Sigma(Q) \in \left(-\infty, \frac{2}{n-1}\ln 2\right), \tag{6.95}$$

Such a range can remind us the Beltrami pseudosphere 3.2:

$$\sigma_B(u) \in \left(-\infty, \ln\frac{r}{c}\right]. \tag{6.96}$$

Another work should be to examine this Ansatz and whether equations (6.95) and (6.96) can be understood together to see the effect of singularities on the convergence/divergence of the geometric series. This task seems to us now to be quite challenging and it would take more time than we had for this thesis.

7. Conclusions

In Introduction, we stated the general goal of this thesis, to investigate the Weyl anomaly from a new perspective, and proposed a model which has the potential to make our analysis simple and transparent. We considered the Weyl transformation of the Dirac massless field and the conformally flat metric, mapping the dynamics of the field in a curved space into a flat space, since the transformation is a symmetry of Dirac massless action.

In chapters 1 and 2, we discussed the model in details, investigating the Weyl symmetry and conformally flat spacetimes.

In chapter 3 we studied the surfaces of constant Ricci curvature. We put emphasis on description of their singularities and geometry. We were looking for the coordinate transformations from (u, v) to $(\tilde{x}, \tilde{y})$, while doing so we also found the local conformal factors. We found, for the first time, the tractable isothermal coordinates and conformal factors for the Elliptic pseudosphere and the Dini surface. Moreover, we added appendix A to this chapter where we described the conical singularity.

In chapter 4 we studied the conformally flat spacetimes of the dimension $n = 3$, which the surfaces of constant Ricci curvature produce (when we add flat time to their line element). We have investigated the solutions Σs of the modified Liouville equation of the first form and searched for the coordinate transformations between (T, X, Y) and $(t, \tilde{x}, \tilde{y})$. The results of both chapters 3 and 4 are important in order to understand the quantum inequivalence we were investigating. These results are also relevant for further work in [18] and [7], for instance.

In chapter 5, we introduced the canonical transformations, the Bogoliubov transformation, the concept of unitarity (in)equivalence of the CCRs and Fock spaces. We discussed both QM and QFT systems.

In chapter 6 we introduced the operator W, implementing the Weyl transformation at the quantum level. To be a symmetry, this transformation should be unitary as the quantum theory states. The main question we are addressing here is whether the operator W is irregular, then the expected quantum symmetry would be just formal and no matter how this operator is related to a symmetry operator, this would be the case of violation of the quantum symmetry.

Given the real nature of the Weyl transformation, the simplest solution for W is to relax the request for unitarity in favor of that of hermiticity. Although not a standard choice, it is a fully legitimate procedure when considered as the quantum procedure that generates the *Weyl transformation*. Of course it raises questions on how such transformation is in direct contact with what is customarily considered a unitary quantum symmetry. Nonetheless, this makes possible to link the singularities of the conformal factor to singular (irregular) behaviors of the transformation, which is the most crucial point of our work.

Under special considerations, we were able to see a point where this operator becomes singular due to the properties of the function Σ. This is still not the end as our Ansatz for C (6.81) needs to be further investigated. After that we can confront (6.95) with (6.96).

Let us add that this last chapter is supplemented with two technical appendices B and C.

A. Conical singularity

The cone is not a regular surface, because of its vertex, where no tangent space is defined. Therefore the standard way how to compute the scalar curvature is not applicable at this point. Here, we shall sketch one way how the concept of curvature can be extended to cover conical singularities. The presented method is based on the distribution theory and was in details described in [27]. We sketch this procedure here.

A.1 Basic concepts of the Distribution theory

Let us remind necessary background of the distribution theory. We consider a two-dimensional manifold S on which a local coordinate system (u, v) is defined. Next we define a function $\phi \equiv \phi(u, v) \in C^\infty$ with compact support on S. This is called the test-function. A continuous linear functional (distribution) F^* maps the test-function ϕ into real numbers, $(F^*, \phi) \in \mathcal{R}$, in the following way:

1. Linearity condition: for real numbers a_1, a_2 it holds:

$$(F^*, a_1\phi_1 + a_2\phi_2) = a_1(F^*, \phi_1) + a_2(F^*, \phi_2), \tag{A.1}$$

2. Continuity condition: if a sequence of test-functions ϕ_i, $i = 1, \ldots, n$, tends uniformaly to zero, then (F^*, ϕ_i) tends to zero as well.

The distribution F^* is said to be regular if there exists a locally integrable function F such that:

$$(F^*, \phi) = \int_U F(u, v)\phi(u, v)\sqrt{g}dudv, \tag{A.2}$$

where $U \subset S$ is a compact domain, g is the determinant of the metric tensor g_{ij} on S. If F^* is not expressable as an integral, then the distribution is called singular.

The product of the coefficient $\alpha(x) \in C^\infty$ with the distribution $F*$ is the distribution αF^*, defined as:

$$(\alpha F^*, \phi) = (F^*, \alpha\phi). \tag{A.3}$$

Moreover, we can define the derivative of F^* as a functional $\delta F/\delta u$, which is related to F^* in following way:

$$\left(\frac{\delta F^*}{\delta u}, \phi\right) = -\left(F^*, \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\phi\right)}{\partial u}\right), \tag{A.4}$$

where the $\sqrt{g}$, $\frac{1}{\sqrt{g}}$ must be C^∞, otherwise we must find another coordinate system where these functions are C^∞.

A.2 Generalized concept of curvature

Let us write the metric in the following form:

$$dl^2 = E(u,v)du^2 + F(u,v)dudv + G(u,v)dv^2, \tag{A.5}$$

where E, F, G are the coefficients of the first fundamental form [22].

Then one can compute the Ricci scalar in the same way as we did for (2.14). Then the Ricci scalar can be obtained as:

$$R = \frac{1}{\sqrt{g}}\left(\frac{\partial P}{\partial v} - \frac{\partial Q}{\partial u}\right), \tag{A.6}$$

where P and Q are functions:

$$\begin{aligned} P &= \frac{1}{\sqrt{g}}\left(\frac{\partial F}{\partial u} - \frac{\partial E}{\partial v} - \frac{1}{2}\frac{F}{E}\frac{\partial E}{\partial u}\right), \\ Q &= \frac{1}{\sqrt{g}}\left(\frac{\partial G}{\partial u} - \frac{1}{2}\frac{F}{E}\frac{\partial E}{\partial v}\right),. \end{aligned} \tag{A.7}$$

where $g = EG - \frac{1}{4}F^2$.

The formula (A.6) can be used to compute the Ricci scalar if the manifold is smooth everywhere - it is a differential manifold.

However, this is not the case of the cone because it is not smooth at its apex. The approach we follow generalizes the concept of curvature by promoting R to distributions:

$$R^* = \frac{1}{\sqrt{g}}\left(\frac{\delta P^*}{\delta v} - \frac{\delta Q^*}{\delta u}\right), \tag{A.8}$$

where P^* and Q^* are the regular distributions corresponding to P and Q, respectively.

The curvature-distribution is related to functions P and Q in the following way:

$$(R^*, \phi) = -\left(P^*, \frac{1}{\sqrt{g}}\frac{\partial \phi}{\partial v}\right) + \left(Q^*, \frac{1}{\sqrt{g}}\frac{\partial \phi}{\partial u}\right) = \int\int_S \left(-P\frac{\partial \phi}{\partial v} + Q\frac{\partial \phi}{\partial u}\right) dudv. \tag{A.9}$$

A.3 Curvature at cone's apex

As we showed in (3.12) the cone's metric can be written as:

$$dl^2 = dr^2 + \sin^2 \alpha r^2 d\phi^2. \tag{A.10}$$

However, such a parametrization is not suitable for description of the cone, e.g. $1/\sqrt{g}$ is not C^∞ function everywhere.

The cone is a two-dimensional surface embeded into the three-dimensional eucledian space with cartesian coordinates (x, y, z), where z-coordinate satisfies: $z = a\sqrt{x^2 + y^2}$, where $a \equiv 1/\tan \phi$. Parametrizating the surface by (x, y), the metric becomes:

$$dl^2 = \left(1 + \frac{a^2x^2}{x^2 + y^2}\right) dx^2 + \left(1 + \frac{a^2y^2}{x^2 + y^2}\right) dy^2 + \frac{2a^2xy}{x^2 + y^2} dxdy. \tag{A.11}$$

Then the functions P and Q become:

$$\begin{aligned} P &= \frac{2a^2}{\sqrt{1+a^2}} \frac{y^3}{(x^2+y^2)(x^2+y^2+a^2x^2)} \\ Q &= -\frac{2a^2}{\sqrt{1+a^2}} \frac{xy^2}{(x^2+y^2)(x^2+y^2+a^2x^2)}, \end{aligned} \tag{A.12}$$

where $\sqrt{g} = \sqrt{1+a^2}$. It might seem that both functions P and Q are not locally integrable, because they are not bounded. However, if we change coordinates from the cartesian one (x, y) to polar one (R, θ), $x = R\cos\theta$, $y = R\sin\theta$, Jacobian regularizes the singularity at $R = 0$, so integrals are finite.

Now, let us consider a disk with radius ϵ with center at the apex. Then we remove the disk from the manifold S and denote the remaing part of S as S_ϵ. Because P and Q are locally integrable functions, the formula for the curvature (A.9) can be rewritten as follows:

$$(R^*, \phi) = \lim_{\epsilon\to 0} \int\int_{S_\epsilon} \left(-P\frac{\partial\phi}{\partial y} + Q\frac{\partial\phi}{\partial x} \right) dxdy. \tag{A.13}$$

Applying the Green theorem, we obtain:

$$(R^*, \phi) = \lim_{\epsilon\to 0} \int\int_{S_\epsilon} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \phi dxdy - \lim_{\epsilon\to 0} \int_{\partial S_\epsilon} (Pdx + Qdy)\, \phi. \tag{A.14}$$

The integrand in the first integral is the formula for the Ricci scaler (upto the test-function ϕ), see (A.6). Because the integral is computed in S_ϵ for the given ϵ, it is zero in value. The second term in (A.14) is the key one. It was shown in [27] and [47] that the second term becomes:

$$(R^*, \phi) = 2\Delta\chi\phi(0), \tag{A.15}$$

where $\Delta\chi$ is the angular deficit of the cone, $\Delta\chi = 2\pi(1 - \sin\alpha)$. Then it holds:

$$(R^*, \phi) = 4\pi(1 - \sin\alpha)\phi(0). \tag{A.16}$$

This can be rewritten in more familiar way, with help of the delta fuction as:

$$R^* = 4\pi(1 - \sin\alpha)\delta^{(2)}(r), \tag{A.17}$$

where the delta function is defined as:

$$\int \delta^{(2)}(r)\sqrt{r}drd\phi = 1. \tag{A.18}$$

In (3.14) we omitted the 'star' in the label and we work with the delta function as we are used to do in physics literature.

B. Ladder operators $b, d, b^\dagger, d^\dagger$

In this appendix, we would like to find how the (anti)particle creation and annihilation operators, $b, b^\dagger, d, d^\dagger$, transform.

We know how the Dirac field is transformed:

$$W(T,\Sigma)\psi_\alpha(Q)W(T,\Sigma)^{-1} = e^{-\frac{n-1}{2}\Sigma(Q)}\psi_\alpha(Q), \tag{B.1}$$

where $Q^\mu \equiv (T, \vec{Q})$, n is dimension of spacetime and $W(T,\Sigma)$:

$$W(T,\Sigma) \equiv \exp B(T,\Sigma) \equiv \exp\left(\frac{n-1}{2}\int d\vec{P}\Sigma(P)\psi^\dagger_\beta(P)\psi_\beta(P)\right), \tag{B.2}$$

where $P^\mu \equiv (T, \vec{P})$. For the sake of brevity, we shall write only W in what follows, but mean $W(T,\Sigma)$. Similarly, we shall denote $B(T,\Sigma)$ simply as B. For instance:

$$W\psi_\alpha(Q)W^{-1} = e^{-\frac{n-1}{2}\Sigma(Q)}\psi_\alpha(Q), \tag{B.3a}$$

$$B = \frac{n-1}{2}\int d\vec{P}\Sigma(P)\psi^\dagger_\beta(P)\psi_\beta(P). \tag{B.3b}$$

Also, $W(T,\Sigma)^{-1} = W(T,-\Sigma)$ will be denoted as W^{-1}.

In order to find the transformation relations for the ladder operators, we start with the the following commutators:

$$\left[W, d^\dagger(\vec{l}, s')\right] = -N_l v^\dagger_\alpha(\vec{l}, s')\int d\vec{P}\left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right)e^{-ilP}U\psi_\alpha(P), \tag{B.4a}$$

$$\left[W, b(\vec{l}, s')\right] = -N_l u^\dagger_\alpha(\vec{l}, s')\int d\vec{P}\left(e^{\frac{n-1}{2}\Sigma(P)} - 1\right)e^{ilP}U\psi_\alpha(P), \tag{B.4b}$$

where the commutator (B.4a) was derived before, see (6.59), and the other can be derived in a similar way. From (B.4) follows the transformation relations for $b, d, b^\dagger, d^\dagger$:

$$Wb(\vec{l}, s')W^{-1} = b(\vec{l}, s') - N_l u^\dagger_\alpha(\vec{l}, s')\int d\vec{P}\left(1 - e^{-\frac{n-1}{2}\Sigma(P)}\right)e^{ilP}\psi_\alpha(P), \tag{B.5a}$$

$$Wd(\vec{l}, s')W^{-1} = d(\vec{l}, s') - N_l v_\alpha(\vec{l}, s')\int d\vec{P}\left(1 - e^{\frac{n-1}{2}\Sigma(P)}\right)e^{ilP}\psi^\dagger_\alpha(P), \tag{B.5b}$$

$$Wb^\dagger(\vec{l}, s')W^{-1} = b^\dagger(\vec{l}, s') - N_l u_\alpha(\vec{l}, s')\int d\vec{P}\left(1 - e^{\frac{n-1}{2}\Sigma(P)}\right)e^{-ilP}\psi^\dagger_\alpha(P), \tag{B.5c}$$

$$Wd^\dagger(\vec{l}, s')W^{-1} = d^\dagger(\vec{l}, s') - N_l v^\dagger_\alpha(\vec{l}, s')\int d\vec{P}\left(1 - e^{-\frac{n-1}{2}\Sigma(P)}\right)e^{-ilP}\psi_\alpha(P). \tag{B.5d}$$

Let us recall the following CARs:

$$\left\{b(\vec{k}, s), b^\dagger(\vec{l}, s')\right\} = \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'}, \tag{B.6a}$$

$$\left\{d(\vec{k}, s), d^\dagger(\vec{l}, s')\right\} = \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'} \tag{B.6b}$$

and the remaining CARs for $b, d, b^\dagger, d^\dagger$ are zero. If we act by W from the left and W^{-1} from the right each of these CARS, they can be absorbed into the CARs, for instance:

$$\{Wb(\vec{k}, s)W^{-1}, Wb^\dagger(\vec{l}, s')W^{-1}\} = \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'} \tag{B.7}$$

and similarly for the remaining CARs.

It is a good check of consistency to compute an anticommutator:

$$\{Wb(\vec{k}, s)W^{-1}, Wb^\dagger(\vec{l}, s')W^{-1}\}, \tag{B.8}$$

using (B.5a) and (B.5c). Then we should obtain the same right hand side as the equation (B.7) has:

$$\{Wb(\vec{k}, s)W^{-1}, Wb^\dagger(\vec{l}, s')W^{-1}\} \overset{(B.5)}{=} \cdots \overset{?}{=} \delta^{(n-1)}(\vec{k} - \vec{l})\delta_{ss'}. \tag{B.9}$$

We shall not elaborate detailed calculations here, which are straightforward and lengthy, but we conclude that all the CARs, computated like (B.9), are in good agreement with expectations (it means (B.7), and similarly others).

C. Fields ψ_α and $\psi_{\Sigma\alpha}$

In the appendix B, we found the transformation relations for the ladder operators, see (B.5). Now, we would like to apply these relations to transform the Dirac field ψ_α. It is another consistency check. Let us add that we shall use the same notation as we did in B.

Let us recall the Dirac field $\psi_\beta(Q)$, expanded into a series of ladder operators:

$$\psi_\beta(Q) = \int d\vec{k} N_{\vec{k}} \sum_s \left(b(\vec{k},s) u_\beta(\vec{k},s) e^{-ikQ} + d^\dagger(\vec{k},s) v_\beta(\vec{k},s) e^{ikQ} \right). \tag{C.1}$$

The Dirac field is transformed as:

$$\psi_{\Sigma\alpha} = W \psi_\alpha W^{-1}, \tag{C.2}$$

which leads to the following result:

$$\begin{aligned} \psi_{\Sigma\beta}(Q) &= \int d\vec{k} N_{\vec{k}} \sum_s \left(W b(\vec{k},s) W^{-1} u_\alpha(\vec{k},s) e^{-ikQ} + W d^\dagger(\vec{k},s) W^{-1} v_\alpha(\vec{k},s) e^{ikQ} \right) \\ &= \psi_\beta(Q) - \int d\vec{P} \left(1 - e^{-\frac{n-1}{2}\Sigma(P)}\right) \left[\bar{C}_{\alpha\beta}(-P+Q) + C_{\alpha\beta}(-P+Q) \right] \psi_\alpha(P) \end{aligned} \tag{C.3}$$

with $Q \equiv (T, \vec{Q})$, $P \equiv (T, \vec{P})$, and we used our results from appendix B, see (B.5), and defined two matrices:

$$\begin{aligned} C_{\alpha\beta}(-P+Q) &\equiv \int d\vec{k} N^2_{\vec{k}} \sum_s v^\dagger_\alpha(\vec{k},s) v_\beta(\vec{k},s) e^{ik(-P+Q)}, \\ &= \int d\vec{k} N^2_{\vec{k}} \sum_s v^\dagger_\alpha(\vec{k},s) v_\beta(\vec{k},s) e^{-i\vec{k}(-\vec{P})+\vec{Q}}, \\ \bar{C}_{\alpha\beta}(-P+Q) &\equiv \int d\vec{k} N^2_{\vec{k}} \sum_s u^\dagger_\alpha(\vec{k},s) u_\beta(\vec{k},s) e^{-ik(-P+Q)}, \\ &= \int d\vec{k} N^2_{\vec{k}} \sum_s u^\dagger_\alpha(\vec{k},s) u_\beta(\vec{k},s) e^{i\vec{k}(-\vec{P}+\vec{Q})}. \end{aligned} \tag{C.4}$$

The sum of the matrices $C_{\alpha\beta}(-P+Q)$ and $\bar{C}_{\alpha\beta}(-P+Q)$ in (C.3) can be computed using the following CARs:

$$\{\psi_\beta(Q), \psi^\dagger_\alpha(P)\}_{E.T.} = \delta_{\alpha\beta} \delta^{(n-1)}(\vec{Q} - \vec{P}). \tag{C.5}$$

Now, we act by $\langle 0|$ from the left and $|0\rangle$ from the right:

$$\langle 0|\psi_\beta(Q)\psi^\dagger_\alpha(P)|0\rangle + \langle 0|\psi^\dagger_\alpha(P)\psi_\beta(Q)|0\rangle = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}). \tag{C.6}$$

Using (C.1), it is straightforward to show that:

$$\langle 0|\psi_\beta(Q)\psi^\dagger_\alpha(P)|0\rangle = \bar{C}_{\alpha\beta}(-P+Q), \quad \langle 0|\psi^\dagger_\alpha(P)\psi_\beta(Q)|0\rangle = C_{\alpha\beta}(-P+Q), \tag{C.7}$$

which leads to:

$$\bar{C}_{\alpha\beta}(-P+Q) + C_{\alpha\beta}(-P+Q) = \delta_{\alpha\beta}\delta^{(n-1)}(\vec{Q} - \vec{P}). \tag{C.8}$$

Thanks to (C.8), the transformed field $\psi_{\Sigma\beta}$ (C.3) becomes:

$$\psi_{\Sigma\beta}(Q) = e^{-\frac{n-1}{2}\Sigma(Q)} \psi_\beta(Q). \tag{C.9}$$

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