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ANALYSIS FOR THE INTERACTIONS BETWEEN FLUIDS AND SOLIDS

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I declare that I carried out this habilitation thesis independently, and only with the cited sources, literature and other professional sources.

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Abstract:

The study on systems of partial differential equations (PDE) that are coupled via a common interface is one of the important challenges of nowadays mathematics. This is due to the overwhelming number of real live applications in which two different materials interact; for example fluid-structure interactions like a blood flow in a vessel or an airplane flaying in the air.

The thesis covers recent progress on coupled systems with an emphasis on fluid-structure interactions. It is biased on the projects that have been achieved by the support of two grants of the author. On the one hand the junior grant GJ19-11707Y of the Czech national grant agency (GAČR), on the other hand the Primus research programme PRIMUS/19/SCI/01 of Charles University. Further the projects where supported by the University Centre UNCE/SCI/023.

The chapters after the introduction are each dedicated to a (submitted) preprint. The chapters 2–4 are on the existence, regularity and uniqueness of fluids interacting with thin objects. Namely shells interacting with incompressible, compressible and/or heat conducting fluids. Chapter 5–7 are about variational methods for coupled systems of fluid-structure interaction type involving bulk elastic solids. Finally Chapter 8 is on the bouncing of (elastic) solids of a rigid wall.

Chapter 2 **Existence and regularity of weak solutions for a fluid interacting with a non-linear shell.** This work was achieved in collaboration with Boris Muha [142]. It is about an *existence* and *regularity* theory for *weak solutions* on fluid-structure interactions.

Chapter 3 **Navier-Stokes-Fourier fluids interacting with elastic shells.** This is a work that has been achieved in collaboration with Dominic Breit on the existence of compressible and heat conducting fluids interacting with elastic non-linear shells [23]. It relies on the *regularity estimate* from chapter 2 and an earlier work on compressible fluid [22].

Chapter 4 **Weak-strong uniqueness for an elastic plate interacting with the Navier Stokes equation.** This is the first *weak-strong uniqueness* result for fluid-structure interactions involving deformable solids. It is a work achieved in collaboration with Matthias Sroczinski [169].

Chapter 5 **A variational approach to fluid-structure interactions.** This is the first part of the joint work with Barbora Benešová and Malte Kampschulte [13]. Here we introduce a methodology how to produce coupled systems via minimizing movements–this is applied to fluid-structure interactions.

Chapter 6 **A variational approach to hyperbolic evolutions.** This is taken from the second part of the joint work with Barbora Benešová and Malte Kampschulte [13]. It is only for elastic solid motions including inertia terms. Here a generalization of De Giorgis minimizing movements to hyperbolic PDEs is performed.

Chapter 7 **Bulk elastic solids interacting with Navier-Stokes fluids.** This is taken from the fhird and last part of the joint work with Barbora Benešová and Malte Kampschulte [13]. Here the hyperbolic variational strategy is extended to a bulk elastic solid interacting with an incompressible fluid.

Chapter 8 **Contactless rebound of elastic bodies in a viscous incompressible fluid.** This is a joint interdisciplinary work with Giovanni Gravina, Karel Tůma and Ondřej Souček [91]. The final chapter is about the challenging question on contact and bouncing of solids in viscous incompressible fluids.

Further the introduction contains some parts of the survey article [14].

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Chapter 1

Introduction

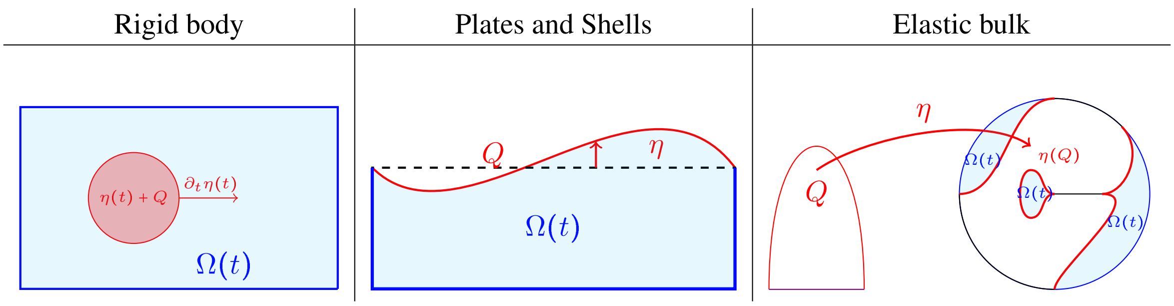
Due to its significance for various applications, the research on fluid-structure interactions is rich and diverse with contributions from many different scientific disciplines including mathematics. While most of the mathematical literature on fluid-structure interactions concerns *numerical analysis and simulations*, see for example the recent monograph [165], there is also an ever increasing effort on the rigorous analysis in the field [27].

In this habilitation thesis we wish to unify the effort of the last two years of the working group on the analysis of the interaction of Fluids and Solids. The aim was to advance the existence theory of weak solutions, the related theory on the regularity of weak solutions, the theory on its uniqueness and a mathematical study on specific behavior when solids and fluids are interacting. All effort was driven by two controversial scientific ambitions. The first direction was to make the theory of weak solutions accessible to more applications. This includes on the one hand more general solid materials, such as *largely deformable bulk solids* (see chapter 7) or *non-linear elastic shells* (see chapter 2), or on the other hand more general fluids as *heat conducting gases* (see chapter 3). The second direction was to produce a quantitative and accurate (local) description on the peculiar phenomena that appear when fluids are interacting with solids; to understand it *quantitatively* as well as *locally* (see chapter 8).

1.0.1 The setup of fluid-structure interactions

Typical for fluid-structure interaction is that the geometry of the Eulerian domain of definition for the fluid motion is a *part of the solution* and variable in time. We introduce $\Omega \subset \mathbb{R}^n$ as the Eulerian domain in which the interaction between the fluid and the solid (the structure) is happening.

The problems studied in the field of fluid structure interaction generally can be divided into three different classes by the types of solid used: Rigid objects, elastic shells which can either be a (moving) part of the boundary or a thin object in between two fluids and elastic objects with the same dimension as the fluid.



Please observe the *characteristic difficulty for fluid-structure interactions* stemming from the presence of two different coordinate systems. The *Eulerian coordinates*, natural to be used for the fluid and the *Lagrangian coordinates*, natural to be used for the solid. Respectively, the solid is commonly characterized via a coordinate map from its reference configuration Q. $\eta : [0,T] \times Q \to \Omega$ The fluid is commonly characterized by its velocity v and its pressure p. Both are defined in Eulerian coordinates over the possibly time dependent domain $\Omega(t) \subset \Omega$. This means in particular that $v : [0,T] \times \Omega(t) \to \mathbb{R}^n$ and $p : [0,T] \times \Omega(t) \to \mathbb{R}$. Typically they satisfy Navier-Stokes equations over the variable fluid domain. In the case of an incompressible Newtonain fluid this means that

$$\begin{aligned} \rho_f(\partial_t v + [\nabla v]v) &= \nu \Delta v - \nabla p + \rho_f f \quad &&\text{on } \Omega(t), \\ \mathrm{div} v &= 0 &&\text{on } \Omega(t), \end{aligned} \tag{1.0.1}$$

for all $t \in [0,T]$.

The coupling between fluid and solid is happening at the interface $\partial\Omega(t) \cap \overline{\eta(Q)}$ via *implicit boundary conditions*.

In this thesis we will assume no-slip boundary conditions at the interface, which reads

$$v(t, \eta(x)) = \partial_t \eta(t,x) \text{ for all contact points } \eta(t,x) \in \Omega. \tag{1.0.2}$$

Further, the forces of the fluid acting on the solid have to be in equilibrium with the respecting forces of the solid. This could mean, as is the case with rigid motions, that the total force of the fluid acting on the rigid body is equal to its momentum and as such summarizes in the (macroscopical) motion of the rigid body. In contrast, in the case of an elastic bulk solid the stresses of the fluid would equalize pointwisely on the interface the respective stresses of the elastic solid.

1.0.2 An overview on the related literature

Let us review some progress in fluid-structure interaction that was relevant for the work summarized in this thesis. We split the overview into several research sub-areas, as the progress in each of them is at different stages. we begin with the analysis for the three regimes of fluid-structure interactions, rigid bulk solid, elastic plates and elastic bulk solids. This follows some references on numerical results and some results in a fourth type of fluid-structure interaction, the filed of homogenization that is related to rigid body motions.The three analytic regimes of bulk, shell and rigid solids are all covered in the thesis. The references in the field of numerics as well as homogenization are included in order to give some examples of the variety of mathematical research questions related to fluid-structure interactions.

Literature on the analysis for rigid body motions: For rigid body motions in fluids many results, dating back to Archimedes, have been achieved already. We refer to [68, 76, 71, 178, 72, 73] for results on the existence of (periodic) weak solutions and some regularity estimates. See [177, 178, 84, 32] for some results concerning uniqueness questions. Interesting are the qualitative results [186, 100, 81, 104, 105] where it is shown that the contact between smooth rigid bodies in an incompressible fluid endowed with no-slip boundary conditions may not happen. In contrast in case of slip-boundary conditions or compressible fluids contacts may happen [81, 32]. Only recently it was shown that if elastic (bulk) solids are considered it is likely that solids do bounce off each other even in the absence of a topological contact which is discussed in chapter 8.

Literature on the analysis for elastic shells or plates: This is a very popular setting in fluid-structure interactions. The solid is assumed to be a shell or a plate that is modeled as a thin object of one dimension less than the fluid. For related up-to-date *modeling and model reductions* on *plates* and *shells* see [38] and the references therein. Well-posedness results for *weak solutions* with a *fixed prescribed scalar direction of displacement of the shell* is commonly shown *until a self-touching* of the solid is approached. For *incompressible Newtonian fluids* see [52, 53, 127, 153, 88, 149, 29] and for *incompressible non-Newtonian fluids* see [126]. In this work we discuss the *regularity for the solid deformations* in chapter 2. The first existence result of weak solutions in the compressible regime was shown in [22]. Built on chapter 2 and [22] is the work presented in chapter 3 where existence of a weak solution for heat conducting compressible fluids interacting with non-linear shells are constructed.

Quite recent is also the first weak-strong uniqueness result involving elastic plates which is discussed here in chapter 4. Additionally, we would like to mention that there are numerous existence and uniqueness results for short times [43, 12, 17] and some global results for small initial data [36, 111].

Literature on the analysis for elastic bulk solids: There is very little literature avaliable about this theme. Up to very recently, results have been restricted to *small deformations of the elastic solid*; see [86, 74] for *steady problems* and [164, 136] for *unsteady problems*. Only due to a recent effort that will be recapitulated in chapter 5–chapter 7 a theory involving *large deformations* was optained in [13] and is the content of chapter 5–chapter 7.

Literature on numerical methods in fluid-structure interactions: The literature on numerical strategies are numerous and we will only give a few examples with no ambition of completeness. For an overview we recommend the monograph [165]. Further numerical results (involving different strategies to approximate fluid-structure interactions) can be found here [158, 93, 162, 185, 167, 65, 168].

Literature on homogenization and fluid-structure interactions: Homogenization results involving fluids are limits of fluid-structure interactions. Overlap between the community on homogenisation and on fluid-structure interactions is via the case of rigid bodys interacting with fluids. Indeed, most of the literature considers domains where equisized obstacles are distributed periodically; hence the solid objects are merrily resisting the fluid without moving. The homogenization is then performed by simultaneously increasing the number of obstacles while decreasing their sizes. This represents a porous medium. See [179, 41, 2, 3, 61] for incompressible fluids and [60, 56, 132, 107] for compressible fluids. Further results include randomly distributed particles and possibly moving particles. See [166, 55, 83, 28, 106] and the references therein. For results concerning the motion of a viscous incompressible fluid interacting with a (fixed) periodically perforated wall (or sieve) see [41] and the references therein. See also the recent preprint [14], where a fluid-structure interaction modelling the flow of air through a porous mask is shown. Here the interaction at the elastic (porous) solid with the fluid is given as a Brinkman type friction term.

In the following we introduce the scientific content of the upcoming chapters.

1.1 Existence, uniqueness and regularity for weak solutions to fluid-structure interactions involving shells or plates chapter 2–chapter 4

In the first part of the habilitation we consider the solid to be deforming in one dimension less. Central for all three results is to use the dissipative properties of the fluid for the elastic solid (see Theorem 1.1.2 below). This key regularity estimate is demonstrated in Section 2.4 and relies on solenoidal extension operators and difference quotient techniques. It is remarkable that only the coupling to the fluid allows to deduce existence of the non-linear hyperbolic PDEs that are modelling the solid evolutions. Indeed, for some of the here considered non-linear hyperbolic PDEs the existence of a weak solution (not interacting with a fluid) has not been shown to exists even for smooth data.

This extra regularity is used in chapter 2 to show the existence of a solution to non-linear solid evolutions coupled to Navier-Stokes fluid. In chapter 3 it is used to produce an energy equality (which is a speciality to hold for this closed system of weak solutions). Finally in chapter 4 it is needed to show the a-priori finiteness of several appearing error terms.

In chapter 2 and chapter 3 we can treat the following non-linear model of an elastic shell.

1.1.1 The non-linear Koiter shell

In this work we consider the classic (non-linear) Koiter shell model (see e.g. [38, 117]) which describes the evolution of the elastic boundary of the fluid domain. Let $\Omega \subset \mathbb{R}^3$ be a domain such that its boundary $\Gamma = \partial\Omega$ is parameterized by a C^3 injective mapping $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$, where $\omega \subset \mathbb{R}^2$. [1]

We denote the tangential vectors at any point $\boldsymbol{\varphi}(y)$ in the following way:

$$\mathbf{a}_\alpha(y) = \partial_\alpha \boldsymbol{\varphi}(y),\ \alpha = 1, 2,\ y \in \omega.$$

[1]To simplify notation we assume that the boundary of Ω can be parameterized by a flat torus $\omega = \mathbb{R}^2/\mathbb{Z}^2$ which corresponds to the assumption of periodic boundary conditions for the structure displacement. We consider the periodic boundary conditions just to avoid unnecessary technical complications.

The unit normal vector is given by $\nu(y) = \dfrac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \times \mathbf{a}_2(y)|}$. The surface area element of $\partial\Omega$ is given by $dS = |\mathbf{a}_1(y) \times \mathbf{a}_2(y)|dy$. We assume that the domain deforms only in the normal direction and denote by $\eta(t, y)$ the magnitude of the displacement. This reflects the situation when the fluid pressure is the dominant force acting on the structure in which case it is reasonable to assume that the shell is deforming in normal direction. In this case the deformed boundary can be parameterized by the following coordinates:

$$\boldsymbol{\varphi}_\eta(t, y) = \boldsymbol{\varphi}(y) + \eta(t, y)\nu(y),\ t \in (0, T),\ y \in \omega. \tag{1.1.1}$$

We wish to emphasize that this restriction is rather standard in the majority of mathematical works on the analysis of weak solutions, mainly due to severe technical difficulties associated with the analysis of the case where the full displacement is taken into account. The deformed boundary is denoted by $\Gamma_\eta(t) = \boldsymbol{\varphi}_\eta(t, \omega)$. It is a well known fact from differential geometry (see e.g. [125]) that there exist $\alpha(\Omega), \beta(\Omega) > 0$ such that for $\eta(y) \in (\alpha(\Omega), \beta(\Omega))$, $\boldsymbol{\varphi}_\eta(t, .)$ is a bijective parameterization of the surface $\Gamma_\eta(t)$ and it defines a domain $\Omega_\eta(t)$ in its interior such that $\partial\Omega_\eta(t) = \Gamma_\eta(t)$. Moreover, there exists a bijective transformation $\boldsymbol{\psi}_\eta(t, .) : \Omega \to \Omega_\eta(t)$.[2]

We denote the moving domain in the following way:

$$(0, T) \times \Omega_\eta(t) := \bigcup_{t \in (0,T)} \{t\} \times \Omega_\eta(t).$$

The non-linear Koiter model is given in terms of the differences of the first and the second fundamental forms of $\Gamma_\eta(t)$ and Γ which represent membrane forces and bending forces respectively. These forces are summarized in its potential - the Koiter energy $\mathcal{E}_K(t, \eta)$. The definition of the potential is taken from [38, Section 4]. For a precise definition and the derivation of the energy for our coordinates see (2.2.14) below. Let $\mathcal{L}_K\eta$ be the L^2-gradient of the Koiter energy $\mathcal{E}_K(t, \eta)$, h be the (constant) thickness of the shell and ϱ_s the (constant) density of the shell. Then the respective momentum equation for the shell reads

$$\varrho_s h \partial_t^2 \eta + \mathcal{L}_K \eta = g, \tag{1.1.2}$$

where g are the momentum forces of the fluid acting on the shell.

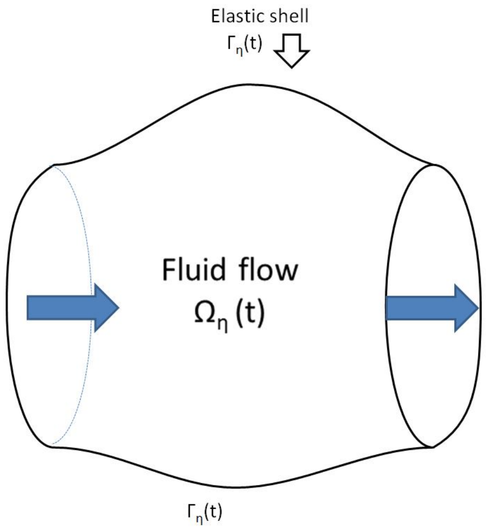


Figure 1.1: An example of the deformed cylindrical domain.

For more details and some model examples please see Subsection 2.2.2.

[2]For more details on the geometry see Section 2.2.2 and Definition 2.2.1.

1.1.2 Existence for non-linear shells and regularity for the shell evolution–chapter 2

This chapter is based on the work in collaboration with B. Muha from the University of Zagreb [142]. In this chapter we consider the non-linear Koiter shell model introduced above and couple it with a fluid flow that is governed by the incompressible Navier-Stokes equations:

$$\varrho_f\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\big)\mathbf{u} = \operatorname{div}\boldsymbol{\sigma}(\mathbf{u},p) \qquad \text{in } (0,T)\times\Omega_\eta(t), \tag{1.1.3}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } (0,T)\times\Omega_\eta(t), \tag{1.1.4}$$

where $\boldsymbol{\sigma}(\mathbf{u},p) = -p\mathbb{I} + 2\mu \operatorname{sym}\nabla\mathbf{u}$ is the fluid stess tensor and ϱ_f the (constant) density of the fluid.

The fluid and the structure are coupled via kinematic and dynamic coupling conditions. We prescribe the no-slip kinematic coupling condition which means that the fluid and the structure velocities are equal on the elastic boundary:

$$\mathbf{u}(t,\boldsymbol{\varphi}_\eta(t,y)) = \partial_t\eta(t,y)\nu(y), \quad y\in\omega. \tag{1.1.5}$$

The dynamic boundary condition states that the total force in the normal direction on the boundary is zero:

$$g(t,y) = -\boldsymbol{\sigma}(\mathbf{u},p)(t,\boldsymbol{\varphi}_\eta(t,y))\nu(\eta(t,y))\cdot\nu(y), \quad y\in\omega, \tag{1.1.6}$$

where $\nu(\eta(t,y)) = \partial_1\boldsymbol{\varphi}_\eta(t,y)\times\partial_2\boldsymbol{\varphi}_\eta(t,y)$ is defined as a weighted vector pointing in the direction of the outer normal to the deformed domain at point $\boldsymbol{\varphi}_\eta(t,y)$; the weight is exactly the Jacobian of the change of variables from Eulerian to Lagrangian coordinates.

We may summarize and state the full fluid-structure interaction problem.
Find $(\mathbf{u},\eta)$ such that

$$\begin{aligned}
\varrho_f\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\big)\mathbf{u} &= \operatorname{div}\boldsymbol{\sigma}(\mathbf{u},p) && \text{in } (0,T)\times\Omega_\eta(t),\\
\operatorname{div}\mathbf{u} &= 0 && \text{in } (0,T)\times\Omega_\eta(t),\\
\varrho_s h\partial_t^2\eta + \mathcal{L}_K\eta &= -(\boldsymbol{\sigma}(\mathbf{u},p)\circ\boldsymbol{\varphi}_\eta)\nu(\eta)\cdot\nu && \text{in } (0,T)\times\omega,\\
\mathbf{u}\circ\boldsymbol{\varphi}_\eta &= \partial_t\eta\nu && \text{in } (0,T)\times\omega,\\
\mathbf{u}(0,.) &= \mathbf{u}_0 && \text{in } \Omega_\eta(0),\\
\eta(0) = \eta_0,\ \partial_t\eta(0) &= \eta_1 && \text{in } \omega.
\end{aligned} \tag{1.1.7}$$

The solution of the above coupled system formally satisfies the following energy equality:

$$\frac{d}{dt}\Big(\frac{\varrho_f}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega_\eta(t))} + \frac{h\varrho_s}{2}\|\partial_t\eta(t)\|^2_{L^2(\omega)} + \mathcal{E}_K(t,\eta)\Big) = -2\mu\int_{\Omega_\eta(t)}|\operatorname{sym}\nabla\mathbf{u}|^2. \tag{1.1.8}$$

The main result of the chapter is the existence of a weak solution:

Theorem 1.1.1. *Assume that $\partial\Omega\in C^3, \eta_0\in H^2(\omega), \eta_1\in L^2(\omega)$ and $\mathbf{u}_0\in L^2(\Omega_{\eta_0})$, and η_0 is such that Γ_{η_0} has no self-intersection and $\overline{\gamma}(\eta_0)\neq 0$. Moreover, we assume that the compatibility condition $\mathbf{u}_0|_{\Gamma_{\eta_0}} = \eta_1\nu$ is satisfied. Then there exists a weak solution $(\mathbf{u},\eta)$ on the time interval $(0,T)$ to (1.1.7) in the sense of Definition 2.2.3.*

Furthermore, one of the following is true: either $T=+\infty$, or the structure self-intersect, or $\overline{\gamma}(\eta)\neq 0$, i.e. the H^2-coercivity of the structure energy degenerates, where $\overline{\gamma}$ is defined in Definition 2.2.1 below.

The second main theorem says that all possible solutions in the natural existence class satisfy better structural regularity properties.

Theorem 1.1.2. *Let $(\mathbf{u},\eta)$ be a weak solution to (1.1.7)[3]. Then the solution has the additional regularity property[4] $\eta\in L^2(0,T;H^{2+s}(\omega))$ and $\partial_t\eta\in L^2(0,T;H^s(\omega))$ for $s\in(0,\frac{1}{2})$. Moreover, it satisfies the following regularity estimate*

$$\|\eta\|_{L^2(0,T;H^{2+s}(\omega))} + \|\partial_t\eta\|_{L^2(0,T;H^s(\omega))} \le C_1$$

with C_1 depending on $\partial\Omega$, C_0 and the H^2-coercivity size $\overline{\gamma}(\eta)$.

[3] in the sense of Definition 2.2.3

[4] For the definition of the fractional Sobolev spaces $H^s(\omega)$ see Subsection 2.2.5

Remark 1.1.3 (Coercivity and non-linearity)**.** *Due to the fact that the Koiter shell equation is non-linear—more precisely since the curvature change is measured w.r.t. the deformed geometry—the H^2-coercivity of the Koiter energy can become degenerate. This is quantified by the estimate that is shown in Lemma 2.4.3 below. At such degenerate instant the given existence and regularity proofs break down. This is a phenomenon purely due to the non-linearity of the Koiter shell equations. Indeed, in case when the leading order term of the elastic energy is quadratic (i.e. the equation is linear or semi-linear), this loss of coercivity is a-priori excluded.*

Novelty & Significance

The main novelty is the improved regularity of the elastic displacement. In particular it allows to overcome the border between Lipschitz and non-Lipschitz domains. This critical step has caused a significant amount of effort in previous works [31, 87, 127, 144, 153]. The regularity uses classical differential quotient techniques applied to the non-linear structure equation. Problematic is the impact of the fluid on the structure which is rather implicit in the frame-work of weak equations. Here newly developed extension-operators are developed that are certainly of independent value (see Proposition 2.3.3). Of critical technical difficulty are commutator estimates for the time dependent extension of a difference quotient (see Lemma 2.3.5).

The power of the newly introduced method to gain higher regularity for the structure allows to prove the existence of weak solutions for fluids interacting with non-linear Koiter shells. These more physical models have not been in reach for the theory of weak solutions that may exists for arbitrary long times. The mathematical reason is that the respective energies are highly non-linear and non-convex. The extra regularity estimate however, allows to derive sequences that converge strongly to the solution w.r.t the highest order of the operator. That is the reason why no *linearity or convexity assumptions* are needed anymore to pass to the limit with in the non-linear stress tensor of the structure equation.

For previous results, the limit passage of the convective term in the Navier Stokes equation was the main effort [31, 87, 127, 144, 153]. The limit passage usually relies on compactness results of Aubin-Lions type. The variable geometry make its application highly technical. In Section 2.5 we rewrite the celebrated result in a form that we believe to be suitable for coupled systems (see Theorem 2.5.1). Indeed, it can be applied to systems where the solution space depends on the solution itself. This section can be seen as the second main technical novelty.

The interval of existence is potentially arbitrary large. The interval of existence is restricted to cases when the geometry degenerates. However, the minimal interval of existence depends on the reference geometry (which defined the shell model) and can be arbitrary large for some commonly used models. We demonstrate this by providing explicit bounds for two popular reference geometries in Subsection 2.2.2; namely the case when the reference geometry is a sphere or a cylinder.

The method seems very suitable to be adapted for further interaction problems. Possible future applications for fluid structure interactions problems are in the field of membrane energies, compressible fluids, tangential displacements, uniqueness issues and/or numerical analysis. In two space dimensions or in the regime of low Reynolds numbers the method inherits great potential to further improve the regularity theory for the fluid structure interactions problems [88]. However, due to the lack of the global regularity result for the fluid equations in three dimensions, we do not expect it is possible to further improve the regularity of the shell. In this sense our regularity result for the shell displacement can be viewed as optimal.

1.1.3 Existence for heat-conducting fluids interacting with non-linear shells–chapter 3

In this chapter the solid evolution is assumed to be the same as in the previous chapter. The fluid, however follows the Navier-Stokes-Fourier system: We then seek the velocity field $\mathbf{u} : I \times \Omega_\eta \to \mathbb{R}^3$, the density

$\varrho : I \times \Omega_\eta \to \mathbb{R}$ and the temperature $\vartheta : I \times \Omega_\eta \to \mathbb{R}$ solving the following system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \qquad \text{in } I \times \Omega_\eta, \tag{1.1.9}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div}\mathbf{S}(\vartheta, \nabla \mathbf{u}) - \nabla p(\varrho, \vartheta) + \varrho \mathbf{f} \qquad \text{in } I \times \Omega_\eta, \tag{1.1.10}$$

$$\begin{aligned}\partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}(\varrho e(\varrho, \vartheta)\mathbf{u}) &= \mathbf{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p(\varrho, \vartheta)\operatorname{div}\mathbf{u} \\ &\quad - \operatorname{div}\mathbf{q}(\vartheta, \nabla \vartheta) + \varrho H \qquad \text{in } I \times \Omega_\eta,\end{aligned} \tag{1.1.11}$$

$$\mathbf{u}(t, x + \eta(t,x)\nu(x)) = \partial_t \eta(t,x)\nu(x) \qquad \text{in } I \times \omega, \tag{1.1.12}$$

$$\partial_{\nu_\eta} \vartheta = 0 \qquad \text{on } I \times \Omega_\eta, \tag{1.1.13}$$

$$\varrho(0) = \varrho_0, \quad (\varrho \mathbf{u})(0) = \mathbf{q}_0, \quad \vartheta(0) = \vartheta_0 \qquad \text{in } \Omega_{\eta_0}. \tag{1.1.14}$$

Here we consider the volume force $\mathbf{f} : I \times \Omega_\eta \to \mathbb{R}^3$ and the heat source $H : I \times \Omega_\eta \to \mathbb{R}$. In (1.1.10) we suppose Newton's rheological law

$$\mathbf{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\Big(\frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2} - \frac{1}{3}\operatorname{div}\mathbf{u}\mathcal{I}\Big) + \lambda(\vartheta)\operatorname{div}\mathbf{u}\mathcal{I}$$

with strictly positive viscosity coefficients μ, λ (see Remark 1.3 in [22] for the case $\lambda \geq 0$). The internal energy (heat) flux is determined by Fourier's law

$$\mathbf{q}(\vartheta, \nabla \vartheta) = -(\vartheta)\nabla \vartheta = -\nabla \mathcal{K}(\vartheta), \quad \mathcal{K}(\vartheta) = \int_0^\vartheta (z)\ \mathrm{d}z \tag{1.1.15}$$

with strictly positive heat-conductivity . The thermodynamic functions p and e are related to the (specific) entropy s through Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta)D\Big(\frac{1}{\varrho}\Big) \quad \text{for all} \quad \varrho, \vartheta > 0. \tag{1.1.16}$$

The model case is given by

$$p(\varrho, \vartheta) = \varrho^\gamma + \frac{a}{3}\vartheta^4, \quad e(\varrho, \vartheta) = \frac{1}{\gamma - 1}\varrho^{\gamma - 1} + a\frac{\vartheta^4}{\varrho}, \quad s(\varrho, \vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\varrho},$$

for $a > 0$ and $\gamma > 1$. In view of Gibb's relation (1.1.16), the internal energy equation (1.1.11) can be rewritten in the form of the entropy balance

$$\partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}(\varrho s(\varrho, \vartheta)\mathbf{u}) = -\operatorname{div}\Big(\frac{\mathbf{q}(\vartheta, \nabla \vartheta)}{\vartheta}\Big) + \sigma + \varrho\frac{H}{\vartheta} \tag{1.1.17}$$

with the entropy production rate

$$\sigma = \frac{1}{\vartheta}\Big(\mathbf{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta}{\vartheta}\Big). \tag{1.1.18}$$

In the weak formulation (1.1.17) will be replaced by a variational inequality.
The shell should response optimally with respect to the forces, which act on the boundary. Therefore we have

$$\varepsilon_0 \varrho_S \partial_t^2 \eta + K'(\eta) = g + \nu \cdot \mathbf{F} \quad \text{in} \quad I \times \omega, \tag{1.1.19}$$

where $\varrho_S > 0$ is the density of the shell. Here, $g : I \times \omega \to \mathbb{R}$ is a given force and $\mathbf{F}$ is given by

$$\mathbf{F} := \big(-\boldsymbol{\tau}\nu_\eta\big) \circ \boldsymbol{\varphi}_{\eta(t)}|\det D\boldsymbol{\varphi}_{\eta(t)}|, \quad \boldsymbol{\tau} := \mathbf{S}(\nabla \mathbf{u}) - p(\varrho, \vartheta)\mathcal{I}.$$

Here, $\boldsymbol{\varphi}_{\eta(t)} : \omega \to \partial\Omega_{\eta(t)}$ is the change of coordinates from (2.2.1) and $\boldsymbol{\tau}$ is the Cauchy stress. To simplify the presentation in (1.1.19) we will assume

$$\varepsilon_0 \varrho_S = 1$$

throughout the paper. We assume the following boundary and initial values for η

$$\eta(0,\cdot) = \eta_0, \quad \partial_t\eta(0,\cdot) = \eta_1 \quad \text{in} \quad \omega, \tag{1.1.20}$$

where $\eta_0, \eta_1 : \omega \to \mathbb{R}$ are given functions. Here, we assume that

$$\mathrm{Im}(\eta_0) \subset (a,b).$$

In view of (1.1.12) we have to suppose the compatibility condition

$$\eta_1(y)\nu(y) = \frac{\mathbf{q}_0}{\varrho_0}(y + \eta(y)\nu(y)) \quad \text{in} \quad \omega. \tag{1.1.21}$$

Our main result is the following existence theorem. The system (1.1.9)–(1.1.21) can be written in a natural way as a weak solution. The concept is introduced in the next section, (see (3.2.14)–(3.2.17)), where also the precise formulation of our main result is presented (see Theorem 3.2.14). It is concerned with the existence of a weak solution up to degeneracy of the geometry and reads in a simplified version as follows.

Theorem 1.1.4. *Under natural assumptions on the data there exists a weak solution* $(\eta, \mathbf{u}, \varrho, \vartheta)$ *to* (1.1.9)–(1.1.20) *with satisfies the energy balance*

$$\begin{aligned}
\mathcal{E}(t) &= \mathcal{E}(0) + \int_{\Omega_\eta} \varrho H \,\mathrm{d}x + \int_{\Omega_\eta} \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x + \int_\omega g\,\partial_t\eta\ \mathrm{d}y,\\
\mathcal{E}(t) &= \int_{\Omega_\eta(t)} \Big(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)e(\varrho(t),\vartheta(t))\Big)\mathrm{d}x + \int_\omega \frac{|\partial_t\eta(t)|^2}{2}\ \mathrm{d}y + K(\eta(t)).
\end{aligned} \tag{1.1.22}$$

The interval of existence is of the form $I = (0,t)$*, where* $t < T$ *only in case* $\Omega_{\eta(s)}$ *approaches a self-intersection when* $s \to t$ *or the Koiter energy degenerates (namely, if* $\lim_{s\to t}\overline{\gamma}(s,y) = 0$ *for some point* $y \in \omega$*).*

The function space of existence for a weak solution to (1.1.9)–(1.1.20) is determined by the total energy $\mathcal{E}$ in (1.1.22) as well as the quantity σ in (1.1.18) taking into account the variable domain.

Novelty & significance

Theorem 3.2.14 extends the results from [22] to the case of a heat-conducting fluid but also applies to nonlinear structure equations. As in the case of fixed domains studied in [63] the heat-conducting model allows (different to the isentropic equations) the striking feature of an energy equality. Energy, which is lost by dissipation, is transfered into heat, cf. (1.1.11). There are few results for compressible fluids interacting with solids. In [22] the authors of the present chapter showed the existence of a weak solution to the compressible Navier–Stokes equations coupled with a linear elastic shell of Koiter type. Eventually, a similar result has been shown by a time-stepping method [184], where the interaction of a compressible fluid with a thermoelastic plate is studied (compare also with with the numeric results from [168]). Results on the short-time existence of strong solutions for compressible fluid models coupled with one-dimensional linear elastic structures can be found in [133, 141]. In [18] the author studies an elastic structure (with a regularised elasticity law) which is immersed into a compressible fluid and proves the existence of weak solutions to the underlying system. Results concerning the long-time existence of weak solutions about structure interactions with heat conducting are missing so far - even in the incompressible case. The existence of a unique local-in-time strong solution to compressible Navier–Stokes–Fourier system coupled with a damped linear plate equation has been established very recently in [134]. Of independent significance are regularity results for variable in time versions of damped equations for density and temperature (see Section 3.3).

1.1.4 Weak-Strong uniqueness for fluid-structure interactions involving elastic solids–chapter 4

The subject is technically quite involving. In particular in the incompressible set up. This is due to the fact that two domains of two solution velocities are different. In order to show that there difference is zero on has to do a change of variables from one domain to the other. However in order to not involving the pressure this change of variables has to be done in a way such that solenoidality is conserved. This can be done using a corrector function (a so-called Bogovskiĭ operator) or (what might be more natural) by the use of the Piola transform. This transform however is in general not so easy to handle. For that reason in this chapter we restrict ourselves to the case of an *elastic plate* which means that the reference geometry is flat and the direction of deformation orthogonal to the flat configuration. Please see Subsection 4.1.1 for the precise set up of this chapter.

The main result of the chapter consists in the *weak-strong uniqueness* of solutions for a flow in a variable 3D (or 2D) domain interacting with a 2D (or 1D) plate (see Theorem 4.1.2). While the regularity of the weak solutions that we use are known to be satisfied for all weak solutions we assume additional regularity of the *velocity* of the strong solution, that can be related (via its index) to the celebrated Ladyzhenskaya-Prodi-Serrin conditions [161, 171, 172, 121]. These are conditions for solutions to Navier-Stokes equations in a fixed domain that imply their smoothness and uniqueness.

Please observe, that we do not assume any additional regularity of the solid displacement; in particular the domain of the strong fluid-velocity is not even assumed to be uniformly Lipschitz continuous. In order to handle the limited regularity assumptions (on the strong solution) rather complex estimates where necessary. Some of them depend sensitively on *a-priori estimates* for the solid deformation shown in chapter 2.

To measure the distance between two solutions it is necessary to introduce a change of variables as the domains of the two velocity fields depend on the solution itself. Moreover, since the solid deformation is governed by a hyperbolic equation a mollification in time is unavoidable. In this paper a methodology is introduced that overcomes both obstacles with operators that conserve the property of solenoidality (see Lemma 4.2.6).

The weak-strong uniqueness result is a consequence of the following stability estimate.

Theorem 1.1.5. *Let* (v_2, η_2) *be weak solutions to the fluid-structure interaction* (4.1.1)-(4.1.9) *on* $[0,T]$*, such that* $\min_{[0,T]\times\omega} \eta_2 > 0$ *and that additionally* $v_2 \in L^r(0,T; W^{1,s}(\Omega_{\eta_2}))$ *and* $\partial_t v_2 \in L^2(0,T; \tilde{W}^{-1,r}(\Omega_{\eta_2}))$ *for any* $s > 3$ *and any* $r > 2$*. If* (v_1, η_1) *is a weak solution to* (4.1.1)-(4.1.9) *on* $[0,T]$*, then for* $\tilde{v_2}(t,x,y) = v_2(t,x,y\frac{\eta_1(t,x)}{\eta_2(t,x)})$ *we find that*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|(v_1 - \tilde{v}_2)(t)\|^2_{L^2(\Omega_{\eta_1(t)})} + \|\partial_t(\eta_1 - \eta_2)(t)\|^2_{L^2(\omega)} + \|(\eta_1 - \eta_2)(t)\|^2_{H^2(\omega)} \\
&\quad + \int_0^T \|(v_1 - \tilde{v}_2)(\tau)\|^2_{H^1(\Omega_{\eta_1(\tau)})} d\tau \\
&\le C(\|v_1^0 - \tilde{v}_2^0\|^2_{L^2(\Omega_{\eta_1^0})} + \|\eta_1^* - \eta_2^*\|^2_{L^2(\omega)}) + \|(\eta_1^0 - \eta_2^0)\|^2_{H^2(\omega)} \\
&\quad + C\int_0^T \|(f_1 - \tilde{f}_2)(\tau)\|^2_{H^1(\Omega_{\eta_1(\tau)})} + \|(g_1 - g_2)(\tau)\|^2_{L^2(\omega)} d\tau,
\end{aligned}
$$

where the constant depends on ω, T*, the assumed bounds on* v_2*, the* L^2*-bounds of* f_1, f_2 *and (symmetrically) on the two deformations* η_1, η_2 *via the bounds related to the energy estimates and via Theorem 4.2.2.*

In particular, the constant C can be bounded a-priori in dependence of ω, T, the assumed bounds on v_2 and the right hand side of the energy inequality (4.1.10) for both solutions. In some situations strong solutions are known to exist. In this case we proved that they are unique in the class of all weak solutions. (See Theorem 4.1.3).

Analytical strategy & technical novelties Usually for uniqueness (or stability estimates) one takes the difference of the two solutions or, in case of a hyperbolic evolution, its time-derivative as a test function. We wish to emphasize that due to the variable geometry *depending on the solution*, even uniqueness of strong solutions for longer times (provided they exist) does not follow in a straight forward manner. An additional difficulty regarding weak-strong uniqueness results is that the regularity of one solution is too low to be used as a test function. We follow the approaches developed in [177, 24, 32]. The idea is to resolve the difference of the systems tested by the difference of solutions into the energy inequality of the weak solution and terms containing a coupling where at least one function is sufficiently regular.

In order to make one fluid velocity a test function for the other equation we follow the methodology introduced in [92] where a change of variables from one geometry to the other is introduced that conserves the solenoidality property. This suffices to circumvent the weak regularity properties of the pressure in case of incompressible fluids.[5] What can not be circumvented is the *weak regularity of the time-derivative of the involved test-functions*. The *technical highlight* is a mollification-in-time operator that conserves solenoidality in the variable domains and the coupling of the boundary conditions. Moreover, it does not reduce the regularity (in space) significantly. The operator is introduced in Lemma 4.2.6. A result that might be of independent interest is that this mollification can be used to show that all weak solutions do indeed have a distributional time derivative in a Bochner space involving negative Sobolev spaces (see Proposition 4.2.7). Finally, of further use in the future might be the estimates (especially on the convective term) which were necessary in order to stay with our assumptions in the regime of the Ladyzhenskaya-Prodi-Serrin conditions.

In case of *compressible flows* weak-strong uniqueness is much less involving. Please see [183] for more details.

1.2 Bulk elastic solids. Variational strategies for fluid-structure interactions chapter 5–chapter 7

In this section we will discuss some results that have been achieved recently in [13]. Even so the results are all about the existence of weak solutions we hope that it will be clear from the constructive nature of the approach that it potentially is quite appropriate for numerical schemes and respective analysis.

1.2.1 Energies and dynamics

There are many complementary ways to perform modelling of dynamical continuum-mechanical systems. A common way to do so in modern mathematical treatments of the topic centers on the balance of forces/momentum. Beginning with Newton's second law, one adds up all the forces acting on each point of each object one seeks to describe. These then equal the change of momentum. Together with boundary data and possibly some auxiliary equations, such as conservation of mass, this balance then forms a system of PDEs. For solutions to this system one then seeks to derive properties, such as an energy (in)-equality.

In this article, we advocate for a different approach, beginning instead with an energy balance as our primary tool of modeling and deriving the balance of forces from it.[6] Consequently, variational methods can be applied replacing PDE-arguments. In particular in contrast to many of the methods employed when dealing with PDEs, these variational methods generally do not rely on linearity of forces or convexity of the admissible set of configurations.

Specifically, we are concerned with energy balances of the form

$$E_{\text{pot}}(T) + E_{\text{kin}}(T) + \int_0^T W_{\text{diss}}(t)dt = E_{\text{pot}}(0) + E_{\text{kin}}(0) + \int_0^T W_{\text{ext}}(t)dt$$

where we consider four quantities: Potential energy E_{pot}, kinetic energy E_{kin}, energy lost through dissipation W_{diss} and work done by external forces W_{ext}. The kinetic energy and the external forces will generally each always have a similar form, independent of the considered model. More interesting and highly dependent on the model one considers are the other two terms.

However, we are not entirely free in the choice of the two terms. At any given time, the current status of the system is given by the values of its variables. For dynamical problems in continuum mechanics, it helps to roughly split this into two parts, the state and the rate variables.

The *state variables* are those that describe the state of the continuum at any given time instant with examples being the deformation, density or pressure. Such variables are well defined without introducing a continuous time such as are time-derivatives. The dynamic, *rate variables* in contrast, consist of terms involving velocities

[5] In unsteady incompressible problems the pressure is known to be hard to control w.r.t. the time variable even in the simplest case of Stokes equation in a fixed (smooth) geometry [116].

[6] In modeling "energetic" approach has been advocated by many authors; e.g. [95] for solids or [192, 193, 191] for fluids. In the analysis an energetic approach is primal in many applications concerning solid materials; see e.g. the monographs [139, 119].

and accelerations. While the rate variables generally can be sought in a linear-space (possibly depending on the values of the state variables), the state variables in generally can not be expected to form a linear space.[7]

Now, by its very nature, the potential energy will only depend on the state variables. It can do so rather freely, usually only restricted due to mathematical reasons. The potential energy also induces an associated force, which is the negative formal gradient with respect to changes of the state; it "resists moving uphill".

In full contrast, the dissipation results in a force resisting movement. In the problems we consider, it is given as the formal gradient (with respect to the rate variables) of a dissipation functional. This functional can also depend on the current state variables, but more importantly for each fixed static state, it usually is a convex function of the rate variables with a global minimum in zero.[8]

Starting with the energy balance allows to use a method that begins at the same place. We construct a time-discrete iteration using a minimization involving the energy and the dissipation functional; consequently one gains a discrete estimate and an Euler-Lagrange equation. The latter will naturally become a discretized version of the momentum balance, with which one eventually may pass to the limit. Thus the momentum-equation is the result of a minimization and not a directly constructed object.

Certainly variational methods are not new in the context of continuum-mechanics. Whenever one is looking for stable stationary configurations, one is looking for local minimizers of the potential energy. Similarly, for quasi-static problems, where inertial effects are ignored, one can consider consecutive minimization of the sum of the potential energy and a "dissipation distance" to the last step. This is known as the method of minimizing movements (see e.g [48, 139, 119]). Both have been studied for a wide class of problems and since our method can be seen as an extension of the latter method, we can in fact build on these results.

Finally it should be noted that in all the methods described above, including ours, the minimization and thus the variational aspect happens in space, for a fixed instant of time. There are variational methods that work on a functional in space-time which could be seen as the grandfather of all motion by the principle of least action or more precisely principle of stationary action. In fact, in the case of continuum mechanics, the action generally cannot be expected to be minimal, but rather of saddle-point structure, which prevents the use of in-time variational methods.[9]

1.2.2 Time-delayed problems for hyperbolic systems–a simple example

The method for existence proofs we presented in chapter 5–chapter 7 can be explained in different ways. At one extreme it represents a way of a reduction principle. This means that its goal is to approximate the solutions to a second-order-in-time (hyperbolic) problem using solutions to a related curious first-order-in-time, (parabolic) problem. At the other extreme, it is a two-time scale extension of the well known time-incremental method of minimizing movements [48].

Fundamentally though, we need the following central observation about the energy balance for discrete-in-time approximations in case of a non-linear energy. Consider a simple toy model involving a single unit mass particle with position $x(t) \in \mathbb{R}^n$ and a potential energy $E(x(t))$. Hence, we seek the solution to the following hyperbolic ODE: $\partial_t^2 x = -\nabla E(x)$ with initial data $x(0) = x_0$ and $\partial_t x(0) = x^*$. The naive ansatz is to consider a time-discretization with step-size τ in order to approximate the solution. This provides the following implicit

[7] Note that any time-dependent change of a state variable involves a rate variable one, but there are dynamic phenomena involving rate variables, such as incompressible fluid flows, that might not change any state variables at all. The rate variables thus includes a "tangential space" of the state, but the two are not equivalent.

[8] In fact we will only consider the case where the dissipation functional is 2-homogeneous, as this is the most common case and avoids additional terms. Specifically if $R(\lambda b) = \lambda^2 b$, we know that $\dot{x} \cdot DR(\dot{x}) = 2R(\dot{x})$ and thus for such a functional the dissipated energy corresponds to twice the dissipation functional. For general convex dissipation functionals one would need to use the Legendre-transformation and gain an additional term.

[9] An easy way to see this is by noting that the action functional roughly consists of the difference between kinetic and potential energy. For a Lagrangian solid with deformation η, the kinetic energy generally will only depend coercively on the time derivative $\partial_t \eta$ and the potential energy only on the spatial derivatives $\nabla \eta$. So neither can be estimated against the other and adding small, quick oscillations in space/time can potentially greatly increase one of them without influencing the other by much.

equation

$$0 = \nabla E(x_{k+1}) + \frac{\frac{x_{k+1}-x_k}{\tau} - \frac{x_k - x_{k-1}}{\tau}}{\tau}. \tag{1.2.1}$$

But what can we say about the energy balance here? In particular, can we say anything about $E(x_{k+1}) - E(x_k)$? The standard approach would be to test with the discretized time derivative. This yields $\left\langle \nabla E(x_{k+1}), \frac{x_{k+1}-x_k}{\tau} \right\rangle$. If E is convex, this term can be used for a-priori estimates. But if it is not convex it can be that only in the limit $\tau \to 0$ we find an admissible term for estimates, namely $\frac{d}{dt}E(x) = \langle \nabla E(x), \partial_t x \rangle$. As long as $\tau > 0$, it is not clear whether and how this ”chain rule” can be imitated.

For the *parabolic* situation, when considering the first, instead of the second time-derivative, this problem has been solved by rewriting the implicit equation as a minimization of a functional. Instead of just solving $0 = \nabla E(x_{k+1}) + \frac{x_{k+1}-x_k}{\tau}$, one defines x_{k+1} to be the minimizer of the functional $\mathcal{F}_k : x \mapsto E(x) + \frac{1}{2\tau}|x - x_k|^2$. Then instead of using the equation, one can compare the values of this functional at x_{k+1} and x_k to get the summable estimate

$$E(x_{k+1}) + \frac{\tau}{2}\left|\frac{x_{k+1} - x_k}{\tau}\right|^2 = \mathcal{F}_k(x_{k+1}) \le \mathcal{F}_k(x_k) = E(x_k).$$

In the here considered *hyperbolic* case one first attempt would be to minimize the functional

$$\mathcal{F}_k(x) := E(x) + \frac{1}{2}\left|\frac{x - x_k}{\tau} - \frac{x_k - x_{k-1}}{\tau}\right|^2, \tag{1.2.2}$$

which is easily checked to have (1.2.1) as Euler-Lagrange equation. But here the estimate does not work as we get

$$E(x_{k+1}) + \frac{\tau^2}{2}\left|\frac{\frac{x_{k+1}-x_k}{\tau} - \frac{x_k - x_{k-1}}{\tau}}{\tau}\right|^2 = \mathcal{F}_k(x_{k+1}) \le \mathcal{F}_k(x_k) = E(x_k) + \frac{1}{2}\left|\frac{x_k - x_{k-1}}{\tau}\right|^2$$

with a term on the right hand side that turns out to have entirely the wrong scaling to estimate.[10]

The solution to this quandry is to note that there is no need for the two difference quotients in (1.2.1) to employ the same τ. We thus keep our step size τ in the first derivative and add an independent time-scale $h \gg \tau$. Accordingly we introduce the following two-scale minimization. The trick is to first construct an approximation $x^h : [0, h] \to \mathbb{R}$ as a gradient flow (under forcing) satisfying

$$\nabla E(x(t)) = -\frac{\partial_t x(t) - x^*}{h}, \quad x(0) = x_0.$$

Indeed, we may minimize iteratively for $k \in \{0, ..., \lfloor \frac{h}{\tau} \rfloor\}$

$$\mathcal{F}_k^0(x) := E(x) + \frac{\tau}{2h}\left|\frac{x - x_k^0}{\tau} - x^*\right|^2.$$

The above implies a uniform-in-τ estimate by a telescope sum

$$E(x_{k+1}^0) + \frac{\tau}{2h}\left|\frac{x_{k+1}^0 - x_k^0}{\tau} - x^*\right|^2 \le E(x_k^0) + \frac{\tau}{2h}|x^*|^2.$$

This allows to pass with $\tau \to 0$ and to construct x^h over the first h-interval $[0, h]$. Then we can interatively prolong x^h from $[0, (\ell - 1)h]$ to $[0, \ell h]$ by minimizing

$$\mathcal{F}_k^\ell(x) := E(x) + \frac{\tau}{2h}\left|\frac{x - x_k^\ell}{\tau} - \fint_{k\tau}^{(k+1)\tau} \partial_t x^h((\ell - 1)h + s)\, ds\right|^2$$

[10] This is not surprising, as we are comparing a proper approximately inertial solution with one that suddenly stops. A better competitor might be the “straight continuation” $x_k + \tau(x_k - x_{k+1})$, but then the estimate again requires convexity to deal with the energy-term.

where we use $\fint_A \cdot\, dt = \frac{1}{|A|}\int_A \cdot\, dt$ as the mean value integral for sets of finite measure A.

As a result we get what we will call a time-delayed solution, satisfying

$$\nabla E(x(t)) = -\frac{\partial_t x(t) - \partial_t x(t-h)}{h} \tag{1.2.3}$$

for $t \in [0,T]$. We can test this time-delayed solution with $\partial_t x(t)$ and find the *hyperbolic a-priori estimate*

$$\begin{aligned} E(x(b)) - E(x(a)) &= -\int_a^b \left\langle \frac{\partial_t x(t) - \partial_t x(t-h)}{h}, \partial_t x(t) \right\rangle dt \\ &\le -\frac{1}{2}\fint_{b-h}^{b} |\partial_t x(t)|^2 dt + \frac{1}{2}\fint_{a-h}^{a} |\partial_t x(t)|^2 dt, \end{aligned}$$

whenever the solution was constructed over $[a-h, b]$.[11]

The above gives us a good estimate on $E(x(t))$ and an averaged time-derivative, independent of h which allows for sending $h \to 0$ in (1.2.3).

The here explained approach, turns out to be admissible for infinite-dimensional spaces instead of $\mathbb{R}^n$ and even coupling between Eulerian and Lagrangian coordinates. For that Eulerian-Lagrangian coupling however additional difficulties appear that require some novel ideas on its own. This will be discussed in the forthcoming sections.

Remark 1.2.1 (Numerical use of the method)**.** *Since numerous numerical schemes for minimization (over discrete spaces) are available the above methodology might also be attractive for computational mathematics. The idea would here be to do a two-scale approximation: This means that once $x_{k+1}^{\ell-1}, x_k^{\ell-1}$ and x_k^{ℓ} are constructed. Then we can define x_{k+1}^{ℓ} as the minimizer of*

$$\mathcal{F}_k^{\ell}(x) := E(x) + \frac{\tau}{2h}\left|\frac{x - x_k^{\ell}}{\tau} - \frac{x_{k+1}^{\ell-1} - x_k^{\ell-1}}{\tau}\right|^2.$$

In order to pass to the limit it is in general unavoidable to use the hyperbolic structure on a time-continuous level. This means that first $\tau \to 0$ and only afterwards $h \to 0$. The question is how much smaller *does τ needs to be? One observes quickly, that in case E is convex τ and h can be chosen arbitrarily. Hence, the smallness of τ in relation should depend on the* non-convexity *of the assumed energies. In a forthcoming paper we hope to investigate this issue further.*

1.2.3 A quasistatic fluid-structure interaction, chapter 5

This is a simplified fluid-structure interaction that was treated in [13] and involves largely deformable elastic bulk solids. This means $Q \subset \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$. We look for $\eta : [0,T] \times Q \to \Omega$, $\Omega(t) = \Omega \setminus \eta(t,Q)$, $v : [0,T] \times \Omega(t) \to \mathbb{R}^n$ and $p : [0,T] \times \Omega(t) \to \mathbb{R}$ satisfying

$$\begin{aligned} \operatorname{div}\sigma(\eta) &= \rho_s f \circ \eta && \text{in } Q, && (1.2.4)\\ 0 &= \nu\Delta v - \nabla p + \rho_f f && \text{on } \Omega(t), && (1.2.5)\\ \operatorname{div} v &= 0 && \text{on } \Omega(t), && (1.2.6) \end{aligned}$$

Here, σ is the *first Piola–Kirchhoff stress tensor* of the solid, ν is the *viscosity constant* of the fluid, ρ_s and ρ_f are the *densities* of the solid and fluid respectively and f is the actual applied force in the current (Eulerian) configuration. Thus, the fluid is assumed to be Newtonian with the *steady Stokes equation* modeling its behavior. For the solid, we consider a material for which the first Piola–Kirchhoff stress tensor σ can be derived from underlying *energy and dissipation potentials*; i.e.

$$\operatorname{div}\sigma := DE(\eta) + D_2R(\eta, \partial_t \eta) \tag{1.2.7}$$

[11] Actually in the construction procedure the hypoerbolic a-priori estimate should be used for each ℓ to guarantee that the h-dependent a-priori estimate possesses a uniform upper bound.

with E being the energy functional describing the elastic properties while R is the dissipation functional used to model the viscosity of the solid. Here D denotes the Fréchet derivative and D_2 the Fréchet derivative with respect to the second argument. Such materials are often called *generalized standard materials* [95, 155, 119]. The results in [13] is for quite general forms of E and R. The assumptions are inspired by the following prototypical examples for E and R:

$$R(\eta, \partial_t \eta) := \int_Q |(\nabla \partial_t \eta)^T \nabla \eta + (\nabla \eta)^T (\nabla \partial_t \eta)|^2 dx = \int_Q |\partial_t (\nabla \eta^T \nabla \eta)|^2 dx \tag{1.2.8}$$

$$E(\eta) := \begin{cases} \int_Q \frac{1}{8} |\nabla \eta^T \nabla \eta - I|_{\mathcal{C}} + \frac{1}{(\det \nabla \eta)^a} + \frac{1}{q} |\nabla^2 \eta|^q dx & \text{if } \det \nabla \eta > 0 \text{ a.e. in } Q \\ +\infty & \text{otherwise} \end{cases} \tag{1.2.9}$$

where we use the notation $|\nabla \eta^T \nabla \eta - I|_{\mathcal{C}} := \big(\mathcal{C}(\nabla \eta^T \nabla \eta - I)\big) \cdot \big(\nabla \eta^T \nabla \eta - I\big)$, with $\mathcal{C}$ being a positive definite tensor of elastic constants, $q > n$ and $a > \frac{qn}{q-n}$.

Notice that in (1.2.9) the first term corresponds to the Saint Venant-Kirchhoff energy, the second models the resistance of the solids to infinite compression and the last is a regularisation term.

The coupling of the fluid and the solid are via their common interface. We consider no-slip boundary conditions and a force balance

$$v(t, \eta(x)) = \partial_t \eta(t, x) \qquad \text{in } [0, T] \times \partial Q, \tag{1.2.10}$$

$$\sigma(t, x) n(x) = \big(\nu \varepsilon v(t, \eta(t, x)) + p(t, \eta(t, x)) I\big) \hat{n}(t, \eta(t, x)) \qquad \text{in } [0, T] \times \partial Q, \tag{1.2.11}$$

where $n(x)$ is the unit normal to Q while $\hat{n}(t, \eta(t, x)) := cof(\nabla \eta(t, x)) n(x)$ is the weighted normal transformed into to the Eulerian configuration and $\varepsilon v := \nabla v + (\nabla v)^T$ is the symmetrized gradient.[12] Additionally, there are second order Neumann-type boundary conditions for the deformation η arising from the second order gradient in its energy.

The main result of the chapter is the following theorem.

Theorem 1.2.2. *Under appropriate initial condition η_0 there exists a weak solution of* (1.2.4)–(1.2.11). *This means in particular that there are η, v, p which satisfy*

$$\begin{aligned} \int_0^T \langle DE(\eta), \phi \rangle + \langle D_2 R(\eta, \partial_t \eta), \phi \rangle + \langle \varepsilon v, \varepsilon \xi \rangle_{\Omega(t)} - \langle p, \operatorname{div} \xi \rangle \, dt \\ = \int_0^T \rho_f \langle f, \xi \rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \phi \rangle_Q \, dt \end{aligned} \tag{1.2.12}$$

for all $\phi \in C^\infty([0, T] \times Q; \mathbb{R}^n)$ and $\xi \in C_0^\infty([0, T] \times \Omega; \mathbb{R}^n))$ such that $\phi(t, x) = \xi(t, \eta(t, x))$ on $[0, T] \times Q$.

Moreover, $\lim_{t \to 0} \eta(t) = \eta_0$ in an appropriate sense. Here the maximal time of existence T is restricted only in case a collision of the solid with itself or $\partial \Omega$ is approached, or if the elastic energy approaches infinity in finite time (a phenomenon that can not be excluded a-priori in the quasi-steady case).

The key part here is to show how geometrically coupled PDEs with coupled Dirichlet boundary values can be approximated variationally. The construction of a weak solution to (1.2.4)-(1.2.6) is via an implicit-explicit time-discretization scheme that exploits the *variational structure* of the problem. Recall that we consider the steady Stokes operator whose solutions are minimizers over the square integral of the symmetric gradient over solenoidal functions. Hence both the stress tensor for the solid as well as of the fluid are related to some potential.

Indeed, let us split $[0, T]$ into N equidistant time steps of length τ. Assume, for $k \in \{0, \ldots, N-1\}$, that η_k is given and denote $\Omega_k = \Omega \setminus \eta_k(Q)$. We then define η_{k+1}, v_{k+1} to be the solution of the following minimization problem

$$E(\eta) + \tau R\left(\eta_k, \frac{\eta - \eta_k}{\tau}\right) + \frac{\tau \nu}{2} \|\nabla v\|_{\Omega_k}^2 - \tau \rho_s \left\langle f \circ \eta_k, \frac{\eta - \eta_k}{\tau} \right\rangle_Q - \tau \rho_f \langle f, v \rangle_{\Omega_k} \longrightarrow \min. \tag{1.2.13}$$

[12] Here we consider a solid floating in a fluid. This is merely for the sake of abbreviation. It is however possible to assume that parts of the solid are fixed or even attached to the boundary of Ω at some parts.

This is the place to specify the *space* over which the minimization needs to be performed. It turns out to be a sensitive point, which we want to discuss in a bit more detail here. Essentially we have to take into account the following three aspects:

(i) The potential energy of the solid E, defined in (1.2.9) has to be well defined. Hence as an underlying function space one should consider $W^{2,q}(Q)$. However, this does not suffice. We further have to take into account the important non-convex restriction that the determinant should not be negative; in our case this means that we will only consider states of finite energy. What turns out to be a good choice for the state space of η is the following set that is in coherence with the celebrated *Ciarlet-Nečas condition* proposed in [39]:

$$\mathcal{E} := \left\{ \eta \in W^{2,q}(Q;\Omega) \,:\, E(\eta) < \infty,\, |\eta(Q)| = \int_Q \det \nabla\eta \, dx \right\}.$$

Here, the finite energy guarantees local injectivity and that any C^1-local homeomorphism is globally injective except for possible touching at the boundary. This space is also a valid state space for the dissipation potential R of the solid deformation.

(ii) The dissipation potential for the fluid is the standard Stokes potential. Hence the natural space is the space of divergence free functions $\mathcal{V}_k := \{v \in W^{1,2}(\Omega_k) \,:\, \mathrm{div} v = 0,\, v|_{\partial\Omega} = 0\}$. The respective fluid pressure then appears as the according Lagrange multiplier.

(iii) Finally the coupling condition at the common interface has to be captured. It turns out that it suffices to prescribe coupling of the common Dirichlet boundary values (1.2.10). Indeed, the second coupling condition is then an automatic consequence of the coupled weak formulation (certainly in a weak sense only). Consequently we require that $(\eta, v) \in \mathcal{E} \times \mathcal{V}_k$, such that the time-discrete speed of the deformation equals to the velocity of the fluid:

$$v(\eta_k(x)) = \frac{\eta(x) - \eta_k(x)}{\tau} \text{ for a.e. } x \in \partial Q.$$

It turns out that the minimization implies a natural approximation of (1.2.12). Indeed, let (η_{k+1}, v_{k+1}) be a minimizer and $\phi \in C^\infty(Q;\mathbb{R}^n)$ as well as $\xi \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$ with $\mathrm{div}\xi = 0$. Then we can use the perturbation $(\eta_{k+1}^{(\tau)} + \varepsilon\phi, v_{k+1}^{(\tau)} + \varepsilon\xi/\tau)$ as a competitor provided that $\xi \circ \eta_k = \phi$. This implies the following Euler-Lagrange equation:

$$\begin{aligned} &\langle DE(\eta_{k+1}), \phi\rangle + \left\langle D_2 R\left(\eta_k, \frac{\eta_{k+1} - \eta_k}{\tau}\right), \phi \right\rangle + \nu \left\langle \varepsilon v_{k+1}, \nabla\xi\right\rangle_{\Omega_k} \\ &= \rho_f \left\langle f, \xi\right\rangle_{\Omega_k} + \rho_s \left\langle f \circ \eta_k, \phi\right\rangle_Q . \end{aligned} \tag{1.2.14}$$

for any $\phi \in C^\infty(Q;\mathbb{R}^n)$ and $\xi \in C^\infty(\Omega_k;\mathbb{R}^n)$, $\mathrm{div}\xi = 0$, $\xi|_{\partial\Omega} = 0$ such that $\xi \circ \eta = \phi$ in ∂Q. The fact that $DE(\eta_{k+1})$ is in a valid negative Sobolev space is certainly relying on estimates of Healy-Krömer type [96] (which again relies on the choices of a, q).

Searching for time-discrete approximations of the weak solution to (1.2.4) alone via a minimization problem similar to the one above is actually well known and heavily used in the mathematics of continuum mechanics of solids (see e.g. [119]). The method is known as the *method of minimizing movements* or, in particularly in the engineering literature, also called the *time-incremental problem.* As far as the authors are aware this method has not been applied to the theory of fluid-structure interaction problems before.

The advantage of the variational approach in contrast to directly solving the corresponding Euler-Lagrange equations is twofold. Not only do we deal with the non-convexity of E and the underlying non-convex space $\mathcal{E}$ in a natural way, but also we automatically gain an *energetic a-prori estimate*. Indeed comparing the value of the functional in (1.2.13) in (η_{k+1}, v_{k+1}) with its value for $(\eta_k, 0)$ and iterating, we get the following (quantitatively optimal) estimate of energy and dissipation

$$\underbrace{E(\eta_{k+1})}_{\text{Final energy}} + \underbrace{\sum_{l=0}^{k} \tau \left[R\left(\eta_l, \frac{\eta_{l+1} - \eta_l}{\tau}\right) + \frac{\nu}{2} \|\varepsilon v_{l+1}\|^2_{\Omega_k} \right]}_{\text{1/2 of Dissipation}} \tag{1.2.15}$$

$$\leq \underbrace{E(\eta_0)}_{\text{Initial energy}} + \underbrace{\sum_{l=0}^{k} \tau \left[\rho_s \left\langle f \circ \eta_l, \frac{\eta_{l+1} - \eta_l}{\tau} \right\rangle_Q + \rho_f \langle f, v_{l+1} \rangle_{\Omega_k} \right]}_{\text{Work from forces}}.$$

This estimate suffices to pass to the limit after overcomming certain technical difficulties. In particular a Korn's inequaliy estimating fluid and solid velocity simultaneously is introduced and a subtle approximation of test-functions. The latter is necessary due to the fact that the fluid-domain (the part where the test-function is supposed to be solenoidal) is a part of the solution. This allows to pass to the limit with (1.2.14) and derive (1.2.12).

1.2.4 Minimizing movements for solids involving inertia, chapter 6

In this part we extend the example given in Subsection 1.2.2 to infinite dimensions. Let us consider a mere solid deformation that is not interacting with a fluid. Namely we seek $\eta : [0,T] \times Q \to \mathbb{R}^n$, evolving according to

$$-DE(\eta) - D_2R(\eta, \partial_t \eta) - f \circ \eta = \rho_s \partial_t^2 \eta \tag{1.2.16}$$

with appropriate prescribed boundary and initial conditions

Following the finite dimensional example, we aim to turn this hyperbolic problem into a sequence of short time consecutive parabolic problems. First one replaces $\partial_t^2 \eta$ with a difference quotient and solving what we call the *time-delayed problem*

$$DE(\eta(t)) + D_2R(\eta(t), \partial_t \eta(t)) - f \circ \eta(t) = \frac{\partial_t \eta(t) - \partial_t \eta(t-h)}{h} \tag{1.2.17}$$

for a given $h > 0$.

Considered on a short interval of length h, the term $\partial_t \eta(t-h)$ can be seen as fixed given data. Then on this interval the problem is parabolic and is solved using the minimizing movements approximation as to what was described before; meaning for fixed h we pick $\tau << h$ and iteratively minimize

$$E(\eta) + \tau R\left(\eta_k, \frac{\eta - \eta_k}{\tau}\right) - \tau \rho_s \left\langle f \circ \eta_k, \frac{\eta - \eta_k}{\tau} \right\rangle_Q + \frac{1}{2h} \left\| \frac{\eta - \eta_k}{\tau} - \partial_t \eta(\tau k - h) \right\|^2 \longrightarrow \text{min.}$$

Upon sending $\tau \to 0$ using the same techniques as before, we then obtain a weak solution to (1.2.17) on $[0,h]$ which can be used as data on $[h, 2h]$ and so on, until we have derived a time-delayed solution on $[0,T]$.

It is important to note that the a-priori estimate obtained via the minimization is dependent on h. Only after passing with $\tau \to 0$ the chain rule can be used, to provide an h-independent a-priori estimate. In particular one has to ensure that the following chain rule can be made rigorous $\partial_t E(\eta) = \langle DE(\eta), \partial_t \eta \rangle_Q$ which then leads to the following a-priori estimate

$$E(\eta(t)) + \rho_s \fint_{t-h}^{t} \frac{\|\partial_t \eta(s)\|_Q^2}{2} ds + \int_0^t 2R(\eta, \partial_t \eta) ds \leq E(\eta_0) + \rho_s \frac{\|\eta_*\|_Q^2}{2} + \int_0^t \langle f \circ \eta, \partial_t \eta \rangle_Q \, ds,$$

where η_0 and η^* are the given initial conditions for η and $\partial_t \eta$ respectively.

The estimate then allows to construct a weak solution to (1.2.16), via weak compactness, Aubin-Lions lemma and the Minty method.

1.2.5 Bulk elastic solids coupled to Navier-Stokes equations, chapter 7

In this chapter we describe the main result of [13] which is the existence of a bulk solid interacting with the incompressible system of Navier-Stokes equations:

$$\begin{aligned} \rho_s \partial_t^2 \eta + \operatorname{div}\sigma &= \rho_s f \circ \eta && \text{in } Q, \\ \rho_f(\partial_t v + [\nabla v] v) &= \nu \Delta v - \nabla p + \rho_f f && \text{on } \Omega(t), \\ \operatorname{div} v &= 0 && \text{on } \Omega(t) \end{aligned}$$

with coupling conditions (1.2.10) and (1.2.11) and σ satisfying (1.2.7).

Theorem 1.2.3. *Under appropriate conditions on the initial values v_0, η_0, η^* for $v, \eta, \partial_t\eta$ the right hand side f and the domains Ω and Q, there exists a weak solution to the above until the point of collision of the solid with itself or $\partial\Omega$. This means in particular that there are η, v, p satisfying*

$$\begin{aligned} &\int_0^T -\rho_s \langle \partial_t \eta, \partial_t \phi \rangle_Q - \rho_s \langle v, \partial_t \xi - v \cdot \nabla \xi \rangle_{\Omega(t)} + \langle DE(\eta), \phi \rangle + \langle D_2 R(\eta, \partial_t \eta), \phi \rangle + \nu \langle \varepsilon v, \varepsilon \xi \rangle_{\Omega(t)} \, dt \\ &= \int_0^T \langle p, \operatorname{div} \xi \rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \phi \rangle_Q + \rho_f \langle f, \xi \rangle_{\Omega(t)} \, dt - \rho_s \langle \eta_*, \phi(0) \rangle_Q - \rho_f \langle v_0, \xi(0) \rangle_{\Omega(0)} \end{aligned}$$

for all $(\phi, \xi) \in C^\infty([0,T] \times Q) \times C^\infty([0,T]; C_0^\infty(\Omega; \mathbb{R}^n))$ satisfying $\xi(T) = 0$, $\phi(t) = \xi(t) \circ \eta(t)$ on Q for all $t \in [0,T]$ and $\partial_t \eta(t) = v(t) \circ \eta(t)$ on ∂Q.

The main obstacle here lies in the Eulerian description of the fluid. Here it turns out to be natural to approximate the *material derivative of the fluid velocity* $(\partial_t v + [\nabla v] v)$ by a time-discrete difference quotient. This is done by subsequently introducing a flow map $\Phi_s(t) : \Omega(t) \to \Omega(t+s)$ fulfilling $\partial_s \Phi_s(t,y) = v(t+s, \Phi_s(t,y))$ (resp. a discrete version of this) and $\Phi_0(t,y) = y$ in both the discrete and the time-delayed approximation layers. This means that Φ *transports* the domain of the fluid along with its velocity.

In particular the fluid analogue of the difference quotient in the time-delayed problem will be a "material difference quotient" in the size of the acceleration scale h, which is essentially of the form

$$\frac{v(t, \Phi_h(t-h, y)) - v(t-h, y)}{h}.$$

As Φ and v are inseparably linked we need to construct their discrete counterparts alongside each other already in the τ scale. This discrete construction of the highly nonlinear Φ and its subsequent convergence are explained in more detail in the existence proof shown in chapter 7.

We wish to mention that the minimizing movements method has been previously used to show existence of solutions to the Navier-Stokes equation for fixed domains. In particular we want to highlight [82] as an inspiration. There the authors also employ flow maps to obtain the material derivative, but as they work on a fixed domain, they do not need to construct them iteratively but can instead rely on the respective existence theory for the Stokes-problem. As an indirect consequence, their minimization happens on what we would consider the h-level, which makes it incompatible with our way of handling the solid evolution. Thus, the scheme proposed here is more of an improvement of the numerical scheme [159] which has been developed much earlier. Also the recent variational work on compressible Euler equation [30] is related. The latter might be a starting point to show fluid-structure interactions involving bulk solids also in the compressible regime.

1.2.6 Outlook: Variational approaches for shells interacting with fluids

In order to provide a connection between the chapters 2–4 with the chapters 5–7 we discuss a work which is currently in preperation [115]. There we use variational methods to construct solutions for a fluid structure interaction involving a plate/shell; which means an elastic object of one dimension less then the fluid. Consequently the fluid stresses become a right hand side for the deformation of the thin object. Let us recall that all previous work on weak solution is restricted to settings, where the fluid domain is a sub-graph for some unchanged reference manifold; most of the works allowing deformation of the structure to be scalar with respect to a given fixed direction (see for instant the recent contributions [153, 29, 142] and chapter 2–chapter 4).

In the work [115] we overcome this obstacle by relying again on the variational method. For the sake of a clearer explanation let us consider the following simple set-up. A beam that forms the top of a 2D canister filled with a fluid governed by the Navier-Stokes equation. The reference cube is hence $[0,\ell]\times[0,1]$.

$$\eta:[0,T]\times[0,\ell]\to\Gamma(t)\subset\mathbb{R}^2 \text{ injective, such that } \eta(t,0)=(0,1)^T=\eta(t,\ell) \text{ and}$$

consequently the variable fluid domain $\Omega(t)$ is the area enclosed by $\Gamma(t)$ an the box underneath. In order to allow for large deformations in all spatial coordinate directions we have to consider an elastic energy that penalizes 1) stretching of the beam, 2) bending of the beam and 3) compression of the beam. The latter one is excluded a-priori in case the deformation is restricted to a fixed coordinate direction. A simple model example that penalizes the deformation with respect to a flat reference configuration is given by

$$\mathcal{E}_K(\eta):=\int_0^l\left(c_{stretch}|\partial_x\eta_1-1|^2+\frac{c_{compr}}{|\partial_x\eta_1|^{2\alpha}}+c_{curv}\frac{|\partial_{xx}\eta_2|^2}{2}\right)dx,$$

with material constants $c_{stretch},c_{compr},c_{curv},\alpha>0$. The fluid structure interaction hence becomes

$$\rho_s\partial_t^2\eta+D\mathcal{E}_K(\eta)=\rho_s f\circ\eta+\big(\nu\varepsilon v(t,\eta(t,x))+p(t,\eta(t,x))I\big)\hat{n}(t,\eta(t,x)) \quad \text{in } [0,T]\times[0,\ell], \quad (1.2.18)$$

$$\rho_f(\partial_t v+[\nabla v]v)=\nu\Delta v-\nabla p+\rho_f f \quad \text{on } \Omega(t), \quad (1.2.19)$$

$$\operatorname{div} v=0 \quad \text{on } \Omega(t), \quad (1.2.20)$$

$$\partial_t\eta=v(t,\eta(t,x)) \quad \text{in } [0,T]\times[0,\ell], \quad (1.2.21)$$

recall here, that $\hat{n}(t,\eta(t,x))$ is the variable in time Eulerian normal-direction on $\Gamma(t)$. This system is closed by equipping it with according initial and boundary conditions.

Unfortunately even for this simple example the elastic energy on its own does not produce enough regularity to show that the respective (step-wise) minimizaters are injective, after very short times. Hence for the time being we have to include a regularizing term in the energy of the type $\epsilon_0\frac{|\partial_{xx}\eta_1|^2}{2}$. However, we do not require any dissipative terms acting on the solid–hence considering (on its own) a hyperbolic solid evolution. Or put in different words the dissipation of the fluid suffices to obtain a weak solution.

Theorem 1.2.4 (To appear in [115])**.** *Assume that the additional regularizer $\epsilon_0\frac{|\partial_{xx}\eta_1|^2}{2}$ is part of the elastic energy $\mathcal{E}_K$, then there exists a weak solution to* (1.2.18)–(1.2.21) *until η touches the bottom of the container.*

This theorem is the starting point for many interesting further developments. First, the extra regularizer is used here merely to guaranty a minimal interval of existence for any injective initial geometry. At this point the question of existence is closely related to regularity results and to no-contact results. We wish to discuss that matter a little here:

As was mentioned before in [88] the authors were able to apply both regularity and no-contact theory (building on [100]) to show that global in time strong solution exists for a 1D beam interacting with a 2D fluid of Navier Stokes type. However, on the one hand the deformation is scalar there [13]. On the other hand the beam was assumed to be dissipative. It is somehow peculiar that the theory of global strong solutions could not be transferred to purely hyperbolic solid evolutions–indeed, for the scalar set-up global weak solutions are known to exists [29] but the regularity is an open problem up to date.

In discussion with B. Muha we came to the following possibility: The regularity could be missing due to the nonphysical restriction of scalar deformation of the structure in a prescribed direction. Hence we finish this section with the following conjecture.

Conjecture: *There exists a global strong solution for the fluid structure interaction* (1.2.18)–(1.2.21).

[13] In explicit $\eta_1(t,x)=x$, hence the deformation of the beam is only with respect to the prescribed x_2 direction.

1.2.7 Further potential applications of the variational method

More general material laws for fluids: Within chapter 5 and chapter 7 we cover incompressible Newtonian fluids. However, more general fluid laws could be considered with the main requirement being that the stress tensor posesses a potential that can be used in place of $\nu\|\varepsilon v\|^2$. Examples include non-Newtonain incompressible fluids (power law fluids), or even Newtonian compressible fluids, for which fluid-structure interactions involving simplified elastic models have alread been studied [126, 22]. The latter feature the added difficulty of density as an additional state variable.

Evolution in solid mechanics including inertia: While the hyperbolic evolution of solid materials including inertia has already been studied [50, 51], the scheme presented here has the potential to provide generalizations of those works. First, in the case when higher gradients are present in the energy, it might be possible to prove existence of injective solutions in full dimension. So far, such results have been limited to special geometries only, like the radial symmetric case [140]. Moreover, it might be possible to prove existence of measure valued solutions to elastodynamics even for quasiconvex energies.

Non-quadratic dissipation of the solid: Similarly as in fluid mechanics, quadratic dissipation potentials are commonly used when modelling solid materials. However, a large class of solid materials are known to behave in a rate-independent manner; i.e. the appropriate dissipation potential R is homogeneous of degree one in the second variable. Such models have already been studied extensively in the case of quasistatic evolution [138]. The main problem there is that their invariance under reparametrisation of the time-scale allows for sudden jumps, something that could be mitigated by considering inertial effects or coupling with a fluid.

Contact conditions: The time-stepping existence scheme by minimization introduced in this work produces results *for arbitrarily large times*, even over the point of self touching. This allows to gain a global-in-time object. However in order to show that this object has a meaning as a solution to the given contact-problem, further work needs to be done. Even for the case of an elastic solid alone, the understanding of self-touching (or alternatively touching the container) is still unclear from a mathematical point of view; in particular understanding the *contact force* on the solid that is preventing the self-penetration. Some contact-forces have been identified in the fully static problem only recently in [157] and a generalization to the quasi-static situation is due to [118]. Generalizations to fluid-structure interaction or even hyperbolic evolution, however, remain widely open.

A different point of view would be to use the fluid as a damping substance. In case the solid surface is smooth it is conjectured that any sort of contact is prevented by the incompressible fluid. Please see [103, 99] or [90] where respective results have been shown. It should be mentioned that the incompressability as well as the no-slip boundary conditions are essential for these results. Certainly, once self-contact of the solid can be excluded a-priori (due to the fluid surrounding it) global solutions are available. One should also mention that due to chapter 8 the no-contact is not necessarily a paradoxical feature since bouncing can be expected to happen.

Limit passage to simpler geometries: Mathematically, fluid-stucture evolution is much better understood if the solid object is of lower dimension (see e.g. [145, 128]).

While passage to lower dimensional objects for solid materials has been studied (see e.g. [123, 67]), coupling with the fluid and passing to the limit in a similar manner is another relevant topic.

Coupling to other physical phenomena: Here we consider a strictly mechanical system but coupling to further physical phenomena such as heat transfer is worthwhile studying. Indeed, as heat transfer happens in the actual configuration even results for the solid alone are sparse, we refer to [137] for a recent work in the quasi-steady case. The methods from chapter 6 in this work might allow a generalization to the case with inertia both for the solid alone and, ultimatively, also fluid-structure interaction.

Analytical bounds on numerical approximations of the scheme: The schemes we are proposing here are constructive in their very nature. Thus one can construct numerical approximations without having to deviate much from the main ideas. There are certain difficulties related to minimizing a non-convex functional and the fact that we are dealing with changing domains, but both have been dealt with before. Instead the main point of interest will be the rate of convergence of any such scheme. Fundamentally our method relies on energy bounds on the acceleration scale which we are only able to obtain after convergence of the velocity scale and which may not hold for any approximation. Proving convergence of a numerical scheme would thus necessitate first finding discrete bounds on that level.

1.3 Contactless rebound in viscous incompressible fluids, chapter 8

The last chapter is dedicated to the important problem of collisions between solids in a viscous fluid. The problem of particle-particle or particle-wall collisions in viscous fluids has important practical applications and has thus been the subject of a plethora of studies, not only experimental and numerical, but also from a purely mathematical standpoint. Yet, the problem is far from being fully resolved and the partial results which are available can often be counterintuitive. An example is given by the simple case of a spherical rigid particle surrounded by a Stokes linear fluid which moves towards a wall. Indeed, it has been known for some time already that if both the particle and the wall are equipped with no-slip boundary conditions, contact cannot take place in a viscous incompressible fluid in finite time without singular forcing. We refer to [25, 42, 59, 79, 80, 88, 98, 101, 105, 102, 187] for the detailed treatment of this issue in several different scenarios. Despite the fact that in an incompressible fluid with no-slip boundary conditions contact seems to be impossible, it has been hypothesized that a particle can rebound provided that it is elastic, or in general, when it admits the storage and release of mechanical energy during the rebound, see e.g. [47].

Apart from that, other physical mechanisms allowing for contact or rebound have been suggested and investigated, such as slip boundary conditions [79], the fluid compressibility [58], pressure-dependent material properties [10], wall roughness [80], etc. (see also [113, 114] and references therein).

In this chapter we consider a solid object (also referred to as particle or structure) that may be elastic or rigid and study its motion when thrown towards a rigid wall in a viscous incompressible liquid environment that adheres to all surfaces (that is, under no-slip boundary conditions). We consider both the two and the three dimensional case. For simplicity, we will assume that the fluid is governed by the steady Stokes equations (8.2.19). We expect, however, that most of our observations should be also relevant in case the fluid is governed by the steady or unsteady Navier–Stokes equations, as analogous conclusions have been drawn by several other authors investigating related issues (see e.g. [79, 80]).

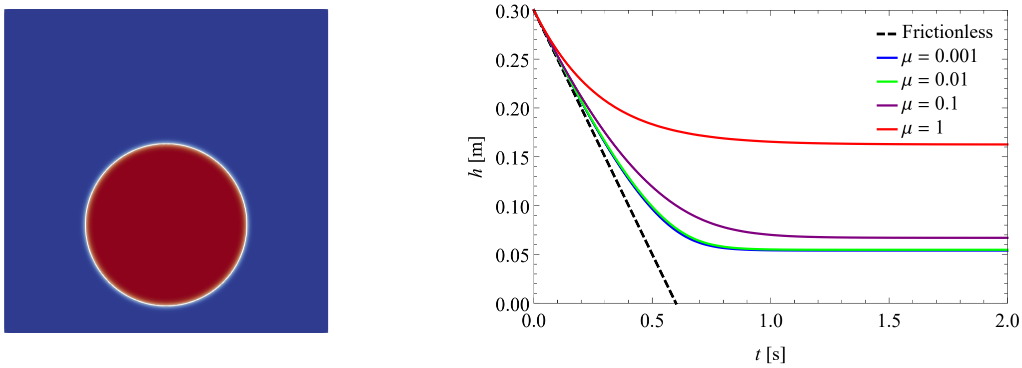
As explained in several of the references given above (see in particular [98, 101, 102]), it has been mathematically proven that the interplay of the *regularity* of all surfaces involved, the *incompressibility* of the fluid, and the prescribed *no-slip* boundary conditions imply that a smooth, rigid body cannot reach any other smooth solid obstacle in finite time. This phenomenon is also known as the no-contact paradox. See Figure 1.2 for a demonstration of this surprising phenomenon in the case of a ball falling towards a flat horizontal ramp.

In this chapter, we aim to advance the understanding of the extent to which the pathological behavior described in the no-contact paradox can affect the dynamics of solid particles in close proximity to the boundary of the container. Throughout the chapter, special emphasis is given to the phenomenon of particle rebound. Indeed, the main question that motivated this work can be formulated as follows:

(Q.1) *Can solid particles rebound in the absence of a topological contact?*

One of our main contributions is that we provide an affirmative answer to (Q.1) in a simplified setting. To be precise, we introduce a system of coupled non-linear ODEs as a toy model approximation for the notoriously challenging fluid-structure interaction problem describing the motion of an *elastic* solid immersed in a viscous incompressible fluid.

Our design of the reduced model is methodologically inspired by the observation that, under certain simplifying assumptions, the motion of a *rigid* body (described by the system of partial differential equations for the coupled fluid-structure interaction problem) can be reduced to a single second order ODE (see [101]; see also subsubsection 8.2.3.1), and conceptually by numerical experiments (see subsubsection 8.4.2.1). In particular, our simplified model (described in detail in subsubsection 8.2.3.2) presents the following two defining features:



(a) (b)

Figure 1.2: (a) A rigid ball falling towards a wall in a fluid environment of viscosity $\mu = 0.1$ Pa s, cf. subsubsection 8.4.2.1. (b) Dependence of the distance from the bottom (denoted by h) on time t for different viscosities.

(i) it allows for the storage and release of mechanical energy to account for an elastic response of the solid (see Figure 8.1);

(ii) it encodes possible deformations of the body.

While property (i) is a rather natural requirement, a few comments on (ii) are in order. Motivated by our numerical experiments, we allow the fluid-solid interaction to affect the shape of the solid object. It is well understood (see, for example, the discussion in Subsection 8.2.4 and the reference therein), that changes in the flatness of the particle in the nearest-to-contact region can have a significant influence on the magnitude of the drag force. Furthermore, since this effect becomes even more dramatic at small distances from other solid objects or from the boundary of the container, we tailor our model to adequately capture this interplay by considering a possible dependence on the deformation parameter in the damping term which represents the drag force. In this simplified setting (see Theorem 8.3.3), we show that rebound is indeed possible for sufficiently small values of the viscosity parameter, provided that the solid experiences a substantial flattening.

Let us mention here that our investigation uncovers a rather surprising "trapping" phenomenon, thus providing further insight into the consequences of the no-contact paradox. In order to illustrate this effect, consider a rigid object, which however allows for the storage and release of (a fraction of) its kinetic energy, as in property (i) above. As a model example, consider a rigid spherical shell with an internal mass-spring energy storing mechanism, as sketched in Figure 8.1, falling towards a horizontal wall. For small values of the dynamic viscosity parameter, the expected dynamics for this particular configuration are as follows: as the outer shell is slowed-down by the viscous forces preventing from collision, part of the kinetic energy of the system is stored in the inner mechanism; the shell can then be expected to rebound once this energy is transferred back to it by the upwards push applied by the mass-spring system. Moreover, one would also anticipate to witness increasingly pronounced rebounds as friction in the fluid is reduced by considering gradually smaller values of the viscosity parameter. However, the analysis of this peculiar fluid-structure interaction performed on our reduced model predicts a rather different behavior. This is made precise in the following corollary.

Corollary 1.3.1. *In the vanishing viscosity limit, the rigid shell system described above falls freely (that is, as it would in the vacuum) towards the wall, to which it then sticks for all times after collision.*

For a proof of the corollary we refer the reader to Section 8.3, and in particular to the proof of Theorem 8.3.2 below, in which we show a more general result.

In view of Theorem 1.3.1, we say that a system does not produce a *physical rebound* if the distance between the body and the wall converges, in the vanishing viscosity limit, to a monotone function of the time variable

t. Thus, for our purposes, a rebound is said to be physical (or physically meaningful) only if it withstands the vanishing viscosity limit.

Obviously, some crucial aspect is missing in the models considered in Theorem 1.3.1 (and Theorem 8.3.2) in order to capture physical bouncing effects. The motion described is not only in clear contrast with our real-world experience of bouncing objects, but also with laboratory experiments and numerical simulations (see, for example, the recent contributions [94, 188]). For more details, we refer to Section 8.4; see also the results presented in [65, 165]. These observations naturally lead to the following question.

(Q.2) *What is the mathematical reason for a physical rebound?*

We present here our scientific progress on this complicated issue. Specifically, our investigations and results prompted us to formulate the following conjecture.

Conjecture: *A qualitative change in the flatness of the solid body as it approaches the wall, together with some elastic energy storage mechanism within the body, can potentially allow for a physically meaningful rebound even for no-slip boundary conditions preventing from topological contact.*

The results presented in this chapter (both analytical and numerical) strongly support our leading conjecture. Indeed, it turns out that our "educated guess" in the design of the reduced model, for which we are able to prove the possibility of a physical rebound, admits solutions that are in striking match with the finite element solutions (FEM solutions) for a full fluid-structure interaction. See indeed **??**, where the motions are compared for several values of the viscosity parameter. We refer to Subsection 8.4.3 for a detailed discussion of the comparison between the numerical simulations.

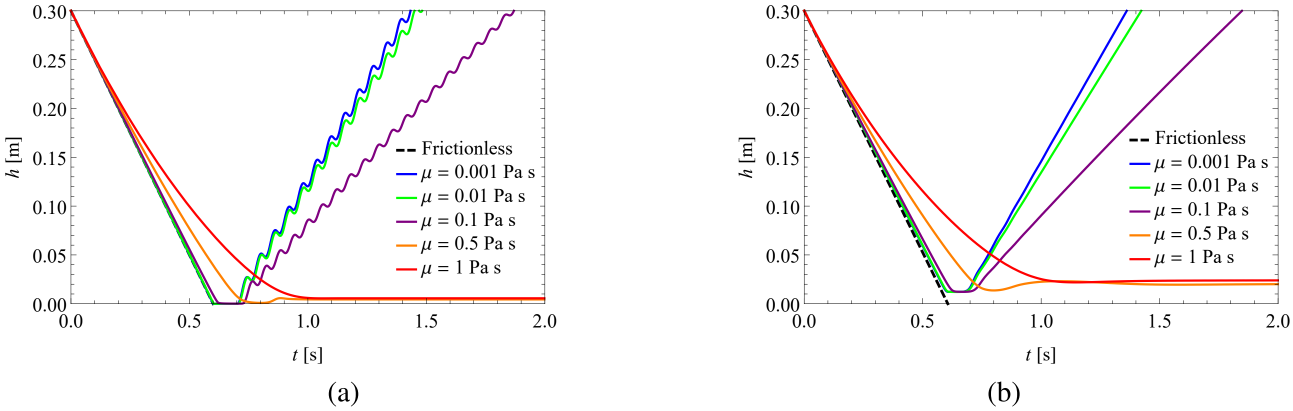


Figure 1.3: Comparison of numerically obtained solutions to the reduced model of ODEs (a) and FEM solutions (b).

While this figure allows to speculate that our reduced model could have indeed potentially captured the essential feature for rebound in the absence of collisions, certainly, a precise connection between the two models is still missing.

It is worth noting, however, that up to now even the existence theory for bulk elastic solids interacting with fluids is rather sparse (see, for example, [86] or chapter 7). We recall that in the existence, regularity and uniqueness results of the previous chapters always some kind of no-contact conditions are required on the solid. (These assumptions are rather standard even in case of rigid body motions.) On the other hand, no-contact results (which can be regarded as the starting point of our investigations) for smooth deformable objects can be expected to be true. An important result in this direction is given by the paper [88], where the authors consider the case of a beam interacting with a viscous fluid.

We conclude by mentioning that special effort is put into keeping the assumptions in the analytical section of the paper as general as possible, without however hindering the tractability of the reduced model. For this reason, in Subsection 8.3.1 we provide an axiomatic set of assumptions which give the reduced model enough flexibility when it comes to fitting it with the full FSI problem.

1.4 Acknowledgments

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Chapter 2

Existence and regularity of weak solutions for a fluid interacting with a non-linear shell

2.1 Introduction

In this chapter we study the coupling of the $3D$ incompressible Navier-Stokes equations with the evolution of the non-linear Koiter shell equation. Our main result is that any finite-energy weak solution to the considered FSI problem satisfies an additional regularity property on its interval of existence (see Theorem 2.2.5). More precisely, we show that the elastic displacement belongs to the following Bochner space $L_t^2(H_x^{2+s}) \cap H_t^1(H_x^s)$ for all $s < \frac{1}{2}$. Here H^s denotes the standard fractional Sobolev space[1]. In particular, due to respective embedding theorems, the elastic displacement is Lipschitz continuous in the space variable for almost every moment of time. We use this result to show the existence of weak solutions to a fluid-non-linear Koiter shell interaction problem (see Theorem 2.2.4). Since the non-linear Kotier shell equations are quasi-linear with non-linear coefficients depending on the terms of leading order in the energy, the additional structure regularity estimate is crucial for the compactness argument in the construction of a weak solution. The main idea behind the regularity theorem is to use the fluid dissipation and the coupling conditions to prove the additional regularity estimate for the structure displacement. The realization of this idea is technically challenging. It includes the development of a comprehensive analysis to construct a solenoidal extension and smooth approximations for the time-changing domain with clear (local) dependence on the regularity of the boundary values and the boundary itself. The approach is quite general and thus seems suitable for further applications related to the analysis of variable geometries. Actually, the present result was already applied, please see the preprints [21, 183] and the next two chapters.

Fluid-structure interaction has been an increasingly active area of research in mathematics in the last 20 years. Due to the overwhelming number of contributions in the area we just mention analytic results that are most relevant for our work in this brief literature review. The existence results for weak solutions for the FSI problems where the incompressible Navier-Stokes equations are coupled with a lower-dimensional elasticity model (e.g. plate or shell laws) have been obtained in [31, 87, 144, 127, 109, 147]. The corresponding existence result for the compressible fluid flow was proved in [22]. All mentioned results on the existence of weak solutions are valid up to time of possible self-intersection of the domain. Up to our knowledge the number of regularity estimates for long time solutions are rather limited. Recently some significant results on strong solutions for large initial data and a $2D$ fluid interacting with a $1D$ solid have been obtained, see [88, 29]. For a three dimensional fluid interacting with a three dimensional elastic body see [110, 111] for the global results with small initial data and structural damping. The theory of local-in-time strong solutions for $3D$-$3D$ FSI problems is rather well developed, see recent results in [120, 163] and references within. We wish to emphasize that in all these works the structure equations were linear. For the FSI problem with non-linear structure the theory is far less developed. The existence of weak solution to the FSI problem with a Koiter membrane energy that includes non-linearities of lower order and a leading order linear regularizing term was proved in [153].

[1] For a precise definition of the fractional Sobolev space see Subsection 2.2.5.

Short time or small data existence result in the context of strong solutions for various non-linear fluid structure models have been obtained in [34, 35, 44]. Finally, we wish to mention some results in the static case that can be found here [75, 86].

The role of the fluid dissipation on the qualitative properties of the solution is one of the central questions in the area of fluid-structure interactions and related systems, and has been studied by many authors, see e.g. [8, 190] and references within. We present here a new technique that allows to transfer dissipation features from the fluid equation to the non-linear hyperbolic elastic displacement. We wish to point out that better regularity can not be expected for a non-linear hyperbolic PDE with arbitrary smooth right hand sides and initial data. It is the coupling with a dissipative equation that allows for this better regularity.

Outline of the chapter: The next section first derives the Koiter energy w.r.t. our chosen coordinates, gives two explicit examples of Koiter energies with respective geometric restrictions on $\alpha(\Omega), \beta(\Omega), \overline{\gamma}(\eta)$, and introduces the definition of a weak solution for fluid-structure interactions. Section 2.3 is the technical heart of the paper since there the solenoidal extension and approximation operators are introduced. In Section 2.4, we give the proof of the regularity result Theorem 2.2.5. Section 2.5 provides a new version of Aubin-Lions compactness result that reveals the connection between the existence of suitable extensions and compactness results for fluid-structure interactions involving elastic shells. Finally, in Section 2.6 we show Theorem 2.2.4; the existence is shown by combining the extra regularity of the shell with the compactness theory.

2.2 Weak solutions

2.2.1 Fluid and interaction

Let $\Omega \subset \mathbb{R}^3$ be a domain such that its boundary $\Gamma = \partial\Omega$ is parameterized by a C^3 injective mapping $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$, where $\omega \subset \mathbb{R}^2$. To simplify notation we assume that the boundary of Ω can be parameterized by a flat torus $\omega = \mathbb{R}^2/\mathbb{Z}^2$ which corresponds to the assumption of periodic boundary conditions for the structure displacement.[2]

We denote the tangential vectors at any point $\boldsymbol{\varphi}(y)$ in the following way:

$$\mathbf{a}_\alpha(y) = \partial_\alpha \boldsymbol{\varphi}(y),\ \alpha = 1, 2,\ y \in \omega.$$

The unit normal vector is given by $\nu(y) = \dfrac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \times \mathbf{a}_2(y)|}$. The surface area element of $\partial\Omega$ is given by $dS = |\mathbf{a}_1(y) \times \mathbf{a}_2(y)| dy$. We assume that the domain deforms only in the normal direction and denote by $\eta(t, y)$ the magnitude of the displacement. This reflects the situation when the fluid pressure is the dominant force acting on the structure in which case it is reasonable to assume that the shell is deforming in normal direction. In this case the deformed boundary can be parameterized by the following coordinates:

$$\boldsymbol{\varphi}_\eta(t, y) = \boldsymbol{\varphi}(y) + \eta(t, y)\nu(y),\ t \in (0, T),\ y \in \omega. \tag{2.2.1}$$

We wish to emphasize that this restriction is rather standard in the majority of mathematical works on the analysis of weak solutions, mainly due to severe technical difficulties associated with the analysis of the case where the full displacement is taken into account. The deformed boundary is denoted by $\Gamma_\eta(t) = \boldsymbol{\varphi}_\eta(t, \omega)$. It is a well known fact from differential geometry (see e.g. [125]) that there exist $\alpha(\Omega), \beta(\Omega) > 0$ such that for $\eta(y) \in (\alpha(\Omega), \beta(\Omega))$, $\boldsymbol{\varphi}_\eta(t, .)$ is a bijective parameterization of the surface $\Gamma_\eta(t)$ and it defines a domain $\Omega_\eta(t)$ in its interior such that $\partial\Omega_\eta(t) = \Gamma_\eta(t)$. Moreover, there exists a bijective transformation $\boldsymbol{\psi}_\eta(t, .) : \Omega \to \Omega_\eta(t)$.[3]

We denote the moving domain in the following way:

$$(0, T) \times \Omega_\eta(t) \coloneqq \bigcup_{t \in (0,T)} \{t\} \times \Omega_\eta(t).$$

[2] We consider the periodic boundary conditions just to avoid unnecessary technical complications.

[3] For more details on the geometry see Section 2.2.2 and Definition 2.2.1.

The non-linear Koiter model is given in terms of the differences of the first and the second fundamental forms of $\Gamma_\eta(t)$ and Γ which represent membrane forces and bending forces respectively. These forces are summarized in its potential - the Koiter energy $\mathcal{E}_K(t,\eta)$. The definition of the potential is taken from [38, Section 4]. For a precise definition and the derivation of the energy for our coordinates see (2.2.14) below. Let $\mathcal{L}_K\eta$ be the L^2-gradient of the Koiter energy $\mathcal{E}_K(t,\eta)$, h be the (constant) thickness of the shell and ϱ_s the (constant) density of the shell. Then the respective momentum equation for the shell reads

$$\varrho_s h\partial_t^2\eta + \mathcal{L}_K\eta = g, \tag{2.2.2}$$

where g are the momentum forces of the fluid acting on the shell.

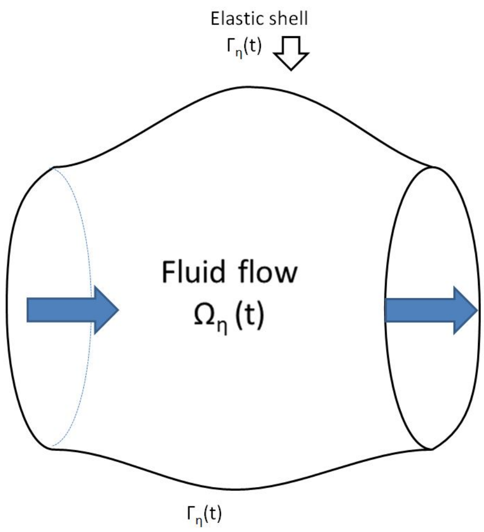


Figure 2.1: An example of the deformed cylindrical domain.

The fluid flow is governed by the incompressible Navier-Stokes equations:

$$\varrho_f\big(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\big)\mathbf{u} = \operatorname{div}\boldsymbol{\sigma}(\mathbf{u},p) \qquad \text{in } (0,T)\times\Omega_\eta(t), \tag{2.2.3}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } (0,T)\times\Omega_\eta(t), \tag{2.2.4}$$

where $\boldsymbol{\sigma}(\mathbf{u},p) = -p\mathbb{I} + 2\mu\operatorname{sym}\nabla\mathbf{u}$ is the fluid stess tensor and ϱ_f the (constant) density of the fluid.

The fluid and the structure are coupled via kinematic and dynamic coupling conditions. We prescribe the no-slip kinematic coupling condition which means that the fluid and the structure velocities are equal on the elastic boundary:

$$\mathbf{u}(t,\boldsymbol{\varphi}_\eta(t,y)) = \partial_t\eta(t,y)\nu(y), \quad y\in\omega. \tag{2.2.5}$$

The dynamic boundary condition states that the total force in the normal direction on the boundary is zero:

$$g(t,y) = -\boldsymbol{\sigma}(\mathbf{u},p)(t,\boldsymbol{\varphi}_\eta(t,y))\nu(\eta(t,y))\cdot\nu(y), \quad y\in\omega, \tag{2.2.6}$$

where $\nu(\eta(t,y)) = \partial_1\boldsymbol{\varphi}_\eta(t,y)\times\partial_2\boldsymbol{\varphi}_\eta(t,y)$ is defined as a weighted vector pointing in the direction of the outer normal to the deformed domain at point $\boldsymbol{\varphi}_\eta(t,y)$; the weight is exactly the Jacobian of the change of variables from Eulerian to Lagrangian coordinates.

We may summarize and state the full fluid-structure interaction problem.
Find $(\mathbf{u},\eta)$ such that

$$\begin{aligned}
\varrho_f\big(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\big)\mathbf{u} &= \operatorname{div}\boldsymbol{\sigma}(\mathbf{u},p) && \text{in } (0,T)\times\Omega_\eta(t),\\
\operatorname{div}\mathbf{u} &= 0 && \text{in } (0,T)\times\Omega_\eta(t),\\
\varrho_s h\partial_t^2\eta + \mathcal{L}_K\eta &= -(\boldsymbol{\sigma}(\mathbf{u},p)\circ\boldsymbol{\varphi}_\eta)\nu(\eta)\cdot\nu && \text{in } (0,T)\times\omega,\\
\mathbf{u}\circ\boldsymbol{\varphi}_\eta &= \partial_t\eta\nu && \text{in } (0,T)\times\omega,\\
\mathbf{u}(0,.) &= \mathbf{u}_0 && \text{in } \Omega_\eta(0),\\
\eta(0) = \eta_0,\ \partial_t\eta(0) &= \eta_1 && \text{in } \omega.
\end{aligned}\tag{2.2.7}$$

2.2.2 The elastic energy

Coordinates.

Here we follow the strategy of [127, Section 2] by introducing the following coordinates attached to the reference geometry Ω which are well defined in the tubular neighborhood of $\partial\Omega$ (see e.g. [125, Section 10] and Figure 2.2 for an illustration).

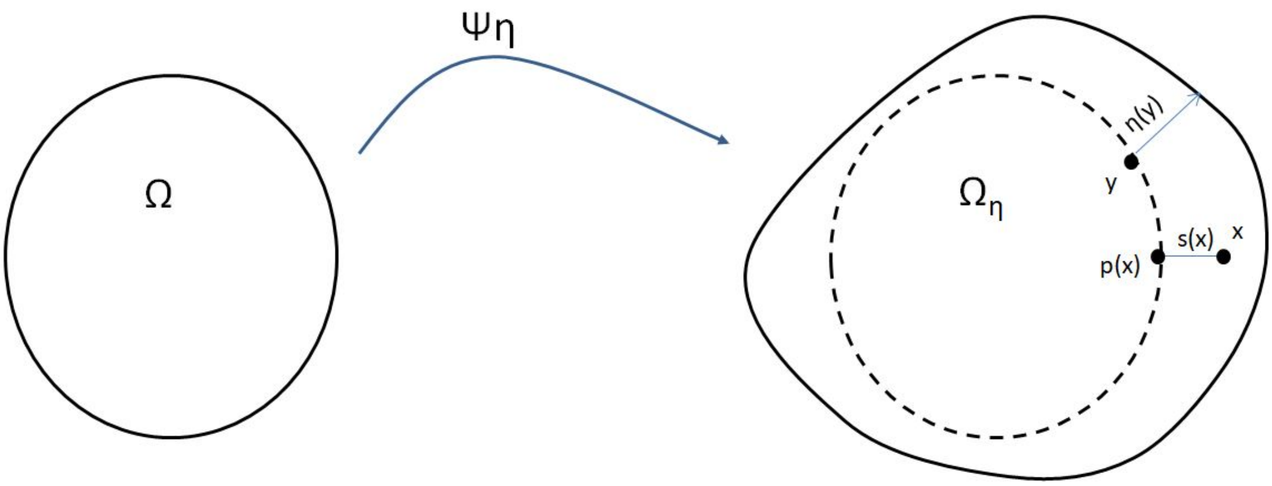


Figure 2.2: Cross section of cylindrical domain, its deformation and corresponding coordinate system.

Definition 2.2.1. *Let x be a point in the neighborhood of $\partial\Omega$. We define the distance parameter with respect to the reference point*

$$y(x) = \text{ arg min}_{y\in\omega}|x-\boldsymbol{\varphi}(y)|,\quad s(x) = (x-y(x))\cdot\nu(x)$$

and the projection $\mathbf{p}(x) = \boldsymbol{\varphi}(y(x))$.

We define the numbers $\alpha(\Omega),\beta(\Omega)$ so that $(\alpha(\Omega),\beta(\Omega))$ is the largest open interval such that numbers $s(x),\mathbf{p}(x),y(x)$ are uniquely defined over $\{\boldsymbol{\varphi}(y)+s\nu : y\in\omega,\ s\in(\alpha(\Omega),\beta(\Omega))\}$.

For $\kappa>0$ we introduce the indicator mapping $\sigma_\kappa\in C^\infty(\mathbb{R})$, such that

$$\sigma_\kappa(s) = 1 \textit{ for } s\in(\alpha(\Omega)+\kappa,\infty),\ \sigma_\kappa(s) = 0 \textit{ for } s\le\alpha(\Omega)+\frac{\kappa}{2} \textit{ and } \sigma_\kappa'\ge 0.$$

We set

$$S_\kappa = \{\boldsymbol{\varphi}(y)+s\nu(y) : (s,y)\in[\alpha(\Omega)+\kappa,\beta(\Omega)-\kappa]\times\omega\}$$

Further

$$Q^\kappa = S_\kappa\cup\Omega \textit{ and } Q_\kappa = \Omega\setminus S_\kappa.$$

In particular we have a clear information on the support of the derivative:

$$\mathcal{A}_\kappa := \{\boldsymbol{\varphi}(y) + s\nu(y) : (s, y) \in [\alpha(\Omega) + \kappa/2, \alpha(\Omega) + \kappa] \times \omega\} \supset \operatorname{supp}(\sigma_\kappa'(s(\cdot))).$$

For $\eta(y) \in (\alpha(\Omega) + \kappa, \beta(\Omega) - \kappa)$, this allows to introduce the mapping $\boldsymbol{\psi}_\eta(t, .) : \Omega \to \Omega_{\eta(t)}$ by

$$z \mapsto \Big(1 - \sigma_\kappa(s(z))\Big)z + \sigma_\kappa(s(z))\Big(\mathbf{p}(z) + \big(\eta(y(z)) + s(z)\big)\nu(y(z))\Big),$$

and $\boldsymbol{\psi}_\eta^{-1}(t, .) : \Omega_{\eta(t)} \to \Omega$ by

$$x \mapsto \Big(1 - \sigma_\kappa(s(x))\Big)x + \sigma_\kappa(s(x))\Big(\mathbf{p}(x) + \big(s(x) - \eta(y(x))\big)\nu(y(x))\Big).$$

Moreover we define

$$\mathbf{\Phi} : (\alpha(\Omega) + \kappa, \beta(\Omega) - \kappa) \times \omega \to S_\kappa, \quad \mathbf{\Phi}(s, y) = \boldsymbol{\varphi}(y) + \nu(y)s.$$

The function is smooth and invertible in dependence of $\boldsymbol{\varphi}$ and κ. This implies that

$$\psi_\eta \circ \mathbf{\Phi} : (\alpha(\Omega) + \kappa, 0) \times \omega \to \Omega_\eta, \quad \psi_\eta \circ \mathbf{\Phi}(s, y) = \boldsymbol{\varphi}(y) + (s + \sigma_\kappa(s)\eta(y))\nu(y)$$

Finally, we define the following geometric quantity depending on $\partial\Omega$ and η:

$$\overline{\gamma}(\eta) = \frac{1}{|a_1 \times a_2|}\Big(|a_1 \times a_2| + \eta(\nu \cdot (\mathbf{a}_1 \times \partial_2\nu + \partial_1\nu \times \mathbf{a}_2)) + \eta^2\nu \cdot (\partial_1\nu \times \partial_2\nu)\Big).$$

Remark 2.2.2. *The numbers $\alpha(\Omega), \beta(\Omega)$ do not have to be small. For example, if Ω is a ball or a cylinder with radius R, then $(\alpha(\Omega), \beta(\Omega)) = (-R, \infty)$. The geometric quantity $\overline{\gamma}(\eta)$ is connected to the H^2-coercivity of the non-linear structure energy and its meaning is clarified in Lemma 2.4.3 and Remark 2.4.4.*

Derivation of the elastic energy.

The non-linear Koiter model is given in terms of the differences of the first and the second fundamental forms of $\Gamma_\eta(t)$ and Γ. The tangent vectors to the deformed boundary are given by:

$$\mathbf{a}_\alpha(\eta) = \partial_\alpha\boldsymbol{\varphi}_\eta = \mathbf{a}_\alpha + \partial_\alpha\eta\nu + \eta\partial_\alpha\nu, \ \alpha = 1, 2. \tag{2.2.8}$$

Therefore, the components of the first fundamental form of the deformed configuration are given by:

$$a_{\alpha\beta}(\eta) = \mathbf{a}_\alpha(\eta) \cdot \mathbf{a}_\beta(\eta) = a_{\alpha\beta} + \partial_\alpha\eta\partial_\beta\eta + \eta\big(\mathbf{a}_\alpha \cdot \partial_\beta\nu + \mathbf{a}_\beta \cdot \partial_\alpha\nu\big) + \eta^2\partial_\alpha\nu \cdot \partial_\beta\nu. \tag{2.2.9}$$

We define the change of metric tensor $\boldsymbol{G}(\eta)$:

$$G_{\alpha\beta}(\eta) = a_{\alpha\beta}(\eta) - a_{\alpha\beta} = \partial_\alpha\eta\partial_\beta\eta + \eta\big(\mathbf{a}_\alpha \cdot \partial_\beta\nu + \mathbf{a}_\beta \cdot \partial_\alpha\nu\big) + \eta^2\partial_\alpha\nu \cdot \partial_\beta\nu. \tag{2.2.10}$$

The normal vector to the deformed configuration is given by:

$$\begin{aligned}\nu(\eta) = \mathbf{a}_1(\eta) \times \mathbf{a}_2(\eta) = |\mathbf{a}_1 \times \mathbf{a}_2|\nu + \partial_2\eta(\mathbf{a}_1 \times \nu + \eta\partial_1\nu \times \nu)\\ +\partial_1\eta(\nu \times \mathbf{a}_2 + \eta\nu \times \partial_2\nu) + \eta(\mathbf{a}_1 \times \partial_2\nu + \partial_1\nu \times \mathbf{a}_2) + \eta^2(\partial_1\nu \times \partial_2\nu).\end{aligned} \tag{2.2.11}$$

Notice that $\nu(\eta)$ is not a unit vector. We follow our reference literature [38] and use the following tensor $\boldsymbol{R}$ (denoted by $\boldsymbol{R}^\#$ in [38, Section 4]) which is some non-normalized variant of the second fundamental form to measure the change of curvature:

$$R_{\alpha\beta}(\eta) = \frac{1}{|\mathbf{a}_1 \times \mathbf{a}_2|}\partial_\alpha\mathbf{a}_\beta(\eta) \cdot \nu(\eta) - \partial_\alpha\mathbf{a}_\beta \cdot \nu, \ \alpha, \beta = 1, 2. \tag{2.2.12}$$

Finally, we define the elasticity tensor in the classical way [38, Theorem 3.2]:

$$\mathcal{A}\mathbf{E} = \frac{4\lambda\mu}{\lambda + 2\mu}(\mathbf{A} : \mathbf{E})\mathbf{A} + 4\mu\mathbf{A}\mathbf{E}\mathbf{A}, \quad \mathbf{E} \in \mathrm{Sym}(\mathbb{R}^{2\times 2}). \tag{2.2.13}$$

Here $\mathbf{A}$ is the contravariant metric tensor associated with $\partial\Omega$ (see e.g. [38, Section 2] for the precise definition of $\mathbf{A}$), and $\lambda > 0$, $\mu > 0$ are the Lamé constants. The Koiter energy of the shell is given by:

$$\mathcal{E}_K(t,\eta) = \frac{h}{4}\int_\omega \mathcal{A}\mathbf{G}(\eta(t,.)) : \mathbf{G}(\eta(t,.))dy + \frac{h^3}{48}\int_\omega \mathcal{A}\boldsymbol{R}(\eta(t,.)) : \boldsymbol{R}(\eta(t,.))dy, \tag{2.2.14}$$

where h is the thickness of the shell. In order to simplify the notation we introduce the following forms connected to the membrane and bending effects in the variational formulation:

$$a_m(t,\eta,\xi) = \frac{h}{2}\int_\omega \mathcal{A}\boldsymbol{G}(\eta(t,.)) : \boldsymbol{G}'(\eta(t,.))\xi\, dy, \tag{2.2.15}$$

$$a_b(t,\eta,\xi) = \frac{h^3}{24}\int_\omega \mathcal{A}\boldsymbol{R}(\eta(t,.)) : \boldsymbol{R}'(\eta(t,.))\xi\, dy, \tag{2.2.16}$$

where $\boldsymbol{G}'$ and $\boldsymbol{R}'$ denote the Fréchet derivatives of $\boldsymbol{G}$ and $\boldsymbol{R}$ respectively. Therefore, the elastodynamics of the shell is given by the following variational formulation:

$$h\varrho_s\frac{d}{dt}\int_\omega \partial_t\eta(t,.)\xi\, dy + a_m(t,\eta,\xi) + a_b(t,\eta,\xi) = \int_\omega g\xi\, dy \text{ on } (0,T),\ \xi \in W^{2,p}(\omega), \tag{2.2.17}$$

where ϱ_s is the structure density, g is the density of area force acting on the structure, and $p > 2$. We denote the elasticity operator by $\mathcal{L}_K$ which is formally given by

$$\langle \mathcal{L}_K\eta, \xi\rangle = a_m(t,\eta,\xi) + a_b(t,\eta,\xi), \quad \xi \in W^{2,p}(\omega). \tag{2.2.18}$$

Next we give some examples for which we can calculate our restrictive numbers $\alpha(\Omega), \beta(\Omega)$ and $\overline{\gamma}(\eta)$.

Example 1: Cylindrical Koiter shell

The parameterization of the reference cylinder is given by $\boldsymbol{\varphi}(\theta,z) = (R\cos\theta, R\sin\theta, z)$, $(\theta,z) \in \omega = (0,2\pi)\times(0,1)$, where $R > 0$ is the radius of the cylinder. We compute

$$\mathbf{a}_1(\theta,z) = (-R\sin\theta, R\cos\theta, 0),\ \mathbf{a}_2(\theta,z) = (0,0,1),\ \nu(\theta,z) = (\cos\theta,\sin\theta,0).$$

The corresponding contravariant metric tensor is given by $A = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & 0 \end{pmatrix}$. The deformation of the cylindrical boundary is given by:

$$\boldsymbol{\varphi}_\eta(\theta,z) = (R\cos\theta + \eta(\theta,z)\cos\theta, R\sin\theta + \eta(\theta,z)\sin\theta, z).$$

Straightforward calculation yields:

$$\mathbf{a}_1(\eta) = (1 + \frac{1}{R})\mathbf{a}_1 + \eta_\theta\nu,\ \mathbf{a}_2(\eta) = \mathbf{a}_2 + \eta_z\mathbf{a}_3,$$

$$\nu(\eta) = (R+\eta)\nu - \eta_z(R+\eta)\mathbf{a}_2 + \frac{\eta_\theta}{R}\mathbf{a}_1.$$

Therefore, the change of metric tensor is given by

$$\boldsymbol{G}(\eta) = \begin{pmatrix} (R+\eta)^2 + \eta_\theta^2 - R^2 & \eta_\theta\eta_z \\ \eta_\theta\eta_z & 1 + \eta_z^2 \end{pmatrix},$$

and the change of curvature tensor by

$$\boldsymbol{R}(\eta) = \begin{pmatrix} (1+\frac{\eta}{R})\eta_{\theta\theta} - \frac{1}{R}(\eta+R)^2 - 2\frac{\eta_\theta^2}{R} + R & (1+\frac{\eta}{R})\eta_{\theta z} - \frac{1}{R}\eta_\theta\eta_z \\ (1+\frac{\eta}{R})\eta_{\theta z} - \frac{1}{R}\eta_\theta\eta_z & (1+\frac{\eta}{R})\eta_{zz} \end{pmatrix}.$$

Here $(\alpha(\Omega),\beta(\Omega)) = (-R,\infty)$ and $\overline{\gamma}(\eta) = 1 + \frac{\eta}{R}$.

Example 2: Spherical shell

Strictly speaking, the sphere does not fit in our framework since it does not have a global parameterization. However, this assumption was introduced just for technical simplicity and can be easily removed by working with local coordinates. In this example we consider an elastic sphere with holes around north and south poles. On these holes we prescribe the boundary condition for the fluid flow, e.g. inflow/outflow or Dirichlet. The shell is clamped on the boundary of the holes (see Figure 2.1 for illustration). More precisely, the parameterization is given by

$$\boldsymbol{\varphi}(\theta,\phi) = R(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi),\ (\theta,\phi)\in w = (0,2\pi)\times(a,\pi-a),$$

where $R > 0$ is the radius of the sphere, and $a > 0$ is the parameter determining the size of the holes. We compute the tangent and normal vectors to the reference configuration

$$\mathbf{a}_1 = -R(\sin\theta\sin\phi, \cos\theta\sin\phi, 0),\ \mathbf{a}_2 = R(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\pi),$$

$$\nu = -(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

The contravariant metric tensor is given by $A = \begin{pmatrix} \frac{1}{R^2\sin^2\phi} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix}$, and the deformation of the cylindrical boundary by

$$\boldsymbol{\varphi}_\eta(\theta,\phi) = (R - \eta(\theta,\phi))(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

We calculate the tangent and normal vectors to the deformed configuration:

$$\mathbf{a}_1(\eta) = (1-\frac{\eta}{R})\mathbf{a}_1 + \eta_\theta\nu,\ \mathbf{a}_2(\eta) = (1-\frac{\eta}{R})\mathbf{a}_2 + \eta_\phi\nu,$$

$$\nu(\eta) = (R-\eta)^2\sin\phi\nu - (1-\frac{\eta}{R})\Big(\frac{\eta_\theta}{\sin\phi}\mathbf{a}_1 + \eta_\phi\sin\phi\mathbf{a}_2\Big).$$

The change of the metric tensor is given by

$$\boldsymbol{G}(\eta) = \begin{pmatrix} \eta_\theta^2 + (\sin\phi)^2\eta(\eta-2R) & \eta_\theta\eta_\phi \\ \eta_\theta\eta_\phi & \eta_\phi^2 + (R-\eta)^2 - R^2 \end{pmatrix}.$$

Finally, the components of the change of curvature tensor are given by

$$\begin{aligned} R_{11}(\eta) = \frac{1}{2R^2}\Big(&-2\eta^3\sin^2\phi + \eta^2(6R\sin^2\phi + \eta_\phi\sin 2\phi + 2\eta_{\theta\theta}) \\ &- 2\eta(3R^2\sin^2\phi + R\eta_\phi\sin 2\phi + 2\eta_\theta^2 + 2R\eta_{\theta\theta}) + R(R\eta_\phi\sin 2\phi + 4\eta_\theta^2 + 2R\eta_{\theta\theta})\Big) \end{aligned}$$

$$R_{12}(\eta) = R_{21}(\eta) = \frac{\eta - R}{R^2}\Big(\eta_\theta(R\cot\phi - \eta\cot\phi - 2\eta_\phi) + \eta_{\theta\phi}(\eta - R)\Big)$$

$$R_{22}(\eta) = \frac{1}{R^2}\Big(-\eta^3 + \eta^2(3R + \eta_{\phi\phi}) + R(2\eta_\phi^2 + R\eta_{\phi\phi}) - \eta(2\eta_\phi^2 + R(3R + 2\eta_{\phi\phi}))\Big).$$

The clamped boundary conditions are $\eta = \partial_\phi\eta = 0$, $\phi = a, \pi - a$. Since we will take finite differences of order less than 1, we can extend η by zero (over the poles) and complete all estimates related to the regularity. Here $(\alpha(\Omega),\beta(\Omega)) = (-\infty, R)$ and $\overline{\gamma}(\eta) = \frac{(\eta-R)^2}{R^2}$.

2.2.3 Weak coupled solutions

We use here the standard notation of Bochner spaces related to Lebesgue and Sobolev spaces. We will use bold letters for vector valued functions in three dimensions. Usually we take $y \in \omega$ to be a two dimensional variable and x as a three dimensional variable. In order to define weak solutions, let us first define the appropriate function spaces:

$$\begin{aligned}
&\mathcal{V}_\eta(t) = \{\mathbf{u} \in H^1(\Omega_\eta(t)) : \mathrm{div}\mathbf{u} = 0\},\\
&\mathcal{V}_F = L^\infty(0,T;L^2(\Omega_\eta(t)) \cap L^2(0,T;V_\eta(t)),\\
&\mathcal{V}_K = L^\infty(0,T;H^2(\omega)) \cap W^{1,\infty}(0,T;L^2(\omega)),\\
&\mathcal{V}_S = \{(\mathbf{u},\eta) \in \mathcal{V}_F \times \mathcal{V}_K : \mathbf{u}(t,\varphi_\eta(t,.)) = \partial_t\eta(t,.)\nu(\eta(t,.))\},\\
&\mathcal{V}_T = \{(\mathbf{q},\xi) \in \mathcal{V}_F \times \mathcal{V}_K : \mathbf{q}(t,\varphi_\eta(t,.)) = \xi(t,.)\nu(\eta(t,.)),\ \partial_t\mathbf{q} \in L^2(0,T;L^2(\Omega_\eta(t))\}.
\end{aligned} \tag{2.2.19}$$

Here $\mathcal{V}_S$ and $\mathcal{V}_F$ are solution and test space respectively. Even though for $\eta \in \mathcal{V}_K$, $\Omega_\eta(t)$ is not necessary a Lipschitz domain, the traces used in definitions (2.2.19) and (2.3.3) are well defined, see Corollary 2.9. from [127] (see also [31, 143]). We introduce the concept of solution which we will consider here. Observe, that from this point on we normalize all physical constants $\rho_s = \rho_f = h = \mu = \lambda = 1$ for notational simplicity since the proofs require just positivity of these constants. We emphasize that the restrictions on existence and regularity are only of geometrical nature. This can be quantified by $\alpha(\Omega)$ and $\beta(\Omega)$ depending only on the reference geometry, and $\overline{\gamma}(\eta)$ depending on the reference geometry and on the particular magnitude and direction of the displacement, but not on the above physical constants.

Definition 2.2.3 (Weak solution)**.** *We call $(\mathbf{u},\eta) \in \mathcal{V}_S$ a weak solution of problem* (2.2.7) *if it satisfies the energy inequality* (2.2.21) *and for every $(\mathbf{q},\xi) \in \mathcal{V}_T$ the following equality holds in $\mathcal{D}'(0,T)$*

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_\eta(t)} \mathbf{u}\cdot\mathbf{q}\,dx + \int_{\Omega_\eta(t)} \Big(-\mathbf{u}\cdot\partial_t\mathbf{q} - \mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{q} + 2\mathrm{sym}\nabla\mathbf{u} : \mathrm{sym}\nabla\mathbf{q}\Big)\,dx\\
+\frac{d}{dt}\int_\omega \partial_t\eta\xi\,dy - \int_\omega \partial_t\eta\partial_t\xi + a_m(t,\eta,\xi) + a_b(t,\eta,\xi)\,dy = 0,
\end{aligned} \tag{2.2.20}$$

Furthermore, the initial values $\eta_0, \eta_1, \mathbf{u}_0$ are attained in the respective weakly continuous sense.

By formally multiplying $(2.2.7)_1$ by $\mathbf{u}$ and $(2.2.7)_2$ by $\partial_t\eta$, integrating over $\Omega_\eta(t)$ and ω respectively, integrating by parts and using the coupling conditions $(2.2.7)_4$, we obtain the energy inequality (see e.g. [31, 144] for details of the computations related to the change of the domain and the convective term):

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega_\eta(t))} + \frac{1}{2}\|\partial_t\eta(t)\|^2_{L^2(\omega)} + \mathcal{E}_K(t,\eta) + 2\int_0^t \int_{(\Omega_\eta(t))} |\mathrm{sym}\nabla\mathbf{u}|^2\,dx\,dt\\
&\quad\le \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})} + \frac{1}{2}\|\eta_1\|^2_{L^2(\omega)} + \mathcal{E}_K(0,\eta_0) =: C_0.
\end{aligned} \tag{2.2.21}$$

2.2.4 Main results

The main result of the chapter is the existence of a weak solution:

Theorem 2.2.4. *Assume that $\partial\Omega \in C^3, \eta_0 \in H^2(\omega), \eta_1 \in L^2(\omega)$ and $\mathbf{u}_0 \in L^2(\Omega_{\eta_0})$, and η_0 is such that Γ_{η_0} has no self-intersection and $\overline{\gamma}(\eta_0) \neq 0$. Moreover, we assume that the compatibility condition $\mathbf{u}_0|_{\Gamma_{\eta_0}} = \eta_1\nu$ is satisfied. Then there exists a weak solution $(\mathbf{u},\eta)$ on the time interval $(0,T)$ to* (2.2.7) *in the sense of Definition 2.2.3.*

Furthermore, one of the following is true: either $T = +\infty$, or the structure self-intersect, or $\overline{\gamma}(\eta) \neq 0$, i.e. the H^2-coercivity of the structure energy degenerates, where $\overline{\gamma}$ is defined in Definition 2.2.1 below.

The second main theorem says that all possible solutions in the natural existence class satisfy better structural regularity properties.

Theorem 2.2.5. *Let $(\mathbf{u}, \eta)$ be a weak solution to* (2.2.7) *in the sense of Definition 2.2.3. Then the solution has the additional regularity property*[4] $\eta \in L^2(0,T;H^{2+s}(\omega))$ *and* $\partial_t\eta \in L^2(0,T;H^s(\omega))$ *for* $s \in (0,\frac{1}{2})$*. Moreover, it satisfies the following regularity estimate*

$$\|\eta\|_{L^2(0,T;H^{2+s}(\omega))} + \|\partial_t\eta\|_{L^2(0,T;H^s(\omega))} \le C_1$$

with C_1 depending on $\partial\Omega$, C_0 and the H^2-coercivity size $\overline{\gamma}(\eta)$.

Remark 2.2.6 (Coercivity and non-linearity)**.** *Due to the fact that the Koiter shell equation is non-linear—more precisely since the curvature change is measured w.r.t. the deformed geometry—the H^2-coercivity of the Koiter energy can become degenerate. This is quantified by the estimate that is shown in Lemma 2.4.3 below. At such degenerate instant the given existence and regularity proofs break down. This is a phenomenon purely due to the non-linearity of the Koiter shell equations. Indeed, in case when the leading order term of the elastic energy is quadratic (i.e. the equation is linear or semi-linear), this loss of coercivity is a-priori excluded.*

2.2.5 Fractional spaces

In the paper, we use the standard definitions of Bochner spaces related to Lebesgue and Sobolev spaces. In particular, we consider *fractional Sobolev spaces* and *Nikolskii spaces*. We recall their definitions here.

For $\alpha \in (0,1)$ (the order of derivative) and $q \in [1,\infty)$ (the exponent of integrability) we say that $g \in W^{\alpha,q}(A)$, for a domain $A \subset \mathbb{R}^d$ if its norm

$$\|g\|_{W^{\alpha,q}(A)}^q := \left(\int_A\int_A \frac{|g(x)-g(y)|^q}{|x-y|^{n+\alpha q}}dx\,dy\right)^{\frac{1}{q}} + \left(\int_A |g(x)|^q dx\right)^{\frac{1}{q}}$$

is finite. Fractional Sobolev spaces can be extended to higher order. For $\alpha \in [k,k+1)$ with $k \in \mathbb{N}$ it is said that $g \in W^{\alpha,q}(A)$, if all partial derviatives of order up to k are in $W^{\alpha-k,q}(A)$. In the particular case $q = 2$ we use the abbreviation

$$H^s(A) \equiv W^{s,2}(A) \text{ for } s \in [0,\infty).$$

We say that $g \in N^{\alpha,q}(A)$ if its norm

$$\|g\|_{N^{\alpha,q}(A)} := \sup_{i\in\{1,\dots,d\}}\sup_{h\neq 0}\left(\int_{A_h}\left|\frac{g(x+he_i)-g(x)}{|h|^{\alpha-1}h}\right|^q dx\right)^{\frac{1}{q}} + \left(\int_A |g(x)|^q dx\right)^{\frac{1}{q}},$$

where e_i is the i-th unit vector and $A_h = \{x \in A : \operatorname{dist}(x,\partial A) > h\}$, is finite. Nikolskii spaces are closely related to fractional Sobolev spaces $W^{\alpha,q}(A)$. Let us just mention that for $0 < \alpha < \beta < 1$ and a bounded domain A we have

$$W^{\beta,q}(A) \subset N^{\beta,q}(A) \subset W^{\alpha,q}(A).$$

Recall also that for fractional Sobolev spaces an embedding theorem is available for a Lipschitz domain $A \subset \mathbb{R}^n$ and $g \in N^{\beta,q}(A)$ and $0 < \alpha < \beta < 1$ we have for $\alpha q < n$ that

$$\|g\|_{L^{\frac{nq}{n-\alpha q}}(A)} \le c_1\|g\|_{W^{\alpha,q}(A)} \le c_2\|g\|_{N^{\beta,q}(A)}, \tag{2.2.22}$$

and for $\alpha q > n$

$$\|g\|_{C^{\alpha-\frac{n}{q}}(A)} \le c_1\|g\|_{W^{\alpha,q}(A)} \le c_2\|g\|_{N^{\beta,q}(A)}. \tag{2.2.23}$$

For the above estimates and more detailed study on the given function spaces we refer to [1, Chapter 7] and [181]. The Nikolskii spaces are very popular in the analysis of PDE, since their definition via difference quotients is rather easy to handle. Namely we introduce for $g \in L^1(\omega)$ and $h \neq 0$

$$D^s_{h,e}(g)(x) := \frac{g(x+he)-g(x)}{|h|^{s-1}h} \text{ for any (unit) vector } e \in \mathbb{R}^2.$$

[4]For the definition of the fractional Sobolev spaces $H^s(\omega)$ see Subsection 2.2.5

In the following, we will omit mentioning the direction e since it is never of relevance and write $D_h^s(q)(y) := D_{h,e}^s(q)(y)$ for an arbitrary direction e. At this point we just wish to mention that these expressions satisfy the following summation-by-parts formula

$$\int_\omega D_{h,e}^s(g)(y)q(y)\,dy = -\int_\omega g(y)D_{-h,e}^s(q)(y)\,dy$$

for all periodic functions $g \in L^p(\omega)$ and $q \in L^{p'}(\omega)$ with $p \in [1,\infty]$.

For a vector field $\mathbf{g} : A \to \mathbb{R}^d$, we say that $\mathbf{g} \in W^{\alpha,p}(A)$, if $\mathbf{g}^i \in W^{\alpha,p}(A)$ for all $i \in \{1,...,d\}$. Finally we denote by

$$W^{\alpha,p}_{\mathrm{div}}(A) = \{\mathbf{g} \in W^{\alpha,p}_{\mathrm{div}}(A) : \mathrm{div}(\mathbf{g}) = 0 \text{ in a distributional sense}\}.$$

2.3 Solenoidal extensions and smooth approximations

In this section we construct a divergence free extension operator from $(0,T) \times \partial\Omega$ to $(0,T) \times \Omega_\eta(t)$. The construction is based on the ideas of the construction in [127, Prop. 2.11]. In contrast to the approach there we will use the celebrated Bogovskiĭ theorem in place of the steady Stokes operator. We use the following theorem that can be found in [77, Section 3.3], and in [63, Appendix 10.5].

Theorem 2.3.1. *Let Ω be uniformly Lipschitz. There exists a linear operator $\mathcal{B} : \hat{C}_0^\infty(\Omega) \to C_0^\infty(\Omega)^d$ which extends from $\hat{W}_0^{k-1,p}(\Omega) \to W_0^{k,p}(\Omega)$ for $1 < p < \infty$ and $k \in \mathbb{Z}$, such that*

$$\|\mathcal{B}(f)\|_{W^{k,p}(\Omega)} \le C\|f\|_{\hat{W}^{k-1,p}(\Omega)}, \quad k \in \mathbb{Z}, \tag{2.3.1}$$

where C is an absolute constant depending only on the Lipschitz constant. Here we use the notation $\hat{C}_0^\infty(\Omega) = \{f \in C_0^\infty(\Omega) : \int_\Omega f\,dx = 0\}$, and for $l \ge 0$, $\hat{W}_0^{l,p}(\Omega) = \{f \in W_0^{l,p}(\Omega) : \int_\Omega f\,dx = 0\}$, $\hat{W}_0^{-l,p}(\Omega) = \{f \in \mathring{W}^{-l,p}(\Omega) : \langle f, 1\rangle = 0\}$, where $\mathring{W}^{-l,p}(\Omega)$ is defined via the norm

$$\|f\|_{\mathring{W}^{-l,p}(\Omega)} = \sup_{\{\phi \in W^{l,p'}(\Omega) : \|\phi\|_{W^{l,p'}} = 1\}} \langle f, \phi\rangle.$$

Within this section we assume that $\eta : [0,T] \times \omega \to \mathbb{R}$ is such that there exists α_η, β_η such that

$$\alpha(\Omega) + \kappa \le \alpha_\eta \le \eta(t,y) \le \beta_\eta \le \beta(\Omega) - \kappa \text{ for all } (t,y) \in [0,T] \times \omega. \tag{2.3.2}$$

Moreover, in this section we use c or C as generic constants which may change their sizes in different instances. Since their dependence on the geometry is relevant for our arguments, it will always be given explicitly in the statements of the results.

The first step is to introduce a solenoidal extension operator. However, since all functions defined on the boundary do not necessary allow for a solenoidal extension, we first need to construct a suitable corrector. We use the coordinates introduced in Definition 2.2.1. For ease of readability we define for a function $\xi : \omega \to \mathbb{R}$

$$\tilde{\xi} : \partial\Omega \to \mathbb{R} \text{ with } \tilde{\xi}(\mathbf{p}(x)) = \tilde{\xi}(x) := \xi(y(x)).$$

In our solenoidal extension the Bogovskiĭ theorem will be applied to

$$S_{\frac{\kappa}{2}} \setminus S_\kappa =: \mathcal{A}_\kappa.$$

Observe, that $\mathcal{A}_\kappa$ is a C^2 domain that contains the support of the function $(\mathbf{p}, s) \mapsto \sigma_\kappa'(\mathbf{p} + s\tilde{\nu}(\mathbf{p}))$ from Definition 2.2.1.

Next we introduce the following weighted mean-value over that set. Let $\lambda \in L^\infty(\mathcal{A}_\kappa)$, $\lambda \ge 0$, and $\int_{\mathcal{A}_\kappa} \lambda(x)\,dx > 0$ be a given weight. Then

$$\langle\psi\rangle_\lambda := \frac{\int_{\mathcal{A}_\kappa} \psi(x)\lambda(x)\,dx}{\int_{\mathcal{A}_\kappa} \lambda(x)\,dx} \text{ for } \psi \in L^1(\mathcal{A}_\kappa).$$

We will denote

$$\lambda_\eta(t,x) := e^{(s(x)-\eta(t,y(x)))\mathrm{div}(\nu(\mathbf{p}(x)))}\sigma'_\kappa(s(x)) \geq 0, \tag{2.3.3}$$

which has compact support in $\mathcal{A}_\kappa$ and satisfies (uniformly in t)

$$c_1 \leq \|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)} \leq c_2\|\lambda_\eta\|_{L^\infty(\mathcal{A}_\kappa)} \leq c_3$$

for some positive constants $c_1 \leq c_2 \leq c_3$ depending just on κ and the upper and lower bounds of η.

Corollary 2.3.2 (Corrector)**.** *Let* (2.3.2) *be satisfied. Then the corrector map*

$$\mathcal{K}_\eta : L^1(\omega) \to \mathbb{R}, \quad \xi \mapsto \mathcal{K}_\eta(\xi) = \langle\tilde{\xi}\rangle_{\lambda_\eta} = \frac{\int_{\mathcal{A}_\kappa} \tilde{\xi}(\mathbf{p}(x))\lambda_\eta(t,x)\,dx}{\int_{\mathcal{A}_\kappa} \lambda_\eta(t,x)\,dx},$$

satisfies the following estimates for $q \in [1,\infty]$:

$$\|\mathcal{K}_\eta(\xi)\|_{L^q(0,T)} \leq C\|\xi\|_{L^q(0,T;L^1(\omega))}, \tag{2.3.4}$$

$$\|\partial_t\mathcal{K}_\eta(\xi)\|_{L^q(0,T)} \leq C\Big(\|\partial_t\xi\|_{L^q(0,T;L^1(\omega))} + \|\xi\partial_t\eta\|_{L^q(0,T;L^1(\omega))}\Big), \tag{2.3.5}$$

whenever the right hand side is finite. Here C depends only on α_η, β_η, and κ.

Proof. The estimates in $L^q(0,T)$ are immediate by the uniform bounds of λ_η and σ. In order to estimate the time-derivative, we use the calculation

$$\begin{aligned}\partial_t\langle\xi(t)\rangle_{\lambda_\eta(t)} = &-\frac{1}{\|\lambda_\eta(t)\|_{L^1}^2}\int_{\mathcal{A}_\kappa}\partial_t\lambda_\eta(t)\,dx\int_{\mathcal{A}_\kappa}\tilde{\xi}(t)\lambda_\eta(t)\,dx + \frac{1}{\|\lambda_\eta(t)\|_{L^1}}\int_{\mathcal{A}_\kappa}\partial_t\tilde{\xi}(t)\lambda_\eta(t)\,dx\\ &+\frac{1}{\|\lambda_\eta(t)\|_{L^1}}\int_{\mathcal{A}_\kappa}\tilde{\xi}(t)\partial_t\lambda_\eta(t)\,dx.\end{aligned}$$

The estimate now follows using $\partial_t\lambda_\eta = -\partial_t\eta\lambda_\eta$ and by the uniform bounds of λ_η and σ.

□

Proposition 2.3.3 (Solenoidal extension)**.** *Let* (2.3.2) *be satisfied and $\eta \in L^\infty(0,T;W^{1,2}(\omega))$. Then there exists a linear solenoidal extension operator*

$$\mathcal{F}_\eta : \{\xi \in L^1(0,T;W^{1,1}(\omega)) : \mathcal{K}_\eta(\xi) \equiv 0\} \to L^1(0,T;W^{1,1}(Q^{\frac{\kappa}{2}})),$$

such that $\mathrm{div}\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi)) = 0$ *for all $\xi \in L^1(0,T;W^{1,1}(\omega))$ and $(\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi)), \xi - \mathcal{K}_\eta(\xi)) \in \mathcal{V}_T$ for $\xi \in \mathcal{V}_K$.*

Moreover, $\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))(t,x) = 0$ for $(t,x) \in (0,T) \times Q_{\frac{\kappa}{2}}$ and it satisfies the following estimates for $q \in [1,\infty]$, $p \in (1,\infty)$ and $l \in \mathbb{N}$.

$$\|\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;L^p(Q^{\frac{\kappa}{2}}))} \leq C\|\xi\|_{L^q(0,T;L^p(\omega))}, \tag{2.3.6}$$

$$\|\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;W^{1,p}(Q^{\frac{\kappa}{2}}))} \leq C\Big(\|\xi\|_{L^q(0,T;W^{1,p}(\omega)} + \|\xi\nabla\eta\|_{L^q(0,T;L^p(\omega)}\Big), \tag{2.3.7}$$

$$\|\partial_t\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;L^p(Q^{\frac{\kappa}{2}}))} \leq C\Big(\|\partial_t\xi\|_{L^q(0,T;L^p(\omega)} + \|\xi\partial_t\eta\|_{L^q(0,T;L^p(\omega))}\Big), \tag{2.3.8}$$

$$\begin{aligned}\|\nabla^2\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;L^p(Q^{\frac{\kappa}{2}}))} &\leq C\Big(\|\nabla^2\xi\|_{L^q(0,T;L^p(\omega)} + \|\xi\nabla^2\eta\|_{L^q(0,T;L^p(\omega))}\Big)\\ &\quad + C\Big(\||\nabla\xi||\nabla\eta|)\|_{L^q(0,T;L^p(\omega))} + \|\xi|\nabla\eta|^2\|_{L^q(0,T;L^p(\omega))}\Big),\end{aligned}$$

$$\begin{aligned}\|\partial_t\nabla\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;L^p(Q^{\frac{\kappa}{2}}))} &\leq C\Big(\|\partial_t\nabla\xi\|_{L^q(0,T;L^p(\omega)} + \|\xi\partial_t\nabla\eta\|_{L^q(0,T;L^p(\omega))}\Big)\\ &\quad + C\||\partial_t\xi||\nabla\eta| + |\nabla\xi||\partial_t\eta| + |\xi\partial_t\eta||\nabla\eta|\|_{L^q(0,T;L^p(\omega))},\end{aligned}$$

$$\|\partial_\nu^l\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))\|_{L^q(0,T;W^{1,p}(Q^{\frac{\kappa}{2}}))} \leq C\|\mathcal{F}_\eta(\xi)\|_{L^q(0,T;W^{1,p}(Q^{\frac{\kappa}{2}}))}, \tag{2.3.9}$$

whenever the right hand side is finite. Here C depends only on α_η, β_η, and κ.

Proof. **Construction:**
The construction relies exclusively on the reference geometry, namely on S_κ defined in Definition 2.2.1. Hence to keep the notation compact we will omit the dependence on the time variable t. Moreover, without loss of generality we assume that $\mathcal{K}_\eta(\xi) = 0$, since otherwise we replace ξ by $\xi - \mathcal{K}_\eta(\xi)$, for which we have

$$\mathcal{K}_\eta(\xi - \mathcal{K}_\eta(\xi)) = \frac{\int_{\mathcal{A}_\kappa}(\tilde{\xi} - \mathcal{K}_\eta(\xi))\lambda_\eta\,dx}{\int_{\mathcal{A}_\kappa}\lambda_\eta\,dx} = \frac{\int_{\mathcal{A}_\kappa}\tilde{\xi}\lambda_\eta\,dx}{\int_{\mathcal{A}_\kappa}\lambda_\eta\,dx} - \mathcal{K}_\eta(\xi) = 0.$$

Hence, once the estimates are valid for ξ, such that $\mathcal{K}_\eta(\xi) = 0$ the estimates follow by Corollary 2.3.2 also for the case $\mathcal{K}_\eta(\xi) \neq 0$.

First observe, that for the coordinates $s(x), \mathbf{p}(x)$ introduced in Definition 2.2.1 we find

$$\nabla s(x) = \partial_\nu s(x)\nu \text{ and } \nabla \mathbf{p}(x) = (\partial_{\boldsymbol{\tau}_i(\mathbf{p}(x))}\mathbf{p}(x))_{i=1,\dots,d-1}$$

and (independent of $s(x)$)

$$\operatorname{div}(\nu(\mathbf{p})) = \sum_{i=1}^{d-1}\partial_{\boldsymbol{\tau}_i(\mathbf{p})}\nu(\mathbf{p})\cdot\boldsymbol{\tau}_i(\mathbf{p}).$$

For $y \in \omega$ and $x \in S_\kappa$ we find by the assumption on Ω, that $y = y(x)$ if and only if $\mathbf{p}(x) = \boldsymbol{\varphi}(y)$ and so (wherever well defined)

$$\partial_{\nu(\mathbf{p})}y(x) = 0 \text{ and so } \partial_{\nu(\mathbf{p})}\xi(y(x)) \equiv 0.$$

Next we introduce the operator:

$$\overline{\mathcal{F}_\eta}(\xi)(x) := e^{(s(x)-\eta(y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\sigma_\kappa(s(x))\nu(\mathbf{p}(x)).$$

Observe, that for $x \in \Omega_\eta \cap S_\kappa$, we find

$$\overline{\mathcal{F}_\eta}(\xi)(x) = e^{(s(x)-\eta(y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\nu(\mathbf{p}(x)).$$

In particular, for $x \in \partial\Omega_\eta$, we find $s(x) = \eta(y(x))$ and hence

$$\overline{\mathcal{F}_\eta}(\xi)(x) = \nu(\mathbf{p}(x))\xi(y(x)),\ x \in \partial\Omega_\eta.$$

Using that $\partial_{\nu(\mathbf{p}(x))}f(x) = -\partial_s f(\mathbf{p}, s)$ we find for $x \in Q^{\frac{\kappa}{2}} \cap S_\kappa$

$$\begin{aligned}\operatorname{div}(\overline{\mathcal{F}_\eta}(\xi)(x)) &= \nabla\Big((e^{(s(x)-\eta(y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\Big)\cdot\nu(\mathbf{p}(x))\\ &\quad + e^{(s(x)-\eta(t,y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\operatorname{div}(\nu(\mathbf{p}(x)))\\ &= -\partial_s\Big((e^{(s(x)-\eta(y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\Big)\\ &\quad + e^{(s(x)-\eta(y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\operatorname{div}(\nu(\mathbf{p}(x)))\\ &= 0.\end{aligned}$$

On $\mathcal{A}_\kappa$ we find (by the same calculations) that

$$\operatorname{div}(\overline{\mathcal{F}_\eta}(\xi)(x)) = -e^{(s(x)-\eta(t,y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}\tilde{\xi}(\mathbf{p}(x))\sigma_\kappa'(s(x)),$$

which has compact support in $\mathcal{A}_\kappa$. Moreover,

$$\int_{\mathcal{A}_\kappa}\operatorname{div}(\overline{\mathcal{F}_\eta}(\xi)(x))\,dx = -\int_{\mathcal{A}_\kappa}\lambda_\eta(x)\tilde{\xi}(\mathbf{p}(x))\,dx = 0.$$

Since $\mathcal{A}_\kappa$ is by assumption a C^2 domain we can apply the Bogovskiĭ operator on this domain which we denote by $\mathcal{B}_\kappa$. We define

$$\mathcal{F}_\eta(\xi)(x) := \overline{\mathcal{F}_\eta}(\xi)(x) - \mathcal{B}_\kappa(\operatorname{div}(\overline{\mathcal{F}_\eta}(\xi)))(x).$$

Estimates:
The estimates are quite standard relaying on the regularity of φ, namely on the C^2-regularity of $\partial\Omega$. We give some details on the estimate in order show a clear dependence on η.

We start with the estimates of the time derivative of $\overline{\mathcal{F}_\eta}(\xi)$. We calculate

$$\partial_t \mathcal{F}_\eta(\xi) = \partial_t \overline{\mathcal{F}_\eta}(\xi) - \mathcal{B}_\kappa(\partial_t \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi))).$$

The Bogovskiĭ operator is well defined due to the fact, that (formally)

$$\int_{\Omega_{\eta(t)}} \partial_t \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi))\,dx = \int_{\mathcal{A}_\kappa} \partial_t \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi))\,dx = \partial_t\bigg(\int_{\mathcal{A}_\kappa} \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi))\,dx\bigg) = 0.$$

We calculate further

$$\begin{aligned}\partial_t \overline{\mathcal{F}_\eta}(\xi)(t,x) &= e^{(s(x)-\eta(t,y(x)))\mathrm{div}(\nu(\mathbf{p}(x)))}\partial_t\xi(t,y(x))\nu(\mathbf{p}(x))\\ &\quad - \mathrm{div}(\nu(\mathbf{p}(x)))\partial_t\eta(t,y(x))e^{(s(x)-\eta(t,y(x)))\mathrm{div}(\nu(\mathbf{p}(x)))}\xi(t,y(x))\nu(\mathbf{p}(x)),\end{aligned}$$

which implies the pointwise estimates for $\partial_t \overline{\mathcal{F}_\eta}(\xi)$:

$$|\partial_t \overline{\mathcal{F}_\eta}(\xi)(t,x)| \le c(|\partial_t\xi(t,\mathbf{p}(x))| + |\partial_t\eta(t,\mathbf{p}(x))||\xi(t,\mathbf{p}(x))|), \tag{2.3.10}$$

where the constant only depends on $\kappa, \alpha_\eta, \beta_\eta$ and the C^2-regularity of $\partial\Omega$. For the sake of better understanding we demonstrate that the assumption $\mathcal{K}_\eta(\xi) = 0$ is indeed without loss of generality. We estimate

$$\begin{aligned}&|\partial_t \overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))(t,x)|\\ &\le c\big(|\partial_t\xi(t,\mathbf{p}(x))| + |\partial_t\eta(t,\mathbf{p}(x))|(|\xi(t,\mathbf{p}(x))| + \|\xi(t)\|_{L^1}) + \|\partial_t\eta(t)\xi(t)\|_{L^1}\big).\end{aligned} \tag{2.3.11}$$

In order to estimate the Bogovskiĭ part we find by Theorem 2.3.1 (with a constant just depending on the Lipschitz constant of $\mathcal{A}_\kappa$) that

$$\begin{aligned}\|\mathcal{B}_\kappa(\partial_t \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))))\|_{L^p(\Omega_\eta)} &= \|\mathcal{B}_\kappa(\partial_t \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))))\|_{L^p(\mathcal{A}_\kappa)}\\ &\le c\|\mathrm{div}(\partial_t \overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))))\|_{\hat{W}^{-1,p}(\mathcal{A}_\kappa)}\\ &= c\|\partial_t \overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))\|_{L^p(\mathcal{A}_\kappa)}.\end{aligned}$$

and so the estimate on $\partial_t \mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))$ follows by (2.3.11).

The estimates on $\nabla \mathcal{F}_\eta(\xi - \mathcal{K}_\eta\xi), \nabla^2 \mathcal{F}_\eta(\xi - \mathcal{K}_\eta\xi)$ and $\partial_t \nabla \mathcal{F}_\eta(\xi - \mathcal{K}_\eta\xi)$ are analogous and we skip the details here. Observe that due to the compact support of $\mathrm{div}(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))))$, by Gauss theorem we find that

$$\int_{\Omega_{\eta(t)}} \nabla^l \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta\xi))\,dx = 0 = \int_{\Omega_{\eta(t)}} \partial_t \nabla^l \mathrm{div}(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta\xi))\,dx;$$

hence $\mathcal{B}_\kappa$ is always well defined.

Clearly the normal derivatives of the constructed function $\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi)))$ depend on the estimates of the derivatives of σ_κ and not on the regularity of the derivatives of η. Since the Bogovskiĭ theorem transfers the regularity to $\mathcal{F}_\eta(\xi - \mathcal{K}_\eta(\xi))$ with no further loss, (2.3.9) follows with according dependences on the higher order derivatives of σ_κ.

□

We include the following corollary that will be necessary for our compactness result (See Section 2.5).

Corollary 2.3.4 (Smooth Solenoidal Extension)**.** *Let* $a,\ r \in [2,\infty]$, $p,\ q \in (1,\infty)$ *and* $s \in [0,1]$. *Assume that* $\eta \in L^r([0,T]; W^{2,a}(\omega)) \cap W^{1,r}([0,T]; L^a(\omega))$, *such that* $\alpha(\Omega) + \kappa \le \alpha_\eta \le \eta \le \beta_\eta \le \beta(\Omega) - \kappa$.

Let $b \in W^{s,p}(\omega))$ and take $(b)_\delta$ as a smooth approximation of b in ω. Then $E_{\eta,\delta}(b) := \mathcal{F}_\eta((b)_\delta - \mathcal{K}_\eta((b)_\delta))$ satisfies all the regularity of Proposition 2.3.3. In particular

$$\|E_{\eta(t),\delta}(b) - \mathcal{F}_\eta(b - \mathcal{K}_\eta(b))\|_{L^p(Q^{\frac{\kappa}{2}})} \le c\|(b)_\delta - b\|_{L^p(\omega)}$$

and

$$\|\partial_t E_{\eta(t),\delta}(b)\|_{L^r(0,T;L^a(Q^{\frac{\kappa}{2}})} \le c\|(b)_\delta \partial_t \eta(t)\|_{L^r(0,T;L^a(\omega)}$$

uniformly in $t \in (0,T)$.

We include the following technical lemma, that will be necessary for the regularity result.

Lemma 2.3.5. *Let $p,\ \tilde{a} \in (1,\infty)$ such that $p' < \tilde{a} \le \frac{dp'}{d-p'}$ if $p' < d$, and $p' < \tilde{a} < \infty$ otherwise, and let the assumptions of Proposition 2.3.3 be satisfied. Assume additionally that $\eta \in C^{0,\theta}(\omega) \cap W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)$ and $\mathbf{u} \in W^{1,p'}(\Omega_\eta)$ then the above constructed test function satisfies*

$$\left|\int_{\Omega_\eta} \mathbf{u}\cdot\mathcal{F}_\eta(D^s_{h,e}\xi - \mathcal{K}_\eta(D^s_{h,e}\xi))\,dx\right| \le c(h^{\theta-s} + \|D^s_{h,e}\eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)})\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}\|\xi\|_{L^p(\omega)} \tag{2.3.12}$$

and in case $\partial_t\xi \in L^p(\omega)$

$$\left|\int_{\Omega_\eta} \mathbf{u}\cdot\partial_t\mathcal{F}_\eta(D^s_{h,e}\xi - \mathcal{K}_\eta(D^s_{h,e}\xi))\,dx\right| \tag{2.3.13}$$

$$\le c\Big(\big(h^{\theta-s} + \|D^s_{h,e}\eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)}\big)\|\partial_t\xi\|_{L^p(\omega)}\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} + \|D^s_{h,e}\xi\partial_t\eta\|_{L^{\tilde{a}'}(\omega)}\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}\Big). \tag{2.3.14}$$

The constants are only depending on α_η, β_η, κ and (linearly) on $\|\eta\|_{C^{1,\theta}(\omega)}$.

Proof. In the following we use the abbreviation $\delta_h f(y) = (f(y+e_i h) - f(y))$ for $i = 1,2$. Moreover, since all estimates are done pointwise in time, we omit the dependence on t of η and Ω_η. First, since the support of $\mathcal{F}_\eta(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))$ is $S_{\frac{\kappa}{2}}$, we can use the coordinates $(\mathbf{p},s)$ on the full support of $\mathcal{F}_\eta(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))$. We will use the following change of coordinates $\psi_\eta\circ\mathbf{\Phi} : \omega\times(\alpha+\kappa/2,0] \to \Omega_\eta$ in order to be able to do integration by parts. Hence

$$\begin{aligned}&\int_{\Omega_\eta} \mathbf{u}\cdot\mathcal{F}_\eta(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))\,dx\\ &\quad= \int_{\alpha+\kappa/2}^0\int_\omega (\mathbf{u}\cdot\mathcal{F}_\eta(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi)))\circ\psi_\eta\circ\mathbf{\Phi}|\det(\nabla(\psi_\eta\circ\mathbf{\Phi})|\,dy\,ds\end{aligned}$$

We will use the following abbreviations for the sake of a better overview:

$$\alpha = \alpha(\Omega), \quad \tilde{\gamma}(s,y) := |\det(\nabla(\psi_\eta\circ\mathbf{\Phi})(s,y)| \text{ and } \tilde{\lambda}_\eta(s,y) := e^{(s-\eta(y)\mathrm{div}_x(\nu(y))}\sigma'_\kappa(s)\tilde{\gamma}(s,y).$$

Hence, we calculate

$$\begin{aligned}\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}\mathcal{K}_\eta(\delta_h\xi) &= \int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega \lambda_\eta\circ\psi_\eta\circ\mathbf{\Phi}\delta_h\xi|\det(\nabla(\psi_\eta\circ\mathbf{\Phi})|\,dy\,ds\\ &=: \int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega \delta_h\xi(y)\tilde{\lambda}_\eta(s,y)\,dy\,ds = \int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega \xi(y)\delta_h\tilde{\lambda}_\eta(s,y)\,dy\,ds.\end{aligned}$$

where we used summation by parts formula for finite differences. Therefore,

$$\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi) = \delta_h(\xi - \mathcal{K}_\eta(\xi)) - \frac{\int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega \xi(y)\delta_h(\tilde{\lambda}_\eta)\,dy}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}}.$$

And so

$$\begin{aligned}
&\overline{\mathcal{F}_\eta}(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi)) \circ \psi_\eta \circ \mathbf{\Phi}(s,y) = \delta_h(\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi)) \circ \psi_\eta \circ \mathbf{\Phi}(s,y))\\
&\quad - \delta_h\Big(e^{(s-\eta(y)\mathrm{div}_x(\nu(y))}\Big)\sigma_\kappa(s)\Big(\xi - \langle\xi\rangle_{\lambda_\eta}\Big)\nu(y)\\
&\quad - e^{(s-\eta(y)\mathrm{div}_x(\nu(y))}\sigma_\kappa(s)\Big(\xi - \langle\xi\rangle_{\lambda_\eta}\Big)\delta_h(\nu(y))\\
&\quad - e^{(s-\eta(y)\mathrm{div}_x(\nu(y))}\sigma_\kappa(s)\frac{\int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega \xi(y)\delta_h(\tilde\lambda_\eta)\,dy\,ds}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}}\nu(y)\\
&=: \delta_h(\overline{\mathcal{F}_\eta}(\xi) \circ \psi_\eta \circ \mathbf{\Phi}(s,y)) + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.
\end{aligned}$$

The partial summation and the Hölder's inequality imply that

$$\begin{aligned}
&\left|\int_{\Omega_\eta} \mathbf{u}\cdot\mathcal{F}_\eta(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))\,dx\right|\\
&\quad\le \left|\int_{\alpha+\kappa/2}^0\int_\omega (\mathbf{u}\circ\psi_\eta\circ\mathbf{\Phi}\cdot\delta_h\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))\circ\psi_\eta\circ\mathbf{\Phi}\,\tilde\gamma\,dy\,ds\right|\\
&\qquad + \left|\int_{\alpha+\kappa/2}^0\int_\omega (\mathbf{u}\circ\psi_\eta\circ\mathbf{\Phi}\cdot(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3))\tilde\gamma\,dy\,ds\right|\\
&\qquad + \left|\int_{\mathcal{A}_\kappa}\mathbf{u}\cdot\mathcal{B}_\kappa(\mathrm{div}(\overline{\mathcal{F}_\eta}(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))\,dx\right|\\
&\quad= (I) + (II) + (III).
\end{aligned}$$

Recall, that $p' < \tilde a \le \frac{dp'}{d-p'}$ (if $p < d$ and no upper bound otherwise). Observe, that

$$|\delta_h\eta| \le ch^\theta, \quad |\tilde\gamma| + |\nabla(\psi_\eta\circ\mathbf{\Phi})| \le c(1+|\nabla\eta|) \text{ and } |\delta_h\tilde\gamma| \le c|\delta_h\nabla\eta|$$

and

$$\begin{aligned}
|\delta_h\mathbf{u}\circ\psi_\eta\circ\mathbf{\Phi}| &\le \left|\frac{\mathbf{u}(\psi_\eta\circ\mathbf{\Phi}(x+h)) - \mathbf{u}(\psi_\eta\circ\mathbf{\Phi}(x))}{\psi_\eta\circ\mathbf{\Phi}(x+h) - \psi_\eta\circ\mathbf{\Phi}(x)}\right||\psi_\eta\circ\mathbf{\Phi}(x+h) - \psi_\eta\circ\mathbf{\Phi}(x)|\\
&\le ch^\theta \fint_{\psi_\eta\circ\mathbf{\Phi}(x)}^{\psi_\eta\circ\mathbf{\Phi}(x+h)} |\nabla u|\,ds.
\end{aligned}$$

We estimate (I) using partial integration, the above inequality, Hölder's inequality for $\frac{1}{p} + \frac{1}{\tilde a} + \frac{\tilde a p - p - \tilde a}{\tilde a p} = 1$ and Sobolev embedding:

$$\begin{aligned}
(I) &= \left|\int_{\alpha+\kappa/2}^0\int_\omega (\delta_h(\mathbf{u}\circ\psi_\eta\circ\mathbf{\Phi})\cdot\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))\circ\psi_\eta\circ\mathbf{\Phi})\,\tilde\gamma\right.\\
&\qquad\left. + (\mathbf{u}\circ\psi_\eta\circ\mathbf{\Phi})\cdot\overline{\mathcal{F}_\eta}(\xi - \mathcal{K}_\eta(\xi))\circ\psi_\eta\circ\mathbf{\Phi})\,\delta_h\tilde\gamma\,dy\,ds\right|\\
&\le ch^\theta\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}\|\xi\|_{L^p(\omega)} + c\|\mathbf{u}\|_{L^{\tilde a}(\Omega_\eta)}\|\xi\|_{L^p(\omega)}\|\delta_h\eta\|_{W^{1,\frac{\tilde a p}{\tilde a p - \tilde a - p}}(\omega)}\\
&\le c\big(h^\theta + \|\delta_h\eta\|_{W^{1,\frac{\tilde a p}{\tilde a p - \tilde a - p}}(\omega)}\big)\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}\|\xi\|_{L^p(\omega)}.
\end{aligned} \tag{2.3.15}$$

We further estimate (II) by using, in a rather straightforward manner, the fact that $|\delta_h g| \le h^\theta\|g\|_C^{0,\theta}$

$$\begin{aligned}
|\mathcal{T}_1\tilde\gamma| &\le c(h + |\delta_h\eta|)(|\xi|\tilde\gamma + |\langle\xi\rangle_{\tilde\lambda_\eta}\tilde\lambda_\eta|) \le ch^\theta(|\xi|\tilde\gamma + |\langle\xi\rangle_{\tilde\lambda_\eta}\tilde\lambda_\eta|),\\
|\mathcal{T}_2\tilde\gamma| &\le ch(|\xi|\tilde\gamma + |\langle\xi\rangle_{\tilde\lambda_\eta}\tilde\lambda_\eta|),\\
|\mathcal{T}_3\tilde\gamma| &\le c\frac{|\tilde\lambda_\eta|}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}}\|\xi\|_{L^p(\omega)}\|\delta_h\eta\|_{W^{1,p'}(\Omega)}.
\end{aligned}$$

This implies

$$\sum_{i=1}^{2}\int_{\alpha+\kappa/2}^{0}\int_{\omega}|\mathcal{T}_i\tilde{\gamma}|^p\,dy,\,ds \le ch^{\theta p}\|\xi\|_{L^p(\omega)}^p\left(1+\frac{\|\tilde{\lambda}_\eta\|_{L^{p'}(\mathcal{A}_\kappa)}^p}{\|\tilde{\lambda}_\eta\|_{L^1(\mathcal{A}_\kappa)}^p}\right),$$
$$\int_{\alpha+\kappa/2}^{0}\int_{\omega}|\mathcal{T}_3\tilde{\gamma}|^p\,dy,\,ds \le c\frac{\|\tilde{\lambda}_\eta\|_{L^p([\alpha+\kappa/2,0]\times\omega)}^p}{\|\lambda_\eta\|_{L^1(\mathcal{A}_\kappa)}^p}\|\xi\|_{L^p(\omega)}^p\|\delta_h\eta\|_{W^{1,p'}(\omega)}^p.$$

Hence, we find by Hölder's and Poincaré's inequality that

$$(II) \le c\|\mathbf{u}\|_{L^{p'}(\Omega_\eta)}\sum_{i=1}^{3}\|\mathcal{T}_i\tilde{\gamma}\|_{L^p((\alpha+\kappa/2,0)\times\omega)} \le c(h^\theta+\|\delta_h\eta\|_{W^{1,p'}(\omega)})\|\xi\|_{L^p(\omega)}\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}. \tag{2.3.16}$$

The estimates on (I) and (II) allow to estimate the Bogovskiĭ term (III). This is possible since due to Theorem 2.3.1 and due to the compact support of σ' in $\mathcal{A}_\kappa$ we find

$$\begin{aligned}
(III) &:= |\langle \mathbf{u}, \mathcal{B}_\kappa(\operatorname{div}(\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))))\rangle| \\
&\le \|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)}\|\mathcal{B}_\kappa(\operatorname{div}(\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))))\|_{\mathring{W}^{-1,p}(\mathcal{A}_\kappa)} \\
&\le c\|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)}\|\operatorname{div}(\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi)))\|_{\mathring{W}^{-2,p}(\mathcal{A}_\kappa)} \\
&\le c\|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)}\|\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))\|_{\mathring{W}^{-1,p}(\mathcal{A}_\kappa)}.
\end{aligned}$$

Now take $q\in W_0^{1,p'}(\mathcal{A}_\kappa)$, with $\|q\|_{W^{1,p'}(\mathcal{A}_\kappa)}\le 1$ arbitrary. From the calculations above, i.e. by replacing $\mathbf{u}$ by q in (2.3.15) and (2.3.16) we find

$$\begin{aligned}
&\langle \overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi)),q\rangle \\
&= \int_{\alpha+\kappa/2}^{0}\int_\omega(\delta_h(\overline{\mathcal{F}_\eta}(\xi-\mathcal{K}_\eta(\xi)))\cdot q)\circ\psi_\eta\circ\mathbf{\Phi}\tilde{\gamma}\,dy\,ds+\sum_{i=1}^{3}\int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega\mathcal{T}_i\cdot q\circ\psi_\eta\circ\mathbf{\Phi}\tilde{\gamma}\,dy\,ds \\
&= \int_{\alpha+\kappa/2}^{0}\int_\omega(\overline{\mathcal{F}_\eta}(\xi-\mathcal{K}_\eta(\delta_h\xi))\circ\psi_\eta\circ\mathbf{\Phi}\cdot\delta_h(q\circ\psi_\eta\circ\mathbf{\Phi}\tilde{\gamma})\,dy\,ds \\
&+\sum_{i=1}^{3}\int_{\alpha+\kappa/2}^{\alpha+\kappa}\int_\omega\mathcal{T}_i\cdot q\circ\psi_\eta\circ\mathbf{\Phi}\tilde{\gamma}\,dy\,ds \le c(h^\theta+\|\delta_h\eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)})\|\xi\|_{L^p(\omega)}.
\end{aligned}$$

But so

$$\begin{aligned}
(III) &\le c\|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)}\|\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))\|_{\mathring{W}^{-1,p}(\mathcal{A}_\kappa)} \\
&\le c\|\mathbf{u}\|_{W^{1,p'}(\mathcal{A}_\kappa)}\Big(h^\theta+\|\delta_h\eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\omega)}\Big)\|\xi\|_{L^p(\omega)}.
\end{aligned}$$

This finishes the proof of (2.3.12). For the time derivative we use the fact that

$$\begin{aligned}
\partial_t(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi)) &= (\delta_h\partial_t\xi-\mathcal{K}_\eta(\delta_h\partial_t\xi))-\frac{\langle\delta_h\tilde{\xi}(t)\rangle_{\lambda_\eta}}{\|\lambda_\eta(t)\|_{L^1}}\int_{\mathcal{A}_\kappa}\partial_t\lambda_\eta(t)\,dx \\
&+\frac{1}{\|\lambda_\eta(t)\|_{L^1}}\int_{\mathcal{A}_\kappa}\delta_h\tilde{\xi}(t)\partial_t\lambda_\eta(t)\,dx =: \mathcal{K}_\eta(\delta_h\partial_t\xi))+K(t),
\end{aligned} \tag{2.3.17}$$

and hence

$$\begin{aligned}
&\partial_t\overline{\mathcal{F}_\eta}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))(t,x) \\
&\quad= \sigma(s(x))e^{(s(x)-\eta(t,y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}(\delta_h(\partial_t\xi)-\mathcal{K}_\eta(\delta_h(\partial_t\xi))(t,y(x))\nu(\mathbf{p}(x)) \\
&\quad-\sigma(s(x))\operatorname{div}(\nu(\mathbf{p}(x)))\partial_t\eta(t,y(x))e^{(s(x)-\eta(t,y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}(\delta_h\xi-\mathcal{K}_\eta(\delta_h\xi))(t,y(x))\nu(\mathbf{p}(x)) \\
&\quad-\sigma(s(x))\operatorname{div}(\nu(\mathbf{p}(x)))e^{(s(x)-\eta(t,y(x)))\operatorname{div}(\nu(\mathbf{p}(x)))}K(t)\nu(\mathbf{p}(x)) \\
&= (A)+(B)+(C).
\end{aligned}$$

The estimates on (A) follows by (2.3.12). We proceed with the straightforward estimates

$$|(B)| \le c|\partial_t\eta(t,y(x))|(|\delta_h(\xi(t,y(x))| + \|\delta_h(\xi(t))\|_{L^1(\mathcal{A}_\kappa)}),$$

and

$$|(C)| \le c\|\delta_h\xi(t)\partial_t\eta(t)\|_{L^1(\omega)}.$$

Hence, we find by (2.3.12), the estimates on (B) and (C), Hölder's inequality and Sobolev embedding that

$$\begin{aligned}
&\left|\int_{\Omega_\eta} \mathbf{u}\cdot\partial_t\overline{\mathcal{F}_\eta}(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi))\,dx\right| \\
&\le \Big(h^\theta + \|\delta_h\eta\|_{W^{1,\frac{\tilde a p}{\tilde a p-\tilde a-p}}(\omega)}\Big)\|\partial_t\xi\|_{L^p(\omega)}\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)} + c\|\mathbf{u}\|_{L^{\tilde a}(\Omega_\eta)}\|\delta_h\xi\partial_t\eta\|_{L^{\tilde a'}(\omega)} \\
&\le c\Big(\Big(h^\theta + \|\delta_h\eta\|_{W^{1,\frac{\tilde a p}{\tilde a p-\tilde a-p}}(\omega)}\Big)\|\partial_t\xi\|_{L^p(\omega)} + \|\delta_h\xi\partial_t\eta\|_{L^{\tilde a'}(\omega)}\Big)\|\mathbf{u}\|_{W^{1,p'}(\Omega_\eta)}.
\end{aligned}$$

The Bogovskiĭ part will be estimated once more in form of negative norms using that

$$\begin{aligned}
&\sup_{\|q\|_{W^{1,p'}(\mathcal{A}_\kappa)}\le 1} \langle\partial_t\overline{\mathcal{F}_\eta}(\delta_h\xi - \mathcal{K}_\eta(\delta_h\xi)))),q\rangle \\
&\qquad\le c\Big(\Big(h^\theta + \|\delta_h\eta\|_{W^{1,\frac{\tilde a p}{\tilde a p-\tilde a-p}}(\omega)}\Big)\|\partial_t\xi\|_{L^p(\omega)} + \|\delta_h\xi\partial_t\eta\|_{L^{\tilde a'}(\omega)}\Big),
\end{aligned}$$

which finishes the proof. □

2.4 The regularity result

2.4.1 Estimates for the structure

In this section we explore the consequences of the energy inequality (2.2.21).

Lemma 2.4.1 (Uniform Korn's inequality)**.** *For every $\mathbf{u}\in\mathcal{V}_F$ such that $\mathbf{u}(t,\varphi_\eta(t,.)) = \xi\nu$ the following Korn's equality holds:*

$$\|\nabla\mathbf{u}\|^2_{L^2\Omega_\eta(t)} = 2\|\mathrm{sym}\nabla\mathbf{u}\|^2_{L^2\Omega_\eta(t)}. \tag{2.4.1}$$

Proof. We follow the idea from [31, Lemma 6] and compute:

$$\int_{\Omega_{\eta(t)}} |\mathrm{sym}\nabla\mathbf{u}|^2\,dx = \frac{1}{2}\Big(\int_{\Omega_{\eta(t)}}|\nabla u|^2\,dx + \int_{\Omega_\eta(t)}\nabla^T\mathbf{u}:\nabla\mathbf{u}\,dx\Big).$$

Therefore it remains to show that the second term is zero:

$$\begin{aligned}
\int_{\Omega_\eta(t)}\nabla^T\mathbf{u}:\nabla\mathbf{u}\,dx &= \sum_{i,j=1}^2\int_{\Omega_\eta(t)}\partial_j u_i\partial_i u_j\,dx \\
&= -\sum_{i,j=1}^2\int_{\Omega_\eta(t)}\partial_j\partial_i u_i u_j\,dx + \int_{\partial\Omega_\eta(t)}\partial_j u_i n_i u_j dS. = \int_{\partial\Omega_\eta(t)}(\nabla\mathbf{u})\nu\cdot\mathbf{u}dS
\end{aligned}$$

Now using the no-slip condition (2.2.5) and the incompressibility condition we deduce $\int_{\partial\Omega_\eta(t)}(\nabla\mathbf{u})\nu\cdot\mathbf{u}dS = 0$ (see [127, Lemma A.5]) and therefore the Korn's equality holds. □

In the following we exploit the energy estimate (2.2.21). In particular, the number C_0, which depends only on the initial conditions, always refers to this energy bound.

Lemma 2.4.2. *Let $(\mathbf{u},\eta)$ be such that energy inequality* (2.2.21) *is satisfied. Then $\eta\in L^\infty(0,T;W^{1,4}(\omega))$ and $\|\eta\|_{L^\infty_t W^{1,4}_x}\le cC_0$, where c depends only on $\boldsymbol{\varphi}$.*

Proof. The boundedness of $\|\eta\|_{L_t^\infty L_x^2}$ follows directly from the energy inequality (2.2.21). Now, we use [38, Theorem 3.3-2.] to conclude that by the definition of $\mathcal{A}$ and (2.2.21):

$$\int_\omega |\boldsymbol{G}(\eta(t,.))|^2\,dy \le c\int_\omega \mathcal{A}\boldsymbol{G}(\eta(t,.)) : \boldsymbol{G}(\eta(t,.))\,dy \le cC_0,$$

here the constant c just depends on the Lamé constants and the geometry of $\partial\Omega$. If $\partial_\alpha\nu \neq 0$ we my use the bound for $G_{\alpha\alpha}(\eta)$ and (2.2.10) to get the bounds for $\|\partial_\alpha\eta(t)\|_{L^4(\omega)}$ and $\|\eta(t)\|_{L^4(\omega)}$ uniform in t. Using these bounds, again (2.2.10) and the bound for $G_{\beta\beta}(\eta)$ above for $\beta \neq \alpha$ we finish the proof.

If $\partial_1\nu = \partial_2\nu = 0$, we get the bound for $\|\nabla\eta\|_{L^4}$ directly from (2.2.10) and the boundedness of $\int_\omega |\boldsymbol{G}(\eta(t,.))|^2$. However, since $\|\eta\|_{L_t^\infty L_x^2}$ is also bounded (using the bounds on $\partial_t\eta$ in (2.2.21)), the Lemma follows also by the Poincaré inequality. □

Lemma 2.4.3. *Let $(\mathbf{u},\eta)$ be such that energy inequality (2.2.21) is satisfied. Then if $\overline{\gamma}(\eta) \neq 0$ we have $\eta(t) \in H^2(\omega)$. Moreover,*

$$\sup_{t\in[0,T]}\int_\omega \overline{\gamma}^2(\eta)|\nabla^2\eta|^2\,dy \le cC_0.$$

where c depends only on $\boldsymbol{\varphi}$.

Proof. We can again use Theorem 3.3-2. from [38] and work with bounds on $\mathbf{R}$. From (2.2.8) we compute:

$$\partial_\beta\mathbf{a}_\alpha(\eta) = \partial^2_{\alpha\beta}\boldsymbol{\varphi} + \partial^2_{\alpha\beta}\eta\nu + \partial_\alpha\eta\partial_\beta\nu + \partial_\beta\eta\partial_\alpha\nu + \eta\partial^2_{\alpha\beta}\nu,\ \alpha,\beta = 1,2. \tag{2.4.2}$$

Using (2.2.8), (2.2.11), (2.2.12) and the definition of $\overline{\gamma}$ from Definition 2.2.1 we have

$$\begin{aligned} R_{\alpha\beta}(\eta) = \frac{1}{|\mathbf{a}_1\times\mathbf{a}_2|}\partial^2_{\alpha\beta}\eta\Big(|a_1\times a_2| + \eta(\nu\cdot(\mathbf{a}_1\times\partial_2\nu + \partial_1\nu\times\mathbf{a}_2)) + \eta^2\nu\cdot(\partial_1\nu\times\partial_2\nu)\Big)\\ +P_0(\eta,\nabla\eta) =: \overline{\gamma}(\eta)\partial^2_{\alpha\beta}\eta + P_0(\eta,\nabla\eta), \end{aligned} \tag{2.4.3}$$

where P_0 is a polynomial of order three in η and $\nabla\eta$ such that all terms are at most quadratic in $\nabla\eta$, and the coefficients of P_0 depend on $\boldsymbol{\varphi}$.

From Lemma 2.4.2 we gain in particular by Sobolev embedding that $\|\eta\|_{L_t^\infty L_x^\infty}$ and $\|\nabla\eta\|_{L_t^\infty L_x^4}$ are bounded by the energy. Therefore

$$\sup_{t\in[0,T]}\int_\omega \overline{\gamma}^2(\eta)|\nabla^2\eta|^2\,dy \le c(\|\mathbf{R}\|_{L_t^\infty L_x^2} + \|P_0(\eta,\nabla\eta)\|_{L_t^\infty L_x^2}) \le cC_0.$$

□

Remark 2.4.4. *By definition we know that $\overline{\gamma} > 0$, as long as*

$$\eta(\nu\cdot(\mathbf{a}_1\times\partial_2\nu + \partial_1\nu\times\mathbf{a}_2)) + \eta^2\nu\cdot(\partial_1\nu\times\partial_2\nu) > -\frac{1}{|\mathbf{a}_1\times\mathbf{a}_2|}. \tag{2.4.4}$$

Therefore it can be easily checked that there exists a c_2 (depending on $\boldsymbol{\varphi}$ only) such that if $\|\eta\|_{L_t^\infty L_x^\infty} \le c_2$, then (2.4.4) is satisfied and hence $\overline{\gamma}(\eta) > 0$. Finally, the energy estimate allows to deduce, that in dependence of the initial configuration there is a minimal time interval $(0,T)$ for which $\|\eta\|_{L_t^\infty L_x^\infty} \le c_2$ is always satisfied.

Similarly as in previous Lemma, let us write form a_b defined by (2.2.15) as a sum of the bilinear form in second derivatives plus the remainder. We calculate the Fréchet derivative of $\mathbf{R}$:

$$R_{\alpha\beta}(\eta)\xi = \overline{\gamma}(\eta)\partial^2_{\alpha\beta}\xi + \overline{\gamma}(\xi)\partial^2_{\alpha\beta}\eta + P_0'(\eta,\nabla\eta)\xi.$$

Therefore we have

$$
\begin{aligned}
a_b(t,\eta,\xi) &= \frac{h^3}{24}\int_w \Big[\mathcal{A}\big(\overline{\gamma}(\eta)\nabla^2\eta\big):\big(\overline{\gamma}(\eta)\nabla^2\xi\big)+\mathcal{A}\big(\overline{\gamma}(\eta)\nabla^2\eta\big):\big(\overline{\gamma}(\xi)\nabla^2\eta\big)\\
&+\Big(\mathcal{A}\big(\overline{\gamma}(\eta)\nabla^2\eta\big):P_0'(\eta,\nabla\eta)\xi+\mathcal{A}\big(P_0(\eta,\nabla\eta)\big):\big(\overline{\gamma}(\xi)\nabla^2\eta\big)\Big)\\
&+\mathcal{A}\big(P_0(\eta,\nabla\eta)\big):\big(\overline{\gamma}(\eta)\nabla^2\xi\big)+\mathcal{A}\big(P_0(\eta,\nabla\eta)\big):P_0'(\eta,\nabla\eta)\xi\Big]dy\\
&= a_b^1(\eta;\nabla^2\eta,\nabla^2\xi)+a_b^2(\eta,\nabla^2\eta;\xi)+a_b^3(\eta,\nabla\eta,\nabla^2\eta;\xi,\nabla\xi)\\
&+a_b^4(\eta,\nabla\eta;\nabla\xi,\nabla^2\xi)+a_b^5(\eta,\nabla\eta,\xi,\nabla\xi).
\end{aligned}
\tag{2.4.5}
$$

We take $\xi = D^s_{-h}D^s_h\eta$, $0<s<1/2$, and obtain the following estimates.

Lemma 2.4.5. *Let $\eta\in H^2(\omega)$ such that $\overline{\gamma}(\eta)\neq 0$. Then for every $h>0$, $0<s<1/2$ the following inequality holds:*

$$a_b(t,\eta,D_h^{-s}D_h^s\eta)\geq \|D_h^s\nabla^2\eta\|_{L^2(\omega)}-C(\|\eta\|_{H^2(\omega)}).$$

Proof. Since all estimates in this lemma are uniform in t for simplicity of notation, we omit the t variable in this proof. First we use the fact that since $\omega\subset\mathbb{R}^2$ Sobolev embedding implies $\|D_h^s\eta\|_{L^\infty}\leq c\|\eta\|_{H^2(\omega)}$ and $\|D^s_{-h}D^s_h\eta\|_{L^\infty(\omega)}\leq c\|\eta\|_{H^2(\omega)}$. Due to Sobolev embedding the estimate is uniform in h for all $s\in(0,1/2)$. This and the integration by parts formula for the finite differences can be used to estimate a_b^1:

$$
\begin{aligned}
a_b^1(\eta;\nabla^2\eta,\nabla^2 D^s_{-h}D^s_h\eta) &\geq C\int_\omega |D^s_h\nabla^2\eta|^2\,dy - C\|D^s_h\overline{\gamma}(\eta)^2\|_{L^\infty}\|\nabla^2\eta\|_{L^2}\|D^s_h\nabla^2\eta\|_{L^2}\\
&\geq \frac{C}{2}\|D^s_h\nabla^2\eta\|^2_{L^2}-C\|D^s_h\overline{\gamma}(\eta)^2\|^2_{L^\infty}\|\nabla^2\eta\|^2_{L^2}\geq\frac{C}{2}\|D^s_h\nabla^2\eta\|^2_{L^2}-C(\|\eta\|_{H^2(\omega)}).
\end{aligned}
$$

Similarly, since $\|D^s_{-h}D^s_h\eta\|_{L^\infty(\omega)}\leq\|\eta\|_{H^2(\omega)}$ uniformly, we estimate

$$|a_b^2(\eta,\nabla^2\eta,D^s_{-h}D^s_h\eta)|\leq C(\|\eta\|_{H^2(\omega)}).$$

To estimate a_b^3 we first notice that $\|P_0(\eta,\nabla\eta)\|_{L^2}\leq C\|\eta\|_{L^\infty}\|\nabla\eta\|^2_{L^4}\leq C(\|\eta\|_{H^2(\omega)})$. Moreover,

$$\|P_0'(\eta,\nabla\eta)D^s_h\eta\|_{L^2}\leq C\|\eta\|_{L^\infty}\|\nabla\eta\|_{L^4}\|\nabla D^s_h\eta\|_{L^4}\leq C(\|\eta\|_{H^2(\omega)}).$$

Now we can use integration by parts and Young's inequality in the same way as in the estimate for a_b^1 to get

$$|a_b^3(\eta,\nabla\eta,\nabla^2\eta;D^{1/2}_{-h}D^{1/2}_h\eta,\nabla,D^{1/2}_{-h}D^{1/2}_h\eta)\leq\frac{C}{8}\|D^s_h\nabla^2\eta\|^2_{L^2}+C(\|\eta\|_{H^2(\omega)}).$$

Estimate for a_b^4 is done in analogous way by integration by parts and using:

$$\|D^s_hP_0(\eta,\nabla\eta)\|_{L^2}\leq\|\eta\|_{L^\infty}\|\eta\|_{W^{1,4}\nabla D^s_h\eta}\|_{L^4}\leq C(\|\eta\|_{H^2(\omega)}).$$

Hence,

$$|a_b^4(\eta,\nabla\eta;\nabla D^s_{-h}D^s_h\eta,\nabla^2D^s_{-h}D^s_h\eta)|\leq\frac{C}{8}\|D^s_h\nabla^2\eta\|^2_{L^2}+C(\|\eta\|_{H^2(\omega)}).$$

Finally, the last term a_b^5 is a lower order term and is easily estimated using the same inequalities:

$$|a_b^5(\eta,\nabla\eta,D^s_{-h}D^s_h\eta,\nabla D^s_{-h}D^s_h\eta)|\leq C(\|\eta\|_{H^2(\omega)}).$$

□

2.4.2 Closing the estimates–Proof of Theorem 2.2.5

In this section we finish the proof of Theorem 2.2.5. Please observe first, that due to the Sobolev embedding theorem and due to the trace theorem [22, Lemma 2.4] we find for all $\theta \in (0,1)$ and all $s \in (0, \frac{1}{2})$

$$\|\eta\|_{L^\infty(0,T;C^{0,\theta}(\omega))} \le c\|\eta\|_{L^\infty(0,T;H^2(\omega))} \text{ and } \|\partial_t\eta\|_{L^2(0,T;H^s(\omega))} \le c\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega_\eta))}.$$

Assume that $s \in (0, \frac{1}{2})$ and take

$$\Big(\mathcal{F}_\eta\big(D^s_{-h}D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta)\big), D^s_{-h}D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta)\Big)$$

as a test function in (2.2.20) and integrate from 0 to T. The test function is admissible by construction, see Proposition 2.3.3. The estimates on the forms a_m and a_b connected to the elastic energy follow directly by Lemma 2.4.5. Indeed, since $\mathcal{K}_\eta(D^s_{-h}D^s_h\eta)$ is constant in space direction and hence does not change the estimate on the derivatives of η we find (using the uniform bounds on λ_η) that

$$\inf_\omega (\overline{\gamma}^2(\eta))\, a_b(t,\eta,(D^{-s}_h D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta)) \ge \|D^s_h\nabla^2\eta\|_{L^2(\omega)} - C(\|\eta\|_{H^2(\omega)}).$$

Hence we are left to estimate the term coming from the structure inertia. Using partial integration and Corollary 2.3.2, we find

$$\begin{aligned}
\int_0^T \left|\int_\omega \partial_t\eta\partial_t(D^s_{-h}D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta)\,dy\right| dt &= \int_0^T \left|\int_\omega \partial_t\eta D^s_{-h}(\partial_t(D^s_h\eta - \mathcal{K}_\eta(D^s_h\eta))\,dy\right| dt\\
&= \int_0^T \left|\int_\omega (D^s_h\partial_t\eta)^2 - D^s_h\partial_t\eta\partial_t\mathcal{K}_\eta(D^s_h\eta)\,dy\right| dt\\
&\le c\|\partial_t\eta\|^2_{L^2(0,T;H^s(\omega))} + \int_0^T \|\partial_t\eta\|_{W^{1,s}(\omega))}(\|\partial_t\eta\|_{W^{1,s}(\omega))} + \|\partial_t\eta\|_{L^2(\omega))}\|\nabla\eta\|_{L^2(\omega)})\,dt\\
&\le c\|\partial_t\eta\|^2_{L^2(0,T;H^s(\omega))} + cT\|\partial_t\eta\|^2_{L^\infty(0,T;L^2(\omega))}\|\nabla\eta\|^2_{L^\infty(0,T;L^2(\omega))}\\
&\le c\|\mathbf{u}\|^2_{L^2(0,T;H^1(\Omega_\eta(t))} + cT\|\partial_t\eta\|^2_{L^\infty(0,T;L^2(\omega))}\|\nabla\eta\|^2_{L^\infty(0,T;L^2(\omega))} \le cC_0^2.
\end{aligned}$$

Here in the last estimate we used the trace theorem [22, Lemma 2.4] and the coupling condition (2.2.5). Notice that this term cannot be estimated in a purely hyperbolic problem and that here it is essential to use the coupling and the fluid dissipation.

Let us next prove the estimates related to the fluid part. From Proposition 2.3.3 and the energy inequality (2.2.21) we have the following estimate

$$\begin{aligned}
&\|\nabla\mathcal{F}_\eta(D^s_{-h}D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta))\|_{L^\infty(0,T;L^2(\Omega_\eta(t))}\\
&\quad\le C\big(\|D^s_{-h}D^s_h\eta\|_{L^\infty(0,T;H^1(\omega))} + \|(D^s_{-h}D^s_h\eta)\nabla\eta\|_{L^\infty(0,T;L^2(\omega))}\big)\\
&\quad\le C\big(\|\eta\|_{L^\infty(0,T;H^2(\omega))} + \|D^s_{-h}D^s_h\eta\|_{L^\infty(0,T;L^\infty(\omega))}\|\nabla\eta\|_{L^\infty(0,T;L^2(\omega))}\big)\\
&\quad\le C\big(\|\eta\|_{L^\infty(0,T;H^2(\omega))} + \|\eta\|^2_{L^\infty(0,T;H^2(\omega))}\big) \le C(C_0 + C_0^2).
\end{aligned}$$

This allows to estimate the integrals:

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega_\eta(t)}(-\mathbf{u}\otimes\mathbf{u} : \nabla\mathcal{F}_\eta(D^s_{-h}D^s_h\eta - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta)) + \operatorname{sym}\nabla\mathbf{u} : \operatorname{sym}\nabla\mathcal{F}_\eta D^s_{-h}D^s_h\eta)\,dx\right|\\
&\le \|\nabla\mathcal{F}_\eta D^s_{-h}D^s_h\eta\|_{L^\infty(0,T;L^2(\Omega_\eta(t))}(\|\nabla\mathbf{u}\|^2_{L^2(0,T;L^2(\Omega_\eta(t)} + \|\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega_\eta(t))})) \le C(C_0 + C_0^2)^2.
\end{aligned}$$

The most difficult estimate is the estimate involving the distributional time-derivative of v. It can be estimated using Lemma 2.3.5; indeed by defining $p = 2 = p'$ and $\tilde{a} = 6$ we get that $\frac{\tilde{a}p}{\tilde{a}p - \tilde{a} - p} = 3$. Hence using the fact that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and

$$\frac{3}{2} - \frac{2}{3} = \frac{5}{6} < 1 = 2 - \frac{2}{2} \text{ and so } W^{\frac{3}{2},3}(\omega) \subset W^{2,2}(\omega),$$

we find by Hölder's inequality and Sobolev embedding that for every $\theta \in (0,1)$ there is a constant c, such that

$$
\begin{aligned}
(I) &= \left| \int_0^T \int_{\Omega_\eta(t)} \mathbf{u} \cdot \partial_t \mathcal{F}_\eta (D^s_{-h} D^s_h \eta - \mathcal{K}_\eta (D^s_{-h} D^s_h \eta))\, dx \right| \\
&\le c\Big(h^{\theta - s} + \|D^s_h \eta\|_{L^\infty(0,T;W^{1,3}(\omega))}\Big) \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|\partial_t D^s_h \eta\|_{L^2(0,T;L^2(\omega))} \\
&\quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|D^s_{-h} D^s_h \eta \partial_t \eta\|_{L^2(0,T;L^{6/5}(\omega))}.
\end{aligned}
$$

Hence choosing $\theta = s$, we find

$$
\begin{aligned}
(I) &\le c\Big(1 + \|\eta\|_{L^\infty(0,T;W^{3/2,3}(\omega))}\Big) \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|\partial_t D^s_h \eta\|_{L^2(0,T;L^2(\omega))} \\
&\quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega)} \|D^s_{-h} D^s_h \eta\|_{L^2(0,T;L^3(\omega)} \\
&\le c\Big(1 + \|\eta\|_{L^\infty(0,T;W^{2,2}(\omega))}\Big) \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|\partial_t D^s_h \eta\|_{L^2(0,T;L^2(\omega))} \\
&\quad + c\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega_\eta(t))} \|\partial_t \eta\|_{L^\infty(0,T;L^2(\omega)} \|\eta\|_{L^2(0,T;W^{1,3}(\omega))} \\
&\le cC_0(C_0 + C_0^2),
\end{aligned}
$$

and the estimate on the term of the time-derivative is complete. The result follows by combining the obtained estimates.

2.5 Compactness rewritten

We introduce the following version of the celebrated Aubin-Lions compactness lemma [7, 112]. The version below is tailored to be applicable for the coupled systems of PDE like the fluid-structure interaction which we we study in this paper. The key point is to fully decouple the *compactness assumption* in space and the *compactness assumption* in time. We emphasize this fact by showing that under appropriate conditions the product of two weak convergent sequences decouple in the limit. It in some sense unifies ideas from time-space decoupling with compensated compactness approaches of div-curl type (see e.g. [26] for some further discussion on that matter).

In this context, the most difficult property to capture is the compactness in time assumption. Commonly it is given in the form of a uniform bound on time-derivative is certain dual space or more precisely a uniform continuity assumption in time. What turned out to be the key observation is that it suffices only to extract the uniform continuity properties over a suitable approximation of its argument. In the theorem below requirement (3) summarizes the time-compactness assumption. As can be seen, no function space is appearing. The assumption is that the pairing of the continuity in time for g_n is uniform with respect to a given suitable approximation of f_n.

This non-function space type requirement is necessary for the application in the context of fluid-structure interactions. Indeed, the weak time derivative of an approximate sequence $\partial_t \eta_\varepsilon, v_\varepsilon$ is defined merely over a non-linear coupled space that changes both with respect to time and with respect to the approximation parameter ϵ itself.

Theorem 2.5.1. *Let X, Z be two Banach spaces, such that $X' \subset Z'$. Assume that $f_n : (0,T) \to X$ and $g_n : (0,T) \to X'$. Moreover assume the following:*

1. *The* weak convergence*: for some $s \in [1,\infty]$ we have that $f_n \overset{*}{\rightharpoonup} f$ in $L^s(X)$ and $g_n \overset{*}{\rightharpoonup} g$ in $L^{s'}(X')$.*
2. *The* approximability-condition *is satisfied: For every $\delta \in (0,1]$ there exists a $f_{n,\delta} \in L^s(0,T;X) \cap L^1(0,T;Z)$, such that for every $\epsilon \in (0,1)$ there exists a $\delta_\epsilon \in (0,1)$ (depending only on ϵ) such that*

$$\|f_n - f_{n,\delta}\|_{L^s(0,T;X)} \le \epsilon \text{ for all } \delta \in (0,\delta_\epsilon]$$

and for every $\delta \in (0,1]$ there is a $C(\delta)$ such that

$$\|f_{n,\delta}\|_{L^1(0,T;Z)}\,dt \le C(\delta).$$

Moreover, we assume that for every δ there is a function f_δ, and a subsequence such that $f_{n,\delta} \overset{*}{\rightharpoonup} f_\delta$ in $L^s(X)$.

3. *The* equi-continuity *of g_n. For every $\epsilon > 0$ and $\delta > 0$ that there exist a $n_{\epsilon,\delta}$ and a $\tau_{\epsilon,\delta} > 0$, such that for all $n \ge n_{\varepsilon,\delta}$ and all $\tau \in (0, \tau_{\varepsilon,\delta}]$*

$$\int_0^{T-\tau}\Big|\fint_0^\tau \langle g_n(t) - g_n(t+s), f_{n,\delta}(t)\rangle_{X',X}\,ds\Big|\,dt \le \epsilon.$$

4. *The* compactness assumption *is satisfied: $X' \hookrightarrow\hookrightarrow Z'$. More precisely, every uniformly bounded sequence in X' has a strongly converging sub-sequence in Z'.*

Then there is a subsequence, such that

$$\int_0^T \langle f_n, g_n\rangle_{X,X'}\,dt \to \int_0^T \langle f, g\rangle_{X,X'}\,dt.$$

Remark 2.5.2 (Modification for applications)**.** *With regard of our application it seems somehow natural to replace (3) by the following condition*

(3') *The* equi-continuity *of g_n. We require that there exists an $\alpha \in (0,1]$ a sequence A_n that is uniformly bounded in $L^1([0,T])$, such that for every $\delta > 0$ that there exist a $C(\delta) > 0$ and an $n_\delta \in \mathbb{N}$ such that for $\tau > 0$ and a.e. $t \in [0, T-\tau]$*

$$\sup_{n\ge n_\delta}\Big|\fint_0^\tau \langle g_n(t) - g_n(t+s), f_{n,\delta}(t)\rangle_{X',X}\,ds\Big| \le C(\delta)\tau^\alpha(A_n(t)+1).$$

Here (3') implies (3) by integration over $[0, T-\tau]$ and an appropriate choice of $\tau_{\delta,\epsilon}$.

Remark 2.5.3 (Classic Aubin-Lions lemma)**.** *Let us explain how Theorem 2.5.1 relates to the classic Aubin-Lions lemma. The simplest case is when Z is a compact subspace of X, $f \in L^2(Z)$ and $\partial_t g_n \in L^2(Z')$. In this case one may take $f_{n,\delta} = f_n$ and finds for $s < t$*

$$|\langle g_n(t) - g_n(s), f_m(t)\rangle| = \Big|\int_s^t \langle \partial_t g_n(\tau)\rangle f_m(t)\,d\tau\Big| \le \|f_m(t)\|_Z \int_s^t \|\partial_t g_n(\tau)\|_{Z'}\,d\tau \le c|t-s|^{\frac12}\|f_m(t)\|_Z,$$

with $c = \|\partial_t g_n\|_{L^2(0,T;Z')}$. The classic Gelfand triple is then the particular case when X is a Hilbert space. Since then $Z \subset\subset X \subset Z'$ implies the same argument as above.

The generalization to allow that Z is independent of the regularity of f_n is essentially some hidden interpolation result (also known as Ehrling property). Here classically one can use convolution estimates to show that a mollifier in one space is uniformly close, while in the other (smaller space) they are merely bounded. One standard example is the periodic solutions over the torus Q and $X = H^a_{per}(Q)$ and $Z = H^c_{per}(Q)$, such that $f_n \in H^b_{per}(Q)$ uniformly with $a < b < c$. Then convolution with the standard mollifying kernel ψ_δ implies that $\|f - f * \psi_\delta\|_{H^r_{per}(Q)} \le C\delta^{s-r}\|f\|_{H^s_{per}(Q)}$ which implies precisely the wanted properties in (2) above. This shows that condition (2) can be seen as a "spatial compactness" condition in the Aubin-Lions lemma (or more generally in Simon's compactness theorem [175]). The condition (3) could be viewed as a "temporal compactness", i.e. as equi-continuity of time shifts in the weaker space Z'.

Remark 2.5.4 (Function spaces)**.** *Since we do not assume that Z is dense in X, an additional clarification of condition (5) is required. Let $\overline{Z}^{\|\cdot\|_X} = \overline{X \cap Z}^{\|\cdot\|_X}$ be the closure of Z w.r.t. X and $(\overline{Z}^{\|\cdot\|_X})'$ its dual (w.r.t. X as pivot space). Then condition (5) has a meaning in the following sense $(\overline{Z}^{\|\cdot\|_X})' \hookrightarrow\hookrightarrow Z'$.*

Proof of Theorem 2.5.1. In this prove we will produce for every $\epsilon > 0$ an $n_\epsilon \in \mathbb{N}$, a $\tau_\epsilon > 0$ with $\tau_\epsilon \to 0$ for $\epsilon \to 0$ and a subsequence of $\langle f_n, g_n \rangle_{X,X'}$ such that

$$\Big| \int_0^{T-\tau_\epsilon} \langle f_n, g_n \rangle_{X,X'} - \langle f_m, g_m \rangle_{X,X'} \, dt \Big| \leq \epsilon$$

for all $n, m \geq n_\epsilon$. This then allows to construct the desired converging subsequence by taking a discrete sequence $\epsilon_i \to 0$ and a respective diagonal argument.

Hence let $\epsilon > 0$, we may choose δ_ϵ in such a way, that for all $\delta \in (0, \delta_\epsilon]$,

$$\|f_n - f_{n,\delta}\|_{L^s(0,T;X)} \leq \epsilon \tag{2.5.1}$$

Next we fix $\tau_{\epsilon,0} > 0$ and $n_{\epsilon,0}$, such that for all $\tau \in (0, \tau_{\epsilon,0}]$

$$\sup_{n \geq n_{\epsilon,0}} \int_0^{T-\tau} \Big| \fint_0^\tau \langle g_n(t) - g_n(s), f_{n,\delta}(t) \rangle_{X',X} \, ds \Big| \, dt \leq \epsilon. \tag{2.5.2}$$

Fix $N \in \mathbb{N}$ such that $\tau_\epsilon := \frac{T}{N} \leq \tau_{\epsilon,0}$. For $k \in \{0, ..., N-1\}$ and $n \in \mathbb{N}$ we define

$$g_n^k = \fint_{k\tau}^{(k+1)\tau} g_n(s) \, ds.$$

This implies by Jensen's inequlaity

$$\|g_n^k\|_{X'} \leq \fint_{k\tau}^{(k+1)\tau} \|g_n(s)\|_{X'} \, ds,$$

and so we define for the given τ_ϵ

$$g_n^{\tau_\epsilon}(t) := g_n^k \text{ for } t \in [k\tau_\epsilon, (k+1)\tau_\epsilon).$$

Since

$$\sup_{[k\tau_\epsilon,(k+1)\tau_\epsilon] \subset [0,T]} \sup_{n \in \mathbb{N}} \|g_n^k)\|_{X'} \leq \frac{C}{\tau_\epsilon},$$

we find by the *compactness assumption,* that we can find a subsequence for which there exists a $n_{\epsilon,1}$, such that

$$\sup_k \|g_n^k - g_m^k\|_{Z'} \leq \epsilon_0 \text{ for all } n, m > n_{\epsilon,1}. \tag{2.5.3}$$

In particular there exists g^{τ_ϵ} and a subsequence, such that$g_n^{\tau_\epsilon} \to g^{\tau_\epsilon}$ strongly in $L^\infty(0,T;Z')$. Clearly, by the uniform bounds we find that $g^\tau \in L^{s'}(X')$.

At this point τ_ϵ and δ_ϵ are fixed. Hence we may define

$$\epsilon_0 := \frac{\epsilon}{C(\delta_\epsilon)},$$

where $C(\delta_\epsilon)$ is defined via (2). Therefore, we find an $n_\epsilon \in \mathbb{N}$, such that for all $n, m \geq n_\epsilon$

$$\Big| \int_0^T \langle f_{n,\delta_\epsilon}, g_n^{\tau_\epsilon} - g_m^{\tau_\epsilon} \rangle \, dt \Big|_{X,X'} \leq \|f_{n,\delta_\epsilon}\|_{L^1(0,T;Z)} \|g_n^{\tau_\epsilon} - g_m^{\tau_\epsilon}\|_{L^\infty(0,T;Z'} \leq \epsilon. \tag{2.5.4}$$

Now all preparations have been made in order to estimate:

$$
\begin{aligned}
&\Big|\int_0^{T-\tau_\epsilon} \langle f_n(t), g_n(t)\rangle_{X,X'} - \langle f_m(t), g_m(t)\rangle_{X,X'}\,dt\Big| \\
&\quad\le \Big|\int_0^{T-\tau_\epsilon} \langle f_{n,\delta_\epsilon}(t), g_n(t)\rangle_{X,X'} - \langle f_{m,\delta_\epsilon}(t), g_m(t)\rangle_{X,X'}\Big| \\
&\qquad + \Big|\int_0^{T-\tau_\epsilon} \langle f_n(t) - f_{n,\delta_\epsilon}(t), g_n(t)\rangle_{X,X'} - \langle f_m(t) - f_{m,\delta_\epsilon}(t), g_m(t)\rangle_{X,X'}\,dt\Big| \\
&\quad\le \Big|\int_0^{T-\tau_\epsilon} \langle f_{n,\delta_\epsilon}(t), g_n(t)\rangle_{X,X'} - \langle f_{m,\delta_\epsilon}(t), g_m(t)\rangle_{X,X'}\Big| + 2\epsilon.
\end{aligned}
$$

We estimate the left

$$
\begin{aligned}
(I) :&= \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\epsilon}(t), g_n(t)\rangle_{X,X'} - \langle f_{m,\delta_\epsilon}(t), g_m(t)\rangle_{X,X'}\,dt \\
&= \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\epsilon}(t), g_n(t) - g_n^{\tau_\varepsilon}(t)\rangle_{X,X'}\,dt + \int_0^{T-\tau_\varepsilon} \langle f_{n,\delta_\epsilon}(t), g_n^{\tau_\varepsilon}(t) - g_n^{\tau_\varepsilon}(t)\rangle_{X,X'}\,dt \\
&\quad + \int_0^{T-\tau_\varepsilon} \langle f_{m,\delta_\epsilon}(t) - f_{n,\delta_\epsilon}(t), g_m^{\tau_\varepsilon}(t)\rangle_{X,X'}\,dt - \int_0^{T-\tau_\varepsilon} \langle f_{m,\delta_\epsilon}(t), g_m(t) - g_n^{\tau_\varepsilon}(t)\rangle_{X,X'}\,dt \\
&= \sum_{k=0}^{N-2}\int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \fint_0^{\tau_\varepsilon} \langle f_{n,\delta_\epsilon}(t), g_n(t) - g_n(s)\rangle_{X,X'}\,ds\,dt + \sum_{k=0}^{N-2}\int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \langle f_{n,\delta_\epsilon}(t), g_n^k - g_m^k\rangle_{X,X'}\,dt \\
&\quad + \sum_{k=0}^{N-2}\int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \langle f_{m,\delta_\epsilon}(t) - f_{n,\delta_\epsilon}(t), g_m^k\rangle_{X,X'}\,dt - \sum_{k=0}^{N-2}\int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} \fint_0^{\tau_\varepsilon} \langle f_{m,\delta_\epsilon}(t), g_m(t) - g_m(s)\rangle_{X,X'}\,ds\,dt \\
&= (II) + (III) + (IV) + (V).
\end{aligned}
$$

First observe that (II) and (V) can be estimated using the *equi-continuity condition*, namely (2.5.2). Term (III) is estimated using the *compactness condition*, namely (2.5.4). Finally for (IV) we deviate and apply (2.5.4) a second and third time

$$
\begin{aligned}
(IV) &= \int_0^T \langle f_{m,\delta_\epsilon} - f_{n,\delta_\epsilon}, g_m^{\tau_\epsilon}\rangle_{X,X'}\,dt \\
&= \int_0^T \langle f_{m,\delta_\epsilon} - f_{n,\delta_\epsilon}, g^{\tau_\epsilon}\rangle_{X,X'} + \langle f_{m,\delta_\epsilon} - f_{n,\delta_\epsilon}, g_m^{\tau_\epsilon} - g^{\tau_\epsilon}\rangle_{X,X'}\,dt \\
&\le \Big|\int_0^T \langle f_{m,\delta_\epsilon} - f_{n,\delta_\epsilon}, g^{\tau_\epsilon}\rangle_{X,X'}\Big| + 2\epsilon.
\end{aligned}
$$

Now we take another subsequence of $f_{n,\delta_\epsilon} \in L^s(X)$ that converges weakly*. Hence we may eventually increase n_ϵ one last time (in dependence of g^{τ_ϵ}) and find that for this subsequence and $n, m \ge n_\epsilon$

$$
|(IV)| \le 3\epsilon.
$$

This finishes the proof.

□

2.6 The existence result

2.6.1 The approximate system

In this section we construct approximate solutions $(\mathbf{u}^\varepsilon, \eta^\varepsilon) \in \mathcal{V}_S$, $\eta^\varepsilon \in L^\infty(0,T;H^3(\omega))$ which satisfy the following weak formulation:

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_\eta(t)} \mathbf{u}^\varepsilon\cdot\mathbf{q}\,dx + \int_{\Omega_\eta(t)} \Big(-\mathbf{u}^\varepsilon\cdot\partial_t\mathbf{q} - \mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon : \nabla\mathbf{q} + \mathrm{sym}\nabla\mathbf{u}^\varepsilon : \mathrm{sym}\nabla\mathbf{q}\Big)\,dx \\
&+\frac{d}{dt}\int_\omega \partial_t\eta^\varepsilon\xi\,dy - \int_\omega \partial_t\eta^\varepsilon\partial_t\xi\,dy + a_m(t,\eta^\varepsilon,\xi) + a_b(t,\eta^\varepsilon,\xi) + \varepsilon\int_\omega \nabla_x^3\eta^\varepsilon : \nabla_x^3\xi\,dy = 0,
\end{aligned}
\tag{2.6.1}
$$

where $\varepsilon > 0$ is a regularizing parameter and with initial conditions $\eta_0, \eta_1, \mathbf{u}_0$. In this section we prove the following Theorem:

Theorem 2.6.1. *There exists a $T > 0$ just depending on $\partial\Omega$ and the initial data, such that for every $\varepsilon \in (0,1]$ there exists a weak solution $(\mathbf{u}^\varepsilon, \eta^\varepsilon)$ to the regularized problem* (2.6.1). *Moreover, the weak solution satisfies the following uniform in ε estimate:*

$$\|\mathbf{u}^\varepsilon\|_{\mathcal{V}_F} + \|\eta^\varepsilon\|_{\mathcal{V}_S} + \|\eta^\varepsilon\|_{L^2(0,T;N^{s,2}(\omega))} \le C, \tag{2.6.2}$$

for every (fixed) $s < \frac{5}{2}$, with C just depends on $\partial\Omega$ and the initial conditions.

The existence of regularized solutions can be proved following the ideas and techniques introduced in [153]. The problem solved in [153] is actually very similar to the regularized system above since there the existence of a solution to a FSI problem with a structure being an elastic shell with a non-linear Koiter membrane energy without bending energy, but with a (linear) regularization term of fourth order is shown. In order to be able to treat the non-linear bending energy in an analogous way we have to include a sixth order regularization term. Another difference comes from the fact that in [153] cylindrical geometry is considered. Nevertheless the introduced existence scheme does not depend on the geometry of the problem and more general geometries can be handled by combining the existence proof with the estimates in this paper and in [127]. To avoid lengthy repetitions of the arguments analogous to [153] here, we summarize the main steps of the construction of a weak solutions with emphasis on the differences coming from the non-linear bending term and the setting of more general geometries. The main steps of the construction are:

1. **Arbitrary Lagrangian-Eulerian (ALE) formulation**. We reformulate the problem in a fixed reference domain Ω using suitable change of variables. This approach is popular in numerics and the change of variables is called Arbitrary Lagrangian-Eulerian (ALE) mapping. The formulation in the fixed reference domain is called ALE formulation of the FSI problem. We use the mapping ψ_η (introduced in Definition 2.2.1) as an ALE mapping.

2. **Construction of the approximate solutions**. We construct the approximate solutions using time-discretizations and operator splitting methods. We use the Lie splitting strategy (also known as Marchuk-Yanenko splitting) to decouple the FSI problem.

3. **Uniform estimates**. Let $\Delta t > 0$ be the time-discretization parameter. We show that the constructed approximate solutions satisfy uniform bounds w.r.t. Δt (and ε) in the energy function spaces. We identify weak and weak* limits.

4. **Compactness**. We prove that the set of approximate solutions is compact in suitable norms. By using the compactness we prove that a limit of the sequence of approximate solutions is a weak solution to the regularized FSI problem. Here we use a generalization of the Aubin-Lions-Simon lemma for discrete in time solutions adapted to the moving domain problems from [151].

Since a solution is constructed by decoupling the problem, the largest difference from [153] is in the second step where in the structure sub-problem we include also the non-linear bending energy. However, we will show that the bending term can be discretized in an analogous way as the membrane term. Other steps are analogous as in [153] using the sixth order regularization. For the convenience of the reader we will describe the details of the time-discretization of the structure sub-problem with the corresponding uniform estimates in the time-discretization parameter Δt. We conclude this chapter with the description of the compactness step. Generally, for more details on the procedure we refer the reader to [153].

In the rest of the subsection we fix the regularizing parameter ε and drop superscripts ε in $(\mathbf{u}^\varepsilon, \eta^\varepsilon)$ since there is no chance of confusion.

Construction of discrete approximations

The main problem in the construction of approximate solution is how to discretize the Fréchet derivatives of $\mathbf{G}$ and $\mathbf{R}$ to obtain the discrete analogue of $\mathbf{R}'(\eta)\partial_t\eta = \partial_t\mathbf{R}(\eta)$. In [153] this was achieved by using the fact that the first fundamental form was polynomial of order two of η and $\nabla\eta$ which was a consequence of the cylindrical geometry. Here we consider a more general geometry so we need to develop a more general approach.

For a given end-time T, we fix Δt as the time step, such that $[0,T] = [0,N\Delta t]$ for some $N \in \mathbb{N}$. Now let $(\eta^n)_{n=1}^N$ be a given time-discrete solution and $\tilde{\eta}$ be the piece-wise linear function in time such that $\tilde{\eta}(n\Delta t) = \eta^n$. Then we have

$$\mathbf{R}'(\tilde{\eta})\frac{\eta^{n+1}-\eta^n}{\Delta t} = \mathbf{R}'(\tilde{\eta})\partial_t\tilde{\eta} = \partial_t\mathbf{R}(\tilde{\eta}) \text{ on } [n\Delta t,(n+1)\Delta t].$$

Notice that the expression $\mathbf{R}'(\tilde{\eta})\partial_t\tilde{\eta}$ is a third order polynomial in the t variable so we can compute its integral $\int_{n\Delta t}^{(n+1)\Delta t}$ by using Newton-Cotes formula. Hence, by defining $\overline{\eta}^{n+1} := \frac{\eta^{n+1}+\eta^n}{2}$ we find the approximation of $\mathbf{G}'(\eta)\xi$ and $\mathbf{R}'(\eta)\xi$ in the following way:

$$\mathbf{G}'(\eta^{n+1},\eta^n)\xi := \frac{1}{3}\Big(\mathbf{G}'(\eta^n) + 4\mathbf{G}'(\overline{\eta}^{n+1}) + \mathbf{G}^{n+1}\Big)\xi \tag{2.6.3}$$

and

$$\mathbf{R}'(\eta^{n+1},\eta^n)\xi := \frac{1}{3}\Big(\mathbf{R}'(\eta^n) + 4\mathbf{R}'(\overline{\eta}^{n+1}) + \mathbf{R}^{n+1}\Big)\xi. \tag{2.6.4}$$

By straightforward calculation it follows that

$$\begin{aligned}\mathbf{G}'(\eta^{n+\frac{1}{2}},\eta^n)\frac{\eta^{n+\frac{1}{2}}-\eta^n}{\Delta t} &= \Delta t\int_{n\Delta t}^{(n+1)\Delta t}\frac{d}{dt}\mathbf{G}(\tilde{\eta})\\ &= \frac{1}{\Delta t}(\mathbf{G}(\eta^{n+\frac{1}{2}})-\mathbf{G}(\eta^n))\end{aligned}$$

which is the correct substitute for "$\partial_t\mathbf{G}(\eta) = \mathbf{G}'(\eta)\partial_t\eta$". Analogously we find as substitute for "$\partial_t\mathbf{R}(\eta) = \mathbf{R}'(\eta)\partial_t\eta$"

$$\mathbf{R}'(\eta^{n+\frac{1}{2}},\eta^n)\frac{\eta^{n+\frac{1}{2}}-\eta^n}{\Delta t} = \frac{1}{\Delta t}(\mathbf{R}(\eta^{n+\frac{1}{2}})-\mathbf{R}(\eta^n)).$$

These identities will be used to derive a semi-discrete uniform energy inequality. First we define the sequence of approximate solutions by solving the following problems.

Structure sub-problem.
Find $(v^{n+\frac{1}{2}},\eta^{n+\frac{1}{2}}) \in \big(H_0^2(\omega)\cap H^3(\omega)\big)^2$ such that:

$$\begin{aligned}&\int_\omega\frac{\eta^{n+\frac{1}{2}}-\eta^n}{\Delta t}\phi\,dy = \int_\omega v^{n+\frac{1}{2}}\phi\,dy,\\ &\int_\omega\frac{v^{n+\frac{1}{2}}-v^n}{\Delta t}\psi\,dy + \frac{1}{2}\int_\omega\mathcal{A}\mathbf{G}(\eta^{n+\frac{1}{2}}):\mathbf{G}'(\eta^{n+\frac{1}{2}},\eta^n)\psi\,dy\\ &\quad+\frac{1}{24}\int_\omega\mathcal{A}\mathbf{R}(\eta^{n+\frac{1}{2}}):\mathbf{R}'(\eta^{n+\frac{1}{2}},\eta^n)\psi\,dy + \varepsilon\int_\omega\nabla^3\eta^{n+\frac{1}{2}}\nabla^3\psi\,dy = 0,\end{aligned}\tag{2.6.5}$$

for all $(\phi,\psi) \in L^2(\omega)\times\big(H_0^2(\omega)\cap H^3(\omega)\big)$.

The existence of a solution to the above problem follows by Schaefer's Fixed Point Theorem as it was demonstrated in [153, Proposition 4]).

Fluid sub-problem.
The fluid problem stays the same as in [153] (which is the advantage of the operator splitting method). Since the domain deformation is calculated in the structure sub-problem and does not change in the fluid sub-problem we set $\eta^{n+1} = \eta^{n+\frac{1}{2}}$, and define $(\mathbf{u}^{n+1},v^{n+1}) \in \mathcal{V}_F^{\eta^n}\times L^2(\omega)$ by requiring that for all $(\mathbf{q},\xi) \in \mathcal{V}_F^{\eta^n}\times L^2(\omega)$ such

that $\mathbf{q}_{|\Gamma} = \xi\nu$, the following weak formulation holds:

$$\begin{aligned}
&\int_\Omega J^n\Bigg(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\cdot\mathbf{q} + \frac{1}{2}\Big[(\mathbf{u}^n - \mathbf{w}^{n+\frac{1}{2}})\cdot\nabla^{\eta^n}\Big]\mathbf{u}^{n+1}\cdot\mathbf{q}\\
&\quad -\frac{1}{2}\Big[(\mathbf{u}^n - \mathbf{w}^{n+\frac{1}{2}})\cdot\nabla^{\eta^n}\Big]\mathbf{q}\cdot\mathbf{u}^{n+1}\Bigg)\,dx\\
&\quad +\frac{1}{2}\int_\Omega \frac{J^{n+1}-J^n}{\Delta t}\mathbf{u}^{n+1}\cdot\mathbf{q}\,dx + 2\int_\Omega J^n\mathbf{D}^{\eta^n}(\mathbf{u}):\mathbf{D}^{\eta^n}(\mathbf{q})\,dx\\
&\quad +\int_\omega \frac{v^{n+1}-v^{n+\frac{1}{2}}}{\Delta t}\xi\,dy = 0\\
&\text{with } \nabla^{\eta^n}\cdot\mathbf{u}^{n+1} = 0,\quad \mathbf{u}^{n+1}_{|\Gamma} = v^{n+1}\nu.
\end{aligned}$$

Here ∇^η is the transformed gradient, $\mathbf{w}^{n+1/2}$ is the ALE velocity (i.e. the time discretization of $\partial_t\boldsymbol{\psi}_{\eta^n}$ (see Definition 2.2.1)), and $J^n = \det\nabla\boldsymbol{\psi}_{\eta^n}$ is the Jacobian of the transformation from Ω_{η^n} to the reference configuration Ω. Please observe that the above system is a linear equation on a fixed domain and it is solvable as long as $J^n > 0$ by the Lax-Milgram Lemma. One can see that no self-intersection implies $J^n > 0$.

Now we define the approximate solutions as a piece-wise constant functions in time:

$$\mathbf{u}_{\Delta t}(t,.) = \mathbf{u}^n_{\Delta t},\ \eta_{\Delta t}(t,.) = \eta^n_{\Delta t},\ v_{\Delta t}(t,.) = v^n_{\Delta t},\ v^*_{\Delta t}(t,.) = v^{n-\frac{1}{2}}_{\Delta t}\ \text{for } t\in[n\Delta t,(n+1)\Delta t]. \tag{2.6.6}$$

Uniform estimates in Δt.

The following proposition gives us the uniform boundedness of the approximate solutions defined by (2.6.6). It is a consequence of [153, Lemma 8] combined with Lemma 2.4.2 and Lemma 2.4.3.

Proposition 2.6.2. *Let $\Delta t > 0$. Then the approximate solutions defined by* (2.6.6) *satisfy the following estimate:*

$$\|\mathbf{u}_{\Delta t}\|_{L^\infty_t L^2_x} + \|\mathbf{u}_{\Delta t}\|_{L^2_t H^1_x} + \|\eta_{\Delta t}\|_{L^\infty_t H^2_x} + \|v_{\Delta t}\|_{L^\infty_t L^2_x} + \|v^*_{\Delta t}\|_{L^\infty_t L^2_x} + \sqrt{\varepsilon}\|\eta_{\Delta t}\|_{L^\infty_t H^3_x} \le C, \tag{2.6.7}$$

where C depends on the data only. Moreover, there exists a $T > 0$ independent of Δt such that no self-intersection is approached.

Proof. The proof can be directly adapted from [153, Lemma 8] combined with Lemma 2.4.2 and Lemma 2.4.3. In particular, we find by the uniform $L^\infty_t H^2_x$ estimates on $\eta_{\Delta t}$ that $\|\eta_{\Delta t}\|_{L^\infty_t(L^\infty_x)}$ is uniformly bounded with constants just depending on $\partial\Omega$ and the initial condition. Moreover, since $v^*_{\Delta t}$ is bounded in $L^\infty_t L^2_x$ we can use the interpolation inequality for Sobolev spaces to show that there exists $T > 0$ such that $\eta_{\Delta t}$ satisfies (2.4.4) and $J_{\Delta t} > 0$ in $[0,T]$, uniformly in Δt (and ε), cf. [153, Proposition 9]. In particular $\partial\Omega_{\eta_{\Delta t}}$ has no self-intersection on $[0,T]$. □

Let us denote by $\mathbf{u}$, η, v and v^* the corresponding weak or weak* limits of $\Delta t \to 0$. From [153, Lemma 11] it follows that $v = v^*$.

Compactness for $\Delta t \to 0$.

First, we prove the strong convergence of the sequence $\eta_{\Delta t}$. This is a consequence of the uniform boundedness of the discrete time derivatives $\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|_{L^2(\omega)}$ and the boundedness of $\eta_{\Delta t}$ in $L^\infty(0,T;H^3(\omega))$. By using the classical Arzelá-Ascoli theorem for the piece-wise affine interpolation we get that

$$\eta_{\Delta t} \to \eta \text{ in } L^\infty(0,T;H^s(\omega)) \text{ for } s\in(0,3).$$

This is enough to pass to the limit in the terms connected to the elastic energy. In order to pass to the limit in the convective term and the terms connected to the moving boundary we need strong L^2 convergence of $(\mathbf{u}_{\Delta t}, v_{\Delta t})$. This is the most delicate part of the existence proof where one has to use the uniform convergence of $\eta_{\Delta t}$ and the fact that the fluid dissipates higher frequencies of the structure velocities.

In the current case this follows by a version of the Aubin-Lions-Simon lemma adapted for the problems with moving boundary [151, Theorem 3.1. and Section 4.2]. Hence, by passing to the limit we find a $T > 0$ such that for every fixed $\varepsilon > 0$ there exists a weak solution to (2.6.1).

2.6.2 Proof of Theorem 2.2.4.

In this subsection we first collect the necessary a-priori estimates (which essentially follow from the regularity theorem) and then pass to the limit with $\varepsilon \to 0$. Here the establishment of the non-linearity in the convective term is (as usually) the most delicate part.

Uniform estimates in ε.
We use the test function:
$(\mathbf{q},\xi) = \Big(\mathcal{F}_\eta\big(D^s_{-h}D^s_h\eta^\varepsilon - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta^\varepsilon)\big), D^s_{-h}D^s_h\eta^\varepsilon - \mathcal{K}_\eta(D^s_{-h}D^s_h\eta^\varepsilon)\Big)$ in (2.6.1) in an analogous way as in the proof of Theorem 2.2.5. In combination with the energy estimates ee obtain the following uniform regularity estimate for all (fixed) $s < \frac{1}{2}$:

$$\|\eta^\varepsilon\|_{L^\infty(0,T;(H^2\cap\sqrt{\varepsilon}H^3)(\omega))} + \|\partial_t\eta^\varepsilon\|_{L^\infty(0,T;L^2(\omega))} + \|\eta^\varepsilon\|_{L^2(0,T;N^{2+s,2}(\omega))} + \|\partial_t\eta^\varepsilon\|_{L^2(0,T;N^{s,2}(\omega))} \leq C. \quad (2.6.8)$$

Passing with $\varepsilon \to 0$.
From the (classic) Aubin-Lions lemma we obtain

$$\eta^\varepsilon \to \eta \text{ in } L^2(0,T;H^s(\omega)) \cap L^\infty(0,T;H^{s-1/2}(\omega)) \text{ for } s < 5/2. \quad (2.6.9)$$

In particular, $\eta^\varepsilon \to \eta$ in $L^2(0,T;H^2(\omega)) \cap L^\infty(0,T;L^\infty(\omega))$ which is enough to pass to the limit in the elastic terms, see (2.4.5) for the highest order terms. The existence result is completed once we can show that (for a sub-sequence)$(\partial_t\eta^\varepsilon, \mathbf{u}^\varepsilon) \to (\partial_t\eta, \mathbf{u})$, since this allows to establish all nonlinearities in the limit equation and the existence is complete.

The proof of the L^2 convergence of the velocities is known to be the most delicate part of the construction of weak solutions in the framework of FSI in the incompressible regime, see [87, 127, 151]. Here we present a more universal approach based on the reformulation of the Aubin-Lions lemma (Theorem 2.5.1) combined with the extension operator presented in Corollary 2.3.4.

Lemma 2.6.3. *There exists a strongly converging subsequence*

$$(\partial_t\eta^\varepsilon, \mathbf{u}^\varepsilon) \to (\eta, \mathbf{u})$$

in $L^2(0,T;L^2(\Omega_\eta(t)) \times L^2(\omega))$.

Proof. The strong convergence follows from Theorem 2.5.1. Actually we will apply the theorem two times. First for the boundary compactness and the other time for the interior compactness. For that please note that since $\operatorname{div}\mathbf{u}^{\varepsilon_n} = 0$ and $\mathbf{u}^{\varepsilon_n}(x) = \partial_t\eta^{\varepsilon_n}(y(x))\nu(\mathbf{p}(x))$, for $x \in \partial\Omega_\eta$, it follows that

$$0 = \int_{\partial\Omega_\eta} \partial_t\eta^{\varepsilon_n}(y(x))\,dx.$$

Moreover, since $\operatorname{div}\mathcal{F}_\eta(\partial_t\eta^{\varepsilon_n} - \mathcal{K}_\eta(\partial_t\eta^{\varepsilon_n})) = 0$ we find that

$$|\partial\Omega_\eta|\mathcal{K}_\eta(b) = \int_{\partial\Omega_\eta} \partial_t\eta^{\varepsilon_n}(y(x)))\,dx = 0.$$

Hence, $\mathcal{F}_{\eta^{\varepsilon_n}}(\partial_t\eta^{\varepsilon_n})$ is well defined:

$$\begin{aligned}\int_0^T \|\partial_t\eta^{\varepsilon_n}\|^2_{L^2(\omega)} + \|\mathbf{u}^{\varepsilon_n}\|^2_{L^2(\Omega_{\eta^{\varepsilon_n}}}\,dt &= \int_0^T \langle\partial_t\eta^{\varepsilon_n}, \partial_t\eta^{\varepsilon_n}\rangle + \langle\mathbf{u}^{\varepsilon_n}, \mathcal{F}_{\eta^{\varepsilon_n}}(\partial_t\eta^{\varepsilon_n})\rangle\,dt \\ &\quad + \int_0^T \langle\mathbf{u}^{\varepsilon_n}, \mathbf{u}^{\varepsilon_n} - \mathcal{F}_{\eta^{\varepsilon_n}}(\partial_t\eta^{\varepsilon_n})\rangle\,dt =: (I_n) + (II_n).\end{aligned}$$

For both terms we will show the convergence separately by using Theorem 2.5.1 The convergence implies that

$$\|\partial_t\eta^{\varepsilon_n}\|_{L^2_tL^2_x} + \|\mathbf{u}^{\varepsilon_n}\chi_{\Omega_{\eta^{\varepsilon_n}}}\|_{L^2_tL^2_x} \to \|\partial_t\eta\|_{L^2_tL^2_x} + \|\mathbf{u}\chi_{\Omega_\eta}\|_{L^2_tL^2_x},$$

which implies, by the uniform convexity of L^2, the strong convergence $(\partial_t\eta^\varepsilon, \mathbf{u}^\varepsilon) \to (\eta, \mathbf{u})$ and the Lemma is proved.

For the first term (I_n) we define $g_n = (\partial_t\eta^{\varepsilon_n}, \mathbf{u}^{\varepsilon_n}\chi_{\Omega_{\eta^{\varepsilon_n}}})$ and $f_n = (\partial_t\eta^{\varepsilon_n}, \mathcal{F}_{\eta^{\varepsilon_n}}(\partial_t\eta^{\varepsilon_n}))$, and apply Theorem 2.5.1 using the spaces $X = L^2(\omega) \times H^{-s}(Q^\kappa)$ and consequently $X' = L^2(\omega) \times H^s(Q^\kappa)$. The space $Z = H^{s_0}(\omega) \times H^{s_0}(Q^\kappa)$ for $0 < s < s_0 < \frac{1}{4}$. Also with respect to time we work in setting of Hilbert spaces, which means that all Lebesgue exponents are equal to two. Further we recall the smooth extension $E_{\eta^{\varepsilon_n}(t),\delta}$ introduced in Corollary 2.3.4 and denote by $(\partial_t\eta^{\varepsilon_n})_\delta := \partial_t\eta^{\varepsilon_n} * \psi_\delta$. This allows to define

$$f_{n,\delta} := ((\partial_t\eta^{\varepsilon_n})_\delta, E_{\eta^{\varepsilon_n},\delta}(\partial_t\eta^{\varepsilon_n}))$$

Next, let us check the assumptions of Theorem 2.5.1 and Remark 2.5.2. First observe that [127, Proposition 2.28] and (2.6.8) implies that g_n is uniformly bounded in $L^2_t(H^s_x)$ (for $s \leq \frac{1}{4}$). Hence the assumptions (1) follows in a rather straightforward manner by weak compactness in Hilber spaces. Next (2) follows by Corollary 2.3.4 and the standard estimates for mollifiers:

$$\|f_n - f_{n,\delta}\|_{L^2(Q^\kappa)} \leq c\|\partial_t\eta^{\varepsilon_n} * \psi_\delta - \partial_t\eta^{\varepsilon_n}\|_{L^2(\omega)} \leq C\delta^s\|\partial_t\eta^{\varepsilon_n}\|_{H^s(\omega)}.$$

Hence we are left to check (3'). As usual for equi-continuity in time, this is a consequence of the weak formulation of the problem (2.6.1). For $\sigma \in [t, t+\tau]$ (using the solenoidality and the matching of the extension) we have

$$\begin{aligned}
|\langle g_n(t) - g_n(\sigma), f_{n,\delta}(t)\rangle| &= \bigg|\int_{Q^\kappa} \Big(\mathbf{u}^{\varepsilon_n}(t)\chi_{\Omega_{\eta^{\varepsilon_n}(t)}} - \mathbf{u}^{\varepsilon_n}(\sigma)\chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}}\Big) \cdot E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t\eta^{\varepsilon_n}(t))\, dx \\
&\quad + \int_\omega \big(\partial_t\eta^{\varepsilon_n}(t) - \partial_t\eta^{\varepsilon_n}(\sigma)\big)(\partial_t\eta^{\varepsilon_n}(t))_\delta)\, dy\bigg| \\
&= \bigg|\int_{Q^\kappa} \mathbf{u}^{\varepsilon_n}(t)\chi_{\Omega_{\eta^{\varepsilon_n}(t)}} \cdot E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t\eta^{\varepsilon_n}(t)) - \mathbf{u}^{\varepsilon_n}(\sigma)\chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot E_{\eta^{\varepsilon_n}(\sigma),\delta}(\partial_t\eta^{\varepsilon_n}(t))dx \\
&\quad + \int_\omega \big(\partial_t\eta^{\varepsilon_n}(t) - \partial_t\eta^{\varepsilon_n}(\sigma)\big)(\partial_t\eta^{\varepsilon_n}(t))_\delta)\, dy\bigg| \\
&\quad + \bigg|\int_{Q^\kappa} \mathbf{u}^{\varepsilon_n}(\sigma)\chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot \Big(E_{\eta^{\varepsilon_n}(\sigma),\delta}(\partial_t\eta^{\varepsilon_n}(t)) - E_{\eta^{\varepsilon_n}(t),\delta}(\partial_t\eta^{\varepsilon_n}(t))\Big)dx\bigg| \\
&=: (A_1) + (A_2)
\end{aligned}$$

First observe that by Corollary 2.3.4

$$\begin{aligned}
(A_2) &\leq \int |\mathbf{u}^{\varepsilon_n}(\sigma)|\chi_{\Omega_{\eta^{\varepsilon_n}(\sigma)}} \cdot \int_t^\sigma |\partial_s E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t\eta^{\varepsilon_n}(t))|\, ds\, dx \\
&\leq c\|\mathbf{u}^{\varepsilon_n}(\sigma)\|_{L^2(\Omega_{\eta^{\varepsilon_n}(\sigma)})}\|\partial_t\eta^{\varepsilon_n}(\partial_t\eta^{\varepsilon_n})_\delta\|_{L^\infty_t(L^2_x)}|\sigma - t|^{\frac{1}{2}} \\
&\leq C(\delta)\tau^{\frac{1}{2}},
\end{aligned}$$

where we used the uniform $L^\infty_t L^2_x$ estimate of $\mathbf{u}^{\varepsilon_n}$ and $\partial_t\eta^{\varepsilon_n}$ multiple times. Second, by the weak formulation (2.6.1), we find

$$\begin{aligned}
(A_1) &= \bigg|\int_\sigma^t \int_{\Omega_\eta} -\mathbf{u}^{\varepsilon_n}(s) \cdot \partial_s E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t\eta^{\varepsilon_n}(t)) \\
&\quad + (\mathrm{sym}\nabla\mathbf{u}^{\varepsilon_n}(s) - \mathbf{u}^{\varepsilon_n}(s) \otimes \mathbf{u}^{\varepsilon_n}(s)) : \nabla E_{\eta^{\varepsilon_n}(s),\delta}(\partial_t\eta^{\varepsilon_n}(t))\, dx \\
&\quad + \int_\omega a_m(t, \eta^{\varepsilon_n}, (\partial_t\eta^{\varepsilon_n}(t))_\delta) + a_b(t, \eta^{\varepsilon_n}, (\partial_t\eta^{\varepsilon_n})_\delta) + \varepsilon_n\nabla^3\eta^{\varepsilon_n}(s) : \nabla^3(\partial_t\eta^{\varepsilon_n}(t))_\delta\, dy\, ds\bigg| \\
&\leq C(\delta)\Big((\|\mathbf{u}^{\varepsilon_n}\|_{L^2(t,t+\tau;W^{1,2}+1)(\Omega_{\eta^{\varepsilon_n}})}(\|\partial_t\eta^{\varepsilon_n}\|_{L^\infty_t(L^2_x)} + \|\eta^{\varepsilon_n}\|_{L^\infty_t(H^2_x+\varepsilon H^3_x)}) \\
&\quad + \|\eta^{\varepsilon_n}\|_{L^2(t,t+\tau;W^{1,\infty}(\omega))}\|\mathbf{u}^{\varepsilon_n}\|^2_{L^\infty(0,T;L^2(\Omega_{\eta^{\varepsilon_n}}))}\Big)|t - \sigma|^{\frac{1}{2}} \\
&\leq C(\delta)\tau^{\frac{1}{2}}.
\end{aligned}$$

This implies (3'), namely

$$\Big|\fint_t^{t+\tau} \langle g_n(t) - g_n(\sigma), f_{n,\delta}(t)\rangle \, d\sigma\Big| \le C(\delta)\tau^{\frac{1}{2}}.$$

This finishes the proof of the convergence of $(I)_n$ term. For the second term (II_n) we again apply Theorem 2.5.1. Here we set $g_n = \mathbf{u}^{\varepsilon_n}\chi_{\Omega_{\eta^{\varepsilon_n}}}$ and $f_n = (\mathbf{u}^{\varepsilon_n} - \mathcal{F}_{\eta^{\varepsilon_n}}(\partial_t\eta^{\varepsilon_n}))\chi_{\Omega_{\eta^{\varepsilon_n}}}$ We apply Theorem 2.5.1 with the following spaces $X = H^{-s}(Q^\kappa)$ and consequently $X' = H^s(Q^\kappa)$ for some $s \in (0, \frac{1}{4})$. Further we define $Z = L^2(Q^\kappa)$. Please observe that we may extend all involved quantities by zero to be functions over Q^κ. Finally, we again set all Lebesgue exponents to two. Similarly as for the first term, again the main effort is the construction of the right mollification. Indeed the assumptions (1) and (4) follow by standard compactness arguments. In particular, for assumption (1) it has been shown in [127, Proposition 2.28] that g_n is uniformly bounded in $H^s(Q^\kappa)$ (if $s \le \frac{1}{4}$). For (2) we use the fact that f_n has a zero trace on $\partial\Omega_{\eta^{\varepsilon_n}(t)}$. First, let $\delta > 0$ be given. We take n_δ large enough and $\tau_\delta > 0$ small enough, such that

$$\sup_{n\ge n_\delta}\ \sup_{\tau\in(t-\tau_\delta,t+\tau_\delta)\cap[0,T]} \|\eta(t,x) - \eta^{\varepsilon_n}(\tau,x)\|_\infty \le \delta. \tag{2.6.10}$$

Second, we fix $0 < s_0 < s$ and $\epsilon > 0$. By [127, Lemma A.13] there exits a σ_ϵ and a sequence $\tilde{f}_{n,\delta}$, such that $supp(\tilde{f}_{n,\delta}(t)) \subset \Omega_{\eta^{\varepsilon_n}(t)-3\delta}$ for all $3\delta \le \sigma_\epsilon$, that is divergence free and $\|f - \tilde{f}_{n,\delta}\|_{(H^{-s_0}(Q^\kappa))} \le \epsilon\|f\|_{L^2(Q^\kappa)}$. We mollify this solenoidal function to define

$$f_{n,\delta} = \tilde{f}_{n,\delta} * \psi_\delta \text{ where } \psi \text{ is the standard mollifier in space.}$$

Then this definition implies (3') by standard mollification estimate. Indeed, choosing

$$\begin{aligned}\|f_n - f_{n,\delta}\|_{H^{-s}(Q^\kappa)} &\le \|\tilde{f}_{n,\delta} - f_{n,\delta}\|_{H^{-s}(Q^\kappa)} + \|\tilde{f}_{n,\delta} - f_n\|_{H^{-s_0}(Q^\kappa)}\\ &\le c\delta^{s-s_0}\|\tilde{f}_{n,\delta}\|_{H^{-s_0}(Q^\kappa)} + \epsilon\|f_n\|_{L^2(Q^\kappa)} \le c\epsilon\|f_n\|_{L^2(Q^\kappa)},\end{aligned}$$

for δ small enough in dependence of $s - s_0$. Please observe that by the properties of the mollification and (2.6.10) that $supp(f_{n,\delta}) \subset \Omega_{\eta^{\varepsilon_n}(t)-2\delta} \subset \Omega_{\eta^{\varepsilon_m}}(\tau)$ which implies that $f_{n,\delta}(t)$ can be used as a testfunction on the fluid equation alone over the interval $[t, t+\tau]$ for $t \in [0, T-\tau]$. Hence (3') follows exactly along the lines of the above estimates using the weak formulation (2.6.1). Notice that here we will work just with the fluid equation since the traces of test function are zero at the moving interface. □

End of the proof of Theorem 2.2.4

The a-priori estimates and the above compactness arguments guarantee that for given initial conditions there is a minimal time interval $T > 0$ for which a weak solution exists (see Remark 2.4.4). Once the solution is established we can repeat the argument (by using $\eta(T), \partial_t\eta(T), \mathbf{u}(T)$ as initial conditions) until either a self-intersection is approached or a degeneracy of the H^2-coercivity is violated (namely if $\overline{\overline{\gamma}}(\eta(t,x)) \to 0$ for some $t \to T$).

Chapter 3

Navier–Stokes–Fourier fluids interacting with elastic shells

3.1 Introduction

Nowadays there exists a vast body of literature on incompressible fluid structure interaction, where a part of the boundary of the underlying domain is the mid-section of a flexible shell.
The mathematical analysis of continuum mechanical models in fluid mechanics reaches back to the pioneering work of Leray on the existence of weak solutions for the incompressible Navier–Stokes equations [129]. Based on this, various fluid-structure interaction results have been achieved already; we will explain this in more detail below. A similar foundational work in the compressible case is due to Lions [130] with important extensions by Feireisl et al. [64]. Compressible fluids are important for applications in aero-dynamics and mathematical results on their interactions with elastic structures appeared in this context recently in [22, 184]. A next natural step is to study the thermodynamics of fluid structure interactions. In fact, the assumption that a physical process is isentropic can only be valid for a very short period of time. In general it is indispensable to take into account the transfer of heat. Similarly, the linearisation of the shell model, often applied in the mathematical literature, looses its validity as soon as the displacement of the boundary is not on a small scale any more. The treatment of non-linear shell models in the context of weak solutions is very recent [142] (see also chapter 2) and (up to date) only available for incompressible fluids. In this work we progress on the theory of weak solutions by showing the existence for systems that take into account 1) heat conduction and compression effects for the fluid and 2) a non-linear elastic respond for the solid. More specifically, we use the classical model by Koiter to describe the shell movement which yields a fully nonlinear fourth order hyperbolic equation with a non-convex energy. The main result of this paper is the existence of a global-in-time weak solution to the Navier–Stokes–Fourier system coupled to the motion of a solid shell of Koiter type. This means that a fourth order PDE for the solid is coupled (via the geometry) to a viscous fluid. A special feature of the Navier–Stokes–Fourier system is that even weak solutions can satisfy an energy equality. We produce a respective equality for the energy of the coupled fluid-structure interaction; this includes the full Koiter energy of the solid deformation. In this context it is noteworthy that we consider perfect elastic shells. This means that no heat is produced by the solid, or reversely entropy is only increased via the fluid. Still some viscous effects can be shown to hold for the elastic solid due to the tight coupling between the solid and the fluid. It is this key observation (and the respective estimate in Subsection 3.5.2) that allows to show that the elastic part of the energy has the necessary compactness in order to prove that the system is indeed closed (energy is preserved). We note that the interval of existence for our weak solutions could be arbitrarily large. In fact, the time of existence is only restricted once either the topology of the fluid domain changes, namely if a self-intersection of the variable boundary (of the elastic shell) is approached, or if the solid energy reaches a point of degeneracy.

Incompressible viscous fluids interacting with lower-dimensional linear elastodynamic equations were studied, for instance, in [31, 87, 144, 127, 109, 147]. All are concerned with the existence of weak solutions which exist as long as the moving part of the structure does not touch the fixed part of the fluid boundary. There are much less results concerning the compressible case. In [22] the authors of the present chapter showed the existence of a weak solution to the compressible Navier–Stokes equations coupled with a linear elastic shell of Koiter type. Eventually, a similar result has been shown by a time-stepping method [184], where the interaction of a compressible fluid with a thermoelastic plate is studied (compare also with with the numeric results from [168]). Results concerning the long-time existence of weak solutions about structure interactions with heat conducting are missing so far - even in the incompressible case. The existence of a unique local-in-time strong solution to compressible Navier–Stokes–Fourier system coupled with a damped linear plate equation has been established very recently in [135].

3.1.1 Overview of the chapter

In Section 3.2 we present basics concerning variable domains as well as the functional analytic set-up. In its last subsection the concept of weak solutions for the coupled system and the main theorem are introduced. The preliminary section is rather significant. Indeed, many standard tools of the analysis need an appropriate adaptation to the variable geometry set-up, as well as to the particular non-linear coupling of the PDE system. In particular, in Subsection 3.2.3 we introduce two different extension operators that are needed for the analysis performed later. In Section 3.3 we study the (regularized) continuity equation as well as the (regularized) internal energy equation in a time dependent domain. These are non-trivial extensions from the analysis presented in [22, Section 3]. In particular, we provide regularity estimates and minimum and maximum principles. Section 3.4 is dedicated to the construction of an approximate solution. Different to previous fixed point approaches (see e.g. [22] and [127]) we construct a fixed point on the Galerkin level which we believe to be appropriate also for future applications. A further achievement is the derivation of the entropy inequality which sensitively relies on Section 3.3. Finally, in Section 3.5 the two limit passages $\varepsilon \to 0$ and $\delta \to 0$ are performed which leads to the proof of Theorem 3.2.14 and the existence of a weak solution is shown. Of particular importance is here Subsection 3.4.4 where the derivation of an *energy equality* is performed. Critical is the strong convergence of the elastic energy of the solid deformation. Here we adapt a recent regularity argument for the shell displacement derived in [142]. As shown in [142] these estimates are crucial to involve non-linear Koiter shell laws in the weak existence theory for incompressible fluids. In the here considered Navier-Stokes-Forier system the regularity is needed even for linear shell models. Since, even for linear Koiter shell models an energy equality cannot be derived without additional regularity estimates and the related compactness properties.

3.2 Preliminaries

3.2.1 Structural and constitutive assumptions

We impose several restrictions on the specific shape of the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$ and $s = s(\varrho, \vartheta)$ which are in line with Gibbs' relation (1.1.16). We consider the pressure p in the form

$$p(\varrho, \vartheta) = p_M(\varrho) + p_R(\vartheta), \quad p_M(\vartheta) = \varrho^\gamma, \quad p_R(\vartheta) = \frac{a}{3}\vartheta^4, \ a > 0, \tag{3.2.1}$$

the specific internal energy

$$e(\varrho, \vartheta) = e_M(\varrho) + e_R(\varrho, \vartheta), \quad e_M(\varrho, \vartheta) = \frac{1}{\gamma - 1}\varrho^{\gamma-1}, \quad e_R(\vartheta, \varrho) = a\frac{\vartheta^4}{\varrho}, \tag{3.2.2}$$

and the specific entropy

$$s(\varrho, \vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\varrho}. \tag{3.2.3}$$

This is slightly more restrictive than the assumptions in [63, Chapter 1], to which we refer for the physical background and the relevant discussion. In fact, in [63, Chapter 1] a weak temperature dependence of p_M is allowed which vanishes asymptotically such that $p_M(\varrho, \vartheta) \sim \varrho^\gamma$ for large ϱ.

The viscosity coefficients μ, λ are continuously differentiable functions of the absolute temperature ϑ, more precisely $\mu,\ \lambda \in C^1([0,\infty))$, satisfying

$$\underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \tag{3.2.4}$$

$$\sup_{\vartheta\in[0,\infty)} \big(|\mu'(\vartheta)| + |\lambda'(\vartheta)|\big) \leq \overline{m}, \tag{3.2.5}$$

$$\underline{\lambda}(1+\vartheta) \leq \lambda(\vartheta) \leq \overline{\lambda}(1+\vartheta), \tag{3.2.6}$$

with positive constants $\underline{\mu}, \overline{\mu}, \overline{m}, \underline{\lambda}, \overline{\lambda}$. The heat conductivity coefficient $\in C^1[0,\infty)$ satisfies

$$0 < (1+\vartheta^3) \leq (\vartheta) \leq (1+\vartheta^3) \tag{3.2.7}$$

with some positive ,. We introduce the following regularizations

$$\begin{aligned} p_\delta(\varrho,\vartheta) &= p(\varrho,\vartheta) + \delta\varrho^\beta, \quad e_\delta(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{\delta}{\beta-1}\varrho^\beta, \\ {}_\delta(\vartheta) &= (\vartheta) + \delta\Big(\vartheta^\beta + \frac{1}{\vartheta}\Big),\ \mathcal{K}_\delta(\vartheta) = \int_0^\vartheta {}_\delta(z)\,\mathrm{d}z, \\ \mathbf{S}^\varepsilon(\vartheta,\nabla\mathbf{u}) &= \mathbf{S}(\vartheta,\nabla\mathbf{u}) + \varepsilon(1+\vartheta)|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}, \end{aligned} \tag{3.2.8}$$

for some $p > 2$.

3.2.2 Function spaces on variable domains

We use the notation introduced in autorefsec:shell. Here the spatial domain Ω is assumed to be a non-empty bounded subset of $\mathbb{R}^3$ with C^4-boundary and an outer unit normal ν. We recall from Subsection 2.2.2 that we assume that $\partial\Omega$ can be parametrised by an injective mapping $\boldsymbol{\varphi} \in C^4(\omega;\mathbb{R}^3)$ such that for all points $y = (y_1, y_2) \in \omega$, the pair of vectors $\partial_i\boldsymbol{\varphi}(y)$, $i = 1, 2$, are linearly independent. For a point x in the neighbourhood or $\partial\Omega$ we can define

$$y(x) = \arg\min_{y\in\omega} |x - \boldsymbol{\varphi}(y)|, \quad s(x) \text{ is defined such that } s(x)\nu(y(x)) + y(x) = x.$$

Moreover, we define the projection $\mathbf{p}(x) = \boldsymbol{\varphi}(y(x))$. We define $L > 0$ to be the largest number such that s, y and $\mathbf{p}$ are well-defined on S_L, where

$$S_L = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\Omega) < L\}, \tag{3.2.9}$$

see also Remark 3.2.17 in connection with this. We remark that due to the C^2 regularity of Ω for L small enough we find that $|s(x)| = \min_{y\in\omega}|x - \boldsymbol{\varphi}(y)|$ for all $x \in S_L$. This implies that $S_L = \{s\nu(y) + y : (s,y) \in [-L, L] \times \omega\}$. For a given function $\eta : I \times \omega \to \mathbb{R}$ we parametrise the deformed boundary by

$$\boldsymbol{\varphi}_\eta(t,y) = \boldsymbol{\varphi}(y) + \eta(t,y)\nu(y), \quad y \in \omega,\ t \in I,$$

and the deformed space-time cylinder $I \times \Omega_\eta = \bigcup_{t\in I}\{t\} \times \Omega_{\eta(t)}$ through

$$\partial\Omega_{\eta(t)} = \{\boldsymbol{\varphi}(y) + \eta(t,y)\nu(y) : y \in \omega\}.$$

The corresponding function spaces for variable domains are defined as follows.

Definition 3.2.1. *(Function spaces) For $I = (0,T)$, $T > 0$, and $\eta \in C(\overline{I} \times \omega)$ with $\|\eta\|_{L^\infty_{t,x}} < L$ we set $I \times \Omega_\eta := \bigcup_{t\in I}\{t\} \times \Omega_{\eta(t)} \subset \mathbb{R}^4$. We define for $1 \leq p, r \leq \infty$*

$$\begin{aligned} L^p(I;L^r(\Omega_\eta)) &:= \big\{v \in L^1(I \times \Omega_\eta) :\ v(t,\cdot) \in L^r(\Omega_{\eta(t)}) \text{ for a.e. } t,\ \|v(t,\cdot)\|_{L^r(\Omega_{\eta(t)})} \in L^p(I)\big\}, \\ L^p(I;W^{1,r}(\Omega_\eta)) &:= \big\{v \in L^p(I;L^r(\Omega_\eta)) :\ \nabla v \in L^p(I;L^r(\Omega_\eta))\big\}. \end{aligned}$$

For various purposes it is useful to relate the time dependent domains and the fixed domain. This can be done by the means of the Hanzawa transform. Its construction can be found in [127, pages 210, 211]. Note that variable domains in [127] are defined via functions $\zeta : \partial\Omega \to \mathbb{R}$ rather than functions $\eta : \omega \to \mathbb{R}$ (clearly, one can link them by setting $\zeta = \eta \circ \varphi^{-1}$). For any $\eta : \omega \to (-L, L)$ we define the Hanzawa transform $\mathbf{\Psi}_\eta : \Omega \to \Omega_\eta$ by

$$\mathbf{\Psi}_\eta(x) = \begin{cases} \mathbf{p}(x) + \Big(s(x) + \eta(y(x))\phi(s(x))\Big)\nu(y(x)), & \text{if } dist(x, \partial\Omega) < L, \\ x, & \text{elsewhere} \end{cases}. \tag{3.2.10}$$

Here $\phi \in C^\infty\big((-\frac{3L}{4}, \infty), [0,1]\big)$ is such that $\phi \equiv 0$ in $[-\frac{3L}{4}, -\frac{L}{2}]$ and $\phi \equiv 1$ in $[-\frac{L}{4}, \infty)$. Due to the size of L, we find that $\mathbf{\Psi}_\eta$ is a homomorphism such that $\mathbf{\Psi}_\eta|_{\Omega \setminus S_L}$ is the identity. Moreover, $\eta \in C^k(\omega)$ for $k \in \mathbb{N}$ implies that $\mathbf{\Psi}_\eta$ is a C^k-diffeomorphism.

We collect a few properties of the above mapping $\mathbf{\Psi}_\eta$.

Lemma 3.2.2. *Let $1 < p \le \infty$ and $\sigma \in (0, 1]$.*

a) If $\eta \in W^{2,2}(\omega)$ with $\|\eta\|_{L^\infty_x} < L$, then the linear mapping $\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta$ ($\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta^{-1}$) is continuous from $L^p(\Omega_\eta)$ to $L^r(\Omega)$ (from $L^p(\Omega)$ to $L^r(\Omega_\eta)$) for all $1 \le r < p$.

b) If $\eta \in W^{2,2}(\omega)$ with $\|\eta\|_{L^\infty_x} < L$, then the linear mapping $\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta$ ($\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta^{-1}$) is continuous from $W^{1,p}(\Omega_\eta)$ to $W^{1,r}(\Omega)$ (from $W^{1,p}(\Omega)$ to $W^{1,r}(\Omega_\eta)$) for all $1 \le r < p$.

c) If $\eta \in C^{0,1}(\omega)$with $\|\eta\|_{L^\infty_x} < L$, then the linear mapping $\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta$ ($\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta^{-1}$) is continuous from $W^{\sigma,p}(\Omega_\eta)$ to $W^{\sigma,p}(\Omega)$ (from $W^{\sigma,p}(\Omega)$ to $W^{\sigma,p}(\Omega_\eta)$).

d) If $\eta \in W^{2,2}(\partial\Omega)$with $\|\eta\|_{L^\infty_x} < L$, then the linear mapping $\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta$ ($\mathbf{v} \mapsto \mathbf{v} \circ \mathbf{\Psi}_\eta^{-1}$) is continuous from $W^{\sigma,p}(\Omega_\eta)$ to $W^{\theta,r}(\Omega)$ (from $W^{\sigma,p}(\Omega)$ to $W^{\theta,r}(\Omega_\eta)$) for all $\theta \in (0, \sigma)$ and all $1 < r < p$.

The continuity constants depend only on $\Omega, p, r, \sigma, \theta$, the respective norms of η.

The following lemma is a modification of [127, Cor. 2.9].

Lemma 3.2.3. *Let $1 < p < 3$, $\sigma \in (\frac{1}{p}, 1]$ and $\eta \in W^{2,2}(\omega)$ with $\|\eta\|_{L^\infty_x} < L$. The linear mapping $tr_\eta : v \mapsto v \circ \mathbf{\Psi}_\eta \circ \varphi|_{\partial\Omega}$ is well defined and continuous from $W^{\sigma,p}(\Omega_\eta)$ to $W^{\sigma - \frac{1}{r}, r}(\omega)$ for all $r \in (\frac{1}{\sigma}, p)$ and well defined and continuous from $W^{\sigma,p}(\Omega_\eta)$ to $L^q(\omega)$ for all $1 < q < \frac{2p}{3 - \sigma p}$. The continuity constants depend only on Ω, p, σ, and $\|\eta\|_{W^{2,2}_x}$.*

Remark 3.2.4. *If $\eta \in L^\infty(I; W^{2,2}(\omega))$ we obtain non-stationary variants of the results stated above.*

It will be convenient for our purposes to extend $\mathbf{\Psi}_\eta$, originally defined only on

$$\Omega_{(L-\eta)_+} = \Omega \cup \{x \in S_L : s(x) < \min\{L, L - \eta(y(x))\}\}),$$

to $\Omega_L = \Omega \cup S_L$ by setting

$$\overline{\mathbf{\Psi}}_\eta(x) = \begin{cases} \mathbf{p}(x) + \Big(s(x) + \eta(y(x))\phi(s(x))\Big)\nu(y(x)), & \text{if } dist(x, \partial\Omega) < L,\ s(x) + \eta(\mathbf{p}(x)) < L, \\ x, & \text{elsewhere.} \end{cases}$$

All the above statements are also true for $\mathbf{v} \mapsto \mathbf{v} \circ \overline{\mathbf{\Psi}}_\eta$ and $\mathbf{v} \mapsto \mathbf{v} \circ \overline{\mathbf{\Psi}}_\eta^{-1}$ on their respective domains.

3.2.3 Extensions on variable domains

Since Ω is assumed to be sufficiently smooth, it is well-know that there is an extension operator $\mathscr{F}_\Omega$ which extends functions from $\partial\Omega$ to $\mathbb{R}^3$ and satisfies

$$\mathscr{F}_\Omega : W^{\sigma,p}(\partial\Omega) \to W^{\sigma+1/p,p}(\mathbb{R}^3)$$

for all $p \in (1,\infty)$ and $\sigma \in [0,1]$, all as well as $\mathscr{F}_\Omega v|_{\partial\Omega} = v$. Now we define $\mathscr{F}_\eta$ by

$$\mathscr{F}_\eta b = \mathscr{F}_\Omega((b\nu)\circ\boldsymbol{\varphi}^{-1})\circ\overline{\boldsymbol{\Psi}}_\eta^{-1}, \quad b \in W^{\sigma,p}(\omega), \tag{3.2.11}$$

where $\boldsymbol{\varphi}$ is the C^4-function in the parametrisation of Ω. If η is smooth $\mathscr{F}_\eta$ behaves as a classical extension by Lemma 3.2.2. The following properties can all be easily derived from the formulas

$$\begin{aligned}
\nabla\mathscr{F}_\eta b &= \nabla\mathscr{F}_\Omega((b\nu)\circ\boldsymbol{\varphi}^{-1})\circ\overline{\boldsymbol{\Psi}}_\eta^{-1}\nabla\overline{\boldsymbol{\Psi}}_\eta^{-1},\\
\nabla^2\mathscr{F}_\eta b &= \nabla^2\mathscr{F}_\Omega((b\nu)\circ\boldsymbol{\varphi}^{-1})\circ\overline{\boldsymbol{\Psi}}_\eta^{-1}\nabla\overline{\boldsymbol{\Psi}}_\eta^{-1}\nabla\overline{\boldsymbol{\Psi}}_\eta^{-1} + \nabla\mathscr{F}_\Omega((b\nu)\circ\boldsymbol{\varphi}^{-1})\circ\overline{\boldsymbol{\Psi}}_\eta^{-1}\nabla^2\overline{\boldsymbol{\Psi}}_\eta^{-1},\\
\partial_t\mathscr{F}_\eta b &= \nabla\mathscr{F}_\Omega((b\nu)\circ\boldsymbol{\varphi}^{-1})\circ\overline{\boldsymbol{\Psi}}_\eta^{-1}\partial_t\overline{\boldsymbol{\Psi}}_\eta^{-1},
\end{aligned}$$

where $\nabla\overline{\boldsymbol{\Psi}}_\eta^{-1}$, $\nabla^2\overline{\boldsymbol{\Psi}}_\eta^{-1}$ and $\partial_t\overline{\boldsymbol{\Psi}}_\eta^{-1}$ behave as $\nabla\eta$, $\nabla^2\eta$ and $\partial_t\eta$ respectively.

Lemma 3.2.5. *Let $\eta \in C^{0,1}(\omega)$ with $\|\eta\|_{L^\infty_x} < \alpha < L$.*

(a) The operator $\mathscr{F}_\eta$ defined in (3.2.11) *satisfies for all $p \in [1,\infty)$ and $\sigma \in [0,1]$*

$$\mathscr{F}_\eta : W^{\sigma,p}(\omega) \to W^{\sigma+1/p,p}(\Omega \cup S_\alpha)$$

and $tr_\eta\mathscr{F}_\eta b = b\nu$ for all $b \in W^{1,p}(\omega)$. In particular, we have

$$\|\mathscr{F}_\eta b\|_{W^{\sigma+1/p,p}(\Omega\cup S_\alpha)} \le c\,\|b\|_{W^{\sigma,p}(\omega)}$$

for all $b \in W^{1,p}(\omega)$, where the constant c depends only on Ω, p, σ, $\|\nabla\eta\|_{L^\infty_x}$ and $L-\alpha$.

(b) If $p = \infty$ we have

$$\|\mathscr{F}_\eta b\|_{W^{1,\infty}(\Omega\cup S_\alpha)} \le c(1 + \|\nabla\eta\|_{L^\infty(\omega)})\,\|b\|_{W^{1,\infty}(\omega)}$$

for all $b \in W^{1,\infty}(\omega)$, where c depends only on Ω, p and $L-\alpha$.

Corollary 3.2.6. *Let $\eta \in C^1(\overline{I}\times\omega)$ with $\|\eta\|_{L^\infty_x} < \alpha < L$. Then we have for all $q < \infty$*

$$\sup_{t\in I}\|\partial_t\mathscr{F}_\eta b\|_{L^q(\Omega\cup S_\alpha)} \le c\,\|b\|_{W^{1,q}(\omega)}\|\partial_t\eta\|_{L^\infty(I\times\omega)}$$

for all $b \in W^{1,q}(\omega)$, where the constant c depends only on Ω, p and $L-\alpha$.

We now turn to the case of a less regular function η and analyse the properties of $\mathscr{F}_\eta$ given by (3.2.11) in this case.

Lemma 3.2.7. *Let $p \in [1,\infty)$ and $\eta \in W^{2,2}(\omega)$ with $\|\eta\|_{L^\infty_x} < \alpha < L$ and let the operator $\mathscr{F}_\eta$ by defined by* (3.2.11).

(a) We have for all $p \in (1,\infty)$ and $\sigma \in (0,1]$

$$\mathscr{F}_\eta : W^{\sigma,p}(\omega) \to W^{\sigma,q}(\Omega \cup S_\alpha)$$

for all $q < \frac{3}{2}p$ and $tr_\eta\mathscr{F}_\eta b = b\nu$ for all $b \in W^{1,p}(\omega)$. In particular, we have

$$\|\mathscr{F}_\eta b\|_{W^{\sigma,p}(\Omega\cup S_\alpha)} \le c\,\|b\|_{W^{\sigma,p}(\omega)}$$

for all $b \in W^{1,p}(\omega)$.

(b) *We have for all $r < 2$*

$$\mathscr{F}_\eta : W^{2,2}(\omega) \to W^{2,r}(\Omega \cup S_\alpha)$$

and $tr_\eta \mathscr{F}_\eta b = b\nu$ for all $b \in W^{2,2}(\omega)$. In particular, we have

$$\|\mathscr{F}_\eta b\|_{W^{2,r}(\Omega \cup S_\alpha)} \le c\,\|b\|_{W^{2,2}(\omega)}$$

for all $b \in W^{2,2}(\omega)$.

The constants in (a) and (b) depend only on Ω, p, q, $\|\eta\|_{W^{2,2}_x}$ and $L - \alpha$.

Corollary 3.2.8. *Let $\eta \in L^2(I; W^{2,2}(\partial\Omega))$ with $\|\eta\|_{L^\infty_{t,x}} < \alpha < L$. Suppose that $\partial_t \eta \in L^q(I \times \omega)$ for some $q > 1$. Then we have uniformly in time*

$$\|\partial_t \mathscr{F}_\eta b\|_{L^r(\Omega \cup S_\alpha)} \le c\,\|b\|_{W^{1,p}(\omega)} \|\partial_t \eta\|_{L^q(\omega)}$$

for all $b \in W^{1,p}(\omega)$, provided $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \le 1$. The constant c depends only on Ω, p and $L - \alpha$.

3.2.4 Convergence in variable domains.

Due to the variable domain the framework of Bochner spaces is not available. Hence, we cannot use the classical Aubin-Lions compactness theorem. In this subsection we are concerned with the question of how to get compactness anyway. We start with the following definition of convergence in variable domains.

Definition 3.2.9. *Let $(\eta_i) \subset C(\overline{I} \times \omega; [-\theta L, \theta L])$, $\theta \in (0,1)$, be a sequence with $\eta_i \to \eta$ uniformly in $\overline{I} \times \omega$. Let $p, q \in [1, \infty]$ and $k \in \mathbb{N}_0$.*

1. *We say that a sequence $(g_i) \subset L^p(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^p(I, L^q(\Omega_{\eta_i}))$ strongly with respect to (η_i), in symbols $g_i \to^\eta g$ in $L^p(I, L^q(\Omega_{\eta_i}))$, if*

$$\chi_{\Omega_{\eta_i}} g_i \to \chi_{\Omega_\eta} g \quad \text{in} \quad L^p(I, L^q(\mathbb{R}^3)).$$

2. *Let $p, q < \infty$. We say that a sequence $(g_i) \subset L^p(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^p(I, L^q(\Omega_{\eta_i}))$ weakly with respect to (η_i), in symbols $g_i \rightharpoonup^\eta g$ in $L^p(I, L^q(\Omega_{\eta_i}))$, if*

$$\chi_{\Omega_{\eta_i}} g_i \rightharpoonup \chi_{\Omega_\eta} g \quad \text{in} \quad L^p(I, L^q(\mathbb{R}^3)).$$

3. *Let $p = \infty$ and $q < \infty$. We say that a sequence $(g_i) \subset L^\infty(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^\infty(I, L^q(\Omega_{\eta_i}))$ weakly* with respect to (η_i), in symbols $g_i \rightharpoonup^{*,\eta} g$ in $L^\infty(I, L^q(\Omega_{\eta_i}))$, if*

$$\chi_{\Omega_{\eta_i}} g_i \rightharpoonup^* \chi_{\Omega_\eta} g \quad \text{in} \quad L^\infty(I, L^q(\mathbb{R}^3)).$$

Note that in the case of one single η (i.e. not a sequence) the space $L^p(I, L^q(\Omega_\eta))$ (with $1 \le p < \infty$ and $1 < q < \infty$) is reflexive and we have the usual duality pairing

$$L^p(I, L^q(\Omega_\eta)) \cong L^{p'}(I, L^{q'}(\Omega_\eta)) \tag{3.2.12}$$

provided η is smooth enough. Definition 3.2.9 can be extended in a canonical way to Sobolev spaces: A sequence $(g_i) \subset L^p(I, W^{1,q}(\Omega_{\eta_i}))$ converges to g in $L^p(I, W^{1,q}(\Omega_{\eta_i}))$ strongly with respect to (η_i), in symbols

$$g_i \to^\eta g \quad \text{in} \quad L^p(I, W^{1,p}(\Omega_{\eta_i})),$$

if both g_i and ∇g_i converges (to g and ∇g respectively) in $L^p(I, L^q(\Omega_{\eta_i}))$ strongly with respect to (η_i) (in the sense of Definition 3.2.9 a)). We also define weak and weak* convergence in Sobolev spaces with respect to (η_i) with an obvious meaning. Note that also an extension to higher order Sobolev spaces is possible but not needed for our purposes.

For the next compactness lemma (see [22, Lemma 2.8]) we require the following assumptions on the functions describing the boundary

(A1) The sequence $(\eta_i) \subset C(\overline{I} \times \omega; [-\theta L, \theta L])$, $\theta \in (0,1)$, satisfies

$$\eta_i \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I, W^{2,2}(\omega)),$$
$$\partial_t \eta_i \rightharpoonup^* \partial_t \eta \quad \text{in} \quad L^\infty(I, L^2(\omega)).$$

(A2) Let (v_i) be a sequence such that for some $p, s \in [1,\infty)$ and $\alpha \in (0,1]$ we have

$$v_i \rightharpoonup^\eta v \quad \text{in} \quad L^p(I; W^{\alpha,s}(\Omega_{\eta_i})).$$

(A3) Let (r_i) be a sequence such that for some $m, b \in [1,\infty)$ we have

$$r_i \rightharpoonup^\eta r \quad \text{in} \quad L^m(I; L^b(\Omega_{\eta_i})).$$

Assume further that there are sequences $(\mathbf{H}_i^1)$, $(\mathbf{H}_i^2)$ and (h_i), bounded in $L^m(I; L^b(\Omega_{\eta_i}))$, such that $\partial_t r_i = \mathrm{div}\mathrm{div}\mathbf{H}_i^1 + \mathrm{div}\mathbf{H}_i^2 + h_i$ in the sense of distributions, i.e.,

$$\int_I \int_{\Omega_{\eta_i}} r_i \, \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t = \int_I \int_{\Omega_{\eta_i}} \mathbf{H}_i^1 : \nabla^2 \phi \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_{\Omega_{\eta_i}} \mathbf{H}_i^2 \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_{\Omega_{\eta_i}} h_i \, \phi \, \mathrm{d}x \, \mathrm{d}t$$

for all $\phi \in C_c^\infty(I \times \Omega_{\eta_i})$.

In [22, Lemma 2.8] the corresponding version of (A2) assumes $\alpha = 1$. But the very same argument is also valid in case $\alpha \in (0,1)$ due to compact embeddings for fractional Sobolev spaces.

Lemma 3.2.10. *Let (η_i), (v_i) and (r_i) be sequences satisfying* (A1)–(A3) *where $\frac{1}{s^*} + \frac{1}{b} = \frac{1}{a} < 1$ (with $s^* = \frac{3s}{3-s\alpha}$ if $s \in (1, 3/\alpha)$ and $s^* \in (1,\infty)$ arbitrarily otherwise) and $\frac{1}{m} + \frac{1}{p} = \frac{1}{q} < 1$. Then there is a subsequence with*

$$v_i r_i \rightharpoonup^\eta vr \text{ weakly in } L^q(I, L^a(\Omega_{\eta_i})). \tag{3.2.13}$$

Corollary 3.2.11. *In the case $r_i = v_i$ we find that*

$$v_i \to^\eta v \text{ strongly in } L^2(I, L^2(\Omega_{\eta_i})).$$

We finish this section be repeating the following Aubin-Lions type lemma which is shown in [142, Theorem 5.1. & Remark 5.2.].

3.2.5 Weak solutions and main theorem

In accordance with the current state of the art for weak solutions of Navier-Stokes-Fourier law fluids and fluid-structure interactions, we introduce here our concept of weak solutions. For that we introduce the following function spaces:

(S1) For the solid deformation $\eta : I \times \omega \to \mathbb{R}$, $Y^I := \{\zeta \in W^{1,\infty}(I; L^2(\omega)) \cap L^\infty(I; W^{2,2}(\omega))\}$.

(S2) For the fluid velocity $\mathbf{u} : I \times \Omega_\eta \to \mathbb{R}^d$, $d = 2,3$, $X_\eta^I := L^2(I; W^{1,2}(\Omega_\eta))$.

(S3) For the fluid density $\varrho : I \times \Omega_\eta \to [0,\infty)$, $W_\eta^I := C_w(\overline{I}; L^\gamma(\Omega_\eta))$, where the subscript w refers to continuity with respect to the weak topology.

(S4) For the temperature $\vartheta : I \times \Omega_\eta \to [0,\infty)$, $Z_\eta^I = L^2(I; W^{1,2}(\Omega_\eta)) \cap L^\infty(I; L^4(\Omega_\eta))$.

The definition of the function spaces above depending on η only make sense provided $\|\eta\|_{L^\infty_{t,x}} < L$.

Definition 3.2.12. *A weak solution to* (4.2.2)–(1.1.20) *is a quadruplet $(\eta, \mathbf{u}, \varrho, \vartheta) \in \times Y^I \times X_\eta^I \times W_\eta^I \times Z_\eta^I$, which satisfies the following.*

(O1) The momentum equation is satisfied in the sense that

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_\eta}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_\eta} \mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta} p(\varrho,\vartheta)\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\bigg(\frac{\mathrm{d}}{dt}\int_\omega \partial_t\eta b\ \mathrm{d}y - \int_\omega \partial_t\eta\,\partial_t b\ \mathrm{d}y + \int_\omega K'(\eta)\,b\ \mathrm{d}y\bigg)\,dt \\
&= \int_I\int_{\Omega_\eta} \varrho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\omega g\,b\ \mathrm{d}y\,dt
\end{aligned} \tag{3.2.14}$$

holds for all $(b,\boldsymbol{\phi}) \in C^\infty(\omega)\times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\phi} = b\nu$. Moreover, we have $(\varrho\mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t\eta(0) = \eta_1$. The boundary condition $\mathrm{tr}_\eta\mathbf{u} = \partial_t\eta\nu$ holds in the sense of Lemma 3.2.3.

(O2) The continuity equation is satisfied in the sense that

$$\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.2.15}$$

holds for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ and we have $\varrho(0) = \varrho_0$.

(O3) The entropy balance

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho s(\varrho,\vartheta)\,\psi\,\mathrm{d}x\,\mathrm{d}t &- \int_I\int_{\Omega_\eta}\big(\varrho s(\varrho,\vartheta)\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_I\int_{\Omega_\eta} \frac{1}{\vartheta}\Big(\mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} + \frac{(\vartheta)}{\vartheta}|\nabla\vartheta|^2\Big)\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_\eta} \frac{(\vartheta)\nabla\vartheta}{\vartheta}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_\eta}\frac{\varrho}{\vartheta}H\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.2.16}$$

holds for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\psi \geq 0$. Moreover, we have $\lim_{r\to 0}\varrho s(\varrho,\vartheta)(t) \geq \varrho_0 s(\varrho_0,\vartheta_0)$ and $\partial_{\nu_\eta}\vartheta|_{\partial\Omega_\eta} \leq 0$.

(O4) The total energy balance

$$\begin{aligned}
-\int_I \partial_t\psi\,\mathcal{E}\,dt = \psi(0)\mathcal{E}(0) &+ \int_I\psi\int_\Omega \varrho H\,\mathrm{d}x\,\mathrm{d}t + \int_I\psi\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\psi\int_\omega g\,\partial_t\eta\ \mathrm{d}y\,dt
\end{aligned} \tag{3.2.17}$$

holds for any $\psi \in C_c^\infty([0,T))$. Here, we abbreviated

$$\mathcal{E}(t) = \int_{\Omega_{\eta(t)}}\Big(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)e(\varrho(t),\vartheta(t))\Big)\,\mathrm{d}x + \int_\omega \frac{|\partial_t\eta(t)|^2}{2}\ \mathrm{d}y + K(\eta(t)).$$

As will be apparent by the analysis we will show that the renormalized continuity equation in the sense of DiPerna and Lions is satisfied, cf. [46, 130].

Definition 3.2.13 (Renormalized continuity equation)**.** *Let $\eta \in Y^I$ and $\mathbf{u} \in X^I_\eta$. we say that rhe function $\varrho \in W^I_\eta$ solves the continuity equation* (4.2.2) *in the renormalized sense if we have*

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \theta(\varrho)\psi\,\mathrm{d}x\,\mathrm{d}t &- \int_I\int_{\Omega_\eta}\Big(\theta(\varrho)\partial_t\psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_I\int_{\Omega_\eta}(\varrho\theta'(\varrho) - \theta(\varrho))\mathrm{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.2.18}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ and all $\theta \in C^1(\mathbb{R})$ with $\theta(0) = 0$ and $\theta'(z) = 0$ for $z \geq M_\theta$.

We are now ready to formulate our main result.

Theorem 3.2.14. *Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). Assume that we have*

$$\frac{|\mathbf{q}_0|^2}{\varrho_0} \in L^1(\Omega_{\eta_0}),\ \varrho_0 \in L^\gamma(\Omega_{\eta_0}),\ \vartheta_0 \in L^4(\Omega_{\eta_0}),\ \eta_0 \in W^{2,2}(\omega),\ \eta_1 \in L^2(\omega),$$
$$\mathbf{f} \in L^2(I; L^\infty(\mathbb{R}^3)),\ g \in L^2(I \times \omega),\ H \in L^\infty(I \times \mathbb{R}^3),\ H \geq 0 \textit{ a.e.}$$

Furthermore suppose that $\varrho_0 \geq 0$ a.e., $\vartheta_0 \geq 0$ a.e. and that (1.1.21) *is satisfied. Then there exists a weak solution $(\eta, \mathbf{u}, \varrho, \vartheta)$ to* (4.2.2)–(1.1.20) *in the sense of Definition 3.2.12. The interval of existence is of the form $I = (0, t)$, where $t < T$ only in case $\Omega_{\eta(s)}$ approaches a self-intersection when $s \to t$ or the Koiter energy degenerates (namely, if $\lim_{s\to t} \overline{\gamma}(s, y) = 0$ for some point $y \in \omega$). Moreover, the continuity equation is satisfied in the renormalized sense as specified in Definition 3.2.13.*

Remark 3.2.15 (Minimal interval of existence)**.** *Let us mention that for any admissible initial conditions there is a minimal positive interval of existence. It follows from the fact that η (and consequently also $\overline{\gamma}$, cf. Theorem 2.2.1) can be shown to be uniformly continuous in space-time (with bounds depending on the data only). Consequently, for some non-empty open time-interval no self-touching or point of degeneracy can be approached a-priori.*

Remark 3.2.16 (Elastic solids vs. damped elastic solids)**.** *Equation* (1.1.19) *implies that we consider a perfectly elastic solid. In particular, no damping is assumed and hence dissipative effects are not assumed to act on the solid. Consequently, no effects stemming from the solid are appearing in the entropy relation. While this decoupling might seem like an advantage at first glance the opposite is actually true. Indeed, a damped solid naturally yields better a priori estimates for the solid evolution. As a consequence the damping effects would simplify the convergence analysis of the energy equality in the latter case.*

In conclusion, the strategy developed here assumes with no loss of generality that the solid has no damping effect. Indeed, it seems to be perfectly applicable for damped shells or plates as well. These damping terms would, however, create naturally non-homogeneous Neumann boundary values for the temperature since the motion of the solid could produce heat.[1]

Remark 3.2.17 (Simplification of notation)**.** *We remark that we will assume without further mentioning that the initial conditions for the elastic deformation are within a neighbourhood of the reference configurations. This simplification is, however, without loss of generality. Indeed, by rephrasing the reference geometry accordingly, the existence procedure can be prolonged until a point of self touching or degeneracy (in case of non-linear Koiter energies) is approached.*

3.2.6 Mathematical strategy

In this paragraph we provide an overview of the developed methodologies. Further we aim to explain all technical novelties and their potential significance.
As is common in the existence theory for weak solutions, the first step is to understand how to prove sequential compactness. Let us assume there is a given sequence of weak solutions $(\eta_n, \mathbf{u}_n, \varrho_n, \vartheta_n)$ to (4.2.2)–(1.1.20) possessing suitable regularity properties. Deriving a priori estimates using the entropy balance one can control, in addition to the total energy defined in (1.1.22), first order spatial derivative of $\mathbf{u}_n$ and ϑ_n using (1.1.17) and (1.1.18). Unlike in the steady domain case, these estimates are not sufficient to show that a subsequence is again converging to a solution. One problem is to derive energy equality (that is expected for closed systems like the Navier–Stokes–Fourier equations considered here). Critical are the *kinetic and elastic part of the solid energy*. To prove their compactness which does not follow from the energy estimates. In fact, the functional K is not even well-defined on $W^{2,2}(\omega)$ (compare with chapter 2). Our strategy is to derive fractional estimates for

[1] The Neumann boundary values would naturally be preserved in the limit passage and lead to a respective inequality for the weak solutions considered here.

$\nabla^2\eta_n$ as a consequence of a testing procedure for (1.1.19) with difference quotients. Testing the shell equation with suitable test-functions requires in the weak formulation to choose an appropriate test-function for the full momentum equation as well. Technically, this means we have to "extend" functions defined on ω to functions defined on the time dependent domain Ω_{η_n}. An obstacle here is that the pressure is only expected to belong to L^1 in space near the moving boundary (compare with [22]).[2] To circumvent the irregularity of the pressure we work with a solenoidal extension $\mathscr{F}^{\mathrm{div}}_{\eta_n}$ that was recently constructed in [142] (that is chapter 2).
A second related problem is the strong convergence of $\partial_t\eta_n$ (which is a part of the kinetic energy). Here we use a modified version of the classical argument by Aubin-Lions. Critical is the uniform continuity in time of the underlying sequence, which relies on the weak coupled momentum equation. Again a carefully chosen test-function is needed. Here, however, we use an extension which has (different to $\mathscr{F}^{\mathrm{div}}_{\eta_n}$) a regularizing effect but no solenoidality is needed. What turns out to be the critical point is that the extension is depending on the variable geometry. In particular, the extension of a constant in time function still possesses a non-trivial time-derivative. The essential term is

$$\int_I\int_{\Omega_{\eta_n}} \varrho_n\mathbf{u}_n\cdot\partial_t(\mathscr{F}_{\eta_n}b)\,\mathrm{d}x\,\mathrm{d}t$$

using the notation from the next section. We observe that $\partial_t(\mathscr{F}_{\eta_n}b)$ (the time-derivative of the extension) is expected to behave like $\partial_t\eta_n$. Based on the a priori estimates $\varrho_n\in L^\infty_t(L^\gamma_x)$, $\mathbf{u}_n\in L^2_t(L^6_x)$, we find that $\partial_t\eta_n\in L^2_t(L^r_x)$ for all $r<4$ uniformly by the trace theorem (see Lemma 3.2.3). Consequently the bound $\gamma>\frac{12}{7}$ naturally appears. It is interesting to note that the same bound was needed in [22, Lemma 7.4] in order to avoid concentrations of the approximate pressure at the boundary (an argument that we will use later in Lemma 3.5.6).
In order to prove Theorem 3.2.14 we have to work with a multi-layer approximation scheme. As is nowadays standard in the theory of compressible fluids we follow [63] and use an artificial pressure (replace $p(\varrho,\vartheta)$ by $p_\delta(\varrho,\vartheta)=p(\varrho,\vartheta)+\delta\varrho^\beta$ where β is chosen large enough) as well as an artificial viscosity (add $\varepsilon\Delta\varrho$ to the right-hand side of (4.2.2)). The resulting system is solved by means of a Galerkin approximation. More specifically, we have to solve a finite-dimensional system of ODEs and eventually pass to the limit in the dimension N. It turns out that existence on the basic level, where the parameters ε and δ are fixed, is quite involving. Troublesome is the derivation of the entropy balance (1.1.17) (in form of a variational inequality): Though it is suitable to pass to the limit it is not appropriate for the direct construction of solutions due to its highly involving non-linearities. Hence the entropy balance is derived a posteriori by dividing the internal energy equation (1.1.11) by ϑ. In order to do this rigorously it has to be shown that the temperature is strictly positive - a property which can only be expected from strong solutions to (1.1.11). One of the main efforts of this paper is consequently to construct strong solutions to (1.1.11) for regularized velocity and smooth pressure. New a priori estimates for (1.1.11) and (4.2.2) in variable domains are shown that go well beyond the results from [22, Sec. 3] and form one if the main achievements of this paper. Finally, we wish to note that we can shorten the approach from [22] considerably. Different to [22] we decouple the geometry from the fluid system on the Galerkin level and apply the fixed point argument to the resulting semi-discrete problem directly. This allows to remove one regularization level in which the moving boundary and the convective terms are regularised by a parameter κ.

3.3 Equations for density and temperature in variable domains

In this section we study the continuity equation (with artificial viscosity) as well as the internal energy equation in variable domains. In Theorems 3.3.3 and 3.3.4 we prove the existence of classical solutions to both equations under the assumption the data (the velocity field as well and the variable boundary) are smooth. In particular, we prove that the temperature stays strictly positive on the regularised level. This is a key ingredient for the remainder of the paper.

[2] As is explained in [22] the usual test with the Bogovskiĭ-operator (that implies higher integrability of the density) fails and we are only able to prove uniform integrability, cf. Lemmas 3.5.4 and 3.5.6.

3.3.1 The continuity equation

In this subsection we are concerned with the regularised continuity equation in a (given) variable domain. We assume that the moving boundary is prescribed by a function $\zeta : \overline{I} \times \omega \to \mathbb{R}$. For a given function $\mathbf{w} \in L^2(I; W^{1,2}(\Omega_\zeta))$ with $tr_\zeta \mathbf{w} = \partial_t \zeta \nu$ and $\varepsilon > 0$ we consider the equation

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{w}) = \varepsilon \Delta \varrho \quad \text{in} \quad I \times \Omega_\zeta, \\ &\varrho(0) = \varrho_0 \text{ in } \Omega_{\zeta(0)}, \quad \partial_{\nu_\zeta} \varrho\big|_{\partial \Omega_\zeta} = 0 \quad \text{on} \quad I \times \partial \Omega_\zeta. \end{aligned} \tag{3.3.1}$$

A weak solution to (3.3.1) satisfies

$$\int_I \frac{\mathrm{d}}{dt} \int_{\Omega_\zeta} \varrho \psi \, \mathrm{d}x \, dt - \int_I \int_{\Omega_\zeta} \big(\varrho \partial_t \psi + \varrho \mathbf{w} \cdot \nabla \psi\big) \, \mathrm{d}x \, \mathrm{d}t = -\int_I \int_{\Omega_\zeta} \varepsilon \nabla \varrho \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t \tag{3.3.2}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$. The following result has been proved in [22, Thm. 3.1](for the analogous results for fixed in time domains see [64, section 2.1]).

Theorem 3.3.1. *Let $\zeta \in C^{2,\alpha}(\overline{I} \times \omega, [\frac{L}{2}, \frac{L}{2}])$ with $\alpha \in (0,1)$ be the function describing the boundary. Assume that $\mathbf{w} \in L^2(I; W^{1,2}(\Omega_\zeta)) \cap L^\infty(I \times \Omega_\zeta)$ with $tr_\zeta \mathbf{w} = \partial_t \zeta \nu$ and $\varrho_0 \in L^2(\Omega_{\zeta(0)})$.*

a) There is a unique weak solution ϱ to (3.3.1) *such that*

$$\varrho \in L^\infty(I; L^2(\Omega_\zeta)) \cap L^2(I; W^{1,2}(\Omega_\zeta)).$$

b) Let $\theta \in C^2(\mathbb{R}_+; \mathbb{R}_+)$ be such that $\theta'(s) = 0$ for large values of s and $\theta(0) = 0$.Then the following holds, for the canonical zero extension of $\varrho \equiv \varrho \chi_{\Omega_\zeta}$:

$$\begin{aligned} &\int_I \frac{\mathrm{d}}{dt} \int_{\mathbb{R}^3} \theta(\varrho) \, \psi \, \mathrm{d}x \, \mathrm{d}t - \int_{I \times \mathbb{R}^3} \theta(\varrho) \, \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = -\int_{I \times \mathbb{R}^3} \big(\varrho \theta'(\varrho) - \theta(\varrho)\big) \operatorname{div} \mathbf{w} \, \psi \, \mathrm{d}x + \int_{I \times \mathbb{R}^3} \theta(\varrho) \mathbf{w} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_{I \times \mathbb{R}^3} \varepsilon \chi_{\Omega_\zeta} \nabla \theta(\varrho) \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t - \int_{I \times \mathbb{R}^3} \varepsilon \chi_{\Omega_\zeta} \theta''(\varrho) |\nabla \varrho|^2 \psi \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.3.3}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$.

c) Assume that $\varrho_0 \geq 0$ a.e. in $\Omega_{\zeta(0)}$. Then we have $\varrho \geq 0$ a.e. in $I \times \Omega_\zeta$.

Remark 3.3.2. *Observe that:*

- *The statement in [22] holds without the assumption $tr_\zeta \mathbf{w} = \partial_t \zeta \nu$ under the boundary condition $\partial_{\nu_\zeta} \varrho\big|_{\partial \Omega_\zeta} = \frac{1}{\varepsilon} \varrho (\mathbf{w} - (\partial_t \zeta \nu) \circ \boldsymbol{\varphi}_\zeta^{-1}) \cdot \nu_\zeta$.*
- *Theorem 3.1 in [22] is formulated with the stronger assumption $\zeta \in C^3(\overline{I} \times \omega, [\frac{L}{2}, \frac{L}{2}])$. However, it can be checked that the condition $\zeta \in C^{2,\alpha}(\overline{I} \times \omega, [\frac{L}{2}, \frac{L}{2}])$ is sufficient for the proof.*

In the following we improve the result from Theorem 3.3.1 and obtain a classical solution to (3.3.1).

Theorem 3.3.3. *Let the assumptions of Theorem 3.3.1 be satisfied and suppose additionally that $\partial_t \nabla^2 \zeta$ as well as $\nabla^3 \zeta$ belong to the class $C^\alpha(\overline{I} \times \omega)$. Furthermore we assume that $J_\zeta := \det \nabla \boldsymbol{\Psi}_\zeta$ is strictly positive, $\varrho_0 \in C^{2,\alpha}(\overline{\Omega}_{\zeta(0)})$ and $\mathbf{w} \in C^{1,\alpha}(\overline{I} \times \overline{\Omega}_\zeta)$ such that $\partial_t \nabla \mathbf{w}$ and $\nabla^2 \mathbf{w}$ belong to the class $C^\alpha(\overline{I} \times \overline{\Omega}_\zeta)$.*

1. The solution ϱ from Theorem 3.3.1 satisfies (3.3.1) *in the classical sense and belongs to the regularity class*

$$\mathcal{Z}_\zeta^I := \big\{ z \in C^1(\overline{I} \times \overline{\Omega}_\zeta) : \; \nabla^2 z \in C(\overline{I} \times \overline{\Omega}_\zeta) \big\}.$$

In particular, we have

$$\|\varrho\|_{C^1_{t,x}} + \|\nabla^2\varrho\|_{C_{t,x}} \leq c\Big(\varrho_0, \zeta, \sup J_\zeta^{-1}, \mathbf{w}\Big),$$

with dependence via the (semi-)norms in the affirmative function spaces.

2. *Suppose that* $\varrho_0 \geq 0$*. Then we have the estimate*

$$C^{-1} \min_{\Omega_{\zeta(0)}} \varrho_0 \leq \max_{I\times\Omega_\zeta} \varrho \leq C \max_{\Omega_{\zeta(0)}} \varrho_0,$$

where $C = C(\zeta, \sup J_\zeta^{-1}, \mathbf{w})$ *with dependence via the (semi-)norms in the affirmative function spaces.*

Proof. We start by transforming (3.3.2) to the reference domain. For $\overline{\psi} \in C^\infty(\overline{I} \times \mathbb{R}^3)$ we set $\psi = \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1}$. Defining similarly $\overline{\varrho} = \varrho \circ \mathbf{\Psi}_\zeta$ and $\overline{\mathbf{w}} = \mathbf{w} \circ \mathbf{\Psi}_\zeta$ we obtain from (3.3.2)

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{dt} \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}\, \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\, dt &= \int_I \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} \big(\partial_t \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} + \nabla\overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1}\big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_I \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} \overline{\mathbf{w}} \circ \mathbf{\Psi}_\zeta^{-1} \cdot (\nabla \mathbf{\Psi}_\zeta)^{-1} \nabla\overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_I \int_{\Omega_\zeta} \varepsilon \big(\nabla \mathbf{\Psi}_\zeta^{-1}\big)^T \nabla \mathbf{\Psi}_\zeta^{-1} \nabla\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} \cdot \nabla\overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}$$

such that

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{dt} \int_\Omega J_\zeta \overline{\varrho}\,\overline{\psi} \,\mathrm{d}x\, dt &= \int_I \int_\Omega J_\zeta \overline{\varrho} \big(\partial_t \overline{\psi} + \nabla\overline{\psi} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta\big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_I \int_\Omega J_\zeta \overline{\varrho}\overline{\mathbf{w}} \cdot \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_I \int_\Omega \varepsilon J_\zeta \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla\overline{\varrho} \cdot \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}$$

where $J_\zeta = \det \nabla \mathbf{\Psi}_\zeta$. Finally, we replace $\overline{\psi}$ by $\overline{\psi}/J_\zeta$ to obtain

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{dt} \int_\Omega \overline{\varrho}\,\overline{\psi} \,\mathrm{d}x\, dt &= \int_I \int_\Omega \big(\overline{\varrho}\partial_t \overline{\psi} + \overline{\varrho} J_\zeta \partial_t J_\zeta^{-1} \overline{\psi}\big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_I \int_\Omega \big(\overline{\varrho}\partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta \cdot \nabla\overline{\psi} + \overline{\varrho} J_\zeta \nabla J_\zeta^{-1} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta \overline{\psi}\big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_I \int_\Omega \overline{\varrho}\overline{\mathbf{w}} \cdot \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_\Omega \overline{\varrho}\overline{\mathbf{w}} \cdot J_\zeta \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla J_\zeta^{-1} \overline{\psi} \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_I \int_\Omega \varepsilon \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla\overline{\varrho} \cdot \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_I \int_\Omega \varepsilon J_\zeta \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla\overline{\varrho} \cdot \nabla J_\zeta^{-1} \overline{\psi} \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

Now we set

$$\begin{aligned}
g_\zeta &= J_\zeta \partial_t J_\zeta^{-1} + J_\zeta \nabla J_\zeta^{-1} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta + \overline{\mathbf{w}} \cdot J_\zeta \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla J_\zeta^{-1} \\
\mathbf{g}_\zeta &= -\varepsilon J_\zeta \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla J_\zeta^{-1}, \\
\mathbf{f}_\zeta &= \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta + \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \overline{\mathbf{w}}, \quad \mathbf{A}_\zeta = \varepsilon \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1},
\end{aligned}$$

such that the equation reads as

$$\begin{aligned}
-\int_I \int_\Omega \overline{\varrho}\, \partial_t \overline{\psi} \,\mathrm{d}x\, dt &= \int_I \int_\Omega \overline{\varrho}\, g_\zeta \overline{\psi} \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_\Omega \nabla\overline{\varrho} \cdot \mathbf{g}_\zeta \overline{\psi} \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_\Omega \overline{\varrho}\mathbf{f}_\zeta \cdot \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_I \int_\Omega \mathbf{A}_\zeta \nabla\overline{\varrho} \cdot \nabla\overline{\psi} \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}$$

for any $\overline{\psi}$ with $\overline{\psi}(0) = \overline{\psi}(T) = 0$. Choosing $\overline{\psi} \in C_c^\infty(I \times \Omega)$ arbitrarily we obtain

$$\begin{aligned}\partial_t\overline{\varrho} &= \overline{\varrho} g_\zeta + \nabla\overline{\varrho}\cdot\mathbf{g}_\zeta - \operatorname{div}(\overline{\varrho}\mathbf{f}_\zeta) + \operatorname{div}\big(\mathbf{A}_\zeta\nabla\overline{\varrho}\big)\\ &= \overline{\varrho}\big(g_\zeta - \operatorname{div}\mathbf{f}_\zeta\big) + \nabla\overline{\varrho}\cdot\big(\mathbf{g}_\zeta - \mathbf{f}_\zeta\big) + \operatorname{div}\big(\mathbf{A}_\zeta\nabla\overline{\varrho}\big)\end{aligned}$$

and we have the boundary condition

$$\nu\cdot\mathbf{A}_\zeta\nabla\overline{\varrho} = \overline{\varrho}\mathbf{f}_\zeta\cdot\nu = 0.$$

Here we use that $\partial_t\mathbf{\Psi}_\zeta^{-1}\circ\mathbf{\Psi}_\zeta = -\nabla\mathbf{\Psi}_\zeta^{-T}\overline{\mathbf{w}}$ on $\partial\Omega$ due to the assumption $tr_\zeta\mathbf{w} = \partial_t\zeta\nu$. In fact, we have

$$0 = \partial_t\big(\mathbf{\Psi}_\zeta\circ\mathbf{\Psi}_\zeta^{-1}\big) = \partial_t\mathbf{\Psi}_\zeta\circ\mathbf{\Psi}_\zeta^{-1} + \nabla\mathbf{\Psi}_\zeta^T\circ\mathbf{\Psi}_\zeta^{-1}\partial_t\mathbf{\Psi}_\zeta^{-1}$$

such that

$$\nabla\mathbf{\Psi}_\zeta^T\partial_t\mathbf{\Psi}_\zeta^{-1}\circ\mathbf{\Psi}_\zeta = -\partial_t\mathbf{\Psi}_\zeta = -(\partial_t\zeta\nu)\circ\boldsymbol{\varphi} = -\mathbf{w}\circ\mathbf{\Psi}_\zeta = -\overline{\mathbf{w}}$$

on $I\times\partial\Omega$ due to the definition of $\mathbf{\Psi}_\zeta$ from (3.2.10).
We can rewrite the equation further as

$$\partial_t\overline{\varrho} = \overline{\varrho}\big(g_\zeta - \operatorname{div}\mathbf{f}_\zeta\big) + \nabla\overline{\varrho}\cdot\big(\mathbf{g}_\zeta - \mathbf{f}_\zeta\big) + \operatorname{div}\mathbf{A}_\zeta\cdot\nabla\overline{\varrho} + \mathbf{A}_\zeta : \nabla^2\overline{\varrho}$$

such that we finally obtain

$$\begin{aligned}\partial_t\overline{\varrho} + b_\zeta(t,x,\overline{\varrho},\nabla\overline{\varrho}) &= \mathbf{A}_\zeta : \nabla^2\overline{\varrho} \quad \text{in} \quad I\times\Omega,\\ \nu\mathbf{A}_\zeta\cdot\nabla\overline{\varrho} &= 0 \quad \text{on} \quad I\times\partial\Omega,\end{aligned} \tag{3.3.4}$$

where

$$b_\zeta(t,x,u,\mathbf{U}) = -u\big(g_\zeta - \operatorname{div}\mathbf{f}_\zeta\big) + \mathbf{U}\cdot\big(\mathbf{f}_\zeta - \mathbf{g}_\zeta - \operatorname{div}\mathbf{A}_\zeta\big).$$

By the classical theory from [122, Thm.'s 7.2, 7.3 & 7.4, Chapter V] the claim of part (a) follows if we can control the following quantities:[3]

- The $C^{2,\alpha}$-norm of ϱ_0;
- The α-Hölder-semi-norms of $\nabla_x b_\zeta, \partial_u b_\zeta$ and $\partial_{\mathbf{U}} b_\zeta$ with respect to x; the constants in

$$\begin{aligned}-ub_\zeta(t,x,u,\mathbf{U}) &\le c_0u^2 + c_1|\mathbf{U}|^2 + c_2 \quad \forall(t,x,u,\mathbf{U})\in I\times\Omega\times\mathbb{R}\times\mathbb{R}^3;\\ |b_\zeta(t,x,u,\mathbf{U})| + |\nabla_{(t,u)}b_\zeta(t,x,u,\mathbf{U})| + (1+|\mathbf{U}|)|\nabla_{\mathbf{U}}b_\zeta(t,x,u,\mathbf{U})| &\le c_3(1+u^2+|\mathbf{U}|^2);\end{aligned}$$

- The coercivity constant of $\mathbf{A}_\zeta$ and its upper bound; the α-Hölder-semi-norm of $\nabla_x\mathbf{A}_\zeta$ and $\partial_u\mathbf{A}_\zeta$ with respect to x.

One readily checks that all these quantities can be controlled in terms of $\|\varrho_0\|_{C_x^{2,\alpha}}$, $\|\zeta\|_{C_{t,x}^{2,\alpha}}$, $\|\partial_t\nabla^2\zeta\|_{C_{t,x}^\alpha}$, $\|\nabla^3\zeta\|_{C_{t,x}^\alpha}$, $\sup J_\zeta^{-1}$, $\|\mathbf{w}\|_{C_{t,x}^{1,\alpha}}$ and $\|\nabla^2\mathbf{w}\|_{C_{t,x}^\alpha}$.
In order to prove (b) we argue similarly to the classical arguments from [122, Thm. 7.3 Chapter V] and set

$$v(t,x) := \varphi(x)e^{-\lambda_1 t}\overline{\varrho}$$

[3]Note that this gives the required regularity for $\overline{\varrho}$; however, transforming back by means of $\mathbf{\Psi}_\zeta^{-1}$ does not alter it due to the regularity of ζ.

to verify the estimate from above. Here $\varphi \in C^\infty(\overline{\Omega})$ is constructed such that it satisfies

$$\varphi(x) \geq 1 \quad \text{in} \quad \overline{\Omega}, \tag{3.3.5}$$
$$\frac{\nabla\varphi \cdot \mathbf{A}_\zeta \nu}{\varphi} < 0 \quad \text{on} \quad \overline{I} \times \partial\Omega. \tag{3.3.6}$$

Such a function φ can be defined using the distance function to the boundary with respect to the direction $\mathbf{A}_\zeta \nu$. By the assumption that $\|\zeta\|_{L^\infty_{t,x}} \leq \frac{L}{2}$ and the C^2 regularity of ζ this is a well defined function. Note that φ is chosen independently of λ_1 (which we will fix below). We have by (3.3.4) using the linearity of b

$$\begin{aligned}
\partial_t v &= \varphi e^{-\lambda_1 t}\partial_t \overline{\varrho} - \lambda_1 v = -\varphi e^{-\lambda_1 t} b_\zeta(t, x, \overline{\varrho}, \nabla \overline{\varrho}) + \varphi e^{-\lambda_1 t}\mathbf{A}_\zeta : \nabla^2 \overline{\varrho} - \lambda_1 v \\
&= -b_\zeta(t, x, \varphi e^{-\lambda_1 t}\overline{\varrho}, \varphi e^{-\lambda_1 t}\nabla \overline{\varrho}) + \varphi e^{-\lambda_1 t}\mathbf{A}_\zeta : \nabla^2 \overline{\varrho} - \lambda_1 v \\
&= -b_\zeta(t, x, v, \nabla v - \overline{\varrho} e^{-\lambda_1 t}\nabla\varphi) + \mathbf{A}_\zeta : \nabla^2 v - \mathbf{A}_\zeta : \big(2e^{-\lambda_1 t}\nabla\overline{\varrho} \otimes^{sym} \nabla\varphi + \overline{\varrho} e^{-\lambda_1 t}\nabla^2\varphi\big) - \lambda_1 v.
\end{aligned}$$

Let us assume that there is a point $(t_0, x_0) \in I \times \Omega$ with $v(t_0, x_0) = \max_{t,x} v(t, x)$. We obtain in this point

$$\begin{aligned}
0 &= -b_\zeta(t, x, v, -\overline{\varrho} e^{-\lambda_1 t_0}\nabla\varphi) + \mathbf{A}_\zeta : \nabla^2 v + \mathbf{A}_\zeta : 2e^{-\lambda_1 t_0}\overline{\varrho}\frac{\nabla\varphi}{\varphi} \otimes \nabla\varphi - \mathbf{A}_\zeta : \overline{\varrho} e^{-\lambda_1 t_0}\nabla^2\varphi - \lambda_1 v \\
&\leq -b_\zeta(t, x, v, \overline{\varrho} e^{-\lambda_1 t_0}\nabla\varphi) + \overline{\varrho} e^{-\lambda_1 t_0}\mathbf{A}_\zeta : \Big(2\frac{\nabla\varphi}{\varphi} \otimes \nabla\varphi - \nabla^2\varphi\Big) - \lambda_1 v \\
&= \overline{\varrho} e^{-\lambda_1 t_0}\Big(\varphi\big(g_\zeta - \operatorname{div}\mathbf{f}_\zeta\big) - \lambda_1\varphi + \nabla\varphi \cdot \big(\mathbf{g}_\zeta - \mathbf{f}_\zeta + \operatorname{div}\mathbf{A}_\zeta\big) + \mathbf{A}_\zeta : \Big(2\frac{\nabla\varphi}{\varphi} \otimes \nabla\varphi - \nabla^2\varphi\Big)\Big).
\end{aligned}$$

If we choose λ_1 large (depending on $\|g_\zeta\|_{L^\infty_{t,x}}, \|\mathbf{g}_\zeta\|_{L^\infty_{t,x}}, \|\nabla\mathbf{f}_\zeta\|_{L^\infty_{t,x}}, \|\nabla\mathbf{A}_\zeta\|_{L^\infty_{t,x}}$ and φ) this leads to a contradiction (note that $\overline{\varrho}$ is non-negative by Theorem 3.3.1 (b)).
Let us now assume that $x_0 \in \partial\Omega$ and $t > 0$. Then since $\mathbf{A}_\zeta(t_0, x_0)\nu(x_0)$ points outside Ω we have

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}s} v\big(t_0, x_0 + s\mathbf{A}_\zeta(t_0, x_0)\nu(x_0)\big)\Big|_{s=0} = \nabla v(t_0, x_0) \cdot \mathbf{A}_\zeta(t_0, x_0)\nu(x_0).$$

By (3.3.4) this implies

$$\begin{aligned}
0 &\leq e^{-\lambda t_0}\big(\overline{\varrho}(t_0, x_0)\nabla\varphi(x_0) \cdot \mathbf{A}_\zeta(t_0, x_0)\nu(x_0) + \varphi\nabla\overline{\varrho}(x_0) \cdot \mathbf{A}_\zeta(t_0, x_0)\nu(x_0)\big) \\
&= \varphi(x_0)\overline{\varrho}(t_0, x_0)e^{-\lambda t_0}\frac{\nabla\varphi(x_0) \cdot \mathbf{A}_\zeta(t_0, x_0)\nu(x_0)}{\varphi(x_0)},
\end{aligned} \tag{3.3.7}$$

which yields a contradiction by (3.3.6). We conclude that the maximum of v is attained at $(0, x_0)$ for some $x_0 \in \overline{\Omega}$. By (3.3.5) the estimate for the maximum follows.
Unfortunately, the approach above used for the maximum principle does not work to achieve a minimim principle. The reason is that we do not know a priori if $\overline{\varrho}$ is strictly positive at a potential minimum of v at the boundary. We multiply (3.3.4) by $-m(\xi + \overline{\varrho})^{-m+1}$ where $0 < \xi \ll 1$ and $m \gg 1$. This yields

$$\begin{aligned}
\partial_t(\xi + \overline{\varrho})^{-m} = &-m(g_\zeta - \operatorname{div}\mathbf{f}_\zeta)\overline{\varrho}(\xi + \overline{\varrho})^{-m+1} - m\nabla\overline{\varrho} \cdot (\mathbf{g}_\zeta - \mathbf{f}_\zeta)(\xi + \overline{\varrho})^{-m+1} \\
&- m\operatorname{div}\big(\mathbf{A}_\zeta\nabla\overline{\varrho}\big)(\xi + \overline{\varrho})^{-m+1}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{dt}\int_\Omega (\xi + \overline{\varrho})^{-m}\,\mathrm{d}x + m(m-1)\int_\Omega (\xi + \overline{\varrho})^{-m}\mathbf{A}_\zeta\big(\nabla\overline{\varrho}, \nabla\overline{\varrho}\big)\,\mathrm{d}x \\
&\quad = -m\int_\Omega (g_\zeta - \operatorname{div}\mathbf{f}_\zeta)\overline{\varrho}(\xi + \overline{\varrho})^{-m+1}\,\mathrm{d}x - m\int_\Omega (\mathbf{g}_\zeta - \mathbf{f}_\zeta) \cdot \nabla\overline{\varrho}(\xi + \overline{\varrho})^{-m+1}\,\mathrm{d}x = (I) + (II)
\end{aligned}$$

using $(3.3.4)_2$. Using the boundedness of $\overline{\varrho}$ from (b) we obtain

$$(I) \leq cm\int_\Omega \overline{\varrho}(\xi + \overline{\varrho})^{-m+1}\,\mathrm{d}x \leq cm\int_\Omega \big(\xi + \overline{\varrho}\big)^{-m}\,\mathrm{d}x.$$

The constant c depends on $\|\zeta\|_{C^2_{t,x}}, \sup J_\zeta^{-1}$, $\|\mathbf{w}\|_{L^\infty_{t,x}}$ and $\|\nabla\mathbf{w}\|_{L^\infty_{t,x}}$. Similarly, we have for any $\kappa > 0$

$$\begin{aligned}(II) &\leq cm\int_\Omega |\nabla\overline{\varrho}|(\xi+\overline{\varrho})^{-m+1}\,\mathrm{d}x\\ &\leq \kappa m(m-1)\int_\Omega |\nabla\overline{\varrho}|^2(\xi+\overline{\varrho})^{-m}\,\mathrm{d}x + c(\kappa)\int_\Omega(\xi+\overline{\varrho})^{-m}\,\mathrm{d}x\\ &\leq c\kappa m(m-1)\int_\Omega \mathbf{A}_\zeta(\nabla\overline{\varrho},\nabla\overline{\varrho})(\xi+\overline{\varrho})^{-m}\,\mathrm{d}x + c\int_\Omega(\xi+\overline{\varrho})^{-m}\,\mathrm{d}x\end{aligned}$$

with $c = c(\|\zeta\|_{C^{2,\alpha}_{t,x}}, \|\mathbf{g}_\zeta\|_{L^\infty_{t,x}}, \|\mathbf{f}_\zeta\|_{L^\infty_{t,x}})$. If we absorb now the terms containing $\mathbf{A}_\zeta$ and apply Gronwall's lemma we obtain

$$\int_\Omega(\xi+\overline{\varrho}(t)))^{-m}\,\mathrm{d}x \leq e^{Cm}\int_\Omega(\xi+\overline{\varrho}_0)^{-m}\,\mathrm{d}x.$$

The constant C depends on $\|\zeta\|_{C^2_{t,x}}, \|\partial_t\nabla^2\zeta\|_{C^\alpha_{t,x}}, \|\nabla^3\zeta\|_{C^\alpha_{t,x}}, \sup J_\zeta^{-1}$, $\|\mathbf{w}\|_{C^{1,\alpha}_{t,x}}$, $\|\partial_t\nabla\mathbf{w}\|_{C^\alpha_{t,x}}$ and $\|\nabla^2\mathbf{w}\|_{C^\alpha_{t,x}}$, but is independent of m. Taking the m-th root shows

$$\bigg(\int_\Omega\Big(\frac{1}{\xi+\overline{\varrho}(t)}\Big)^{-m}\,\mathrm{d}x\bigg)^{\frac{1}{m}} \leq e^C\bigg(\int_\Omega(\xi+\overline{\varrho}_0)^{-m}\,\mathrm{d}x\bigg)^{\frac{1}{m}}.$$

Passing with $m\to\infty$ implies

$$\sup_\Omega\frac{1}{\xi+\overline{\varrho}(t)} \leq e^C\sup_\Omega\frac{1}{\xi+\overline{\varrho}_0}$$

or, equivalently,

$$e^C\inf_\Omega(\xi+\overline{\varrho}_0) \leq \inf_\Omega\big(\xi+\overline{\varrho}(t)\big).$$

Consequently, passing with $\xi\to 0$ we have $e^C\min_\Omega\overline{\varrho}_0 \leq \overline{\varrho}(t,x)$ for all $(t,x)\in\overline{I}\times\overline{\Omega}$. Thus, transforming back to ϱ, (c) is shown and the proof is complete. □

3.3.2 The internal energy equation

The regularized internal energy equation reads as

$$\begin{aligned}&\partial_t\big(\varrho e(\varrho,\vartheta)\big)+\mathrm{div}\big(\varrho e(\varrho,\vartheta)\mathbf{w}\big)-\mathrm{div}\big(_\delta(\vartheta)\nabla\vartheta\big)\\ &\quad= \mathbf{S}^\varepsilon(\vartheta,\nabla\mathbf{w}):\nabla\mathbf{w}-p_\delta(\varrho,\vartheta)\mathrm{div}\mathbf{w}\\ &\quad+\varepsilon\varrho^{\beta-2}|\nabla\varrho|^2+\delta\frac{1}{\vartheta^2}-\varepsilon\vartheta^5+\varrho H \quad \text{in} \quad I\times\Omega_\zeta,\\ &\partial_{\nu_\zeta}\vartheta\big|_{\partial\Omega_\zeta}=0 \quad \text{on} \quad I\times\partial\Omega_\zeta\end{aligned} \tag{3.3.8}$$

and we have $\vartheta(0)=\vartheta_0$ (note that $_\delta$ and $\mathbf{S}^\varepsilon$ are defined in (3.2.8)). Similar to Theorem 3.3.3 we have the following result concerning a classical solution to (3.3.8).

Theorem 3.3.4. *Let $\zeta\in C^{2,\alpha}(\overline{I}\times\omega,[\frac{L}{2},\frac{L}{2}])$ with $\alpha\in(0,1)$ be the function describing the boundary. Suppose additionally that $\partial_t\nabla^2\zeta$ as well as $\nabla^3\zeta$ belong to the class $C^\alpha(\overline{I}\times\omega)$ and suppose that $J_\zeta := \det\nabla\mathbf{\Psi}_\zeta$ is strictly positive. Assume that $\mathbf{w}\in C^{1,\alpha}(\overline{I}\times\overline{\Omega}_\zeta)$ such that $\partial_t\nabla\mathbf{w}$ and $\nabla^2\mathbf{w}$ belong to the class $C^\alpha(\overline{I}\times\overline{\Omega}_\zeta)$ and $tr_\zeta\mathbf{w}=\partial_t\zeta\nu$. Assume further that $\vartheta_0\in C^{2,\alpha}(\overline{\Omega}_{\zeta(0)})$, ϑ_0 strictly positive, $\varrho, H\in C^{1,\alpha}(\overline{I}\times\overline{\Omega}_\zeta)$ and that $\varrho, H\geq 0$.*

1. *There is a unique classical solution* ϑ *to* (3.3.8) *which belongs to the regularity class*

$$\mathcal{Z}_\zeta^I := \big\{z \in C^1(\overline{I} \times \overline{\Omega}_\zeta) : \nabla^2 z \in C(\overline{I} \times \overline{\Omega}_\zeta)\big\}.$$

   In particular, we have

$$\|\vartheta\|_{C^1_{t,x}} + \|\nabla^2 \vartheta\|_{C_{t,x}} \leq c\Big(\vartheta_0, \zeta, \sup J_\zeta^{-1}, \mathbf{w}, \varrho, H\Big),$$

   with dependence via the (semi-)norms in the affirmative function spaces.

2. *We have the estimate*

$$\min\Big\{C^{-1} \min_{\Omega_{\zeta(0)}} \vartheta_0, 1\Big\} \leq \min_{I\times\Omega_\zeta} \vartheta \leq \max_{I\times\Omega_\zeta} \vartheta \leq \max\Big\{C \max_{\Omega_{\zeta(0)}} \vartheta_0, 1\Big\},$$

   where $C = C(\zeta, \sup J_\zeta^{-1}, \mathbf{w}, \varrho, H)$ *with dependence via the (semi-)norms in the affirmative function spaces.*

Proof. Equation (3.3.8) contains several nonlinear terms which blow up for small or large values of ϑ. Hence we replace them with regularized versions. Let $\chi_\ell \in C^\infty([0,\infty))$ with $\chi_\ell(Z) = Z$ for $Z \in [1/\ell, \ell]$ and $c\ell^{-1} \leq \chi_\ell \leq C\ell$ for some positive constants c, C and $\ell \gg 1$. We define

$$\mathbf{S}^{\varepsilon,\ell}(\vartheta, \nabla\mathbf{w}) = \mathbf{S}^{\varepsilon}(\chi_\ell(\vartheta^4)^{1/4}, \nabla\mathbf{w}),\ {}^{\ell}_{\delta}(\vartheta) = \frac{\delta(\sqrt[4]{\chi_\ell(\vartheta^4)})}{\chi_\ell(\vartheta^4)^{3/4}}\vartheta^3,$$

and consider the equation

$$\begin{aligned} &\partial_t\big(\varrho e(\varrho,\vartheta)\big) + \operatorname{div}\big(\varrho e(\varrho,\vartheta)\mathbf{w}\big) - \operatorname{div}\big({}^{\ell}_{\delta}(\vartheta)\nabla\vartheta\big) \\ &\quad = \mathbf{S}^{\varepsilon,\ell}(\vartheta,\nabla\mathbf{w}) : \nabla\mathbf{w} - p(\varrho,\vartheta)\operatorname{div}\mathbf{w} + \varepsilon\delta\beta\varrho^{\beta-2}|\nabla\varrho|^2 \\ &\quad + \delta\chi_\ell(\vartheta^4)^{-1/2} - \varepsilon\chi_\ell(\vartheta^4)^{\frac{5}{4}} + \varrho H \quad \text{in} \quad I\times\Omega_\zeta, \\ &\partial_{\nu_\zeta}\vartheta\big|_{\partial\Omega_\zeta} = 0 \quad \text{on} \quad I\times\partial\Omega_\zeta \end{aligned} \tag{3.3.9}$$

and we have $\vartheta(0) = \vartheta_0$. We will show that the solution to equation (3.3.9) satisfies

$$\max_{I\times\Omega_\zeta} \vartheta \leq \max\Big\{C \max_{\Omega_{\zeta(0)}} \vartheta_0, 1\Big\} \tag{3.3.10}$$

as well as

$$\min_{I\times\Omega_\zeta} \vartheta \geq \min\Big\{C^{-1} \min_{\Omega_{\zeta(0)}} \vartheta_0, 1\Big\} \tag{3.3.11}$$

with $C = C(\|\zeta\|_{C^{2,\alpha}_{t,x}}, \|(\partial_t\nabla^2\zeta, \nabla^3\zeta)\|_{C^\alpha_{t,x}}, \sup J_\zeta^{-1}, \|\varrho\|_{C^1_{t,x}}, \|\mathbf{w}\|_{C^{1,\alpha}_{t,x}}, \|\nabla^2\mathbf{w}\|_{C^\alpha_{t,x}}, \|H\|_{L^\infty_{t,x}})$ independent of ℓ. Consequently, the cut-offs in (3.3.9) are not seen for ℓ are enough and we obtain the result for the original problem (3.3.8). Arguing as in the proof of Theorem 3.3.3 we can transform (3.3.9) to the reference domain. For this purpose it is useful to work with the weak formulation

$$\begin{aligned} \int_I \bigg(\frac{\mathrm{d}}{dt}\int_{\Omega_\zeta} \varrho e(\varrho,\vartheta)\psi\,\mathrm{d}x - \int_{\Omega_\zeta}\Big(\varrho e(\varrho,\vartheta)\,\partial_t\psi + \varrho e(\varrho,\vartheta)\mathbf{w}\cdot\nabla\psi\Big)\,\mathrm{d}x\bigg)\,dt& \\ + \int_I\int_{\Omega_\zeta} {}^{\ell}_{\delta}(\vartheta)\nabla\vartheta\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t& \\ = \int_I\int_{\Omega_\zeta}\Big[\mathbf{S}^{\varepsilon,\ell}(\vartheta,\nabla\mathbf{w}) : \nabla\mathbf{w} - p(\varrho,\vartheta)\operatorname{div}\mathbf{w}\Big]\psi\,\mathrm{d}x\,\mathrm{d}t& \\ + \int_I\int_{\Omega_\zeta}\Big[\varepsilon\delta\beta\varrho^{\beta-2}|\nabla\varrho|^2 + \delta\chi_\ell(\vartheta^4)^{-1/2} - \varepsilon\chi^\ell(\vartheta^4)^{5/4} + \varrho H\Big]\psi\,\mathrm{d}x\,\mathrm{d}t& \end{aligned}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$. Setting $\psi = \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1}$ for some $\overline{\psi} \in C^\infty(\overline{I} \times \mathbb{R}^3)$, $\overline{\varrho} = \varrho \circ \mathbf{\Psi}_\zeta$, $\overline{\mathbf{w}} = \mathbf{w} \circ \mathbf{\Psi}_\zeta$, $\overline{H} = H \circ \mathbf{\Psi}_\zeta$ and $\overline{\vartheta} = \vartheta \circ \mathbf{\Psi}_\zeta$ this is equivalent to

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt} \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} e(\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}, \overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}) \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} e(\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}, \overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}) \Big(\partial_t \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} + \nabla \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1}\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_{\Omega_\zeta} \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1} e(\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}, \overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}) \overline{\mathbf{w}} \circ \mathbf{\Psi}_\zeta^{-1} \cdot \nabla \mathbf{\Psi}_\zeta^{-1} \nabla \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\, dt \\
&+ \int_I \int_{\Omega_\zeta} {}^{\ell}_{\delta}(\overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}) \big(\nabla \mathbf{\Psi}_\zeta^{-1}\big)^T \nabla \mathbf{\Psi}_\zeta^{-1} \nabla \overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1} \cdot \nabla \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I \int_{\Omega_\zeta} (\nabla \mathbf{\Psi}_\zeta^{-1})^T \mathbf{S}^{\varepsilon,\ell}(\overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}, \nabla \mathbf{\Psi}_\zeta^{-1} \nabla \overline{\mathbf{w}} \circ \mathbf{\Psi}_\zeta^{-1}) \nabla \overline{\mathbf{w}} \circ \mathbf{\Psi}_\zeta^{-1} \, \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_{\Omega_\zeta} p(\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}, \overline{\vartheta} \circ \mathbf{\Psi}_\zeta^{-1}) \nabla \overline{\mathbf{w}} \circ \mathbf{\Psi}_\zeta^{-1} : \big(\nabla \mathbf{\Psi}_\zeta^{-1}\big)^T \, \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_{\Omega_\zeta} \Big[\varepsilon \delta \beta \Big(\overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}\Big)^{\beta-2} |\nabla \mathbf{\Psi}_\zeta^{-1} \nabla \overline{\varrho} \circ \mathbf{\Psi}_\zeta^{-1}|^2\Big] \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_{\Omega_\zeta} \Big[\delta \chi^\ell \big(\overline{\vartheta}^4 \circ \mathbf{\Psi}_\zeta^{-1}\big)^{-1/2} - \varepsilon \chi_\ell \big(\overline{\vartheta}^4 \circ \mathbf{\Psi}_\zeta^{-1}\big)^{5/4} + \overline{\varrho} \overline{H}\Big] \overline{\psi} \circ \mathbf{\Psi}_\zeta^{-1} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

and, setting $J_\zeta = \det \nabla \mathbf{\Psi}_\zeta$,

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt} \int_\Omega J_\zeta \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega J_\zeta \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \overline{\psi} + \nabla \overline{\psi} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_\Omega J_\zeta \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \overline{\mathbf{w}} \cdot \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\psi} \,\mathrm{d}x\, dt \\
&+ \int_I \int_\Omega J_\zeta {}^{\ell}_{\delta}(\overline{\vartheta}) \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\vartheta} \cdot \nabla \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I \int_\Omega J_\zeta (\nabla \mathbf{\Psi}_\zeta)^{-T} \mathbf{S}^{\varepsilon,\ell}(\overline{\vartheta}, (\nabla \mathbf{\Psi}_\zeta)^{-1} \nabla \mathbf{w}) : \nabla \overline{\mathbf{w}} \, \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_\Omega J_\zeta p(\overline{\varrho}, \overline{\vartheta}) \nabla \overline{\mathbf{w}} : \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \, \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_\Omega J_\zeta \Big[\varepsilon \delta \beta \overline{\varrho}^{\beta-2} |\big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\varrho}|^2 + \delta \chi_\ell \big(\overline{\vartheta}^4\big)^{-1/2} - \varepsilon \chi_\ell \big(\overline{\vartheta}^4\big)^{5/4} + \overline{\varrho} \overline{H}\Big] \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for all $\overline{\psi} \in C^\infty(\overline{I} \times \overline{\Omega})$. Again we replace $\overline{\psi}$ by $\overline{\psi}/J_\zeta$ to obtain

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt} \int_\Omega \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \, \partial_t \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I \int_\Omega \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \, \nabla \overline{\psi} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1} \circ \mathbf{\Psi}_\zeta \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \overline{\mathbf{w}} \cdot \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\psi} \,\mathrm{d}x\, dt \\
&- \int_I \int_\Omega {}^{\ell}_{\delta}(\overline{\vartheta}) \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\vartheta} \cdot \nabla \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_\Omega J_\zeta {}^{\ell}_{\delta}(\overline{\vartheta}) \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\vartheta} \cdot \nabla J_\zeta^{-1} \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_\Omega (\nabla \mathbf{\Psi}_\zeta)^{-T} \mathbf{S}^{\varepsilon,\ell}(\overline{\vartheta}, (\nabla \mathbf{\Psi}_\zeta)^{-1} \nabla \overline{\mathbf{w}}) : \nabla \overline{\mathbf{w}} \, \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_\Omega p(\overline{\varrho}, \overline{\vartheta}) \nabla \overline{\mathbf{w}} : \big(\nabla \mathbf{\Psi}_\zeta\big)^{-T} \, \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_\Omega \Big[\varepsilon \delta \beta \overline{\varrho}^{\beta-2} |\big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla \overline{\varrho}|^2 + \delta \chi_\ell \big(\overline{\vartheta}^4\big)^{-1/2} - \varepsilon \chi_\ell \big(\overline{\vartheta}^4\big)^{5/4} + \overline{\varrho} \overline{H}\Big] \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_\Omega J_\zeta \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t J_\zeta^{-1} + \nabla J_\zeta^{-1} \cdot \partial_t \mathbf{\Psi}_\zeta^{-1}\Big) \overline{\psi} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_\Omega J_\zeta \overline{\varrho} e(\overline{\varrho}, \overline{\vartheta}) \overline{\mathbf{w}} \cdot \big(\nabla \mathbf{\Psi}_\zeta\big)^{-1} \nabla J_\zeta^{-1} \overline{\psi} \,\mathrm{d}x\, dt.
\end{aligned}$$

Now we set

$$\begin{aligned}
\tilde{g}^\ell_\zeta(\overline{\vartheta}) &= (\nabla\mathbf{\Psi}_\zeta^{-1})^T\mathbf{S}^{\varepsilon,\ell}(\overline{\vartheta},\nabla\mathbf{\Psi}_\zeta^{-1}\nabla\overline{\mathbf{w}}):\nabla\overline{\mathbf{w}} - p(\overline{\varrho},\overline{\vartheta})\nabla\overline{\mathbf{w}}:\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\\
&\quad + \varepsilon\delta\beta\overline{\varrho}^{\beta-2}|\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\nabla\overline{\varrho}|^2 + \delta\chi_\ell\big(\overline{\vartheta}^4\big)^{-1/2} - \varepsilon\chi_\ell\big(\overline{\vartheta}^4\big)^{5/4} + \overline{\varrho}\overline{H}\\
&\quad + J_\zeta\overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\Big(\partial_t J_\zeta^{-1} + \nabla J_\zeta^{-1}\cdot\partial_t\mathbf{\Psi}_\zeta^{-1}\Big) - J_\zeta\overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\overline{\mathbf{w}}\cdot\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\nabla J_\zeta^{-1},\\
\tilde{\mathbf{g}}^\ell_\zeta(\overline{\vartheta}) &= -J_\zeta{}^\ell_\delta(\overline{\vartheta})\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\nabla J_\zeta^{-1},\\
\tilde{\mathbf{f}}^\ell_\zeta(\vartheta) &= \overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\cdot\partial_t\mathbf{\Psi}_\zeta^{-1}\circ\mathbf{\Psi}_\zeta - \overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\overline{\mathbf{w}}\\
\tilde{\mathbf{A}}^\ell_\zeta(\vartheta) &= {}^\ell(\overline{\vartheta})\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}
\end{aligned}$$

such that the equation reads as

$$\begin{aligned}
\int_I\frac{\mathrm{d}}{dt}\int_\Omega\overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\,\overline{\psi}\,\mathrm{d}x\,dt &= \int_I\int_\Omega\tilde{g}^\ell_\zeta(\overline{\vartheta})\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega\nabla\overline{\vartheta}\cdot\tilde{\mathbf{g}}^\ell_\zeta(\overline{\vartheta})\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I\int_\Omega\tilde{\mathbf{f}}^\ell_\zeta(\overline{\vartheta})\cdot\nabla\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_\Omega\tilde{\mathbf{A}}^\ell_\zeta(\overline{\vartheta})\nabla\overline{\vartheta}\cdot\nabla\overline{\psi}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{3.3.12}$$

for any $\overline{\psi}$ with $\overline{\psi}(0) = \overline{\psi}(T) = 0$. Choosing $\overline{\psi}$ from $C_c^\infty(I\times\Omega)$ shows

$$\partial_t\big(\overline{\varrho}e(\overline{\varrho},\overline{\vartheta})\big) = \tilde{g}^\ell_\zeta(\overline{\vartheta}) + \nabla\overline{\vartheta}\cdot\tilde{\mathbf{g}}^\ell_\zeta(\overline{\vartheta}) - \operatorname{div}(\tilde{\mathbf{f}}^\ell_\zeta(\overline{\vartheta})) + \operatorname{div}\big(\tilde{\mathbf{A}}^\ell_\zeta(\overline{\vartheta})\nabla\overline{\vartheta}\big)$$

and we have the boundary conditions

$$\nu\tilde{\mathbf{A}}^\ell_\zeta(\overline{\vartheta})\cdot\nabla\overline{\vartheta} = 0$$

as in (3.3.4). Recalling (3.2.1) and (3.2.2) we define $Z = \overline{\vartheta}^4$ and set

$$\begin{aligned}
g^\ell_\zeta(Z) &= (\nabla\mathbf{\Psi}_\zeta)^{-T}\mathbf{S}^{\varepsilon,\ell}(\sqrt[4]{\chi_\ell(Z)},(\nabla\mathbf{\Psi}_\zeta)^{-1}\nabla\overline{\mathbf{w}}):\nabla\overline{\mathbf{w}} - \Big(p_M(\overline{\varrho}) + \frac{a}{3}Z\Big)\nabla\overline{\mathbf{w}}:\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\\
&\quad + \varepsilon\delta\beta\overline{\varrho}^{\beta-2}|\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\nabla\overline{\varrho}|^2 - \partial_t(\overline{\varrho}e_M(\overline{\varrho})) + \delta\chi_\ell(Z)^{-1/2} - \varepsilon\chi_\ell(Z)^{5/4} + \overline{\varrho}\overline{H}\\
&\quad + \Big(\overline{\varrho}e_M(\overline{\varrho}) + aZ\Big)\Big(\partial_t J_\zeta^{-1} + \nabla J_\zeta^{-1}\cdot\partial_t\mathbf{\Psi}_\zeta^{-1} - \overline{\mathbf{w}}\cdot\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\nabla J_\zeta^{-1}\Big)J_\zeta,\\
\mathbf{g}^\ell_\zeta(Z) &= -J_\zeta\frac{\delta(\chi_\ell(Z)^{1/4})}{4\chi_\ell(Z)^{3/4}}\nabla J_\zeta^{-1}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1},\\
\mathbf{f}^\ell_\zeta(Z) &= aZ\Big(\partial_t\mathbf{\Psi}_\zeta^{-1} - \big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\overline{\mathbf{w}}\Big) + \overline{\varrho}e_M(\overline{\varrho})\Big(\partial_t\mathbf{\Psi}_\zeta^{-1} - \big(\nabla\mathbf{\Psi}_\zeta\big)^{-1}\overline{\mathbf{w}}\Big),\\
\mathbf{A}^\ell_\zeta(Z) &= \frac{\delta(\chi_\ell(Z)^{1/4})}{4\chi_\ell(Z)^{3/4}}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-T}\big(\nabla\mathbf{\Psi}_\zeta\big)^{-1},
\end{aligned}$$

such that the equation becomes

$$\partial_t aZ = g^\ell_\zeta(Z) + \nabla Z\cdot\mathbf{g}^\ell_\zeta(Z) - \operatorname{div}(\mathbf{f}^\ell_\zeta(Z)) + \operatorname{div}\big(\mathbf{A}^\ell_\zeta(Z)\nabla Z\big)\quad\text{in}\quad I\times\Omega,\tag{3.3.13}$$

$$\nu\cdot\mathbf{A}^\ell_\zeta(Z)\nabla Z = 0\quad\text{on}\quad I\times\partial\Omega.\tag{3.3.14}$$

It can be written as

$$\begin{aligned}
\partial_t Z + b^\ell_\zeta(t,x,Z,\nabla Z) &= \mathbf{A}^\ell_\zeta(Z):\nabla^2 Z\quad\text{in}\quad I\times\Omega,\\
\nu\mathbf{A}^\ell_\zeta(Z)\cdot\nabla Z &= 0\quad\text{on}\quad I\times\partial\Omega,
\end{aligned}\tag{3.3.15}$$

where

$$\begin{aligned}
ab^\ell_\zeta(t,x,u,\mathbf{U}) &= -g^\ell_\zeta(u) + \operatorname{div}_x\mathbf{f}^\ell_\zeta(u) + \big(-\mathbf{g}^\ell_\zeta(u) + \partial_u\mathbf{f}^\ell_\zeta(u) - \operatorname{div}_x\mathbf{A}^\ell_\zeta(u)\big)\cdot\mathbf{U}\\
&\quad - \partial_u\mathbf{A}^\ell_\zeta(u)(\mathbf{U},\mathbf{U}).
\end{aligned}$$

As in proof of Theorem 3.3.3 (a) we can use the theory from [122, Thm. 7.2, 7.3 & 7.4 Chapter V] to infer the existence of a unique classical solution Z_ℓ to (3.3.15) with the required regularity. Transforming back yields the regularity of ϑ claimed in (a), provided we can show that the cut-offs in (3.3.9) are not seen. As far as (3.3.10) is concerned we argue as in the proof of Theorem 3.3.3 and set

$$v(t,x) := \varphi(x) e^{-\lambda_1 t} Z$$

where $\varphi \in C^\infty(\overline{I} \times \overline{\Omega})$ is such that

$$\varphi(x) \geq 1 \quad \text{in} \quad \overline{I} \times \overline{\Omega}, \tag{3.3.16}$$

$$\frac{\nabla\varphi \cdot \mathbf{A}^\ell_\zeta(Z)\nu}{\varphi} < 0 \quad \text{on} \quad \overline{I} \times \partial\Omega. \tag{3.3.17}$$

Note that we have $\frac{\delta(\sqrt[4]{|Z|})}{4\sqrt[4]{|Z|}^3} \geq \frac{}{4}$ by (3.2.7) such that the coercivity constant of $\mathbf{A}^\ell_\zeta(Z)$ can be bounded from below independently of ℓ. Consequently, the function φ can also be chosen independently of ℓ. In an interior maximum point $(t_0, x_0) \in I \times \Omega$ of Z we have again

$$\begin{aligned}
0 &= -e^{-\lambda_1 t}\varphi b^\ell_\zeta\big(t, x, Z, \nabla Z\big) + \mathbf{A}^\ell_\zeta(Z_L) : \nabla^2 v - \lambda_1 v \\
&\quad - \mathbf{A}^\ell_\zeta(Z) : \big(2e^{-\lambda_1 t_0}\frac{Z}{\varphi}\nabla\varphi \otimes \nabla\varphi + Ze^{-\lambda_1 t_0}\nabla^2\varphi\big) \\
&\leq -e^{-\lambda_1 t}\varphi b^\ell_\zeta\Big(t_0, x, Z, -\frac{Z}{\varphi}\nabla\varphi\Big) - \lambda_1 v + c(\varphi)e^{-\lambda_1 t_0}(1+Z) \\
&\leq ce^{-\lambda_1 t_0}(1+Z) + \frac{\delta\varphi e^{-\lambda_1 t_0}}{\chi_\ell(Z)^{1/2}} - \lambda_1 e^{-\lambda_1 t_0}\varphi Z,
\end{aligned} \tag{3.3.18}$$

where

$$c = c(\varphi, \|\zeta\|_{C^{2,\alpha}_{t,x}}, \|(\partial_t\nabla^2\zeta, \nabla^3\zeta)\|_{C^\alpha_{t,x}}, \|\overline{\varrho}\|_{C^1_{t,x}}, J_\zeta^{-1}, \|\mathbf{w}\|_{C^{1,\alpha}_{t,x}}, \|\nabla^2\mathbf{w}\|_{C^\alpha_{t,x}}, \|H\|_{L^\infty_{t,x}}) \tag{3.3.19}$$

is independent of ℓ. Note that we used that the coefficients in the definition of b^ℓ_ζ have linear growth uniformly in ℓ except for $\delta\chi_\ell(u)^{-1/2}$, $-\varepsilon\chi_\ell(u)^{5/4}$ and $\partial_u\mathbf{A}^\ell_\zeta(u)(\mathbf{U},\mathbf{U})$. Fortunately, the first two terms have the correct sign, whereas the second one is evaluated at $\mathbf{U} = -\frac{Z}{\varphi}\nabla\varphi$. Now we distinguish two cases. If $Z(t_0,x_0) \leq 1$ there is nothing to show. Otherwise, $\frac{\delta}{\chi_\ell(Z(t_0,x_0))^{1/2}}$ is bounded (independent of ℓ) such that we obtain a contradiction in (3.3.18) by choosing λ_1 large (depending on the quantities in (3.3.19)). The case $x_0 \in \partial\Omega$ and $t_0 > 0$ can be ruled out again as in (3.3.7). Hence (3.3.10) follows with a constant independent of ℓ.

In order to prove (3.3.11) we first establish a lower bound which depends on ℓ. Choosing first ℓ large enough and than $\underline{Z} \in (0, \inf Z_0)$ small enough (depending on ℓ) we have $g^\ell_\zeta(\underline{Z}) - \operatorname{div}\big(\mathbf{f}^\ell_\zeta(\underline{Z})\big) \geq 0$. This is thanks to the term $\delta\chi_\ell(Z)^{-1/2}$ in the definition of g^ℓ_ζ. Consequently, we obtain from (3.3.13)

$$\begin{aligned}
\partial_t a(Z - \underline{Z}) &\geq g^\ell_\zeta(Z) - g^\ell_\zeta(\underline{Z}) + \nabla Z \cdot \mathbf{g}^\ell_\zeta(Z) - \nabla\underline{Z} \cdot \mathbf{g}^\ell_\zeta(\underline{Z}) - \operatorname{div}\big(\mathbf{f}^\ell_\zeta(Z) - \mathbf{f}^\ell_\zeta(\underline{Z})\big) \\
&\quad + \operatorname{div}\big(\mathbf{A}^\ell_\zeta(Z)\nabla(Z - \underline{Z})\big).
\end{aligned}$$

Multiplying by $(Z - \underline{Z})^-$ and integrating over Ω implies

$$\begin{aligned}
&\frac{a}{2}\frac{\mathrm{d}}{dt}\int_\Omega ((Z-\underline{Z})^-)^2\,\mathrm{d}x + \int_\Omega \mathbf{A}^\ell_\zeta(Z)\big(\nabla(Z-\underline{Z})^-, \nabla(Z-\underline{Z})^-\big)\,\mathrm{d}x \\
&\quad \leq \int_\Omega \big(g^\ell_\zeta(\underline{Z}) - g^\ell_\zeta(Z)\big)(Z-\underline{Z})^-\,\mathrm{d}x + \int_\Omega \big(\nabla\underline{Z}\cdot\mathbf{g}^\ell_\zeta(\underline{Z}) - \nabla Z\cdot\mathbf{g}^\ell_\zeta(Z)\big)(Z-\underline{Z})^-\,\mathrm{d}x \\
&\quad + \int_\Omega \big(\mathbf{f}^\ell_\zeta(Z) - \mathbf{f}^\ell_\zeta(\underline{Z})\big)\nabla(Z-\underline{Z})^-\,\mathrm{d}x
\end{aligned}$$

using also (3.3.14). By the Lipschitz continuity of g_ζ^ℓ, $\mathbf{g}_\zeta^\ell$ and $\mathbf{f}_\zeta^\ell$ in Z and (3.3.10) we obtain

$$\frac{\mathrm{d}}{dt}\int_\Omega \frac{a}{2}|(Z-\underline{Z})^-|^2\,\mathrm{d}x + \int_\Omega \mathbf{A}_\zeta^\ell(Z_\ell)\big(\nabla Z_\ell^-, \nabla Z_\ell^-\big)\,\mathrm{d}x \\ \leq \xi\int_\Omega |\nabla(Z-\underline{Z})^-|^2\,\mathrm{d}x + c(\xi,\ell)\int_\Omega |(Z-\underline{Z})^-|^2\,\mathrm{d}x$$

for all $\xi > 0$. Due to (3.2.7) the first term can be absorbed for ξ small enough, whereas the second one can be handled by Gronwall's lemma and $\vartheta_0 > 0$. We conclude that

$$Z \geq \underline{Z} > 0 \quad \text{in} \quad \overline{I}\times\overline{\Omega}. \tag{3.3.20}$$

Recall that $\underline{Z}$ depends on ℓ. We are now going to prove a uniform lower bound. Similarly to (3.3.16) and (3.3.17) we consider a function $\varphi \in C^\infty(\overline{I}\times\overline{\Omega})$ satisfying

$$\varphi(x) \geq 1 \quad \text{in} \quad \overline{I}\times\overline{\Omega}, \tag{3.3.21}$$

$$\frac{\nabla\varphi\cdot\mathbf{A}_\zeta^\ell(Z)\nu}{\varphi} \geq 1 \quad \text{on} \quad \overline{I}\times\partial\Omega. \tag{3.3.22}$$

Let us first assume that the minimum of $v = \varphi e^{\lambda_1 t}Z$ is attained in an interior point $(t_0, x_0) \in I\times\Omega$. We obtain similarly to (3.3.18)

$$0 \geq -ce^{\lambda_1 t_0}(1+Z) + \frac{\delta\varphi e^{\lambda_1 t_0}}{\chi_\ell(Z)^{1/2}} - \varepsilon\chi_\ell(Z)^{5/2} + \lambda_1 e^{\lambda_1 t_0}\varphi Z. \tag{3.3.23}$$

An appropriate choice of λ_1 contradicts (3.3.23). In the case of $x_0 \in \partial\Omega$ and $t_0 > 0$ we have similarly to the proof of (b)

$$0 \geq Ze^{-\lambda_1 t_0}\nabla\varphi\cdot\mathbf{A}_\zeta^\ell(t_0,x_0)\nu(x_0).$$

This gives a contradiction by (3.3.20), (3.3.21) and (3.3.22). Consequently, the minimum of Z is attained in a point $(0, x_0)$ for some $x_0 \in \overline{\Omega}$. This gives the claim of (b) since λ_1 is independent of ℓ. □

3.4 Construction of an approximate solution

In this section we construct an approximation of the system, where the continuity equation contains an artificial diffusion (ε-layer) and the pressure is stabilised by a high power of the density (δ-layer). Following [62] we add various regularizing terms depending on ε and δ to the equations to preserve the energy balance. One of the regularizing terms can only be shown to belong to L^1, which is not enough to conclude uniform continuity in time needed for the application of Theorem 2.5.1. To overcome this peculiarity we include a further diffusion term of the fluid velocity which is non-linear and of p-growth with $p > 2$. It vanishes in the limit but improves the time integrability mentioned before. Additionally, we regularize the shell equation by replacing the operator K with

$$K_\varepsilon(\eta) = K(\eta) + \varepsilon\mathcal{L}(\eta), \quad \mathcal{L}(\eta) = \frac{1}{2}\int_\omega |\nabla^3\eta|^2\,\mathrm{d}y,$$

defined for $\eta \in W^{3,2}(\omega)$. Thanks to this we can prove compactness of the shell energy in the Galerkin limit.

Remark 3.4.1. *We observe that adding dissipative regularization terms to the shell equation is not possible. This is a special feature for energetically closed systems and in contrast to other fluid systems [88]. Indeed, a dissipation term in the solid creates heat on the surface, which consequently effects the temperature. In the case of shells this yields a non-homogeneous Neumann boundary value for the temperature variable. This non-homogeneity naturally possesses the "wrong sign" in order to attain in the limit the boundary values for the temperature that are in accordance with the concept of weak solutions. In the case of visco-elastic solids, where dissipative terms such as an additional heat source are included (they are physical and not only relaxation terms) our approximation would yield the correct non-homogeneous boundary values. However, we considered here perfectly elastic solids. Hence all energy is supposed to be stored in the elastic potential.*

In contrast to [22] and [127] we construct the fixed point on the Galerkin level. This allows to remove one regularization level for the boundary and the convective term that was needed there. The formulation of the Galerkin approximation in our case is more involved since the basis functions are defined on the a priori unknown time dependent domain. The fixed point argument (which is now applied on the Galerkin level) is, however, much easier. After constructing a solution on the basic level, we prove in Subsection 3.4.2 the energy equality and derive further estimates through the Helmholz-function. In particular, we derive the approximate system and the a-priori estimates. They are essential for the remainder of the paper and are preserved in all limit procedures.
For the original system we seek a solution of the shell in the class

$$Y^I := W^{1,\infty}(I; L^2(\omega)) \cap L^\infty(I; W^{2,2}(\omega)).$$

However, in this section we are dealing with a regularised system where instead solutions are located in

$$\tilde{Y}^I := \cap W^{1,\infty}(I; L^2(\omega)) \cap L^\infty(I; W^{3,2}(\omega)).$$

For $\zeta \in \tilde{Y}^I$ with $\|\zeta\|_{L^\infty_{t,x}} \leq \frac{L}{2}$ we consider

$$\tilde{X}^I_\zeta := L^p(I; W^{1,p}(\Omega_{\zeta(t)})), \quad \tilde{X}^I_\zeta := L^2(I; W^{1,2}(\Omega_{\zeta(t)})) \quad Z^I_\zeta \quad = L^2(I; W^{1,2}(\Omega_\zeta)) \cap L^\infty(I; L^4(\Omega_\zeta)).$$

A solution to the regularized system, in the weak formulation, is a quadruplet $(\eta, \mathbf{u}, \varrho, \vartheta) \in \tilde{Y}^I \times \tilde{X}^I_\eta \times X^I_\eta \times Z^I_\eta$ that satisfies the following.

(K1) The regularized weak momentum equation

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\partial_t\boldsymbol{\phi}\,\mathrm{d}x\,dt \\
&- \int_I\int_{\Omega_\eta} \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,dt + \int_I\int_{\Omega_\eta} \mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I\int_{\Omega_\eta} p_\delta(\varrho,\vartheta)\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_\eta} \varepsilon\nabla\varrho\nabla\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\left(\frac{\mathrm{d}}{dt}\int_\omega \partial_t\eta\, b\,\,\mathrm{d}y - \int_\omega \partial_t\eta\,\partial_t b\,\,\mathrm{d}y + \int_\omega K'_\varepsilon(\eta)\,b\,\,\mathrm{d}y\right)dt \\
&+ \int_I\int_{\Omega_\eta} \varepsilon(1+\vartheta)\overline{\mathbf{P}} : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I\int_{\Omega_\eta} \varrho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\omega g\,b\,\,\mathrm{d}x\,dt
\end{aligned} \tag{3.4.1}$$

holds for all test-functions $(b, \boldsymbol{\phi}) \in C^\infty(\omega) \times C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\phi} = b\nu$ and for some $\overline{\mathbf{P}} \in L^{p'}(I \times \Omega_\eta)$. Moreover, we have $(\varrho\mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t\eta(0) = \eta_1$. The boundary condition $\mathrm{tr}_\eta\mathbf{u} = \partial_t\eta\nu$ holds in the sense of Lemma 3.2.3.

(K2) The regularized continuity equation

$$\partial_t\varrho + \mathrm{div}\big(\varrho\mathbf{u}\big) = \varepsilon\Delta\varrho \tag{3.4.2}$$

holds in $I \times \Omega_\eta$ and we have $\partial_{\nu_\eta}\varrho|_{\partial\Omega_\eta} = 0$ as well as $\varrho(0) = \varrho_0$.

(K3) The entropy balance

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho s(\varrho,\vartheta)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\big(\varrho s(\varrho,\vartheta)\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \int_I\int_{\Omega_\eta}\frac{1}{\vartheta}\Big[\mathbf{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u} + \varepsilon(1+\vartheta)\max\big\{|\overline{\mathbf{P}}|^{p'},|\nabla\mathbf{u}|^p\big\}\Big]\psi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_I\int_{\Omega_\eta}\frac{1}{\vartheta}\Big[\frac{\delta}{2}\big(\vartheta^{\beta-1}+\frac{1}{\vartheta^2}\big)\Big)|\nabla\vartheta|^2 + \delta\frac{1}{\vartheta^2}\Big]\psi\,\mathrm{d}x\,\mathrm{d}t \qquad (3.4.3)\\
&-\int_I\int_{\Omega_\eta}\Big(\frac{(\vartheta)}{\vartheta} + \delta\big(\vartheta^{\beta-1}+\frac{1}{\vartheta^2}\big)\Big)\nabla\vartheta\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_\eta}\frac{\varrho}{\vartheta}H\psi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_I\int_{\Omega_\eta}\varepsilon\left[\frac{\delta}{2\vartheta}\beta\varrho^{\beta-2}|\nabla\varrho|^2 - \vartheta^4\right]\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

holds for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\psi \geq 0$. Moreover, we have $\lim_{r\to 0}\varrho s(\varrho,\vartheta)(t) \geq \varrho_0 s(\varrho_0,\vartheta_0)$ and $\partial_{\nu_\eta}\vartheta|_{\partial\Omega_\eta} \leq 0$.

(K4) The total energy balance

$$\begin{aligned}
-\int_I \partial_t\psi\,\mathcal{E}_{\varepsilon,\delta}\,dt &= \psi(0)\mathcal{E}_{\varepsilon,\delta}(0) + \int_I\psi\int_{\Omega_\eta}\Big(\frac{\delta}{\vartheta^2} - \varepsilon\vartheta^5\Big)\,\mathrm{d}x\,dt + \int_I\psi\int_{\Omega_\eta}\varrho H\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \int_I\psi\int_M g\,\partial_t\eta\,\,\mathrm{d}y\,dt
\end{aligned} \qquad (3.4.4)$$

holds for any $\psi \in C_c^\infty([0,T))$. Here, we abbreviated

$$\mathcal{E}_{\varepsilon,\delta}(t) = \int_{\Omega_{\eta(t)}}\Big(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)e_\delta(\varrho(t),\vartheta(t))\Big)\,\mathrm{d}x + \int_M\frac{|\partial_t\eta(t)|^2}{2}\,\,\mathrm{d}y + K_\varepsilon(\eta(t)).$$

Remark 3.4.2. *In order to deal with the term $\int_I\int_{\Omega_\eta}\varepsilon\nabla\varrho\nabla\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t$ (appearing in* (3.4.1) *to balance the artificial viscosity term in* (3.4.2)*) in the proof of* (3.4.33) *we need higher integrability of $\nabla\mathbf{u}$ in time. This is achieved by introducing an artificial p-Laplacian term $\varepsilon(1+\vartheta)(1+|\nabla\mathbf{u}|)^{p-2}\nabla\mathbf{u}$ for some $p>2$ on the Galerkin approximation in the next section. It gives the additional term $\varepsilon(1+\vartheta)\max\big\{|\overline{\mathbf{P}}|^{p'},|\nabla\mathbf{u}|^p\big\}$ in* (3.4.3)*. The term $\varepsilon(1+\vartheta)\overline{\mathbf{P}}$ in* (3.4.1) *is the weak limit of the p-Laplacian term and can be seen as the defect in the strong convergence of $\nabla\mathbf{u}$. It disappears in the limit $\varepsilon\to 0$.*

The rest of this section is dedicated to the proof of the following existence theorem.

Theorem 3.4.3. *Assume that we have for some $\alpha\in(0,1)$*

$$\begin{aligned}
&\frac{|\mathbf{q}_0|^2}{\varrho_0}\in L^1(\Omega_{\eta_0}),\ \varrho_0,\vartheta_0\in C^{2,\alpha}(\overline{\Omega}_{\eta_0}),\ \eta_0\in W^{3,2}(\omega;[-\tfrac{L}{4},\tfrac{L}{4}]),\ \eta_1\in L^2(\omega),\\
&\mathbf{f}\in L^2(I;L^\infty(\mathbb{R}^3)),\ g\in L^2(I\times\omega),\ H\in C^{1,\alpha}(\overline{I}\times\mathbb{R}^3),\ H\geq 0.
\end{aligned} \qquad (3.4.5)$$

Furthermore suppose that ϱ_0 and ϑ_0 are strictly positive and that (1.1.21) *is satisfied. Then there exists a solution $(\eta,\mathbf{u},\varrho,\vartheta)\in\tilde{Y}^I\times X^I_\eta\times Z^I_\eta\times Z^I_\eta$ to (K1)–(K4). Here, we have $I=(0,T_*)$, where $T_*<T$ only if $\lim_{t\to T^*}\|\eta(t,\cdot)\|_{L^\infty_x}=\frac{L}{2}$ or the Koiter energy degenerates (namely, if $\lim_{s\to t}\overline{\gamma}(s,y)=0$ for some point $y\in\omega$).*

We prove Theorem 3.4.3 in two steps. First we construct a finite dimensional Galerkin approximation to (K1)–(K3) in the next subsection. Then we derive the energy balance, prove uniform a priori estimates and pass to the limit.

3.4.1 Galerkin approximation

By solving respective eigenvalue problems we construct a smooth orthogonal basis $(\tilde{\mathbf{X}}_k)_{k\in\mathbb{N}}$ of $W_0^{1,2}(\Omega)$ that is orthogonal in $L^2(\Omega)$ and a smooth orthonormal basis $(\tilde{Y}_k)_{k\in\mathbb{N}}$ of $W^{3,2}(\omega)$ which is orthogonal in $L^2(\omega)$. We define vector fields $\tilde{\mathbf{Y}}_k$ by setting $\tilde{\mathbf{Y}}_k = \mathscr{F}_\Omega((\tilde{Y}_k\nu)\circ\boldsymbol{\varphi}^{-1})$, where $\mathscr{F}_\Omega$ is the extension operator used in Section 3.2.3. We recall that $\mathscr{F}_\Omega : W^{k,2}(\omega) \to W^{k,2}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ such that the $\tilde{\mathbf{Y}}_k$'s are smooth. Now we choose an enumeration $(\tilde{\boldsymbol{\omega}}_k)_{k\in\mathbb{N}}$ of $(\tilde{\mathbf{X}}_k)_{k\in\mathbb{N}} \cup (\tilde{\mathbf{Y}}_k)_{k\in\mathbb{N}}$. In return we associate $w_k := (\tilde{\boldsymbol{\omega}}_k|_{\partial\Omega}\nu)\circ\boldsymbol{\varphi}$. Obviously, we obtain a basis $(\tilde{\boldsymbol{\omega}}_k)_{k\in\mathbb{N}}$ of $W_0^{1,2}(\Omega)$ and a basis $(w_k)_{k\in\mathbb{N}}$ of $W^{3,2}(\omega)$. We define $\mathcal{P}_N$, as the orthogonal projection (in space)

$$\mathcal{P}_N(\phi) := \sum_{k=1}^N P_N^k(\phi) w_k := \sum_{k=1}^N \langle \phi, w_k \rangle_{W^{3,2}(\omega)} w_k,$$

which satisfies the expected stability and convergence properties in all spaces relevant for the analysis. Next we seek for a couple of discrete solutions $(\eta_N, \mathbf{u}_N)$ of the form

$$\eta_N = \mathcal{P}_N\eta_0 + \sum_{k=1}^N \int_0^t \alpha_{kN} w_k \,\mathrm{d}\sigma, \quad \mathbf{u}_N = \sum_{k=1}^N \alpha_{kN}\tilde{\boldsymbol{\omega}}_k \circ \boldsymbol{\Psi}_{\eta_N}^{-1},$$

with time-dependent coefficients $\boldsymbol{\alpha}_N = (\alpha_{kN})_{k=1}^N$, which solve the following discrete version of (3.4.1):

$$\begin{aligned}
&\int_{\Omega_{\eta_N}} \varrho_N(t)\mathbf{u}_N(t)\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}(t)\,\mathrm{d}x \\
&- \int_0^t \int_{\Omega_{\eta_N}} \Big(\varrho_N\mathbf{u}_N\cdot\partial_t\Big(\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}\Big) + \varrho_N\mathbf{u}_N\otimes\mathbf{u}_N : \nabla\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^t \int_{\Omega_{\eta_N}} \Big(\mathbf{S}^\varepsilon(\vartheta_N,\nabla\mathbf{u}_N) : \nabla\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&- \int_0^t \int_{\Omega_{\eta_N}} \Big(p_\delta(\varrho_N,\vartheta_N)\,\mathrm{div}\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1} + \varepsilon\nabla\varrho_N\nabla\mathbf{u}_N\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&+ \int_0^t \int_\omega \Big(K_\varepsilon'(\eta_N)w_k - \partial_t\eta_N\,\partial_t w_k\Big)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_\omega \partial_t\eta_N(t)w_k\,\,\mathrm{d}y\,\mathrm{d}\sigma \\
&= \int_0^t \int_{\Omega_{\eta_N}} \varrho_N\mathbf{f}\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}\,\mathrm{d}x\,\mathrm{d}\sigma + \int_0^t \int_\omega g\,w_k\,\,\mathrm{d}y\,\mathrm{d}\sigma \\
&+ \int_{\Omega_{\eta_N(0)}} \mathbf{q}_0\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\eta_N}^{-1}(0,\cdot)\,\mathrm{d}x + \int_\omega \eta_1\,w_k\,\,\mathrm{d}y.
\end{aligned} \tag{3.4.6}$$

Here $\varrho_N = \varrho(\eta_N, \mathbf{u}_N)$ and $\vartheta_N = \vartheta(\eta_N, \mathbf{u}_N, \varrho_N)$ are the unique solutions from Theorems 3.3.3 and 3.3.4 subject to the initial data ϱ_0 and ϑ_0, where $\zeta \equiv \eta_N$ and $\mathbf{w} \equiv \mathbf{u}_N$. Note that by construction we have $tr_{\eta_N}\mathbf{u}_N = \partial_t\eta_N\nu$ and that we can choose $\alpha_{kN}(0)$ in a way that $\mathbf{u}_N(0)$ converges to $\mathbf{q}_0/\varrho_0$. In order to solve (3.4.6) we decouple the nonlinearities. Consider a given couple of discrete functions $(\zeta_N, \mathbf{v}_N)$ of the form

$$\zeta_N = \mathcal{P}_N\eta_0 + \sum_{k=1}^N \int_0^t \beta_{kN} w_k \,\mathrm{d}\sigma, \quad \mathbf{v}_N = \sum_{k=1}^N \beta_{kN}\tilde{\boldsymbol{\omega}}_k \circ \boldsymbol{\Psi}_{\zeta_N}^{-1}, \tag{3.4.7}$$

with time-dependent coefficients $\boldsymbol{\beta}_N = (\beta_{kN})_{k=1}^N$. By construction they satisfy $tr_{\zeta_N}\mathbf{v}_N = \partial_t\zeta_N\nu$. We aim to solve

$$\begin{aligned}
&\int_{\Omega_{\zeta_N}} \varrho_N(t)\mathbf{u}_N(t)\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}(t)\,\mathrm{d}x\\
&-\int_0^t\int_{\Omega_{\zeta_N}}\Big(\varrho_N\mathbf{u}_N\cdot\partial_t\Big(\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)+\varrho_N\mathbf{v}_N\otimes\mathbf{u}_N:\nabla\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^t\int_{\Omega_{\zeta_N}}\Big(\mathbf{S}^\varepsilon(\vartheta_N,\nabla\mathbf{u}_N):\nabla\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\\
&-\int_0^t\int_{\Omega_{\zeta_N}}\Big(p_\delta(\varrho_N,\vartheta_N)\,\mathrm{div}\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}+\varepsilon\nabla\varrho_N\nabla\mathbf{u}_N\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\int_0^t\int_\omega\Big(K_\varepsilon'(\eta_N)w_k-\partial_t\eta_N\,\partial_t w_k\Big)\,\mathrm{d}y\,\mathrm{d}\sigma+\int_\omega\partial_t\eta_N(t)w_k\,\mathrm{d}y\\
&=\int_0^t\int_{\Omega_{\zeta_N}}\varrho_N\mathbf{f}\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^t\int_\omega g\,w_k\,\mathrm{d}y\,\mathrm{d}\sigma\\
&+\int_{\Omega_{\zeta_N(0)}}\mathbf{q}_0\cdot\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}(0,\cdot)\,\mathrm{d}x+\int_\omega\eta_1\,w_k\,\mathrm{d}y.
\end{aligned}\tag{3.4.8}$$

Here $\varrho_N = \varrho(\zeta_N, \mathbf{v}_N)$ and $\vartheta_N = \vartheta(\zeta_N, \mathbf{v}_N, \varrho_N)$ are the unique solutions from Theorems 3.3.3 and 3.3.4 subject to the initial data ϱ_0 and ϑ_0, where $\zeta \equiv \zeta_N$ and $\mathbf{w} \equiv \mathbf{v}_N$. Note that this is possible since $\|P_N\eta_0\|_{L^\infty_x} \leq \frac{L}{3}$ for N large enough, which implies $\|\zeta_N\|_{L^\infty_{t,x}} \leq \frac{L}{2}$ for T_* small enough. The system (3.4.8) is equivalent to a system of integro-differential equations for the vector $\boldsymbol{\alpha}_N = (\alpha_{kN})_{k=1}^N$. It reads as

$$\mathcal{A}(t)\boldsymbol{\alpha}_N(t) = \int_0^t\mathcal{B}(\sigma)\boldsymbol{\alpha}_N(\sigma)\,\mathrm{d}\sigma+\int_0^t\tilde{\mathcal{B}}\Big(\sigma,\boldsymbol{\alpha}_N(\sigma),\int_0^\sigma\boldsymbol{\alpha}_N(s)\,\mathrm{d}s\Big)\,\mathrm{d}\sigma+\int_0^t\mathbf{c}(\sigma)\,\mathrm{d}\sigma+\tilde{\mathbf{c}},\tag{3.4.9}$$

with

$$\begin{aligned}
\mathcal{A}_{ij} &= \int_{\Omega_{\zeta_N}}\varrho_N(t)\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}(t)\cdot\tilde{\boldsymbol{\omega}}_j\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}(t)\,\mathrm{d}x+\int_\omega w_iw_j\,\mathrm{d}y\\
\mathcal{B}_{ij} &= \int_{\Omega_{\zeta_N}}\Big(\varrho_N\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\cdot\partial_t\Big(\tilde{\boldsymbol{\omega}}_j\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)+\varrho_N\mathbf{v}_N\otimes\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}:\nabla\tilde{\boldsymbol{\omega}}_j\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\Big)\,\mathrm{d}x\\
&\quad-\int_{\Omega_{\zeta_N}}\varepsilon\nabla\varrho_N\nabla\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\cdot\tilde{\boldsymbol{\omega}}_j\circ\boldsymbol{\varphi}_{\zeta_N}^{-1}\,\mathrm{d}x\,\mathrm{d}\sigma-\int_\omega w_i\,\partial_t w_j\,\mathrm{d}y\\
\tilde{\mathcal{B}}_j &= \int_\omega K_\varepsilon'\Big(\mathcal{P}_N\eta_0+\textstyle\sum_{k=1}^N\displaystyle\int_0^\sigma\alpha_{kN}(s)w_k\,\mathrm{d}s\Big)w_j\,\mathrm{d}y\\
&\quad+\int_{\Omega_{\zeta_N}}\mathbf{S}^\varepsilon\Big(\vartheta_N,\sum_{k=1}^N\nabla\big(\alpha_{kN}\tilde{\boldsymbol{\omega}}_k\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\big)\Big):\nabla\tilde{\boldsymbol{\omega}}_j\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x\\
c_i &= \int_{\Omega_{\zeta_N}}p_\delta(\varrho_N,\vartheta_N)\,\mathrm{div}\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x+\int_{\Omega_{\zeta_N}}\varrho_N\mathbf{f}\cdot\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x\,\mathrm{d}t+\int_\omega g\,w_i\,\mathrm{d}y\\
\tilde{c}_i &= \int_{\Omega_{\zeta_N(0)}}\mathbf{q}_0\cdot\tilde{\boldsymbol{\omega}}_i\circ\boldsymbol{\Psi}_{\zeta_N}^{-1}(0,\cdot)\,\mathrm{d}x+\int_\omega\eta_1\,w_i\,\mathrm{d}y.
\end{aligned}$$

The matrix $\mathcal{A}_{ij}$ is invertible and all non-linear quantities are locally Lipschitz continuous in $\boldsymbol{\alpha}_N$ (compare also with [22, Thm. 4.4]). Also our analysis from Section 3.3 shows that ϑ_N and ϱ_N depend in a smooth way on $\mathbf{v}_N$ and ζ_N. By the Picard-Lindelöf theorem there is a unique solution in short time. It can be extended to a global-in-time solution by means of some a priori estimates which we derive below in (3.4.13). Consequently, we obtain a solution $(\eta_N, \mathbf{u}_N)$ to (3.4.8) which satisfies the following energy balance (testing (3.4.8) by $(\mathbf{u}_N, \partial_t \eta_N)$ and (3.3.1) by $\frac{1}{2}|\mathbf{u}_N|^2)$

$$
\begin{aligned}
&-\int_I \bigg(\int_{\Omega_{\zeta_N}} \varrho_N \frac{|\mathbf{u}_N|^2}{2}\,\mathrm{d}x + \int_\omega \frac{|\partial_t \eta_N|^2}{2}\,\mathrm{d}y + K_\varepsilon(\eta_N) \bigg) \partial_t \psi\, dt \\
&+ \int_I \psi \int_{\Omega_\zeta} \mathbf{S}^\varepsilon(\vartheta_N, \nabla \mathbf{u}_N) : \nabla \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t \\
&= \psi(0) \bigg(\int_{\Omega_{\zeta_N(0)}} \frac{|\mathbf{q}_0|^2}{2\varrho_0}\,\mathrm{d}x + \int_\omega \frac{|\eta_0|^2}{2}\,\mathrm{d}y + \int_\omega \frac{|\eta_1|^2}{2}\,\mathrm{d}y + K_\varepsilon(\eta_0) \bigg) \\
&+ \int_I \psi \int_{\Omega_{\zeta_N}} \varrho_N \mathbf{f} \cdot \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t + \int_I \psi \int_\omega g \partial_t \eta_N \,\mathrm{d}y\, dt \\
&+ \int_I \psi \int_{\Omega_{\zeta_N}} p_\delta(\varrho_N, \vartheta_N)\, \mathrm{div} \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all $\psi \in C_c^\infty([0,T))$. Testing further the continuity equation by $\delta \varrho^\beta$ yields

$$
\begin{aligned}
&-\int_I \bigg(\int_{\Omega_{\zeta_N}} \varrho_N \frac{|\mathbf{u}_N|^2}{2}\,\mathrm{d}x + \int_{\Omega_{\zeta_N}} \frac{\delta}{\beta-1} \varrho_N^\beta \,\mathrm{d}x + \int_\omega \frac{|\partial_t \eta_N|^2}{2}\,\mathrm{d}y + K_\varepsilon(\eta_N) \bigg) \partial_t \psi\, dt \\
&+ \int_I \psi \int_{\Omega_\zeta} \mathbf{S}^\varepsilon(\vartheta_N, \nabla \mathbf{u}_N) : \nabla \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t + \int_I \psi \int_{\Omega_\zeta} \varepsilon \delta \beta \varrho_N^{\beta-2} |\nabla \varrho_N|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \psi(0) \bigg(\int_{\Omega_{\zeta_N(0)}} \frac{|\mathbf{q}_0|^2}{2\varrho_0}\,\mathrm{d}x + \int_\omega \frac{|\eta_0|^2}{2}\,\mathrm{d}y + \int_\omega \frac{|\eta_1|^2}{2}\,\mathrm{d}y + K_\varepsilon(\eta_0) \bigg) \\
&+ \int_I \psi \int_{\Omega_{\zeta_N}} \varrho_N \mathbf{f} \cdot \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t + \int_I \psi \int_\omega g \partial_t \eta_N \,\mathrm{d}y\, dt \\
&+ \int_I \psi \int_{\Omega_{\zeta_N}} p(\varrho_N, \vartheta_N)\, \mathrm{div} \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{3.4.10}
$$

for all $\psi \in C_c^\infty([0,T))$ using the definition $p_\delta(\varrho, \vartheta) = p(\varrho, \vartheta) + \delta \varrho^\beta$.
We consider the mapping

$$
F : D \to F(D), \quad \boldsymbol{\beta} \mapsto \boldsymbol{\alpha}, \quad D = \Big\{ \boldsymbol{\beta} \in C^{1,\alpha}(\overline{I}_*, \mathbb{R}^N) : \sup_{I_*} \|\boldsymbol{\beta}'\|_\alpha \le K^* \Big\}
$$

where $I_* = (0, T_*)$ and $\alpha \in (0,1)$. We will choose K^* sufficiently large. In dependence of K^* we find T_* (sufficiently small) but uniform to solve the above ODE uniquely on I_*. Note that we may take T^* small enough (in dependence of K^*) such that ζ_N (defined via $\boldsymbol{\beta}$ by (3.4.7)) satisfies $\|\zeta_N\|_{L^\infty_{t,x}} \le \frac{L}{2}$ for any $\boldsymbol{\beta} \in D$. We are going to prove that F has a fixed point. Let us first note that F is upper-semicontinuous. Indeed, if we have a sequence $(\boldsymbol{\beta}^j)$ which converges in $C^{1,\alpha}(\overline{I})$ to some $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha}^j = F(\boldsymbol{\beta}^j)$ converges in $C^{1,\alpha}(\overline{I})$ to some $\boldsymbol{\alpha}$, we have $\boldsymbol{\alpha} = F(\boldsymbol{\beta})$. This is due to the unique solvability of (3.4.8) and the continuity of the coefficients $\mathcal{A}$, $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathbf{c}$. In fact, the continuity of $\mathcal{A}$, $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathbf{c}$ (with respect to $\boldsymbol{\beta}$) can be shown by transforming the integrals to the reference domain and using (3.2.10) similarly to the proofs of Theorems 3.3.3 and 3.3.4. The regularity and continuity of ϱ_N and ϑ_N then implies the continuity of the coefficients.

Next we aim to show that $F(D) \subset D$. The internal energy equation (3.3.9) for ϑ_N yields

$$
\begin{aligned}
&-\int_I \int_{\Omega_{\zeta_N}} \varrho_N e(\varrho_N, \vartheta_N)\, \partial_t \psi \,\mathrm{d}x\, dt - \psi(0) \int_{\Omega_{\zeta_N(0)}} \varrho_0 e(\varrho_0, \vartheta_0)\,\mathrm{d}x \\
&= \int_I \int_{\Omega_{\zeta_N}} \Big[\mathbf{S}^\varepsilon(\vartheta_N, \nabla \mathbf{v}_N) : \nabla \mathbf{v}_N - p(\varrho_N, \vartheta_N) \mathrm{div} \mathbf{v}_N \Big] \psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \int_{\Omega_{\zeta_N}} \Big[\varepsilon \delta \beta \varrho_N^{\beta-2} |\nabla \varrho_N|^2 + \frac{\delta}{\vartheta_N^2} - \varepsilon \vartheta_N^5 \Big] \psi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all $\psi \in C_c^\infty([0,T))$. Combining this with (3.4.10) implies

$$\begin{aligned} -\int_I \partial_t\psi\, \mathcal{E}^N_{\varepsilon,\delta}\, dt &= \psi(0)\mathcal{E}^N_{\varepsilon,\delta}(0) + \int_I \psi \int_{\Omega_{\zeta_N}} \left(\frac{\delta}{\vartheta_N^2} - \varepsilon\vartheta_N^5\right) \mathrm{d}x\, dt \\ &\quad + \int_I \psi \int_{\Omega_\zeta} \Big(\mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{v}_N) : \nabla\mathbf{v}_N - \mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{u}_N) : \nabla\mathbf{u}_N\Big) \mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_I \psi \int_{\Omega_\zeta} p(\varrho_N, \vartheta_N)\big(\mathrm{div}\mathbf{u}_N - \mathrm{div}\mathbf{v}_N\big)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_I \psi \int_{\Omega_\eta} \big(\varrho_N H + \varrho_N \mathbf{f}\cdot\mathbf{u}_N\big)\, \mathrm{d}x\, \mathrm{d}t + \int_I \psi \int_\omega g\, \partial_t\eta_N \;\mathrm{d}y\, dt \end{aligned} \tag{3.4.11}$$

with

$$\begin{aligned} \mathcal{E}^N_{\varepsilon,\delta}(t) &= \int_{\Omega_{\zeta_N(t)}} \Big(\frac{1}{2}\varrho_N(t)|\mathbf{u}_N(t)|^2 + \varrho_N(t)e_\delta(\varrho_N(t), \vartheta_N(t))\Big)\, \mathrm{d}x \\ &\quad + \int_\omega \frac{|\partial_t\eta_N(t)|^2}{2}\; \mathrm{d}y + K_\varepsilon(\eta_N(t)). \end{aligned}$$

By choosing $\psi = \mathbb{I}_{(0,t)}$, we find that (3.4.11) implies uniform a-priori estimates. Note that we can apply Young's inequality to the forcing terms in (3.4.11) and absorb terms containing the unknowns in the left-hand side. Moreover, by Theorem 3.3.4 we obtain bounds for θ_N (in dependence of $\varepsilon, \delta, N, K^*$) from below such that

$$\int_{I_*} \int_{\Omega_{\zeta_N}} \frac{\delta}{\vartheta_N^2}\, \mathrm{d}x\, \mathrm{d}t \le c(\varepsilon, \delta, N, K^*)T^* \le 1$$

for T^* small enough. So, in order to apply the Gronwall lemma it is enough to control the error term

$$\begin{aligned} &\int_{I_*} \int_{\Omega_{\zeta_N}} \Big(\mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{v}_N) : \nabla\mathbf{v}_N - \mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{u}_N) : \nabla\mathbf{u}_N\Big)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_{I_*} \int_{\Omega_\zeta} p(\varrho_N, \vartheta_N)\big(\mathrm{div}\mathbf{u}_N - \mathrm{div}\mathbf{v}_N\big)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad \le \int_{I_*} \int_{\Omega_{\zeta_N}} \Big(\mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{v}_N) : \nabla\mathbf{v}_N + p(\varrho_N, \vartheta_N)(|\nabla\mathbf{u}_N| + |\nabla\mathbf{v}_N|)\Big)\, \mathrm{d}x\, \mathrm{d}t. \end{aligned}$$

Using Theorem 3.3.3 and 3.3.4 we can bound ϱ_N and ϑ_N in terms of K such that the above is bounded by

$$\begin{aligned} &\le c(K) \int_{I_*} \int_{\Omega_{\zeta_N}} \big(1 + |\nabla\mathbf{v}_N|^p\big)\, \mathrm{d}x\, \mathrm{d}t + c(K) \int_{I_*} \int_{\Omega_{\zeta_N}} |\nabla\mathbf{u}_N|^2\, \mathrm{d}x\, \mathrm{d}t \\ &\le c(K,N)\, T^*\Big(1 + \sup_{I^*} |\boldsymbol{\beta}_N|^p\Big) + c(K,N)T^* \sup_{I_*} \int_{\Omega_{\zeta_N}} |\mathbf{u}_N|^2\, \mathrm{d}x \\ &\le c(K,N)\, T^* + c(K,N)T^* \sup_{I_*} \int_{\Omega_{\zeta_N}} \varrho_N |\mathbf{u}_N|^2\, \mathrm{d}x. \end{aligned}$$

We choose $T^* = T^*(\varepsilon, N, K^*)$ small enough such that $c(K,N)\, T^* \le \frac{1}{2}$ and and obtain

$$\sup_{I_*} E^N_{\varepsilon,\delta} \le c(\mathbf{f}, H, g, \mathbf{q}_0, \eta_0, \eta_1, \varrho_0). \tag{3.4.12}$$

In particular, we have

$$\sup_{I_*} \int_{\Omega_{\zeta_N}} |\mathbf{u}_N|^2\, \mathrm{d}x + \sup_{I_*} \int_\omega \frac{|\partial_t\eta_N|^2}{2}\; \mathrm{d}y + \sup_{I_*} K_\varepsilon(\eta_N) \le\; c(\mathbf{f}, H, g, \mathbf{q}_0, \eta_0, \eta_1, \varrho_0). \tag{3.4.13}$$

recalling the lower bound for ϱ_N from Theorem 3.3.3 (b) (which depends on N here). Consequently, we see that the mapping $\boldsymbol{\beta} \mapsto \boldsymbol{\alpha}$ satisfies $F(D) \subset D$, for K^* large enough.

Now, we need to prove compactness of F with respect to the $C^{1,\alpha}(\overline{I})$ topology. First we find by Leibnitz rule that

$$\partial_t \boldsymbol{\alpha}_N = \mathcal{A}^{-1}\Big(\partial_t(\mathcal{A}\boldsymbol{\alpha}_N) - \partial_t \mathcal{A}\boldsymbol{\alpha}_N\Big).$$

Due to (3.4.9) and the regularity of ϱ_N and ϑ_N from Theorems 3.3.3 and 3.3.4 we have $\partial_t(\mathcal{A}\boldsymbol{\alpha}_N) \in C^1(\overline{I}_*)$. This can be easily seen by transforming the integrals in the definitions of the coefficients $\mathcal{A}$, $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\mathbf{c}$ to the reference domain and recalling from (3.2.10) that $\boldsymbol{\Psi}_{\zeta_N}$ and $\boldsymbol{\Psi}_{\zeta_N}^{-1}$ have the same regularity as ζ_N. Also note that $\boldsymbol{\beta}_N \in C^{1,\alpha}(\overline{I}_*)$ implies $\zeta_N \in C^{2,\alpha}(\overline{I}_*)$ by construction. Similarly, we are going to prove that $\partial_t \mathcal{A}_{i,j} \in C^1(\overline{I}_*)$. By taking the test function $\tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\varphi}_{\zeta_N}^{-1} \cdot \tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\varphi}_{\zeta_N}^{-1}$ in the continuity equation we find that

$$\begin{aligned}
\partial_t \mathcal{A}_{i,j} &= \frac{\mathrm{d}}{dt}\int_{\Omega_{\zeta_N}} \varrho_N\, \tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \cdot \tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x \\
&= \int_{\partial\Omega_\zeta} \partial_t \zeta_N \nu \circ \boldsymbol{\varphi}_{\zeta_N}^{-1} \varrho_N \tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \cdot \tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \nu_{\Omega_\zeta}\;\mathrm{d}y \\
&+ \int_{\Omega_\zeta} \varrho_N \mathbf{v}_N \cdot \nabla(\tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \cdot \tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1})\,\mathrm{d}x \\
&+ \varepsilon \int_{\Omega_\zeta} \nabla\varrho_N \cdot \nabla(\tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \cdot \tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\Psi}_{\zeta_N}^{-1})\,\mathrm{d}x \\
&+ \int_{\Omega_\zeta} \varrho_N \partial_t\big(\tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1}\big) \cdot \tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\Psi}_{\zeta_N}^{-1}\,\mathrm{d}x \\
&+ \int_{\Omega_\zeta} \varrho_N \tilde{\boldsymbol{\omega}}_i \circ \boldsymbol{\Psi}_{\zeta_N}^{-1} \cdot \partial_t\big(\tilde{\boldsymbol{\omega}}_j \circ \boldsymbol{\Psi}_{\zeta_N}^{-1}\big)\,\mathrm{d}x.
\end{aligned}$$

The last two terms containing the time-derivative behave as $\boldsymbol{\beta}_N$ which is bounded in $C^{1,\alpha}(\overline{I}_*)$. Consequently, we find that $\partial_t \boldsymbol{\alpha}_N \in C^1(\overline{I}_*)$ with bound depending only on K (and N). So, the mapping F is compact by Arcelá-Ascoli's theorem. Consequently, there is a fixed point $\boldsymbol{\alpha}^*$ which gives rise to the solution to (3.4.6) if T^* is sufficiently small (depending on δ, ε, K^* and N). The general case follows by iterating the procedure and gluing the solutions together.

3.4.2 Total energy balance

At this stage ϑ_N is still strictly positive by Theorem 3.3.4 (with a bound depending on N) so we can devide energy by ϑ_N to obtain the entropy balance

$$\begin{aligned}
\partial_t(\varrho_N s(\varrho_N,\vartheta_N)) + \operatorname{div}(\varrho_N s(\varrho_N,\vartheta_N)\mathbf{u}_N) - \operatorname{div}\Big[\Big(\frac{(\vartheta_N)}{\vartheta_N} + \delta(\vartheta_N^{\beta-1} + \frac{1}{\vartheta_N^2})\Big)\nabla\vartheta_N\Big] \\
= \frac{1}{\vartheta_N}\Big[\Big(\frac{\delta(\vartheta_N)}{\vartheta_N} + \delta(\vartheta_N^{\beta-1} + \frac{1}{\vartheta_N^2})\Big)|\nabla\vartheta_N|^2 + \delta\frac{1}{{\vartheta_N}^2}\Big] \\
+ \frac{1}{\vartheta_N}\mathbf{S}^\varepsilon(\vartheta_N, \nabla\mathbf{u}_N) : \nabla\mathbf{u}_N + \frac{\varepsilon\delta}{\vartheta_N}\beta\varrho_N^{\beta-2}|\nabla\varrho_N|^2 - \varepsilon\vartheta_N^4
\end{aligned} \tag{3.4.14}$$

satisfied in $I \times \Omega_{\eta_N}$, together with the boundary condition $\nabla\vartheta_N \cdot \nu_{\eta_N}|_{\partial\Omega_{\eta_N}} = 0$. In the weak form it reads as

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_{\eta_N}} \varrho_N s(\varrho_N,\vartheta_N)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_{\eta_N}} \big(\varrho s(\varrho_N,\vartheta_N)\partial_t\psi + \varrho_N s(\varrho_N,\vartheta_N)\mathbf{u}_N\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_I\int_{\Omega_{\eta_N}} \frac{1}{\vartheta_N}\mathbf{S}^\varepsilon(\vartheta_N,\nabla\mathbf{u}_N) : \nabla\mathbf{u}_N\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_{\eta_N}} \frac{1}{\vartheta_N}\Big[\Big(\frac{(\vartheta_N)}{\vartheta_N} + \frac{\delta}{2}(\vartheta_N^{\beta-1} + \frac{1}{\vartheta_N^2})\Big)|\nabla\vartheta_N|^2 + \delta\frac{1}{\vartheta_N^2}\Big]\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_{\eta_N}} \Big(\frac{(\vartheta_N)}{\vartheta_N} + \delta(\varrho_N^{\beta-1} + \frac{1}{\vartheta_N^2})\Big)\nabla\vartheta_N\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_{\eta_N}} \frac{\varrho_N}{\vartheta_N}H\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_{\eta_N}} \varepsilon\Big[\frac{\delta}{2\vartheta_N}\beta\varrho_N^{\beta-2}|\nabla\varrho_N|^2 - \vartheta_N^4\Big]\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\psi \geq 0$. We combine this with the energy balance proved in (3.4.11) which reads as (note that in the fixed point we have $\zeta_N = \eta_N$ and $\mathbf{v}_N = \mathbf{u}_N$)

$$
\begin{aligned}
-\int_I \partial_t \psi \, \mathcal{E}_\delta^N \, dt &= \psi(0)\mathcal{E}_\delta^N(0) + \int_I \psi \int_{\Omega_{\eta_N}} \left(\frac{\delta}{\vartheta_N^2} - \varepsilon \vartheta_N^5 \right) \mathrm{d}x \, dt + \int_I \psi \int_{\Omega_{\eta_N}} \varrho_N H \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_I \int_{\Omega_{\eta_N}} \varrho_N \mathbf{f} \cdot \mathbf{u}_N \, \mathrm{d}x \, \mathrm{d}t + \int_I \psi \int_\omega g \, \partial_t \eta_N \, \mathrm{d}\mathcal{H}^2 \, dt
\end{aligned}
\tag{3.4.15}
$$

with

$$
\begin{aligned}
\mathcal{E}_\delta^N(t) &= \int_{\Omega_{\eta_N(t)}} \Big(\frac{1}{2} \varrho_N(t) |\mathbf{u}_N(t)|^2 + \varrho_N(t) e_\delta(\varrho_N(t), \vartheta_N(t)) \Big) \mathrm{d}x \\
&\quad + \int_\omega \frac{|\partial_t \eta_N(t)|^2}{2} \, \mathrm{d}y + K(\eta_N(t)).
\end{aligned}
$$

We introduce the *ballistic free energy* for some parameter value $\Theta > 0$

$$
H_\Theta(\varrho, \vartheta) = \varrho \left(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \right), \; H_{\delta,\Theta}(\varrho, \vartheta) = \varrho \left(e_\delta(\varrho, \vartheta) - \Theta s_\delta(\varrho, \vartheta) \right),
$$

cf. [63, Chapter 2, Section 2.2.3], and obtain

$$
\begin{aligned}
-\int_I \partial_t \psi \left(\mathcal{E}_{\delta,\varepsilon}^N - \Theta \varrho s(\varrho, \vartheta) \right) dt + \Theta \int_{\Omega_{\eta_N}} \sigma_{\varepsilon,\delta} \, \mathrm{d}x \, \mathrm{d}t + \int_I \psi \int_{\Omega_{\eta_N}} \Big(\varepsilon \vartheta^5 - \frac{\delta}{\vartheta^2} \Big) \mathrm{d}x \, \mathrm{d}t \\
= \psi(0) \big(\mathcal{E}_{\delta,\varepsilon}^N - \Theta \varrho s(\varrho, \vartheta) \big)(0) + \Theta \int_I \psi \int_{\Omega_{\eta_N}} \varepsilon \vartheta^4 \, \mathrm{d}x \, \mathrm{d}t \\
+ \int_I \psi \int_{\Omega_{\eta_N}} \varrho H \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_{\Omega_{\eta_N}} \varrho \mathbf{f} \cdot \mathbf{u}_N \, \mathrm{d}x \, \mathrm{d}t + \int_I \psi \int_\omega g \, \partial_t \eta_N \, \mathrm{d}y \, dt,
\end{aligned}
\tag{3.4.16}
$$

where

$$
\begin{aligned}
\sigma_{\varepsilon,\delta}^N &= \frac{1}{\vartheta_N} \Big[\mathbf{S}(\vartheta_N, \nabla \mathbf{u}_N) : \nabla \mathbf{u}_N + \varepsilon (1 + \vartheta_N) |\nabla \mathbf{u}_N|^p \Big] \\
&\quad \frac{1}{\vartheta_N} \Big[\frac{(\vartheta_N)}{\vartheta_N} |\nabla \vartheta_N|^2 + \frac{\delta}{2} \Big(\varrho^{\beta-1} + \frac{1}{\vartheta_N^2} \Big) |\nabla \vartheta_N|^2 + \delta \frac{1}{\vartheta_N^2} \Big] + \frac{\varepsilon \delta}{2 \vartheta_N} \beta \varrho_N^{\beta-2} |\nabla \varrho_N|^2.
\end{aligned}
$$

Consequently, we obtain the estimates

$$
\begin{aligned}
&\sup_I \int_{\Omega_{\eta_N}} \varrho_N |\mathbf{u}_N|^2 \, \mathrm{d}x + \sup_I \int_{\Omega_{\eta_N}} \varrho_N^\beta \, \mathrm{d}x + \int_I \int_{\Omega_{\eta_N}} |\nabla \mathbf{u}_N|^p \, \mathrm{d}x \, \mathrm{d}t \leq c, \\
&\varepsilon \sup_I \int_\omega |\nabla^3 \eta_N|^2 \, \mathrm{d}y + \sup_I \int_\omega \frac{|\partial_t \eta_N|^2}{2} \, \mathrm{d}y + \sup_I K(\eta_N) \leq c, \\
&\sup_I \int_{\Omega_{\eta_N}} \vartheta_N^4 \, \mathrm{d}x + \int_I \int_{\Omega_{\eta_N}} \frac{1}{\vartheta_N} \Big(\frac{\delta(\vartheta_N)}{\vartheta_N} + \delta \big(\vartheta_N^{\beta-1} + \frac{1}{\vartheta_N^2} \big) \Big) |\nabla \vartheta_N|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c,
\end{aligned}
$$

where $c = (\mathbf{f}, H, g, \mathbf{q}_0, \eta_0, \eta_1, \varrho_0)$ is independent of N. The first estimate together with Poincaré's inequality, the boundary condition $tr_{\eta_N} \mathbf{u}_N$ and bound for $\partial_t \eta_N$ from the second estimate impliesthat $\mathbf{u}_N$ is bounded in

$L^p(I;L^p(\Omega_{\eta_N}))$. So, we may choose a subsequence such that

$$\eta_N \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I,W^{3,2}(\omega)), \tag{3.4.17}$$
$$\partial_t\eta_N \rightharpoonup^* \partial_t\eta \quad \text{in} \quad L^\infty(I,L^2(\omega)), \tag{3.4.18}$$
$$\mathbf{u}_N \rightharpoonup^\eta \mathbf{u} \quad \text{in} \quad L^p(I;L^p(\Omega_{\eta_N})), \tag{3.4.19}$$
$$\nabla\mathbf{u}_N \rightharpoonup^\eta \nabla\mathbf{u} \quad \text{in} \quad L^p(I;L^p(\Omega_{\eta_N}))), \tag{3.4.20}$$
$$|\nabla\mathbf{u}_N|^{p-2}\nabla\mathbf{u}_N \rightharpoonup^\eta \overline{\mathbf{P}} \quad \text{in} \quad L^{p'}(I;L^{p'}(\Omega_{\eta_N}))), \tag{3.4.21}$$
$$\varrho_N \rightharpoonup^{\eta,*} \varrho \quad \text{in} \quad L^\infty(I;L^\beta(\Omega_{\eta_N})), \tag{3.4.22}$$
$$\vartheta_N \rightharpoonup^{\eta,*} \vartheta \quad \text{in} \quad L^\infty(I;L^4(\Omega_{\eta_N})), \tag{3.4.23}$$
$$\vartheta_N \rightharpoonup^\eta \vartheta \quad \text{in} \quad L^\beta(I;L^{3\beta}(\Omega_{\eta_N})), \tag{3.4.24}$$
$$\nabla\vartheta_N \rightharpoonup^\eta \nabla\vartheta \quad \text{in} \quad L^2(I;L^2(\Omega_{\eta_N}))), \tag{3.4.25}$$

for some $\overline{\mathbf{P}} \in L^{p'}(I\times\Omega_\eta)$. This implies

$$\eta_N \to \eta \quad \text{in} \quad C(\overline{I}\times\omega). \tag{3.4.26}$$

Compactness of ϑ_N can be shown as in [63, Chapter 3, Section 3.5.3.] using (3.4.14). It is based on local arguments, which are not effected by the moving shell. Consequently we have

$$\vartheta_N \to^\eta \vartheta \quad \text{in} \quad L^4(I;L^4(\Omega_{\eta_N})). \tag{3.4.27}$$

In order to pass to the limit in various terms in the equations we are concerned with the compactness of ϱ_N. Using Theorem 3.3.1 (b) with $\theta(s)=s^2$ (which is admissible by approximation) we obtain

$$\begin{aligned}\int_{\Omega_{\eta_N}} \varrho_N^2\,\mathrm{d}x + \int_0^t\int_{\Omega_{\eta_N}} 2\varepsilon|\nabla\varrho_N|^2\,\mathrm{d}x\,\mathrm{d}\sigma \\ = \int_{\Omega_{\eta_N(0)}} \varrho_0^2\,\mathrm{d}x - \int_0^t\int_{\Omega_{\eta_N}} 2\varrho_N\mathrm{div}\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}\sigma.\end{aligned} \tag{3.4.28}$$

Due to (3.4.20) and (3.4.22) we conclude (for a non-relabelled subsequence)

$$\nabla\varrho_N \rightharpoonup^\eta \nabla\varrho \quad \text{in} \quad L^2(I;L^2(\Omega_{\eta_N}))). \tag{3.4.29}$$

Applying Corollary 3.2.11 yields

$$\varrho_N \to^\eta \varrho \quad \text{in} \quad L^2(I;L^2(\Omega_{\eta_N})). \tag{3.4.30}$$

To improve the integrability we use (3.4.10) with $\mathbf{v}_N=\mathbf{u}_N$ and $\xi_N=\eta_N$ and $\psi=\mathbb{I}_{(0,t)}$. The forcing terms can be controlled by a Gronwall-argument such that we obtain

$$\begin{aligned}\int_I\int_{\Omega_{\eta_N}} \varrho_N^{\beta-2}|\nabla\varrho_N|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le c(\mathbf{f},g,\mathbf{q}_0,\varrho_0,\eta_0,\eta_1)\bigg(1+\int_I\int_{\Omega_{\eta_N}} p(\varrho_N,\vartheta_N)\mathrm{div}\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}t\bigg)\end{aligned} \tag{3.4.31}$$

neglecting various non-negative terms on the left-hand side. The constant depends on ε and δ but is independent of N. Using the inform bounds from (3.4.20), (3.4.22) and (3.4.23) we obtain

$$\int_I\int_{\Omega_{\eta_N}} p(\varrho_N,\vartheta_N)\mathrm{div}\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}t \le c\int_I\int_{\Omega_{\eta_N}}\Big(\varrho_N^{2\gamma}+\vartheta_N^8+|\nabla\mathbf{u}_N|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le c$$

choosing β large enough. In combination with (3.4.30) we have (after passing to a subsequence)

$$\varrho_N \to^\eta \varrho \quad \text{in} \quad L^q(I_*;L^q(\Omega_{\eta_N})), \tag{3.4.32}$$

for some $q > \beta$. This is enough to pass to the limit in the nonlinear pressure. We are, however, still concerned with the term

$$\varepsilon \int_I \int_{\Omega_{\eta_N}} \nabla \varrho_N \nabla \mathbf{u}_N \cdot \boldsymbol{\phi} \, \mathrm{d}x \, \mathrm{d}t,$$

which requires compactness of $\nabla \varrho_N$. As for (3.4.28) we have

$$\int_{\Omega_{\eta_N}} \varrho^2 \, \mathrm{d}x + \int_0^t \int_{\Omega_\eta} 2\varepsilon |\nabla \varrho|^2 \, \mathrm{d}x \, \mathrm{d}\sigma = \int_{\Omega_{\eta_N(0)}} \varrho_0^2 \, \mathrm{d}x - \int_0^t \int_{\Omega_{\eta_N}} 2\varrho_N \mathrm{div} \mathbf{u}_N \, \mathrm{d}x \, \mathrm{d}\sigma.$$

applying Theorem 3.3.1 (b) to the limit version. Due to (3.4.26), (3.4.32) and the strong convergence of ϱ_N we can pass to the limit in all terms in (3.4.28) expect for the one containing $\nabla \varrho_N$. Consequently,

$$\int_0^t \int_{\Omega_{\eta_N}} |\nabla \varrho_N|^2 \, \mathrm{d}x \, \mathrm{d}\sigma = \int_0^t \int_{\Omega_\eta} |\nabla \varrho|^2 \, \mathrm{d}x \, \mathrm{d}\sigma$$

for all $t \in I$ such that

$$\lim_{N \to \infty} \int_I \int_{\Omega_{\eta_N}} \nabla \varrho_N \nabla \mathbf{u}_N \cdot \boldsymbol{\phi} \, \mathrm{d}x \, \mathrm{d}t = \int_I \int_{\Omega_\eta} \nabla \varrho \nabla \mathbf{u} \cdot \boldsymbol{\phi} \, \mathrm{d}x \, \mathrm{d}t$$

using also (3.4.20). We conclude that (3.4.1) holds.

3.4.3 Compactness of $\partial_t \eta_N$

The effort of this subsection is to prove that

$$\partial_t \eta_N \to \partial_t \eta \quad \text{in} \quad L^2(I; L^2(\omega)). \tag{3.4.33}$$

We will show this convergence in the generality we will need also in the subsequent limit procedures in the next section. In particular, we will not make use of any higher regularity beyond $L_t^\infty(L_x^\gamma)$ with $\gamma > \frac{12}{7}$ for the density.

The following aim is establishing

$$\int_I \int_{\Omega_{\eta_N}} |\sqrt{\varrho_N} \mathbf{u}_N|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_\omega |\partial_t \eta_N|^2 \, \mathrm{d}y \, dt \longrightarrow \int_I \int_{\Omega_\eta} |\sqrt{\varrho} \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_\omega |\partial_t \eta|^2 \, \mathrm{d}y \, dt, \tag{3.4.34}$$

which implies the strong convergence (3.4.33) by convexity of the L^2-norm. Relation (3.4.34) will be a consequence of

$$\int_I \int_{\Omega_{\eta_N}} \varrho_N \mathbf{u}_N \cdot \mathscr{F}_{\eta_N} \partial_t \eta_N \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_\omega |\partial_t \eta_N|^2 \, \mathrm{d}y \, dt \longrightarrow \int_I \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \mathscr{F}_\eta \partial_t \eta \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_\omega |\partial_t \eta|^2 \, \mathrm{d}y \, dt \tag{3.4.35}$$

and

$$\int_I \int_{\Omega_{\eta_N}} \varrho_N \mathbf{u}_N \cdot (\mathbf{u}_N - \mathscr{F}_{\eta_N} \partial_t \eta_N) \, \mathrm{d}x \, \mathrm{d}t \longrightarrow \int_I \int_{\Omega_\eta} \varrho \mathbf{u} \cdot (\mathbf{u} - \mathscr{F}_\eta \partial_t \eta) \, \mathrm{d}x \, \mathrm{d}t. \tag{3.4.36}$$

First observe that (due to the trace theorem Lemma 3.2.3) we find that $\partial_t \eta_N$ possesses some compactness in space. To be precise, we have

$$\|\partial_t \eta_N\|_{L^2(I, W^{1-\frac{1}{r}, r}(\omega))} + \|\partial_t \eta_N\|_{L^2(I, L^\ell(\omega))} \leq c \tag{3.4.37}$$

for all $r < 2$ and $\ell < 4$. The bounds only depend on the $L^2_t(W^{1,2}_x)$ bounds of $\mathbf{u}_N$ and hence are uniform by estimates (3.4.19) and (3.4.20). We define the projection

$$\mathcal{P}_N w = \sum_{k=1}^N \alpha_k(w) w_k, \quad \mathcal{P}_N^\zeta w = \sum_{k=1}^N \alpha_k(w) \tilde{\boldsymbol{\omega}}_k \circ \boldsymbol{\Psi}_\zeta^{-1},$$

where $\alpha_k(w) = \langle w, w_k \rangle_{W^{3,2}(\omega)}$ if $w_k = \tilde{Y}_\ell$ for some $\ell \in \mathbb{N}$ and $\alpha_k(w) = 0$ otherwise. Obviously, we have $tr_\zeta \mathcal{P}_N^\zeta w = \mathcal{P}_N w$ for any $w \in W^{3,2}(\omega)$. We have by definition,

$$\|\mathcal{P}_N w\|^2_{W^{3,2}(\omega)} \le \|w\|^2_{W^{3,2}(\omega)} \quad \forall w \in W^{3,2}(\omega). \tag{3.4.38}$$

The eigenvalue equation for the basis vectors implies additionally that

$$\|\mathcal{P}_N w\|^2_{L^2(\omega)} \le c\|w\|^2_{L^2(\omega)} \quad \forall w \in L^2(\omega). \tag{3.4.39}$$

Moreover, by definition of $\tilde{\mathbf{Y}}_k$ and $\mathscr{F}_\zeta$ (see Section 3.2.3) we have

$$\mathcal{P}_N^\zeta w = \mathscr{F}_\zeta(\mathcal{P}_N w) \tag{3.4.40}$$

for all $w \in W^{3,2}(\omega)$. Finally, we note that $\mathcal{P}_N \eta_N = \eta_N$ such that $(\eta_N, \mathscr{F}_{\eta_N}\eta_N)$ is admissible in (3.4.6). Due to the uniform a priori bounds from the last subsection and the respective embeddings, we find that the convergence in (3.4.36) follows directly from Lemma 3.2.10 with the choices $v_N = \mathbf{u}_N - \mathscr{F}_{\eta_N}\partial_t\eta_N$, $r_N = \mathcal{P}_N^{\eta_N}(\varrho_N \mathbf{u}_N)$ (which solves the projected equation (3.4.6) in the domain Ω_{η_N}) and the continuity of the projection operator $\mathcal{P}_N^{\eta_N}$ defined above (recall also (3.4.40)). The corresponding uniform estimates are given in the previous subsection and the weak convergence of $\mathscr{F}_{\eta_N}\partial_t\eta_N$ follows from (3.4.17), (3.4.18), Lemma 3.2.7 and Corollary 3.2.8.
In order to prove (3.4.35) we need to make use of the coupled momentum equation using Theorem 2.5.1. We define $g_N = (\partial_t\eta_N, \varrho_N \mathbf{u}_N \mathbb{I}_{\Omega_{\eta_N}})$ and $f_N = (\partial_t\eta_N, \mathscr{F}_{\eta_N}\partial_t\eta_N)$ noticing that (by construction) $\Omega_{\eta_N} \subset \Omega \cup S_{L/2}$ as well as for all $s < \frac{1}{2}$ and $q < 3$

$$\mathscr{F}_{\eta_N}\partial_t\eta_N \in L^2(I; W^{s,q}(\Omega \cup S_{L/2})$$

uniformly in N. The last observation is a consequence of (3.4.37) and Lemma 3.2.7 (a). In particular, we have

$$f_N \rightharpoonup f \quad \text{in} \quad L^2(I; X), \tag{3.4.41}$$

where $f = (\partial_t\eta, \mathscr{F}_\eta\partial_t\eta)$ and

$$g_N \rightharpoonup g \quad \text{in} \quad L^2(I; X'), \tag{3.4.42}$$

where $X = L^2(\omega) \times W^{s_x,q}(\Omega \cup S_{L/2})$ with $s_x < s_y < \frac{1}{2}$ (such that $X' = L^2(\omega) \times W^{-s_x,q'}(\Omega \cup S_{L/2})$), since[4]

$$\varrho_N \mathbf{u}_N \rightharpoonup^\eta \varrho\mathbf{u} \quad \text{in} \quad L^2(I; L^{\frac{6\gamma}{\gamma+6}}(\Omega_{\eta_N})) \tag{3.4.43}$$

and $L_x^{\frac{6\gamma}{\gamma+6}} \hookrightarrow W_x^{-s_x,q'}$ due to $\gamma > \frac{12}{7}$ (choosing s_x sufficiently close to $1/2$ and q close to 3). Further we define

$$Z = W^{1,2}(\omega) \times W^{1,q}(\Omega \cup S_{L/2})$$

Boundedness of g_N in $L^\infty(I; Z')$ follows now from, (3.4.18), $\varrho_N \mathbf{u}_N \in L^2_t(L_x^{\frac{2\beta}{\beta+1}})$ uniformly and the embedding $L_x^{\frac{2\beta}{\beta+1}} \hookrightarrow W_x^{-1,2} \hookrightarrow W_x^{-1,q}$ for $\beta > \frac{3}{2}$ and $q \ge 2$. The conditions 1. in Theorem 2.5.1 follow now from (3.4.41)

[4]Here, this follows easily from (3.4.32), but it will be critical in the final limit $\delta \to 0$.

and (3.4.42) by weak compactness. For 2. we observe that we may assume that a regularizer $b \mapsto (b)_\kappa$ exists such that for any $s, a \in \mathbb{R}$ and $p \in [1, \infty)$

$$\|b - (b)_\kappa\|_{W^{a,p}(\omega)} \le c\kappa^{s-a}\|b\|_{W^{s,p}(\omega)}, \quad b \in W^{s,p}(\omega). \tag{3.4.44}$$

The estimate is well-known for $a, s \in \mathbb{N}_0$, while the general case follows by interpolation and duality. Moreover, since we use standard Fourier bases in $W^{3,2}(\omega)$ for the discretisation of η_N, we find by interpolation that the projection error satisfies the following stability estimates for all $s \in [0, 3]$

$$\|\mathcal{P}^N b\|_{W^{s,2}(\omega)} \le c\|b\|_{W^{s,2}(\Omega)}. \tag{3.4.45}$$

Next we introduce the mollification operator on $\partial_t \eta_N$ by considering for $\kappa > 0$ and $N \in \mathbb{N}$ $\mathcal{P}^N((\partial_t \eta_N)_\kappa)$ and set

$$f_{N,\kappa}(t) := (\mathcal{P}^N((\partial_t \eta_N(t))_\kappa), \mathscr{F}_{\eta_N(t)}(\mathcal{P}^N((\partial_t \eta_N(t))_\kappa))).$$

We find by the continuity of the mollification operator from (3.4.44), the continuity of the projection operator from (3.4.45) and the estimate for the extension operator (due to (3.4.17) and Lemma 3.2.7) that for a.e. $t \in (0, T)$

$$\|f_{N,\kappa} - f_N\|_{L^2(\omega) \times W^{s_x,q}(\Omega \cup S_{L/2})} \le c\kappa^{s_y - s}\|\partial_t \eta_N\|_{W^{s_y,2}(\omega)}, \tag{3.4.46}$$

which can be made arbitrarily small in L^2 choosing κ appropriately, cf. (3.4.37). Similarly, we have

$$\|f_{N,\kappa}\|_{W^{1,2}(\omega) \times W^{1,q}(\Omega \cup S_{L/2})} \le c\kappa^{-1}\|\partial_t \eta_N\|_{L^2(\omega)}.$$

Moreover, by (3.4.41) we clearly can deduce a converging subsequence such that $f_{N,\kappa} \rightharpoonup f_\kappa$ (for some f_κ) in $L^2(I; X)$ for any $\kappa > 0$, which implies (b).
For 3. have to control $\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle$ and hence decompose

$$\begin{aligned}
&\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle \\
&= \big(\langle g_N(t), (\mathcal{P}^N((\partial_t \eta_N(t))_\kappa), \mathscr{F}_{\eta_N(t)}(\mathcal{P}^N((\partial_t \eta_N(t))_\kappa)))\rangle \\
&\quad - \langle g_N(s), (\mathcal{P}^N((\partial_t \eta_N(t))_\kappa), \mathscr{F}_{\eta_N(s)}(\mathcal{P}^N((\partial_t \eta_N(t))_\kappa)))\rangle\big) \\
&\quad + \langle g_N(s), (0, \mathscr{F}_{\eta_N(t)}(\mathcal{P}^N((\partial_t \eta_N(t))_\kappa)) - \mathscr{F}_{\eta_N(s)}(\mathcal{P}^N((\partial_t \eta_N(t))_\kappa)))\rangle =: (I) + (II).
\end{aligned}$$

We begin estimating (II) using Corollary 3.2.8 to find that

$$\begin{aligned}
(II) &= \int_s^t \int_{\Omega_{\eta_N(s)}} \varrho_N(s)\mathbf{u}_N(s) \cdot \partial_\theta \mathscr{F}_{\eta_N(\theta)}(\mathcal{P}^N((\partial_t \eta_N)_\kappa)(t)\,\mathrm{d}x\,\mathrm{d}\theta \\
&\le c\|\varrho_N \mathbf{u}_N(s)\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_{\eta_N(s)})}|s-t|^{\frac{1}{2}}\bigg(\int_I \|\partial_t \eta_N(\theta)\|_{L^\ell(\omega)}^2\bigg)^{\frac{1}{2}}\|\mathcal{P}^N((\partial_t \eta_N))_\delta(t)\|_{L^\infty(\omega)}
\end{aligned}$$

for some $\ell < 4$ (recall that $\gamma > \frac{12}{7}$). By Sobolev's embedding's, (3.4.44) and (3.4.45) the last term can be estimated by

$$\begin{aligned}
\|\mathcal{P}^N((\partial_t \eta_N)_\kappa)(t)\|_{L^\infty(\omega)} &\le c\|\mathcal{P}^N((\partial_t \eta_N)_\kappa(t)\|_{W^{3,2}(\omega)} \le c\|(\partial_t \eta_N)_\kappa(t)\|_{W^{3,2}(\omega)} \\
&\le c\kappa^{-3}\|\partial_t \eta_N(t)\|_{L^2(\omega)},
\end{aligned}$$

which is bounded to to (3.4.18). Using also (3.4.37) we conclude

$$(II) \le c(\kappa)\|\varrho_N \mathbf{u}_N(s)\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_{\eta_N(s)})}|s-t|^{\frac{1}{2}}$$

The term (I) is estimated using the test-function $\mathbb{I}_{(s,t)} f_{N,\kappa}$ in (3.4.6). One obtains the uniform Hölder estimate in a similar sense as for (II) using the various estimates on the extension, projections, embeddings and Hölder's inequality. We explain here in detail only the two most complicated terms stemming from the time derivative and the pressure. All other terms can be estimated analogously by simpler means. First, we consider the term acting on the time derivative. Observe that this term only appears due to the time-dependent extension. We choose a such that $\frac{1}{a}+\frac{1}{\gamma}+\frac{1}{6}=1$. Then by the assumption $\gamma>\frac{12}{7}$, we find that $a<4$. Hence we can choose $a_0\in(a,4)$ and $\chi\in(0,1)$ such that $\frac{1}{a}=\frac{\chi}{2}+\frac{1-\chi}{a_0}$ and

$$\|\partial_t\eta_N\|_{L^a(\omega)}\leq\|\partial_t\eta_N\|^{\chi}_{L^2(\omega)}\|\partial_t\eta_N\|^{1-\chi}_{L^{a_0}(\omega)}.$$

Using Corollary 3.2.6, (3.4.44) and (3.4.45) we obtain

$$\begin{aligned}
&\Big|\int_s^t\int_{\Omega_{\eta_N(\theta)}}\varrho_N\mathbf{u}_N\cdot\partial_\theta\mathscr{F}_{\eta_N(\theta)}(\mathcal{P}^N((\partial_t\eta_N)_\kappa)(t)\,\mathrm{d}x\,\,\mathrm{d}\theta\Big|\\
&\leq c\int_s^t\|\varrho_N\|_{L^\gamma(\Omega_{\eta_N})}\|\mathbf{u}_N\|_{L^6(\Omega_{\eta_N})}\|\mathscr{F}_{\eta_N(\theta)}(\mathcal{P}^N((\partial_t\eta_N)_\kappa)\|_{L^a(\Omega_{\eta_N})}\,\mathrm{d}\theta\\
&\leq c\int_s^t\|\varrho_N\|_{L^\gamma(\Omega_{\eta_N})}\|\mathbf{u}_N\|_{L^6(\Omega_{\eta_N})}\|\partial_t\eta_N(\theta)\|_{L^a(\omega)}\|\mathcal{P}^N((\partial_t\eta_N)_\kappa)\|_{L^\infty(\omega)}\,\mathrm{d}\theta\\
&\leq c\|\varrho_N\|_{L^\infty(I;L^\gamma)}\|\partial_t\eta_N\|^{1-\chi}_{L^\infty(I;L^2(\omega)}\|\mathcal{P}^N((\partial_t\eta_N))_\kappa(t)\|_{W^{3,2}(\omega)}\int_s^t\|\partial_t\eta_N\|^{\chi}_{L^{a_0}(\omega)}\|\mathbf{u}_N\|_{L^6(\Omega_{\eta_N})}\,\mathrm{d}\theta\\
&\leq c\kappa^{-3}|s-t|^{\frac{1-\chi}{2}}\|\partial_t\eta_N\|^{\chi}_{L^2(I;L^{a_0}(\omega)}\|\mathbf{u}_N\|_{L^2(I;L^6(\Omega_{\eta_N})}\leq c\kappa^{-3}|s-t|^{\frac{1-\chi}{2}},
\end{aligned}$$

where the constant depends on the a priori estimates only. As far as the pressure is concerned, Hölder's inequality and Lemma 3.2.5 (b) imply

$$\begin{aligned}
&\Big|\int_s^t\int_{\Omega_{\eta_N(\theta)}}p_\delta(\varrho_N,\vartheta_N)\operatorname{div}\mathscr{F}_{\eta_N(\theta)}(\mathcal{P}^N((\partial_t\eta_N)_\kappa)(t)\,\mathrm{d}x\,\,\mathrm{d}\theta\Big|\\
&\leq c\|p_\delta(\varrho_N,\vartheta_N)\|_{L^\infty(I;L^1(\Omega_{\eta_N}))}\int_s^t\|\nabla\mathscr{F}_{\eta_N(\theta)}(\mathcal{P}^N((\partial_t\eta_N)_\kappa)\|_{L^\infty(\Omega_{\eta_N})}\,\mathrm{d}\theta\\
&\leq c\int_s^t(1+\|\nabla\eta_N\|_{L^\infty(\omega)})\|\mathcal{P}^N((\partial_t\eta_N)_\kappa)\|_{W^{1,\infty}(\omega)}\,\mathrm{d}\theta\\
&\leq c\int_s^t(1+\|\nabla\eta_N\|_{L^\infty(\omega)})\|(\partial_t\eta_N)_\kappa)\|_{W^{3,2}(\omega)}\,\mathrm{d}\theta\\
&\leq c\kappa^{-3}\|\partial_t\eta_N\|_{L^\infty(I;L^2(\omega))}\int_s^t(1+\|\nabla\eta_N\|_{L^\infty(\omega)})\,\mathrm{d}\theta\\
&\leq c\kappa^{-3}|t-s|^{\frac{1}{2}}\bigg(\int_I(1+\|\nabla\eta_N\|^2_{L^\infty(\omega)})\,\mathrm{d}\theta\bigg)^{\frac{1}{2}}\leq c\kappa^{-3}|t-s|^{\frac{1}{2}}
\end{aligned}$$

provided that we have

$$p_\delta(\varrho_N,\vartheta_N)\in L^\infty(I;L^1(\Omega_{\eta_N})),\quad \partial_t\eta_N\in L^\infty(I;L^2(\omega)),\tag{3.4.47}$$

$$\nabla\eta_N\in L^2(I;L^\infty(\omega)),\tag{3.4.48}$$

uniformly in N. While (3.4.47) follows hear and on the subsequent directly form the energy estimates, we need some further regularity for (3.4.48). On this level it follows from the regularisation of the shell equation, cf. (3.4.17).

In conclusion, we can now choose $\alpha\in(0,1)$ close enough to one and conclude that for $\tau>0$ and $t\in[0,T-\tau]$

$$\Big|\fint_0^\tau\langle g_N(t)-g_N(t+s),f_{N,\kappa}(t)\rangle\,\mathrm{d}s\Big|\leq c\kappa^{-3}\tau^{1/2}\big(A_N(t)+1\big),$$

where

$$\begin{aligned}
A_N(t)&=\|g_N(t)\|^2_{X'}+\|f_N(t)\|^2_X+\|\varrho_N\mathbf{u}_N(t)\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_{\eta_N})}\\
&\quad+\fint_0^\tau\Big(\|g_N(s)\|^2_{X'}+\|f_N(s)\|^2_X+\|\varrho_N\mathbf{u}_N(s)\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_{\eta_N})}\Big)\,\mathrm{d}s
\end{aligned}$$

uniformly bounded in $L^1(I)$ due to (3.4.41) and (3.4.42) and (3.4.43).
Finally, the condition on 4. follows by the usual compactness in (negative) Sobolev spaces.

3.4.4 Compactness of the shell energy

In order to complete the proof of Theorem 3.4.3 it remains to justify the limit in the shell energy. Since we have a regularized system (3.4.17) yields for any $p < \infty$

$$\eta_N \to \eta \quad \text{in} \quad L^p(I; W^{2,p}(\omega)), \tag{3.4.49}$$

which is enough to conclude

$$\lim_{N\to\infty} \int_I \psi K(\eta_N)\, dt = \int_I \psi K(\eta) dt \tag{3.4.50}$$

for all $\psi \in C_c^\infty(I)$ (this step will be much harder on the subsequent levels, see Section 3.5.2). It remains to show the convergence of the regularizer

$$\lim_{N\to\infty} \int_I \psi \mathcal{L}(\eta_N)\, dt = \int_I \psi \mathcal{L}(\eta)\, dt \tag{3.4.51}$$

First of all, we can assume that

$$\partial_t \eta_N \rightharpoonup \partial_t \eta \quad \text{in} \quad L^2(I; W^{1-1/r,r}(\omega)), \tag{3.4.52}$$

for all $r < 2$ due to (3.4.37). We infer from (3.4.6) using $(\psi\eta_N, \psi\mathscr{F}_{\eta_N}(\eta_N))$ as a test-function

$$\begin{aligned}
\int_I \psi \int_\omega K_\varepsilon'(\eta_N)\eta_N \;\mathrm{d}y\, dt &= \int_I \int_{\Omega_{\eta_N}} \varrho_N \mathbf{u}_N \cdot \partial_t\Big(\psi \mathscr{F}_{\eta_N}(\eta_N)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \psi \int_{\Omega_{\eta_N}} \varrho_N \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathscr{F}_{\eta_N}(\eta_N)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I \psi \int_{\Omega_{\eta_N}} \mathbf{S}^\varepsilon(\vartheta_N, \nabla \mathbf{u}_N) : \nabla \mathscr{F}_{\eta_N}(\eta_N)\,\mathrm{d}x\, dt \\
&+ \int_I \psi \int_{\Omega_{\eta_N}} \Big(p_\delta(\varrho_N, \vartheta_N)\,\mathrm{div}\mathscr{F}_{\eta_N}(\eta_N) + \varepsilon \nabla\varrho_N \nabla \mathbf{u}_N \mathscr{F}_{\eta_N}(\eta_N)\Big)\,\mathrm{d}x\, dt \\
&+ \int_I \psi \int_\omega \partial_t \eta_N\, \partial_t(\psi\eta_N)\;\mathrm{d}y\,\mathrm{d}\sigma \\
&+ \int_I \psi \int_{\Omega_{\eta_N}} \varrho_N \mathbf{f}\cdot \mathscr{F}_{\eta_N}(\eta_N)\,\mathrm{d}x\,\mathrm{d}t + \int_I \psi \int_\omega g\,\eta_N \;\mathrm{d}y\, dt \\
&+ \psi(0) \int_{\Omega_{\eta_N(0)}} \mathbf{q}_0 \cdot \mathscr{F}_{\eta_N}(\eta_N)(0,\cdot)\,\mathrm{d}x + \psi(0)\int_\omega \eta_1\,\eta_N \;\mathrm{d}y.
\end{aligned} \tag{3.4.53}$$

The terms on the right-hand side related to the shell clearly converge to their expected limits because of (3.4.17) and (3.4.33). On account of Lemma 3.2.7 and Corollary 3.2.8 we have

$$\|\partial_t(\mathscr{F}_{\eta_N}(\eta_N))\|_{L^2_t L^{q_1}_x} + \|\mathscr{F}_{\eta_N}(\eta_N))\|_{L^\infty_t W^{1,q_2}_x} + \|\mathscr{F}_{\eta_N}(\eta_N))\|_{L^\infty_t W^{2,q_3}_x} \le c$$

uniformly in N for all $q_1 < 4$, $q_2 < \infty$ and $q_3 < 2$, cf. (3.4.17) and (3.4.33). In particular, applying standard compact embeddings we can choose a subsequence (not relabelled) such that

$$\begin{aligned}
\partial_t(\mathscr{F}_{\eta_N}(\eta_N)) &\rightharpoonup \partial_t(\mathscr{F}_\eta(\eta)) \quad \text{in} \quad L^2(I; L^{q_1}(\Omega \cup S_{L/2})), \\
\mathscr{F}_{\eta_N}(\eta_N) &\to \mathscr{F}_\eta(\eta) \quad \text{in} \quad L^{q_2}(I; W^{1,q_2}(\Omega \cup S_{L/2})), \\
\mathscr{F}_{\eta_N}(\eta_N) &\to \mathscr{F}_\eta(\eta) \quad \text{in} \quad L^\infty(I; L^\infty(\Omega \cup S_{L/2})),
\end{aligned}$$

for all $q_1 < 4$ and $q_2 < \infty$. Combining these convergences with the convergences form the last subsection we can pass to the limit in the terms on the right-hand side of (3.4.53) related to the fluid system as well. On the other hand, the resulting expression coincides with $\int_I \psi K(\eta)\,dt$ as can be seen from testing the limit system with $(\psi\eta, \psi\mathscr{F}_\eta(\eta))$. We conclude that

$$\varepsilon \int_I \psi\mathcal{L}(\eta_N)\,dt = \frac{\varepsilon}{2}\int_I \psi\mathcal{L}'(\eta_N)\eta_N\,dt = \frac{1}{2}\int_I \psi K_\varepsilon'(\eta_N)\eta_N\,dt - \frac{1}{2}\int_I \psi K'(\eta_N)\eta_N\,dt$$
$$\longrightarrow \frac{1}{2}\int_I \psi K_\varepsilon'(\eta)\eta\,dt - \frac{1}{2}\int_I \psi K'(\eta)\eta\,dt = \varepsilon\int_I \psi\mathcal{L}(\eta)\,dt$$

as $N \to \infty$ due to (3.4.50). Combing this with (3.4.49) shows that (3.4.51) must be true. Combining all the convergences proven above allows us to pass to the limit in the energy balance (3.4.15) and to conclude that

$$-\int_I \partial_t\psi\,\mathcal{E}_{\varepsilon,\delta}\,dt = \psi(0)\mathcal{E}_{\varepsilon,\delta}(0) + \int_I \psi\int_{\Omega_\eta}\left(\frac{\delta}{\vartheta^2} - \varepsilon\vartheta^5\right)\mathrm{d}x\,dt$$
$$+ \int_I \psi\int_{\Omega_\eta}\varrho H\,\mathrm{d}x\,\mathrm{d}t + \int_I \psi\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \int_I\psi\int_\omega g\,\partial_t\eta\;\mathrm{d}y\,dt$$

with

$$\mathcal{E}_{\varepsilon,\delta}(t) = \int_{\Omega_{\eta(t)}}\left(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)e_\delta(\varrho(t),\vartheta(t))\right)\mathrm{d}x + \int_\omega \frac{|\partial_t\eta(t)|^2}{2}\;\mathrm{d}y + K_\varepsilon(\eta(t)).$$

The proof of Theorem 3.4.3 is hereby complete.

3.5 Construction of a solution.

In this section we pass to the limit in the approximate equations. For technical reasons the limits $\varepsilon \to 0$ and $\delta \to 0$ have to be performed independently from each other. For the greater part of this Section we study the limit $\varepsilon \to 0$ in the approximate system (K1)–(K4) and only highlight the difference in the δ-limit.

3.5.1 The limit system for $\varepsilon \to 0$

We wish to establish the existence of a weak solution $(\eta, \mathbf{u}, \varrho, \vartheta)$ to the system with artificial pressure in the following sense: We define

$$\widetilde{W}_\eta^I = C_w(\overline{I}; L^\beta(\Omega_\eta))$$

as the function space for the density, whereas the other function spaces are defined in Section 3.2.5. A weak solution is a quadruplet $(\eta, \mathbf{u}, \varrho, \vartheta) \in Y^I \times X_\eta^I \times \widetilde{W}_\eta^I \times Z_\eta^I$ that satisfies the following.

(D1) The momentum equation holds in the sense that

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta}\varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x - \int_{\Omega_\eta}\left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\phi}\right)\mathrm{d}x\,\mathrm{d}t\\
&+ \int_I\int_{\Omega_\eta}\mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta} p_\delta(\varrho,\vartheta)\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_I\left(\frac{\mathrm{d}}{dt}\int_\omega \partial_t\eta b\;\mathrm{d}y - \int_\omega \partial_t\eta\,\partial_t b\;\mathrm{d}y + \int_\omega K'(\eta)\,b\;\mathrm{d}y\right)dt\\
&= \int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\omega g\,b\;\mathrm{d}x\,dt
\end{aligned} \tag{3.5.1}$$

for all $(b, \boldsymbol{\phi}) \in C^\infty(\omega) \times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\phi} = b\nu$. Moreover, we have $(\varrho\mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t\eta(0) = \eta_1$. The boundary condition $\mathrm{tr}_\eta\mathbf{u} = \partial_t\eta\nu$ holds in the sense of Lemma 3.2.3.

(D2) The continuity equation holds in the sense that

$$\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta}\varrho\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\left(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\right)\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.5.2}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ and we have $\varrho(0) = \varrho_0$.

(D3) The entropy balance

$$\begin{aligned}\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho s_\delta(\varrho,\vartheta)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\big(\varrho s_\delta(\varrho,\vartheta)\partial_t\psi + \varrho s_\delta(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t\\ \geq \int_I\int_{\Omega_\eta}\frac{1}{\vartheta}\Big[\mathbf{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u} + \Big(\frac{(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\beta-1}+\frac{1}{\vartheta^2})\Big)|\nabla\vartheta|^2 + \delta\frac{1}{\vartheta^2}\Big]\psi\,\mathrm{d}x\,\mathrm{d}t\\ - \int_I\int_{\Omega_\eta}\Big(\frac{(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1}+\frac{1}{\vartheta^2})\Big)\nabla\vartheta\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_\eta}\frac{\varrho}{\vartheta}H\psi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.5.3}$$

holds for all $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\psi\geq 0$. Moreover, we have $\lim_{r\to 0}\varrho s(\varrho,\vartheta)(t)\geq\varrho_0 s(\varrho_0,\vartheta_0)$ and $\partial_{\nu_\eta}\vartheta|_{\partial\Omega_\eta}\leq 0$.

(D4) The total energy balance

$$\begin{aligned}-\int_I\partial_t\psi\,\mathcal{E}_\delta\,dt = \psi(0)\mathcal{E}_\delta(0) + \int_I\psi\int_{\mathbb{R}^3}\frac{\delta}{\vartheta^2}\,\mathrm{d}x\,dt + \int_I\psi\int_\Omega\varrho H\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\ + \int_I\psi\int_\omega g\,\partial_t\eta\;\mathrm{d}y\,dt\end{aligned} \tag{3.5.4}$$

holds for any $\psi\in C_c^\infty([0,T))$. Here, we abbreviated

$$\mathcal{E}_\delta(t) = \int_{\Omega_\eta(t)}\Big(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)e_\delta(\varrho(t),\vartheta(t))\Big)\,\mathrm{d}x + \int_\omega\frac{|\partial_t\eta(t)|^2}{2}\;\mathrm{d}y + K(\eta(t)).$$

Theorem 3.5.1. *Assume that we have for some $\alpha\in(0,1)$ and $s>0$*

$$\begin{aligned}\frac{|\mathbf{q}_0|^2}{\varrho_0}\in L^1(\Omega_{\eta_0}),\ \varrho_0,\vartheta_0\in C^{2,\alpha}(\overline{\Omega}_{\eta_0}),\ \eta_0\in W^{3+s,2}(\omega),\ \eta_1\in L^2(\omega),\\ \mathbf{f}\in L^2(I;L^\infty(\mathbb{R}^3)),\ g\in L^2(I\times\omega),\ H\in C^{1,\alpha}(\overline{I}\times\mathbb{R}^3),\ H\geq 0.\end{aligned}$$

Furthermore suppose that ϱ_0 and ϑ_0 are strictly positive and that (1.1.21) *is satisfied. There is a solution $(\eta,\mathbf{u},\varrho,\vartheta)\in Y^I\times X^I_\eta\times\widetilde{W}^I_\eta\times Z^I_\eta$ to (D1)–(D4). Here, we have $I=(0,T_*)$, where $T_*<T$ only if $\lim_{t\to T^*}\|\eta(t,\cdot)\|_{L^\infty_x}=\frac{L}{2}$ or the Koiter energy degenerates (namely, if $\lim_{s\to t}\overline{\gamma}(s,y)=0$ for some point $y\in\omega$).*

Lemma 3.5.2. *Under the assumptions of Theorem 3.5.1 the continuity equation holds in the renormalized sense as specified in Definition 3.2.13.*

The proof of the above theorem and lemma will be split in several parts. For a given ε we obtain a solution $(\eta_\varepsilon,\mathbf{u}_\varepsilon,\varrho_\varepsilon)$ to (K1)–(K4) by Theorem 3.4.3. As in the preceding Section we can combine the total energy balance (3.4.4) with the entropy balance (3.4.3) to obtain the total dissipation balance

$$\begin{aligned}&\int_{\Omega_{\eta_\varepsilon}}\Big[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + H_{\delta,\Theta}(\varrho_\varepsilon,\vartheta_\varepsilon)\Big]\mathrm{d}x + \int_\omega\frac{|\partial_t\eta_N|^2}{2}\;\mathrm{d}y + K(\eta_N)\\ &+\Theta\int_0^\tau\int_{\Omega_{\eta_\varepsilon}}\sigma_{\varepsilon,\delta}\,\mathrm{d}x + \int_0^\tau\int_{\Omega_{\eta_\varepsilon}}\varepsilon\vartheta_\varepsilon^5\,dt\\ &\leq\int_{\Omega_{\eta_\varepsilon}}\Big[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H_{\delta,\Theta}(\varrho_0,\vartheta_0)\Big]\mathrm{d}x + \int_\omega\frac{|\partial_t\eta_1|^2}{2}\;\mathrm{d}y + K(\eta_0)\\ &+\int_0^\tau\int_{\Omega_{\eta_\varepsilon}}\left(\frac{\delta}{\vartheta_\varepsilon^2} + \varepsilon\Theta\vartheta_\varepsilon^4\right)\mathrm{d}x\,dt\end{aligned} \tag{3.5.5}$$

for any $0\leq\tau\leq T$. Here $H_{\delta,\Theta}(\varrho,\vartheta) = \varrho\,(e_\delta(\varrho,\vartheta) - \Theta s(\varrho,\vartheta))$ for some $\Theta>0$ and

$$\begin{aligned}\sigma_{\varepsilon,\delta} = \frac{1}{\vartheta_\varepsilon}\Big[\mathbf{S}(\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon + \varepsilon(1+\vartheta_\varepsilon)\max\{|\overline{\mathbf{P}}_\varepsilon|^{p'},|\nabla\mathbf{u}_\varepsilon|^p\}\Big] + \frac{\varepsilon\delta}{2\vartheta_\varepsilon}\beta\varrho_\varepsilon^{\beta-2}|\nabla\varrho_\varepsilon|^2\\ + \frac{1}{\vartheta_\varepsilon}\Big[\frac{(\vartheta_\varepsilon)}{\vartheta_\varepsilon}|\nabla\vartheta_\varepsilon|^2 + \frac{\delta}{2}\Big(\vartheta_\varepsilon^{\beta-1} + \frac{1}{\vartheta_\varepsilon^2}\Big)|\nabla\vartheta_\varepsilon|^2 + \delta\frac{1}{\vartheta_\varepsilon^2}\Big].\end{aligned}$$

Absorbing the final term on the left-hand side of (3.5.5) into the left-hand side we deduce the bounds

$$\sup_{t\in I}\int_{\Omega_{\eta_\varepsilon}}\Big[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+H_{\delta,\Theta}(\varrho_\varepsilon,\vartheta_\varepsilon)\Big]\,\mathrm{d}x\leq c \tag{3.5.6}$$

$$\sup_I\int_\omega\frac{|\partial_t\eta_\varepsilon|^2}{2}\,\mathrm{d}y+\sup_I K(\eta_\varepsilon)+\varepsilon\sup_I\mathcal{L}(\eta_\varepsilon)\leq c. \tag{3.5.7}$$

In particular, we have

$$\sup_{t\in I}\|\varrho_\varepsilon\|^\beta_{L^\beta(\Omega_{\eta_\varepsilon})}+\sup_{t\in I}\|\varrho_\varepsilon\mathbf{u}_\varepsilon\|^{\frac{2\beta}{\beta+1}}_{L^{\frac{2\beta}{\beta+1}}(\Omega_{\eta_\varepsilon})}+\sup_{t\in I}\|\vartheta_\varepsilon\|^4_{L^4(\Omega_{\eta_\varepsilon})}\leq c. \tag{3.5.8}$$

Moreover, boundedness of the entropy production rate

$$\|\sigma_{\varepsilon,\delta}\|_{L^1(I\times\Omega_{\eta_\varepsilon})}\leq c \tag{3.5.9}$$

gives rise to

$$\varepsilon\|\nabla\mathbf{u}_\varepsilon\|^p_{L^p(I\times\Omega_{\eta_\varepsilon})}+\varepsilon\|\overline{\mathbf{P}}_\varepsilon\|^{p'}_{L^{p'}(I\times\Omega_{\eta_\varepsilon})}\leq c, \tag{3.5.10}$$

$$\|\nabla\mathbf{u}_\varepsilon\|^2_{L^2(I\times\Omega_{\eta_\varepsilon})}+\|\nabla\vartheta_\varepsilon^{\beta/2}\|^2_{L^2(I\times\Omega_{\eta_\varepsilon})}+\|\nabla\vartheta_\varepsilon\|^2_{L^2(I\times\Omega_{\eta_\varepsilon})}\leq c; \tag{3.5.11}$$

whence, by Poincare's inequality and (3.5.8),

$$\|\mathbf{u}_\varepsilon\|^2_{L^2(I;W^{1,2}(\Omega_{\eta_\varepsilon}))}+\|\vartheta_\varepsilon\|^2_{L^2(I;W^{1,2}(\Omega_{\eta_\varepsilon}))}\leq c. \tag{3.5.12}$$

Finally, we deduce from the equation of continuity (3.4.2) (using the renormalized formulation from Theorem 3.3.1 (b) with $\theta(z)=z^2$ and testing with $\psi\equiv 1$)) that

$$\int_{\Omega_{\eta_\varepsilon(t)}}\varrho_\varepsilon(t,\cdot)\,\mathrm{d}x=\int_{\Omega_{\eta_\varepsilon(0)}}\varrho_0\,\mathrm{d}x,\ \|\sqrt{\varepsilon}\nabla\varrho_\varepsilon\|_{L^2(I\times\Omega_{\eta_\varepsilon})}\leq c. \tag{3.5.13}$$

Note that all estimates are independent of ε. Hence, we may take a subsequence such that for some $\alpha\in(0,1)$ we have

$$\eta_\epsilon\rightharpoonup^*\eta\quad\text{in}\quad L^\infty(I;W^{2,2}(\omega)), \tag{3.5.14}$$

$$\varepsilon\eta_\epsilon\to 0\quad\text{in}\quad L^\infty(I;W^{3,2}(\omega)), \tag{3.5.15}$$

$$\eta_\epsilon\rightharpoonup^*\eta\quad\text{in}\quad W^{1,\infty}(I;L^2(\omega)), \tag{3.5.16}$$

$$\eta_\epsilon\to\eta\quad\text{in}\quad C^\alpha(\overline{I}\times\omega)), \tag{3.5.17}$$

$$\mathbf{u}_\epsilon\rightharpoonup^\eta\mathbf{u}\quad\text{in}\quad L^2(I;W^{1,2}(\Omega_{\eta_\epsilon})), \tag{3.5.18}$$

$$\varepsilon\mathbf{u}_\epsilon\to^\eta 0\quad\text{in}\quad L^p(I;W^{1,p}(\Omega_{\eta_\epsilon})), \tag{3.5.19}$$

$$\varepsilon\overline{\mathbf{P}}_\epsilon\to^\eta 0\quad\text{in}\quad L^{p'}(I;L^{p'}(\Omega_{\eta_\epsilon})), \tag{3.5.20}$$

$$\varrho_\epsilon\rightharpoonup^{*,\eta}\varrho\quad\text{in}\quad L^\infty(I;L^\beta(\Omega_{\eta_\epsilon})), \tag{3.5.21}$$

$$\varepsilon\nabla\varrho_\varepsilon\to^\eta 0\quad\text{in}\quad L^2(I\times\Omega_{\eta_\epsilon}), \tag{3.5.22}$$

$$\vartheta_\varepsilon\rightharpoonup^{*,\eta}\vartheta\quad\text{in}\quad L^\infty(I;L^4(\Omega_{\eta_\varepsilon})), \tag{3.5.23}$$

$$\vartheta_\varepsilon\rightharpoonup^\eta\vartheta\quad\text{in}\quad L^\beta(I;L^{3\beta}(\Omega_{\eta_\varepsilon})), \tag{3.5.24}$$

$$\vartheta_\varepsilon\rightharpoonup^\eta\vartheta\quad\text{in}\quad L^2(I;W^{1,2}(\Omega_{\eta_\varepsilon})). \tag{3.5.25}$$

We observe that the a-priori estimates (3.5.8) imply uniform bounds of $\varrho_\epsilon\mathbf{u}_\epsilon$ in $L^\infty(I,L^{\frac{2\beta}{\beta+1}})$. Therefore, we may apply Lemma 3.2.10 with the choice $v_i\equiv\mathbf{u}_\epsilon$, $r_i=\varrho_\epsilon$, $p=s=2$, $b=\beta$ and m sufficiently large to obtain

$$\varrho_\varepsilon\mathbf{u}_\varepsilon\rightharpoonup^\eta\varrho\mathbf{u}\qquad\text{in}\qquad L^q(I,L^a(\Omega_{\eta_\epsilon})), \tag{3.5.26}$$

where $a \in (1, \frac{2\beta}{\beta+1})$ and $q \in (1,2)$. We apply Lemma 3.2.10 once more with the choice $v_i \equiv \mathbf{u}_\epsilon$, $r_i = \varrho_\epsilon \mathbf{u}_\epsilon$, $p = s = 2$, $b = \frac{2\beta}{\beta+1}$ and m sufficiently large to find that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup^\eta \varrho \mathbf{u} \otimes \mathbf{u} \qquad \text{in} \qquad L^1(I \times \Omega_{\eta_\epsilon}). \tag{3.5.27}$$

We also obtain

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to^\eta \varrho \mathbf{u} \qquad \text{in} \qquad L^q(I, L^q(\Omega_{\eta_\epsilon})), \tag{3.5.28}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup^{\eta,*} \varrho \mathbf{u} \qquad \text{in} \qquad L^\infty(I, L^{\frac{2\beta}{\beta+1}}(\Omega_{\eta_\epsilon})), \tag{3.5.29}$$

for all $q < \frac{6\beta}{\beta+6}$. Moreover, we have as a consequence of (3.5.18) and (3.5.23)

$$\mathbf{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) \rightharpoonup^\eta \overline{\mathbf{S}} \qquad \text{in} \qquad L^{4/3}(I, L^{4/3}(\Omega_{\eta_\varepsilon})) \tag{3.5.30}$$

for some limit function $\overline{\mathbf{S}}$. The convergence (3.5.14) and the assumption on K yields

$$K'(\eta_\varepsilon) \rightharpoonup^* \overline{K}' \quad \text{in} \quad L^\infty(I; W^{-2,r}(\omega)) \tag{3.5.31}$$

for any $r < 2$ with some limit quantity $\overline{K}$.
At this stage of the proof the pressure is only bounded in L^1, so we have to exclude its concentrations. The standard approach from [63, Chapter 3, Section 3.6.3] only works locally where the moving shell is not seen (see Lemma 3.5.3 below). The problem can be circumvented by excluding concentrations at the boundary (see Lemma 3.5.4 which is inspired by [120]). The proof is exactly as in [22, Lemma 6.4].

Lemma 3.5.3. *Let $Q = J \times B \Subset I \times \Omega_\eta$ be a parabolic cube. The following holds for any $\epsilon \leq \epsilon_0(Q)$*

$$\int_Q p_\delta(\varrho_\varepsilon, \vartheta_\varepsilon)\varrho_\varepsilon \, dx \, dt \leq C(Q) \tag{3.5.32}$$

with a constant independent of ϵ.

Lemma 3.5.4. *Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that we have for all $\varepsilon \leq \varepsilon_0(\kappa)$*

$$\int_{I \times \mathbb{R}^3 \setminus A_\kappa} p_\delta(\varrho_\varepsilon, \vartheta_\varepsilon)\varrho_\varepsilon \chi_{\Omega_{\eta_\epsilon}} \, dx \, dt \leq \kappa. \tag{3.5.33}$$

We connect Lemma 3.5.3 and Lemma 3.5.4 to obtain the following corollary.

Corollary 3.5.5. *Under the assumptions of Theorem 3.5.1 there exists a function $\overline{p}$ such that*

$$p_\delta(\varrho_\varepsilon, \vartheta_\varepsilon) \rightharpoonup^\eta \overline{p} \text{ in } L^1(I; L^1(\Omega_{\eta_\epsilon})),$$

at least for a subsequence. Additionally, for $\kappa > 0$ arbitrary, there is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that $\overline{p}\varrho \in L^1(A_\kappa)$ and

$$\int_{(I \times \Omega_\eta) \setminus A_\kappa} \overline{p} \, dx \, dt \leq \kappa. \tag{3.5.34}$$

Combining Corollary 3.5.5 with the convergences (3.5.14)–(3.5.31) we can pass to the limit in (3.4.1) and (3.4.2) and obtain the following. There is

$$(\eta, \mathbf{u}, \varrho, \vartheta, \overline{p}) \in Y^I \times X^I_\eta \times \widetilde{W}^I_\eta \times Z^I_\eta \times L^1(I \times \Omega_\eta)$$

that satisfies

$$\mathbf{u}(\cdot, \cdot + \eta\nu) = \partial_t \eta \nu_\eta \quad \text{in} \quad I \times \omega,$$

the continuity equation

$$\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.5.35}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ and the coupled weak momentum equation

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\Omega_\eta}\overline{\mathbf{S}} : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\overline{p}\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_I\frac{\mathrm{d}}{dt}\int_\omega \partial_t\eta b\;\mathrm{d}y - \int_\omega \partial_t\eta\,\partial_t b\;\mathrm{d}y + \int_\omega \overline{K}'\,b\;\mathrm{d}y\,dt\\
&=\int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\omega g\,b\;\mathrm{d}x\,dt
\end{aligned} \tag{3.5.36}$$

for all $(b,\boldsymbol{\phi}) \in C^\infty(\omega)\times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\phi} = b\nu$. It remains to show strong convergence of ϑ_ε, ϱ_ε and $\nabla^2\eta_\varepsilon$. The convergence proof for ϑ_ε is entirely based on local arguments. Consequently the shell is not seen and we can follow the arguments in [63, Chapter 3, Section 3.7.3] to conclude

$$\vartheta_\varepsilon \to^\eta \vartheta \text{ in } L^4(I\times\Omega_{\eta_\varepsilon}). \tag{3.5.37}$$

This yields $\overline{\mathbf{S}} = \mathbf{S}(\vartheta,\nabla\mathbf{u})$ in (3.5.36). Additionally we can pass to the limit in the entropy balance (3.4.3) using lower semi-continuity. The remainder of this subsection is dedicated to the proof of $\overline{p} = p(\varrho,\vartheta)$. Eventually, we will pass to the limit in the shell energy in Section 3.5.2 which will finish the proof of Theorem 3.5.1.

The proof of strong convergence of the density is based on the effective viscous flux identity introduced in [130] and the concept of renormalized solutions from [46]. Arguing locally, there is no difference to the standard setting and we can follow the arguments in [63, Chapter 3, Section 3.6.5]. We consider a parabolic cube $\tilde{Q} = \tilde{J}\times\tilde{B}$ with $Q \Subset \tilde{Q} \Subset I\times\Omega_\eta$. Due to (3.5.17) we can assume that $\tilde{Q} \Subset I\times\Omega^I_{\eta_\varepsilon}$ (by taking ε small enough). For $\psi \in C_c^\infty(\tilde{Q})$ we obtain

$$\begin{aligned}
&\int_{I\times\mathbb{R}^3}\psi^2\big(p_\delta(\varrho_\varepsilon,\vartheta_\varepsilon) - (\lambda(\vartheta_\varepsilon) + 2\mu(\vartheta_\varepsilon))\mathrm{div}\mathbf{u}_\varepsilon\big)\,\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\longrightarrow \int_{I\times\mathbb{R}^3}\psi^2\big(\overline{p} - (\lambda(\vartheta) + 2\mu(\vartheta))\mathrm{div}\mathbf{u}\big)\,\varrho\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.5.38}$$

as $\varepsilon \to 0$ (note that the term related to $\overline{\mathbf{P}}_\varepsilon$ disappears due to (3.5.19) provided we choose β large enough). The proof of Lemma 3.5.2 follows exactly as in [22, Lemma 6.2]. So, for $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ we have

$$\begin{aligned}
&\int_I\frac{\mathrm{d}}{dt}\int_{\mathbb{R}^3}\theta(\varrho)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{I\times\mathbb{R}^3}\theta(\varrho)\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{I\times\mathbb{R}^3}\big(\varrho\theta'(\varrho) - \theta(\varrho)\big)\mathrm{div}\mathscr{E}_\eta\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_{I\times\mathbb{R}^3}\theta(\varrho)\mathscr{E}_\eta\mathbf{u}\cdot\nabla\psi.
\end{aligned} \tag{3.5.39}$$

Here $\mathscr{E}_\eta : W^{1,2}(\Omega_\eta) \to W^{1,p}(\mathbb{R}^3)$ is the extension operator from [22, Lemma 2.5] where $1 < p < 2$ (but may be chosen close to 2). In order to deal with the local nature of (3.5.38) we use ideas from [63]. First of all, by the monotonicity of the mapping $\varrho \mapsto p(\varrho,\vartheta)$, we find for arbitrary non-negative $\psi \in C_c^\infty(\tilde{Q})$

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{I\times\mathbb{R}^3}\psi(\lambda(\vartheta) + 2\mu(\vartheta))\big(\mathrm{div}\mathbf{u}_\epsilon\,\varrho_\epsilon - \mathrm{div}\mathbf{u}\,\varrho\big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\liminf_{\varepsilon\to 0}\int_{I\times\mathbb{R}^3}\psi\Big((\lambda(\vartheta_\varepsilon) + 2\mu(\vartheta_\varepsilon))\mathrm{div}\mathbf{u}_\epsilon\,\varrho_\epsilon - (\lambda(\vartheta) + 2\mu(\vartheta))\mathrm{div}\mathbf{u}\,\varrho\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\liminf_{\varepsilon\to 0}\int_{I\times\Omega_{\eta_\epsilon}}\psi\Big(\big(\overline{p} - (\lambda(\vartheta) + 2\mu(\vartheta))\mathrm{div}\mathbf{u}\big)\varrho - \big(p(\varrho_\varepsilon,\vartheta_\varepsilon) - (\lambda(\vartheta_\varepsilon) + 2\mu(\vartheta_\varepsilon))\mathrm{div}\mathbf{u}_\varepsilon\big)\,\varrho_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\liminf_{\varepsilon\to 0}\int_{I\times\Omega_{\eta_\epsilon}}\psi\big(p(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon - \overline{p}\varrho\big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\liminf_{\varepsilon\to 0}\int_{I\times\Omega_{\eta_\epsilon}}\psi\big(p(\varrho_\varepsilon,\vartheta_\varepsilon) - \overline{p}\big)\big(\varrho_\varepsilon - \varrho\big)\,\mathrm{d}x\,\mathrm{d}t \geq 0
\end{aligned}$$

using (3.5.38) as well as (3.5.37) (together with (3.2.4) and the uniform bounds (3.5.8) and (3.5.11)). As ψ is arbitrary and μ strictly positive by (3.2.4) we conclude

$$\overline{\mathrm{div}\mathbf{u}\,\varrho} \geq \mathrm{div}\mathbf{u}\,\varrho \quad \text{a.e. in} \quad I \times \Omega_\eta, \tag{3.5.40}$$

where

$$\mathrm{div}\mathbf{u}_\epsilon\,\varrho_\epsilon \rightharpoonup^\eta \overline{\mathrm{div}\mathbf{u}\,\varrho} \quad \text{in} \quad L^1(\Omega; L^1(\Omega_{\eta_\varepsilon})),$$

recall (3.5.18) and (3.5.21). Now, we compute both sides of (3.5.40) by means of the corresponding continuity equations. Due to Theorem 3.3.1 (b) with $\theta(z) = z \ln z$ and $\psi = \mathbb{I}_{(0,t)}$ we have

$$\int_0^t \int_{\mathbb{R}^3} \mathrm{div}\mathbf{u}_\epsilon\,\varrho_\epsilon \,\mathrm{d}x\,\mathrm{d}\sigma \leq \int_{\mathbb{R}^3} \varrho_0 \ln(\varrho_0)\,\mathrm{d}x - \int_{\mathbb{R}^3} \varrho_\varepsilon(t) \ln(\varrho_\varepsilon(t))\,\mathrm{d}x. \tag{3.5.41}$$

Similarly, equation (3.5.39) yields

$$\int_0^t \int_{\mathbb{R}^3} \mathrm{div}\mathbf{u}\,\varrho \,\mathrm{d}x\,\mathrm{d}\sigma = \int_{\mathbb{R}^3} \varrho_0 \ln(\varrho_0)\,\mathrm{d}x - \int_{\mathbb{R}^3} \varrho(t) \ln(\varrho(t))\,\mathrm{d}x. \tag{3.5.42}$$

Combining (3.5.40)–(3.5.42) shows

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \varrho_\varepsilon(t) \ln(\varrho_\varepsilon(t))\,\mathrm{d}x \leq \int_{\mathbb{R}^3} \varrho(t) \ln(\varrho(t))\,\mathrm{d}x$$

for any $t \in I$. This gives the claimed convergence $\varrho_\varepsilon \to \varrho$ in $L^1(I \times \mathbb{R}^3)$ by convexity of $z \mapsto z \ln z$. Consequently, we have $\overline{p} = p(\varrho, \vartheta)$.

3.5.2 Compactness of the shell energy

All the forthcoming effort is to prove

$$\lim_{\varepsilon\to 0} \int_I \int_\omega |\partial_t \eta_\varepsilon(t)|^2 \,\mathrm{d}y\,dt = \int_I \int_\omega |\partial_t \eta(t)|^2 \,\mathrm{d}y\,dt, \tag{3.5.43}$$

$$\lim_{\varepsilon\to 0} \int_I K_\varepsilon(\eta_\varepsilon(t))\,dt = \int_I K(\eta(t))\,dt, \tag{3.5.44}$$

as $\varepsilon \to 0$ at least for a subsequence. This will allow us to pass to the limit in the energy balance as well as in the nonlinear term of the shell equation. In the following we derive a framework to prove (3.5.44) based on fractional estimates. The same approach will be subsequently used in the limit passage $\delta \to 0$ in Section 3.5.3. The difference is that the bounds on the density will be more restrictive. We develop the theory here using only these weaker estimates to have it ready for the final limit procedure as well.
A first observation is that $tr_{\eta_\varepsilon}(\mathbf{u}_\varepsilon) = \partial_t \eta_\varepsilon \nu$ implies

$$\partial_t \eta_\varepsilon \rightharpoonup \partial_t \eta \quad \text{in} \quad L^2(I; W^{1-1/r,r}(\omega)), \tag{3.5.45}$$

for all $r < 2$ by (3.5.11) in combination with Lemma 3.2.3. In the following we are going to prove that

$$\int_I \|\eta_n\|^2_{W^{2+s,2}(\omega)}\,dt \tag{3.5.46}$$

is uniformly bounded for some $s > 0$ using an appropriate test-function in the shell equation. On account of the coupling we need a suitable test-function for the momentum equation for it as well. Hence we set

$$(\boldsymbol{\phi}_\varepsilon, \phi_\varepsilon) = \big(\mathcal{F}^{div}_{\eta_\varepsilon}(\Delta^s_{-h}\Delta^s_h \eta_\varepsilon - \mathscr{K}_{\eta_\varepsilon}(\Delta^s_{-h}\Delta^s_h \eta_\varepsilon)), \Delta^s_{-h}\Delta^s_h \eta_\varepsilon - \mathscr{K}_{\eta_\varepsilon}(\Delta^s_{-h}\Delta^s_h \eta_\varepsilon)\big),$$

where $\mathcal{F}_\eta^{\mathrm{div}}$ and $\mathscr{K}_\eta$ have been introduced in Proposition 2.3.3. Here $\Delta_s^h v(y) = h^{-s}(v(y+he_\alpha) - v(y))$ is the fractional difference quotient in direction e_α for $\alpha \in \{1,2\}$. We obtain

$$\begin{aligned}
&\int_I K_\varepsilon'(\eta_\varepsilon)\,\phi_\varepsilon\,dt\\
&\quad= \int_I\int_{\Omega_{\eta_\varepsilon(t)}} \big(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mathbf{S}(\vartheta_\varepsilon,\mathbf{u}_\varepsilon) - \varepsilon\overline{\mathbf{P}}_\varepsilon\big) : \nabla\boldsymbol{\phi}_\varepsilon + \mathbf{f}\cdot\boldsymbol{\phi}_\varepsilon\big)\,\mathrm{d}x\,dt\\
&\quad+ \int_I\int_{\Omega_{\eta\varrho(t)}} \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\boldsymbol{\phi}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_I \frac{\mathrm{d}}{dt}\bigg(\int_{\Omega_{\eta_\varepsilon(t)}} \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\boldsymbol{\phi}_n\,\mathrm{d}x + \int_\omega \partial_t\eta_\varepsilon\,\phi_\varepsilon\;\mathrm{d}y\bigg)\,dt\\
&\quad+ \int_I\int_\omega \big(\partial_t\eta_\varepsilon\,\partial_t\phi_\varepsilon + g\,\phi_\varepsilon\big)\;\mathrm{d}y\,dt =: (I)_\varepsilon + (II)_\varepsilon + (III)_\varepsilon + (IV)_\varepsilon
\end{aligned}$$

recalling that the function $\boldsymbol{\phi}_\varepsilon$ is divergence-free such that the pressure term disappears. Since $\eta_\varepsilon \in L^\infty(I, W^{2,2}(\omega))$ uniformly, we have

$$\int_I \|\Delta_h^s \nabla^2\eta_\varepsilon\|_{L^2(\omega)}^2\,dt \lesssim 1 + \int_I K'(\eta_\varepsilon)\,\phi_\varepsilon\,dt$$

for every $h > 0$ and $s \in (0, \frac{1}{2})$ due to [142, Lemma 4.5]. Consequently, it holds

$$\int_I \|\Delta_h^s \nabla^2\eta_\varepsilon\|_{L^2(\omega)}^2\,dt + \varepsilon\int_I \|\Delta_h^s \nabla^3\eta_\varepsilon\|_{L^2(\omega)}^2\,dt \lesssim 1 + (I)_\varepsilon + (II)_\varepsilon + (III)_\varepsilon + (IV)_\varepsilon$$

and our task consists in establishing uniform estimates for the terms $(I)_\varepsilon, \dots, (IV)_\varepsilon$. As far as $(I)_n$ is concerned the most critical term is the convective term $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ with integrability $\frac{6\gamma}{\gamma+6} > 1$. By Theorem 2.3.3 and (3.5.14)

$$\begin{aligned}
\|\boldsymbol{\phi}_\varepsilon\|_{L^q(I;W^{1,p}(\omega))} &\le \|\Delta_{-h}^s\Delta_h^s\eta_\varepsilon\|_{L^q(I;W^{1,p}(\omega))} + \|\Delta_{-h}^s\Delta_h^s\eta_\varepsilon\nabla\eta_\varepsilon\|_{L^q(I;L^p(\omega))}\\
&\le \|\eta_\varepsilon\|_{L^q(I;W^{1+2s,p}(\omega))} + \|\Delta_{-h}^s\Delta_h^s\eta_\varepsilon\|_{L^\infty(I;L^{2p}(\omega))}\|\nabla\eta_\varepsilon\|_{L^\infty(I;L^{2p}(\omega))}\\
&\le \|\eta_\varepsilon\|_{L^q(I;W^{1+2s,p}(\omega))} + \|\eta_\varepsilon\|_{L^\infty(I;W^{2s,2p}(\omega))}\|\nabla\eta_\varepsilon\|_{L^\infty(I;L^{2p}(\omega))}\\
&\le \|\eta_\varepsilon\|_{L^q(I;W^{1+2s,p}(\omega))} + c_p
\end{aligned} \tag{3.5.47}$$

for all $s < \frac{1}{2}$, $p < \infty$ and $q \in [1,\infty]$. For $p = \frac{6\gamma}{6\gamma-\gamma-6}$ we can choose $s > 0$ small enough such that $W^{2,2}(\omega) \hookrightarrow W^{1+2s,p}(\omega)$. Using (3.5.14) again implies that $\boldsymbol{\phi}_\varepsilon$ is uniformly bounded in $L_t^\infty(W_x^{1,p})$. We conclude that $(I)_\varepsilon$ is uniformly bounded in ε and h if we choose s small enough. The most critical term is in fact $(II)_\varepsilon$. We note that (3.5.18) and (3.5.21) imply

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \in L^2(I; L^{q_3}(\Omega_{\eta_n}))$$

uniformly for all $q_3 < \frac{6\gamma}{\gamma+6}$. Due to the assumption $\gamma > \frac{12}{7}$ we can choose in the above $q_3 > \frac{4}{3}$. On the other hand we have

$$\begin{aligned}
\|\partial_t\boldsymbol{\phi}_\varepsilon\|_{L^2(I;L^{q_3'}(S_{L/2}\cup\Omega))} &\lesssim \|\partial_t\Delta_{-h}^s\Delta_h^s\eta_n\|_{L^2(I;L^{q_3'}(\omega))} + \|\Delta_{-h}^s\Delta_h^s\eta_n\partial_t\eta_\varepsilon\|_{L^2(I;L^{q_3'}(\omega))}\\
&\lesssim \|\partial_t\eta_\varepsilon\|_{L^2(I;W^{2s,q_3'}(\omega))} + \|\Delta_{-h}^s\Delta_h^s\eta_\varepsilon\|_{L^\infty(I\times\omega)}\|\partial_t\eta_\varepsilon\|_{L^2(I;L^{q_3'}(\omega))}.
\end{aligned}$$

Thus, we can choose s small enough such that $\partial_t\boldsymbol{\phi}_\varepsilon$ is uniformly bounded in $L_t^2(L_x^{q_3'})$ thanks to (3.5.14) and (3.5.45) (together with Sobolev's embedding and $q_3' < 4$). We conclude boundedness of $(II)_\varepsilon$. As far as $(III)_\varepsilon$ is concerned, uniform bounds for the first term are easily obtained from (3.5.47) (choosing $p > 2$ and using Sobolev's embedding) in combination with (3.5.26). For the second term we use

$$\|\phi_\varepsilon\|_{L^2(I;L^2(\omega))} \lesssim \|\eta_\varepsilon\|_{L^2(I;W^{2s,2}(\omega)} \lesssim \|\eta_\varepsilon\|_{L^2(I;W^{1,2}(\omega))}$$

together with (3.5.14). The second term in $(IV)_\varepsilon$ is analogous. Finally, we can use again (3.5.45) to control the first term in $(IV)_\varepsilon$ and the proof of (3.5.44) is complete. Moreover we have shown

$$\varepsilon \int_I \|\eta_n\|^2_{W^{3+s,2}(\omega)}\, dt \leq c$$

uniformly in ε. This, interpolated with (3.5.46), yields $\varepsilon\mathcal{L}(\eta_\varepsilon) \to 0$ as $\varepsilon \to 0$, which completes the proof of (3.5.44).

Finally we observe that the convergence in (3.5.43) follows exactly as was done in Subsection 3.4.3 (in particular, (3.5.46) implies the required to obtain a counterpart of (3.4.48)). The proof is even slightly simpler since we do not need to project into a discrete space when proving the equi-continuity.

3.5.3 Proof of Theorem 3.2.14.

In this section we are ready to prove the main result of this paper by passing to the limit $\delta \to 0$ in the system (D1)–(D4) from Section 3.5.1. Large parts of the proof are very similar to their counterparts in the limit $\varepsilon \to 0$. In particular, the compactness arguments from 3.5.2 and 3.4.3 have been written in such a way that they are directly adaptable for the final layer here (using only the more restrictive bounds on γ). The main exception is the analysis related to the limit passage in the molecular pressure. This can, however, be adapted from [22, Section 7]. As there, we can localise the argument for fixed boundaries from [63]. Consequently, parts of the argument are independent from the variable domain and the fluid-structure interaction. Nevertheless we sketch the main steps of the proof for the convenience of the reader.

Given initial data $(\mathbf{q}_0, \varrho_0, \vartheta_0)$ and H belonging to the function spaces stated in Theorem 3.2.14 it is standard to find regularized versions $(\mathbf{q}_{0,\delta}, \varrho_{0,\delta}, \vartheta_{0,\delta})$ and H_δ such that for all $\delta > 0$

$$\varrho_{0,\delta}, \vartheta_{0,\delta} \in C^{2,\alpha}(\overline{\Omega}_{\eta_0}),\ \varrho_{0,\delta}, \vartheta_{0,\delta} \text{ strictly positive},\ H_\delta \in C^{1,\alpha}(\overline{I} \times \mathbb{R}^3),\ H_\delta \geq 0,$$

as well as

$$\int_{\Omega_{\eta_0}} \Big(\frac{1}{2}\frac{|\mathbf{q}_{0,\delta}|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta} e(\varrho_{0,\delta}, \vartheta_{0,\delta})\Big)\,\mathrm{d}x \to \int_{\Omega_{\eta_0}} \Big(\frac{1}{2}\frac{|\mathbf{q}_0|^2}{\varrho_{0,\delta}} + \varrho_0 e(\varrho_0, \vartheta_0)\Big)\,\mathrm{d}x,$$
$$H_\delta \to H \text{ in } L^\infty(\overline{I} \times \mathbb{R}^3),$$

as $\delta \to 0$. For a given δ we gain a weak solution $(\eta_\delta, \mathbf{u}_\delta, \varrho_\delta, \vartheta_\delta)$ to (3.5.1)–(3.5.2) with this data by Theorem 3.5.1. It is defined in the interval $(0, T_*)$, where T_* is restricted by the data only. The counterpart of the total dissipation balance from (3.5.5), that can be derived exactly as in Section 3.5.1, provides the following uniform bounds:

$$\sup_{t\in I} \|\partial_t \eta_\delta\|^2_{L^2(\omega)} + \sup_{t\in I} \|\eta_\delta\|^2_{W^{2,2}(\omega)} \leq c, \tag{3.5.48}$$

$$\sup_{t\in I} \|\varrho_\delta\|^\gamma_{L^\gamma(\Omega_{\eta_\delta})} + \sup_{t\in I} \delta \|\varrho_\delta\|^\beta_{L^\beta(\Omega_{\eta_\delta})} \leq c, \tag{3.5.49}$$

$$\sup_{t\in I} \big\|\varrho_\delta |\mathbf{u}_\delta|^2\big\|_{L^1(\Omega_{\eta_\delta})} + \sup_{t\in I} \|\varrho_\delta \mathbf{u}_\delta\|^{\frac{2\gamma}{\gamma+1}}_{L^{\frac{2\gamma}{\gamma+1}}(\Omega_{\eta_\delta})} \leq c, \tag{3.5.50}$$

$$\|\mathbf{u}_\delta\|^2_{L^2(I;W^{1,2}(\Omega_{\eta_\delta}))} \leq c, \tag{3.5.51}$$

$$\sup_{t\in I} \|\vartheta_\delta\|^4_{L^4(\Omega_{\eta_\delta})} + \|\nabla \vartheta_\delta\|^2_{L^2(I\times\Omega_{\eta_\delta})} \leq c, \tag{3.5.52}$$

$$\left\|\frac{\delta(\vartheta_\delta)}{\vartheta_\delta} \nabla \vartheta_\delta\right\|^2_{L^2(I\times\Omega_{\eta_\delta})} \leq c. \tag{3.5.53}$$

Finally, we report the conservation of mass principle

$$\|\varrho_\delta(\tau,\cdot)\|_{L^1(\Omega_{\eta_\delta})} = \int_{\Omega_{\eta_\delta}} \varrho(\tau,\cdot)\,\mathrm{d}x = \int_\Omega \varrho_0\,\mathrm{d}x \quad \text{for all } \tau \in [0,T]. \tag{3.5.54}$$

Hence we may take a subsequence, such that for some $\alpha \in (0,1)$ we have

$$\eta_\delta \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I; W^{2,2}(\omega)) \tag{3.5.55}$$
$$\eta_\delta \rightharpoonup^* \eta \quad \text{in} \quad W^{1,\infty}(I; L^2(\omega)), \tag{3.5.56}$$
$$\eta_\delta \to \eta \quad \text{in} \quad C^\alpha(\overline{I} \times \omega), \tag{3.5.57}$$
$$\mathbf{u}_\delta \rightharpoonup^\eta \mathbf{u} \quad \text{in} \quad L^2(I; W^{1,2}(\Omega_{\eta_\delta})), \tag{3.5.58}$$
$$\varrho_\delta \rightharpoonup^{*,\eta} \varrho \quad \text{in} \quad L^\infty(I; L^\gamma(\Omega_{\eta_\delta})), \tag{3.5.59}$$
$$\vartheta_\delta \rightharpoonup^{*,\eta} \vartheta \quad \text{in} \quad L^\infty(I; L^4(\Omega_{\eta_\delta})), \tag{3.5.60}$$
$$\vartheta_\delta \rightharpoonup^\eta \vartheta \quad \text{in} \quad L^2(I; W^{1,2}(\Omega_{\eta_\delta})). \tag{3.5.61}$$

By Lemma 3.2.10, arguing as in Sections 3.4.2 and 3.5.1, we find for all $q \in (1, \frac{6\gamma}{\gamma+6})$ that

$$\varrho_\delta \mathbf{u}_\delta \rightharpoonup^\eta \varrho \mathbf{u} \quad \text{in} \quad L^2(I, L^q(\Omega_{\eta_\delta})) \tag{3.5.62}$$
$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to^\eta \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{in} \quad L^1(I; L^1(\Omega_{\eta_\delta})). \tag{3.5.63}$$
$$\sqrt{\varrho_\delta} \mathbf{u}_\delta \to^\eta \sqrt{\varrho} \mathbf{u} \quad \text{in} \quad L^1(I; L^1(\Omega_{\eta_\delta})). \tag{3.5.64}$$

As in Section 3.5 we also obtain again

$$\mathbf{S}(\vartheta_\delta, \nabla \mathbf{u}_\delta) \rightharpoonup^\eta \overline{\mathbf{S}} \quad \text{in} \quad L^{4/3}(I, L^{4/3}(\Omega_{\eta_\delta})) \tag{3.5.65}$$
$$K'(\eta_\delta) \rightharpoonup^* \overline{K}' \quad \text{in} \quad L^\infty(I; W^{-2,r}(\omega)) \tag{3.5.66}$$

for any $r < 2$ with some limit objects $\overline{\mathbf{S}}$ and $\overline{K}$. As before in Proposition 3.5.3 we have higher integrability of the density (see [22, Lemma 7.3] for the proof).

Lemma 3.5.6. *Let $\gamma > \frac{3}{2}$ ($\gamma > 1$ in two dimensions). Let $Q = J \times B \Subset I \times \Omega_\eta$ be a parabolic cube and $0 < \Theta \le \frac{2}{3}\gamma - 1$. The following holds for any $\delta \le \delta_0(Q)$*

$$\int_Q p_\delta(\varrho_\delta, \vartheta_\delta) \varrho_\delta^\Theta \, \mathrm{d}x \, \mathrm{d}t \le C(Q) \tag{3.5.67}$$

with constant independent of δ.

Similarly to [22, Lemma 7.4] we can exclude concentrations of the pressure at the moving boundary. Here, we need the assumption $\gamma > \frac{12}{7}$.

Lemma 3.5.7. *Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that we have for all $\delta \le \delta_0$*

$$\int_{I \times \mathbb{R}^3 \setminus A_\kappa} p_\delta(\varrho_\delta, \vartheta_\delta) \chi_{\Omega_{\eta_\delta}} \, \mathrm{d}x \, \mathrm{d}t \le \kappa. \tag{3.5.68}$$

Lemma 3.5.6 and Lemma 3.5.7 imply equi-integrability of the sequence $p_\delta(\varrho_\delta, \vartheta_\delta)\chi_{\Omega_{\eta_\delta}}$. This yields the existence of a function $\overline{p}$ such that (for a subsequence)

$$p_\delta(\varrho_\delta, \vartheta_\delta) \rightharpoonup \overline{p} \quad \text{in} \quad L^1(I \times \mathbb{R}^3), \tag{3.5.69}$$
$$\delta \varrho_\delta^\beta \to 0 \quad \text{in} \quad L^1(I \times \mathbb{R}^3). \tag{3.5.70}$$

Similarly to Corollary 3.5.5 we have the following.

Corollary 3.5.8. *Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that*

$$\int_{I \times \mathbb{R}^3 \setminus A_\kappa} \overline{p} \, \mathrm{d}x \, \mathrm{d}t \le \kappa. \tag{3.5.71}$$

Using (3.5.69) and the convergences (3.5.55)–(3.5.66) we can pass to the limit in (3.5.1) and (3.5.2) and obtain

$$\begin{aligned}&\int_I \frac{\mathrm{d}}{dt}\int_{\Omega_\eta}\varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x-\int_{\Omega_\eta}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\phi}+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_I\int_{\Omega_\eta}\overline{\mathbf{S}}:\nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\overline{p}\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\&+\int_I\bigg(\frac{\mathrm{d}}{dt}\int_\omega\partial_t\eta b\;\mathrm{d}y-\int_\omega\partial_t\eta\,\partial_t b\;\mathrm{d}y+\int_\omega\overline{K}'\,b\;\mathrm{d}y\bigg)\,dt\\&=\int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t+\int_I\int_\omega g\,b\;\mathrm{d}x\,dt\end{aligned}\tag{3.5.72}$$

for all test-functions $(b,\boldsymbol{\phi})$ with $\mathrm{tr}_\eta\boldsymbol{\phi}=\partial_t\eta\nu$, $\boldsymbol{\phi}(T,\cdot)=0$ and $b(T,\cdot)=0$. Moreover, the following holds

$$\int_I\int_{\Omega_\eta}\varrho\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\mathrm{div}(\varrho\,\mathbf{u})\,\psi\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_{\eta_0}}\varrho_0\,\psi(0,\cdot)\,\mathrm{d}x\tag{3.5.73}$$

for all $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3)$. It remains to show strong convergence of ϑ_δ, ϱ_δ and $\nabla^2\eta_\delta$. As in the last section the proof of the convergence of ϑ_δ is entirely based on local arguments. Consequently the shell is not seen and we can follow the arguments in [63, Chapter 3, Section 3.7.3] to conclude

$$\vartheta_\delta\to^\eta\vartheta\quad\text{in}\quad L^4(I;L^4(\Omega_\delta)).\tag{3.5.74}$$

Consequently we have $\overline{\mathbf{S}}=\mathbf{S}(\vartheta,\nabla\mathbf{u})$ in (3.5.72). Moreover, we can pass to the limit in the entropy balance and obtain (O3). Next we aim to prove strong convergence of the density. We define the L^∞-truncation

$$T_k(z):=k\,T\Big(\frac{z}{k}\Big)\quad z\in\mathbb{R},\ k\in\mathbb{N}.\tag{3.5.75}$$

Here T is a smooth concave function on $\mathbb{R}$ such that $T(z)=z$ for $z\le 1$ and $T(z)=2$ for $z\ge 3$. Now we have to show that

$$\begin{aligned}&\int_{I\times\Omega_{\eta_\delta}}\big(a\varrho_\delta^\gamma+\delta\varrho_\delta^\beta-(\lambda(\vartheta)+2\mu(\vartheta))\mathrm{div}\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t\\&\qquad\longrightarrow\int_{I\times\Omega_\eta}\big(\overline{p}-(\lambda(\vartheta)+2\mu(\vartheta))\mathrm{div}\mathbf{u}\big)\,T^{1,k}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.5.76}$$

For this step we are able to use the theory established in [130] on a local level. Similarly to [22, Subsection 7.1] (see [63, Chapter 3, Section 3.7.4] about how to include the temperature) we first prove a localised version of (3.5.76) and then use Lemma 3.5.7 and Corollary 3.5.8 to deduce the global version. The next aim is to prove that ϱ is a renormalized solution (in the sense of Definition 3.2.13). In order to do so it suffices to use the continuity equation and (3.5.76) again on the whole space. Following line by line the arguments from [22, Subsection 7.2] we have

$$\partial_t T^{1,k}+\mathrm{div}\big(T^{1,k}\mathbf{u}\big)+T^{2,k}=0\tag{3.5.77}$$

in the sense of distributions on $I\times\mathbb{R}^3$. Note that we extended ϱ by zero to $\mathbb{R}^3$. The next step is to show

$$\limsup_{\delta\to0}\int_{I\times\mathbb{R}^3}|T_k(\varrho_\delta)-T_k(\varrho)|^q\,\mathrm{d}x\,\mathrm{d}t\le C,\tag{3.5.78}$$

where C does not depend on k and $q>2$ will be specified later. The proof of (3.5.78) follows exactly the arguments from the classical setting with fixed boundary (see [63, Chapter 3, Section 3.7.5]) using (3.5.76) and the uniform bounds on $\mathbf{u}_\delta$ (with the only exception that we do not localise). Using (3.5.78) and arguing as in [22, Sec. 7.2] we obtain the renormalised continuity equation. As in [22, Sec. 7.3] we can use the latter one to show strong convergence of the density. The limit passage in the shell energy can be performed using the method from Section 3.5. At this stage the estimates become absolutely critical. The crucial points are to show that

$$\int_{\Omega_{\eta_\delta}} \varrho_\delta \mathbf{u}_\delta \cdot \partial_t \big(\mathscr{F}_{\eta_\delta}(b) \big) \,\mathrm{d}x \in L^\chi(I)$$

for b smooth and some $\chi > 1$ uniformly in δ as well as the uniform bounds of

$$\int_I \int_{\Omega_{\eta_\delta}} \varrho_\delta \mathbf{u}_\delta \cdot \partial_t \Big(\psi \mathscr{F}^{\mathrm{div}}_{\eta_\delta} (\Delta_h^{-s}(\Delta_h^s(\eta_\delta))) \Big) \,\mathrm{d}x \,\mathrm{d}t$$

for some positive $s > 0$. As seen in Section 3.5 it requires the bound $\gamma > \frac{12}{7}$, cf. (3.5.62). This finishes the proof of Theorem 3.2.14 for the time interval $[0, T_*]$, with T_* depending on the data only (such that $\|\eta(t)\|_\infty < \frac{L}{2}$ in $(0, T^*)$). As in [22, Sec. 7.4] the existence interval can be extended until a self intersection is reached which finishes the proof of Theorem 3.2.14.

Chapter 4

Weak-strong uniqueness for an elastic plate interacting with the Navier Stokes equation

4.1 Introduction

The chapter investigates the interaction between an elastic solid plate and a viscous incompressible fluid. For the fluid we will consider the three (or two) dimensional *Navier-Stokes equations* [69, 129]. For the solid we consider a shell or a plate that is modeled as a thin object of one dimension less than the fluid and which is assumed to be fixed on the top of a container (See Figure 1). For modeling on *elastic plates* see [37, 38] and the references therein. The fluid and the plate interact via a kinematic and a dynamic coupling condition on the moving interface.

The main result consists in the *weak-strong uniqueness* of solutions for a flow in a variable 3D (or 2D) domain interacting with a 2D (or 1D) plate (see Theorem 4.1.2). While the regularity of the weak solutions that we use are known to be satisfied for all weak solutions we assume additional regularity of the *velocity* of the strong solution, that can be related (via its index) to the celebrated Ladyzhenskaya-Prodi-Serrin conditions [161, 171, 172, 121]. These are conditions for solutions to Navier-Stokes equations in a fixed domain that imply their smoothness and uniqueness.

Please observe, that we do not assume any additional regularity of the solid displacement; in particular the domain of the strong fluid-velocity is not even assumed to be uniformly Lipschitz continuous. In order to handle the limited regularity assumptions (on the strong solution) rather complex estimates where necessary. Some of them depend sensitively on *a-priori estimates* for the solid deformation shown in [142].

To measure the distance between two solutions it is necessary to introduce a change of variables as the domains of the two velocity fields depend on the solution itself. Moreover, since the solid deformation is governed by a hyperbolic equation a mollification in time is unavoidable. In this paper a methodology is introduced that overcomes both obstacles with operators that conserve the property of solenoidality (see Lemma 4.2.6).

While the existence theory for weak solutions describing flexible (thin) shells interacting with fluids has been flourishing in the past years [52, 53, 19, 87, 70, 144, 127, 126, 153, 147, 88, 22, 142] the uniqueness and stability questions are rather untouched. The only available result for an elastic plate seems to be the work of [92]; it treats a 1D elastic beam interacting with a 2D fluid whith slip-boundary conditions at the interface.[1] Otherwise, the only weak-strong uniqueness results for fluid-structure interactions are for non-elastic solids, namely rigid objects [177, 84, 32, 20]. For fluid-structure interactions involving elastic materials there are some existence results where the uniqueness of strong solutions (in the class of strong solutions) is inherited from the methodology of existence. These are short time uniqueness results for strong solutions [43, 44, 12, 17, 89], global uniqueness results of strong solutions for small data [36, 111] and the uniqueness for arbitrary times of strong solutions for a 1D visco-elastic plate interacting with a 2D fluid [88]. As a consequence of our estimates

[1] Actually some conditions in [92] could be missing, as the estimate in formula (6.33) on page 25 seems sensitively incorrect. The estimate would only be correct if the distributional time-derivative was in the dual of a Sobolev space and not merely in the dual of its solenoidal subspace.

all constructed strong solutions (involving elastic plates) are unique within the class of *weak solutions*.

The *applications* within this framework consist in fluids interacting with various thin materials. Of particular interest are those in medicine and biology for arteries or the trachea [162, 15, 108]. These fields relay strongly on robust computer simulations, many of which are built along the concept of *weak solutions* [97, 189, 165]. Stability results as the one presented here are very suitable to be adapted to such *numerical approximations*. We plan to perform that in a future paper.

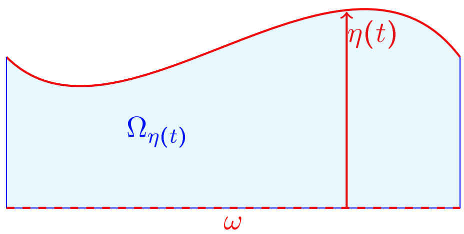


Figure 4.1: 1D plate interacting with a 2D fluid

4.1.1 Formulation of the problem

We consider a $3D$ container whose top wall consist of a $2D$ Koiter type plate (or a $2D$ container whose walls consist of a $1D$ Koiter type plate). As is common for the analysis on plates we assume that the plate can move only upwards and downwards. The deformation of the plate is described by a bounded function $\eta : [0,T] \times \omega \to (\delta, \infty)$ for some time interval $[0,T]$, some bounded domain $\omega \subset \mathbb{R}^2$ (or $\omega \subset \mathbb{R}$) that has a Lipschitz boundary and some $\delta \in (0,1)$. The time-dependent fluid domain is defined by

$$\Omega_\eta(t) := \{(x,y) \in \omega \times (0,\infty) : 0 \le y \le \eta(t,x)\},\ t \in [0,T].$$

Here and in the following x denotes a $2D$ (or $1D$), y a $1D$ and $z = (x,y)$ a $3D$ (or $2D$) variable. With some misuse of notation we consider the space-time domain

$$[0,T] \times \Omega_\eta(t) := \bigcup_{t\in[0,T]} \{t\} \times \Omega_\eta(t).$$

The motion of the fluid is described by the incompressible Navier-Stokes equations

$$\rho_f(\partial_t v + [\nabla v]v) = \mu_f \Delta v - \nabla p + \rho_f f \qquad \text{on } [0,T] \times \Omega_\eta(t), \tag{4.1.1}$$

$$\mathrm{d}v = 0 \qquad \text{on } [0,T] \times \Omega_\eta(t), \tag{4.1.2}$$

where the fluid's velocity field v and the pressure p are the unknown quantities, ρ_f is the fluid density, μ_f the fluid viscosity and f is a given outer force (e.g. gravity). By $\sigma(v,p) = 2\mu_f \varepsilon v - p\mathbb{I}$ we denote the fluid stress tensor, where $\varepsilon v := \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the symmetric part of the gradient and $\mathbb{I}$ denotes the identity matrix in 3D, (2D). The incompressibility condition implies that the pressure is determined by the velocity field. On the non-moving parts of the container $B_c = \omega \times \{0\} \cup \partial\omega \times [0,1]$ we assume no-slip boundary conditions

$$v = 0 \text{ on } [0,T] \times B_c. \tag{4.1.3}$$

The moving part of the shell satisfies a linearized plate equation of Koiter type with a source term stemming from the forces the fluid exerts on the shell

$$\rho_s h_0 \partial_t^2 \eta + \mathcal{L}(\eta, \partial_t \eta) = \mathbf{F}(u,p,\eta) + \rho_s g,\ \text{on } [0,T] \times \omega, \tag{4.1.4}$$

with Dirichlet boundary conditions

$$\eta = 1,\ \nabla\eta = \Delta\eta = 0 \text{ on } (0,T) \times \partial\omega. \tag{4.1.5}$$

Here η is the (scalar valued) unknown deformation, ρ_s is the solid density, h_0 is the thickness of the plate, $\mathcal{L}(\eta, \partial_t \eta)$ is the L^2 gradient of the elastic and dissipative potentials of the deformation of the plate, $\mathcal{F}$ are forces stemming from the fluid and g is a given outer force. Due to the troubles between hyperbolic equations and non-linearities we have to assume that $\mathcal{L}(\eta, \partial\eta)$ is of the following form

$$\mathcal{L}(\eta, \partial_t \eta) := \alpha \Delta^2 \eta - \tilde{\beta} \Delta \eta - \tilde{\gamma} \Delta \partial_t \eta$$

with $\alpha > 0$ and $\tilde{\beta}, \tilde{\gamma} \geq 0$. Note that the equations for the fluid are stated in Eulerian coordinates while the equations for the solid are stated in Lagrangian coordinates.

The fluid and the shell are coupled via a kinematic and a dynamic coupling condition on the moving interface. For expressing the coupling condtions we define the variable transform from Langrangian to Eulerain coordinates

$$\psi : [0, T] \times \omega \to [0, T] \times \mathbb{R}^3, \quad (t, x) \mapsto (t, x, \eta(t, x)).$$

The dynamic coupling condition states that the total force in normal direction at the interface is zero

$$\mathbf{F}(v, \eta, p) = -(0, 1)^t ((\nabla v - p\mathbb{I}) \circ \psi) n \cdot n \text{ on } [0, T] \times \omega, \tag{4.1.6}$$

where $n(t, x) = (-\nabla \eta, 1)/(1 + |\nabla \eta|^2)^{\frac{1}{2}}$ is the outer normal of $\Omega_\eta(t)$ at the point $(x, \eta(x))$.

We assume a no slip kinematic boundary condition, i.e. the fluid and the structure velocity are equal at the interface

$$v \circ \psi = (0, \partial_t \eta)^T \text{ on } [0, T] \times \omega, \tag{4.1.7}$$

To complete the equations we impose initial conditions

$$v(0) = v_0 \text{ on } \Omega_\eta(0), \tag{4.1.8}$$

$$\eta(0) = \eta_0, \ \partial_t \eta(0) = \eta^* \text{ on } \omega. \tag{4.1.9}$$

Within the chapter we will refer to (4.1.1)-(4.1.9) as *FSI*.

By formally multiplying equation (4.1.1) by v, (4.1.4) by $\partial_t \eta$ and integrating over $\Omega_\eta(t)$, ω and $(0, t)$ we get (using Korn's identity Lemma 4.2.1 and Absorption) the energy inequality

$$\begin{aligned} &\|v(t)\|^2_{L^2(\Omega_{\eta(t)})} + \|\partial_t \eta(t)\|^2_{L^2(\omega)} + \|\nabla^2 \eta(t)\|^2_{L^2(\omega)} + \int_0^t \|\nabla v(\tau)\|^2_{L^2(\Omega_{\eta(\tau)})} d\tau \\ &\leq c\left(\|v_0\|^2_{L^2(\Omega_{\eta_0})} + \|\eta_1\|^2_{L^2(\omega)} + \|\nabla^2 \eta_0\|^2_{L^2(\omega)} + + \int_0^t \|f(\tau)\|^2_{L^2(\Omega_{\eta(\tau)})} + \|g(\tau)\|_{L^2(\omega)} \, d\tau \right). \end{aligned} \tag{4.1.10}$$

In the paper we use the standard notation for Lebesgue and Sobolev spaces. The weak solutions to *FSI* are defined in the following function spaces.

$$\begin{aligned} \mathcal{V}_\eta(t) &= \{v \in H^1(\Omega_\eta(t)) : \mathrm{d}v = 0 \text{ in } \Omega_{\eta(t)}, v = 0 \text{ on } B_c\}, \\ \mathcal{V}_F &= L^\infty((0, T), L^2(\Omega_\eta(t)) \cap L^2((0, T), \mathcal{V}_\eta(t)), \\ \mathcal{V}_K &= W^{1,\infty}([0, T], L^2(\omega)) \cap \{\eta \in L^\infty([0, T], H^2(\omega)) : \eta = 1, \ \nabla \eta = \Delta \eta = 0 \text{ on } \partial\omega\} \\ \mathcal{V}_S &= \{(v, \eta) \in \mathcal{V}_F \times \mathcal{V}_K : v \circ \psi = \partial_t \eta\}, \\ \mathcal{V}_T &= \{(w, \xi) \in \mathcal{V}_F \times \mathcal{V}_K : w \circ \psi = \xi, \ \partial_t w \in L^2(0, T; L^2(\Omega_\eta(t)))\} \end{aligned}$$

For the distributional time derivative we introduce the following space

$$\tilde{W}^{-l,p'}(\Omega) := (\{f \in W^{1,p}(\Omega) \ : \ f = 0 \text{ on } B_c\})^*.$$

Definition 4.1.1. *Let $f \in L^2([0, T] \times \omega \times \mathbb{R})$, $g \in L^2([0, T] \times \omega$, $\eta_0 \in H^2_0(\omega)$, $\eta^* \in L^2(\omega)$ and $v_0 \in L^2(\Omega_{\eta_0})$. Then we call a pair $(v, \eta) \in \mathcal{V}_S$ a weak solution to FSI if it satisfies the energy inequality* (4.1.10)*, if*

$$\begin{aligned} &\frac{d}{dt}\left(\rho_f \int_{\Omega_\eta(t)} v \cdot w dz\right) - \rho_f \int_{\Omega_\eta(t)} v \cdot \partial_t w - 2\mu \varepsilon v : \varepsilon w + \rho_f (v \otimes v) : \nabla w dz \\ &+ h_0 \rho_s \partial_t \left(\int_\omega \partial_t \eta \xi \, dx\right) - h_0 \rho_s \int_\omega \partial_t \eta \xi_t \, dx + \langle \mathcal{L}(\eta, \partial_t \eta), \xi \rangle = \rho_f \int_{\Omega_\eta} f \cdot w \, dz + \rho_s \int_\omega g \xi \, dx \end{aligned} \tag{4.1.11}$$

for all $(w,\xi)\in\mathcal{V}_T$ as an equation in $\mathcal{D}'(0,T)$ and if it attains the initial conditions in the sense of the L^2 weak convergence.

4.1.2 Main results

The main result of the chapter are the following.[2]

Theorem 4.1.2. *In case that $\omega\subset\mathbb{R}^2$ let $r>2$ and $s>3$ and in case that $\omega\subset\mathbb{R}$ let $r=2$ and $s=2$. Assume that (v_2,η_2) is a weak solutions to FSI on $[0,T]$, such that $\min_{[0,T]\times\omega}\eta_2>0$ and additionally that $v_2\in L^r(0,T;W^{1,s}(\Omega_{\eta_2}))$ and $\partial_t v_2\in L^2(0,T;\tilde{W}^{-1,r}(\Omega_{\eta_2}))$. Then this solution is unique in the class of weak solutions.In particular, if (v_1,η_1) is any weak solution to FSI on $[0,T_0]$ (for any $T_0>0$) and if $v_1(0)=v_2(0)$, $\eta_1(0)=\eta_2(0)$, $\partial_t\eta_1(0)=\partial_t\eta_2(0)$, than $(v_1,\eta_1)\equiv(v_2,\eta_2)$ as an equation in $\mathcal{V}_S$ on $[0,T_0]$.*

In some situations strong solutions are known to exist. In particular, in the case of $\omega=[0,L]$ and $\tilde{\gamma}>0$ strong solutions exist for arbitrary times [88]. This means that our result implies the following corollary.

Corollary 4.1.3. *In the 2D case ($\omega=[0,L]$) with $\tilde{\gamma}>0$ and smooth initial values, there exists a strong solution to FSI which is unique in the class of weak solutions.*

Remark 4.1.4 (Minimality of the regularity assumptions on v_2.)**.** *Let us compare our assumptions to the case of a non-moving domain for a 3D fluid; i.e. $\eta\equiv\eta_c$ and therefore $\Omega_{\eta_c}\subset\mathbb{R}^3$ is constant in time and v_1, $v_2\in\mathcal{V}_F$ are weak (Leray-Hopf) solutions. If additionally v_2 satisfies the Ladyzhenskaya-Prodi-Serrin condition, namely $v_2\in L^r(0,T;L^q(\Omega))$ for $\frac{3}{q}+\frac{2}{r}=1$, then from the well known regularity and uniqueness result [161, 171, 172, 121] on the Navier-Stokes equations it follows:*

$$\|w(t)\|^2\le C\|w(0)\|\exp\left(c\int_0^t\|v_2\|_{L^q}^r dy\right)$$

which in particular, implies the weak-strong uniqueness. In order to obtain the above estimate a regularity theory *for solutions satisfying the Ladyzhenskaya-Prodi-Serrin condition is used.*

For the here considered fluid-structure interactions a regularity theory for weak solutions satisfying the Ladyzhenskaya-Prodi-Serrin condition is not known to be satisfied up to date. Actually, it is debatable whether such a theory can expected to be true. (This counts even for 2D fluid-structure interactions in case when $\tilde{\gamma}=0$.) However, the borders for the exponents in our assumptions have the same index as in the exponents in the Ladyzhenskaya-Prodi-Serrin condition. We briefly explain this here: We assume in 3D that the stronger solution satisfies $v_2\in L^r(0,T;W^{1,s}(\Omega))$ for some $s>3$ and $r>2$. As $W^{1,s}(\Omega)\hookrightarrow L^\infty(\Omega)$ for all $s>3$ the corresponding borderline exponent for v_2 is the one of $L^2(0,T;L^\infty(\Omega))$ which has the index $\frac{3}{q}+\frac{2}{r}=\frac{3}{\infty}+\frac{2}{2}=1$.

Please observe that in $2D$ no further assumption on the gradient are necessary beyond its energy estimate. Our stronger assumptions on the weak time-derivative are necessary both in 2D and 3D. This is again due to the fact that a regularity theory for solutions satisfying the Ladyzhenskaya-Prodi-Serrin condition might not be valid. While the bounds on the index *for the spaces we request for the weak time-derivative (of the stronger solution) are in coherence with weak solution, we have to assume that the negative space is considerably smaller; i.e. the dual of the Sobolev space and not the dual of its solenoidal subspace.*

Further we prove the following stability estimate.

Theorem 4.1.5. *Let (v_2,η_2) be weak solutions to FSI on $[0,T]$, such that $\min_{[0,T]\times\omega}\eta_2>0$ and that additionally $v_2\in L^r(0,T;W^{1,s}(\Omega_{\eta_2}))$ and $\partial_t v_2\in L^2(0,T;\tilde{W}^{-1,r}(\Omega_{\eta_2}))$ for any $s>3$ and any $r>2$. If (v_1,η_1) is a*

[2] For the notation please see the next section.

weak solution to FSI on $[0,T]$*, then for* $\tilde{v_2}(t,x,y) = v_2(t,x,y\frac{\eta_1(t,x)}{\eta_2(t,x)})$ *we find that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|(v_1-\tilde{v}_2)(t)\|^2_{L^2(\Omega_{\eta_1(t)})} + \|\partial_t(\eta_1-\eta_2)(t)\|^2_{L^2(\omega)} + \|(\eta_1-\eta_2)(t)\|^2_{H^2(\omega)} \\
&\quad + \int_0^T \|(v_1-\tilde{v}_2)(\tau)\|^2_{H^1(\Omega_{\eta_1(\tau)})} d\tau \\
&\le C(\|v_1^0-\tilde{v}_2^0\|^2_{L^2(\Omega_{\eta_1^0})} + \|\eta_1^*-\eta_2^*\|^2_{L^2(\omega)}) + \|(\eta_1^0-\eta_2^0)\|^2_{H^2(\omega)} \\
&\quad + C\int_0^T \|(f_1-\tilde{f}_2)(\tau)\|^2_{H^1(\Omega_{\eta_1(\tau)})} + \|(g_1-g_2)(\tau)\|^2_{L^2(\omega)} d\tau,
\end{aligned}$$

where the constant depends on ω, T*, the assumed bounds on* v_2*, the* L^2*-bounds of* f_1, f_2 *and (symmetrically) on the two deformations* η_1, η_2 *via the bounds related to the energy estimates and via Theorem 4.2.2.*

In particular, the constant C can be bounded a-priori in dependence of ω, T, the assumed bounds on v_2 and the right hand side of the energy inequality (4.1.10) for both solutions.

4.2 Notation & preliminary results

4.2.1 Simplifications

In order to simplify the quite technical argument below we assume in the following that $\mathcal{L}(\eta, \partial_t\eta) \equiv \Delta^2\eta$; as the argument can be adapted to more general $\mathcal{L}$ in a straight forward manner. Moreover we will assume in the following that we have a fluid in $3D$. In particular we assume that $\omega \subset \mathbb{R}^2$. The adaption of the proof for $\omega \subset \mathbb{R}$ implies only simplifications and no further complications. Finally we set all constants in the equations to one (i.e. both densities, the thickness of the plate, the viscosity of the fluid).

For vector valued functions $u : \Omega_\eta \to \mathbb{R}^3$ we use $u = (u', u^3)^T = (u^1, u^2, u^3)^T$. The constants $c, c_1, ...$ are used as a constants that are independent of η, while the constants $C, C_1, ...$ are used as constants that may depend on bounded quantities of the deformations. Both letters c, C may change there actual value with every instance. Moreover, we use the notation $a \sim b$, if there are constants c, c_1 such that $|a| \le c|b| \le c_1|a|$.

4.2.2 Identities & Estimates

We will use Reynold's transport theorem which for plates reads (using the fact that the third component of the outer normal times the Jacobian of the change of variables is one) as for all $u \in W^{1,1}(0,T;\Omega_\eta)$ with $u'(x,\eta(x)) = 0$ for all x, we find

$$\partial_t\bigg(\int_{\Omega_\eta} u(t,z)\cdot\phi(t,z)\,dz\bigg) = \int_{\Omega_\eta} \partial_t(u\cdot\phi)\,dz + \int_\omega u^3(t,x,\eta(x))\phi^3(t,x,\eta(x)))\partial_t\eta(t,x)\,dx,$$

for all ϕ, η for which the above expression is well defined.

Next due to the zero boundary conditions of v' on $\partial\Omega$ we actually may use Korn's identity which is done throughout the paper.

Lemma 4.2.1. *Let* $u \in H^1(\Omega_\eta)$ *such that* $u = 0$ *on* B_c *and* $u'(x,\eta(x)) = 0$*, than*

$$\|u\|_{H^1(\Omega_\eta)} \sim \|\nabla u\|_{L^2(\Omega_\eta)} = 2\|\varepsilon u\|_{L^2(\Omega_\eta)}.$$

Proof. The fact that $\|u\|_{H^1(\Omega_\eta)} \sim \|\nabla u\|_{L^2(\Omega_\eta)}$ follows by Poincaré's inequality as all components have zero boundary values on large parts of the boundary and the inequality is a straight consequence of the fundamental theorem of calculus. Korn's identity follows by [142, Lemma 4.1]. □

Our proof makes use of the following additional regularity result that has been shown in [142, Theorem 1.2]:

Theorem 4.2.2. *For any weak solution to FSI we find that as long as $\eta > 0$ in $[0,T]\times\omega$ that $\eta \in L^2(0,T;H^{2+\sigma}(\omega))$ and $\partial_t\eta \in L^2(0,T;H^\sigma(\omega))$ for all $\sigma < \frac{1}{2}$.*

An adaption of [142, Theorem 1.2] is the following corollary.
We will need the following interpolation estimate:

1. $L^{a'}(L^a) \subset L^\infty(L^1) \cap L^2(L^2)$ for all $a \in [1,2]$.

Lemma 4.2.3. *For $Y \subset \mathbb{R}^2$. If $b \in L^\infty(0,T;L^2(Y))$ and $\phi \in L^2(0,T;W^{1,a}(Y))$ for all $a \in (1,2)$, then $|b||\phi| \in L^2(0,T;L^p(Y))$ for all $p \in (1,2)$.*

Proof. The result follows by Sobolev embedding and Hölder's inequality. □

Very often we will have the product of a function defined on ω with a function defined on Ω_η. We will integrate such products over Ω_η where one of the two functions is than constant in the variable direction. In some cases this allows to improve the regularity. In particular we will need the following extra information on the weak solution that will be used upon the convective term:

Lemma 4.2.4. *Let (η, v) be a weak solution to FSI. Then we find that $\int_0^{\eta(t,x)} |v|\,dy \in L^2(0,T;H^1(\omega))$*

$$\left\| \int_0^{\eta(t,x)} |v|\,dy \right\|_{L^2(0,T;H^1(\omega))} \le c\|v\|_{L^2(0,T;H^1(\Omega_\eta))}\|\eta\|_{L^\infty(0,T;H^2(\omega))}$$

and $\int_0^{\eta(t,x)} |v|^2\,dy \in L^2(0,T;W^{1,1}(\omega))$

$$\begin{aligned}\left\| \int_0^{\eta(t,x)} |v(t)|^2\,dy \right\|_{L^2(0,T;W^{1,1}(\omega))} &\le \|v\|_{L^2(0,T;L^2(\Omega_\eta))} + 2\|v\|_{L^\infty(0,T;L^2(\Omega_\eta))}\|\nabla v\|_{L^2([0,T]\times\omega)} \\ &\quad + \|\partial_t\eta\|^2_{L^\infty(0,T;L^2(\omega))}\|\nabla\eta\|_{L^1(0,T;L^\infty(\omega))}.\end{aligned}$$

This implies in particular that $\int_0^{\eta(t,x)} |v|^2\,dy \in L^2([0,T]\times\omega)$.

Proof. For the first statement we calculate

$$\nabla_x \int_0^{\eta(t,x)} |v(x,y)|\,dy = \int_0^{\eta(t,x)} \nabla_x|v(x,y)|\,dy + \nabla_x\eta(t,x)|\partial_t\eta|$$

which is uniformly bounded in $L^2([0,T]\times\omega)$ since $v_1 \in L^2(0,T;H^1(\Omega_\eta))$, $\partial_t\eta \in L^2(0,T;L^3(\omega))$ and $\nabla\eta \in L^\infty(0,T;L^6(\omega))$. The estimate follows using Sobolev embedding and the trace Theorem [22, Lemma 6].

For the second statement we calculate

$$\begin{aligned}\nabla_x \int_0^{\eta(t,x)} |v|^2\,dy &= \int_0^{\eta(t,x)} 2[\nabla v]v\,dy + |v(\eta(t,x))|^2\nabla\eta(t,x) \\ &= \int_0^{\eta(t,x)} 2[\nabla v]v\,dy + |\partial_t\eta(t,x)|^2\nabla\eta(t,x) =: I_1 + I_2\end{aligned}$$

Due to Holeder's inequality

$$\int_\omega I_1 \le 2\|v\|_{L^2(\Omega_\eta)}\|\nabla v\|_{L^2(\Omega_\eta)}.$$

And it is also straightforward to see

$$\int_\omega I_2 \le \|\partial_t\eta\|^2_{L^2(\omega)}\|\nabla\eta\|_{L^\infty(\omega)}.$$

Thus the statement follows since $v \in L^\infty(0,T;L^2(\Omega_\eta)) \cap L^2(0,T;H^1(\Omega_\eta))$, $\partial_t\eta \in L^\infty(0,T,L^2(\omega))$ and by Theorem 4.2.2 $\eta \in L^2(0,T;H^{2+\sigma}(\omega)) \hookrightarrow L^2(0,T;W^{1,\infty}(\omega))$ for all $\sigma > 0$. □

4.2.3 Convolution

Since the regularity in space of $\partial_t \eta$ and the regularity in time for v a a test function is formally not sufficient to use the couple as a test function we have to introduce a mollification in time. Unfortunately, it was not possible to use the mollification introduced [142] and we have to introduce a new version. Already here the *regularity of the deformation influences the regularity of the mollification sensitively* due to the fact that a change of variables will be a part of the convolution kernel.

First a technical Lemma. Here we will use a mollifier with respect to time. As is the standard procedure, choose a function $j \in C_0^\infty(\mathbb{R})$ which is positive, even, has support in $(-1,1)$ and satifies $\int_{\mathbb{R}} j\, dt = 1$, $\frac{d}{dt} j(-t) \geq 0$, $\frac{d}{dt} j(t) \leq 0$ for $t \geq 0$. For $\delta > 0$ define $j_\delta(t) \equiv \delta^{-1} j(t/\delta)$. Then j_δ has support in $(-\delta,\delta)$ and otherwise the same properties as j.

Let $(H,(\cdot,\cdot))$ be a Hilbert space, $T > 0$. Let $u \in L^\infty(0,T;H)$ be continues w.r.t. the weak topology on H and assume that the limits $u(0) := \lim_{t\to 0} u(t)$, $u(T) := \lim_{t\to T} u(t)$ exist in the weak topology of H. In the following we will call the space of all such functions $C_w(0,T;H)$. Define the extension $\bar{u} \in L^\infty(\mathbb{R},H)$ by

$$\bar{u}^T(t) = \begin{cases} u(t), & t \in (0,T), \\ u(0), & t \in (-\infty,0], \\ u(T), & t \in [T,\infty]. \end{cases} \tag{4.2.1}$$

Now for all $\delta > 0$, $t \in [0,T]$ set

$$u_\delta^T(t) = \int_{\mathbb{R}} j_\delta(\tau - s)\bar{u}^T(s) ds.$$

It is well known that $u_\delta^T \in C^\infty([0,T],H)$ and $\lim_{\delta\to 0} u_\delta = u$ in $L^p(0,T;H)$ for all $1 \leq p < \infty$. Furthermore the following holds

Lemma 4.2.5. *Let $u, v \in C_w(0,T;H)$ and $t \in (0,T]$. Then for all $t \in [0,T]$*

$$\lim_{\delta\to 0} \int_0^t (u, v_\delta^T) - (u_\delta^T, v) d\tau = 0 \tag{4.2.2}$$

and

$$\lim_{\delta\to 0} \int_0^T \left(u, \frac{d}{dt} v_\delta^T\right) + \left(\frac{d}{dt} u_\delta^T, v\right) d\tau = (u(T), v(T)) - (u(0), v(0))$$

Proof. In the following we omit the superscript T. The first assertion holds since

$$(u, v_\delta) - (u_\delta, v) = (u, v_\delta - v) + (v, u - u_\delta).$$

and the weak continuity in time.

To prove the second assertion note that $\partial_t j_\delta$ is an odd function and therefore

$$\int_0^T \int_0^T \frac{d}{d\tau} j_\delta(\tau - s)(v(s), u(\tau))\, ds d\tau = -\int_0^T \int_0^T \frac{d}{d\tau} j_\delta(\tau - s)(u(s), v(\tau))\, d\tau ds.$$

Hence

$$\begin{aligned}
&\int_0^T \left(u, \frac{d}{dt} v_\delta\right) + \left(\frac{d}{dt} u_\delta, v\right) d\tau \\
&= \int_0^T \left(u(\tau), \int_{-\infty}^0 \frac{d}{d\tau} j_\delta(\tau - s)\bar{v}(s) ds\right) d\tau + \int_0^T \left(u(\tau), \int_T^\infty \frac{d}{d\tau} j_\delta(\tau - s)\bar{v}(s) ds\right) d\tau \\
&\quad + \int_0^T \left(v(\tau), \int_{-\infty}^0 \frac{d}{d\tau} j_\delta(\tau - s)\bar{u}(s) ds\right) d\tau + \int_0^T \left(v(\tau), \int_T^\infty \frac{d}{d\tau} j_\delta(\tau - s)\bar{u}(s) ds\right) d\tau \\
&:= R_1(\delta) + R_2(\delta) + R_3(\delta) + R_4(\delta).
\end{aligned}$$

By symmetry it suffices to prove $R_1(\delta) \to -\frac{1}{2}(u(0), v(0))$ and $R_2(\delta) \to \frac{1}{2}u(t)v(t)$. As $\bar{v}(s) \equiv v(0)$ for all $s < 0$ and j_δ has support in $(-\delta, \delta)$ we get

$$\begin{aligned} R_1(\delta) &= \int_0^T (v(0), u(\tau)) \int_\tau^\infty \frac{d}{ds} j_\delta(s)\, ds d\tau = \int_0^\delta (v(0), u(\tau)) \int_\tau^\delta \frac{d}{ds} j_\delta(s)\, ds d\tau \\ &= \int_0^\delta (v(0), u(\tau))(j_\delta(\delta) - j_\delta(\tau))\, d\tau = -\frac{1}{\delta} \int_0^\delta (v(0), u(\tau)) j\left(\frac{\tau}{\delta}\right)\, d\tau \\ &= -\int_0^1 (v(0), u(\delta\tau)) j(\tau) d\tau. \end{aligned}$$

By weak continuity we get

$$\lim_{\delta \to 0} (v(0), u(\delta\tau)) j(\tau) = (v(0), u(0)) j(\tau).$$

As $u \in L^\infty(0, T; H)$ we get by dominated convergence

$$\lim_{\delta \to 0} R_1(\delta) = -(v(0), u(0)) \int_0^1 j(\tau)\, d\tau = -\frac{1}{2}(v(0), u(0)).$$

The convergence of $R_2(\delta)$ is analogous. □

Here and in the following we will always consider the extension $\overline{u}$ introduced above implicitly. Meaning, that when ever necessary we extend any function to a global in (positive and negative) time object. In order to treat distributional time derivatives we will use the notation of the dual product over a variable domain by

$$\int_0^T \langle f, \phi \rangle_\eta\, dt := \int_0^T \langle f(t), \phi(t) \rangle_{\Omega_{\eta(t)}}\, dt,$$

where $\langle f, \phi \rangle_{\Omega_{\eta(t)}}$ is the dual product over function spaces over $\Omega_{\eta(t)}$ which are assumed to be bilinear mappings that map into measurable functions in time.

For our case of moving boundaries we will need the following convolution result that allows to con volute with respect to the moving geometry by keeping the solenoidality.

Lemma 4.2.6. *Let $\eta \in \mathcal{V}_K$, such that η is bounded uniformly from below. Let $\phi \in L^\kappa(0, T; L^q(\Omega_\eta)) \cap L^\alpha(0, T; W^{1,a}(\Omega_\eta))$ for some $a > 1$ and $\alpha, \kappa, q \geq 1$. Let $b \in L^2(0, T; L^1(\omega))$ with $\phi(t, x, \eta(x)) = (0, b(t, x))$ on $[0, T] \times \omega$ (in the sense of traces).*

Set $K : [0, T] \times [0, T] \times \mathbb{R} \times \omega \to \mathbb{R}^{3\times 3}$

$$K(s, t, y, x) = \begin{pmatrix} \frac{\eta(s,x)}{\eta(t,x)} & 0 & 0 \\ 0 & \frac{\eta(s,x)}{\eta(t,x)} & 0 \\ -y\partial_{x_1}\left(\frac{\eta(s,x)}{\eta(t,x)}\right) & -y\partial_{x_2}\left(\frac{\eta(s,x)}{\eta(t,x)}\right) & 1 \end{pmatrix}$$

For each $\delta > 0$ define $b_\delta = b * j_\delta$ and

$$\phi_\delta(t, x, y) = \int_0^T K(s, t, y, x) \phi\left(s, x, y\frac{\eta(s, x)}{\eta(t, x)}\right) j_\delta(t - s)\, ds.$$

Then it holds for $\kappa < \infty$ that

$$\mathrm{d}\phi_\delta = 0, \quad \phi_\delta(t, x, \eta(x)) = b_\delta(t, x)$$

and $\phi_\delta \to \phi$ strongly $L^\kappa(0, T; L^p(\Omega_\eta(t)))$ for all $p \in [1, \kappa)$.

Moreover,

1. *if $\phi \in L^2(0, T; W^{1,a}(\Omega_\eta))$ for all $a \in (1, 2)$, then $\phi_\delta \to \phi$ converges weakly in $L^2(0, T; W^{1,p}(\Omega_\eta(t))$ for all $p \in [1, 2)$.*

2. *if $\phi \in L^2(0, T; W^{1,a}(\Omega_\eta))$ for $a > 3$ than $\phi_\delta \to \phi$ converges weakly in $L^2(0, T; H^1(\Omega_\eta(t)))$.*

3. *if* $\phi \in H^1(0,T;\tilde{W}^{-1,p'}(\Omega_\eta)) \cap L^2(0,T;W^{1,a}(\Omega_\eta))$ *for some* $a > 3$ *and some* $p \in (1,2)$ *then* $\partial_t\phi_\delta \to \partial_t\phi$ *converges weakly in* $L^2(0,T;\tilde{W}^{-1,p'}(\Omega_\eta)))$.

Proof. We define

$$\phi(s,t,x,y) = K(s,t,x,y)\phi\left(s,x,y\frac{\eta(s,x)}{\eta(t,x)}\right)$$

If we show that $\mathrm{d}\phi(t,s,x,y) \equiv 0$ then clearly also $\mathrm{d}\phi_\delta = 0$. We get

$$\begin{aligned}\mathrm{d}\phi &= \left(\frac{\eta(s,x)}{\eta(t,x)}\right)\phi^1 + \frac{\eta(s,x)}{\eta(t,x)}\mathrm{d}_x\phi^1 + y\partial_y\phi^1\nabla\left(\frac{\eta(s,x)}{\eta(t,x)}\right)\frac{\eta(s,x)}{\eta(t,x)} - \nabla\left(\frac{\eta(s,x)}{\eta(t,x)}\right)\phi^1\\ &- y\nabla\left(\frac{\eta(s,x)}{\eta(t,x)}\right)\frac{\eta(s,x)}{\eta(t,x)}\partial_y\phi^1 + \frac{\eta(s,x)}{\eta(t,x)}\partial_y\phi^2 = \frac{\eta(s,x)}{\eta(t,x)}(\partial_y\phi^2 + \mathrm{d}_x\phi^1) = 0,\end{aligned}$$

where we used in the last line that $\mathrm{d}\phi = 0$. Now as $\phi(t,x,\eta(t,x)) = (0,b(t,x))$ we get

$$\phi(s,t,x,\eta(t,x)) = \phi(s,x,\eta(s,x)) = (0,b(s)).$$

Thus

$$\phi_\delta(t,x,\eta(t,x)) = \int_0^T b(s)j_\delta(t-s)ds = b_\delta(t,x).$$

For the convergence result we introduce the function on the reference domain

$$\phi_0 : [0,T]\times\omega\times[0,1] \to \mathbb{R}^3, \quad (t,x,y) \mapsto \phi(t,x,y\eta(t,x)).$$

Let $p \in [1,\kappa)$. First we estimate $\phi^1_\delta - \phi^1$ in $L^\kappa(0,T;L^p(\Omega_\eta(t)))$. We have

$$(\phi^1_\delta - \phi^1)(t,x,y) = \int_0^T \left(\frac{\eta(s,x)}{\eta(t,x)}\phi^1\left(s,x,y\frac{\eta(s,x)}{\eta(t,x)}\right) - \phi^1(t,x,y)\right)j_\delta(t-s)ds$$

Hence (by a change of variables) we find

$$\begin{aligned}&\int_0^T\left(\int_{\Omega_\eta(t)}|(\phi^1_\delta - \phi^1)(t,x,y)|^p dxdy\right)^{\frac{\kappa}{p}}dt\\ &= \int_0^T\left(\int_{\omega\times[0,1]}\left|\int_0^T(\eta(s,x)\phi^1(s,x,y\eta(s,x)) - \eta(t,x)\phi^1(t,x,y\eta(t,x)))j_\delta(t-s)ds\right|^p dz\right)^{\frac{\kappa}{p}}dt\\ &= \|\varphi_\delta - \varphi\|_{L^\kappa(0,T;L^p(\omega\times[0,1]))}\end{aligned}$$

for $\varphi(t,x,y) = \eta(t,x)\phi^1_0(t,x,y)$. As $\eta \in L^\infty(0,T;L^\infty(\omega))$ and $\phi \in L^\kappa(0,T;L^q(\Omega_\eta))$ this converges to 0 by standard convolution estimates. Next note by a similar argument that

$$\begin{aligned}&\int_0^T\left(\int_{\Omega_\eta(t)}\left|\int_0^T\left(\phi^2\left(s,x,y\frac{\eta(s,x)}{\eta(t,x)}\right) - \phi^2(t,x,y)\right)j_\delta(t-s)\,ds\right|^p dz\right)^{\frac{\kappa}{p}}dt\\ &\le \|\eta\|_{L^\infty_t(0,T;L^\infty(\omega))}\|\phi^2_{0,\delta} - \phi^2_0\|_{L^\kappa(0,T;L^p(\omega\times[0,1]))},\end{aligned}$$

which also converges to 0. Lastly

$$\begin{aligned}&\int_0^T\left(\int_{\Omega_\eta(t)}\left|\int_0^T y\nabla\left(\frac{\eta(s)}{\eta(t)}\right)\phi^1\left(s,x,y\frac{\eta(s)}{\eta(t)}\right)j_\delta(t-s)ds\right|^p dz\right)^{\frac{\kappa}{p}}dt\\ &= \int_0^T\left(\int_{\omega\times[0,1]}\left|\int_0^T y\eta(t)\nabla\left(\frac{\eta(s)}{\eta(t)}\right)\phi^1_0(s,x,y)j_h(t-s)ds\right|^p dz\right)^{\frac{s}{p}}dt\end{aligned}$$

As j_δ has unit integral we can compute

$$\int_0^T \eta(t)\nabla\left(\frac{\eta(s)}{\eta(t)}\right)\phi_0^1(s)j_\delta(t-s)ds = \int_0^T \phi_0^1(s)j_\delta(t-s)(\nabla\eta(s)-\nabla\eta(t)) + \frac{\nabla\eta(t)}{\eta(t)}(\eta(t)-\eta(s)))\,ds$$
$$= \int_0^T j_\delta(t-s)(\phi_0(s)\nabla\eta(s)-\phi_0(t)\nabla\eta(t)) + j_\delta(t-s)\frac{\nabla\eta(t)}{\eta(t)}(\phi_0(s)\eta(s)-\phi_0(t)\eta(t))$$
$$+ 2j_\delta(t-s)\nabla\eta(t)(\phi_0(t)-\phi_0(s))\,ds$$

Thus

$$\int_0^T\left(\int_{\Omega_\eta(t)}\left|\int_0^T y\nabla\left(\frac{\eta(s)}{\eta(t)}\right)\phi^1\left(s,x,y\frac{\eta(s)}{\eta(t)}\right)j_\delta(t-s)ds\right|^p dz\right)^{\frac{\kappa}{p}}dt$$
$$\leq \|(\nabla\eta\phi_0^1)_\delta - \nabla\eta\phi_0^1\|_{L^2(0,T;L^p(\Omega_\eta))} + \int_0^T\left(\int_{\omega\times[0,1]}\frac{|\nabla\eta(t)|^p}{|\eta(t)|^p}|(\eta\phi_0^1)_\delta(t)-\eta(t)\phi_0^1(t)|^p dz\right)^{\frac{\kappa}{p}}dt$$
$$+\int_0^T\left(\int_{\omega\times[0,1]}|\nabla\eta(t)|^p|\phi_{0,\delta}^1(t)-\phi_0^1(t)|^p dz\right)^{\frac{\kappa}{p}}dt$$

The first term converges to 0 by standard convolution. The third term we can estimate as $p<q$

$$\int_0^T\left(\int_{\omega\times[0,1]}|\nabla\eta|^p|\phi_0^1 * j_\delta - \phi_0^1|^p\,dz\right)^{\frac{v}{p}}dt \leq \|\nabla\eta\|_{L^\infty(0,T;L^{q*}(\omega))}\|\phi_{0,\delta}^1-\phi_0^1\|_{L^\kappa(0,T;L^q(\omega\times[0,1]))}.$$

Hence this term converges to 0 as well. The third term can be estimated analogously using the assumed uniform lower bounds on η.

As we have shown strong convergence in $L^2(0,T;L^p(\Omega_\eta(t)))$ it suffices to show that $\nabla\phi_\delta$ is bounded in $L^2(0,T;L^p(\Omega_\eta(t)))$ to prove weak convergence. The estimate on the gradient is a standard exercise combining the bounds of η and ϕ via Hölder's inequality. For that reason we omit here most of the details and only mention the critical terms that appear in the estimates. One critical term appearing in the estimates for (1), (2), (3) can be estimated using

$$|\nabla\phi||\nabla\eta| \in L^2(0,T;L^p(\Omega_\eta) \text{ for all } p\in[1,a).$$

Moreover, one needs

for (1) $|\phi||\nabla^2\eta| \in L^2([0,T];L^p(\Omega_\eta))$ for all $p\in[1,2)$ by Lemma 4.2.3.

for (2) $|\phi||\nabla^2\eta| \in L^2([0,T];L^2(\Omega_\eta))$ as $\phi\in L^2(L^\infty)$ by Sobolev embedding.

Next let us consider the weak time derivative. Let us take $\psi\in\tilde{W}^{1,p'}([0,T]\times\omega\times\mathbb{R}))$, such that $\psi(t,x,y)=0$ for all $x\in B_c$ and $\|\psi\|_{W^{1,p'}([0,T]\times\omega\times\mathbb{R})}\leq 1$ to find that

$$\int_0^T\langle\partial_t\phi_\delta,\psi\rangle = \int_0^T\int_0^T\langle\partial_t K(s,t,y,x)\phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big)j_\delta(t-s),\psi(t,z)\rangle\,ds\,dt$$
$$+\int_0^T\int_0^T\int_{\Omega_\eta}K(s,t,y,x)\phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big)\partial_t j_\delta(t-s)\cdot\psi(t,z)\,dz\,ds\,dt$$
$$-\int_0^T\int_0^T\int_{\Omega_\eta}K(s,t,y,x)\partial_y\phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big)y\frac{\eta(s)}{\eta^2(t)}\partial_t\eta(t)j_\delta(t-s)\cdot\psi(t,z)\,dz\,ds\,dt$$
$$=(I)+(II)+(III)$$

The expression (I) can be transferred into an integral by using partial integration in x_i and the fact that $\phi^i(t,x,\eta(t,x)) = 0$ for $i \in \{1,2\}$ and $(t,x) \in [0,T] \times \omega$:

$$
\begin{aligned}
(I) &= \sum_{i=1}^{2} \int_0^T \int_0^T \Big(- \langle y \partial_t \partial_{x_i} \Big(\frac{\eta(s,x)}{\eta(t,x)}\Big) \phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big), \psi^3(t) \rangle \, ds \, dt \\
&\quad + \int_{\Omega_{\eta(t)}} \partial_t \Big(\frac{\eta(s,x)}{\eta(t,x)}\Big) \phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) \cdot \psi^i(t,z) \, dz \Big) j_\delta(t-s) \, ds \, dt \\
&= \sum_{i=1}^{2} \int_0^T \int_0^T \int_\omega \partial_t \Big(\frac{\eta(s,x)}{\eta(t,x)}\Big) \Big(\partial_{x_i} \int_0^{\eta(t,x)} y \phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) \cdot \psi^3(t,x,y) \, dy \\
&\quad + \int_0^{\eta(t,x)} y \phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) \cdot \psi^i(t,x,y) \, dy \Big) dx j_\delta(t-s) \, ds \, dt \\
&= \sum_{i=1}^{2} \int_0^T \int_0^T \int_\omega \partial_t \Big(\frac{\eta(s,x)}{\eta(t,x)}\Big) \Big(\int_0^{\eta(t,x)} y \partial_{x_i} \Big(\phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) \cdot \psi^3(t,x,y) \Big) dy \\
&\quad + \int_0^{\eta(t,x)} y \phi^i\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) \cdot \psi^i(t,x,y) \, dy \Big) dx j_\delta(t-s) \, ds \, dt.
\end{aligned}
$$

But these expression can be estimated using that $p^* = \frac{3p}{3-p}$ can be assumed to be close enough to 6 such that

$$
(I) \le C \int_0^T \|\partial_t \eta\|_{L^2(\omega)} \Big((\|\phi\|_{W^{1,s}(\Omega_{\eta(t)})} + \||\nabla\phi||\nabla\eta|\|_{L^{3+(3-s)/2}(\Omega_\eta)}) \|\Psi\|_{L^{p^*}(\Omega_\eta)} + \|\Psi\|_{W^{1,p}(\Omega_\eta)} \Big) \, dt.
$$

This expression is bounded as $\partial_t \eta \in L^\infty(0,T;L^2(\omega))$, $|\nabla\eta||\nabla\phi| \in L^2(0,T;L^q(\Omega_\eta))$ for all $q \in [3,s)$. The estimate on (III) is analogous (but simpler).

For (II) we use $\partial_t j_\delta(t-s) = \partial_s j_\delta(t-s)$ to find (using the 0-trace of $j_\delta(t-s)$ that)

$$
\begin{aligned}
(II) &= \int_0^T \int_0^T \partial_s \langle (K(s,t,y,x) \phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) j_\delta(t-s), \psi(t,z) \rangle \, ds \, dt \\
&\quad - \int_0^T \int_0^T \langle \partial_s K(s,t,y,x) \phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) j_\delta(t-s), \psi(t,z) \rangle \, ds \, dt \\
&\quad - \int_0^T \int_0^T \int_{\Omega_\eta} K(s,t,y,x) \partial_y \phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) y \frac{\partial_s \eta(s)}{\eta(t)} j_\delta(t-s) \cdot \psi(t,z) \, dz \, ds \, dt \\
&\quad - \int_0^T \int_0^T \langle K(s,t,y,x) \partial_s \phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big) j_\delta(t-s), \psi(t,z) \rangle \, ds \, dt, \\
&=: II_1 + II_2 + II_3 + II_4.
\end{aligned}
$$

First observe, that$II_1 = 0$. The estimates on II_2, II_3 are similar to the estimate of (I) above. Now, finally II_4 is estimated using the assumption on $\partial_t \phi$. We define $\hat{K}^T(s,t,y,x)$ in such a way that

$$
\begin{aligned}
II_4 &= - \int_0^T \int_0^T \langle \partial_s \phi\Big(s,x,y\frac{\eta(s)}{\eta(t)}\Big), K^T(s,t,y,x) \psi(t,z) \rangle_{\Omega_{\eta(t)}} j_\delta(t-s) \, ds \, dt \\
&= - \int_0^T \int_0^T \langle \partial_s \phi(s,z), \hat{K}^T(t,s,y,x) \psi\Big(s,x,y\frac{\eta(t)}{\eta(s)}\Big) \rangle_{\Omega_{\eta(s)}} j_\delta(t-s) \, ds \, dt.
\end{aligned}
$$

This implies that

$$
II_4 \le \int_0^T \int_0^T \|\partial_t \phi(s)\|_{\tilde{W}^{-1,p'}(\Omega_\eta)} \Big\| \hat{K}^T(t,s,y,x) \psi\Big(s,x,y\frac{\eta(t)}{\eta(s)}\Big) \Big\|_{W^{1,p}(\Omega_\eta)} j_\delta(t-s) \, ds \, dt,
$$

which is uniformly bounded using $|\nabla\psi||\nabla\eta|^2 \in L^2(0,T;L^p(\Omega_\eta))$ and $|\psi||\nabla^2\eta| \in L^2([0,T];L^p(\Omega_\eta))$ for all $p \in [1,2)$.

□

4.2.4 The distributional time derivatives.

En passant we include here a result that is independent of our main result but might be important for further use. Here a meaning is given to the distributional time derivative of solutions.

Proposition 4.2.7. *Let (v, p, η) be a weak solution satisfying* (4.1.11)*, then if $v \in L^2(0, T; W^{1,s}(\Omega_\eta))$ for $s \geq 2$, than*

$$\partial_t v + [\nabla v]v \in L^2(0, T; (W^{1,q}_{0,\mathrm{d}}(\Omega_\eta)^*),$$

for any $q \in (2, \infty)$ if $s = 2$ and $q = 2$ if $s > 2$.

This means[3] *that for $\phi \in L^2(0, T; W^{1,q}_{0,\mathrm{d}}(\Omega_\eta))$ we find that*

$$\int_0^T \langle \partial_t v + [\nabla v]v, \phi \rangle_\eta \, dt = -\int_0^T \int_{\Omega_\eta} \nabla v \cdot \nabla \phi \, dx \, dt. \tag{4.2.3}$$

Moreover, $(\partial_t v + [\nabla v]v, \partial_t^2 \eta) \in L^2(0, T; \mathcal{W}^*)$ for

$$\mathcal{W} = \{(\phi, b) \in W^{1,q}_{\mathrm{d}}(\Omega_\eta) \times H^2(\omega) \, : \, \phi(t, x, \eta(x)) = b(t, x)\}$$

for any $q \in (2, \infty)$ if $s = 2$ and $q = 2$ if $s > 2$.

In particular, for all $(\phi, b) \in \mathcal{W}$ we find that

$$\int_0^T \langle \partial_t v + [\nabla v]v, \phi \rangle_\eta + \langle \partial_t^2 \eta, b \rangle \, dt = -\int_0^T \int_{\Omega_\eta} \nabla v \cdot \nabla \phi \, dx \, dt + \int_0^T \int_\omega \nabla^2 \eta \cdot \nabla^2 b \, dx \, dt.$$

Proof. Let $\phi \in L^2(0, T; W^{1,q}_{0,\mathrm{d}}(\Omega_\eta))$. First observe, that if (additionally) $\partial_t \phi \in L^2([0, T] \times \Omega_\eta)$ and $\nabla \phi \in L^\infty(0, T; L^2(\Omega_\eta))$, than (as $|v|^2 \in L^1_t(L^2_z)$) we find

$$\begin{aligned}\int_0^T \langle \partial_t v + (v \cdot \nabla)v, \phi \rangle_\eta &:= \int_{\Omega_{\eta(T)}} v(T) \cdot \phi(T) \, dz - \int_{\Omega_{\eta_0}} v^0 \cdot \phi(0) \, dz - \int_0^T \int_{\Omega_\eta} v \cdot \partial_t \phi + v \otimes v \cdot \nabla \phi \, dz \, dt \\ &= -\int_0^T \int_{\Omega_\eta} \nabla v \cdot \nabla \phi \, dz \, dt.\end{aligned}$$

Hence, by taking the mollification introduced in Lemma 4.2.6 (here $b \equiv 0$), we find that

$$\int_0^T \langle \partial_t v + (v \cdot \nabla)v, \phi_\delta \rangle_\eta = -\int_0^T \int_{\Omega_\eta} \nabla v \cdot \nabla \phi_\delta \, dz \, dt,$$

which implies the result by passing with $\delta \to 0$ by the convergence result of Lemma 4.2.6. This allows to give the left hand side a well defined meaning; hence the domain of the left hand side can accordingly be extended. The proof of the second identity is analogous.

□

4.3 Proof of the main result

4.3.1 The set-up

Throughout this section let (v_1, η_1), (v_2, η_2) be weak solutions to *FSI* for initial conditions $v_1(0) = v_{1,0}$, $v_2(0) = v_{2,0}$, $\eta_1(0) = \eta_{1,0}$ $\eta_2(0) = \eta_{2,0}$ and $\partial_t \eta_1(0) = \eta^*_{1,0}, = \partial_t \eta_2(0) = \eta^*_{2,0}$. Let v_2 satisfy the additional regularity assumption $v_2 \in L^r(0, T; W^{1,s}(\Omega_{\eta_2}))$, $\partial_t v_2 \in L^2(0, T; W^{-1,r}(\Omega_{\eta_2}))$ for some $s > 3$, $r > 2$. Note that as $\partial_t \eta_1 = tr_{\eta_1}(v_1)$ and $\partial_t \eta_2 = tr_{\eta_2}(v_2)$ we have by the trace theorem for moving boundaries (see [22, Lemma 6]])

$$\partial_t \eta_1 \in L^2(0, T; H^l(\omega)), \quad \partial_t \eta_2 \in L^r(0, T; W^{\frac{3}{2},3}(\omega))$$

[3]The expression (4.2.3) seems to be the appropriate definition of a weak time derivative in the setting of fluid-structure interaction.

for all $l \in (0,1/2)$. By Theorem 4.2.2 we find additionally that

$$\eta_1 \in L^2(0,T;H^{2+l}(\omega)), \quad \eta_2 \in L^r(0,T;H^{2+l}(\omega)),\ \ l \in (0,1/2).$$

We define the variable in time domains

$$\Omega_1 := \Omega_{\eta_1} \text{ and } \Omega_2 := \Omega_{\eta_2}.$$

Since most of the computations will be given on the domain of the weak solution Ω_1 we introduce for $u : [0,T] \times \Omega_1 \to \mathbb{R}^3$ the notation

$$\|u(t)\|_{k,p} := \|u(t)\|_{W^{k,p}(\Omega_{\eta_1(t)})}, \quad \|u(t)\| := \|u(t)\|_{L^2(\Omega_{\eta_1(t)})} \text{ and } (u(t),w(t)) := \langle u(t),w(t)\rangle_{\eta_1},$$

whenever well defined. Recall also, that in case a function $b : [0,T] \times \omega \to \mathbb{R}$ we will extend it constantly to $b : [0,T] \times \omega \times \mathbb{R} \to \mathbb{R}$ without further notice. For such function we use

$$\|b(t)\|_{k,p} := \|b(t)\|_{W^{k,p}(\omega)}, \quad \|b(t)\| := \|b(t)\|_{L^2(\omega)} \text{ and } (u(t),w(t)) := \langle u(t),w(t)\rangle_{\omega}.$$

The first step of the proof is to introduce a diffeomorphism $\psi : \Omega_1 \to \Omega_2$ to compare the velocity fields on the same domain. We define such a ψ explicitly by

$$\gamma : \omega \to (0,\infty), \quad x \mapsto \frac{\eta_2(x)}{\eta_1(x)},$$
$$\psi : [0,T] \times \omega \times \mathbb{R} \to [0,T] \times \omega \times \mathbb{R} \quad (t,x,y) \mapsto (t,x,\gamma(t,x)y).$$

Then $\psi(\{t\} \times \Omega_1) = \{t\} \times \Omega_2$ for all $t \in [0,T]$. Note however that this transformation does not conserve the property of vanishing divergence. For that we follow the approach in [92]. Define the 3×3 matrix[4]

$$J(t,x,y) = D_z\psi(t,x,y) = \begin{pmatrix} \mathbb{I}_2 & 0 \\ y\nabla\gamma(t,x) & \gamma(t,x) \end{pmatrix},$$
$$\tilde{J} = J \circ \psi^{-1} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ y\gamma^{-1}\nabla\gamma & \gamma(t,x) \end{pmatrix}.$$

Now for $w : [0,T] \times \Omega_2 \to \mathbb{R}^3$ set $\hat{w} = \gamma J^{-1}(w \circ \psi)$ and for $u : [0,T] \times \Omega_1 \to \mathbb{R}^3$ set $\check{u} = \gamma^{-1}\tilde{J}u \circ \psi^{-1}$. The next lemma shows that $(\hat{w},\xi)$ is an admissible and solenoidal test function for (v_1,η_1) if (w,ξ) is an admissible and solenoidal test function for (v_2,η_2) and $(\check{u},\xi)$ is an admissible and solenoidal test function for (v_1,η_1) if (u,ξ) is an admissible and solenoidal for (v_2,η_2).

Lemma 4.3.1. *Let $w \in L^1(0,T;W^{1,q}(\Omega_2;\mathbb{R}^3))$, $u : [0,T] \to \Omega_1$ (sufficiently smooth). The following holds*

1. *If $\mathrm{d}w = \mathrm{d}u = 0$ then $\mathrm{d}\hat{w} = \mathrm{d}\check{u} = 0$.*
2. *$u^3(t,x,\eta_2(t,x)) = \hat{u}^3(t,x,\eta_1(x))$, $u^3(t,x,\eta_1(x)) = \check{u}^3(t,x,\eta_2(x))$.*
3. *$(u - \hat{w}) \circ \psi^{-1} = \gamma\tilde{J}^{-1}(\check{u} - w)$ and $(\check{u} - w) \circ \psi = \gamma^{-1}J(u - \hat{w})$*

Proof. We calculate

$$\gamma J^{-1} = \begin{pmatrix} \gamma\mathbb{I}_2 & 0 \\ -y\nabla\gamma & 1 \end{pmatrix}, \quad \gamma^{-1}\tilde{J} = \begin{pmatrix} \gamma^{-1}\mathbb{I}_2 & 0 \\ y\gamma^{-2}\nabla\gamma & 1 \end{pmatrix} = \begin{pmatrix} \gamma^{-1}\mathbb{I}_2 & 0 \\ -y\nabla(\gamma^{-1}) & 1 \end{pmatrix}.$$

Thus it is sufficient to prove (1) and (2) for $\hat{w}$ as for $\check{u}$ we just have to replace γ by γ^{-1} everywhere. We get

$$\hat{w} = (\gamma w' \circ \psi, -y\nabla\gamma \cdot w' \circ \psi + w^2 \circ \psi),$$

[4] Here and in the following we use $(\mathbb{I}_2, 0)$ for $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

As $\psi(x,\eta_1) = (x,\eta_2)$ this directly yields the second assertion. For the divergence we find

$$\mathrm{d}_x\hat{w}' = \nabla\gamma\cdot w'\circ\psi + \gamma\mathrm{d}_x(w'\circ\psi) = \nabla\gamma\cdot w'\circ\psi + \gamma((\mathrm{d}_x w')\circ\psi + (\partial_y w')\circ\psi)\cdot y\nabla\gamma)$$

and using $\partial_y(w\circ\psi) = \gamma(\partial_y w)\circ\psi$

$$\partial_y\hat{w}^2 = -\nabla\gamma\cdot w'\circ\psi + \gamma(-y\nabla\gamma\cdot(\partial_y w')\circ\psi + (\partial_y w^2)\circ\psi).$$

Thus $\mathrm{d}w_1 = 0$ gives $\mathrm{d}\hat{w} = \gamma(\mathrm{d}_x w)\circ\psi = 0$. For (3) note first that

$$J^{-1}\circ\psi^{-1} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ -y\gamma^{-2}\nabla\gamma & \gamma^{-1}\end{pmatrix} = \tilde{J}^{-1}$$

This gives

$$(u-\hat{w})\circ\psi^{-1} = u\circ\psi^{-1} - \gamma(J^{-1}\circ\psi^{-1})w = \gamma\tilde{J}^{-1}(\gamma^{-1}\tilde{J}u\circ\psi^{-1} - w) = \gamma\tilde{J}^{-1}(\check{u}-w).$$

Lastly

$$(\check{u}-w)\circ\psi = \gamma^{-1}Ju - w\circ\psi = \gamma^{-1}J((u-\hat{w})).$$

□

For notational purposes set

$$\eta_1 - \eta_2 = \eta, \quad w_1 = v_1 - \hat{v}_2, \quad w_2 = \check{v}_1 - v_2.$$
$$v_2\circ\psi = \tilde{v}_2, \quad v_1\circ\psi^{-1} = \tilde{v}_1, \quad w_2\circ\psi = \tilde{w}_2, \quad w_1\circ\psi^{-1} = \tilde{w}_1, \quad \tilde{f}_2 = f_2\circ\psi$$

Note that by Lemma 4.3.1

$$\tilde{w}_2 = \gamma^{-1}Jw_1, \quad \tilde{w}_1 = \gamma\tilde{J}^{-1}w_2, \tag{4.3.1}$$

and with a slight missuse of notation.

$$\check{v}_{1,\delta} = \gamma^{-1}\tilde{J}v_{1,\delta}\circ\psi^{-1}, \quad \hat{v}_{2,\delta} = \gamma J^{-1}v_{2,\delta}\circ\psi, \quad w_{2,\delta} = \check{v}_{1,\delta} - v_{2,\delta}, \quad w_{1,\delta} = v_{1,\delta} - \hat{v}_{2,\delta}.$$

Note that by Lemma 4.2.6 $\mathrm{d}v_{2,\delta} = \mathrm{d}v_{1,\delta} = 0$ and $v_{2,\delta}(x,\eta_2(x)) = (0,\partial_t\eta_{2,\delta})$, $v_{1,\delta} = (0,\partial_t\eta_{1,\delta})$. Thus by Lemma 4.3.1 $\mathrm{d}\hat{v}_{2,\delta} = \mathrm{d}\check{v}_{1,\delta} = 0$ and $\hat{v}_{2,\delta}(x,\eta_1(x)) = \partial_t\eta_{2,\delta}$, $\check{v}_1(x,\eta_2(x)) = \partial_t\eta_{1,\delta}$ as well as $\mathrm{d}w_{1,\delta} = \mathrm{d}w_{2,\delta} = 0$ and $w_{1,\delta}(x,\eta_1(x)) = w_{2,\delta}(x,\eta_2(x)) = \partial_t\eta_\delta$.

4.3.2 A-priori estimates

Before we turn to the main argument we collect some results that show that our test-functions are admissible and that the error terms due to the geometric convolution in time are converging to 0.

Remark 4.3.2. *The following estimates we will use frequently in the following. They are consequences of Hölder's inequality and the imbeddings $H^1(\omega)\hookrightarrow L^p(\omega)$ ($p\in[1,\infty)$) and in case $q<3$, that $W^{1,q}(\Omega_i)\hookrightarrow L^r(\Omega_i)$ for all $r<3q/(3-q)$) ($i=1,2$ here and in the following). See [127] for a reference.*

1. *For all $s\in(1,\infty)$, $p\in[1,s)$ and $f\in L^s(\Omega_i)$, $g\in H^1(\omega)$*

$$\|fg\|_{L^p(\Omega_i)} \le C\|f\|_{L^s(\Omega_i)}\|g\|_{H^1(\omega)}.$$

2. *For all $p\in(1,2)$, $q\in(6p/(6-p),3)$, $f\in W^{1,q}(\Omega_i)$ and $g\in L^2(\Omega_i)$*

$$\|fg\|_{L^p(\Omega_i)} \le C\|f\|_{W^{1,q}(\Omega_i)}\|g\|_{L^2(\Omega_i)}.$$

3. *If p,q, f are as above and $g\in H^2(\omega)$ 1. and 2. give in particular*

$$\|fg\|_{W^{1,p}(\Omega_i)} \le C\|f\|_{W^{1,q}(\Omega_i)}\|g\|_{H^2(\omega)}.$$

Lemma 4.3.3. *Let $(v_1,\eta_1),(v_2,\eta_2)\in\mathcal{V}_S$ weak solutions of FSI, (v_2,η_2) satisfying the additional regularity assumptions. Then*

1. *γ satisfies the following estimates for a.e. $t\in[0,T]$.*
$$\|\gamma(t)-1\|_{H^2(\omega)}\leq C\|\eta(t)\|_{H^2(\omega)}\quad \|\partial_t\gamma(t)\|_{L^2(\omega)}\leq C\|\partial_t\eta(t)\|_{L^2(\omega)}+C\|\eta(t)\|_{L^2(\omega)}.$$
The same estimates hold for γ^{-1}.
2. *$\nabla\gamma\in L^\infty(0,T;L^q(\omega))$ for all $q\in[1,\infty)$*
$$\|\nabla\gamma(t)\|_{L^q(\omega)}\leq C\|\eta(t)\|_{H^2(\omega)}$$
and the same holds for γ^{-1}.
3. *$\hat{v}_1\in L^\infty(0,T;L^p(\Omega_2))\cap L^2(0,T;W^{1,p}(\Omega_2))$ for all $p\in(1,2)$ and $\|\hat{v}_1\|_{W^{1,p}(\Omega_2)}\leq C\|v_1\|_{1,2}$ for all $p\in[1,2)$.*
4. *$\partial_t\hat{v}\in L^2(0,T;\tilde{W}^{-1,p'}(\Omega_1))$ for all $p'\in[1,r)$,*

Proof. (1) and (2):
It holds
$$\begin{aligned}\gamma-1&=\frac{\eta_2-\eta_1}{\eta_1}\leq C|\eta|\\ \gamma_t&=\frac{\partial_t\eta_2\eta}{\eta_1^2}-\frac{\eta_2\partial_t\eta}{\eta_1^2}\leq C(|\partial_t\eta_2||\eta|+|\partial_t\eta|),\\ \nabla\gamma&=\frac{\nabla\eta_2\eta}{\eta_1^2}-\frac{\eta_2\nabla\eta}{\eta_1^2}\leq C(|\nabla\eta_2||\eta|+|\nabla\eta|),\\ \partial^2_{x_ix_j}\gamma&=\eta_1^{-2}(\partial^2_{x_ix_j}\eta_2\eta+\partial_{x_j}\eta_2\partial_{x_i}\eta-\partial_{x_i}\eta_2\partial_{x_j}\eta-\eta_2\partial^2_{x_ix_j}\eta)-2\frac{\partial_{x_i}\eta_1}{\eta_1^3}\partial_{x_j}\gamma\\ &\leq C(|\nabla^2\eta_2||\eta|+|\nabla\eta_2||\nabla\eta|+|\nabla^2\eta|+|\nabla\eta_1|(|\nabla\eta_2||\eta|+|\nabla\eta|)\end{aligned}$$
(1) and (2) now follow from the embeddings $H^2(\omega)\hookrightarrow W^{1,q}(\omega)\hookrightarrow L^\infty(\omega)$ for all $q\in[1,\infty)$. The results for γ^{-1} follow by replacing the roles of η_1 and η_2

Proof of (3):
We calculate
$$\partial_{x_i}(\gamma^{-1}\tilde{J}) = \begin{pmatrix} \mathbb{I}_2\partial_{x_i}(\gamma^{-1}) & 0 \\ -y\partial_{x_i}\nabla(\gamma^{-1}) & 0 \end{pmatrix}, \quad \partial_y(\gamma^{-1}\tilde{J}) = \begin{pmatrix} \mathbb{I}_2 0 & 0 \\ -\nabla(\gamma^{-1}) & 0 \end{pmatrix}$$
Hence
$$|\partial_{x_i}(\gamma^{-1}\tilde{J})| + |\partial_y(\gamma^{-1}\tilde{J})| \le C(|\nabla(\gamma^{-1})| + |\nabla(\gamma^{-1})|^2 + |y\nabla^2(\gamma^{-1})|) \tag{4.3.2}$$
Observe further, that by Lemma 4.2.4 $\int_0^{\eta_1(t,x)} |v|\,dy \in L^2(0,T;L^q(\omega))$ for all $q \in [1,\infty)$, which implies (using also (2)) that
$$|\nabla^2\gamma||v_1| \in L^2(0,T;L^p(\Omega_1)) \text{ and } |\nabla^2\gamma||\tilde{v}_1| \in L^2(0,T;L^p(\Omega_2)) \text{ for all } p \in [1,2) \tag{4.3.3}$$

Now by (4.3.2)
$$|\partial_{z_i}(\gamma^{-1}J\tilde{v}_1)| \le C(|\nabla(\gamma^{-1})| + |\nabla(\gamma^{-1})| + |\nabla^2(\gamma^{-1})||\tilde{v}_1| + |\nabla(\gamma^{-1})||(\nabla v_1)\circ\psi^{-1}|$$
Thus the assertion for $\hat{v}_1$ follows using also (1), (2) and Remark 4.3.2.
Proof of (4):
This estimate is analogous to (3) in Lemma 4.2.6: Let us take $\psi \in \tilde{W}^{1,p'}(\omega\times\mathbb{R})$, such that $\psi(t,x,y) = 0$ for all $x \in B_c$ and $\|\psi\|_{W^{1,p'}([0,T]\times\omega\times\mathbb{R})} \le 1$ to find that
$$\int_0^T (\partial_t\hat{v}_2,\psi)\,dt = \int_0^T (\partial_t(\gamma J^{-1})\tilde{v}_2,\psi)\,dt + \int_0^T \langle J^{-1}\partial_t v_2,\psi\rangle_{\eta_2}\,dt + \int_0^T\int_{\Omega_1}\gamma J^{-1}\partial_3 v_2\partial_t\gamma\cdot\psi\,dz\,dt$$
The estimates on the first and the third term are now straight forward using the assumptions on v_2. In the first term it is important to observe that the terms involving $\partial_t\nabla\gamma$ are always coupled to v_2'. Using the fact that $v_2'(t,x,\eta_2(t,x)) = 0$ for all $(t,x) \in [0,T]\times\omega$, we may use integration by parts in x direction and find
$$(\partial_t(\gamma J^{-1})\tilde{v}_2,\psi) \le C\int_{\Omega_1}|\partial_t\gamma|(|\nabla\gamma||\nabla\tilde{v}_2||\psi| + \|v_2\|_{L^\infty(\Omega_2)}|\tilde{v}_2||\nabla\psi|),$$
But these expression can be estimated using that $p^* = \frac{3p}{3-p}$ can be assumed to be close enough to 6 such that
$$\begin{aligned}&\int_0^T (\partial_t(\gamma J^{-1})\tilde{v}_2,\psi)\,dt \\ &\le C\int_0^T \|\partial_t\gamma\|\big(\||\nabla\tilde{v}_2||\nabla\gamma|\|_{3+(3-s)/2}\|\psi\|_{p^*} + \|v_2\|_{W^{1,s}(\Omega_2)}\|\psi\|_{1,p}\big)\,dt.\end{aligned}$$
This expression is bounded since $\partial_t\eta \in L^\infty(L^2)$ and $|\nabla\gamma||\nabla\tilde{v}_2| \in L^2(0,T;L^q(\Omega_1))$ for all $q \in [3,s)$. □

At this point we choose $t \in [0,T]$ such that all involved quantities do have a Lebesgue point at this time instance. Without any further notice we extend all quantities via (4.2.1) constant on $(-\infty,0]$ and $[t,\infty)$.

Next we take the convolution introduced in Lemma 4.2.6 on w_2 and $\hat{v}_2$. We will need the following convergences:

Lemma 4.3.4. *The following expressions are all well defined and convergence to zero with $\delta \to 0$:*
$$\int_0^t \langle\partial_t v_2, w_2 - w_{2,\delta}\rangle_{\eta_2} + \langle[\nabla v_2]v_2, w_2 - w_{2,\delta}\rangle_{\eta_2} + \langle\varepsilon v_2, \varepsilon w_2 - \varepsilon w_{2,\delta})\rangle_{\eta_2}\,dt0 \tag{4.3.4}$$
$$\int_0^t (v_1\otimes v_1, \nabla\hat{v}_2 - \nabla\hat{v}_{2,\delta})\,dt \tag{4.3.5}$$
$$(v_1(t), \hat{v}_2(t) - \hat{v}_{2,\delta}(t)) - \int_0^t (v_1, \partial_t\hat{v}_2 - \partial_t\hat{v}_{2,\delta}) - (\varepsilon v_1, \varepsilon\hat{v}_2 - \varepsilon\hat{v}_{2,\delta})\,dt. \tag{4.3.6}$$
Moreover, $(\partial_t\eta_\delta, \hat{v}_{2,\delta})$ is a valid testfunction for the weak formulation of (η_1, v_1) and the terms $\langle\partial_t v_2, w_{2,\delta}\rangle_{\eta_2}$, $\langle\varepsilon v_2, \varepsilon w_{2,\delta}\rangle_{\eta_2}$, $\langle[\nabla v_2]v_2, w_{2,\delta}\rangle_{\eta_2} \in L^1(0,T)$ uniformly in δ.

Proof. For (4.3.4) we know that $w_2 \in L^2(0,T;W^{1,p}(\Omega_2)$ for all $p \in [1,2)$ by Lemma 4.3.3. Hence by Lemma 4.2.6 $w_2 - w_{2,\delta} \to 0$ weakly in $L^2(0,T;W^{1,p}(\Omega_2))$ for all $p \in [1,2)$. Since it is a valid argument for $\partial_t v_2 \in L^2(0,T;\tilde{W}^{-1,p'}(\Omega_2))$ and since $\nabla v_2 \in L^2(0,T;W^{1,s}(\Omega_2))$ for $s > 3$ it yields the convergence of the first and third term. Moreover, it was shown in Lemma 4.3.3 (6) that $[\nabla v_2]v_2 \in L^2(0,T;L^q(\Omega_2)$ for some $q > (6/5)$. Since we may assume $p \in [1,2)$ such that $W^{1,p}(\Omega_2) \hookrightarrow L^{q'}$ the convergence of the second term follows again from the weak convergence of $w_{2,\delta}$ in $L^2(0,T;W^{1,p}(\Omega_2))$.

In (4.3.5) we will show that all involved terms are uniformly bounded. The uniform bounds imply that all weakly converging sub-sequences converge to 0, by the uniqueness of the weak limits. The critical term here is $\int_0^T \int_{\Omega_{\eta_1}} |v_1 \otimes v_1 \cdot \nabla(\partial_{x_i}\gamma\tilde{v}_{2,\delta})|\,dz\,dt$. All other terms can be estimated in a straight forward manner and we skip the details. Using the uniform bounds on $\eta_1, \eta_2, \frac{1}{\eta_1}, \frac{1}{\eta_2}$ we find

$$\begin{aligned}
&\int_{\Omega_1} |v_1 \otimes v_1 \cdot \nabla(\partial_{x_i}\gamma\tilde{v}_2)|\,dz\,dt \\
&\le C\int_\omega \int_0^{\eta_1(t,x)} |v_1|^2|\tilde{v_2}|\,dy\Big((|\nabla\eta_1| + |\nabla\eta_2|)(1 + |\nabla\eta_2| + |\nabla^2\eta_2|) + |\nabla\eta_2||\nabla^2\eta_1|\Big)\,dx \\
&+ C\int_{\Omega_1} |v_1|^2|\nabla\tilde{v_2}|\,dy\Big(1 + |\nabla\eta_1|^2 + |\nabla\eta_2|^2\Big)\,dz =: I_1 + I_2.
\end{aligned}$$

Using Lemma 4.2.4 and Hölder's inequality in space we can estimate

$$\begin{aligned}
I_1 &\le C\|v_2\|_{L^\infty(\Omega_{\eta_2})}\int_\omega \int_0^{\eta_1(t,x)} |v_1|^2\,dy(|\nabla\eta_1| + |\nabla\eta_2|)\Big(1 + |\nabla^2\eta_1| + |\nabla^2\eta_2|\Big)\,dx \\
&\le C\|v_2\|_{L^\infty(\Omega_{\eta_2})}(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty})(\|\eta_1\|_{2,2} + \|\eta_2\|_{2,2} + 1)\Big\|\int_0^{\eta_1(t,x)} |v_1|^2\,dy\Big\| \\
&\le C\|v_2\|_{L^\infty(\Omega_{\eta_2})}(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty})(\|v_1\|^2 + \|v_1\|\|\nabla v_1\| + \|\partial_t\eta_1\|\|\nabla\eta_1\|_\infty) \\
&\le C(\|v_2\|_{W^{1,s}(\Omega_2)} + 1)^2(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} + 1)^2(\|v_1\|^2 + \|\partial_t\eta_1\|^2) + C\|v_1\|_{1,2}^2
\end{aligned}$$

Since $v_2 \in L^r(0,T;W^{1,s}(\Omega_{\eta_2}))$ for some $r > 2$ and $\eta_1, \eta_2 \in L^q(0,T;W^{1,\infty}(\omega))$ for all $q < \infty$ (Theorem 4.2.2) $\|v_2\|_{L^\infty(\Omega_{\eta_2})}(\|\eta_1\|_{1,\infty} + \|\eta_1\|_{1,\infty} \in L^2([0,T])$. As additionally $v_1 \in L^\infty(0,T;L^2(\Omega_1)) \cap L^2(0,T,H^2(\Omega_1))$ and $\partial_t\eta_1 \in L^\infty(0,T;L^2(\omega))$ the last term is bounded in time.

To estimate I_2 note that as $v_1 \in L^2(0,T;H^1(\Omega_1)) \hookrightarrow L^2(0,T;L^\alpha(\Omega_1))$ for all $\alpha \in [1,6)$ we find for all $a < 3/2$ (i.e. $(\frac{2}{a})' < 4$)

$$\|v_1\|_a \le \|v_1\|_2\|v_1\|_{(\frac{2}{a})'} \le \|v_1\|\|v_1\|_{1,2}$$

Now choose $p > 1$, $q > 3$ such that $qp < s$ and $pq' < 3/2$.

$$\begin{aligned}
I_2 &\le C(1 + \|\nabla\eta_1\|_{2p'} + \|\nabla\eta_2\|_{2p'})\||\nabla\tilde{v}_2||v_1|^2\|_p \\
&\le C(\|\eta_1\|_{2,2} + \|\eta_2\|_{2,2})\|\nabla\tilde{v}_2\|_{pq}\||v_1|^2\|_{pq'} \le C\|v_2\|_{W^{1,s}(\Omega_2)}\|v_1\|\|v_1\|_{1,2}
\end{aligned}$$

which is bounded in time due to the regularities on v_2 and v_1. We continue with (4.3.6). We write

$$\begin{aligned}
\int_0^t (v_1, \partial_t\hat{v}_2 - \partial_t\hat{v}_{2,\delta})\,dt &= \int_0^t (\gamma J^{-T}v_1, \partial_t\tilde{v}_2 - \partial_t\tilde{v_{2,\delta}})\,dt + \int_0^t (v_1, \partial_t(\gamma J^{-1})(\tilde{v}_2 - \tilde{v}_{2,\delta}))\,dt \\
&= \int_0^t \langle J^{-T}\tilde{v}_1, \partial_t v_2 - \partial_t v_{2,\delta}\rangle_{\eta_2}\,dt + \sum_{i=1}^2 \int_0^t (v_1^i, \partial_t\gamma\tilde{v}_2^i - \tilde{v}_{2,\delta}^i))\,dt \\
&\quad - \sum_{i=1}^2 \int_0^t (y\partial_{x_i}\partial_t\gamma, v_1^i(\tilde{v}_2^3 - \tilde{v}_{2,\delta}^3))\,dt =: (i) + (ii) + (iii)
\end{aligned}$$

The term (i) converges to 0 by Lemma 4.2.6 using that by an analogous estimate to Lemma 4.3.3, (3) we find that $J^{-T}\tilde{v}_1 \in L^2(W^{1,p}(\Omega_2)$ for all $p \in (1,2)$. The term (ii) converges directly by Lemma 4.2.6 and Lemma 4.3.3. On the term (iii) we integrate by parts to find that

$$|(iii)| \le \int_0^t \int_{\Omega_1} |\partial_t\gamma||\nabla(v_1(\tilde{v}_2^3 - \tilde{v}_{2,\delta}^3)|\,dz\,dt$$

which can be bounded uniformly (using Lemma 4.2.6 and Lemma 4.3.3 again) and therefore converges to 0. The estimate on the part involving symmetric gradients is straight forward using the bounds in Lemma 4.2.6 and Lemma 4.3.3. It remains to show that the first term in (4.3.6) converges. For that we simply use the fact that we chose t to be a Lebesgue point of all involved quantities. Hence by the very definition of $\hat{v}_{2,\delta}$, we find that

$$\lim_{\delta\to 0}(v_1(t),\hat{v}_{2,\delta}(t)) = (v_1(t),\hat{v}_2(t)).$$

For the last statement observe that for all $p \in [1,2)$ by the calculations in Lemma 4.3.3 that $w_2 = v_2 - \hat{v}_1 \in L^2(0,T;W^{1,p}(\Omega_2))$ and therefore by Lemma 4.2.6 $w_{2,\delta} \in L^2(0,T;W^{1,p}(\Omega_2))$. This holds in particular for $p = r'$ which yields that the first two terms are in $L^1(0,T)$. Further, since $\nabla v_2 \in L^2(0,T;L^s(\Omega_2))$ for $s > 3$ Hölder's inequality implies for some $q > \frac{6}{5}$

$$\|[\nabla v_2]v_2\|_{L^q(\Omega_2)} \le \|v_2\|_{L^2(\Omega_2)}\|\nabla v_2\|_{L^{2q'}(\Omega_2)}.$$

Choosing $q > 6/5$ such that $(2/q') < s$ bounds the right hand side in $L^2([0,T])$. As by embedding $w_{2,\delta} \in L^2(0,T;L^a(\Omega_2)$ for all $a \in [1,6)$ we find that $[\nabla v_2]v_2 \cdot w_2 \in L^1(0,T;L^1(\Omega_2))$.

□

4.3.3 The stability estimate (Proof of Theorem 4.1.5)

We have collected all the necessary notations and estimates to start the stability estimate. The estimate is derived by testing first the equation of (v_2,η_2) by $(w_{2,\delta},\partial_t\eta_\delta)$, second the energy inequality for (v_1,η_1) and finally testing (v_1,η_1) with $(\hat{v}_{2,\delta},\partial_t\eta_{2,\delta})$.

Testing the equation of (v_2,η_2) by $(w_{2,\delta},\partial_t\eta_\delta)$, integration by parts and Reynold's transport theorem give

$$\begin{aligned}&\int_0^t \langle\partial_t v_2 + [\nabla v_2]v_2, w_{2,\delta}\rangle_{\eta_2} + \langle\varepsilon v_2,\varepsilon w_{2,\delta}\rangle_{\eta_2} - \langle f_2, w_{2,\delta}\rangle_{\eta_2}\,dt\\ &\quad + (\partial_t\eta_2,\partial_t\eta_\delta) - (\partial_t\eta_{2,0},\partial_t\eta_0) - \int_0^t(\partial_t\eta_2,\partial_t^2\eta_\delta) - (\Delta\eta_2,\Delta\partial_t\eta_\delta) - (g_2,\partial_t\eta_\delta)dt = 0.\end{aligned}\tag{4.3.7}$$

We can write this

$$\begin{aligned}&\int_0^t \langle\partial_t v_2 + [\nabla v_2]v_2, w_2\rangle_{\eta_2} + \langle\varepsilon v_2,\varepsilon w_2\rangle_{\eta_2} - \langle f_2, w_2\rangle_{\eta_2}\,dt\\ &\quad + (\partial_t\eta_2,\partial_t\eta_\delta) - (\partial_t\eta_{2,0},\partial_t\eta_0) - \int_0^t(\partial_t\eta_2,\partial_t^2\eta_\delta) - (\Delta\eta_2,\Delta\partial_t\eta_\delta) - (g_2,\partial_t\eta_\delta)dt = K_{1,\delta}\end{aligned}\tag{4.3.8}$$

where

$$K_{1\,\delta} := \int_0^t \langle\partial_t v_2 + [\nabla v_2]v_2, w_2 - w_{2,\delta}\rangle_{\eta_2} + \langle\varepsilon v_2,\varepsilon(w_2 - w_{2,\delta})\rangle_{\eta_2} - \langle f_2, w_2 - w_{2,\delta}\rangle_{\eta_2}$$

Then $K_{1,\delta}\to 0$ for $\delta\to 0$ by Lemma 4.3.4.

The next step is to transform the equation for v_2,η_2 to the domain Ω_1. In particular we want to prove an estimate for

$$\int_0^t(\partial_t\hat{v}_2 + \nabla\hat{v}_2\hat{v}_2, w_1) + (\nabla\hat{v}_2,\nabla w_1) - (\tilde{f}_2, w_1)\,dt$$

First compute

$$\begin{aligned}(\partial_t\hat{v}_2, w_1) &= (\gamma J^{-1}\partial_t\tilde{v}_2 + \partial_t(\gamma J^{-1})\tilde{v}_2), w_1))\\ &= \langle\tilde{J}^{-1}((\partial_t\tilde{v}_2)\circ\psi^{-1}),\tilde{w}_1\rangle_{\eta_2} + (\partial_t(\gamma J^{-1})\tilde{v}_2, w_1).\end{aligned}$$

By chain rule we get

$$(\partial_t\tilde{v}_2)\circ\psi^{-1} = \partial_t v_2 + y\gamma^{-1}\partial_t\gamma\partial_y v_2,$$

Also using $w_2 = \gamma^{-1}\tilde{J}\tilde{w_1}$ (cf. (4.3.1)) this gives

$$\begin{aligned}\tilde{J}^{-1}(\partial_t\tilde{v}_2)\circ\psi^{-1}\cdot\tilde{w}_1 &= \partial_t v_2\cdot w_2 + \partial_t v_2\cdot(\tilde{J}^{-t}\tilde{w_1} - w_2) + y\gamma^{-1}\partial_t\gamma\tilde{J}^{-1}\partial_y v_2\cdot\tilde{w}_1\\ &= \partial_t v_2\cdot w_2 + \partial_t v_2\cdot(\tilde{J}^{-t} - \gamma^{-1}\tilde{J})\tilde{w}_1 + y\gamma^{-1}\partial_t\gamma\tilde{J}^{-1}\partial_y v_2\cdot\tilde{w}_1,\end{aligned}$$

which yields

$$\begin{aligned}\langle\partial_t v_2, w_2\rangle_{\eta_2} &= (\partial_t\hat{v}_2, w_1) - (\partial_t(\gamma J^{-1})\tilde{v}_2, w_1) - \langle\partial_t v_2, (\tilde{J}^{-t} - \gamma^{-1}\tilde{J})\tilde{w}_1\rangle_{\eta_2}\\ &\quad + \langle y\gamma^{-1}\partial_t\gamma\tilde{J}^{-1}\partial_y v_2, \tilde{w}_1\rangle_{\eta_2} =: (\partial_t\hat{v}_2, w_1) + R_1.\end{aligned} \tag{4.3.9}$$

Estimate of R_1. With similar estimates as in the proof of Lemma 4.3.3 we get

$$|\tilde{J}^{-t} - \gamma^{-1}\tilde{J}| \le C(|1-\gamma| + |\nabla\gamma|), \quad |\nabla(\tilde{J}^{-t} - \gamma^{-1}\tilde{J})| \le C(|\nabla\gamma| + |\nabla^2\gamma|) \tag{4.3.10}$$

Hence as in the proof of Lemma 4.3.3 (1) we have (using also Lemma 4.3.3 (1))

$$\|(\tilde{J}^{-t} - \gamma^{-1}\tilde{J})\tilde{w}_1\|_{W^{1,q}(\Omega_2)} \le C\|\eta\|_{2,2}\|w_1\|_{1,2}$$

for all $q \in [1,2)$. This yields for $p' \in (2,r]$

$$\begin{aligned}\langle\partial_t v_2, (\tilde{J}^{-t} - \gamma^{-1}\tilde{J})\tilde{w}_1\rangle_{\eta_2} &\le \|\partial_t v_2\|_{\tilde{W}^{-1,p'}(\Omega_2)}\|(\tilde{J}^{-t} - \gamma^{-1}\tilde{J})\tilde{w}_1\|_{W^{1,p}(\Omega_2)}\\ &\le C_\epsilon\|\partial_t v_2\|_{-1,r}^2\|\eta\|_{2,2}^2 + \epsilon\|w_1\|_{1,2}^2.\end{aligned}$$

By Remark 4.3.2 we have for $p \in (1,3/2)$, $q \in (p,3/2)$ and $a \in (6q/(6-q),2)$

$$\||\partial_t\gamma||\nabla\gamma||\tilde{w}_1|\|_{L^p(\Omega_2)} \le \|\nabla\gamma\|_{1,2}\|\partial_t\gamma\|\|\tilde{w}_1\|_{L^q(\Omega_2)} \le \|\nabla\gamma\|_{1,2}\ \|\partial_t\gamma\|_2\|\tilde{w}_1\|_{W^{1,a}(\Omega_2)}$$

Thus by (4.3.10) and Lemma 4.3.3, we get for $p = s' \in (1,3/2)$

$$\begin{aligned}\langle y\gamma^{-1}\partial_t\gamma\tilde{J}^{-1}\partial_y v_2, \tilde{w}_1\rangle_{\eta_2} &\le C\|v_2\|_{1,s}\|\ \||\partial_t\gamma||\nabla\gamma||\tilde{w}_1|\|_{1,p}\\ &\le C_\epsilon\|\partial_t\eta\|_2^2\|\eta\|_{2,2}\|v_2\|_{W^{1,s}(\Omega_2)}^2 + \epsilon\|w_1\|_{1,2}^2.\end{aligned}$$

Next compute

$$\partial_t(\gamma J^{-1}) = \begin{pmatrix}\partial_t\gamma & 0\\ -y\partial_t\nabla\gamma & 0.\end{pmatrix}.$$

By Hölder's inequality we get for all $p \in (3,s)$ and $q = 2(p/2)' < 6$

$$\||\nabla\tilde{v}_2||w_1|\| \le \|\nabla\tilde{v}_2\|_p\|w_1\|_q \le \|v_2\|_{W^{1,s}(\Omega_2)}\|w_1\|_{1,2},$$

also

$$\||\tilde{v}_2||\nabla w_1|\|_2 \le \|\tilde{v}_2\|_\infty\|w_1\|_{1,2} \le \|v_2\|_{W^{1,s}(\Omega_2)}\|w_1\|_{1,2}$$

This yields

$$\begin{aligned}(\partial_t(\gamma J^{-1}), \tilde{v}_2, w_1) &\le C\|\partial_t\gamma\|\|\tilde{v}_2\|_\infty\|w_1\|_{1,2} + \|\partial_t\nabla\gamma\|_{-1,2}\|\tilde{v}_2^1 w_1^2\|_{1,2}\\ &\le C_\epsilon\|\partial_t\eta\|^2\|v_2\|_{W^{1,s}(\Omega_2)}^2 + \epsilon\|w_1\|_{1,2}^2.\end{aligned}$$

In conclusion

$$|R_1| \le C_\epsilon(\|\eta\|_{2,2}^2 + \|\partial_t\eta\|^2)(\|v_2\|_{W^{1,s}(\Omega_2)}^2 + \|\partial_t v_2\|_{\tilde{W}^{-1,r}(\Omega_2)}^2) + \epsilon\|w_1\|_{1,2}^2. \tag{4.3.11}$$

□

To symplify Notation in the next step, for a Matrix $A \in \mathbb{R}^{3\times 3}$ we denote the symmetric part of it as $A^s = \frac{1}{2}(A + A^t)$. We get by transformation and chain rule

$$\langle \varepsilon v_2, \varepsilon w_2 \rangle_{\eta_2} = (\gamma(\nabla \tilde{v}_2 J^{-1})^s, (\nabla \tilde{w}_2 J^{-1})^s)$$

By (4.3.1)

$$\begin{aligned}\gamma \nabla \tilde{w}_2 J^{-1} &= \gamma \nabla(\gamma^{-1} J w_1) J^{-1} = J \nabla w_1 J^{-1} + \gamma \nabla(\gamma^{-1} J) w_1 J^{-1} \\ &= \nabla w_1 + \nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla(\gamma^{-1} J) w_1 J^{-1}.\end{aligned}$$

and using $\hat{v}_2 = \gamma J^{-1} v_2$

$$\begin{aligned}\nabla \tilde{v}_2 J^{-1} = \nabla \tilde{v}_2 + \nabla \tilde{v}_2 (J^{-1} - I) &= \nabla \hat{v}_2 + \nabla((I - \gamma J^{-1}) \tilde{v}_2) + \nabla \tilde{v}_2 (J^{-1} - I) \\ &= \nabla \hat{v}_2 + (I - \gamma J^{-1}) \nabla \tilde{v}_2 - \nabla(\gamma J^{-1}) \tilde{v}_2 + \nabla \tilde{v}_2 (J^{-1} - I)\end{aligned}$$

Hence

$$\begin{aligned}&\big(\gamma(\nabla \tilde{v}_2 J^{-1})^s : (\nabla \tilde{w}_2 J^{-1})^s\big) \\ &= \big((\nabla \tilde{v}_2 J^{-1})^s, \varepsilon w_1 + \big[\nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla(\gamma^{-1} J) w_1 J^{-1}\big]^s\big) \\ &= (\varepsilon \hat{v}_2, \varepsilon w_1) + \big((\nabla \tilde{v}_2 J^{-1})^s, \big[\nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1} + \gamma \nabla(\gamma^{-1} J) w_1 J^{-1}\big]^s\big) \\ &\quad + \big(\big[(I - \gamma J^{-1}) \nabla \tilde{v}_2 - \nabla(\gamma J^{-1}) \tilde{v}_2 + \nabla \tilde{v}_2 (J^{-1} - I)\big]^s, \varepsilon w_1\big) \\ &=: (\varepsilon \hat{v}_2 : \varepsilon w_1) + R_2.\end{aligned} \tag{4.3.12}$$

Estimate of R_2. By the definition of J it is straightforward to see that

$$\begin{aligned}|J^{-1}| &\le C(1 + |\nabla \gamma|), \\ |J^{-1} - I| + |J - I| + |\gamma J^{-1} - I| &\le C(|\gamma - 1| + |\nabla \gamma|).\end{aligned}$$

By Hölder's inequality we get $\||\nabla \tilde{v}_2||\nabla w_1|\|_{6/5} \le \|\nabla \tilde{v}_2\|_3 \|\nabla w_1\|_2$ and thus for $p = 6/5$

$$\begin{aligned}\big((\nabla \tilde{v}_2 J^{-1})^s, \big[\nabla w_1 (J^{-1} - I) + (J - I) \nabla w_1 J^{-1}\big]^s\big) + \big(\big[(I - \gamma J^{-1}) \nabla \tilde{v}_2 + \nabla \tilde{v}_2 (J^{-1} - I)\big]^s, \varepsilon w_1\big) \\ \le C \|\nabla \gamma\|_{3p'} \, \||\nabla \tilde{v}_2| \, |\nabla w_1|\|_p \le \|\eta\|_{2,2} (\|\nabla v_2\|_s \|\nabla w_1\|) \\ \le C_\epsilon \|\eta\|_{2,2}^2 \|v_2\|_{W^{1,s}(\Omega_2)}^2 + \epsilon \|\nabla w_1\|^2\end{aligned}$$

Furthermore as in the proof of Lemma 4.3.3 we get for $p \in (3, s)$ (i.e. $p' \in (s', 3/2)$)

$$\begin{aligned}\big((\nabla \tilde{v}_2 J^{-1})^s : (\nabla(\gamma^{-1} J) w_1 J^{-1})^s\big) &\le C \|(1 + |\nabla \gamma| + |\nabla \gamma|^2 + |\nabla \gamma|^3) |\nabla \tilde{v}_2|\|_p \||\nabla^2 \gamma||w_1|\|_{p'} \\ &\le C \|\eta\|_{2,2} \|\nabla v_2\|_{1,s} \|w_1\|_{1,2} \le C_\epsilon \|\eta\|_{2,2}^2 \|v_2\|_{W^{1,s}(\Omega_2)}^2 + \epsilon \|w_1\|_{1,2}^2\end{aligned}$$

and

$$\big((\nabla(\gamma J^{-1}) \tilde{v}_2)^s, \varepsilon w_1\big) \le C \|(|\nabla^2 \gamma| + |\nabla \gamma|) |\tilde{v}_2|\| \, \|\nabla w_1\| \le C_\epsilon \|v_2\|_{W^{1,s}(\Omega_2)}^2 \|\eta\|_{2,2}^2 + \epsilon \|w_1\|_{1,2}^2$$

In conclusion

$$|R_2| \le C_\epsilon \|\eta\|_{2,2}^2 \|v_2\|_{W^{1,s}(\Omega_2)}^2 + \epsilon \|w_1\|_{1,2}^2 \tag{4.3.13}$$

□

Next by chain rule and (4.3.1) we get

$$\begin{aligned}\langle [\nabla v_2] v_2, w_2 \rangle_{\eta_2} &= ([\nabla \tilde{v}_2] \gamma J^{-1} \tilde{v}_2, \gamma^{-1} J w_1) = ([\nabla \tilde{v}_2] \gamma J^{-1} \tilde{v}_2, \gamma^{-1} J w_1) \\ &= ([\nabla \tilde{v}_2] \hat{v}_2, w_1) + ([\nabla \tilde{v}_2] \hat{v}_2, (\gamma^{-1} J - I) w_1) \\ &= ([\nabla \hat{v}_2] \hat{v}_2, w_1) + ([\nabla((I - \gamma^{-1} J) \tilde{v}_2)] \hat{v}_2, w_1) + ([\nabla \tilde{v}_2] \hat{v}_2, (\gamma^{-1} J - I) w_1) \\ &:= ([\nabla \hat{v}_2] \hat{v}_2, w_1) + R_3\end{aligned} \tag{4.3.14}$$

Estimate on R_3. With similar estiamtes as above we can conclude

$$\begin{aligned}([\nabla\tilde{v}_2]\hat{v}_2,(\gamma^{-1}J-I)w_1) &\leq C\|\hat{v}_2\|_\infty\|(\gamma^{-1}-J^t)\nabla\tilde{v}_2\|\|w_1\|\\ &\leq C\|\eta\|_{2,2}\|v_2\|^2_{W^{1,s}(\Omega_2)}\|w_1\| \leq C\|v_2\|^2_{W^{1,s}(\Omega_2)}(\|\eta\|^2_{2,2}+\|w_1\|^2)\end{aligned}$$

Additionally

$$\begin{aligned}([\nabla((I-\gamma^{-1}J)\tilde{v}_2)]\hat{v}_2,w_1) &\leq C\|\hat{v}_2\|_{L^\infty}\|w_1\|(\|\tilde{v}_2\|_\infty\|\nabla(\gamma J^{-1})\|+\|(I-\gamma J^{-1})\nabla\tilde{v}_2\|\\ &\leq C\|v_2\|^2_{W^{1,s}(\Omega_2)}(\|\eta\|^2_{2,2}+\|w_1\|^2)\end{aligned}$$

Thus

$$|R_3| \leq C\|v_2\|^2_{W^{1,s}(\Omega_2)}(\|\eta\|^2_{2,2}+\|w_1\|^2) \tag{4.3.15}$$

□

Lastly by transformation rule and (4.3.1)

$$\langle f_2,w_2\rangle_{\eta_2} = (\tilde{f}_2,\gamma\tilde{w}_2) = (\tilde{f}_2,w_1)-(\tilde{f}_2,(I-J)w_1) \equiv (\tilde{f}_2,w_1)-R_4. \tag{4.3.16}$$

We find for all $p\in(1,\infty)$

$$R_4 \leq C\|(1-\gamma)\|_{p'}\||\tilde{f}_2\|w_1|\|_p \leq C_\epsilon\|f_2\|^2_{L^2(\Omega_2)}\|\eta\|^2_{2,2}+\epsilon\|w_1\|^2_{1,2} \tag{4.3.17}$$

Adding (4.3.9), (4.3.12), (4.3.14), (4.3.16) and integrating over $(0,t)$ we get

$$\begin{aligned}&\int_0^t(\partial_t\hat{v}_2+[\nabla\hat{v}_2]\hat{v}_2,w_1)+(\varepsilon\hat{v}_2,\varepsilon w_1)-(\tilde{f}_2,w_1)\,dt\\ &=\int_0^t\langle\partial_t v_2+[\nabla v_2]v_2,w_2\rangle_{\eta_2}+\langle\varepsilon v_2,\varepsilon w_2\rangle_{\eta_2}-\langle f_2,w_2\rangle\,dt+R,\end{aligned} \tag{4.3.18}$$

where $R=\int_0^t R_1+R_2+R_3+R_4\,dt$. By (4.3.11), (4.3.13), (4.3.15), (4.3.17) we get

$$\begin{aligned}|R| &\leq \int_0^t h_1(t)(\|\eta\|^2_{2,2}+\|\partial_t\eta\|^2_2+\|w_1\|^2_2)+\epsilon\|w_1\|^2_{1,2}\,dt,\\ h_1(t) &= C_\epsilon(\|v_2\|^2_{W^{1,s}(\Omega_2)}+\|\partial_t v_2\|^2_{W^{-1,s}(\Omega_2)}+\|f_2\|^2_{L^2(\Omega_2)})\in L^1([0,T]).\end{aligned} \tag{4.3.19}$$

We can now estimate the differences of the solutions, namely we estimate

$$\begin{aligned}I := {}&\frac{1}{2}\|w_1\|^2+\frac{1}{2}\left(\|\partial_t\eta\|^2+\|\Delta\eta\|^2\right)+\int_0^t\|\varepsilon w_1\|^2dt\\ = {}&\frac{1}{2}\|v_1\|^2+\frac{1}{2}\left(\|\partial_t\eta_1\|^2+\|\Delta\eta_1\|^2\right)\\ &-(v_1(t),\hat{v}_2(t))-(\partial_t\eta_1,\partial_t\eta_2)-(\Delta\eta_1,\Delta\eta_2)\\ &+\frac{1}{2}\left(\|\hat{v}_2\|^2+\|\eta_2\|^2+\|\Delta\eta_2\|^2\right)\\ &+\int_0^t\|\varepsilon v_1\|^2-(\varepsilon v_1,\varepsilon\hat{v}_2)-(\varepsilon\hat{v}_2,\varepsilon w_1)\,dt.\end{aligned}$$

The energy inequality for (v_1,η_1) gives

$$\begin{aligned}I \leq {}&\frac{1}{2}(\|v_{1,0}\|^2+\|\eta^*_{1,0}\|^2+\|\Delta\eta_{1,0}\|^2)+\int_0^t(f_1,v_1)_{\eta_1}+(g_1,\partial_t\eta_1)\,dt\\ &-(v_1(t),\hat{v}_2(t))-\int_0^t(\varepsilon v_1,\varepsilon\hat{v}_2)dt-(\partial_t\eta_1,\partial_t\eta_2)-(\Delta\eta_1,\Delta\eta_2)\\ &+\frac{1}{2}(\|\hat{v}_2\|^2+\|\partial_t\eta_2\|^2+\|\Delta\eta_2\|^2)-\int_0^t(\varepsilon\hat{v}_2,\varepsilon w_1)\,dt\end{aligned} \tag{4.3.20}$$

By (4.3.8) and (4.3.18) we get

$$-\int_0^t (\varepsilon\hat{v}_2, \varepsilon w_1)\,dt = \int_0^t (\partial_t\hat{v}_2 + ([\nabla\hat{v}_2]\hat{v}_2, w_1) - (\tilde{f}_2, w_1)\,dt + (\partial_t\eta_2, \partial_t\eta_\delta) - (\eta^*_{2,0}, \eta^*_0)$$
$$- \int_0^t \int_\omega \partial_t\eta_2\partial_t^2\eta_\delta - (\Delta\eta_2, \Delta\partial_t\eta_\delta) + (g_2, \partial_t\eta_\delta)dt + K^1_\delta + R.$$

Reynold's transport theorem and $\hat{v}_2(x, \eta_1(x)) = \partial_t\eta_2(x)$ gives

$$\int_0^t (\partial_t\hat{v}_2, w_1)\,dt = \int_0^t (\partial_t\hat{v}_2, v_1 - \hat{v}_2)\,dt$$
$$= -\frac{1}{2}(\|\hat{v}_2\|^2 - \|\hat{v}_{2,0}\|^2 - (\partial_t\eta_1, (\partial_t\eta_2)^2)) + \int_0^t (\partial_t\hat{v}_2, v_1)\,dt$$

Inserting this calculation in (4.3.20) yields

$$I \le \frac{1}{2}((\|v_{1,0}\|^2 + \|\hat{v}_{2,0}\|^2) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t\hat{v}_2) - (\varepsilon v_1, \varepsilon\hat{v}_2) + (f_1, v_1) - (\tilde{f}_2, w_1)dt$$
$$+ \frac{1}{2}(\|\eta^*_{1,0}\|^2 + \|\Delta\eta_{1,0}\|^2 + \|\partial_t\eta_2(t)\|^2 + \|\Delta\eta_2(t)\|^2) - (\partial_t\eta_1(t), \partial_t\eta_2(t)) - (\Delta\eta_1(t), \Delta\eta_2(t))$$
$$+ (\partial_t\eta_2(t), \partial_t\eta_\delta(t)) - (\eta^*_{2,0}, \eta^*_0) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_\delta) - (\Delta\eta_2, \Delta\partial_t\eta_\delta) - (g_1, \partial_t\eta_1) + (g_2, \partial_t\eta_\delta)\,dt$$
$$+ \int_0^t ([\nabla\hat{v}_2]\hat{v}_2, w_1) + \frac{1}{2}(\partial_t\eta_1, (\partial_t\eta_2)^2) + K^1_\delta + R$$

We denote the first line of the right hand side as I_1 the second and third line as I_2 and the fourth line as I_3. We calculate that

$$\frac{1}{2}(\|v_{1,0}\|^2 + \|\hat{v}_{2,0}\|^2) + \int_0^t (f_1, v_1) - (\tilde{f}_2, w_1)\,dt = (v_{1,0}, \hat{v}_{2,0}) + \frac{1}{2}\|v_{1,0} - \hat{v}_{2,0}\|^2 + \int_0^t (f_1, \hat{v}_2) + (f_1 - \tilde{f}_2, w_1)\,dt.$$

Thus

$$I_1 = (v_{1,0}, \hat{v}_{2,0}) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t\hat{v}_2) - (\nabla v_1, \nabla\hat{v}_2) + (f_1, \hat{v}_2)\,dt$$
$$+ \frac{1}{2}\|v_{1,0} - \hat{v}_{2,0}\|_2^2 + \int_0^t (f_1 - \tilde{f}_2, w_1)\,dt$$

We write the first line as

$$(v_{1,0}, \hat{v}_{2,0}) - (v_1(t), \hat{v}_2(t)) + \int_0^t (v_1, \partial_t\hat{v}_2) - (\nabla v_1, \nabla\hat{v}_2) + (f_1, \hat{v}_2)\,dt$$
$$= (v_{1,0}, \hat{v}_{2,0}) - (v_1(t), \hat{v}_{2,\delta}(t)) + \int_0^t (v_1, \partial_t\hat{v}_{2,\delta}) - (\nabla v_1, \nabla\hat{v}_{2,\delta})) + (f_1, \hat{v}_{2,\delta})\,dt + K_{2,\delta},$$

with

$$K_{2,\delta} = -(v_1, \hat{v}_2 - \hat{v}_{2,\delta}) + \int_0^t (v_1, \partial_t\hat{v}_2 - \partial_t\hat{v}_{2,\delta}) - (\nabla v_1, \nabla\hat{v}_2 - \nabla\hat{v}_{2,\delta}) + (f_1, \hat{v}_2 - \hat{v}_{2,\delta})\,dt,$$

which converges to zero for $\delta \to 0$ by Lemma 4.3.4. We divide I_2 into the parts that depend solely on η_2 and the rest:

$$I_2 = \frac{1}{2}(\|\Delta\eta_2\|^2 - \|\partial_t\eta_2\|^2) + \|\eta^*_{2,0}\|^2 + \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{2,\delta}) - (\Delta\eta_2, \partial_t\Delta\eta_{2,\delta})\,dt$$
$$+ \frac{1}{2}(\|\eta^*_{1,0}\|^2 + \|\Delta\eta_{1,0}\|^2) - (\partial_t\eta_1(t), \partial_t\eta_2(t)) - (\Delta\eta_1(t), \Delta\eta_2(t))$$
$$+ (\partial_t\eta_2(t), \partial_t\eta_{1,\delta}(t)) - (\eta^*_{2,0}, \eta^*_{1,0}) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{1,\delta}) - (\Delta\eta_2, \Delta\partial_t\eta_{1,\delta}) - (g_1, \partial_t\eta_1) + (g_2, \partial_t\eta_\delta)\,dt$$

We denote the first line by I_{21} and find that

$$\begin{aligned} I_{21} &= \frac{1}{2}(\|\eta_{2,0} *\|^2 + \|\Delta\eta_{2,0}\|^2) \\ &+ \frac{1}{2}(\|\eta^*_{2,0}\|^2 - \|\partial_t\eta_2\|^2 + \|\Delta\eta_2\|^2 - \|\Delta\eta_{2,0}\|^2) + \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{2,\delta}) - (\Delta\eta_2, \partial_t\Delta\eta_{2,\delta})\, dt \\ &=: \frac{1}{2}(\|\eta^*_{2,0}\|^2 + \|\Delta\eta_{2,0}\|^2) + K_{3,\delta} \end{aligned}$$

where $K_{3,\delta} \to 0$ for $\delta \to 0$ by Lemma 4.2.5.

Collecting the above we arrive at

$$\begin{aligned} I &\le (v_{1,0}, \hat{v}_{2,0}) - (v_1(t), \hat{v}_{2,\delta}(t)) + \int_0^t (v_1, \partial_t\hat{v}_{2,\delta}) - (\varepsilon v_1, \varepsilon\hat{v}_{2,\delta})) + (f_1, \hat{v}_{2,\delta})\, dt \\ &+ \frac{1}{2}(\|\eta^*_{1,0}\|^2 + \|\eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0}\|^2 + \|\Delta\eta_{2,0}\|^2) + (\partial_t\eta_{1,\delta}(t) - \partial_t\eta_1(t), \partial_t\eta_2(t)) - (\Delta\eta_1(t), \Delta\eta_2(t)) \\ &- (\eta^*_{2,0}, \eta^*_{1,0}) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{1,\delta}) - (\Delta\eta_2, \Delta\partial_t\eta_{1,\delta}) - (g_1, \partial_t\eta_1) + (g_2, \partial_t\eta_\delta)\, dt \\ &+ \frac{1}{2}\|v_{1,0} - \hat{v}_{2,0}\|_2^2 + \int_0^t (f_1 - \tilde{f}_2, w_1)\, dt + I_3 + K_{2,\delta} + K_{3,\delta}. \end{aligned} \tag{4.3.21}$$

Now we use the equation vor (v_1, η_1) and test it with $\hat{v}_{2,\delta}$:

$$\begin{aligned} &(v_{1,0}, \hat{v}_{2,0}) - (v_1(t), \hat{v}_{2,\delta}(t)) + \int_0^t (v_1, \partial_t\hat{v}_{2,\delta}) - (\varepsilon v_1, \varepsilon\hat{v}_{2,\delta}) + (f_1, \hat{v}_{2,\delta})\, dt \\ &= -\int_0^t (v_1 \otimes v_1, \nabla\hat{v}_{2,\delta})\, dt + (\partial_t\eta_1(t), \partial_t\eta_{2,\delta}(t)) - (\eta^*_{1,0}, \eta^*_{2,0}) \\ &\quad - \int_0^t (\partial_t\eta_1, \partial_t^2\eta_{2,\delta}) - (\Delta\eta_1, \Delta\partial_t\eta_{2,\delta}) + (g_1, \partial_t\eta_{2,\delta})\, dt \end{aligned} \tag{4.3.22}$$

Note that

$$\begin{aligned} &\frac{1}{2}(\|\eta^*_{1,0}\|^2 + \|\eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0}\|^2 + \|\Delta\eta_{2,0}\|^2) \\ &\quad = (\eta^*_{1,0}, \eta^*_{2,0}) + (\Delta\eta_{1,0}, \Delta\eta_{2,0}) + \frac{1}{2}(\|\eta^*_{1,0} - \eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0} - \Delta\eta_{2,0}\|^2) \end{aligned}$$

and

$$\begin{aligned} (g_1, \partial_t\eta_1) - (g_2, \partial_t\eta_\delta) &= (g_1, \partial_t\eta_1) - (g_2, \partial_t\eta) + (g_2, \partial_t\eta - \partial_t\eta_\delta) \\ &= (g_1, \partial_t\eta_2) + (g_1 - g_2, \partial_t\eta) + (g_2, \partial_t\eta - \partial_t\eta_\delta). \end{aligned}$$

This gives

$$\begin{aligned} I &\le -\int_0^t ([\nabla\hat{v}_2]v_1, v_1)\, dt + \frac{1}{2}(\|v_{1,0} - \hat{v}_{2,0}\|_2^2 + \|\eta^*_{1,0} - \eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0} - \Delta\eta_{2,0}\|^2) \\ &\quad + \int_0^t (f_1 - \tilde{f}_2, w_1) + (g_1 - g_2, \partial_t\eta)\, dt + K_{2,\delta} + K_{3,\delta} + K_{4,\delta} + I_3 \end{aligned} \tag{4.3.23}$$

where

$$\begin{aligned} K_{4,\delta} &= (\Delta\eta_{1,0}, \Delta\eta_{2,0}) - (\Delta\eta_1(t), \Delta\eta_2(t)) + \int_0^t (\Delta\eta_2, \partial_t\Delta\eta_{1,\delta}) + (\Delta\eta_1, \partial_t\Delta\eta_{2,\delta})\, dt + (\partial_t\eta_{1,\delta} - \partial_t\eta_1, \partial_t\eta_2) \\ &+ (\partial_t\eta_1(t), \partial_t\eta_{2,\delta}(t)) - (\eta^*_{1,0}, \eta^*_{2,0}) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{1,\delta}) + (\partial_t\eta_1, \partial_t^2\eta_{2,\delta})\, dt \\ &+ \int_0^t (g_1, \partial_t\eta_2 - \partial_t\eta_{2,\delta}) + (g_2, \partial_t\eta - \partial_t\eta_\delta)\, dt \end{aligned}$$

Proof that $K_{4,\delta} \to 0$. The first and third line of $K_{4,\delta}$ converge to 0 again by Lemma 4.2.5. We write the second line as

$$\begin{aligned}&(\partial_t\eta_1(t), \partial_t\eta_{2,\delta}(t)) - (\eta^*_{1,0}, \eta^*_{2,0}) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{1,\delta}) + (\partial_t\eta_1, \partial_t^2\eta_{2,\delta})\, dt\\ &\quad= (\partial_t\eta_1(t), \partial_t\eta_{2,\delta}(t) - \partial_t\eta_2(t)) + (\partial_t\eta_1(t), \partial_t\eta_2(t)) - (\eta^*_{1,0}, \eta^*_{2,0}) - \int_0^t (\partial_t\eta_2, \partial_t^2\eta_{1,\delta}) + (\partial_t\eta_1, \partial_t^2\eta_{2,\delta})\, dt,\end{aligned}$$

which also converges to 0 for $\delta \to 0$ by Lemma 4.2.5. Thus $K_{4,\delta} \to 0$ for $\delta \to 0$. □

We continue by writing

$$\begin{aligned}-\int_0^t (v_1 \otimes v_1, \nabla\hat{v}_{2,\delta})\, dt &= -\int_0^t (v_1 \otimes v_1, \nabla\hat{v}_2)\, dt + \int_0^t (v_1 \otimes v_1, \nabla\hat{v}_2 - \nabla\hat{v}_{2,\delta})\, dt\\ &=: -\int_0^t (v_1 \otimes v_1, \nabla\hat{v}_2)\, dt + K_{5,\delta},\end{aligned}$$

where $K_{5,\delta} \to 0$ by Lemma 4.3.4. Inserting this and the definition of I_3 in (4.3.23) finally yields

$$\begin{aligned}I &\le \int_0^t -(v_1 \otimes v_1, \nabla\hat{v}_2) + ([\nabla\hat{v}_2]\hat{v}_2, w_1) + \frac{1}{2}(\partial_t\eta_1, (\partial_t\eta_2)^2)\, dt\\ &+ \frac{1}{2}(\|v_{1,0} - \hat{v}_{2,0}\|_2^2 + \|\eta^*_{1,0} - \eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0} - \Delta\eta_{2,0}\|^2) + \int_0^t (f_1 - \tilde{f}_2, w_1) + (g_1 - g_2, \partial_t\eta)\, dt\\ &+ R + K_{1,\delta} + K_{2,\delta} + K_{3,\delta} + K_{4,\delta} + K_{5,\delta}\end{aligned} \tag{4.3.24}$$

The first line can be estimated as follows. As $\mathrm{d}v_1 = 0$ we get by Gaußintegral formula

$$([\nabla\hat{v}_2]v_1, \hat{v}_2)_{\eta_1} = \frac{1}{2}((\partial_t\eta_2)^2), \partial_t\eta_1)_\omega$$

Hence

$$(v_1 \otimes v_1, \nabla\hat{v}_2) = ([\nabla\hat{v}_2]v_1, v_1 - \hat{v}_2) + ([\nabla\hat{v}_2]v_1, \hat{v}_2) = ([\hat{v}_2]v_1, w_1) + \frac{1}{2}((\partial_t\eta_2)^2), \partial_t\eta_1)$$

Thus we get

$$\int_0^t -(v_1 \otimes v_1, \nabla\hat{v}_2) + ([\nabla\hat{v}_2]\hat{v}_2, w_1) + \frac{1}{2}(\partial_t\eta_1, (\partial_t\eta_2)^2)\, dt = -\int_0^t ([\nabla\hat{v}_2]w_1, w_1)\, dt$$

We can estimate this term the same way as (4.3.5) in Lemma 4.3.4 by replacing v_1 by w_1 and $\partial_t\eta_1$ by $\partial_t\eta$. We find

$$([\nabla\hat{v}_2]w_1, w_1) \le C_\epsilon(\|v_2\|_{W^{1,s}(\Omega_2)} + 1)^2(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} + 1)^2(\|w_1\|^2 + \|\partial_t\eta\|^2) + \epsilon\|w_1\|_{1,2}^2$$

Thus

$$\begin{aligned}\int_0^t ([\nabla\hat{v}_2]w_1, w_1)\, dt &\le \int_0^t h_2(t)(\|\partial_t\eta\|^2 + \|w_1\|^2) + \epsilon\|\nabla w_1\|_2^2\, dt,\\ h_2(t) &= (\|v_2\|_{W^{1,s}(\Omega_2)} + 1)^2(\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} + 1)^2.\end{aligned}$$

As $v_2 \in L^r(0, T; W^{1,s}(\Omega_2))$ ($r > 2$) and $\eta_1, \eta_2 \in L^p(0, T; W^{1,\infty}(\omega))$ for all $p \in [1, \infty)$ (by Theorem 4.2.2 and interpolation) we get $h_2 \in L^1([0, T])$.

Thus recalling the estimate on R (4.3.19) we get

$$\begin{aligned}\int_0^t ([\nabla\hat{v}_2]w_1, w_1)\, dt + R &\le \int_0^t h(t)(\|\eta\|_{2,2}^2 + \|\partial_t\eta\|^2 + \|w_1\|^2) + \epsilon\|\nabla w_1\|_2^2\, dt,\\ h &= h_1 + h_2 \in L^1([0, T]).\end{aligned}$$

Since $K_{i,\delta} \to 0$ for $\delta \to 0$ ($i = 1, \dots, 5$) the last estimate leads to

$$\begin{aligned}
&\frac{1}{2}(\|w_1\|^2 + \|\partial_t \eta\|^2 + \|\Delta\eta\|^2) + \int_0^t \|\varepsilon w_1\|^2 dt \\
&\leq \frac{1}{2}(\|v_{1,0} - \hat{v}_{2,0}\|_2^2 + \|\eta^*_{1,0} - \eta^*_{2,0}\|^2 + \|\Delta\eta_{1,0} - \Delta\eta_{2,0}\|^2) + \int_0^t \|f_1 - \tilde{f}_2\|_2 + \|g_1 - g_2\|_2^2 \, dt \\
&\quad + \int_0^t h(t)(\|\eta\|_{2,2}^2 + \|\partial_t\eta\|^2 + \|w_1\|^2) + \epsilon\|w_1\|_{1,2}^2 \, dt
\end{aligned}$$

As η is 0 on the boundary $\|\eta\|_{2,2} \sim \|\Delta\eta\|$. Korn's inequality and the 0 trace of w_1 on B_c implies that $\|w_1\|_{1,2} \sim \|\varepsilon w_1\|_2$. Hence choosing $\epsilon < 1$ small enough we can apply Gronwall's Lemma. this implies a stability estimate in terms of w_1. In order to change to $v_1 - \tilde{v}_2$ one uses

$$\|w_1\|_2 \leq \|v_1 - \tilde{v}_2\|_2 + \|\tilde{v}_2 - \hat{v}_2\| \leq \|v_1 - \tilde{v}_2\|_2 + C\|\eta\|_{1,2};$$

the estimate on the gradients is analogous. This finishes the proof of Theorem 4.1.5.

Proof of Theorem 4.1.2. Let (v_1, π_1, η_1) be a weak solution (with $\eta > 0$) on $[0, T]$ for any $T > 0$. Then the stability estimate implies that $\|w_1\| = \|\partial_t\eta\| = \|\eta\|_{2,2} = 0$ a.e. in $[0, T]$. As $\eta = \eta_1 - \eta_2 = 0$ we have $\Omega_1 = \Omega_2$ and in particular the transformation ψ is the identitiy, $\gamma = 1$, $J = \mathbb{I}$. Thus $\hat{v}_2 = v_2$ and $w_1 = 0$ gives $v_1 = v_2$. This proves Theorem 4.1.2. □

Chapter 5

A variational approach to fluid-structure interactions

Within this chapter, we provide existence of weak solutions to the *parabolic* fluid-structure interaction problem, i.e. the inertia-less balances (1.2.4)-(1.2.6) together with the coupling conditions (7.1.4)-(7.1.5) as well as Dirichlet boundary conditions for the deformation and a Navier boundary condition stemming from the higher gradients in the energy. This has the interesting difficulty that in fluid structure interaction a naturally Lagrangian solid needs to be coupled with a naturally Eulerian fluid on a variable domain. We will do so without fixing a reference fluid domain and instead use a variational approach to deal with the fluid directly on a varying domain.

The assumption on the solid are done in an axiomatic fashion as this allows for the most free applicability. We introduce them in the next section. In Section 5.4 we will show that the example energy and dissipation does indeed satisfy the required assumptions.

5.1 Mechanical and analytical restrictions on the energy/dissipation functional

As introduced in Section 7.1.1, we consider solid materials for which the stress tensor can be determined by prescribing two functionals; *the energy and dissipation functional*. Materials admitting such modeling are called *generalized standard materials* [95, 155, 119] and many available rheological models fall into this frame [119]. Nonetheless, the two functionals cannot be chosen completely freely, but have to comply with certain physical requirements. We summarize these at this point.

As in examples (1.2.8) and (1.2.9), we will for the sake of discussion assume that the energy and dissipation functional have a density, i.e.

$$E(\eta) = \int_Q e(\nabla\eta, \nabla^2\eta)dx \qquad R(\eta, \partial_t\eta) = \int_Q r(\nabla\eta, \partial_t\nabla\eta)dx, \tag{5.1.1}$$

for all smooth vector fields $\eta : Q \to \mathbb{R}^n$.

Here, the energy density depends on the first and second gradient[1] of the deformation which puts us into the class of so-called non-simple (or second grade) materials (see the pioneering work [180] as well as [173, 66] for later development). In fact, allowing for the energy density to depend on higher order gradients puts us beyond the standard theory of hyperelasticty but allows us to bring along more regularity to the problem.

Any admissible $e(F,G)$ in (5.1.1) should satisfy the frame-indifference

$$e(RF, GR) = e(F,G) \qquad \forall F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times n\times n},$$

[1]The formalism of our proofs naturally also allow for dependence on material and spatial positions x and $\eta(x)$, but the latter dependence is non-physical and the former does not add much to the discussion. Nevertheless we emphasize that our results hold also for inhomogeneous materials.

for any proper rotation R. In other words, the energy remains unchanged upon a change of observer. Moreover, any physical energy will blow up if the material is to be extended or compressed infinitely i.e.

$$e(F,G) \to \infty \qquad \text{if } |F| \to \infty \text{ or } \det F \to 0$$

and it should prohibit change of orientation for the deformations i.e.

$$e(F,G) = \infty \qquad \text{if } \det F \le 0.$$

From an analytical point of view these basic requirement have the following consequences: $e(F,G)$ *cannot be a convex function* of the first variable and $e(F,G)$ *cannot be bounded.* As a result considering the variational approach in the parabolic fluid-structure interaction as well as the two-scale approximation in the hyperbolic case are essential.

Going even further, while these conditions lead to non-convexities, they are at the same time beneficial. Assuming appropriate growth conditions, in particular for $\det F \to 0$, we will be able to deduce a uniform lower bound on the determinant of $\nabla\eta$ in the style of [96]. This will not only result in a meaningful boundary for the fluid domain, but also help us to readily switch between Lagrangian and Eulerian descriptions of the solid velocity.

As for the dissipation potential, we will also need it to be independent of the observer, i.e. for all smoothly time-varying proper rotations $R(t)$, and all smooth time-dependent $F : [0,T] \to \mathbb{R}^n$

$$r(RF, \partial_t(RF)) = r(F, \partial_t F).$$

This restriction implies [6] that r *cannot* depend only on $\partial_t\eta$ but needs to depend on η, too. This in turn, will require us to use *fine Korn type inequalities* [154, 160] to deduce a-priori estimates (as already in [137]). We also note that both for physical as well as for analytic reasons, r should be non-negative and convex in the second variable. We additionally require R to be a quadratic form in its second variable.[2]

Taking into account these, as well as some analytical requirements, we will now detail our set-up for the deformation. Throughout the paper, for the elastic energy potential we impose the following assumptions.

Assumption 5.1.1 (Elastic energy)**.** *We assume that $Q,\Omega \subset \mathbb{R}^n$, $q > n$ and $E : W^{2,q}(Q;\Omega) \to \overline{\mathbb{R}}$ satisfies:*

S1 Lower bound: There exists a number $E_{min} > -\infty$ such that

$$E(\eta) \ge E_{min} \textit{ for all } \eta \in W^{2,q}(Q;\Omega)$$

S2 Lower bound in the determinant: For any $E_0 > 0$ there exists $\epsilon_0 > 0$ such that $\det\nabla\eta \ge \epsilon_0$ for all $\eta \in \{\eta \in W^{2,q}(Q;\Omega) : E(\eta) < E_0\}$.

S3 Weak lower semi-continuity: If $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\Omega)$ then $E(\eta) \le \liminf_{l\to\infty} E(\eta_l)$.

S4 Coercivity: All sublevel-sets $\{\eta \in \mathcal{E} : E(\eta) < E_0\}$ are bounded in $W^{2,q}(Q;\Omega)$.

S5 Existence of derivatives: For finite values E has a well defined derivative which we will formally denote by

$$DE : \{\eta \in \mathcal{E} : E(\eta) < \infty\} \to (W^{2,q}(Q;\mathbb{R}^n))'$$

Furthermore on any sublevel-set of E, DE is bounded and continuous with respect to strong $W^{2,q}$-convergence.

S6 Monotonicity and Minty type property: If $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\Omega)$, then

$$\liminf_{l\to\infty} \langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \ge 0 \textit{ for all } \psi \in C_0^\infty(Q;[0,1]).$$

If additionally $\limsup_{l\to\infty} \langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \le 0$ then $\eta_l \to \eta$ in $W^{2,q}(Q;\Omega)$.

[2] See Theorem 6.2.4 for possible relaxations of the assumptions on the dissipation potential.

Let us shortly elaborate on the above stated assumptions. As elastic energies are generally bounded from below, assumption S1 is a natural one. Similarly, assumption S5 is to be expected as we need to take the derivative of the energy to determine a weak version of the Piola-Kirchhoff stress tensor. Assumptions S3 and S4 are standard in any variational approach as they open-up the possibility for using the direct method of the calculus of variations. Assumption S6 effectively means that the energy density has to be convex in the highest gradient (but of course not convex overall) and allows us to get weak solutions and not merely measure valued ones (as in the case of a solid material in [51]). Finally, assumption S2 is probably the most restricting one and, to the authors' knowledge, necessitates the use of second-grade elasticity, combined with an energy density e which blows up sufficiently fast as $\det F \to 0$ (see [96]). This is, in particular, the case for the model energy (1.2.9).

Definition 5.1.2 (Domain of definition)**.** *The set of functions in* $W^{2,q}(Q;\Omega)$ *(and satisfying the Dirichlet boundary condition) used for minimization in* (1.2.13) *can be expressed as*

$$\mathcal{E} := \left\{ \eta \in W^{2,q}(Q;\Omega) : E(\eta) < \infty,\, |\eta(Q)| = \int_Q \det \nabla \eta \, dx, \eta|_P = \gamma(x) \right\}. \tag{5.1.2}$$

Here, the finite energy guarantees local injectivity (see Assumption S2) and the equality $|\eta(Q)| = \int_Q \det \nabla \eta \, dx$ is termed the *Ciarlet-Nečas condition* which has been proposed in [40] and, as has also been proved there, it assures that any C^1-local homeomorphism is globally injective except for possible touching at the boundary. Working with this equivalent condition bears the advantage that it is easily seen to be preserved under weak convergence in $W^{2,q}(Q;\Omega)$.

Remark 5.1.3. *Of particular interest is the topology of* $\mathcal{E}$*. It is easy to see that the set is a closed subset of the affine space* $W^{2,q}_\gamma(Q;\Omega)$ *i.e.* $W^{2,q}(Q;\Omega)$ *with fixed boundary conditions. As a subset of this topological space it has both interior points (denoted by* $\operatorname{int}(\mathcal{E})$*) and a boundary* $\partial\mathcal{E}$*. As we construct our approximative solutions by minimization over* $\mathcal{E}$*, it is crucial to know if* $\eta_k \in \operatorname{int}(\mathcal{E})$ *as only then we are allowed to test in all directions and have the full Euler-Lagrange equation we need.*

Luckily however $\operatorname{int}(\mathcal{E})$ *and* $\partial\mathcal{E}$ *are easily quantifiable. As long as* $\det \nabla \eta > 0$*, which is true for finite energy, we are able to vary in all directions, if and only if* $\eta|_M$ *is injective and does not touch* $\partial\Omega$*. Thus the relevant part of* $\partial\mathcal{E}$*, i.e. the deformations with finite energy consists precisely of the* η *which have a collision.*

Finally for the dissipation functional we have the following assumption:

Assumption 5.1.4 (Dissipation functional)**.** *The dissipation* $R : \mathcal{E} \times W^{1,2}(Q;\mathbb{R}^n) \to \mathbb{R}$ *satisfies*

R1 Weak lower semicontinuity: If $b_l \rightharpoonup b$ *in* $W^{1,2}$ *then*

$$\liminf_{l\to\infty} R(\eta, b_l) \geq R(\eta, b)$$

R2 Homogeneity of degree two: The dissipation is homogeneous of degree two in its second argument , i.e.

$$R(\eta, \lambda b) = \lambda^2 R(\eta, b) \qquad \forall \lambda \in \mathbb{R}$$

In particular, this implies $R(\eta, b) \geq 0$ *and* $R(\eta, 0) = 0$.

R3 Energy-dependent Korn-type inequality: Fix $E_0 > 0$*. Then there exists a constant* $c_K = c_K(E_0) > 0$ *such that for all* $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ *with* $E(\eta) \leq E_0$ *and all* $b \in W^{1,2}(Q;\mathbb{R}^n)$ *with* $b|_P = 0$ *we have*

$$c_K \|[\|W^{1,2}(Q)]b^2 \leq R(\eta, b).$$

R4 Existence of a continuous derivative: The derivative $D_2R(\eta, b) \in (W^{1,2}(Q;\mathbb{R}^n))'$ *given by*

$$\frac{d}{d\epsilon}|_{\epsilon=0} R(\eta, b + \epsilon\phi) =: \langle D_2 R(\eta, b), \phi \rangle$$

exists and is weakly continuous in its two arguments. Due to the homogeneity of degree two this in particular implies

$$\langle D_2 R(\eta, b), b \rangle = 2R(\eta, b).$$

Again some remarks are in order. As above assumption R4 is natural as we need do be able to evaluate the actual stress. Assumption R2, on the other hand, reflects the fact that we are considering viscous dissipation. Assumption R1 is again important from the point of view of calculus of variations. Assumption R3 is a coercivity assumption in a sense and needs to be stated in this rather weak form to satisfy frame indifference. Indeed, our model dissipation (1.2.8) satisfies this assumption as shown in, e.g., [137] relying on quite general Korn's inequalities due to [154, 160].

We shall work with the following weak formulation:

Definition 5.1.5 (Weak solution to the parabolic problem)**.** *We call the pair (η, v) a weak solution to the* parabolic fluid structure interaction problem*, if it satisfies*

$$\eta \in L^\infty([0,T];\mathcal{E}), \qquad \partial_t\eta \in L^2([0,T];W^{1,2}(Q;\mathbb{R}^n)),$$
$$v \in L^2([0,T];W^{1,2}(\Omega(t);\mathbb{R}^n)), \qquad \operatorname{div} v(t) = 0 \quad \text{for a.a. } t \in [0,T],$$

as well as $v(t,\eta(t,x)) = \partial_t\eta(t,x)$ for a.a. $t \in [0,T]$ and $x \in \partial Q$ and there exists a $p \in \mathcal{D}'([0,T]\times\Omega)$ with $\operatorname{supp} p \subset [0,T]\times\Omega(t)$, such that they satisfy the weak equation

$$\begin{aligned}\int_0^T \langle DE(\eta),\phi\rangle + \langle D_2R(\eta,\partial_t\eta),\phi\rangle + \langle \varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} - \langle p, \operatorname{div}\xi\rangle \, dt \\ = \int_0^T \rho_f \langle f,\xi\rangle_{\Omega(t)} + \rho_s \langle f\circ\eta,\phi\rangle_Q \, dt\end{aligned} \tag{5.1.3}$$

for all $\phi \in L^2([0,T];W^{2,q}(Q;\mathbb{R}^n))$, with $\phi|_P = 0$ and $\xi \in C_0([0,T];W^{2,q}_0(\Omega;\mathbb{R}^n))$ such that $\phi = \xi\circ\eta$ on Q where, as before, we set $\Omega(t) := \Omega \setminus \eta(t,Q)$. Moreover, the initial condition for η is satisfied in the sense that

$$\lim_{t\to 0}\eta(t) = \eta_0 \quad \text{in } L^2(Q;\mathbb{R}^n).$$

The main goal of this section is to prove existence of weak solutions to the parabolic fluid-structure interaction problem. In particular, we show the following theorem:

Theorem 5.1.6 (Existence of a parabolic fluid structure interaction)**.** *Assume that the energy E fulfills Theorem 5.1.1 and the dissipation R fulfills Theorem 5.1.4. Further let $\eta_0 \in \mathcal{E}$ and $f \in L^\infty(\Omega;\mathbb{R}^n)$. Then there exists a maximal time $T_{\max} > 0$ such that on the interval $[0,T_{\max})$ a weak solution to the parabolic fluid-structure interaction problem in the sense of Theorem 5.1.5 exists.*

For the maximal time, we have $T_{\max} = \infty$, or $\liminf_{t\to T_{\max}} E(\eta(t)) = \infty$, or $T_{\max}$ is the time of the first collision of the solid with either itself or the container, i.e. the continuation $\eta(T_{\max})$ exists and $\eta(T_{\max}) \in \partial\mathcal{E}$. Furthermore we have $p \in L^2([0,T];L^\infty(\Omega(t))) + L^\infty([0,T];L^2(\Omega(t)))$ for all $T < T_{\max}$.

In order to prove Theorem 5.1.6, we shall exploit the natural gradient flow-structure of the parabolic fluid-structure evolution. Indeed, at the heart of the proof is the construction of time-discrete approximations via variational problems inspired by DeGiorgi's *minimizing movements* method [49] given in (5.3.1). We refer to Section 5.3 for a detailed proof of Theorem 5.1.6 and to Section 5.2 for the preliminary material.

Remark 5.1.7 (Maximal existence time)**.** *The maximal existence time in Theorem* (5.1.6) *is not only given by possible collisions but also by a possible blow-up of the energy due to the acting forces. It is quite notable, that such a situation cannot appear in the full (hyperbolic) model Theorem 7.3.3. The reason is that the acting forces can be compared to the intertial term instead of the dissipative one.*

5.2 Preliminary analysis

We will start this section with discussing the relevant geometry of the fluid-solid coupling and derive some necessary properties for the coupled system that will also be of use for the full Navier-Stokes system in chapter 7.

Lemma 5.2.1 (Closedness of $\mathcal{E}$). *Let $(\eta_l)_{l\in\mathbb{N}} \subset \mathcal{E}$ be a sequence such that $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$ and $\sup_{l\in\mathbb{N}} E(\eta_l) < \infty$. Then $\eta \in \mathcal{E}$.*

Proof. The boundary condition holds as $W^{2,q}(Q;\mathbb{R}^n)$ has a continuous trace operator. Similarly the lower semicontinuity of E guarantees $E(\eta) < \infty$. For the Ciarlet-Nečas-condition we refer to [40], but note that, due to the higher regularity we employ, a more direct proof would be feasible as well. □

5.2.1 Injectivity and boundary regularity of the solid

Further, we discuss the injectivity of deformations in $\mathcal{E}$ up-to-the boundary. In fact, any $\eta \in \mathcal{E}$ is injective on Q but not necessarily on $\bar{Q}$, so collisions are in principle possible. Nonetheless, we can exclude them for short times as shown via the following two lemmas as well as Corollary 5.3.8.

Lemma 5.2.2 (Local injectivity of the boundary). *For any $E_0 < \infty$ there exists a $\delta_0 > 0$ such that all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ are locally injective with radius δ_0 at the boundary, i.e.*

$$\eta(x_0) \neq \eta(x_1) \text{ for all } x_0, x_1 \in \partial Q, |x_0 - x_1| < \delta_0.$$

Proof. Assume that there are two points $x_0, x_1 \in \partial Q$ such that $\eta(x_0) = \eta(x_1)$. Now using embedding theorems and Theorem 5.1.1, S2 and S4, $E(\eta) < E_0$ implies that $\nabla\eta$ is uniformly continuous and there exists a uniform lower bound on $\det\nabla\eta$. This also results in a uniform continuity of $(\nabla\eta)^{-1} = \frac{cof\nabla\eta}{\det\nabla\eta}$.

Let A denote the contact plane spanned by $\nabla\eta(x_0)v$, v tangential to ∂Q in x_0 and denote the projection onto this plane using P_A. Then for any $x \in \partial Q$, the linear map $\phi_x : v \mapsto P_A\nabla\eta(x)v$ maps the tangential space $T_x\partial Q$ to A. In particular for x_0, we have $P_A\nabla\eta(x_0) = \nabla\eta(x_0)$ and thus ϕ_{x_0} is an isomorphism with determinant bounded from below. Now from the regularity of ∂Q ($T_x\partial Q$ does not change fast depending on x) and uniform continuity of $\nabla\eta$, we get that the same has to hold in a δ_0 neighborhood of x_0. Here, δ_0 is given just by the uniform continuity function of $\nabla\eta(x_0)$ and thus controlled by E_0. Further, if we orient the tangential spaces through the exterior normal and A through the orientation inherited from $\nabla\eta(x_0)$, then ϕ_x has to be orientation preserving in this neighborhood. So x_1 cannot lie in this neighborhood, as a simple geometrical argument shows that the orientation imparted on A through $\nabla\eta(x_1)$ is opposite to the one chosen though $\nabla\eta(x_0)$. □

Remark 5.2.3. *The preceeding proof is much easier to formulate in the case $n = 2$ as one can deal with tangential vectors directly: Consider the positively oriented unit tangentials τ_x at $x \in \partial Q$. Then $\nabla\eta(x_0)\tau_{x_0}$ and $\nabla\eta(x_1)\tau_{x_1}$ point in opposite directions and their length is bounded from below. But if x_0 and x_1 are close, then so are the τ_{x_i} and the $\nabla\eta(x_i)$, which leads to a contradiction.*

Lemma 5.2.4 (Short time global injectivity preservation). *Fix $E_0 < \infty$ and $\varepsilon_0 > 0$ and let δ_0 be given by the previous lemma. Then there exists a $\gamma_0 > 0$ such that for all $\eta_0 \in \mathcal{E}$ with $E(\eta_0) < E_0$ and*

$$|\eta_0(x_0) - \eta_0(x_1)| > \varepsilon_0 \text{ for all } x_0, x_1 \in \partial Q, |x_0 - x_1| \geq \delta_0 \tag{5.2.1}$$

we have that for all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and $\|\eta_0 - \eta\|_{L^2(Q)} < \gamma_0$ it holds that

$$|\eta(x_0) - \eta(x_1)| > \frac{\varepsilon_0}{2} \text{ for all } x_0, x_1 \in \partial Q, |x_0 - x_1| \geq \delta_0$$

Proof. Let η_0 be as prescribed and pick $\eta \in \mathcal{E}$, $E(\eta) < E_0$ with $|\eta(x_0) - \eta(x_1)| \leq \frac{\varepsilon_0}{2}$ for two points x_0, x_1 with $|x_0 - x_1| \geq \delta_0$. But then

$$|\eta_0(x_0) - \eta(x_0)| + |\eta_0(x_1) - \eta(x_1)| \geq |\eta_0(x_0) - \eta_0(x_1)| - |\eta(x_0) - \eta(x_1)| > \frac{\varepsilon_0}{2}$$

So, without loss of generality, we can assume that $|\eta_0(x_0) - \eta(x_0)| \geq \frac{\varepsilon_0}{4}$. But then since η_0 and η are uniformly continuous with the modulus of continuity depending just on E_0, there exists an $r > 0$ such that $|\eta_0(x) - \eta(x)| \geq \frac{\varepsilon_0}{8}$ for all $x \in B_r(x_0) \cap Q$. Thus

$$\|\eta_0 - \eta\|_{L^2(Q)} \geq \sqrt{\left(\frac{\varepsilon_0}{8}\right)^2 |B_r(x_0) \cap Q|} =: \gamma_0 > 0$$ □

Since we are concerned with variable in-time-domains for the fluid flow, we recall here the quantification of uniform regular domains. Later we will use several analytical results which will be used uniformly with respect to these quantifications.

Definition 5.2.5. *For $k \in \mathbb{N}$ and $\alpha \in [0,1]$. We call $\Omega \subset \mathbb{R}^n$ a $C^{k,\alpha}$–domain with* characteristics L, r, *if for all $x \in \partial\Omega$ there is a $C^{k,\alpha}$-diffeomorphismn $\phi_x : B_1(0) \to B_r(x))$, such that $\phi_x : B_1^-(0) \to B_r(x) \cap \Omega$, $\phi_x : B_1^+(0) \to B_r(x) \cap \Omega^c$ and $\phi_x(0) = x$. We require that it can be written as a graph over a direction $e_x \in \mathcal{S}^{n-1}$. This means that for $(z', z_n) \in B_1(0)$ we may write $\phi_x(z) = \phi_x((z',0)) + re_x z_n$. And that it satisfies the bound:*

$$\|\phi_x\|_{C^{\alpha,k}(B_1(0))} + \left\|\phi_x^{-1}\right\|_{C^{\alpha,k}(B_r(x))} \le L.$$

Collecting the regularity that comes from the energy bounds lead to an important (locally) uniform estimate on the $C^{1,\alpha}$ regularity of the fluid domains.

Corollary 5.2.6 (Uniform $C^{1,\alpha}$ domains)**.** *Fix $E_0 < \infty$, $\eta_0 \in \mathcal{E}$ with $E(\eta_0) < E_0$ and satisfying* (5.2.1) *for some $\varepsilon_0 > 0$. Then for all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and $\|\eta_0 - \eta\|_{L^2(Q)} < \gamma_0$ for γ_0 from Theorem 5.2.4 we have that $\Omega_\eta := \Omega \setminus \eta(Q)$ is a $C^{1,\alpha}$-domain with characteristics L, r depending only on E_0, η_0 and ε_0.*

5.2.2 Global velocity and a global Korn inequality

A useful tool when dealing with fluid structure interaction in the bulk is the global Eulerian velocity field, which is defined on the unchanging domain Ω. In particular this will allow us to circumvent the problem of talking about convergence on a changing domain.

Definition 5.2.7 (The global velocity field)**.** *Let $\eta \in \mathcal{E}$ be a given deformation. Let $v \in W^{1,2}(\Omega;\mathbb{R}^n)$ be a divergence-free fluid velocity and $b \in W^{1,2}(Q;\mathbb{R}^n)$ a solid velocity satisfying the coupling condition, $v \circ \eta = b$ on $\partial Q \setminus P$. Then the corresponding global velocity $u \in W_0^{1,2}(\Omega;\mathbb{R}^n)$ is defined by*

$$u(y) := \begin{cases} v(y) & \text{if } y \in \Omega_\eta := \Omega \setminus \eta(Q) \\ b \circ \eta^{-1}(y) & \text{if } y \in \eta(Q). \end{cases}$$

Note that this definition does not involve a reference time-scale directly. The solid velocity b is equally allowed to be a time derivative $b := \partial_t \eta$ or a discrete derivative $b := \frac{\eta_{k+1} - \eta_k}{\tau}$. Furthermore this definition is invertible. Given u and knowing η, both v and b can be reconstructed and those reconstructed velocities will satisfy the coupling condition as above.

When deriving a-priori estimates, the only bounds on the velocities that will be available to us are in form of a bounded dissipation. As this dissipation is given in form of a symmetrised derivative, we will need to use a Korn-type inequalities, the constants of which generally depend on the domain. However, another benefit of the global velocity and its constant domain is that the Korn-inequalities for the solid and the fluid can be merged into one global Korn-inequality.

Lemma 5.2.8 (Global Korn inequality)**.** *Fix $E_0 > 0$. Then there exists $c_{gK} = c_{gK}(E_0) > 0$ such that for any $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and any $b \in W^{1,2}(Q;\mathbb{R}^n)$ and $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ with $u|_{\partial\Omega} = 0$ and $b|_P = 0$ and satisfying the coupling condition*

$$u \circ \eta = b \text{ in } Q,$$

we have that

$$c_{gK} \|u\|_{W^{1,2}(\Omega)} \le \frac{\nu}{2} \|\varepsilon u\|_{\Omega_\eta} + R(\eta, b).$$

where we define $\Omega_\eta = \Omega \setminus \eta(Q)$.

Proof. On the reference domain Q we have per chain rule $\nabla b = \nabla(u \circ \eta) = (\nabla u) \circ \eta \cdot \nabla \eta$. Using this, we can estimate in analogy to Theorem 5.5.4, as η is a diffeomorphism:

$$\begin{aligned}\int_{\Omega \setminus \Omega_\eta} |\nabla u|^2 dy &= \int_Q |(\nabla u) \circ \eta|^2 \det \nabla \eta dx = \int_Q |(\nabla b) \cdot (\nabla \eta)^{-1}|^2 \det \nabla \eta dx \\ \leq \int_Q |\nabla b|^2 \frac{|cof \nabla \eta|^2}{\det \nabla \eta} dx &\leq \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0} \int_Q |\nabla b|^2 dx \leq \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0} c_K R(\eta, b)\end{aligned}$$

where $\epsilon_0 > 0$ is the uniform lower bound on $\det \nabla \eta$ as given in Assumption S2, $\|\eta\|_{C^1}$ is uniformly bounded by embeddings and we use the Korn-type inequality from Assumption R3.

But now we can apply Korn's inequality to the fixed domain Ω to get a constant c_Ω, for which

$$\begin{aligned}c_\Omega \|u\|_{W^{1,2}(\Omega)}^2 \leq \|\varepsilon u\|_\Omega^2 &= \|\varepsilon u\|_{\Omega_\eta}^2 + \|\varepsilon u\|_{\Omega \setminus \Omega_\eta}^2 \\ &\leq \|\varepsilon u\|_{\Omega_\eta}^2 + \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0} c_K R(\eta, b).\end{aligned}$$

Collecting all the constants then proves the lemma. □

5.3 Proof of Theorem 5.1.6

As mentioned before, we will show Theorem 5.1.6 in several steps using a time-discretisation in the form of a minimizing movements iteration.

5.3.1 Step 1: Existence of the discrete approximation

For this we will fix a time-step size τ. Setting $\eta_0^{(\tau)} := \eta_0$ and assuming $\eta_k^{(\tau)} \in \mathcal{E}$ given we define $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ as solutions to the following problem

$$\begin{aligned}\text{Minimize} \quad & E(\eta) + \tau R\left(\eta_k^{(\tau)}, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right) + \tau \frac{\nu}{2} \|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 \\ & - \rho_s \tau \left\langle f \circ \eta_k^{(\tau)}, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right\rangle_Q - \rho_f \tau \langle f, v\rangle_{\Omega_k^{(\tau)}} \\ \text{subject to} \quad & \eta \in \mathcal{E}, v \in W^{1,2}(\Omega_k^{(\tau)}; \mathbb{R}^n) \text{ with } \operatorname{div} v = 0, v|_{\partial\Omega} = 0 \\ & \text{and } \frac{\eta - \eta_k^{(\tau)}}{\tau} = v \circ \eta_k^{(\tau)} \text{ in } M.\end{aligned} \tag{5.3.1}$$

We then repeat this process until we reach $k\tau > T$.

Notice that in formulating the coupling condition in (5.3.1), we implicitly assumed that the solid is free of collisions, i.e. $\eta_k \notin \partial\mathcal{E}$ or in other words that $\eta_k|_M$ is injective and does not map to $\partial\Omega$. We will show in Theorem 5.3.8 that for small enough T this will always be the case. In principle though, it is possible to extend the coupling condition to situations including contact by using the global velocity field $u : \Omega \to \mathbb{R}^n$ instead.

We will now show that this problem has a (not necessarily unique) solution and that the sought minimizer satisfies an Euler-Lagrange equation which already is a discrete approximation of our problem.

Remark 5.3.1. *In* (5.3.1) *we minimize over the sum of the energy and the dissipation needed to reach the current step from the last one. In this context, we understand the Stokes potential as dissipative damping on the solid. Now, as far as the deformation (that we would consider the single* state variable *of the system) is concerned, the scheme is implicit in the energy and* implicit-explicit *in the dissipation. Since in the Stokes potential the dependence on the deformation manifests itself through the explicitly given domain and implicitly through the coupled boundary values.The implicit/explicit use of state can be motivated from the point of view of* *the Euler-Lagrange equations; as they are, with regards to the dissipation, exactly the Fréchet derivatives with respect to the rate variables that appear there. Explicit-implicit schemes are commonly used in fluid-structure interactions (see e.g. [145]). Moreover, it is the common way to produce solutions in solid mechanics if the dissipation depends on the state variables [119]. Within the proposed variational approach, it is important that we impose a coupling condition as equality of approximate velocities also in an* implicit-explicit *fashion; i.e. we keep the geometry explicit while the rates are implicit. An equality of tractions then needs not be imposed but follows automatically from the variational approach.*

Proposition 5.3.2 (Existence of solutions to (5.3.1))**.** *Assume that $\eta_k^{(\tau)} \in \mathcal{E}$. Then the iterative problem* (5.3.1) *has a minimizer, i.e. $\eta_{k+1}^{(\tau)}$ and $v_{k+1}^{(\tau)}$ are defined. Furthermore if $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$ (i.e. $\eta_{k+1}^{(\tau)}$ is injective on $\overline{Q}$) the minimizers obey the following Euler-Lagrange equation:*

$$\begin{aligned}&\left\langle DE(\eta_{k+1}^{(\tau)}), \phi\right\rangle + \left\langle D_2 R\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right), \phi\right\rangle + \nu\left\langle \varepsilon v_{k+1}^{(\tau)}, \nabla\xi\right\rangle_{\Omega_k^{(\tau)}}\\ &= \rho_f \left\langle f, \xi\right\rangle_{\Omega_k^{(\tau)}} + \rho_s\left\langle f\circ\eta_k^{(\tau)}, \phi\right\rangle_Q .\end{aligned}$$

for any $\phi \in W^{2,q}(Q;\mathbb{R}^n)$, $\phi|_P = 0$ and $\xi \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$, $\operatorname{div}\xi = 0$, $\xi|_{\partial\Omega} = 0$ such that $\xi\circ\eta = \phi$ in ∂M.

Proof. First we investigate existence using the direct method. The class of admissible functions is non-empty, since $(\eta_k^{(\tau)}, 0)$ is a possible competitor with finite energy. Next we show that the functional is bounded from below. As energy and dissipation have lower bounds per assumption, the only problematic terms are those involving the force f. For those we note that per the weighted Young's inequality and using Assumption R3 it holds that

$$\begin{aligned}\left|\left\langle f\circ\eta_k^{(\tau)}, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right\rangle_Q\right| &\leq \frac{\delta}{2}\left\|\frac{\eta-\eta_k^{(\tau)}}{\tau}\right\|_Q^2 + \frac{1}{2\delta}\left\|f\circ\eta_k^{(\tau)}\right\|_Q^2\\ &\leq \frac{\delta}{2c_K} R\left(\eta_k^{(\tau)}, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right) + \frac{1}{2\delta}\left\|f\circ\eta_k^{(\tau)}\right\|_Q^2\end{aligned}$$

and equally, using Theorem 5.2.8

$$|\langle f, v\rangle_{\Omega_k^{(\tau)}}| \leq \frac{\delta}{2}\|v\|_{\Omega_k^{(\tau)}}^2 + \frac{1}{2\delta}\|f\|_{\Omega_k^{(\tau)}}^2 \leq \frac{\delta}{2c_{gK}}\left(\|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 + R\left(\eta_k^{(\tau)}, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right)\right) + \frac{1}{2\delta}\|f\|_{\Omega_k^{(\tau)}}^2 .$$

Now if we choose δ small enough, e.g. $\delta := \frac{\min(c_K, c_{gK})}{2}$, all v and η-dependent terms can be absorbed to get the lower bound

$$\begin{aligned}&E(\eta) + \tau R\left(\eta_k, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right) + \tau\frac{\nu}{2}\|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 - \rho_f\tau\langle f, v\rangle_{\Omega_k^{(\tau)}} - \rho_s\tau\left\langle f\circ\eta_k^{(\tau)}, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right\rangle_Q \qquad (5.3.2)\\ &\geq E(\eta) + \frac{\tau}{2}R\left(\eta_k^{(\tau)}, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right) + \tau\frac{\nu}{4}\|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 - \tau\frac{\max(\rho_f,\rho_s)}{2\delta}\left(\left\|f\circ\eta_k^{(\tau)}\right\|_Q^2 + \|f\|_{\Omega_k^{(\tau)}}^2\right)\\ &\geq E_{\min} - \tau\frac{\max(\rho_f,\rho_s)}{2\delta}\left(\left\|f\circ\eta_k^{(\tau)}\right\|_Q^2 + \|f\|_{\Omega_k^{(\tau)}}^2\right)\end{aligned}$$

Thus a minimizing sequence $\tilde{\eta}_l, \tilde{v}_l$ exists and along that sequence, energy and dissipation are bounded. So by coercivity of the energy we know that $\tilde{\eta}_l$ is bounded in $W^{2,q}(Q;\Omega)$ and using the Banach-Alaoglu theorem along with compact embeddings we may extract a subsequence (not relabeled) and a limit $\eta_{\min}$ for which

$$\tilde{\eta}_l \rightharpoonup \eta_{\min} \qquad \text{in } W^{2,q}(Q;\Omega)$$
$$\tilde{\eta}_l \to \eta_{\min} \qquad \text{in } C^{1,\alpha^-}(Q;\Omega) \text{ for } 0 < \alpha^- < \alpha := 1 - \frac{n}{q}.$$

By Theorem 5.2.1 we know that $\eta_{\min} \in \mathcal{E}$. We also know that E and R are lower semicontinuous with respect to the above convergence by Assumptions S3 and R1 respectively.

Next we pass to the limit with the fluid velocity. With no loss of generality, we may assume $\tilde{v}_l$ to be the minimizer of the functional in (5.3.1) holding the deformation $\tilde{\eta}_l$ fixed. As the functional in (5.3.1) is convex with respect to the velocity, minimizing is *equivalent* to solving the appropriate Euler-Lagrange equation, in other words, it is equivalent to finding a weak solution to the following classical Stokes boundary value problem:

$$\begin{cases} -\nu\Delta\tilde{v}_l + \nabla p = \rho_f f & \text{in } \Omega_k^{(\tau)} \\ \operatorname{div}\tilde{v}_l = 0 & \text{in } \Omega_k^{(\tau)} \\ \tilde{v}_l = g_l := \frac{(\tilde{\eta}_l - \eta_k^{(\tau)})\circ(\eta_k^{(\tau)})^{-1}}{\tau} & \text{in } \partial\Omega_k^{(\tau)} \cap \partial\eta_k^{(\tau)}(Q) \\ \tilde{v}_l = 0 & \text{in } \partial\Omega \end{cases}$$

Now since $\eta_k^{(\tau)}$ is a fixed diffeomorphism, and $\tilde{\eta}_l$ converges uniformly, the boundary data g_l in this problem converges uniformly to $g := \frac{(\eta_{\min}-\eta_k^{(\tau)})\circ(\eta_k^{(\tau)})^{-1}}{\tau}$ as well. Furthermore, the solution operator $L^2(\partial\Omega_k^{(\tau)};\mathbb{R}^n) \to W^{1,2}(\Omega;\mathbb{R}^n)$ associated with this boundary value problem is continuous, which implies the existence of a limit $v_{\min} \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$ with $\tilde{v}_l \to v_{\min}$ in $W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$. Then per construction $(\eta\min, v_{\min})$ satisfy the compatibility condition and since $\|\varepsilon v\|_{\Omega_k^{(\tau)}}$ is lower semicontinuous and all terms involving f are continuous, the pair η, v is indeed a minimizer to the problem.

Next let us derive the Euler-Lagrange equation. Let $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ be a minimizer and $\phi \in C^\infty(Q;\mathbb{R}^n)$ as well as $\xi \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$. We require the perturbation $(\eta_{k+1}^{(\tau)} + \varepsilon\phi, v_{k+1}^{(\tau)} + \varepsilon\xi/\tau)$ to also be admissible[3] for all small enough ε. From this we immediately get the conditions $\operatorname{div}\xi = 0$, $\xi|_{\partial\Omega} = 0$, $\phi_P = 0$ and for the coupling we require

$$\left(v_{k+1}^{(\tau)} + \varepsilon\frac{\xi}{\tau}\right) \circ \eta_k^{(\tau)} = \frac{\eta_{k+1}^{(\tau)} + \varepsilon\phi - \eta_k^{(\tau)}}{\tau}$$

on M which reduces to $\xi \circ \eta_k^{(\tau)} = \phi$.

Now since we assume $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$, for small enough ε, we have $\eta_{k+1}^{(\tau)} + \varepsilon\phi \in \mathcal{E}$. Thus we are allowed to take the first variation with respect to $(\phi, \xi/\tau)$ which immediately results in the weak formulation. □

Now let us give some a-priori estimates on the solutions (5.3.1). Here, we will crucially use that the approximants are constructed as minimizers of an appropriate functional.

Lemma 5.3.3 (Parabolic a-priori estimates)**.** *We have*

$$E(\eta_{k+1}^{(\tau)}) + \tau R\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right) + \tau\nu \left\|\varepsilon v_{k+1}^{(\tau)}\right\|_{\Omega_k^{(\tau)}}^2$$
$$\le E(\eta_k^{(\tau)}) + \tau\rho_f \left\langle f, v_{k+1}^{(\tau)}\right\rangle_{\Omega_k} + \tau\rho_s \left\langle f \circ \eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right\rangle_Q$$

Furthermore take a number E_0 such that $E_0 > E(\eta_0)$. Then there exists a time $T_{E_0} > 0$ depending only on E_0 and the difference $E_0 - E(\eta_0)$ as well as $\|f\|_{L^\infty(\Omega)}$, such that for all $\tau > 0$, and all $N \in \mathbb{N}$ with $N\tau \le T_{E_0}$

[3]The different scaling of ϕ and ξ/τ with respect to τ used here allows us to remove most occurrences of τ in the Euler-Lagrange equation. This does not matter as long as τ is fixed, but it turns out to be the correct scaling when we take the limit $\tau \to 0$.

we have

$$E(\eta_N^{(\tau)}) + \frac{\tau}{2}\sum_{k=1}^{N}\left[\frac{\nu}{2}\left\|\varepsilon v_k^{(\tau)}\right\|^2_{\Omega_{k-1}^{(\tau)}} + R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right)\right] \le E_0$$

Proof. As before, for fixed k, we may compare the value of the cost functional in (5.3.1) at the minimizer with its value for the pair $(\eta_k^{(\tau)}, 0)$. As $R(\eta_k^{(\tau)}, 0) = 0$ and the terms involving v vanish for $v = 0$, the comparison yields the first line.

Now we proceed by induction over N. Assume that $E(\eta_{N-1}^{(\tau)}) \le E_0$ and let c_{gK} be the Korn-constant corresponding to E_0 from Theorem 5.2.8. Using (5.3.2) again, we end up with

$$\begin{aligned} &E(\eta_{k+1}^{(\tau)}) + \frac{\tau}{2}R\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right) + \tau\frac{\nu}{4}\left\|\varepsilon v_k^{(\tau)}\right\|^2_{\Omega_k^{(\tau)}} \\ &\le E(\eta_k^{(\tau)}) + \tau\frac{1}{2\delta}\left(\left\|f\circ\eta_k^{(\tau)}\right\|^2_Q + \|f\|^2_{\Omega_k^{(\tau)}}\right) \le E(\eta_k^{(\tau)}) + \tau\frac{\max(\rho_f,\rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)} \end{aligned} \tag{5.3.3}$$

where for all $k \in \{1, ..., N\}$ the δ does only depend on c_{gK} and c_K and thus only on E_0.

Hence we may sum this estimate over k, yielding

$$\begin{aligned} &E(\eta_N^{(\tau)}) + \frac{\tau}{2}\sum_{l=1}^{N}\left[\frac{\nu}{2}\left\|\varepsilon v_k^{(\tau)}\right\|^2_{\Omega_{k-1}^{(\tau)}} + R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right)\right] \\ &\le E(\eta_0) + N\tau\frac{\max(\rho_f,\rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)} \le E_0 \end{aligned}$$

assuming that $N\tau \le T_{E_0}$ for $T_{E_0} > 0$ given by $\frac{T\max(\rho_f,\rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)} = E_0 - E(\eta_0)$. But then in particular $E(\eta_N^{(\tau)}) \le E_0$ and we can continue the induction until $N\tau$ reaches T_{E_0}. □

Remark 5.3.4. *Clearly, the maximal length of the time-interval on which the a-priori estimates are true depends on the choice of E_0 and could be, thus, optimized. However, we do not enter this investigation here, since, later we may prolong the solution to the maximal existence time.*

Let us also mention that a slightly better single-step estimate could have been gotten by comparing with $(\eta_k, \tilde{v})$, where $\tilde{v}$ is the minimizer of $\frac{\nu}{2}\|\varepsilon v\|^2 + \langle f, v\rangle$ under $\operatorname{div} v = 0$ *and zero boundary conditions.*

5.3.2 Step 2: Time-continuous approximations and their properties

Now as a next step, we use these iterative solutions to construct approximations of the continuous problem. At this point, we will completely switch over to the global velocity u. We will also approximate the deformation η in two different ways, a piecewise constant approximation, which we will need to keep track of the fluid-domain, and a piecewise affine approximation, which will give us the correct time derivative $\partial_t\eta$. To be more precise, we define:

Definition 5.3.5 (Discrete parabolic approximation)**.** *For some $E_0 > E(\eta_0)$ fix $T_{E_0} > 0$ as given by Theorem 5.3.3. We now define the piecewise constant τ-approximation as*

$$\begin{aligned} \eta^{(\tau)}(t,x) &:= \eta_k^{(\tau)}(x) && \text{for } t \in [\tau k, \tau(k+1)), x \in Q \\ u^{(\tau)}(t,y) &:= v_k^{(\tau)}(y) && \text{for } t \in [\tau k, \tau(k+1)), y \in \Omega_k^{(\tau)} \\ u^{(\tau)}(t,y) &:= \frac{(\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)})\circ(\eta_k^{(\tau)})^{-1}(y)}{\tau} && \text{for } t \in [\tau k, \tau(k+1)), y \in \eta_k(\overline{Q}) \\ \Omega^{(\tau)}(t) &:= \Omega_k^{(\tau)} && \text{for } t \in [\tau k, \tau(k+1)) \end{aligned}$$

where $(\eta_k^{(\tau)}, v_k^{(\tau)})$ *is the iterative solution for timestep* τ*. We also define the piecewise affine approximation for* η *as*

$$\tilde{\eta}^{(\tau)}(t,.) := ((k+1) - t/\tau)\eta_k^{(\tau)} - (t/\tau - k)\eta_{k+1}^{(\tau)} \qquad \text{for } t \in [\tau k, \tau(k+1)), x \in Q.$$

Note that $\tilde{\eta}^{(\tau)}$ is Lipschitz-continuous in time, $\tilde{\eta}^{(\tau)}(k\tau) = \eta^{(\tau)}(k\tau)$ for all $k \in \{0, ..., N\}$ and

$$\partial_t \tilde{\eta}^{(\tau)}(t) = \frac{1}{\tau}(\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}) = u^{(\tau)}(t) \circ \eta^{(\tau)}(t)$$

for all $t \in (\tau k, \tau(k+1))$. Also from this point on, we will only work with the global velocity field $u^{(\tau)}$ as given in Theorem 5.2.7

Lemma 5.3.6 (Basic a-priori estimates)**.** *For any fixed* E_0 *and the resulting time* T_{E_0} *from Theorem 5.3.3 there exists a constant* C *independent of* τ *such that*

$$E(\eta^{(\tau)}(t)) + \int_0^t R(\eta^{(\tau)}(t), \partial_t \tilde{\eta}^{(\tau)}(t)) + \frac{\nu}{2} \left\| \varepsilon u^{(\tau)} \right\|_{\Omega^{(\tau)}(t)}^2 dt \leq E_0.$$

for all $t \in [0, T_{E_0}]$ *as well as*

$$\sup_{t \in [0, T_{E_0}]} \left\| \eta^{(\tau)}(t) \right\|_{W^{2,q}(Q)} \leq C, \quad \int_0^{T_{E_0}} \left\| \partial_t \tilde{\eta}^{(\tau)} \right\|_{W^{1,2}(Q)}^2 dt \leq C, \text{ and } \int_0^T \left\| u^{(\tau)} \right\|_{W^{1,2}(\Omega)}^2 dt \leq C.$$

Proof. The first statement is a direct translation of Theorem 5.3.3 while the latter ones follow from this. In particular, since $E(\eta^{(\tau)}(t)) < E_0$ on any of its constant intervals and thus on all of $[0, T_{E_0}]$ its supremum is bounded. Similarly the two integral inequalities follow from the boundedness of the dissipation combined with the Korn-inequalities R2 and Theorem 5.2.8. □

Lemma 5.3.7 (Energy and Hölder-estimates)**.** *For any* E_0 *and the resulting time* T_{E_0} *from Theorem 5.3.3, there exists a constant* C *independent of* $\tau < 1$ *such that we have the following estimates:*

1. *For all* $t \in [0, T_{E_0}]$
$$\left\| \eta^{(\tau)}(t) - \tilde{\eta}^{(\tau)}(t) \right\|_{W^{1,2}(Q)} \leq C\sqrt{\tau}$$

2. $E(\eta^{(\tau)}(t))$ *is nearly monotone, i.e. for any* $t > t_0$*,* $t, t_0 \in [0, T_{E_0}]$ *with* $t - t_0 \geq \tau$ *we have*
$$E(\eta^{(\tau)})(t) - E(\eta^{(\tau)})(t_0) \leq C(t - t_0)$$

3. $\eta^{(\tau)}(t)$ *is nearly Hölder-continuous in* $W^{1,2}(Q)$*, i.e. for any* $t > t_0$*,* $t, t_0 \in [0, T_{E_0}]$ *with* $t - t_0 > \tau$ *we have*
$$\left\| \eta^{(\tau)}(t) - \eta^{(\tau)}(t_0) \right\|_{W^{1,2}(Q)} \leq C\sqrt{t - t_0}$$

Proof. Consider the lower bound on a single step given in (5.3.3). Singling out the dissipation of the solid material, and dropping some terms with compatible sign, we get using the Korn's inequality R3

$$c_K \frac{1}{\tau} \left\| \eta_{k+1}^{(\tau)} - \eta_k^{(\tau)} \right\|_{W^{1,2}(Q)}^2 \leq \tau R\left(\frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau} \right) \leq 2\left(E(\eta_k^{(\tau)}) - E(\eta_{k+1}^{(\tau)}) + \tau \frac{\max(\rho_f, \rho_s)}{2\delta} \|f\|_\infty^2 \right)$$

Now as the energy is bounded uniformly from above and from below, we can derive that

$$\left\| \eta_{k+1}^{(\tau)} - \eta_k^{(\tau)} \right\|_{W^{1,2}(Q)} \leq C\sqrt{\tau}$$

for some constant C depending only on E_0 and f. In particular, due to the definition of $\tilde{\eta}^{(\tau)}(t)$ and $\eta^{(\tau)}(t)$ this implies (1).

Equally, reordering the terms in a different way, we get

$$E(\eta_{k+1}^{(\tau)}) - E(\eta_k^{(\tau)}) \leq \tau \frac{\max(\rho_f, \rho_s)}{2\delta} \|f\|_\infty^2 .$$

Now fix $T \geq t > t_0 \geq 0$ and let $M := \lfloor \frac{t}{\tau} \rfloor, N := \lfloor \frac{t_0}{\tau} \rfloor$. Summing up the inequality yields

$$E(\eta^{(\tau)}(t)) - E(\eta^{(\tau)}(t_0)) \leq \tau(M-N) \frac{\max(\rho_f, \rho_s)}{2\delta} \|f\|_\infty^2$$

Now either $\tau \leq t - t_0 < 2\tau$, in which case $\tau(M-N) < 2\tau < 2(t-t_0)$ or $t - t_0 \geq 2\tau$ and thus $\tau(M-N) < (t-t_0) + \tau < \frac{3}{2}(t-t_0)$, so this estimate proves (2).[4]

Finally we again use the first estimate and Hölder's inequality to sum up the distances:

$$\begin{aligned}
\left\|\eta^{(\tau)}(t) - \eta^{(\tau)}(t_0)\right\|_{W^{1,2}(Q)} &\leq \sum_{k=N}^{M-1} \left\|\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}\right\|_{W^{1,2}(Q)} \\
&\leq \sqrt{\sum_{k=N}^{M-1} \tau} \sqrt{\sum_{k=N}^{M-1} \frac{1}{\tau} \left\|\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}\right\|_{W^{1,2}(Q)}^2} \\
&\leq \sqrt{\tau(M-N)} \sqrt{\sum_{k=N}^{M-1} 2 \left(E(\eta_k^{(\tau)}) - E(\eta_{k+1}^{(\tau)}) + \tau \frac{\max(\rho_f, \rho_s)}{2\delta} \|f\|_\infty^2 \right)} \\
&\leq c\sqrt{t-t_0} \sqrt{E(\eta^{(\tau)}(t_0)) - E(\eta^{(\tau)}(t)) + (t-t_0) \frac{\max(\rho_f, \rho_s)}{2\delta} \|f\|_\infty^2}
\end{aligned}$$

which proves (3). □

A direct consequence of the last estimate is that the solid cannot move much in a short time. In particular, this implies the following result on injectivity:

Corollary 5.3.8 (Short-time collision exclusion)**.** *If $\eta_0 \in \mathcal{E}$ is injective (i.e. $\eta_0 \notin \partial\mathcal{E}$) then there exists $T_{\text{inj}} > 0$ such that for all τ small enough and all $t \in [0, T_{\text{inj}}]$, the deformations $\eta^{(\tau)}(t)$ and $\tilde{\eta}^{(\tau)}(t)$ are injective (i.e. not in $\partial\mathcal{E}$).*

Proof. If we choose T_{inj} small enough, then the near Hölder continuity from Theorem 5.3.7 implies that $\left\|\eta_0 - \eta_k^{(\tau)}\right\|_Q$ is uniformly small. In particular we can choose it to be smaller than the constant γ_0 from Theorem 5.2.4 which then results in injectivity. □

In the following, we take, for η_0, E_0 fixed, $T \leq \min\{T_{\text{inj}}, T_{E_0}\}$. In this way, both the a-priori estimates Theorem 5.3.3 hold and we may assume injectivity.

5.3.3 Step 3: Existence and regularity of limits

As a next step, we will derive limiting objects for $\tau \to 0$ for the deformation and the global velocity, as well as their mode of convergence.

Proposition 5.3.9 (Convergence of the time-discrete scheme)**.** *There exists a (not relabeled) subsequence $\tau \to 0$ and a limit*

$$\eta \in C^{1/2}([0,T]; W^{1,2}(Q;\mathbb{R}^n)) \cap C_w([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap C^0([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n))$$

[4] The lower bound on $t_0 - t$ is somewhat arbitrary and is only due to the jumps in the piece-wise constant approximation. As we are generally interested in $\tau \to 0$ for fixed t, t_0, this will not represent an issue.

for $\alpha = 1 - \frac{n}{q}$ *and* $u \in L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$*, such that*

$$\begin{aligned}\tilde{\eta}^{(\tau)} &\to \eta \text{ in } C^{(1/2)^-}([0,T];W^{1,2}(Q;\mathbb{R}^n))\\ \eta^{(\tau)},\tilde{\eta}^{(\tau)} &\rightharpoonup^* \eta \text{ in } L^\infty([0,T];W^{2,q}(Q;\mathbb{R}^n))\\ u^{(\tau)} &\rightharpoonup u \text{ in } L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))\\ \partial_t\tilde{\eta}^{(\tau)} &\rightharpoonup \partial_t\eta \text{ in } L^2([0,T];W^{1,2}(Q;\mathbb{R}^n)).\end{aligned}$$

Furthermore we have

$$\partial_t\eta = u\circ\eta \text{ in } [0,T]\times Q$$

and that $\eta^{(\tau)}$ *converges uniformly to* η *in the following sense: For all* $r>0$*, there exists a* $\delta_r>0$*, such that for all* $\tau<\delta_r$ *and all* $|x_1-x_2| + |t_1-t_2|^{\frac{1}{\alpha+n}} \le r^{\alpha^-}$ *we have*

$$|\nabla(\eta^{(\tau)}(t_1,x_1)-\eta(t_2,x_2))| + |\eta^{(\tau)}(t_1,x_1)-\eta(t_2,x_2)| \le Cr^{\alpha^-},$$

for all $0<\alpha^-<1-\frac{n}{q}$*. Finally, we obtain that*

$$\tilde{\eta}^{(\tau)} \to \eta \in C^0([0,T];C^{1,\alpha^-}(Q;\mathbb{R}^n)) \text{ and } \eta^{(\tau)} \to \eta \in L^\infty([0,T];C^{1,\alpha^-}(Q;\mathbb{R}^n)). \tag{5.3.4}$$

Proof. We proceed with a weak version of the Arzela-Ascoli theorem. Let $\{t_i\}_{i\in\mathbb{N}} \subset [0,T]$ be a countable dense set. By the upper bound on the energy and its coercivity, we have a uniform bound on $\|\eta^{(\tau)}(t)\|_{W^{2,q}(Q)}$; thus, by a diagonalization argument we can pick a subsequence of τ's (not relabeled) and limits $\eta(t_i)$ such that $\eta^{(\tau)}(t_i) \rightharpoonup \eta(t_i)$ in $W^{2,q}(Q;\mathbb{R}^n)$ and uniformly strongly in $W^{1,2}(Q;\mathbb{R}^n)$ for all $i\in\mathbb{N}$. Then by the convergence of norms, the Hölder-continuity from Theorem 5.3.7 (3) carries over to

$$\|\eta(t_i)-\eta(t_j)\|_{W^{1,2}(Q)} \le C\sqrt{|t_i-t_j|} \quad \forall i,j\in\mathbb{N}.$$

This means that η has a unique extension onto $[0,T]$ in the space $C^{1/2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$.

By the compactness arguments one gets

$$\tilde{\eta}^\tau \to \eta \in C^{1/2^-}([0,T];W^{1,2}(Q;\mathbb{R}^n)).$$

Now pick $t\in[0,T]$ and a new sequence $\{t_i\}_{i\in\mathbb{N}} \subset [0,T]$, $t_i\to t$. Due to the uniform $W^{2,q}(Q;\mathbb{R}^n)$-bounds resulting from the bounded energy, the sequence $\{\eta(t_i)\}_{i\in\mathbb{N}}$ has a weakly converging subsequence which, by the uniqueness of limits, must converge to $\eta(t)$ weakly in $W^{2,q}(Q;\mathbb{R}^n)$. As the original sequence $\{t_i\}_{i\in\mathbb{N}}$ was arbitrary this means that η is weakly continuous in $W^{2,q}(Q;\mathbb{R}^n)$. By the same argument, for any subsequence of τ's there exists a sub-subsequence such that $\eta^{(\tau)}(t) \rightharpoonup \eta(t)$ in $W^{2,q}(Q;\mathbb{R}^n)$ and thus $\eta^{(\tau)} \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$ pointwise. By Theorem 5.3.7 (1), we know that $\tilde{\eta}^{(\tau)}(t)$ converges to the same limit as $\eta^{(\tau)}(t)$ in a $W^{1,2}(Q;\mathbb{R}^n)$-sense. Since $\tilde{\eta}^{(\tau)}(t)$ satisfies the same $W^{2,q}(Q;\mathbb{R}^n)$ bounds, we can then also prove weak $W^{2,q}(Q;\mathbb{R}^n)$ convergence by the same argument.

Next we interpolate in order to prove that $\eta\in C^0([0,T];C^{1,\alpha}(Q;\mathbb{R}^n))$. Actually we show more, namely that $\nabla\eta$ is Hölder continuous in time-space.[5] For that we take $(s_1,x_1),(s_2,x_2)\in[0,T]\times Q$ with $B_r \ni x_1,x_2$ (i.e. $|x_1-x_2|\le r$) and $|s_1-s_2|\le r^{2\alpha+n}$.

$$\begin{aligned}&|\nabla\eta(s_1,x_1)-\nabla\eta(s_2,x_2)|\\ &\le |\nabla\eta(s_1,x_1) - \fint_{B_r}\nabla\eta(s_1)\,dx| + |\fint_{B_r}\nabla(\eta(s_1)-\eta(s_2))\,dx| + |\nabla\eta(s_2,x_2)-\fint_{B_r}\nabla\eta(s_2)\,dx|\\ &\le Cr^\alpha + |s_2-s_2||\fint_{s_1}^{s_2}\fint_{B_r}\partial_t\nabla\eta\,dx\,ds| \le Cr^\alpha + |s_2-s_2|\left(\fint_{s_1}^{s_2}\fint_{B_r}|\partial_t\nabla\eta|^2\,dx\,ds\right)^{1/2}\\ &\le Cr^\alpha + \frac{|s_2-s_2|^{1/2}}{r^{n/2}}\left(\int_{s_1}^{s_2}\int_{B_r}|\partial_t\nabla\eta|^2\,dx\,ds\right)^{1/2} \le Cr^\alpha + C|\frac{s_1-s_2}{r^n}|^{\frac12} \le Cr^\alpha.\end{aligned} \tag{5.3.5}$$

[5]Note that due to the zero boundary values on P the estimates on the continuity of η follow directly by the gradient estimates.

Thus, we proved the uniform Hölder regularity of the gradient.

By similar arguments, we can also prove that $\eta^{(\tau)} \to \eta$. To this end, recall that let $0 < \alpha^- < \alpha = 1 - \frac{n}{q}$. For $r > 0$ we may choose a finite subset $\{t_i\}_{i=1}^{m_r}$ such that for every $t \in [0,T]$ there exists a t_i such that $|t_i - t| \leq r^{2\alpha^- + n}$. Using the compactness result of Arzela-Ascoli we may choose a subsequence of τ's and a $\delta_r > 0$ such that for all $\tau \leq \delta_r$

$$\max_{i \in \{1,\dots,m_r\}} \left\| \eta^{(\tau)}(t_i) - \eta(t_i) \right\|_{C^{1,\alpha^-}(Q)} \leq 1;$$

without loss of generality, we may assume that $\delta_\tau < r^{2\alpha^- + n}$.

Now for all $(s_1, x_1), (s_2, x_2) \in [0,T] \times Q$ with $|x_1 - x_2| \leq r$, a ball B_r of radius r such that $x_1, x_2 \in B_r$ and $|s_1 - s_2| \leq r^{2\alpha^- + n}$ there is a $t_i \in [s_1, s_2]$ and by an analogous calculation to (5.3.5) we obtain

$$\begin{aligned} |\nabla\eta^{(\tau)}(s_1,x_1) - \nabla\eta(s_2,x_2)| &\leq |\nabla\eta^{(\tau)}(s_1,x_1) - \fint_{B_r} \nabla\eta^{(\tau)}(s_1)\,dx| + |\fint_{B_r} \nabla(\eta^{(\tau)}(s_1) - \eta^{(\tau)}(t_i))\,dx| \\ &+ |\fint_{B_r} \nabla(\eta^{(\tau)}(t_i) - \eta(t_i))\,dx| + |\nabla\eta(s_2,x_2) - \fint_{B_r} \nabla\eta(t_i)\,dx| \leq Cr^\alpha, \end{aligned}$$

by using the already obtained Hölder continuity of η.

Having the uniform convergence of $\eta^{(\tau)}$ at hand, we finally deduce the convergence of the global velocity field $u^{(\tau)}$. To do so, we use the uniform $L^2([0,T]; W^{1,2}(\Omega; \mathbb{R}^n))$ bound on $u^{(\tau)}$ derived through Theorem 5.3.3 and Theorem 5.2.8 to extract another subsequence of τ's such that $u^{(\tau)}$ converges weakly in $L^2([0,T]; W^{1,2}(\Omega; \mathbb{R}^n))$ to some limit u. Equally, the uniform $L^2([0,T]; W^{1,2}(Q; \mathbb{R}^n))$ bound on $\partial_t \tilde{\eta}^\tau$ implies that up to a subsequence $\partial_t \tilde{\eta}^\tau$ converges weakly to $\partial_t \eta$ in $L^2([0,T]; W^{1,2}(Q; \mathbb{R}^n))$.

Directly from the definition, we see that

$$\partial_t \tilde{\eta}^\tau(t) = u^\tau(t) \circ \eta^\tau(t)$$

for almost all times t. So in particular for all $\phi \in C_0^\infty([0,T] \times Q; \mathbb{R}^n)$

$$\begin{aligned} \int_0^T \langle \partial_t \tilde{\eta}, \phi \rangle_Q \, dt &\leftarrow \int_0^T \left\langle \partial_t \tilde{\eta}^{(\tau)}, \phi \right\rangle_Q dt = \int_0^T \left\langle u^{(\tau)} \circ \eta^{(\tau)}, \phi \right\rangle_Q dt \\ &= \int_0^T \left\langle u^{(\tau)} \circ \eta, \phi \right\rangle_Q dt + \int_0^T \left\langle u^{(\tau)} \circ \eta^{(\tau)} - u^{(\tau)} \circ \eta, \phi \right\rangle_Q dt \end{aligned}$$

Now the first integral on the last line converges to $\int_0^T \langle u \circ \eta, \phi \rangle_Q \, dt$ as η is a diffeomorphism, while the second vanishes in the limit by the following argument: Let $\pi_s(t,x) := s\eta^{(\tau)}(t,x) + (1-s)\eta(t,x))$. Then

$$\begin{aligned} |u^{(\tau)}(t, \eta^{(\tau)}(t,x)) - u^{(\tau)}(t, \eta(t,x))|^2 &= |\int_0^1 \frac{\partial}{\partial s} u^{(\tau)}(t, \pi_s(t,x)) ds|^2 \\ &\leq \int_0^1 |\nabla u^{(\tau)}(t, \pi_s(t,x)) \cdot (\eta^{(\tau)}(t,x) - \eta(t,x))|^2 ds \\ &\leq \int_0^1 |\nabla u^{(\tau)}(t, \pi_s(t,x))|^2 ds \sup_{t \in [0,T], x \in Q} |\eta^{(\tau)}(t,x) - \eta(t,x)|^2. \end{aligned}$$

Now, as $\eta^{(\tau)}(t)$ and $\eta(t)$ are both diffeomorphisms with lower bound on the determinant and uniformly close gradients, the linear interpolation π_s also has to be a diffeomorphism. So integrating the equation yields

$$\begin{aligned} &\int_0^T \int_Q |u^{(\tau)}(t, \eta^{(\tau)}(t,x)) - u^{(\tau)}(t, \eta(t,x))|^2 dx dt \\ &\leq \int_0^T \int_Q \int_0^1 |\nabla u^{(\tau)}(t, \pi_s(t,x))|^2 ds dx dt \sup_{t \in [0,T], x \in Q} |\eta^{(\tau)}(t,x) - \eta(t,x)|^2 \\ &\leq c \int_0^T \int_\Omega |\nabla u^{(\tau)}|^2 dx dt \sup_{t \in [0,T], x \in Q} |\eta^{(\tau)}(t,x) - \eta(t,x)|^2. \end{aligned}$$

Here the first term is uniformly bounded and the second converges to 0, by the uniform convergence of $\eta^{(\tau)}$ outlined above.

Thus we have $\partial_t \eta = u \circ \eta$ almost everywhere in Q. □

5.3.4 Step 4: Convergence of the equation

Using the convergences we derived in Proposition 5.3.9, we proceed by showing that the discrete Euler-Lagrange equations from Theorem 5.3.2 converge to the equation satisfied by the weak solution. This is not a straightforward task, as we have to deal with coupled pairs of test functions with the coupling being non-linearly dependent on the deformation. We will deal with this issue by focusing on a global test function ξ on Ω from which we derive the test functions on the discrete level. In order to do so, we need to be able to approximate the test functions smoothly while also maintaining the coupling condition. This is shown in Proposition 5.3.11.

For the approximation of test-functions we make use of a Bogovskiĭ-type theorem.

Theorem 5.3.10 (Bogovskiĭ-Operator [16, Theorem 2.4])**.** *Let Ω be a bounded Lipschitz domain, then there is a linear operator $\mathcal{B} : \{g \in C_0^\infty(\Omega) \,|\, \int_\Omega g\,dy = 0\} \to C_0^\infty(\Omega)$, such that*

$$\operatorname{div}\mathcal{B}(g) = g.$$

Moreover, for $k \in \{0,1,2,...\}$ and $a \in (1,\infty)$ the operator extends to Sobolev-spaces in the form of $\mathcal{B} : \{g \in W_0^{k-1,a}(\Omega) \,|\, \int_\Omega g\,dy = 0\} \to W_0^{k,a}(\Omega)$, such that

$$\|\mathcal{B}(g)\|_{W_0^{k,a}(\Omega)} \leq c\,\|g\|_{W_0^{k-1,a}(\Omega)},$$

where the constant just depending on k, a, n and Ω.

Next we introduce the sought approximation result. It is introduced in order to approximate test-functions and later in chapter 7 in order to extend the Aubin-Lions lemma to the variable domain set up. The proof is quite involved and for that reason put in the appendix (see Subsection 5.5.2).

Proposition 5.3.11 (Approximation of test functions)**.** *Fix a function*

$$\eta \in L^\infty([0,T];\mathcal{E}) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)) \text{ with } \sup_{t\in T} E(\eta(t)) < \infty,$$

such that $\eta(t) \notin \partial\mathcal{E}$ for all $t \in [0,T]$. As before we set $\Omega(t) = \Omega \setminus \eta(t,Q)$. Let $\mathcal{T}_\eta$ be the set admissible test functions, which is defined as

$$\begin{aligned}\mathcal{T}_\eta := \{(\phi,\xi) \in W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)) \times L^2([0,T];W_0^{1,2}(\Omega);\mathbb{R}^n))\} \\ \text{s.t. } \phi = \xi\circ\eta \text{ on } [0,T]\times Q \text{ and } \operatorname{div}\xi(t) = 0 \text{ in } \Omega(t)\}.\end{aligned}$$

Then the set

$$\begin{aligned}\tilde{\mathcal{T}}_\eta := \{(\phi,\xi) \in \mathcal{T}_\eta, \xi \in C^\infty([0,T];C_0^\infty(\Omega;\mathbb{R}^n)) \,|\, \operatorname{div}\xi(t,y) = 0 \text{ for all } t \in [0,T] \text{ and all } y \\ \text{with } dist(y,\Omega(t)) < \varepsilon \text{ for some } \varepsilon > 0\}\end{aligned}$$

is dense in $\mathcal{T}_\eta$ in the following sense:

For every ε sufficiently small there exists a linear map $(\phi,\xi) \mapsto (\phi_\varepsilon,\xi_\varepsilon) \in \tilde{\mathcal{T}}_\eta$ such that

$$\operatorname{div}(\xi_\varepsilon(t,y)) = 0 \text{ for all } y \in \Omega \text{ with } dist(y,\Omega(t)) \leq \epsilon.$$

Moreover, if $\xi \in L^b([0,T];W^{k,a}(\Omega))$, for $k \in \mathbb{N}$, $a \in (1,\infty)$ and $b \in [1,\infty]$, then

$$\xi_\varepsilon \to \xi \text{ in } L^b([0,T];W^{k,a}(\Omega)).$$

If additionally $\eta \in L^b([0,T];W^{k,a}(Q;\mathbb{R}^n))$, *with* $a=2$, *if* $k\geq 3$, *then*

$$\phi_\varepsilon \to \phi \text{ in } L^b([0,T];W^{k,a}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)).$$

Further in case, $\partial_t\xi \in L^2([0,T];W^{1,2}(\Omega))$, *we find that* $\partial_t\xi_\varepsilon \to \partial_t\xi$ *in* $L^2([0,T];W^{1,2}(Q))$. *If additionally* $\xi \in L^\infty([0,T];W^{3,a}(\Omega))$ *with* $a>n$ *and* $\partial_t\xi \in L^2([0,T];W^{1,2}(\Omega))$, *we find that* $\partial_t\phi_\varepsilon \to \partial_t\phi$ *in* $L^2([0,T];W^{1,2}(Q))$.

Moreover, the following bounds are satisfied at every time-instant where the right hand side is bounded:

$$\begin{aligned}
\|\xi_\varepsilon(t)\|_{W^{1,2}(\Omega)} &\leq c\,\|\xi(t)\|_{W^{1,2}(\Omega;\mathbb{R}^n))}\\
\|\xi_\varepsilon(t)-\xi(t)\|_{L^2(\Omega)} &\leq c\varepsilon^{\frac{2}{n+2}}\,\|\xi(t)\|_{W^{1,2}(\Omega)}\,,\\
\|\xi_\varepsilon(t)\|_{W^{k,a}(\Omega)} &\leq c(\varepsilon)\,\|\xi(t)\|_{L^2(\Omega;\mathbb{R}^n)}\\
\|\phi_\varepsilon(t)\|_{W^{k,a}(Q)} &\leq c\,\|\xi(t)\|_{C^k(\Omega)}\,\|\eta(t)\|_{W^{k,a}(Q)} \leq c(\varepsilon)\,\|\xi(t)\|_{L^2(\Omega)}\,\|\eta(t)\|_{W^{k,a}(Q)}\,,
\end{aligned}$$

where the constant c *depends on the bounds of* $\eta \in L^\infty([0,T];\mathcal{E}) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$ *and the lower bound on the Jacobian of* η *only. The constant* $c(\varepsilon)$ *depends additionally on* ε.

Having Theorem 5.3.11 at hand, we now pass to the limit in the Euler-Lagrange equation.

Proposition 5.3.12 (Limit-equation)**.** *The limit pair* (η, v) *as obtained in Proposition 5.3.9 satisfies the following:*

$$\begin{aligned}
0 = \int_0^T &\langle DE(\eta(t)),\phi\rangle_Q + \langle D_2R(\eta(t),\partial_t\eta(t)),\phi\rangle_Q + \nu\,\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)} \\
&- \rho_f\,\langle f,\xi\rangle_{\Omega(t)} - \rho_s\,\langle f\circ\eta,\phi\rangle_Q\,dt
\end{aligned} \tag{5.3.6}$$

for all pairs $\phi \in L^2([0,T];W^{2,q}(Q;\mathbb{R}^n))$, $\xi \in L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$ *which satisfy* $\phi(t,.) = \xi\circ\eta(t)$ *on* Q *and* $\operatorname{div}\xi(t) = 0$ *on* $\Omega(t)$.

Proof. First, we use the Minty method to show that $\langle DE(\eta^\tau(t)),\phi^\tau\rangle_Q \to \langle DE(\eta(t)),\phi\rangle_Q$. Fix $t\in[0,T]$ and pick $\psi \in C_0^\infty(Q;[0,1])$. Then, the pair $((\eta^{(\tau)}-\eta)\psi, 0)$ fulfills the coupling condition for the discrete Euler-Lagrange equation and we have

$$\begin{aligned}
&\left\langle DE(\eta^{(\tau)}) - DE(\eta), (\eta^{(\tau)}-\eta)\phi\right\rangle\\
&= -\left\langle DE(\eta),(\eta^{(\tau)}-\eta)\phi\right\rangle - \left\langle D_2R(\eta^{(\tau)},\partial_t\tilde\eta^{(\tau)}),(\eta^{(\tau)}-\eta)\phi\right\rangle + \left\langle f,(\eta^{(\tau)}-\eta)\phi\right\rangle
\end{aligned}$$

As $\eta^{(\tau)}(t) \to \eta(t)$ weakly in $W^{2,q}(Q;\mathbb{R}^n)$ and strongly in $C^{1,\alpha^-}(Q;\mathbb{R}^n)$ and moreover as $\partial_t\tilde\eta^{(\tau)}(t)$ is uniformly bounded in $L^2([0,T]\times Q;\mathbb{R}^n)$ all three terms on the right hand side converge to 0 when integrated in time and thus for almost all $t\in[0,T]$ by Theorem 5.3.9. Hence Theorem 5.1.1, S6 implies the strong convergence of $\eta^{(\tau)}(t) \to \eta(t)$ in $W^{2,q}(Q;\mathbb{R}^n)$ for almost all $t\in[0,T]$.

By Theorem 5.3.11, it is enough to show the limit equation for $\xi \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^n)$, which is divergence free on a slightly larger set than the fluid domain. Fix such a ξ. Then since $\eta^{(\tau)}$ converges uniformly to η, $\operatorname{div}\xi = 0$ on $\Omega^{(\tau)}(t)$ for all τ small enough.

Now we construct the matching $\phi^{(\tau)}(t,x) := \xi(t,\eta^{(\tau)}(t,x))$ and $\phi(t,x) := \xi(t,\eta(t,x))$. Then by Theorem 5.5.2 and Theorem 5.5.4, $\phi^{(\tau)} \in L^\infty([0,T];W^{2,q}(Q;\mathbb{R}^n))$ with uniform bounds. Thus we get by compactness and uniqueness of limits $\phi^{(\tau)}(t) \rightharpoonup \phi(t)$ in $W^{2,q}(\mathbb{R}^n)$.

As constructed, the pairs $(\phi^{(\tau)}(t),\xi(t))$ are admissible in the respective Euler-Lagrange equations from Theorem 5.3.2 and we have

$$\begin{aligned}
0 = &\left\langle DE(\eta^{(\tau)}(t)),\phi^{(\tau)}(t)\right\rangle + \left\langle D_2R\left(\eta^{(\tau)}(t),\partial_t\tilde\eta^{(\tau)}(t)\right),\phi^{(\tau)}(t)\right\rangle\\
&+\nu\left\langle\varepsilon u^{(\tau)}(t),\nabla\xi(t)\right\rangle_{\Omega^{(\tau)}(t)} - \rho_f\,\langle f,\xi(t)\rangle_{\Omega^{(\tau)}(t)} - \rho_s\left\langle f\circ\eta^{(\tau)}(t),\phi^{(\tau)}(t)\right\rangle_Q
\end{aligned}$$

for all $t \in [0,T]$ and τ small enough.

Now we integrate this equation in time and check each of the terms for convergence. For the first term we note that by the strong convergence of $\eta^{(\tau)}$ in $W^{2,q}(Q;\mathbb{R}^n)$ and Theorem 5.1.1, S5 that $DE(\eta^{(\tau)}(t))$ converges strongly in $W^{-2,q}(Q;\mathbb{R}^n)$ for every fixed t. Since $\phi^{(\tau)}(t)$ converges weakly and both terms are uniformly bounded in their respective spaces, we get

$$\int_0^T \left\langle DE(\eta^{(\tau)}(t)), \phi^{(\tau)}(t)\right\rangle dt \to \int_0^T \langle DE(\eta(t)), \phi(t)\rangle\, dt.$$

For the next term we find by Theorem 5.3.9 and the continuity of R in Theorem 5.1.4, R1 that $D_2R\left(\eta^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)}\right)$ converges weakly in $L^2([0,T];W^{-1,2}(Q;\mathbb{R}^n))$ and $\phi^{(\tau)}$ converges strongly in $L^2([0,T];W^{1,2}(Q;\mathbb{R}^n))$ which implies that

$$\int_0^T \left\langle D_2R\left(\eta^{(\tau)}(t), \partial_t\tilde{\eta}^{(\tau)}(t)\right), \phi^{(\tau)}(t)\right\rangle dt \to \int_0^T \langle D_2R\left(\eta(t), \partial_t\eta(t)\right), \phi(t)\rangle\, dt.$$

For the next terms, let us first deal with the variable domain by rewriting the terms using characteristic functions. Denoting the symmetric difference by $A \triangle B := A \setminus B \cup B \setminus A$ we have

$$\int_0^T \left\| \chi_{\Omega^{(\tau)}(t)} - \chi_{\Omega(t)}\right\|_{L^2(\Omega)}^2 dt = \int_0^T |\Omega^{(\tau)}(t) \triangle \Omega(t)| dt \to 0$$

by the uniform convergence of the boundary and can thus conclude

$$\begin{aligned}\int_0^T \left\langle \nabla u^{(\tau)}(t), \nabla\xi(t)\right\rangle_{\Omega^{(\tau)}(t)} dt &= \int_0^T \int_\Omega \chi_{\Omega^{(\tau)}(t)} \nabla u^{(\tau)}(t) : \nabla\xi(t)\, dydt \\ \to \int_0^T \int_\Omega \chi_{\Omega(t)} \nabla u(t) : \nabla\xi(t)\, dydt &= \int_0^T \left\langle \nabla u^{(\tau)}(t), \nabla\xi(t)\right\rangle_{\Omega(\cdot)} dt.\end{aligned}$$

as u converges weakly in $L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$.

The same approach also works for the forces on the fluid, where the domain is the only variable in τ and thus

$$\int_0^T \rho_f \langle f, \xi(t)\rangle_{\Omega^{(\tau)}(t)}\, dt \to \int_0^T \rho_f \langle f, \xi(t)\rangle_{\Omega(t)}\, dt$$

Finally we have the forces acting on the solid. Here both sides converge uniformly:

$$\int_0^T \rho_s \left\langle f \circ \eta^{(\tau)}(t), \phi^{(\tau)}(t)\right\rangle_Q dt \to \int_0^T \rho_s \langle f \circ \eta(t), \phi(t)\rangle_Q\, dt$$

Collecting all the terms then concludes the proof. □

5.3.5 Step 5: Construction of the pressure

Take some arbitrary $s \in (0,T)$. Since we have excluded collisions on $(0,T)$, we know that $\Omega(t)$ is a uniform Lipschitz domain with bounds in the sense of Theorem 5.2.6 for all $t \leq s$. Taking $\psi \in C_0^\infty(\Omega(t))$, such that $\int_{\Omega(t)} \psi dy = 0$, we can use the Bogovskiĭ-operator $\mathcal{B}_t$ defined on $\Omega(t)$ via Theorem 5.3.10 to define

$$\tilde{P}(t)(\psi) = \nu \langle \varepsilon u, \varepsilon\mathcal{B}_t\psi\rangle_{\Omega(t)} - \rho_f \langle f, \mathcal{B}_t\psi\rangle_{\Omega(t)}.$$

This then gives the estimate

$$|\tilde{P}(t)(\psi)| = C\, \|\mathcal{B}_t\|\, \|\psi\|_{L^2(\Omega(t))}$$

where $\|\mathcal{B}_t\|$ is the operator-norm of $\mathcal{B}_t : \{\psi \in L^2(\Omega(t)) : \int_{\Omega(t)} \psi\, dy = 0\} \to W^{1,2}(\Omega(t))$ which is bounded by the Lipschitz constants of $\Omega(t)$ by Theorem 5.3.10. Now since $\{\psi \in L^2(\Omega(t)) : \int_{\Omega(t)} \psi\, dy = 0\}$ is a Hilbert space we find a $\tilde{p}(t)$ in that space such that $\tilde{p}(t) \equiv \tilde{P}(t)$.

We can extend the operator to $L^2(\Omega(t))$ in the following way: Take $\varphi(t) \in C_0^\infty(\Omega(t))$ and $\tilde{\varphi}(t) \in C_0^\infty(\Omega \setminus \Omega(t))$ fixed, such that $\int_\Omega \varphi(t)\,dy = \int_\Omega \tilde{\varphi}(t)\,dy = 1$ for all $t \in [0,s]$. Since the change of domain in time is uniformly continuous, we may assume further that $\varphi, \tilde{\varphi}$ are C^1 smooth in time. Next we define $\mathcal{B}$ to be the operator of Theorem 5.3.10 with respect to the full domain Ω.

By taking the fixed pair of test functions

$$\xi_0(t) := \mathcal{B}(\varphi(t) - \tilde{\varphi}(t)), \quad \phi_0(t,x) := \xi_0(t, \eta(t,x)),$$

we may define

$$\begin{aligned}\hat{p}(t,y) = \Big(&\langle DE(\eta(t)), \phi_0(t)\rangle_Q + \langle D_2R(\eta(t), \partial_t\eta(t)), \phi_0(t)\rangle_Q + \nu \langle \varepsilon u(t), \varepsilon\xi_0(t)\rangle_{\Omega(t)} \\ &- \rho_f \langle f(t), \xi_0(t)\rangle_{\Omega(t)} - \rho_s \langle f(t)\circ\eta(t), \phi_0(t)\rangle_Q \Big)\varphi(t,y)\end{aligned}$$

which satisfies $\|\hat{p}\|_{L^2([0,s];L^\infty(\Omega(t))} \leq C$ with C depending on the energy estimates only. But this allows to introduce the pressure. We define for $\psi \in L^1(\Omega(t))$, $c_\psi(t) = \int_{\Omega(t)} \psi(t)\,dy$. Now, if $\psi \in L^2([0,T], L^1(\Omega(t)))$ we find that $c_\psi \in L^2([0,s])$. Hence we may define

$$P(\psi) = \int_0^T \left\langle \tilde{p}, \psi - c_\psi\varphi \right\rangle\,dt + \int_0^T \int_{\Omega(t)} \hat{p}\,dy c_\psi\,dt$$

Thus $p \in L^\infty(0,s;L^2(\Omega(t)) + L^2(0,s;L^\infty(\Omega(t))$ is well defined via that operator

$$\int_0^T \langle \nabla p, \xi\rangle\,dt := P(\mathrm{div}\xi),$$

and satisfies the proposed regularity.

One can now check that it fulfills the right equations. For that it suffices to see that

$$\xi - \mathcal{B}_t(\mathrm{div}(\xi) - c_{\mathrm{div}(\xi)}\varphi) - c_{\mathrm{div}(\xi)}\mathcal{B}(\varphi - \tilde{\varphi}) = \xi - \mathcal{B}_t(\mathrm{div}(\xi) - c_{\mathrm{div}(\xi)}\varphi) - c_{\mathrm{div}(\xi)}\xi_0$$

is divergence free over $\Omega(t)$. Hence (5.1.3) is satisfied by (5.3.6) using the test-function

$$(\phi - c_{\mathrm{div}(\xi)}\phi_0, \xi - \mathcal{B}_t(\mathrm{div}(\xi) - c_{\mathrm{div}(\xi)}\varphi) - c_{\mathrm{div}(\xi)}\xi_0).$$

This finally allows us to conclude the Theorem:

Proof of Theorem 5.1.6. For any injective η_0 there is a short interval $[0,T]$ such that for all τ small enough all $\eta_k^{(\tau)}$ are injective according to Theorem 5.3.8. Passing to the limit in the sequence of the accordingly constructed $(\eta^{(\tau)}, v^{(\tau)})$'s we find by Theorem 5.3.2 (η, v) that is a weak solution to the parabolic fluid-structure interaction problem.

Now let $[0, T_{\max})$ be a maximal interval on which a solution (η, v) constructed in the previous way exists. If $T_{\max} = \infty$ there is nothing to be shown. The same holds if $T_{\max} < \infty$ and $\liminf_{t\to T_{\max}} E(\eta(t)) = \infty$ or if a self-intersection is approached. Now assume that all of that is not the case. Then there exists a sequence of times $t_i \nearrow T_{\max}$ such that $E(\eta(t_i))$ is bounded and there exists a limit, which we will denote $\eta(T_{\max})$.

Now take $E_0 := \liminf_{t\to T_{\max}} E(\eta(t)) \geq E(\eta(T_{\max}))$ due to lower semicontinuity. Following Theorem 5.3.3 and Theorem 5.3.7, there exists a minimal time T on which the solution any solution starting with energy below $2E_0$ stays below energy $3E_0$ and is Hölder-continuous in time in that time interval. Due to the convergence, we can pick t_i with $T_{\max} - t_i \leq T$ and $E(\eta(t_i)) \leq 2E_0$, which makes the solution Hölder-continuous right until $T_{\max}$ and thus $\lim_{t\nearrow T_{\max}} \eta(t) = \eta(T_{\max})$. But then we can use the short-term existence to construct a solution starting from $\eta(T_{\max})$ and appending this to the previous solution yields a contradiction as $T_{\max}$ cannot be maximal.

□

5.4 The example energy-dissipation pair

Let us now consider the prototypical example we stated in the introduction in the form of (1.2.8) and (1.2.9). In particular, we will prove that this energy-dissipation pair fulfills Assumptions 5.1.1 and 5.1.4. While doing so, we comment in a bit more detail on the meaning of those assumptions and on how they come into play in the course of the construction. Effectively we will prove the following proposition.

Proposition 5.4.1. *The example energy and dissipation given in* (1.2.8) *and* (1.2.9) *fulfill the assumptions S1-S6 and R1-R4 respectively. In particular the resulting fluid-structure interaction problem has a weak solution, under the additional conditions given in Theorem 5.1.6.*

Instead of proving the assumptions in ascending order or order of convenience, we will try to tackle them in the order as they appear in the proof of Theorem 5.1.6. Furthermore, we will roughly group them by some relevant subtopics.

5.4.1 The minimization problem (S1,S3-S4,R1-R2)

We start with the definition of $\eta_{k+1}^{(\tau)}$ in the minimizing movements-scheme in (5.3.1). In order to prove existence of minimizers, we need to invoke the direct method of the calculus of variation. Given a minimizing sequence, we find a converging subsequence and then show that the resulting limit has indeed a minimal value. In other words, we need to show compactness and lower semicontinuity, as well as a lower bound for the functional.

The last one seems to be directly stated in S1 together with the quadratic homogeinity in R2. Of course for our example energy S1 immediately holds, as all terms are non-negative and R2 is similarly obvious, as $\partial_t\eta$ occurs as a quadratic factor. There is however some hidden difficulty in order to find a lower bound for the whole functional, which does not only include energy and dissipation, but also the force terms, which can indeed be negative. To counteract these, we actually use the proper quadratic growth of the dissipation, which is immediate for the fluid through Korn's inequality and a result of the similar Korn-type inequality R3 for the solid. At this point though, as the first argument of the dissipation and the fluid domain are still fixed, there is no need yet, to use R3 to its full extent.

Once a lower bound for our minimizing sequence is established, we need to consider compactness. Here the relevant topology for η is the weak $W^{2,q}(Q;\Omega)$ topology and the relevant assumption for compactness is coercivity, in the form of S4. As we have bounded the other terms in the functional from below without involving the energy E, we know that this energy needs to be bounded from above and thus the coercivity allows us to use the Banach-Alaoglu theorem to extract converging subsequences. In our example, the coercivity is obtained in the most simple way, as $\|\nabla^2\eta\|_{L^q(Q)}^q$ is part of the energy.

As for (weak) lower semicontinuity, we need to verify assumptions S3 and R1 for the example case. First, note that the highest order term in the energy $\|\nabla^2\eta\|_{L^q(Q)}^q$ is weakly lower semi-continuous as it is a convex function of the norm. Second, we find that $q > n$ allows us to pick another subsequence converging in $C^{1,\alpha}$ for some $\alpha < \frac{q-n}{q}$. This allows us to pass to the limit in the terms $\int_Q \frac{1}{8}|\nabla\eta^T\nabla\eta - I|_{\mathcal{C}}dx$ as well as in the terms of R. Finally we can either use polyconvexity or the lower bound on $\det\nabla\eta$ to similarly show convergence of $\frac{1}{(\det\nabla\eta)^a}$ in (1.2.9).

5.4.2 Converting between Lagrangian and Eulerian setting (S2)

Note that as long as we were only discussing the minimization over the solid, the specific choice of $W^{2,q}(Q;\mathbb{R}^n)$ as a space was unimportant and choosing different terms in the integrand might as well have led us to a different space. It however becomes important when adding in the fluid, since it is prescribed w.r.t. Eulerian setting which is again determined by the solid deformation η. The key here is the assumption S2, on the determinant. Not only does this result in physically reasonable injectivity (in conjunction with the Ciarlet-Nečas condition), but it also allows us to convert between Eulerian and Lagrangian quantities as it actually implies that η is a diffeomorphism with uniform bounds. In particular, this will then imply that the fluid domain has a regular enough boundary, which will play an important role in the hyperbolic case.

To prove this property we follow the ideas of [96] where a similar energy was studied. Define $f(x) := \det \nabla\eta$. If $E(\eta)$ is bounded, then f is bounded in $W^{1,q}(Q)$ and $C^\alpha(Q)$. Now for a fixed ϵ_0 assume that there is $x_0 \in Q$ with $f(x_0) = 2\epsilon_0$. Then

$$E(\eta) \geq \int_{B_\delta(x_0)\cap Q} \frac{1}{f(x)^a} dx \geq \int_{B_\delta(x_0)\cap Q} \frac{1}{(f(x_0) + |f(x) - f(x_0)|dx)^a} \geq c\frac{\delta^n}{(2\epsilon_0 + C\delta^\alpha)^a}$$

However if $a\alpha > n$, the right hand side can be arbitrarily large if ϵ_0 and δ are choosen small enough, which is a contradiction.

5.4.3 Uniform bounds (R3)

It has been long known that there is a certain mismatch between physically reasonable and mathematically expedient dissipation functionals (see e.g. [6]). Mathematicians would prefer the dissipation potential to be of the form $\|\partial_t\eta\|^2_{W^{1,2}(Q)}$ and $\|u\|^2_{W^{1,2}(\Omega)}$. This would then lead directly to $L^2([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ and $L^2([0,T]; W^{1,2}(\Omega;\mathbb{R}^n))$-bounds respectively for $\partial_t\eta$ and v as well as their approximations. Instead, for physical reasons we have to consider $R(\eta, \partial_t\eta)$ and $\|\varepsilon v\|^2_{\Omega(t)}$, which are independent of the observer. Thus Korn-type inequalities are required to convert the bounds for latter into bounds for the former.

As the Korn inequality for the fluid is the classic one and the added difficulties due to the changing domain are overcome by Theorem 5.2.8, we only need to focus at the solid. For our example, this inequality and thus R3 follows from the main theorems in [154, 160]. See also the discussion in [137], where these results are coupled with an energy similar to ours in the context of a thermoviscoelastic solid (but without a fluid).

Observe that these inequalities require a certain regularity of the deformation $\nabla\eta$ itself. In fact we need the same properties that allow us to switch between Lagrangian and Eulerian settings, i.e. a uniform lower bound on the determinant $\det \nabla\eta$ and continuity of $\nabla\eta$, as otherwise there are known counterexamples for which the inequality fails.

5.4.4 Weak equations (S5, R4)

Combined, the assumptions so far are enough to construct iterative minimizers and even to have a subsequence converge to a limit object (η, v) in space-time by weak compactness. We are left to show that these function do satisfy a weak coupled PDE. This is where the assumptions S5 and R4 come in. Both of them are two-part in nature, requiring both the existence of a derivative as well as some form of continuity. Both are also immediately shown for the example by just doing the calculation. Let us start with the dissipation, namely

$$\langle D_2R(\eta, b), \phi\rangle = \int_Q 2(\nabla b^T\nabla\eta + \nabla\eta^T\nabla b) \cdot (\nabla\phi^T\nabla\eta + \nabla\eta^T\nabla\phi)\, dx.$$

Since we have $C^{1,\alpha}(Q;\mathbb{R}^n)$-bounds on $\nabla\eta$, the $L^2(Q)$-regularity of $\nabla b(= \nabla\partial_t\eta)$ is enough to make sense of $D_2R(\cdot, b)$ as an operator in $W^{-1,2}(Q;\mathbb{R}^n)$. Similarly, the uniform convergence in some Hölder space for $\nabla\eta$ is enough to give this derivative the required continuity with respect to both b and η.

The calculation for the energy is a bit more involved. Restricting ourselves to deformations η of finite energy and thus positive determinant, we get by a short calculation

$$\begin{aligned}\langle DE(\eta), \phi\rangle = \int_Q &\frac{1}{4}\mathcal{C}(\nabla\eta^T\nabla\eta - I) \cdot (\nabla\phi^T\nabla\eta + \nabla\eta^T\nabla\phi)\\ &- a\frac{cof\nabla\eta}{(\det\nabla\eta)^{a+1}} \cdot \nabla\phi + |\nabla^2\eta|^{q-2}\nabla^2\eta \cdot \nabla^2\phi\, dx\end{aligned}$$

where the scalar products are to be understood over all tensorial dimensions.

Again in order to pass to the limit with the energy, we need to make use of the uniform Hölder-continuity of $\nabla\eta$ to see that the first two terms in $DE(\eta)$ are well defined and continuous with respect to the corresponding convergence. Finally, the last term is well defined since $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ uniformly, but to show that it is also continuous we need to show strong convergence using the convexity of the quantity. This then leads to the final assumption.

5.4.5 Improved convergence (S6)

As the usual compactness methods will only result in weak compactness, and S5 requires strong convergence we need a way to improve upon this. For this we rely on an idea that is most commonly attributed to Minty. While it is certainly not true that our energy is convex, the critical, second order term in its derivative $DE(\eta)$ is monotone and this allows us to improve convergence as desired.

Assume that as stated $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$. Then after possibly extracting another subsequence $\eta_l \to \eta$ in $C^{1,\alpha}(Q;\mathbb{R}^n)$ the first two terms of $DE(\eta_l)$ already converge to their respective limits (using the lower bound on $\det \nabla\eta$ given through S2). As a result, the stated conditions on convergence of $\langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \to 0$ for all cutoffs $\psi \in C_0^\infty(Q;[0,1])$ are equivalent to those for

$$\left\langle |\nabla^2\eta_l|^{q-2}\nabla^2\eta_l - |\nabla^2\eta|^{q-2}\nabla^2\eta, \nabla^2((\eta_l - \eta)\psi)\right\rangle$$

Here the cutoff complicates things slightly, but expanding the right hand side yields terms of lower order ($(\eta_l - \eta)\otimes\nabla^2\psi$ and $\nabla(\eta_l - \eta)\otimes\nabla\psi$) which already converge strongly to 0 and one term of second order, which leaves us with

$$\left\langle |\nabla^2\eta_l|^{q-2}\nabla^2\eta_l - |\nabla^2\eta|^{q-2}\nabla^2\eta, (\nabla^2\eta_l - \nabla^2\eta))\psi\right\rangle$$

where we now can send $\psi \to 1$ by approximation. Now $\eta \mapsto |\nabla^2\eta|^{q-2}\nabla^2\eta$ is a classic example of a monotone operator. Thus the term is bounded from below by 0 and its convergence to 0 implies strong convergence $\eta_l \to \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$, by the fact that for $q \geq 2$ and $a, b \in \mathbb{R}^{n^3}$

$$(|a|^{q-2}a - |b|^{q-2}b)\cdot(a-b) \geq c|a-b|^q.$$

5.5 Appendix of the chapter

5.5.1 Some technical lemmata

Here we gather the proofs of some technical lemmata.

Lemma 5.5.1 (Expansion of the determinant)**.** *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$\det(I + \tau A) = 1 + \tau trA + \sum_{l=2}^{n} \tau^l M_l(A)$$

where $M_l(A)$ is a homogeneous polynomial of degree l in the entries of A. Note that this is a finite sum.

Proof. Consider the Leibniz formula

$$\det(I + \tau A) = \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\prod_{i=1}^{n}(\delta_{i,\pi(i)} + \tau A_{i,\pi(i)})$$

where S_n is the set of permutations of $\{1,...,n\}$. We expand the product and order the terms by the exponent of the factor τ^l and thus by the number of terms $\tau A_{i,\pi(i)}$ that are taken while expanding the product. This will directly yield the homogeneous polynomial $M_l(A)$.

For τ^0 and τ^1, the only non-zero terms occur for $\pi = id$, otherwise there will be at least one factor $\delta_{i,\pi(i)}$ for $i \neq \pi(i)$. For τ^0 this means we only choose the $\delta_{i,i}$ terms and for τ^1 we can choose any one $\tau A_{i,i}$-term. Thus $M_0(A) = 1$ and $M_1(A) = trA$. □

Lemma 5.5.2 (Invertible maps)**.** *Let $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ be injective, such that $\det\nabla\eta > \epsilon_0 > 0$ for some $\epsilon_0 < 1$ and $\eta|_P = \gamma$. Then $\eta^{-1} \in W^{2,q}(\eta(Q);\mathbb{R}^n)$ and $\|[\|W^{2,q}(\eta(Q))]\eta^{-1} \leq c\frac{\|[\|W^{2,q}(Q)]\eta^{2n-1}}{\epsilon_0^2}$ where c depends only on q, Q, γ and n.*

Proof. Due to the condition on the determinant $\nabla\eta$ is invertible and furthermore, we have the well known formula

$$\nabla(\eta^{-1}) = (\nabla\eta)^{-1}\circ\eta^{-1} = \frac{(cof\nabla\eta)^T}{\det\nabla\eta}\circ\eta^{-1}$$

Now we take the derivative of $\nabla(\eta^{-1})\circ\eta$ to get

$$(\nabla^2(\eta^{-1}))\circ\eta\cdot\nabla\eta = \nabla\left(\nabla(\eta^{-1})\circ\eta\right) = \frac{\nabla(cof\nabla\eta)^T}{\det\nabla\eta} - \frac{(cof\nabla\eta)^T\otimes(cof\nabla\eta)}{(\det\nabla\eta)^2}\cdot\nabla^2\eta$$

Integrating then yields

$$\int_{\eta(Q)}|(\nabla^2(\eta^{-1}))|^q dy = \int_Q|(\nabla^2(\eta^{-1}))\circ\eta|^q\det\nabla\eta dx$$
$$= \int_Q\left|\frac{\nabla(cof\nabla\eta)^T}{\det\nabla\eta} - \frac{(cof\nabla\eta)^T\otimes(cof\nabla\eta)}{(\det\nabla\eta)^2}\cdot\nabla^2\eta\right|^q\det\nabla\eta dx$$

Now the determinants in the denominators can be estimated by ϵ_0, while the numerators all consist of one second derivative multiplied with a number of first derivatives, which we can estimate by their supremum.

$$\leq\int_Q C\left(\frac{|\nabla^2\eta|\|[\|\infty]\nabla\eta^{n-2}}{\epsilon_0^{1-1/q}} + \frac{\|[\|\infty]\nabla\eta^{2n-2}|\nabla^2\eta|}{\epsilon_0^{2-1/q}}\right)^q dx \leq C\frac{\|[\|L^q(\eta(Q))]\nabla^2\eta^q\|[\|\infty]\nabla\eta^{q(2n-2)}}{\varepsilon^{2q}}$$

Using the Morrey embedding $\|[\|\infty]\nabla\eta\leq\|[\|C^\alpha]\nabla\eta\leq C\|[\|W^{2,q}(Q)]\eta$ and collecting the terms then shows

$$\|[\|L^q(\eta(Q))]\nabla^2(\eta^{-1})\leq C\frac{\|[\|W^{2,q}(Q)]\eta^{2n-1}}{\epsilon_0^2}.$$

Finally, as we have partially known boundary values $\eta^{-1}|_{\gamma(P)} = \gamma^{-1}$, the lower order estimates follow from a Poincaré-inequality. □

For the next result we an interpolation. We begin by recalling the following result, which follows for instance from the interpolation estimate in [182, Theorem 2.13] which implies combined with the usual Sobolev embeddings [194, Theorem 2.5.1 and Remark 2.5.2] that for all $m\in[0,\infty)$, $\alpha\in[1,\infty)$ and all Lipschitz domains Ω satisfy

$$m\leq l \qquad \text{and} \qquad \frac{1}{\alpha}-\frac{m}{n}\geq\frac{1}{\gamma}-\frac{l}{n} = \frac{k-l}{ka}+\frac{l}{2k}-\frac{l}{n}$$

the estimate

$$\|[\|W^{m,\alpha}]g\leq C\|[\|W^{k,2}]g^{\frac{l}{k}}\|[\|L^a]g^{\frac{k-l}{k}}. \tag{5.5.1}$$

Lemma 5.5.3. *Let $Q\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $q>n$ and the number $k\in\mathbb{N}$ be defined as*

$$k = 2+\frac{n+1}{2}\ \textit{if } n \textit{ is odd},\quad k = 3+\frac{n}{2}\ \textit{if } n \textit{ is even}. \tag{5.5.2}$$

For every $\eta\in W^{2,q}(Q)\cap W^{k,2}(Q)$, there is a constant c depending on Q, n, k and $\|[\|W^{2,q}(Q)]\eta$ such that

$$\sum_{l=1}^k\sum_{a\in\{1,\dots,n\}^l}\|\nabla^{k-l}\Pi_{i=1}^l\partial_{a_i}\eta\|\leq c\|[\|W^{k,2}(Q)]\eta.$$

Proof. Observe, that since $\nabla\eta$ is uniformly bounded by the $W^{2,q}(Q;\mathbb{R}^n)$ norm, we find that

$$\sum_{a\in\{1,\dots,n\}^l}\|\nabla^{k-l}\Pi_{i=1}^l\partial_{a_i}\eta\|\leq c\sum_{\beta\in\mathbb{N}_0^l,\ |\beta|=k-l}\|\Pi_{i=1}^l|\nabla^{\beta_i}\nabla\eta|\|.$$

The estimate for $l=1$ is direct. Next assume, that $l\geq 2$ and $\beta\in\mathbb{N}_0^l$, $|\beta|=k-l$ such that all $\beta_i\neq 1$. Now by Hölder's and Young's inequality

$$\Pi_{i=1}^l\|\nabla^{\beta_i}\nabla\eta\|\leq c\sum_{\beta_i>1}\|[\|\frac{2(k-l)}{\beta_i}]\nabla^{\beta_i-1}\nabla^2\eta^{\frac{k-l}{\beta_i}}$$

Next we seek to interpolate $\nabla^2\eta$ in between $W^{2,q}$ and $W^{k-2,2}$. For that we wish to use (5.5.1). Hence we have to prove that

$$\frac{\beta_i}{2(k-l)}\geq\frac{k-1-\beta_i}{q(k-2)}+\frac{\beta_i-1}{2(k-2)}. \tag{5.5.3}$$

Since $l\geq 2$ we find (by multiplying (5.5.3) with $k-2$) that (5.5.3) holds true whenever

$$\frac{1}{2}\geq\frac{k-\beta_i-1}{n}\Leftrightarrow n\geq 2(k-\beta_i-1),$$

which is satisfied by the definition of k as long as $\beta_i\geq 2$.

Hence we may use (5.5.3)

$$\|[\|\frac{2(k-l)}{\beta_i}]\nabla^{\beta_i-1}\nabla^2\eta^{\frac{k-l}{\beta_i}}\leq c\|[\|L^q(Q)]\nabla^2\eta^{\frac{k-l}{\beta_i}\frac{k-1-\beta_i}{k-2}}\|[\|W^{k-2,2}(Q)]\nabla^2\eta^{\frac{k-l}{\beta_i}\frac{\beta_i-1}{k-2}}\leq c\|[\|W^{k-2,2}(Q)]\nabla^2\eta,$$

using that $\frac{k-l}{\beta_i}\frac{\beta_i-1}{k-2}\leq 1$.

The last case is proved inductively. First with no loss of generality we take $\beta_1=1$. Then $\sum_{i=2}^l\beta_i\leq k-l-1$ and using Hölder's inequality and Sobolev embedding implies

$$\||\nabla^2\eta|\Pi_{i=1}^{l-1}|\nabla^{\beta_i}\eta|\|\leq\|[\|n]\nabla^2\eta\|[\|\frac{2n}{n-2}]\Pi_{i=1}^{l-1}|\nabla^{\beta_i}\eta|\leq c\|[\|W^{1,2}(Q)]\Pi_{i=1}^{l-1}|\nabla^{\beta_i}\eta|.$$

If now $\beta_i\neq 1$ for all $i>1$, the estimate follows by the above case for the pair $\nabla^{k-(l-1)}\Pi_{i=1}^{l-1}\partial_{a_i}\eta$. If not, we may assume that $\beta_2=1$ and can repeat the argument again. After at most l steps (in which case $k\geq 2l$), we get the result.

□

Proposition 5.5.4 (Space isomorphisms). *Let $\eta\in W^{2,q}(Q;\mathbb{R}^n)$ such that $\det\nabla\eta>\epsilon_0>0$ and $\eta|_P=\gamma$. Then the map*

$$\eta^\#:\xi\mapsto\xi\circ\eta;W^{2,q}(\eta(Q);\mathbb{R}^n)\to W^{2,q}(Q;\mathbb{R}^n)$$

is a linear vector space-isomorphism with operator-norm $\|\eta^\#\|\leq C\|[\|W^{2,q}(Q)]\eta^2/\epsilon_0^{1/q}$ where c does only depend on q,Q,γ and n. Moreover, if $q>n$ and additionally $\eta\in W^{k,2}(Q;\mathbb{R}^n)$ and $\xi\in C^k(\eta(Q;\mathbb{R}^n))$, for k defined in (5.5.2)*, then*

$$\|[\|W^{k,2}(Q)]\xi\circ\eta\leq c\|[\|W^{k,2}(Q)]\eta\|[\|C^k(Q)]\xi,$$

where the constant depends on Ω,n,k and $\|[\|W^{2,q}(Q)]\eta$ only.

Proof. Linearity follows immediately from the definition. Now we calculate

$$\begin{aligned}\|[\|L^q(Q)]\nabla^2(\xi\circ\eta)&=\|[\|L^q(Q)]((\nabla^2\xi)\circ\eta\cdot\nabla\eta)\cdot\nabla\eta+(\nabla\xi)\circ\eta\cdot\nabla^2\eta\\&\leq C\left(\|[\|\infty]\nabla\eta^2\|[\|L^q(Q)](\nabla^2\xi)\circ\eta+\|[\|\infty]\nabla\xi\|[\|L^q(Q)]\nabla^2\eta\right).\end{aligned}$$

and use

$$\epsilon_0\|[\|L^q(Q)](\nabla^2\xi)\circ\eta^q \le \int_Q |(\nabla^2\xi)\circ\eta|^q \det\nabla\eta\, dx = \|[\|L^q(\eta(Q))]\nabla^2\xi^q$$

to estimate the first term. Then using a Poincaré's inequality and the usual Morrey's embeddings we get

$$\|[\|W^{2,q}(Q)]\xi\circ\eta \le C\|[\|W^{2,q}(\eta(Q))]\xi\frac{\|[\|W^{2,q}(Q)]\eta^2}{\varepsilon^{1/q}}$$

which proves that $\eta^\#$ is a vector space-homomorphism with given operator-norm. Now as $(\eta^\#)^{-1} = (\eta^{-1})^\#$ we conclude that it is also an isomorphism by the previous lemma.

For the second estimate we observe that

$$\|\nabla^k(\xi\circ\eta)\| \le c\sum_{l=1}^k \sum_{a\in\{1,\dots,n\}^l} \|[\|C^l(\eta(Q))]\xi\|\nabla^{k-l}\Pi_{i=1}^l\partial_{a_i}\eta\|,$$

which finishes the proof by Theorem 5.5.3 □

5.5.2 Proof of Theorem 5.3.11

The proof is split in two parts. The first part constructs an extension of the solenoidality. The second part shows how this extension can than be convoluted. We also will need the following Poincaré type lemma:

Lemma 5.5.5 (Poincaré's lemma for thin regions)**.** *Let $S_0 \subset \mathbb{R}^n$ be an $(n-1)$-dimensional rectifiable set and $\Phi : S_0 \times [0,\varepsilon_0] \to \mathbb{R}^n$ a injective L-bi-Lipschitz function such that $\Phi(\cdot,0) = \mathrm{id}$. Define $S_\varepsilon = \Phi(S_0,[0,\varepsilon])$ for $\varepsilon \in [0,\varepsilon_0]$. Then for all $f \in W^{1,a}(S_\varepsilon)$ with $f|_{S_0} = 0$ in the trace sense we have*

$$\|[\|L^a(S_\varepsilon)]f \le c\varepsilon\|[\|W^{1,a}(S_\varepsilon)]f \text{ for all } f\in W^{1,a}(S_\varepsilon(t);\mathbb{R}^n) \text{ with } f|_{\Omega(t)} = 0. \tag{5.5.4}$$

where c is independent of ε.

Proof. By density arguments it is enough to prove the theorem for smooth functions. Now for $z \in S_0$ and $s_0 \in [0,\varepsilon_0]$ we find:

$$\begin{aligned}|f(\Phi(z,s_0))| &= |f(\Phi(z,s_0)) - f(\Phi(z,0))| = |\int_0^{s_0}\partial_s f(\Phi(z,s))ds|\\ &\le \int_0^{s_0}|(\nabla f)(\Phi(z,s))||\partial_s\Phi(z,s)|ds \le \int_0^{s_0} L|(\nabla f)(\Phi(z,s))|ds\end{aligned}$$

But then integrating over the whole domain gets us

$$\begin{aligned}\int_{S_\varepsilon}|f(y)|^a dy &= \int_{S_0}\int_0^\varepsilon |f(\Phi(z,s_0))|^a|J(z,s_0)|ds_0dz\\ &\le \int_{S_0}\int_0^\varepsilon\left(\int_0^{s_0} L|\nabla f(\Phi(z,s))|ds\right)^a|J(z,s_0)|dsds_0dz\\ &\le \int_{S_0}\int_0^\varepsilon \fint_0^\varepsilon \varepsilon^a L^a|\nabla f(\Phi(z,s))|^a|J(z,s_0)|dsds_0dz\\ &= \varepsilon^a L^a\int_{S_0}\int_0^\varepsilon L^a|\nabla f(\Phi(z,s))|^a|J(z,s)|\|[\|\infty]J\|[\|\infty]J^{-1}dsdz\\ &= L^a\varepsilon^a\|[\|\infty]J\|[\|\infty]J^{-1}\int_{S_\varepsilon}|\nabla f(y)|^a dy\end{aligned}$$

where $J(z,s)$ is the Jacobian of Φ which is bounded from above as well as away from zero because Φ is bi-Lipschitz. □

Lemma 5.5.6 (Extension of the solenoidal region)**.** *Fix a function*

$$\eta \in L^\infty([0,T];\mathcal{E}) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)) \text{ with } \sup_{t\in T} E(\eta(t)) < \infty,$$

such that $\eta(t) \notin \partial\mathcal{E}$ *for all* $t \in [0,T]$. *As before we set* $\Omega(t) = \Omega \setminus \eta(t,Q)$.

Let $\xi \in L^2([0,T];W_0^{1,2}(\Omega;\mathbb{R}^n))$ *such that* $\operatorname{div}\xi(t) = 0$ *on* $\Omega(t)$. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon > 0$, *there exists* ξ_ε *such that* $\operatorname{div}\xi(t,y) = 0$ *for all* $y \in \Omega$ *with* $dist(y,\Omega(t) \cup \partial\Omega) < \varepsilon$ *and there are constants* c *independent of* ξ *such that for a.e.* $t \in [0,T]$

$$\|[\|W^{1,2}(\Omega)]\xi_\varepsilon \le c\|[\|W^{1,2}(\Omega)]\xi \qquad \text{and} \qquad \|[\|L^2(\Omega)]\xi_\varepsilon - \xi \le c\varepsilon^{\frac{2}{n+2}}\|[\|W^{1,2}(\Omega)]\xi.$$

Additionally for any $k \in \mathbb{N}$ *and* $a \in (1,\infty)$ *such that* $\xi \in L^2([0,T];W^{k,a}(Q;\mathbb{R}^n))$ *we also have*

$$\|[\|L^2([0,T];W^{k,a}(\Omega))]\xi_\varepsilon - \xi \to 0 \text{ for } \varepsilon \to 0$$

and similarly if $\xi \in W^{1,2}([0,T];W^{1,\infty}(\Omega;\mathbb{R}^n))$ *then also*

$$\|[\|L^2([0,T];W^{1,2}(\Omega))]\partial_t(\xi - \xi_\varepsilon) \to 0 \text{ for } \varepsilon \to 0. \tag{5.5.5}$$

Proof. We begin by defining

$$S_\varepsilon(t) := \{y \in \Omega \,|\, dist(y,\eta(t,\partial Q)) \le \varepsilon\}$$

and introduce the cutoff function $\psi_\varepsilon : [0,T] \times \Omega \to [0,1]$, such that

$$\chi_{S_\varepsilon(t)} \le \psi_\varepsilon(t) \le \chi_{S_{2\varepsilon}(t)} \text{ and } \|[\|C^l]\psi_\varepsilon(t) \le \frac{c}{\varepsilon^l} \text{ for } l \in \mathbb{N}.$$

Due to the regularity of η, we may assume, that $\partial_t\psi_\varepsilon$ is uniformly bounded, such that

$$\|[\|L^2([0,T]\times\Omega)]\partial_t\psi_\varepsilon \to 0 \text{ with } \varepsilon \to 0. \tag{5.5.6}$$

We also pick $\tilde{\psi} \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^n)$ such that $supp\tilde{\psi}(t) \cap S_{\varepsilon_0}(t) = \emptyset$ for some $\varepsilon_0 > 0$ and $\int_\Omega \tilde{\psi}(t)dy = 1$ for all t. Using this we then define

$$\xi_\varepsilon(t) := \xi(t) - \mathcal{B}\left(\psi_\varepsilon(t)\operatorname{div}\xi(t) - b_\varepsilon(t)\tilde{\psi}(t)\right)$$

where $\mathcal{B}$ is the Bogovskiĭ-operator on Ω and $b_\varepsilon(t) := \int_\Omega \psi_\varepsilon(t)\operatorname{div}\xi(t)\,dy$ is used to keep the mean. Then per definition

$$\operatorname{div}\xi_\varepsilon(t) = (1 - \psi_\varepsilon(t))\operatorname{div}\xi(t) - b_\varepsilon(t)\tilde{\psi}(t)$$

has no support on $S_\varepsilon(t)$, as required and

$$\begin{aligned}\|[\|W^{k,a}(\Omega)]\xi - \xi_\varepsilon &= \|[\|W^{k,a}(\Omega)]\mathcal{B}\left(\psi_\varepsilon(t)\operatorname{div}\xi(t) - b_\varepsilon(t)\tilde{\psi}(t)\right)\\ &\le c\|[\|W^{k-1,a}(\Omega)]\psi_\varepsilon(t)\operatorname{div}\xi(t) - b_\varepsilon(t)\tilde{\psi}(t) \le c\|[\|W^{k-1,a}(\Omega)]\psi_\varepsilon(t)\operatorname{div}\xi(t) + c|b_\varepsilon(t)|\end{aligned}$$

is the main quantity we need to estimate.

Let us begin with the special case $k = 0, a = 2$. Here we use that $L^{\frac{2n}{2+n}}(\Omega;\mathbb{R}^n) \subset W^{-1,2}(\Omega;\mathbb{R}^n)$ and apply Hölder's inequality to show that

$$\begin{aligned}\|[\|L^2(\Omega)]\xi - \tilde{\xi}_\varepsilon &\le c\|[\|W^{-1,2}(\Omega)]\psi_\varepsilon\operatorname{div}(\xi) + c|b_\varepsilon| \le c\|[\|L^{\frac{2n}{n+2}}(\Omega)]\psi_\varepsilon\operatorname{div}(\xi)\\ &\le c\|[\|L^n(S_{2\varepsilon}(t))]\psi_\varepsilon\|[\|L^2(\Omega)]\operatorname{div}\xi \le c|S_{2\varepsilon}|^{\frac{1}{n}}\|[\|W^{1,2}(\Omega)]\xi \le c\varepsilon^{\frac{1}{n}}\|[\|W^{1,2}(\Omega)]\xi.\end{aligned}$$

For $k \geq 1$ we first note that $|b_\varepsilon(t)| \leq c\|[\|L^2(S_\varepsilon(t))]\mathrm{div}\xi \to 0$ for each fixed ξ and that furthermore

$$\|[\|W^{k-1,a}(\Omega)]\psi_\varepsilon(t)\mathrm{div}\xi(t) \leq c\sum_{l=0}^{k-1}\|[\|C^{k-1-l}(\Omega)]\psi_\varepsilon(t)\|[\|L^a(S_{2\varepsilon}(t))]\nabla^l\mathrm{div}\xi$$
$$\leq c\sum_{l=0}^{k-1}\varepsilon^{-(k-1-l)}\|[\|L^a(S_{2\varepsilon}(t))]\nabla^l\mathrm{div}\xi.$$

In particular for $k = 1, a = 2$ we have $k-1-l = 0$ so this immediately proves that $\|[\|W^{1,2}(\Omega)]\xi_\varepsilon \leq c\|[\|W^{1,2}(\Omega)]\xi$ independently of ξ. For $k > 1$ we will apply the Poincaré's inequality Theorem 5.5.5. For this we make use of the fact that $S_{\varepsilon_0} \setminus \Omega(t)$ is a small neighborhood of an uniform Lipschitz boundary and thus can be written in the required way using η itself. Furthermore for any $l < k$ we have $\nabla^l\mathrm{div}\xi = 0$ on $\Omega(t)$ and thus also on $\partial\Omega(t)$ in the trace sense. Now this then gives us $\|[\|L^a(S_{2\varepsilon}(t))]\nabla^l\mathrm{div}\xi \leq c\varepsilon^{k-1-l}\|[\|W^{k,a}(S_{2\varepsilon}(t))]\xi$ which is enough to finish the estimate.

Finally let us consider the time-derivative. As $\mathcal{B}$ is a linear operator, we have

$$\|[\|W^{1,2}(\Omega)]\partial_t(\xi_\varepsilon - \xi) = c\|[\|L^2(\Omega)]\partial_t\left(\psi_\varepsilon(t)\mathrm{div}\xi(t) - b_\varepsilon(t)\tilde{\psi}(t)\right)$$
$$\leq c\|[\|L^2(S_{2\varepsilon}(t))]\partial_t(\psi_\varepsilon(t)\mathrm{div}\xi(t)) + c|\partial_t b_\varepsilon(t)|\|[\|L^2(\Omega)]\tilde{\psi}(t) + c|b_\varepsilon(t)|\|[\|L^2(\Omega)]\partial_t\tilde{\psi}(t)$$

For the last term we have already shown that $|b_\varepsilon(t)| \to 0$ and $\tilde{\Psi}$ does not depend on ε. For the second to last term we note that

$$|\partial_t b_\varepsilon(t)| = |\int_\Omega \partial_t(\psi_\varepsilon(t)\mathrm{div}\xi(t))dy| \leq \|[\|L^2(\Omega)]\partial_t(\psi_\varepsilon(t)\mathrm{div}\xi(t))$$

which is the same as the first term and for which we use the estimate

$$\|[\|L^2(\Omega)]\partial_t(\psi_\varepsilon(t)\mathrm{div}\xi(t)) \leq \|[\|L^2(\Omega)]\partial_t\psi_\varepsilon(t)\|[\|W^{1,\infty}(S_{2\varepsilon}(t))]\xi(t) + \|[\|L^2(\Omega)]\psi_\varepsilon(t)\|[\|W^{1,\infty}(S_{2\varepsilon}(t))]\partial_t\xi(t)$$

which implies (5.5.5) by (5.5.6) and Hölder's inequality.

□

Proof of Theorem 5.3.11. First we apply Theorem 5.5.6 to find a function $\hat{\xi}$ with $\hat{\xi} = 0$ on $\Omega(t)$ and an ε-neighborhood of $\partial(\Omega \setminus \Omega(t))$. Thus taking a convolution with γ_{ε^2} does not intervene with the zero boundary values (if ε is small enough).

We will now apply Theorem 5.5.6 again to $\hat{\xi}_\varepsilon * \gamma_{\varepsilon^2}$ and call the result ξ_ε, a function which is smooth by Theorem 5.3.10. Moreover since all operations are linear we find that $\xi_\varepsilon \in C_0^\infty(\Omega)$ is divergence free in $\Omega(t) \cup S_\varepsilon$. By collecting all the properties of the approximation, we find that for

$$\|[\|W^{l,a}(\Omega)]\xi - \xi_\varepsilon \to 0$$

for $l \leq k-1$. Moreover,

$$\|[\|W^{1,2}(\Omega)]\xi - \xi_\varepsilon \leq c\|[\|W^{1,2}(\Omega)]\xi.$$

and

$$\|[\|L^2(\Omega)]\xi - \xi_\varepsilon \leq c\varepsilon^{\frac{2}{n+2}}\|[\|W^{1,2}(\Omega)]\xi$$

Next we turn to the estimates for

$$\phi_\varepsilon := \xi_\varepsilon \circ \eta.$$

They follow by Theorem 5.5.4, and standard convolution estimates. First, for $k > 2$ we find

$$\|[\|W^{k,2}(Q)]\phi_\varepsilon \leq c\|[\|C^k(\Omega)]\xi_\varepsilon\|[\|W^{k,2}(Q)]\eta \leq c(\varepsilon)\|[\|L^2(\Omega)]\xi\|[\|W^{k,a}(Q)]\eta.$$

Second, in case $\xi \in L^\infty([0,T];W^{2,a}(\Omega))$, we find

$$\|[\|W^{2,a}(Q)]\phi_\varepsilon - \phi \le c\|[\|W^{2,a}(\Omega)]\xi_\varepsilon - \xi \to 0 \text{ with } \varepsilon \to 0.$$

Finally, for the time derivative in case $\partial_t\xi \in L^\infty([0,T];W^{1,2}(\Omega))$ and $\xi \in L^\infty([0,T];W^{3,a}(\Omega))$ with $a > n$ we find by Sobolev embedding that:

$$\begin{aligned}\|[\|W^{1,2}(Q)]\partial_t(\phi_\varepsilon - \phi) &\le c\|[\|W^{1,2}(\Omega)]\partial_t(\xi_\varepsilon - \xi) + c\|[\|W^{1,\infty}(\Omega)]\nabla(\xi_\varepsilon - \xi)\|[\|W^{1,2}(Q)]\partial_t\eta \\ &\le c\|[\|W^{1,2}(\Omega)]\partial_t(\xi_\varepsilon - \xi) + c\|[\|W^{3,a}(\Omega)]\xi_\varepsilon - \xi\|[\|W^{1,2}(Q)]\partial_t\eta\end{aligned}$$

which implies the assertions for the time-derivatives by (5.5.5).

□

Chapter 6

A variational approach to hyperbolic evolutions

In this section, we will introduce a general method for adding inertial effects to continuum mechanical problems, thereby turning them from parabolic to hyperbolic. We will demonstrate this in the purely Lagrangian case of a single viscoelastic solid, but as we will see in the next section, the method turns out to be flexible enough to allow even for problems which are of a mixed Lagrangian/Eulerian type such as fluid-structure interaction. Also note that while this section can be read independently from the previous one, at some places we will use a similar reasoning, which will thus be abridged slightly.

In particular we keep the notation from the previous section. Thus $\eta : Q \to \mathbb{R}^n$, $\eta \in \mathcal{E}$ is the deformation of the solid specimen and E and R are its elastic energy and dissipation. For simplicity we will also use the same set of assumptions (i.e. Assumption 5.1.1 and Assumption 5.1.4, respectively), though many of them could be relaxed, as they are intended for interaction with the fluid. Furthermore, as the only relevant domain Q is kept fixed, we will suppress the dependence of the inner products and the resulting L^2-norms on Q.

The problem we thus want to solve is to find the deformation of the viscoelastic solid specimen moving inertially in space subject to an action of forces. In other words, we need to solve the balance of momentum (Newton's second law) that reads as

$$\rho \partial_t^2 \eta = D_2 R(\eta, \partial_t \eta) + DE(\eta) - f \circ \eta \text{ in } [0,T] \times Q. \tag{6.0.1}$$

where $\rho = \rho_s$ is a constant density and f some, not necessarily conservative, external force. In addition, we will require that $\eta \in \mathcal{E}$ which implies that it satisfies given Dirichlet boundary conditions on P. On the other parts of the boundary $\partial Q \smallsetminus P$ we assume here natural Neumann type (free) boundary conditions that will result from minimization. Finally, we will add appropriate initial conditions to (6.0.1),

$$\eta(0) = \eta_0 \text{ and } \partial_t \eta(0) = \eta^* \text{ in } Q. \tag{6.0.2}$$

As usual, we translate this into a notion of a weak solution.

Definition 6.0.1 (Weak solution to the inertial problem for solids)**.** *We call* $\eta \in L^\infty([0,T];\mathcal{E}) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$*, such that* $\partial_t \eta \in C_w^0([0,T];L^2((Q;\mathbb{R}^n))$ *and* $\eta(0) = \eta_0$ *a weak solution to the inertial problem of the viscoelastic solid* (6.0.1) *with initial conditions* (6.0.2) *if*

$$\int_0^T \langle DE(\eta), \phi \rangle + \langle D_2 R(\eta, \partial_t \eta), \phi \rangle - \langle f \circ \eta, \phi \rangle + \rho \langle \partial_t \eta, \partial_t \phi \rangle \, dt - \rho \langle \eta_*, \phi(0) \rangle_Q = 0$$

for all $\phi \in C^\infty([0,T];C^\infty(Q;\mathbb{R}^n))$ *with* $\phi|_{[0,T]\times P} = 0$ *such that* $\phi(T) = 0$*.*

Observe, that we restrict the solution to the closed set $\mathcal{E}$ and thus will only work with injective deformations on Q. This will be of particular interest to us as this property is relevant for modelling fluid-structure interactions.

The main goal of this section will be to prove the following theorem.

Theorem 6.0.2 (Existence of solutions for solids)**.** *Assume that the conditions from Theorem 5.1.1 (with $\Omega = \mathbb{R}^n$) and Theorem 5.1.4 hold. Assume that the initial data $\eta_0 \in \mathcal{E} \setminus \partial\mathcal{E}$ with $E(\eta) < \infty$, that $\eta_* \in L^2(Q;\mathbb{R}^n)$ and that $f \in C^0([0,\infty) \times \mathbb{R}^n;\mathbb{R}^n)$. Then there exists a weak solution to* (6.0.1) *according to Theorem 6.0.1 on* $[0,T]$. *Furthermore, $T > 0$ can be chosen in such a way that $T = \infty$ or $\eta(T) \in \partial\mathcal{E}$ (see also Theorem 6.2.5).*

The main goal is to approximate the sought solution of the hyperbolic problem by solutions to suitably constructed parabolic problems. For us, this concerns particularly the *method of minimizing movements*. The key is in discretizing the second time derivative in (6.0.1) by a difference quotient w.r.t. the acceleration scale h. We will thus first solve what we will call the time-delayed problem:

$$\rho\frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h} = -DR_2(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)) - DE(\eta^{(h)}(t)) + f \circ \eta^{(h)}(t) \tag{6.0.3}$$

For any fixed h, (6.0.3) has the structure of a gradient flow, yet one with a nonlocality in time in form of the term $\partial_t\eta^{(h)}(t-h)$. Now the important observation is that on the interval $[0,h]$, $\partial_t\eta^{(h)}(t-h)$ is not part of the solution but actually given through the initial data. Thus, on this interval, the problem can be solved using parabolic methods. But then, once we know the solution on $[0,h]$, we can use this as data for the problem on $[h,2h]$ and iterate. To allow for an iteration process, we in particular need to know that the solution obtained from the previous step is admissible to play the role of data in the next step. In other words, we need to assure that that $E(\eta^{(h)}(h))$ is bounded and that $\partial_t\eta^{(h)}$ possesses the necessary integrability. This is guaranteed by proving a suitable energy inequality, *a key element of the proof.* Fundamentally, a gradient flow needs to be viewed in terms of energy and dissipation. In particular there is always an energy balance, which often only takes the form of an inequality. In our case, for the time delayed problem on $[0,h]$, the energy inequality will have the form

$$\begin{aligned} &E(\eta^{(h)}(h)) + \frac{\rho}{2h}\int_0^h \left\|\partial_t\eta^{(h)}(t)\right\|^2 dt + \int_0^h R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))dt \\ &\le E(\eta^{(h)}(0)) + \frac{\rho}{2h}\int_0^h \left\|\partial_t\eta^{(h)}(t-h)\right\|^2 dt + \int_0^h \left\langle f \circ \eta^{(h)}, \partial_t\eta^{(h)}\right\rangle dt \end{aligned}$$

Let us elaborate the terms in this inequality: On the right-hand side, we have the potential energy E of the initial data, as well as the averaged kinetic energy $\frac{\rho}{2}⨍_0^h \left\|\partial_t\eta^{(h)}\right\|^2 dt$ of the "previous step". On the left hand side, we have the potential energy at the end of the step, as well as the averaged kinetic energy of the current step.

So not only have we bounded the initial data for the next step in terms of the initial data of the previous step which allows for an iterative process, we also have an estimate suitable to employ a telescope argument. Indeed, by summing up the estimate, over l time intervals of length h, we will gain a uniform bound on the new endpoint $E(\eta^{(h)}(lh))$ and $\frac{\rho}{2}⨍_{(l-1)h}^{lh} \left\|\partial_t\eta^{(h)}\right\|^2 dt$ only in terms of the given initial data and forces. These uniform bounds for $\eta^{(h)}$ are independent of h, thus they allow us to deduce a-priori estimates and, in turn, to pass to the limit $h \to 0$ in order to obtain a solution to the hyperbolic problem.

Following this approach, we will show the existence of weak solutions for the time-delayed problem in detail in Section 6.1 before proving Theorem 6.0.2 in Section 6.2.

6.1 The time-delayed problem

For all of this subsection we will assume $h > 0$ to be fixed. In order to solve the time-delayed problem, we first need to give a precise definition of its weak formulation.

Definition 6.1.1 (Weak solutions to the time-delayed equation for solids)**.** *Let $w \in L^2([0,h] \times Q;\mathbb{R}^n)$. We call $\eta \in L^\infty([0,h] \times Q;\mathcal{E}) \cap W^{1,2}([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$ a weak solution to the time-delayed equation* (6.0.3) *if $\eta(0) = \eta_0$ and*

$$0 = \int_0^h \langle DE_h(\eta), \phi\rangle + \langle D_2R_h(\eta, \partial_t\eta), \phi\rangle - \langle f \circ \eta, \phi\rangle + \frac{\rho}{h}\langle \partial_t\eta - w, \phi\rangle\, dt. \tag{6.1.1}$$

for all $\phi \in C^\infty([0,h] \times Q;\mathbb{R}^n)$ with $\phi|_{[0,h]\times P} = 0$.

In this definition w will play the role of the given data $\partial_t\eta(t-h)$. In addition, as we assume $h>0$ to be a given constant throughout this subsection, so we will not highlight the h-dependence for any of the given quantities. Note that in Definition 6.1.1 we used the regularized forms of the energy and dissipation potentials that read as

$$E_h(\eta) = E(\eta) + h^{a_0}\left\|\nabla^{k_0}\eta\right\|^2 \quad R_h(\eta,b) := R(\eta,b) + h\left\|\nabla^{k_0}b\right\|^2, \tag{6.1.2}$$

where we choose k_0 large enough, such that $k_0 - \frac{n}{2} \geq 2 - \frac{n}{q}$ which implies that $W^{k_0,2}(Q;\mathbb{R}^n) \subset W^{2,q}(Q;\mathbb{R}^n)$ compactly. This actually has no impact on the existence of time delayed solutions. Instead it is a mollifying strategy which will allow us to test the Euler-Lagrange equation with $\partial_t\eta$ in order to obtain the previously mentioned energy inequality (See also Theorem 6.2.6). A similar term will also help us with some regularity issues in the fluid-structure interaction problem later in Theorem 7.3.3.

Remark 6.1.2 (Properties of the regularizing energy and dissipation)**.** *For all $h>0$, we find that E_h fulfills the properties given in Theorem 5.1.1, replacing $W^{2,q}(Q;\mathbb{R}^n)$ with $W^{k_0,2}(Q;\mathbb{R}^n)$ and R_h fulfills the properties given in Theorem 5.1.4 replacing $W^{1,2}(Q;\mathbb{R}^n)$ by $W^{k_0,2}(Q;\mathbb{R}^n)$ where we may replace $R2$ by*

$$c\left(\|\nabla\lambda\|^2 + h\left\|\nabla^{k_0}\lambda\right\|^2\right) \leq R_h(\eta,\lambda) \leq C\left(\|\nabla\lambda\|^2 + h\left\|\nabla^{k_0}\lambda\right\|^2\right).$$

Now the bulk of this subsection will be devoted to proving the following theorem:

Theorem 6.1.3 (Existence of time delayed solutions for solids)**.** *Let $\eta_0 \in \mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)\setminus\partial\mathcal{E}$, $w \in L^2([0,h]\times Q;\mathbb{R}^n)$ and $f \in C^0([0,h]\times Q;\mathbb{R}^n)$. Then there exists a weak solution to the time delayed equation* (6.0.3) *in the sense of Theorem 6.1.1 or there exists a solution on a shorter interval $[0,h_{\max}]$ such that $\eta(h_{\max}) \in \partial\mathcal{E}$.*[1]

Before we start, let us discuss how the time delayed problem can still be seen as a type of parabolic gradient flow. In particular, let us compare it to the classical parabolic gradient flow problem at its root, which reads

$$DE_h(\eta(t)) = -D_2R_h(\eta(t),\partial_t\eta(t)) + f\circ\eta(t).$$

This problem consists of three components: energy, dissipation and forces. Our goal is to identify each of the two additional terms in the time-delayed problem with one of those three in order to show that we are still solving a similar problem.

Let us start with the delayed time derivative $\frac{\rho}{h}w(t) = \frac{\rho}{h}\partial_t\eta(t-h)$. As we work in the interval $[0,h]$, this is just a given function, not depending on the $\eta|_{[0,h]}$. But then any such function plays the role of a force. In fact, in contrast to the actual forces we consider in the problem, it is a force given in reference configuration and thus even easier to handle.

The other term, $\frac{\rho}{h}\partial_t\eta(t)$ can be seen as stemming from a quadratic dissipation potential

$$\hat{R}(\eta,b) := \hat{R}(b) := \frac{\rho}{2h}\|b\|^2,$$

so that $D_2\hat{R}(\eta(t),\partial_t\eta(t)) = \frac{\rho}{h}\partial_t\eta(t)$.

By this reasoning, we claim that in general, if there is a method to solve the parabolic gradient flow problem, then there the same method can solve the corresponding time delayed problem.

Proof of Theorem 6.1.3. The proof essentially follows the same lines as was done in the last section. We start by a time-discretization; i.e. we fix some time-step size τ by which we discretize the interval $[0,h]$. Given $\eta_k^{(\tau)}$, we recursively solve the following minimization problem to obtain $\eta_{k+1}^{(\tau)}$

$$\text{Minimize} \quad E_h(\eta) + \tau R_h\left(\eta_k^{(\tau)}, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right) - \tau\left\langle f_k^{(\tau)}\circ\eta_k, \frac{\eta-\eta_k^{(\tau)}}{\tau}\right\rangle + \tau\frac{\rho}{2h}\left\|\frac{\eta-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\right\|^2 \tag{6.1.3}$$

$$\text{subject to} \quad \eta\in\mathcal{E}$$

[1] Note that a-posteriori (see Theorem 5.3.8) it will be shown that (in dependence of η_0) there is always a minimal time-length $h_{\min}$ for which it can be guaranteed that $\eta(t)\notin\partial\mathcal{E}$ for $t\in[0,h_{\min}]$.

where $w_k^{(\tau)} = ⨍_{k\tau}^{(k+1)\tau} w\,dt \in L^2(Q;\mathbb{R}^n)$ and $f_k^{(\tau)} = ⨍_{k\tau}^{(k+1)\tau} f\,dt \in L^2(Q;\mathbb{R}^n)$ are in-time averages.

Note that (6.1.3) is not quite in the form suggested by the previous discussion. Instead we deliberately wrote the last term as a quadratic difference, to give the problem a bit more structure. Note that when expanded, the last term is

$$\hat{R}\left(\frac{\eta - \eta_k^{(\tau)}}{\tau}\right) - \frac{\rho}{h}\left\langle w_k, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right\rangle + \frac{\rho}{2h}\left\|w_k^{(\tau)}\right\|^2$$

so these two approaches only differ by a constant, which has no effect on the minimization.

Now using the coercivity of E similar to, but easier as in the proof of Theorem 5.3.2, a (possibly non-unique) minimizer exists and a short calculation shows that it satisfies (assuming that $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$) the Euler-Lagrange equation

$$\begin{aligned}0 = \left\langle DE_h(\eta_{k+1}^{(\tau)}), \phi\right\rangle + \left\langle D_2R_h\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right), \phi\right\rangle - \left\langle f_k^{(\tau)} \circ \eta_k^{(\tau)}, \phi\right\rangle \\ + \frac{\rho}{2h}\left\langle \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}, \phi\right\rangle\end{aligned} \tag{6.1.4}$$

for all $\phi \in W^{2,q}(Q;\mathbb{R}^n)$ with $\phi|_P = 0$.

Next we follow in the steps of Theorem 5.3.3 (see Theorem 6.2.4 for a discussion of some interesting differences) and derive a simple initial energy estimate by comparing the value of the functional in (6.1.3) at the minimizer $\eta_{k+1}^{(\tau)}$ with its value at $\eta_k^{(\tau)}$:

$$\begin{aligned}E_h(\eta_{k+1}^{(\tau)}) + \tau R_h\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right) - \tau\left\langle f_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right\rangle \\ + \tau\frac{\rho}{2h}\left\|\frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\right\|^2 \le E_h(\eta_k^{(\tau)}) + \tau\frac{\rho}{2h}\left\|w_k^{(\tau)}\right\|^2.\end{aligned} \tag{6.1.5}$$

This estimate is can be summed so that, using the triangle and the weighted Young's inequality, we can derive for any N such that $\tau N \le h$

$$\begin{aligned}&E_h(\eta_N) + \sum_{k=0}^{N-1}\tau\left[R_h\left(\eta_k^{(\tau)}, \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right) + c\left\|\frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\right\|^2\right] \\ &\le E_h(\eta_0) + \tau C\sum_{k=0}^{N-1}\left[\left\|w_k^{(\tau)}\right\|^2 + \left\|f_k^{(\tau)}\right\|^2\right] \\ &\le E_h(\eta_0) + C\int_0^h \|w\|^2 + \|f\|^2\,dt\end{aligned}$$

for some $C, c > 0$ depending on h but independent of τ. Further, in the last step we used Jensen's inequality to show

$$\tau\sum_{k=0}^{N-1}\left\|w_k^{(\tau)}\right\|^2 = \tau\sum_{k=0}^{N-1}\left\|⨍_{k\tau}^{(k+1)\tau} w dt\right\|^2 \le \tau\sum_{k=0}^{N-1}⨍_{k\tau}^{(k+1)\tau}\|w\|^2\,dt = \int_0^{N\tau}\|w\|^2\,dt$$

and a similar estimate for f. In particular this also allows us to apply Theorem 5.2.4 to show that $\eta_k^{(\tau)}$ is always injective and the Euler-Lagrange equation is well defined.

If we now define the piecewise constant and piecewise affine approximations

$$\eta^{(\tau)}(t) = \eta_k^{(\tau)} \qquad \text{for } k\tau \le t < (k+1)\tau$$
$$\tilde\eta^{(\tau)}(t) = \left(\frac{t}{\tau} - k\right)\eta_{k+1}^{(\tau)} + \left(k+1-\frac{t}{\tau}\right)\eta_k^{(\tau)} \qquad \text{for } k\tau \le t < (k+1)\tau$$

where in particular

$$\partial_t\tilde\eta^{(\tau)}(t) = \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau} \qquad \text{for } k\tau < t < (k+1)\tau$$

our energy estimate turns into an uniform (in τ and t) bound on $E_h(\eta^{(\tau)}(t))$, as well as a uniform (in τ) bound on $\int_0^h R_h(\eta^{(\tau)}, \partial_t\tilde\eta^{(\tau)}) + c\left\|\partial_t\tilde\eta^{(\tau)}\right\|^2 dt$. Now using the properties of energy and dissipation from our assumptions, this gives an uniform $L^\infty([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ bound on $\eta^{(\tau)}$ and $\tilde\eta^{(\tau)}$ as well as a uniform $L^2([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ bound on $\partial_t\tilde\eta^{(\tau)}$.

Analogously to Theorem 5.3.9, we may extract a converging subsequence and a single limit $\eta \in W^{1,2}([0,T]; W^{k_0,2}(Q)) \cap C^0([0,T]; C^{1,\alpha}(Q))$. In particular we get that

$$\tilde\eta^{(\tau)} \rightharpoonup \eta \text{ in } W^{1,2}([0,T]; W^{k_0,2}(Q;\mathbb{R}^n))$$
$$\eta^{(\tau)} \rightharpoonup^* \eta \text{ in } L^\infty([0,T]; W^{k_0,2}(Q;\mathbb{R}^n))$$
$$\tilde\eta^{(\tau)} \to \eta \text{ in } L^\infty([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n))$$
$$\eta^{(\tau)} \to \eta \text{ in } L^\infty([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n))$$

for all $0 < \alpha^- < \alpha := 1 - \frac{n}{q}$.

This is already enough to pass to the limit in all of the terms in the Euler-Lagrange equation (6.1.4); note that due to the added regularizing terms we use the strong convergence and the linearity in the highest gradient to pass to the limit in the $DE(\eta^{(\tau)})$. □

6.1.1 Time-delayed energy inequality

In the proof of Theorem 6.1.3 we already gave an initial, somewhat crude energy estimate on the discrete level. Now that we have a solution of the time-delayed equation, we can however give the much stronger, "physical" energy inequality, which will turn out to be crucial in what follows.

Lemma 6.1.4 (Time-delayed energy inequality for the solid)**.** *Let the deformation $\eta \in L^\infty([0,h]\times Q;\mathcal{E}) \cap W^{1,2}([0,h]\times Q;\mathbb{R}^n)$ be a weak solution to the time-delayed equation Theorem 6.1.1. Then for all $t \in [0,h]$, we have*

$$E_h(\eta(t)) + \frac{\rho}{2h}\int_0^t \|\partial_t\eta\|^2 dt + \int_0^t 2R_h(\eta,\partial_t\eta)dt \le E_h(\eta_0) + \frac{\rho}{2h}\int_0^t \|w\|^2 dt + \int_0^t \langle f\circ\eta, \partial_t\eta\rangle\, dt.$$

Proof. We use $\chi_{[0,t]}\partial_t\eta$ as a test function in the weak equation.[2] From this we get

$$\begin{aligned} 0 &= \int_0^t \langle DE_h(\eta), \partial_t\eta\rangle + \langle D_2R_h(\eta,\partial_t\eta), \partial_t\eta\rangle - \langle f\circ\eta, \partial_t\eta\rangle + \frac{\rho}{h}\langle \partial_t\eta - w, \partial_t\eta\rangle\, dt \\ &= E_h(\eta(t)) - E_h(\eta(0)) + \int_0^t 2R_h(\eta,\partial_t\eta) - \langle f\circ\eta,\partial_t\eta\rangle + \frac{\rho}{h}\langle\partial_t\eta - w, \partial_t\eta\rangle\, dt \end{aligned}$$

where we in particular used that $\langle D_2R_h(\eta,\partial_t\eta), \partial_t\eta\rangle = 2R_h(\eta,\partial_t\eta)$ by the quadratic nature of R_h. Finally we use Young's inequality on the last term in the form of

$$\langle\partial_t\eta - w, \partial_t\eta\rangle = \|\partial_t\eta\|^2 - \langle w, \partial_t\eta\rangle \ge \|\partial_t\eta\|^2 - \frac{1}{2}\|\partial_t\eta\|^2 - \frac{1}{2}\|w\|^2 = \frac{1}{2}\|\partial_t\eta\|^2 - \frac{1}{2}\|w\|^2.$$

Reordering the terms then closes the proof. □

[2]Note that this is the point where we rely on R_h, since to test $DE(\eta)$, we need $\phi \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n))$, but bounding $R(\eta,\partial_t\eta)$ only gives us a $L^2([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ bound. See also Theorem 6.2.6.

6.2 Proof of Theorem 6.0.2

We will start the proof of the theorem by directly using its two key ingredients, the two results from the previous section. First we iteratively use the existence of time-delayed solutions on the short intervals $[0,h]$ to construct a time-delayed solution on the longer interval $[0,T]$.

6.2.1 Step 1: Iterated time-delayed solutions and energy estimates

For fixed h we start with given initial deformation $\eta_0 \in \mathcal{E}$ and we use the initial velocity as a constant right hand side $w_0(t) = \eta_*$ for $t \in [0,h]$. This allows us to find $\tilde{\eta}_1$ as a solution of the time-delayed problem. Then we continue the constructions iteratively; i.e. when given $\eta_l \in \mathcal{E}$ and $w_l \in L^2([0,h] \times Q; \mathbb{R}^n)$, we find a solution $\tilde{\eta}_{l+1} \in L^\infty([0,h] \times Q; \mathcal{E}) \cap W^{1,2}([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ to the time-delayed equation using Theorem 6.1.3. We then set $\eta_{l+1} = \tilde{\eta}_{l+1}(h)$ and $w_{l+1} = \tilde{\eta}_{l+1}$ as data for the next step for which they are admissible by Theorem 6.1.4.

From these ingredients we construct $\eta^{(h)} : [0,T] \times Q \to \mathbb{R}^n$ using

$$\eta^{(h)}(t,x) := \tilde{\eta}_{l+1}(t - hl) \text{ for } hl \le t \le h(l+1).$$

Thus, directly from the definition we see that $\eta^{(h)}$ fulfills

$$\begin{aligned} 0 = \int_0^T & \left\langle DE_h(\eta^{(h)}(t)), \phi \right\rangle + \left\langle D_2 R_h(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), \phi \right\rangle \\ & - \left\langle f \circ \eta^{(h)}(t), \phi \right\rangle + \frac{\rho}{h} \left\langle \partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h), \phi \right\rangle dt \end{aligned} \tag{6.2.1}$$

for all $\phi \in C^\infty([0,h] \times Q; \mathbb{R}^n)$ with $\phi|_{[0,h]\times P} = 0$. Furthermore, exploiting the energy inequality (Theorem 6.1.4) yields

$$\begin{aligned} & E_h(\eta^{(h)}((l+1)h)) + \frac{\rho}{2} \fint_{lh}^{(l+1)h} \left\| \partial_t \eta^{(h)} \right\|^2 dt + \int_{lh}^{(l+1)h} 2R_h(\eta^{(h)}, \partial_t \eta^{(h)}) dt \\ & \le E_h(\eta^{(h)}(lh)) + \frac{\rho}{2} \fint_{(l-1)h}^{lh} \left\| \partial_t \eta^{(h)} \right\|^2 dt + \int_{lh}^{(l+1)h} \left\langle f \circ \eta, \partial_t \eta^{(h)} \right\rangle dt. \end{aligned}$$

Taking $t \in [lh, (l+1)h]$, we find after summing the above over $1, ..., l$ and adding the energy inequality for $\tilde{\eta}^{l+1}$ from Theorem 6.1.4 the following crucial estimate:

$$\begin{aligned} (E) := E_h(\eta^{(h)}(t)) + \frac{\rho}{2} \fint_{t-h}^{t} \left\| \partial_t \eta^{(h)} \right\|^2 dt + \int_0^t 2R(\eta^{(h)}, \partial_t \eta^{(h)}) dt \\ \le E_h(\eta_0) + \frac{\rho}{2} \|\eta_*\|^2 + \int_0^t \left\langle f \circ \eta, \partial_t \eta^{(h)} \right\rangle dt \end{aligned} \tag{6.2.2}$$

for all $t \in [0,T]$.

Now, as before, we need to estimate the force term using Young's inequality. This gives

$$(E) \le E_h(\eta_0) - E_{min} + \frac{\rho}{2} \|\eta_*\|^2 + \frac{t}{2\delta} \|f\|_{L^\infty}^2 + \frac{\delta}{2} \int_0^t \left\| \partial_t \eta^{(h)} \right\|^2 dt;$$

here recall that E_{min} is defined in Assumption (S1). As all terms involving t on the right hand side have a fixed sign, we extend to $t = T$ and find

$$(E) \le C_0 + C_1 \frac{T}{\delta} + \frac{\delta}{2} \int_0^T \left\| \partial_t \eta^{(h)} \right\|^2 ds$$

for some constants C_0, C_1 resulting from the given data and independent of h. Dropping the positive terms involving E and R_h on the left-hand side, multiplying by h and adding up implies

$$\frac{\rho}{2}\int_0^T \left\|\partial_t \eta^{(h)}\right\|^2 ds = \sum_{l=0}^{N} \frac{\rho}{2}\int_{lh}^{(l+1)h} \left\|\partial_t \eta^{(h)}\right\|^2 ds \le hN\left(C_0 + C_1\frac{T}{\delta} + \frac{\delta}{2}\int_0^T \left\|\partial_t \eta^{(h)}\right\|^2 ds\right)$$

for $hN = T$.[3] Now choosing $\delta := \frac{\rho}{2T}$ allows us to absorb the integral on the right hand side to the left and we end up with an uniform estimate of the form

$$\frac{\rho}{4}\int_0^T \left\|\partial_t \eta^{(h)}\right\|^2 ds \le TC_0 + C_2' T^2,$$

which implies also that

$$(E) \le TC_0 + C_2' T^2$$

Note that in contrast to the parabolic setup from the last section, up to this point there was no need to apply Korn's inequality. In particular, as we used the inertial term to estimate the force term, we obtain a uniform bound on the energy without exploiting the dissipative terms; i.e. we already know that $\sup_{t\in[0,T]} E(\eta^h(t)) \le TC_0 + C_2' T^2$. Now, using this estimate, we may apply Theorem 5.2.8 without restrictions on the final time T, to find that

$$\sup_{t\in[h,T]}\left(\fint_{t-h}^{h} \left\|\partial_t \eta^{(h)}\right\|^2 ds + E(\eta^{(h)}(t)) + h^{a_0}\left\|\nabla^{k_0}\eta^{(h)}(t)\right\|^2\right) \le C \quad \text{and} \tag{6.2.3}$$

$$\int_0^T \left\|\partial_t \eta^{(h)}\right\|^2_{W^{1,2}(Q)} + h\left\|\partial_t \eta^{(h)}\right\|^2_{W^{k_0,2}(Q)} ds \le C$$

are uniformly bounded with constant $C = C(T)$ independent of h. Moreover, it allows to conclude that $\eta^{(h)}(t)$ is always injective by Theorem 5.2.4.

By the same arguments as used in the proof of Lemma 5.3.9, we can now choose a subsequence which converges to a limit function $\eta \in C_w([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n)) \cap C^0([0,T]; C^{1,\alpha}(Q;\mathbb{R}^n))$. In particular we obtain that

$$\begin{aligned}
\eta^{(h)} &\rightharpoonup \eta \text{ in } W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n))\\
\eta^{(h)} &\rightharpoonup^* \eta \text{ in } L^\infty([0,T]; W^{2,q}(Q;\mathbb{R}^n))\\
\eta^{(h)} &\to \eta \text{ in } C^0([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n))
\end{aligned}$$

for all $0 < \alpha^- < \alpha := 1 - \frac{n}{q}$. Moreover, the weak lower semi-continuity implies that

$$\sup_{t\in[0,T]}\left(\|\partial_t \eta(t)\|^2 + E(\eta(t))\right) \le C \text{ and } \int_0^T \|\partial_t \eta\|^2_{W^{1,2}(Q)} ds \le C \tag{6.2.4}$$

with the same constant as before.

6.2.2 Step 2: Improving convergence

Our final goal is to prove convergence of the weak equation (6.2.1), which is satisfied by the time-delayed approximation $\eta^{(h)}$, to the weak hyperbolic inertial equation (Theorem 6.0.1), as this then implies that the limit η is a weak solution. The crucial term here is $DE(\eta^{(h)})$ which requires strong convergence of $\eta^{(h)}$ in $W^{2,q}(Q;\mathbb{R}^n)$. For this we want to use the Minty-type property of the energy, which requires convergence of the other terms in the equation. We achieve this convergence by the Aubin-Lions lemma, for which in turn we need another estimate on the discrete difference quotient.

[3] There is no need to assume that T is a multiple of h, but we will do so for the sake of simplification.

Lemma 6.2.1 (Length h bounds (solid))**.** *Fix $T > 0$. Then there exists a constant C depending only on the initial data and T, such that for $k_0 > 2 + \frac{(q-2)n}{2q}$ the following holds:*

$$\int_0^T \left\| \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h} \right\|_{W^{-k_0,2}(Q)}^2 dt \leq C$$

where $\partial_t \eta$ is extended by η_* for negative times.

Proof. Pick $\phi \in C_0^\infty(Q;\mathbb{R}^n)$. Then, using the time-delayed equation, we have

$$\begin{aligned}
&\rho_s \left| \left\langle \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h}, \phi \right\rangle_Q \right| \leq \left| \left\langle DE(\eta^{(h)}(t)), \phi \right\rangle \right| + h^{a_0} \left| \left\langle \nabla^{k_0} \eta^{(h)}, \nabla^{k_0} \phi \right\rangle \right| \\
&+ \left| \left\langle D_2 R(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), \phi \right\rangle \right| + h \left| \left\langle \nabla^{k_0} \partial_t \eta^{(h)}, \nabla^{k_0} \phi \right\rangle \right| + \left| \langle f(t), \phi \rangle_Q \right| \\
&\leq \left(\left\| DE(\eta^{(h)}(t)) \right\|_{W^{-2,q}(Q)} + h^{a_0} \left\| \nabla^{k_0} \eta^{(h)}(t) \right\|_Q + \|f\|_\infty \right) \|\phi\|_{W^{k_0,2}(Q)} \\
&+ \left(\left\| D_2 R(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)) \right\|_{W^{-1,2}(Q)} + h \left\| \nabla^{k_0} \partial_t \eta^{(h)}(t) \right\|_Q \right) \|\phi\|_{W^{k_0,2}(Q)}
\end{aligned}$$

Now for the first set of terms, we note that they are uniformly bounded by Theorem 5.1.1, S5 and (6.2.3). For the second set, we note that the quadratic growth of $R(\eta,.)$ in $W^{1,2}(Q;\mathbb{R}^n)$ implies a linear growth of $D_2 R$ thus equally (6.2.3) implies boundedness when integrated in time. □

Note that in the previous lemma the h, by which time is shifted, is the same h as the sequence index. Thus even though $\partial_t \eta^{(h)}$ is already continuous, we can only ever compare at fixed distances in the form of multiples of h. This is an unavoidable consequence of the way the estimate is obtained, using the equation. In particular, we cannot use directly the Aubin-Lions lemma to conclude that $\partial_t \eta^{(h)}$ converges strongly in $C([0,T], L^2(Q;\mathbb{R}^n))$. Instead we will prove the strong convergence for averages $\partial_t \eta^{(h)}$ over time-intervals of length h, which turn out to be much more natural in this context and are in fact the same averages that also occur in the energy inequality.

Lemma 6.2.2 (Aubin-Lions (solid))**.** *Let $b^{(h)}(t) := \fint_t^{t+h} \partial_t \eta^{(h)} ds$. We have (for a subsequence $h \to 0$)*

$$b^{(h)} \to \partial_t \eta \text{ in } C^0([0,T]; L^2(Q;\mathbb{R}^n)).$$

Proof. By the fundamental theorem of calculus we have

$$\partial_t b^{(h)} = \frac{\partial_t \eta^{(h)}(t+h) - \partial_t \eta^{(h)}(t)}{h}.$$

Now $b^{(h)}$ is uniformly bounded in $L^\infty([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ by the energy estimate and $\partial_t b^{(h)}$ is uniformly bounded in $L^2([0,T]; W_0^{-k_0,2}(Q;\mathbb{R}^n))$ by the previous lemma. Thus we can apply the classical Aubin-Lions lemma [174], yielding the existence of a converging subsequence in $C^0([0,T]; L^2(Q;\mathbb{R}^n))$. It remains to associate the limit function with $\partial_t \eta$. For that take $h_0 > 0$ and $\phi \in C_0^\infty([h_0, T-h_0] \times Q)$, we find for all $h \in (0,h_0)$ by the weak convergence of $\partial_t \eta^{(h)} \rightharpoonup \partial_t \eta$ (and the Lebesgue point theorem) that

$$\begin{aligned}
\int_0^T \left\langle b^{(h)}, \phi \right\rangle_Q dt &= \fint_0^h \int_0^T \left\langle \partial_t \eta^{(h)}(t+s), \phi(t) \right\rangle_Q dt\, ds = \fint_0^h \int_0^T \left\langle \partial_t \eta^{(h)}(\tau), \phi(\tau - s) \right\rangle_Q d\tau\, ds \\
&\to \int_0^T \langle \partial_t \eta(\tau), \phi(\tau) \rangle_Q d\tau.
\end{aligned}$$
□

Finally we will use a Minty-type argument to improve convergence.

Lemma 6.2.3 (Minty-Trick)**.** *$\eta^{(h)}(t) \to \eta(t)$ strongly in $W^{2,q}(Q;\mathbb{R}^n)$ for a.a $t \in [0,T]$.*

Proof. As in the last section we will rely on Theorem 5.1.1, S6. Let $h_0 > 0$ and $h \in (0, h_0)$. Further take $\psi \in C_0^\infty((h_0, T - h_0) \times Q; \mathbb{R}^+)$ with $dist(supp(\psi), \partial Q) > h_0$. Accordingly we define for $\delta_h = h^{a_1} < h_0$ the approximation $\eta_{\delta_h} := (\eta\chi_{[0,T]\times Q}) * \gamma_{\delta_h}$, where γ_δ is the standard convolution kernel in time-space. This implies that $(\eta^{(h)} - \eta_{\delta_h})\psi$ is a valid test function for (6.2.1). Moreover, we find that by the standard convolution estimates that

$$\|\eta_{\delta_h}\psi\|_{W^{k_0,2}(Q)} \le ch^{a_1(2-\frac{q}{n}-k_0+\frac{2}{n})}\|\eta\|_{W^{2,q}(Q)} \quad \text{and} \tag{6.2.5}$$
$$\|\partial_t\eta_{\delta_h}\psi\|_{W^{k_0,2}(Q)} \le ch^{(1-k_0)a_1}\|\partial_t\eta\|_{W^{1,2}(Q)}.$$

Also we have $\eta_{\delta_h} \to \eta$ strongly as $h \to 0$ in all norms in which η is bounded.

Now we calculate

$$\begin{aligned}
0 &\le \limsup_{h\to 0}\int_0^T \left\langle DE(\eta^{(h)}(t)) - DE(\eta(t)), (\eta^{(h)} - \eta)\psi\right\rangle dt \\
&= \limsup_{h\to 0}\int_0^T \left\langle DE(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt + \limsup_{h\to 0}\Big\langle \underbrace{DE(\eta^{(h)}(t))}_{\text{bd. in } W^{-2,q}(Q)}, \underbrace{(\eta - \eta_{\delta_h})\psi}_{\to 0 \text{ in } W^{2,q}(Q)}\Big\rangle dt \\
&= \limsup_{h\to 0}\int_0^T \left\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle - 2h^{\frac{a_0}{2}}\left\langle h^{\frac{a_0}{2}}\nabla^{k_0}(\eta^{(h)}(t)), \nabla^{k_0}(\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt \\
&\le \limsup_{h\to 0}\int_0^T \left\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt + 4h^{a_0}\int_0^T \left\|\nabla^{k_0}(\eta^{(h)}(t))\right\| \|\eta_{\delta_h}\psi\|_{W^{k_0,2}} dt \\
&\le \limsup_{h\to 0}\int_0^T \left\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle + ch^{\frac{a_0}{2}-(2-\frac{q}{n}-k_0+\frac{2}{n})a_1} dt \\
&= \limsup_{h\to 0}\int_0^T \left\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt
\end{aligned}$$

by (6.2.5) and by choosing a_1 small enough. The final term can be rewritten using (6.2.1) as

$$\begin{aligned}
\int_0^T \left\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt &= -\left\langle D_2R_h(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle \\
+ \left\langle f\circ\eta^{(h)}(t), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle &+ \frac{\rho_s}{h}\left\langle \partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle dt.
\end{aligned}$$

On the right-hand side we may pass to the limit $h \to 0$. In particular observe that

$$\begin{aligned}
\left\langle D_2R_h(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle &= \left\langle D_2R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\right\rangle \\
+ 2h\left\langle \nabla^{k_0}\partial_t\eta^{(h)}, \nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\right\rangle &\to \langle D_2R(\eta(t), \partial_t\eta(t)), (\eta - \eta_{\delta_h})\psi\rangle
\end{aligned}$$

by the strong convergence of $\eta^{(h)}$ in $W^{1,2}(Q;\mathbb{R}^n)$, the weak convergence of $\partial_t\eta^{(h)}$ in $W^{1,2}(Q;\mathbb{R}^n)$ and since

$$h\left|\left\langle \nabla^{k_0}\partial_t\eta^{(h)}, \nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\right\rangle\right| \le h^{\frac{1}{2}-\frac{a_0}{2}}\left\|\sqrt{h}\nabla^{k_0}\partial_t\eta^{(h)}\right\|\left\|h^{\frac{a_0}{2}}\nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\right\|$$

which converges to zero a.e. using the energy estimates and (6.2.5) by choosing $a_0 < 1$ and $a_1 < 1$ accordingly. The term including the right-hand side converges, since all terms involve converge strongly. For the last term, we use the discrete integration by parts in time (i.e. shift the term involving $t - h$) to get

$$\begin{aligned}
&\int_0^T \frac{\rho_s}{h}\left\langle \partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), (\eta^{(h)}(t) - \eta_{\delta_h}(t))\psi(t)\right\rangle dt \\
&= -\rho_s\int_0^T \left\langle \partial_t\eta^{(h)}(t), \left(\frac{\eta^{(h)}(t+h) - \eta^{(h)}(t)}{h} - \frac{\eta_{\delta_h}(t+h) - \eta_{\delta_h}(t)}{h}\right)\psi(t+h)\right\rangle dt \\
&- \rho_s\int_0^T \left\langle \partial_t\eta^{(h)}(t), \left(\eta^{(h)}(t) - \eta_{\delta_h}(t)\right)\frac{\psi(t+h) - \psi(t)}{h}\right\rangle dt.
\end{aligned}$$

Now note that the first difference quotient is equal to $w^{(h)}$ as it was defined in Theorem 6.2.2 and thus converges strongly to $\partial_t \eta$ in $L^2([0,T] \times Q; \mathbb{R}^n)$, while the other difference quotients only involve constant functions and their mollifications and thus also converge in the same space. As a result, all the right hand sides converge strongly to 0 in $L^2([0,T] \times Q; \mathbb{R}^n)$ and the left hand sides are bounded. Thus the total limit is 0 and via Theorem 5.1.1, S6, we have $\eta^{(h)}(t) \to \eta(t)$ in $W^{2,q}(Q; \mathbb{R}^n)$ for almost all $t \in [0,T]$. □

6.2.3 Step 3: Limit equation

With all the necessary ingredients at hand, we can finally consider the weak equation (6.2.1) for arbitrary test functions. For the first three terms we have, as before

$$\int_0^T \left\langle DE_h(\eta^{(h)}(t)), \phi \right\rangle + \left\langle D_2 R_h(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), \phi \right\rangle + \left\langle f \circ \eta^{(h)}(t), \phi \right\rangle dt$$
$$\to \int_0^T \langle DE(\eta(t)), \phi \rangle + \langle D_2 R(\eta(t), \partial_t \eta(t)), \phi \rangle + \langle f \circ \eta(t), \phi \rangle \, dt,$$

where the regularizing terms vanish by the same estimates as the ones used in the proof of Lemma 6.2.3.

This leaves us with the last term, where we shift the discrete derivative to the test function again and get

$$\int_0^T \frac{\rho}{h} \left\langle \partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h), \phi \right\rangle dt = -\rho \int_0^T \left\langle \partial_t \eta^{(h)}(t), \frac{\phi(t+h) - \phi(t)}{h} \right\rangle dt$$
$$\to -\rho \int_0^T \langle \partial_t \eta(t), \partial_t \phi \rangle \, dt.$$

From this, we get solutions on the interval $[0,T]$.

6.2.4 Step 4: Continuation until collision

Using the short term existence, we can now employ a continuation argument: Assume that $\eta : [0, T_{\max}) \to \mathcal{E}$ is a solution on a maximal interval. Then either $T_{\max} = \infty$ or we can use the energy inequality to show existence of a unique limit $\eta(T_{\max})$ similar to as we did at the end of Theorem 5.1.6. Then $\eta(T_{\max}) \notin \partial \mathcal{E}$, would allow us to reapply the short time existence, which would be a contradiction. This finishes the proof of Theorem 6.0.2.

Let us close this section with some remarks on the preceeding proofs.

Remark 6.2.4 (On the need for dissipation)**.** *It is noteworthy, that the a-priori estimates* (6.2.4) *are valid even in case of a purely elastic solid, which means in case $R \equiv 0$. In that case, however, no strategy is known to deal with the non-linearity in $DE(\eta)$ without resorting to a relaxed concept of the solutions such as measure valued solutions. Even for the hyperbolic p-Laplacian $\partial_t^2 \eta - \operatorname{div}(|\nabla \eta|^{p-2} \nabla \eta) = 0$ the existence of weak solutions is a long standing open problem (see e.g. [5] for a discussion). Only in the case $p = 2$, where the elastic energy is quadratic in highest order (and its derivative thus linear) existence of solutions is known.*

Remark 6.2.5 (On collisions and continuation afterwards)**.** *Let us note that understanding self-collisions of a elastic body in detail is a long-standing open problem in the mathematical continuum mechanics of solids [9].*

For the evolution of the solid including inertia we face a similar difficulty as in the previous section when approaching collisions. While the limit object η can be constructed by the above methodology, such that it exists for all times independently of any collision, to get to a limit equation, we need to have an appropriate Euler-Lagrange equation for all $\eta_k^{(\tau)}$. This we only have if $\eta_k^{(\tau)} \notin \partial \mathcal{E}$, i.e. if there is no self-touching of the solid.

We wish to mention that some recent progress for the stationary and quasi-stationary analogue of the above problem has been made by [96] and [118] respectively. The approach is different. Instead of a variational inequality, the authors consider the associated Lagrange-multiplyier and its physical significance.

Remark 6.2.6 (On the proof of the energy inequality)**.** *In the proof of the energy inequality Theorem 6.1.4 we used a regularization term in the dissipation to simplify the proof. Since we will need that term later on* *in the fluid-structure interaction, this only seemed natural, but it should be noted that strictly speaking, it was not necessary. The same result is still true, if we only ever use R. To show this directly, one can use some techniques from the theory of minimizing movements, specifically the so called Moreau-Yosida approximation.*

Chapter 7

Bulk elastic solids interacting with Navier-Stokes fluids

7.1 Introduction

We will now combine the methods developed in the last two sections to show existence of weak solutions (in the sense of Definition 7.1.1) for a general fluid-structure interaction problem. In contrast to previous works (see [90, 145, 146, 148, 150, 128, 33] as well as the discussion in the introduction) we work in arbitrary dimension and consider a bulk solid that can undergo large elastic deformations. But most importantly, we consider the full nonlinear equation, both for the fluid in form of the incompressible Navier-Stokes equation with its transport-term as well as full nonlinear elasticity of the solid.

7.1.1 Setup

We consider the following set-up for the fluid-structure interaction problem: The fluid together with the elastic structure are both confined to a container $\Omega \subset \mathbb{R}^n$ that is fixed in time. The deformation of the solid is at any instant of time t described via the deformation function $\eta(t) := \eta(t,.) : Q \to \Omega$. Here Q is a given reference configuration of the solid. We assume that both $Q, \Omega \subset \mathbb{R}^n$ are Lipschitz-domains. Here, $n \geq 2$ is the dimension of the problem with $n = 2$ corresponding to the planar case and $n = 3$ to the bulk case. The fluid variables are

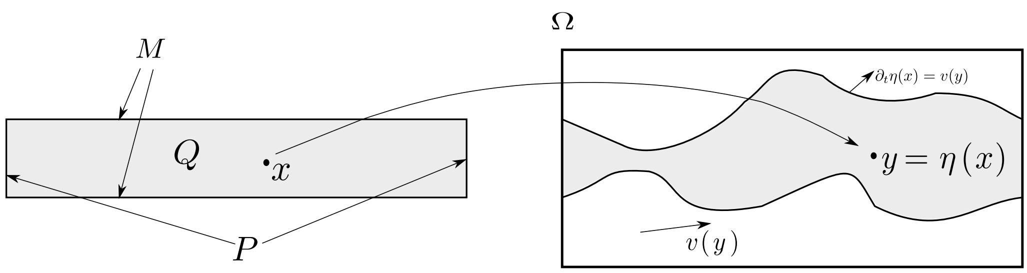


Figure 7.1: A scheme of the geometry of the fluid-structure interaction. The reference configuration is at the left while at the right we depict the situation in a given time instant t (the actual configuration).

defined in the time-dependent domain $\Omega(t) := \Omega \setminus \eta(t, Q)$. The flow of the fluid is determined by its velocity $v(t) : \Omega(t) \to \mathbb{R}^n$ and its pressure $p(t) : \Omega(t) \to \mathbb{R}$. Thus, the solid is described in *Lagrangian* and the fluid in *Eulerian* coordinates. Observe, that a similar configuration has already been studied for linear elasticity in [54].

For setting up the evolution equation, we will need the basic physical balances to be fulfilled. As we are not modeling any thermal effects, this reduces to the balance of momentum for both the fluid and the structure together with suitable conditions on their mutual boundary, as well as conservation of mass. In the interior these balance equations read in strong formulation as

$$\rho_s \partial_t^2 \eta + \mathrm{div}\sigma = \rho_s f \circ \eta \qquad \text{in } Q, \tag{7.1.1}$$

$$\rho_f(\partial_t v + [\nabla v]v) = \nu \Delta v - \nabla p + \rho_f f \qquad \text{on } \Omega(t), \tag{7.1.2}$$

$$\mathrm{div} v = 0 \qquad \text{on } \Omega(t). \tag{7.1.3}$$

Here, σ is the *first Piola–Kirchhoff stress tensor* of the solid, ν is the *viscosity constant* of the fluid, ρ_s and ρ_f are the *densities* of the solid and fluid respectively and f is the actual applied force in the current (Eulerian) configuration. Thus, the fluid is assumed to be Newtonian with the *Navier-Stokes equation* modeling its behavior. For the solid, we will restrict ourselves to materials for which the first Piola–Kirchhoff stress tensor σ can be derived from underlying *energy and dissipation potentials*; i.e.

$$\mathrm{div}\sigma := DE(\eta) + D_2 R(\eta, \partial_t \eta)$$

with E being the energy functional describing the elastic properties while R is the dissipation functional used to model the viscosity of the solid. Here D denotes the Fréchet derivative and D_2 the Fréchet derivative with respect to the second argument. Such materials are often called *generalized standard materials* [95, 155, 119]. For the analysis performed in this paper, quite general forms of E and R can be admitted (see Section 5.1 below).

From an analytical point of view, the potentials (1.2.8)-(1.2.9) already carry the *full difficulty* that we will need to cope with. Namely, any deformation η of finite energy will necessarily be a local diffeomorphism; in fact we will even strengthen this condition in the presented analysis and construct weak solutions to (7.1.1)-(7.1.3) for which η is *globally injective*. This *geometrical restriction* is necessary not only from the point of view of physics of solids per se but also essential to properly set-up the fluid-structure interaction problem. Moreover, $E(\eta)$ from (1.2.9) is neither convex nor quasi-convex. And for physical reasons explained below the dissipation potential R has to depend on the state η and cannot depend just on $\partial_t \eta$. We discuss the modeling issues in Section 5.1 while we explain the mathematical difficulties in Subsection 7.1.2 below.

Additionally, we impose coupling conditions between η and v on their common interface; namely, we will assume the *continuity of deformation* (i.e. no-slip conditions adapted to the moving domain) as well as *traction on the boundary between the fluid and the solid*. We denote by M the portion of the boundary of Q that is mapped to the contact interface between the fluid and the solid. While Q is only assumed to be a Lipschitz-domain, we assume that the pieces of its boundary that belong to M are additionally C^2. The boundary conditions read

$$v(t, \eta(x)) = \partial_t \eta(t, x) \qquad \text{in } [0, T] \times M, \tag{7.1.4}$$

$$\sigma(t, x) n(x) = \big(\nu \varepsilon v(t, \eta(t, x)) + p(t, \eta(t, x)) I\big) \hat{n}(t, \eta(t, x)) \qquad \text{in } [0, T] \times M, \tag{7.1.5}$$

where $n(x)$ is the unit normal to M while $\hat{n}(t, \eta(t, x)) := cof(\nabla \eta(t, x)) n(x)$ is the normal transformed to the actual configuration and $\varepsilon v := \nabla v + (\nabla v)^T$ is the symmetrized gradient. Additionally, there are second order Neumann-type zero boundary conditions for the deformation η arising from the second order gradient in its energy.[1]

Finally, we will prescribe Dirichlet boundary conditions on $P := \partial Q \setminus M$, i.e.

$$\eta(x, t) = \gamma(x) \qquad \text{in } [0, T] \times P \tag{7.1.6}$$

for some fixed boundary displacement $\gamma : P \to \Omega$. Together with the injectivity of deformations, we will encode this condition in the set of admissible deformations $\mathcal{E}$ (See Theorem 5.1.2 for the precise definition).

[1] Specifically, these naturally occur while minimizing the elastic energy and not prescribing boundary values for $\nabla \eta$. It can also be seen as a kind of integrability condition for σ. I.e. for σ to be defined as a measure, we need that $\langle DE(\eta), \phi_\delta \rangle \to 0$ (for $\delta \to 0$) , where ϕ_δ is a regularized version of $\xi \delta (1 - dist(., \partial Q)/\delta)^+$ with $\xi \in C_0^\infty(M)$ extended constantly along the normal direction. For our example energy (1.2.9) this simply reduces to $\frac{\partial^2 \eta}{\partial n^2} = 0$ on M.

We close the system by prescribing initial conditions for $v, \eta, \partial_t\eta$:

$$\begin{aligned} \eta(0,x) &= \eta_0(x) \text{ for } x \in Q \\ \partial_t\eta(0,x) &= \eta_*(x) \text{ for } x \in Q \\ \Omega(0) &= \Omega \setminus \eta_0(Q) \\ v(0,y) &= v_0(y) \text{ for all } y \in \Omega(0). \end{aligned} \tag{7.1.7}$$

7.1.2 Main result

The final objective of this paper is to prove existence of weak solutions to the system (7.1.1)-(7.1.3) subject to the coupling conditions (7.1.4)-(7.1.5) and the remaining boundary and initial conditions detailed in the previous subsection.

As is customary in fluid-structure interaction problems (see e.g. [85, 128]), the weak formulation is designed in such a way that the coupling conditions are realized by choosing *well-fitted test functions*. Indeed, we have the following definition:

Definition 7.1.1. *Let $f \in C^0([0,\infty) \times \Omega)$, $v_0 \in L^2(\Omega)$, $\eta_* \in L^2(Q)$ and $\eta_0 \in \mathcal{E}$ with $\mathcal{E}$ be given as defined in* (5.1.2)*, such that $v_0 \circ \eta_0 = \eta_*$. We call*[2] *$\eta : [0,T] \times Q \to \Omega$, $v : [0,T] \times \Omega(t) \to \mathbb{R}^n$ and $p : [0,T] \times \Omega(t) \to \mathbb{R}$, where $\Omega(t) := \Omega \setminus \eta(t,Q)$, a weak solution to the fluid-structure interaction problem* (7.1.1)–(7.1.2) *and* (7.1.4)–(7.1.7)*, if the following holds:*

The deformation satisfies $\eta \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n))$, $\eta(0) = \eta_0$ such that $\partial_t\eta \in C_w([0,T]; L^2(Q;\mathbb{R}^n))$. The velocity satisfies $v \in L^2([0,T]; W^{1,2}_{\text{div}}(\Omega(\cdot);\mathbb{R}^n))$ and the pressure satisfies[3] *$p \in D'([0,T] \times \Omega)$ with $supp(p) \subset [0,T] \times \overline{\Omega(t)}$.*

For all

$$(\phi,\xi) \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n)) \times C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^n))$$

satisfying $\xi(T) = 0$, $\phi(t) = \xi(t) \circ \eta(t)$ on Q, $\phi(t) = 0$ on P for all $t \in [0,T]$, we require that

$$\begin{aligned} &\int_0^T -\rho_s \langle \partial_t\eta, \partial_t\phi \rangle_Q - \rho_s \langle v, \partial_t\xi - v \cdot \nabla\xi \rangle_{\Omega(t)} + \langle DE(\eta), \phi \rangle + \langle D_2R(\eta, \partial_t\eta), \phi \rangle + \nu \langle \varepsilon v, \varepsilon\xi \rangle_{\Omega(t)} \, dt \\ &= \int_0^T \langle p, \text{div}\xi \rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \phi \rangle_Q + \rho_f \langle f, \xi \rangle_{\Omega(t)} \, dt - \rho_s \langle \eta_*, \phi(0) \rangle_Q - \rho_f \langle v_0, \xi(0) \rangle_{\Omega(0)} \end{aligned}$$

and that $\partial_t\eta(t) = v(t) \circ \eta(t)$ on M, $\eta(t) \in \mathcal{E}$ and $v(t)|_{\partial\Omega} = 0$ for almost all $t \in [0,T]$.

We can then formulate our main theorem as follows:

Theorem 7.1.2 (Existence of weak solutions)**.** *Assume that E satisfies Assumption 5.1.1 and R satisfies Assumption 5.1.4 given in Section 5.1. Then for any $\eta_0 \in \text{int}(\mathcal{E}) = \mathcal{E} \setminus \partial\mathcal{E}$ (see Theorem 5.1.2) with $E(\eta_0) < \infty$, any $\eta_* \in L^2(Q;\mathbb{R}^n)$, $v_0 \in L^2(\Omega(0);\mathbb{R}^n)$ and any right hand side $f \in C^0([0,\infty) \times \Omega;\mathbb{R}^n)$ there exists a $T > 0$ such that a weak solution to* (7.1.1)-(7.1.7) *according to Theorem 7.1.1 exists on $[0,T)$. Here either $T = \infty$ or the time T is the time of the first contact of the free boundary of the solid body either with itself or $\partial\Omega$ (i.e. $\eta(T) \in \partial\mathcal{E}$).*

Moreover, the solution satisfies the energy inequality (7.1.8)*; for additional regularity of the pressure see* (7.3.10).

[2]We use standard notation for Bochner spaces over Lebesgue spaces and Sobolev spaces with time changing domains. By the subscript div we mean the respective solenoidal subspace: $W^{1,2}_{\text{div}}(\Omega(t);\mathbb{R}^n) = \{v \in W^{1,2}(\Omega(t);\mathbb{R}^n) | \text{div}v = 0\}$.

[3]From the given weak formulation one can deduce some more regularity of the pressure. However as is known from the non-variable theory, regularity of the pressure in time can be obtain merely in a negative Sobolev space. See the estimates in Section 7.3, Step 3b, which show that the pressure is in the respective natural class.

Let us remark that the assumptions on the energy and dissipation functional are in particular satisfied by the model case energies (1.2.8)-(1.2.9) (see Section 5.4).

The coupled system possesses a natural energy inequality which reads

$$\begin{aligned} &E(\eta(t)) + \rho_s \int_Q \frac{|\partial_t\eta(t)|^2}{2} dx + \rho_f \int_{\Omega(t)} \frac{|v(t)|^2}{2} dy \\ &\quad + \int_0^t 2R(\eta(s), \partial_t\eta(s)) + \nu \int_{\Omega(s)} |\varepsilon v(s)|^2 \, dy \, ds \\ &\le E(\eta_0) + \rho_s \int_Q \frac{|\eta_*|^2}{2} dx + \rho_f \int_{\Omega(0)} \frac{|v_0|^2}{2} dy \\ &\quad + \int_0^t \rho_s \int_Q f(s) \circ \eta(s) \cdot \partial_t\eta(s) \, dx + \rho_f \int_{\Omega(s)} f(s) \cdot v(s) \, dy \, ds. \end{aligned} \tag{7.1.8}$$

As is usual in evolution-equations, this inequality holds as an equality for sufficiently regular solutions, i.e. if $(\partial_t\eta, v)$ can be used as a pair of test functions.

Before embarking on the technical discussion let us highlight some aspects regarding the convective term in the Navier-Stokes equation. Indeed, the very presence of this term necessitates the use of techniques beyond those presented in the previous two sections. At this point, it is illustrative to recall some of the arguments behind the derivation of the Navier-Stokes equation. The natural way to deal with inertia in a moving fluid is to transport it along the flow of the fluid; usually by employing the well known concept of a *flow-map*.

A flow-map is the the fluid counterpart of the deformation of the solid. Indeed, let, as before, $\Omega(t)$ denote the fluid domain at a given time t. Now for a fixed t_0, a flow map is a family $\Phi_s : \Omega(t_0) \to \Omega$ for $s \in [0, T - t_0]$, which we say is generated by v if $\Phi_0(y) = y$ and $\partial_s\Phi_s(y) = v(t_0 + s, \Phi_s(y))$.

If it exists and has some regularity, it has to be a volume preserving diffeomorphism, which allows us to validly compare $v(t_0 + h, \Phi_h(y))$ and $v(t_0, y)$ for any $y \in \Omega(t_0)$. From this we are able to obtain the material derivative via the chain rule:

$$\lim_{h\searrow 0} \frac{v(t_0, \Phi_h(y)) - v(t_0, y)}{h} = \partial_t v(t_0, y) + \nabla v(t_0, y) \cdot v(t_0, y).$$

This kind of difference quotient will be the Eulerian counterpart to the ordinary difference quotient for $\partial_t\eta$ in the previous section.

Having explained the idea, we immediately have to note that the existence of such a flow map is not guaranteed, even in the case of the Navier-Stokes equation without additional interaction. We will thus additionally use the fact that we no longer need such a flow map in the limit $h \to 0$. This allows us to add a h-dependent regularisation term for the fluid flow, similar to those already introduced for the solid.

Additionally we note that in turn to obtain the proper weak equation, we already need to construct a discretized version of Φ along with our minimization procedure. As an added benefit of this, as we send $\tau \to 0$, we are able to prove convergence of this discretization, directly giving us a flow map for any $h > 0$, without having to resort to additional ODE-arguments.

7.2 An intermediate, time delayed problem

As in the previous section, let us start with deriving a time-delayed equation, similar to Section 6.1.

Definition 7.2.1 (Time-delayed solution). *Let $f \in C^0([0,h] \times \Omega; \mathbb{R}^n)$, $w \in L^2([0,h] \times \Omega; \mathbb{R}^n)$ and $\Omega_0 = \Omega \setminus \eta_0(Q)$. We call the pair $\eta : [0,h] \times Q \to \Omega, u : [0,h] \times \Omega \to \mathbb{R}^n$ a weak solution to the time-delayed inertial equation if it satisfies*

$$\begin{aligned} 0 = \langle DE_h(\eta), \phi\rangle_Q + \langle D_2R_h(\eta, \partial_t\eta), \phi\rangle_Q + \left\langle \rho_s \frac{\partial_t\eta - w \circ \eta_0^{-1}}{h}, \phi \right\rangle_Q - \rho_s \langle f \circ \eta, \phi\rangle_Q \\ + \nu \langle \varepsilon u, \varepsilon\xi\rangle_{\Omega(t)} + h \left\langle \nabla^{k_0} u, \nabla^{k_0}\xi \right\rangle_{\Omega(t)} + \left\langle \rho_f \frac{u \circ \Phi - w}{h}, \xi \circ \Phi \right\rangle_{\Omega_0} - \rho_f \langle f, \xi\rangle_{\Omega(t)} \end{aligned} \tag{7.2.1}$$

for almost all $t \in [0,h]$ *and all* $\phi \in C^0([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$, $\xi \in C^0([0,h];W^{k_0,2}(\Omega;\mathbb{R}^n))$ *satisfying* $\mathrm{div}\xi|_{\Omega(t)} = 0$, $\xi|_{\partial\Omega} = 0$, $\phi|P = 0$ *and the coupling conditions*

$$\xi \circ \eta = \phi \text{ and } u \circ \eta = \partial_t \eta \text{ in } Q.$$

Here we define $\Omega(t) = \Omega \setminus \eta(t,Q)$ *and* $\Phi : [0,h] \times \Omega_0 \to \Omega$ *solves* $\partial_t \Phi = u \circ \Phi$ *and* $\Phi_0(y) = y$.

The construction of the time-delayed solution shares many similarities to that of the weak solution defined in Theorem 5.1.5 combined with ideas from the construction of the time-delayed solutions for solids in Theorem 6.1.1. However an important addition here is the *flow map* Φ. Note that in this subsection, the map will always start at $t = 0$. This allows us to take a temporary Lagrangian point of view, as Ω_0 will play the role of a reference configuration for the fluid.

As in chapter 6, we will construct the time-delayed solutions by time-discretization. Notice that, in the way that Φ is linked with the equation, we already need to begin its construction in the discrete setting. Here, we make use of the additional regularizing dissipation terms for v, as they will allow us to construct Φ in the limit.

In this subsection we will prove the following existence theorem:

Theorem 7.2.2 (Existence of time delayed solutions)**.** *Let* $\eta_0 \in \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n) \setminus \partial\mathcal{E}$, $w \in L^2([0,h] \times Q;\mathbb{R}^n)$ *and* $f \in C^0([0,h] \times Q;\mathbb{R}^n)$. *Then there exists a solution* (η, v) *to the time delayed equation as given in Theorem 7.2.1 on the interval* $[0,h]$, *or there exists a solution on a shorter interval* $[0,h_{\max}]$ *such that* $\eta(h_{\max}) \in \partial\mathcal{E}$.[4] *Furthermore* $\Phi(t,.)$ *is a volume preserving diffeomorphism between* Ω_0 *and* $\Omega(t)$.

Let us now begin with the proof of this theorem. Parts that are identical to one of the previous proofs will only be sketched.

7.2.1 Proof of Theorem 7.2.2, step 1: Constructing an iterative approximation

Fix a step-size $\tau > 0$. We again proceed iteratively, this time constructing both the pair (η, v) as well as Φ. We start with the given $\eta_0^{(\tau)} := \eta_0$ and $\Phi_0^{(\tau)} := id$. Assuming $\eta_k^{(\tau)} \in \mathcal{E}$ and $\Phi_k^{(\tau)} : \Omega_0 \to \Omega_k^{(\tau)}$ given, we define $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ as solution to the following problem

$$\begin{aligned}
\text{Minimize}\quad & E_h(\eta) + \tau R_h\left(\eta_k^{(\tau)}, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right) + \frac{\tau\rho_s}{2h}\left\|\frac{\eta - \eta_k^{(\tau)}}{\tau} - w_k^{(\tau)} \circ \eta_0\right\|^2 - \tau\left\langle f \circ \eta, \frac{\eta - \eta_k^{(\tau)}}{\tau}\right\rangle \qquad (7.2.2)\\
& \tau\frac{\nu}{2}\|\varepsilon v\|_{\Omega_k}^2 + \frac{\tau h}{2}\left\|\nabla^{k_0} v\right\|_{\Omega_k^{(\tau)}}^2 + \frac{\tau\rho_f}{2h}\left\|v \circ \Phi_k^{(\tau)} - w_k^{(\tau)}\right\|_{\Omega_0}^2 - \tau\left\langle f \circ \Phi_k^{(\tau)}, v \circ \Phi_k^{(\tau)}\right\rangle_{\Omega_0}\\
\text{subject to}\quad & \eta \in \mathcal{E}, v \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n) \text{ with } \mathrm{div} v = 0, v|_{\partial\Omega} = 0\\
& \text{and } \frac{\eta - \eta_k^{(\tau)}}{\tau} = v \circ \eta_k^{(\tau)} \text{ in } P.
\end{aligned}$$

Here, as before $\Omega_k^{(\tau)} = \Omega \setminus \eta_k^{(\tau)}(Q)$ and we define $w_k^{(\tau)}(y) = \fint_{k\tau}^{(k+1)\tau} w(t,y)dt$ for all $y \in \Omega$.

Finally we update Φ_k to Φ_{k+1} using

$$\Phi_{k+1}^{(\tau)} := (id + \tau v_{k+1}^{(\tau)}) \circ \Phi_k^{(\tau)}.$$

Note that at this point using the coupling condition, we can immediately derive $\Phi_{k+1}^{(\tau)}(\partial\Omega_0) = \partial\Omega_{k+1}^{(\tau)}$ but we still need to show that a similar property holds in the interior. This will be done in step 2a of the proof. For now we can simply assume $v_{k+1}^{(\tau)}$ to be extended by 0 in the definition of $\Phi_{k+1}^{(\tau)}$.

[4] Note that a-posteriori (see Theorem 7.3.4) it will be shown that (in dependence of η_0) there is always a minimal time-length $h_{\min}$ for which it can be guaranteed that $\eta(t) \notin \partial\mathcal{E}$ for $t \in [0,h_{\min}]$.

Proposition 7.2.3 (Existence of iterative solutions)**.** *The iterative problem* (7.2.2) *has a solution, i.e. $\eta_{k+1}^{(\tau)}$ and $v_{k+1}^{(\tau)}$ are defined. Furthermore the minimizers obey the following equation:*

$$\begin{aligned}
&\left\langle DE_h\left(\eta_{k+1}^{(\tau)}\right),\phi\right\rangle + \left\langle D_2R_h\left(\eta_k^{(\tau)},\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\right),\phi\right\rangle + \frac{\rho_s}{h}\left\langle\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}-w_k^{(\tau)}\circ\eta_0,\phi\right\rangle_Q\\
&+\nu\left\langle\varepsilon v_{k+1}^{(\tau)},\varepsilon\xi\right\rangle_{\Omega_k^\tau} + h\left\langle\nabla^{k_0}v_{k+1}^{(\tau)},\nabla^{k_0}\xi\right\rangle_{\Omega_k^{(\tau)}} + \frac{\rho_f}{h}\left\langle v_{k+1}^{(\tau)}\circ\Phi_k^{(\tau)}-w_k^{(\tau)},\xi\circ\Phi_k^{(\tau)}\right\rangle_{\Omega_0}\\
&=\rho_f\left\langle f\circ\Phi_k^{(\tau)},\xi\circ\Phi_k^{(\tau)}\right\rangle_{\Omega_0}+\rho_s\left\langle f\circ\eta_k^{(\tau)},\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\right\rangle_Q
\end{aligned}$$

where $\phi\in W^{2,q}(Q;\mathbb{R}^n)$, $\phi|_P=0$ and $\xi\in W_0^{1,2}(\Omega;\mathbb{R}^n)$ such that

$$\phi=\xi\circ\eta_k \text{ on } Q \text{ and } \operatorname{div}\xi|_{\Omega_k}=0.$$

Proof. The proof differs from the quasistatic case in Theorem 5.3.2 only in the occurrence of the additional terms for the effects of inertia. As both are non-negative, we still have a minimizing sequence $(\tilde\eta_l,\tilde v_l)$ bounded in the same spaces as in the proof of Theorem 5.3.2. In particular due to the compact embeddings, we can assume that for a sub-sequence both converge strongly in $L^2(Q;\mathbb{R}^n)$ and $L^2(\Omega_k^{(\tau)};\mathbb{R}^n)$ respectively. As the inertial terms are continuous with respect to this convergence, this minimizing sequence will again converge to a minimizer. In fact establishing the lower bound on the sequence is easier in this case, as the two force terms can now be estimated against the inertial terms directly, without having to resort to a potentially energy dependent Korn-inequality. (See the corresponding calculations the proof of Theorem 6.1.3 and Theorem 6.2.4 for more details.)

Further, with regards to the Euler-Lagrange equation, we can treat the additional terms individually. Since both are quadratic functionals of η and v respectively, and neither involve any derivatives, this is straightforward. Note that again we are able to remove a factor of τ from the final term by scaling ϕ and ξ differently than η and v. □

Now as before, our minimization yields a discrete energy inequality by comparing minimizers.

Lemma 7.2.4 (Discrete energy inequality and estimates)**.** *We have*

$$\begin{aligned}
&E_h(\eta_{k+1}^{(\tau)})+\tau R_h\left(\eta_k^{(\tau)},\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\right)+\tau\frac{\rho_s}{2h}\left\|\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}-w_k^{(\tau)}\circ\eta_0\right\|_Q^2\\
&\quad+\tau\frac{\nu}{2}\left\|\varepsilon v_{k+1}^{(\tau)}\right\|_{\Omega_k^{(\tau)}}^2+\frac{\tau h}{2}\left\|\nabla^{k_0}v_{k+1}^{(\tau)}\right\|_{\Omega_k^{(\tau)}}^2+\tau\frac{\rho_f}{2h}\left\|v_{k+1}^{(\tau)}\circ\Phi_k^{(\tau)}-w_k^{(\tau)}\right\|_{\Omega_0}^2\\
&\le E_h(\eta_k^{(\tau)})+\tau\frac{\rho_s}{2h}\left\|w_k^{(\tau)}\circ\eta_0\right\|_Q^2+\tau\frac{\rho_f}{2h}\left\|w_k^{(\tau)}\right\|_{\Omega_0}^2+\tau\rho_f\left\langle f\circ\Phi_k^{(\tau)},v\circ\Phi_k^{(\tau)}\right\rangle_{\Omega_0}\\
&\quad+\tau\rho_s\left\langle f\circ\eta_k^{(\tau)},\frac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\right\rangle_Q
\end{aligned}$$

and there exist $c,C>0$ independent of τ and N (with $N\in\mathbb{N}$ satisfying $N\tau\le h$) such that

$$\begin{aligned}
&E_h(\eta_N^{(\tau)})+\sum_{k=1}^N\tau\Bigg[R_h\left(\eta_{k-1}^{(\tau)},\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right)+c\left\|\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}-w_k^{(\tau)}\circ\eta_0\right\|_Q^2\\
&\quad+\nu\left\|\varepsilon v_k^{(\tau)}\right\|_{\Omega_{k-1}^{(\tau)}}^2+\frac{\tau h}{2}\left\|\nabla^{k_0}v_k^{(\tau)}\right\|_{\Omega_{k-1}^{(\tau)}}^2+c\left\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}-w_k^{(\tau)}\right\|_{\Omega_0}^2\Bigg]\\
&\le E_h(\eta_0)+C\left(\int_0^h\|w\circ\eta_0\|_Q^2\,dt+\int_0^h\|w\|_{\Omega_0}^2\,dt+\|f\|_\infty^2\right).
\end{aligned}$$

Proof. As in Theorem 5.3.3, we compare the minimizer $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ in (7.2.2) with the pair $(\eta_k^{(\tau)}, 0)$ to get the first inequality. For the second we sum up all those inequalites for $k \leq N-1$ to end up with

$$\begin{aligned}
&E_h(\eta_N^{(\tau)}) + \sum_{k=1}^{N} \tau \Bigg[R_h\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right) + \frac{\rho_s}{2h}\left\| \frac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau} - w_k^{(\tau)} \circ \eta_0 \right\|_Q^2 \\
&\quad + \frac{\nu}{2}\left\|\varepsilon v_k\right\|_{\Omega_{k-1}^{(\tau)}}^2 + \frac{\tau h}{2}\left\|\nabla^{k_0} v_{k+1}^{(\tau)}\right\|_{\Omega_k^{(\tau)}}^2 + \frac{\rho_f}{2h}\left\|v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)} - w_{k-1}^{(\tau)}\right\|^2 \Bigg] \\
&\leq E_h(\eta_0) + \sum_{k=1}^{N} \tau \left[\frac{\rho_s}{2h}\left\|w_{k-1}^{(\tau)} \circ \eta_0\right\|_Q^2 + \frac{\rho_f}{2h}\left\|w_{k-1}^{(\tau)}\right\|_{\Omega_0}^2 + \left\langle f, v_k^{(\tau)}\right\rangle_{\Omega_{k-1}} + \left\langle f \circ \eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle_Q \right]
\end{aligned}$$

Now using the definition of $w_k^{(\tau)}$ we note that

$$\sum_{k=1}^{N} \tau \left\|w_{k-1}^{(\tau)}\right\|_{\Omega_0}^2 = \sum_{k=1}^{N} \tau \left\| \fint_{\tau(k-1)}^{\tau k} w\,dt \right\|_{\Omega_0}^2 \leq \sum_{k=1}^{N} \tau \fint_{\tau(k-1)}^{\tau k} \|w\|_{\Omega_0}^2\,dt = \int_0^h \|w\|_{\Omega_0}^2\,dt.$$

The same can be done to show that $\sum_{k=1}^{N} \tau \left\|w_{k-1}^{(\tau)} \circ \eta_0\right\|_Q^2 \leq \int_0^h \|w \circ \eta_0\|_Q^2\,dt$. We are left to estimate the force terms.

As in Theorem 6.1.3 we will absorb them in the inertial terms. This allows us to avoid the use of the Korn's inequality that was needed in Theorem 5.3.3 and we get

$$\begin{aligned}
\left|\left\langle f \circ \Phi_{k-1}^{(\tau)}, v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}\right\rangle_{\Omega_0}\right| &\leq \frac{1}{2\delta}\left\|f \circ \Phi_{k-1}^{(\tau)}\right\|_{\Omega_0}^2 + \frac{\delta}{2}\left\|v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}\right\|_{\Omega_0}^2 \\
&\leq \frac{1}{2\delta}\|f\|_\infty^2 + \frac{\delta}{2}\left\|v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}\right\|_{\Omega_0}^2
\end{aligned}$$

where now for small δ, the last term can be subsumed into the inertial term. From this the estimate follows. □

As before this immediately implies that for h small enough all $\eta_k^{(\tau)}$ will be in $\mathcal{E} \setminus \partial\mathcal{E}$.

7.2.2 Proof of Theorem 7.2.2, step 2: Constructing interpolations

Now we unfix τ and define the follwing interpolants:

$$\begin{aligned}
\eta^{(\tau)}(t,x) &= \eta_k^{(\tau)}(x) && \text{for } \tau k \leq t < \tau(k+1) \\
\tilde{\eta}^{(\tau)}(t,x) &= \frac{\tau(k+1)-t}{\tau}\eta_k^{(\tau)}(x) + \frac{t-\tau k}{\tau}\eta_{k+1}^{(\tau)}(x) && \text{for } \tau k \leq t < \tau(k+1) \\
u^{(\tau)}(t,y) &= v_k^{(\tau)}(y) && \text{for } \tau k \leq t < \tau(k+1), y \in \Omega_k^{(\tau)} \\
u^{(\tau)}(t,y) &= \frac{(\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}) \circ \left(\eta_k^{(\tau)}\right)^{-1}}{\tau} && \text{for } \tau k \leq t < \tau(k+1), y \in \Omega \setminus \Omega_k^{(\tau)} \\
\Phi^{(\tau)}(t,y) &= \Phi_{k-1}^{(\tau)}(y) && \text{for } \tau k \leq t < \tau(k+1) \\
\tilde{\Phi}^{(\tau)}(t,y) &= \frac{\tau(k+1)-t}{\tau}\Phi_{k-1}^{(\tau)}(x) + \frac{t-\tau k}{\tau}\Phi_k^{(\tau)}(x) && \text{for } \tau k \leq t < \tau(k+1)
\end{aligned}$$

as well as $\Omega^{(\tau)}(t) = \Omega_k^{(\tau)}$ for $\tau k \leq t < \tau(k+1)$.

Now using the a-priori estimate Theorem 7.2.4, we derive some uniform bounds on those functions.

Lemma 7.2.5 (Uniform bounds in τ). *The following quantities are bounded independently of τ:*

$$\sup_{t\in[0,h]} E_h(\eta^{(\tau)}(t)), \qquad \sup_{t\in[0,h]} \left\|\eta^{(\tau)}\right\|_{W^{k_0,2}(Q)}, \qquad \sup_{t\in[0,h]} \left\|\tilde{\eta}^{(\tau)}\right\|_{W^{k_0,2}(Q)}$$

$$\int_0^h \left\|\partial_t\tilde{\eta}^{(\tau)}\right\|^2_{W^{k_0,2}(Q)} dt, \qquad \int_0^h \left\|u^{(\tau)}\right\|^2_{W^{k_0,2}(Q)} dt \qquad \int_0^h \left\|u^{(\tau)}\circ\Phi^{(\tau)}\right\|^2_{\Omega_0} dt.$$

Furthermore we have per definition

$$\partial_t\tilde{\Phi}^{(\tau)} = u^{(\tau)}\circ\Phi^{(\tau)}$$

whenever $\Phi^{(\tau)}(t,y)\in\Omega^{(\tau)}(t)$ and $t\notin\tau\mathbb{N}$.

Proof. First we note that the right hand side of the second estimate in Theorem 7.2.4 only depends on the initial data η_0 and w as well as the force f. Then this gives us uniform bounds on $E_h(\eta_k)$ and thus an L^∞ bound on $E_h(\eta^{(\tau)}(t,.))$. By the properties of the energy, Theorem 5.1.1 and its regularized version, this also results in a uniform bound on $\|\eta_k\|_{W^{k_0,2}(Q)}$ and thus in $L^\infty([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$ bounds on $\eta^{(\tau)}$ and $\tilde{\eta}^{(\tau)}$. By the properties of the dissipation, Theorem 5.1.4 using the bound on the energy, we get

$$\begin{aligned} c_K\int_0^h \left\|\partial_t\nabla\tilde{\eta}^{(\tau)}\right\|^2_Q + h\left\|\nabla^{k_0}\partial_t\tilde{\eta}^{(\tau)}\right\|^2_Q dt &= \int_0^h R\left(\eta^{(\tau)}_{k-1},\partial_t\tilde{\eta}^{(h)}\right)dt + c_K\int_0^h h\left\|\nabla^{k_0}\partial_t\tilde{\eta}^{(\tau)}\right\|^2_Q dt \\ &\le c\int_0^h R_h\left(\eta^{(\tau)}_{k-1},\partial_t\tilde{\eta}^{(h)}\right)dt \le c\sum_{k=0}^N \tau R_h\left(\eta^{(\tau)}_{k-1},\frac{\eta^{(\tau)}_k-\eta^{(\tau)}_{k-1}}{\tau}\right) \end{aligned}$$

where we know the right hand side to be bounded. Using Poincare's inequality, as $\partial_t\tilde{\eta}^{(\tau)}|_P = 0$, this then extends into an uniform $L^2([0,T];W^{k_0,2}(Q;\mathbb{R}^n))$ bound on $\partial_t\tilde{\eta}^{(\tau)}$. For the fluid, we use Theorem 5.5.4, as well as the global Korn inequality Theorem 5.2.8 get a $c>0$ such that

$$\begin{aligned} &\int_0^h C_{gK}\left\|u^{(\tau)}\right\|^2_{W^{1,2}(\Omega)} + ch\left\|\nabla^{k_0}u^{(\tau)}\right\|^2_\Omega dt \\ &\le \int_0^h R(\eta^{(\tau)},\tilde{\eta}^{(\tau)}) + \frac{\nu}{2}\left\|\varepsilon u^{(\tau)}(t)\right\|^2_{\Omega^{(\tau)}(t)} dt + h\int_0^h \left\|\partial_t\nabla^{k_0}\tilde{\eta}^{(\tau)}\right\|^2_Q + \left\|\nabla^{k_0}u^{(\tau)}\right\|^2_{\Omega^{(\tau)}(t)} dt \end{aligned}$$

which is uniformly bounded using the energy estimate again. The $L^2([0,h];W^{k_0,2}(\Omega;\mathbb{R}^n)$-estimate then follows by interpolating the missing intermediate derivatives.

For the last estimate, we have

$$\int_0^h \left\|u^{(\tau)}\circ\Phi^{(\tau)}\right\|^2_{\Omega_0} dt = \sum_{k=0}^N \tau\left\|u^{(\tau)}_k\circ\Phi^{(\tau)}_k\right\|^2 \le \sum_{k=0}^N \tau\frac{3}{2}\left(\left\|u^{(\tau)}_k\circ\Phi^{(\tau)}_k - w_k\right\|^2 + \|w_k\|^2\right)$$

which again consists of two bounded sums. □

7.2.3 Proof of Theorem 7.2.2, step 2a: Bounds on $\Phi^{(\tau)}$

We now arrive at a delicate point in the existence proof for the time-delayed problem; namely establishing the properties of and suitable bounds on $\Phi^{(\tau)}$. The challenge here is that $\Phi^{(\tau)}$ is defined via concatenation of an unbounded (for $\tau\to 0$) number of functions and thus is highly nonlinear. As any linearizing would break the coupling properties needed, we will instead rely on using a high enough regularity for the constituting functions.

We start by proving the following:

Proposition 7.2.6 (A higher regularity for the velocity). *There is a $\tau_0>0$ and $\alpha>0$, such that for all $\tau\in(0,\tau_0)$, we have that $\Phi^{(\tau)}_k:\Omega_0\to\Omega_k$ is a diffeomorphism with $\frac{1}{2}\le\det\nabla\Phi^{(\tau)}_k\le 2$ for all $k<\frac{h}{\tau}$ and*

$$\sum_{k=1}^N \tau\left\|v^{(\tau)}_k\right\|^2_{C^{1,\alpha}(\Omega^{(\tau)}_{k-1})} \le \mathcal{K}$$

for any $N<\frac{h}{\tau}$ where $\mathcal{K}$ and τ_0 only depend on $w,h,E(\eta_0)$ and f.

Proof. As k_0 is chosen such that $k_0 - \frac{n}{2} \geq 2 - \frac{n}{q}$, we know that $W_0^{k_0,2}(\Omega;\mathbb{R}^n)$ embeds into $C^{1,\alpha}(\Omega;\mathbb{R}^n)$ for some $\alpha > 0$. Thus

$$\sum_{k=1}^{N} \tau \left\| v_k^{(\tau)} \right\|^2_{C^{1,\alpha}(\Omega_{k-1}^{(\tau)})} \leq \int_0^h \left\| u^{(\tau)} \right\|^2_{C^{1,\alpha}(\Omega)} \leq c \int_0^h \left\| u^{(\tau)} \right\|^2_{W^{k_0,2}(\Omega)}$$

which is uniformly bounded by Theorem 7.2.5.

Now we need to show the properties of Φ_N. By chain rule, the multiplicative nature of the determinant and its expansion (Theorem 5.5.1) we have

$$\det \nabla \Phi_N^{(\tau)} = \prod_{k=1}^{N} \left[\det \left(I + \tau \nabla v_k^{(\tau)} \right) \right] \circ \Phi_{k-1}^{(\tau)}$$
$$= \prod_{k=1}^{N} \left[1 + \tau \underbrace{tr \left(\nabla v_k^{(\tau)} \right)}_{=\operatorname{div} v_k^{(\tau)} = 0} + \sum_{l=2}^{n} \tau^l M_l \left(\nabla v_k^{(\tau)} \right) \right] \circ \Phi_{k-1}^{(\tau)}$$

where M_l are homogeneous polynomials of degree l. By the inequality between the arithmetic and geometric mean, we then have

$$\det \nabla \Phi_N^{(\tau)} \leq \left(\sum_{k=1}^{N} \frac{1}{N} \left(1 + \sum_{l=2}^{n} \tau^l M_l \left(\nabla v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)} \right) \right) \right)^N \leq \left(1 + \frac{1}{N} \sum_{k=1}^{N} \sum_{l=2}^{n} \tau^l c_l Lip(v_k^{(\tau)})^l \right)^N$$

where $Lip(v_k^{(\tau)})$ denotes the Lipschitz constant of $v_k^{(\tau)}$ with respect to its domain $\Omega_{k-1}^{(\tau)}$. Now as $(1 + a/N)^N \to \exp(a)$ is monotone increasing for $a > 0$, we can further estimate

$$\leq \exp \left(\sum_{k=1}^{N} \sum_{l=2}^{n} \tau^l c_l Lip(v_k^{(\tau)})^l \right) = \exp \left(\sum_{l=2}^{n} c_l \tau^{l/2} \sum_{k=1}^{N} \left(\tau Lip(v_k^{(\tau)})^2 \right)^{l/2} \right) \leq \exp \left(\sum_{l=2}^{n} c_l \tau^{l/2} \mathcal{K}^{l/2} \right)$$

where we used that $l \geq 2$ and $\tau Lip(v_{k_0}^{(\tau)})^2 \leq \sum_{k=1}^{N} \tau Lip(v_k^{(\tau)})^2 \leq \sum_{k=1}^{N} \tau \left\| v_k^{(\tau)} \right\|^2_{C^{1,\alpha}(\Omega_{k-1}^{(\tau)})} \leq \mathcal{K}$.

In a similar fashion, we can give a lower estimate

$$\left(\det \nabla \Phi_N^{(\tau)} \right)^{-1} \leq \left(\sum_{k=1}^{N} \frac{1}{N} \left(1 + \sum_{l=2}^{n} \tau^l M_l \left(\nabla v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)} \right) \right)^{-1} \right)^N \leq \exp \left(2 \sum_{l=2}^{n} c_l \tau^{l/2} \mathcal{K}^{l/2} \right)$$

using $\frac{1}{1+a} \leq \frac{1}{1-|a|} \leq 1 + 2|a|$ for $|a|$ small enough. Thus for τ_0 small enough, we have that

$$\frac{1}{2} \leq \det(\nabla \Phi_N^{(\tau)}) \leq 2.$$

Now we know from the boundary condition that $\Phi_N^{(\tau)}$ is an orientation preserving diffeomorpism on $\partial\Omega_0$ as it is given by $\eta_N^{(\tau)} \circ \eta_0^{-1}$ and id at the respective parts of the boundary. We also know that Ω_0 and $\Omega_N^{(\tau)}$ are domains with the same topology as there were no collisions. But then $\Phi_N^{(\tau)}$ has to be a diffeomorphism by a degree argument. □

An immediate consequence of the last proof is the following:

Corollary 7.2.7 (Regularity of $\Phi^{(\tau)}$). *The maps $\Phi^{(\tau)}(t,.)$ are uniformly Lipschitz continuous, i.e. Lipschitz continuous with respect to x such that the constants are bounded independently of τ and t. Furthermore in the limit we have that*

$$\lim_{\tau \to 0} \det \nabla \Phi^{(\tau)} = 1.$$

Proof. By the estimates in the last proof, we find that $\lim_{\tau\to 0}\det\nabla\Phi^{(\tau)} = 1$. What is left, is to prove the Lipschitz regularity.

Here we proceed in the same fashion as in the preceeding proof:

$$\begin{aligned}
Lip(\Phi_N^{(\tau)}) &\le \prod_{l=1}^{N} Lip(\mathrm{id} + \tau v_l^{(\tau)}) \le \prod_{l=1}^{N}(1+\tau Lip(v_l^{(\tau)})) \le \left(\frac{1}{N}\sum_{l=1}^{N}(1+\tau Lip(v_l^{(\tau)}))\right)^N \\
&= \left(1+\frac{1}{N}\sum_{l=1}^{N}\tau Lip(v_l^{(\tau)})\right)^N \le \exp\left(\sum_{l=1}^{N}\tau Lip(v_l^{(\tau)})\right) \\
&\le \exp\left(\sqrt{\sum_{l=1}^{N}\tau}\sqrt{\sum_{l=1}^{N}\tau Lip(v_l^{(\tau)})^2}\right) \le \exp\left(\sqrt{h}\sqrt{\mathcal{K}}\right).
\end{aligned}$$

□

7.2.4 Proof of Theorem 7.2.2, step 3: Convergence of the equation

Relying on the Banach-Alaoglu theorem as well as the classical Aubin-Lions lemma, we pick up subsequence of τ's and find $\eta \in W^{1,2}([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$, $u \in L^2([0,h]; W^{k_0,2}(\Omega;\mathbb{R}^n))$, $\Phi \in C^0([0,h]; W^{1,\infty}(\Omega_0;\mathbb{R}^n))$ such that

$$\begin{aligned}
\eta^{(\tau)}, \tilde{\eta}^{(\tau)} &\rightharpoonup^* \eta && \text{in } L^\infty([0,h]; W^{k_0,2}(Q;\mathbb{R}^n)) \\
\partial_t\tilde{\eta}^{(\tau)} &\rightharpoonup \partial_t\eta && \text{in } L^2([0,h]; W^{k_0,2}(Q;\mathbb{R}^n)) \\
u^{(\tau)} &\rightharpoonup u && \text{in } L^2([0,h]; W^{k_0,2}(\Omega;\mathbb{R}^n)) \\
\Phi^{(\tau)} &\to \Phi && \text{in } C^0([0,h]; C^\alpha(\Omega_0;\mathbb{R}^n))
\end{aligned}$$

and we define $\Omega(t) = \Omega \setminus \eta(t,Q)$. Moreover, due to Theorem 7.2.7 we know that Φ is Lipschitz with constant $\exp(\sqrt{Lh})$ and that $\det\nabla\Phi = 1$ almost everywhere. We also remark that $\Phi(t,.)|_{\partial\Omega_0}$ is injective as long as there is no collision in the solid (which we already excluded), and that again we thus know that $\Phi(t,.) : \Omega_0 \to \Omega(t)$ is a volume preserving diffeomorphism.

Finally we can conclude that

$$\partial_t\Phi = \lim_{\tau\to 0}\partial_t\tilde{\Phi}^{(\tau)} = \lim_{\tau\to 0} u^{(\tau)}\circ\Phi^{(\tau)} = u\circ\Phi$$

almost everywhere.

Then Φ fulfills the requirements in Definition 7.2.1 and v and η are coupled in the right way, as before. What is left is to show that these functions fulfill the weak equation (7.2.1). This is indeed very similar to the proofs of Theorem 5.3.9 and Theorem 6.1.3.

As before, we use Theorem 5.3.11 and pick a test function $\xi \in C_0^\infty([0,h]\times\Omega;\mathbb{R}^n)$ such that $\mathrm{div}\xi = 0$ in a neighborhood of the fluid domain. From this we can construct matching $\phi^{(\tau)} := \xi\circ\eta^{(\tau)}$ and use those to test the discrete Euler-Lagrange equation from Theorem 7.2.3.

For most of the terms including all those related to the solid, we have already dealt with in Theorem 5.3.9 and Theorem 6.1.3. What is left are the additional regularization term, the inertial effects of the fluid and the force term for the fluid which has been slightly modified here.

We start with the latter, where we simply note that $\Phi^{(\tau)}$ converges uniformly and thus any concatenation with a uniformly continuous function such as given by $f\circ\Phi^{(\tau)}$ converges uniformly as well. Therefore

$$\int_0^h \left\langle f\circ\Phi^{(\tau)}, \xi\circ\Phi^{(\tau)}\right\rangle_{\Omega_0} dt \to \int_0^T \langle f\circ\Phi, \xi\circ\Phi\rangle_{\Omega_0} dt = \int_0^h \langle f,\xi\rangle_{\Omega(t)} dt,$$

where the last equality is true as Φ is volume preserving.

Of greater interest is the inertial term of the fluid, where we have

$$\int_0^h \left\langle u^{(\tau)}\circ\Phi^{(\tau)} - w^{(\tau)}, \xi\circ\Phi^{(\tau)}\right\rangle_{\Omega_0} dt \to \int_0^h \langle u\circ\Phi - w, \xi\circ\Phi\rangle_{\Omega_0} dt$$

as the right side of the product converges uniformly and the left side at least weakly in $L^2([0,h] \times \Omega_0; \mathbb{R}^n)$. Here, we introduced the notation

$$w^{(\tau)}(t) := w_k^{(\tau)} \text{ if } \tau k \leq t < \tau(k+1).$$

Then in particular by the Lebesgue differentiation theorem $w^{(\tau)} \to w$ in $L^2([0,h] \times \Omega; \mathbb{R}^n)$ and $w^{(\tau)} \circ \eta_0^{-1} \to w \circ \eta_0^{-1}$ in $L^2([0,h] \times Q; \mathbb{R}^n)$.

Finally we note that as used before $\chi_{\Omega^{(\tau)}(t)} \nabla^{k_0} \xi \to \chi_{\Omega(t)} \nabla^{k_0} \xi$ in $L^2([0,h]; L^2(\Omega; \mathbb{R}^n))$ and thus

$$\int_0^h \left\langle \nabla^{k_0} u^{(\tau)}, \nabla^{k_0} \xi \right\rangle_{\Omega^{(\tau)}(t)} dt \to \int_0^h \left\langle \nabla^{k_0} u, \nabla^{k_0} \xi \right\rangle_{\Omega(t)} dt$$

by the corresponding weak convergence of $u^{(\tau)}$. This finishes the proof. □

7.2.5 A posteriori energy inequality

We close this section with an energy inequality analogous to Theorem 6.1.4. As before, this will be the central estimate that allows us to take the limit $h \to 0$ and pass to the limit with the equation.

Lemma 7.2.8 (Energy inequality for time delayed solutions)**.** *Assume that* (η, v) *is a weak solution to the time delayed equation* (7.2.1)*, as constructed in Theorem 7.2.2. Then we have the following energy inequality*

$$E_h(\eta(h)) + \int_0^h 2R_h(\eta, \partial_t \eta) + \nu \|\varepsilon v\|_{\Omega(t)}^2 + h \left\|\nabla^{k_0} v\right\|_{\Omega(t)}^2 dt + \int_0^h \frac{\rho_f}{2} \|v\|_{\Omega(t)}^2 + \frac{\rho_f}{2} \|\partial_t \eta\|_Q^2 dt$$
$$\leq E_h(\eta(0)) + \int_0^h \rho_f \langle f, v\rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \partial_t \eta\rangle_Q dt + \int_0^h \frac{\rho_f}{2h} \|w\|_{\Omega_0}^2 + \frac{\rho_f}{2h} \left\|w \circ \eta_0^{-1}\right\|_Q^2 dt.$$

Proof. We insert $(\partial_t \eta, v)$ as test functions in (7.2.1). These have the correct coupling and boundary conditions. We need to be careful with regularity here and thus have to rely on the added regularizing terms. From these we know that $\partial_t \eta \in L^2([0,h]; W^{k_0,2}(Q; \mathbb{R}^n))$ and it thus can be used in duality pairing with $DE_h(\eta)$. We hence obtain

$$0 = \int_0^h \langle DE_h(\eta), \partial_t \eta\rangle + \langle D_2 R_h(\eta, \partial_t \eta), \partial_t \eta\rangle + \left\langle \rho_s \frac{\partial_t \eta - w \circ \eta_0^{-1}}{h}, \partial_t \eta \right\rangle_Q$$
$$+ \nu \langle \varepsilon v, \varepsilon v\rangle_{\Omega(t)} + \left\langle \rho_f \frac{v \circ \Phi - w}{h}, v \circ \Phi \right\rangle_{\Omega_0} - \rho_f \langle f, v\rangle_{\Omega_0} - \rho_s \langle f \circ \eta, \partial_t \eta\rangle_Q dt$$

Now the first term is just the time derivative of the energy and thus its integral is $E_h(\eta(h)) - E_h(\eta(0))$ while for the second term we remember that due to the 2-homogeneity of the dissipation $\langle D_2 R_h(\eta, \partial_t \eta), \partial_t \eta\rangle_Q = 2R_h(\eta, \partial_t \eta)$. Finally we estimate the inertial terms using Young's inequality in the form of $\langle a - b, a\rangle = |a|^2 - \langle b, a\rangle \geq \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2$. Reordering terms according to their sign proves the estimate. □

7.3 Proof of Theorem 7.1.2

Similarly to the proof of Theorem 6.0.2 in chapter 6, we will use time-delayed solutions constructed in the previous subsection to approximate weak solutions to the fluid-structure interaction problem (7.1.1)–(7.1.7). The main added difficulty, when compared to chapter 6, is in dealing with the inertial effects of the fluid. A particular problem there is that the flow-map itself will not persist in the limit for $h \to 0$. However since it is only ever needed for a flow of length h, the goal is simply to find the right reformulation such that limit quantities still exist. In particular, the material derivative $\partial_t v + v \cdot \nabla v$ will only be obtained in a weak sense.

Furthermore we note that due to the changing domain, we generally use convergence of u instead of v.

With all this in mind, let us begin with the proof.

7.3.1 Proof of Theorem 7.1.2, step 1: Constructing another iterative approximation

We now iteratively construct an approximative solutions to the to the fluid-structure interaction problem (7.1.1)–(7.1.7) using time-delayed solutions.

For some fixed h assume that η_0 with finite energy $E_h(\eta_0)$, $v_0 : \Omega_0 := \Omega \setminus \eta_0(Q) \to \mathbb{R}^n$ satisfying $\operatorname{div} v_0 = 0$ and $\eta_* : Q \to \mathbb{R}^n$ are given. Set $w_0(t,y) = v_0(y)$ for $y \in \Omega_0$ and $w_0 = \eta_* \circ \eta_0^{-1}$ otherwise.[5]

For $\eta_l : Q \to \Omega$, $w_l : [0,h] \times \Omega \to \mathbb{R}^n$ and $\Omega_l := \Omega \setminus \eta_l(Q)$ given, we rely on Theorem 7.2.2 to construct time-delayed solutions to (7.1.1)–(7.1.7) according to Theorem 7.2.1 on $[0,h]$ with the given data, which we will denote using $\tilde{\eta}_{l+1}, v_{l+1}, \Phi_{l+1}$. Observe in particular, that

$$\Phi_{l+1}(s)(\Omega_l) = \Omega \setminus \tilde{\eta}_{l+1}(s,Q).$$

We then set $\eta_{l+1} := \tilde{\eta}_{l+1}(h,\cdot)$ and $\Omega_{l+1} := \Omega \setminus \eta_{l+1}(Q)$ and construct

$$w_{l+1} : [0,h] \times \Omega \to \mathbb{R}^n, \quad w_{l+1}(t,\cdot) = \begin{cases} v_{l+1}(t,\cdot) \circ \Phi_{l+1}(t,\cdot) \circ \Phi_{l+1}(h,\cdot)^{-1} & \text{on } \Omega_{l+1} \\ \partial_t \eta(t,\cdot) \circ \eta(t,\cdot)^{-1} & \text{on } \Omega \setminus \Omega_{l+1} \end{cases}$$

which will again allow us to find time-delayed solutions according to Theorem 7.2.1. Indeed, $E_h(\eta_{l+1}) < \infty$ by the energy inequality Theorem 7.2.8 and since Φ_{l+1} is volume preserving we have $\int_0^h \|w_{l+1}\|^2_{\Omega_{l+1}}\, dt = \int_0^h \|v_{l+1}\|^2_{\Omega_k(t)}\, dt < \infty$ and a similar estimate for the solid. Hence we can iterate until we reach a collision or until $E(\eta_l)$ or w_l diverge (as we will see by Lemma 7.3.2, neither of the last two can happen in finite time).

Now we construct the h-approximation.

Definition 7.3.1 (h-approximation)**.** *For $h > 0$ and all $l \in \mathbb{N}_0$ such that $lh < T$, let $\tilde{\eta}_l$, v_l and Φ_l be time-delayed solutions as constructed above. Then we define the approximations $\eta^{(h)} : [0,T] \times Q \to \Omega$, $u^{(h)} : [0,T] \times \Omega \to \mathbb{R}^n$ as well as $\Phi_s^{(h)} : [0,T] \times \Omega \to \Omega$ for $s \in [-h,h]$ by*

$$\begin{aligned}
\eta^{(h)}(t,x) &:= \tilde{\eta}_l(t-lh,x) && \text{for } t \in [lh,(l+1)h) \\
\Omega^{(h)}(t) &:= \Omega_l(t-hl) && \text{for } t \in [lh,(l+1)h) \\
v^{(h)}(t,y) &:= v_l(t-lh,y) && \text{for } t \in [lh,(l+1)h), y \in \Omega^{(h)}(t) \\
u^{(h)}(t,y) &:= v^{(h)}(t,y) && \text{for } t \in [0,T), y \in \Omega^{(h)}(t) \\
u^{(h)}(t,y) &:= \partial_t \eta^{(h)}(t,(\eta^{(h)}(t))^{-1}(y)) && \text{for } t \in [0,T), y \in \eta^{(h)}(t,Q) \\
\rho^{(h)}(t,y) &:= \rho_f && \text{for } t \in [0,T), y \in \Omega^{(h)}(t) \\
\rho^{(h)}(t,y) &:= \frac{\rho_s}{\det(\nabla \eta^{(h)}(t,(\eta^{(h)}(t))^{-1}(y)))} && \text{for } t \in [lh,(l+1)h), y \notin \Omega^{(h)}(t)
\end{aligned}$$

Moreover for $y \in \Omega^{(h)}(t)$ and $s \in [-h,h]$ we define for $t \in [lh,(l+1)h)$

$$\begin{aligned}
\Phi_s^{(h)}(t,.) &:= \Phi_l(t+s-lh) \circ (\Phi_l(t-lh))^{-1} && \text{if } t+s \in [lh,(l+1)h) \\
\Phi_s^{(h)}(t,.) &:= \Phi_{l+1}(t+s-(l+1)h) \circ \Phi_l(h) \circ (\Phi_l(t-lh))^{-1} && \text{if } (l+1)h \le t+s < (l+2)h \\
\Phi_s^{(h)}(t,.) &:= \Phi_{l-1}(t+s-(l-1)h) \circ (\Phi_{l-1}(h))^{-1} \circ (\Phi_l(t-lh))^{-1} && \text{if } (l-1)h \le t+s < lh.
\end{aligned}$$

For $y \in \eta^{(h)}(t,Q)$ and $s \in [-h,h]$ we define

$$\Phi_s^{(h)}(t) := \eta^{(h)}(t+s) \circ (\eta^{(h)}(t))^{-1}$$

[5] Note that for this first step, v_0 and η_* do not need to fulfill a coupling condition $\eta_* = v_0 \circ \eta_0$ on $\partial Q \setminus P$ yet. This is completely reasonable from a mathematical point of view, as initial values will only ever be taken in an L^2-sense, so there is no trace-theorem to make sense of this condition.

Note that in contrast to the usage in the proof of Theorem 7.2.2, where the $\Phi(t,.)$ always corresponded to the flow starting from the initial configuration of the fluid, we now use a full flow map $\Phi_s^{(h)}(t,.)$ which corresponds to the flow from time t to time $t+s$. In particular $\Phi_l(r)$ maps the fluid at time lh to the fluid at time $lh+r$ for $r \in [0,h]$, so we always need to use the previous multiples of h as an intermediate steps in defining $\Phi_s^{(h)}$.

This being said, what we will use in the coming proofs is not the definition but the fact that $\Phi_s^{(h)}$ is the flow map of $u^{(h)}$. In particular we will rely on the resulting properties that are shown in the following lemma.

Lemma 7.3.2 (The global flow map)**.** *For all $h > 0$, the flow-map defined above is continuous in space-time and satisfies*

$$\partial_s \Phi_s^{(h)}(t,y) = u^{(h)}(t+s, \Phi_s^{(h)}(t,y)). \tag{7.3.1}$$

Moreover, $\Phi_s^{(h)}(t,.)$ is density preserving, i.e.

$$\det(\nabla\Phi_s^{(h)}(t,y)) = 1 \qquad \text{for } y \in \Omega^{(h)}(t) \quad \text{and}$$

$$\det(\nabla\Phi_s^{(h)}(t,y)) = \frac{\rho^{(h)}(t+s, \Phi_s^{(h)}(t,y))}{\rho^{(h)}(t,y)} \qquad \text{for } y \in \eta^{(h)}(t,Q).$$

The inverse of the flow map is given by $(\Phi_s^{(h)}(t))^{-1} = \Phi_{-s}^{(h)}(t+s)$.

Proof. For all $y \in \Omega^{(h)}(t) \cup \eta^{(h)}(t,Q)$ we find (by chain rule and Theorem 7.2.2) that that

$$\partial_s \Phi_s^{(h)}(t,y) = u^{(h)}(t+s, \Phi_s^{(h)}(t,y)).$$

For $s = 0$ the function $\Phi_0^{(h)}(t) = \mathrm{id}$ is trivially continuous over Ω and by the a-priori estimates also u is uniformly Lipschitz continuous (in dependence of h). Hence by a standard argument for ordinary differential equations $\Phi_s^{(h)}(t,y)$ is continuous over Ω.

The identity of the determinant follows by Theorem 7.2.2 for the fluid part and by chain rule and the definition of $\rho^{(h)}$ for the solid part. Furthermore the inverse of the flow map is given as the respective flow in the opposite direction, which is verified by considering $(\Phi_s^{(h)}(t))^{-1} = \Phi_{-s}^{(h)}(t+s)$. □

While it would also be possible to define $\Phi_s^{(h)}(t)$ for larger s, for the remainder of the proof we only need $s \in [-h,h]$. (See also Theorem 7.3.11 with regards to this).

With the h-approximation defined, (7.2.1) translates to

$$\begin{aligned}
&\int_0^T \left\langle DE_h(\eta^{(h)}), \phi\right\rangle + \left\langle DR_h(\eta^{(h)}, \partial_t\eta^{(h)}), \phi\right\rangle + \rho_s \left\langle \frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h}, \phi\right\rangle_Q \qquad (7.3.2)\\
&+ \left\langle \varepsilon v^{(h)}, \varepsilon\xi\right\rangle_{\Omega^{(h)}(t)} + \rho_f \left\langle \frac{v^{(h)}(t)\circ\Phi_h^{(h)}(t-h) - v^{(h)}(t-h)}{h}, \xi(t)\circ\Phi_h^{(h)}(t-h)\right\rangle_{\Omega^{(h)}(t-h)} dt.\\
&= \int_0^T \rho_s \left\langle f\circ\eta^{(h)}, \phi\right\rangle_Q + \rho_f \left\langle f,\xi\right\rangle_{\Omega^{(h)}(t)} dt
\end{aligned}$$

for all $\phi \in C^0([0,T]; W^{k_0,2}(Q;\mathbb{R}^n))$, $\xi \in C^0([0,T]; W_0^{k_0,2}(\Omega;\mathbb{R}^n))$ satisfying $\mathrm{div}\xi|_{\Omega(t)} = 0$, $\xi|_{\partial\Omega} = 0$, $\phi|P = 0$ and the coupling conditions $\xi\circ\eta = \phi$ and $u\circ\eta = \partial_t\eta$ in Q.

Observe that by the definition of $\rho^{(h)}$ above, we find by a change of variables the following identity for the global momentum:

$$\begin{aligned}&\left\langle \frac{\rho^{(h)}u^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-\rho^{(h)}u^{(h)}(t-h)}{h},\xi(t)\circ\Phi_h^{(h)}(t-h)\right\rangle_\Omega\\&=\rho_f\left\langle \frac{v^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-v^{(h)}(t-h)}{h},\xi(t)\circ\Phi_h^{(h)}(t-h)\right\rangle_{\Omega^{(h)}(t-h)}\\&+\rho_s\left\langle\frac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\phi\right\rangle_Q,\end{aligned}\tag{7.3.3}$$

which holds for the same set of test functions as (7.3.2).

From Theorem 7.2.8 we deduce the following a-priori estimate:

Lemma 7.3.3 (A-priori estimate (full problem))**.** *We have for any $t\in[0,T]$*

$$\begin{aligned}&E_h(\eta^{(h)}(t))+\fint_{t-h}^{t}\frac{\rho_f}{2}\left\|u^{(h)}\right\|^2_{\Omega^{(h)}(t)}+\frac{\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt\\&+\int_0^t R_h(\nabla\eta^{(h)},\partial_t\eta^{(h)})+\nu\left\|\varepsilon u^{(h)}\right\|^2_{\Omega^{(h)}(t)}+h\left\|\nabla^{k_0}u^{(h)}\right\|^2_{\Omega^{(h)}(t)}dt\\&\le E_h(\eta_0)+\frac12\|v_0\|^2_{\Omega_0}+\int_0^t\rho_f\left\langle f,u^{(h)}\right\rangle_{\Omega^{(h)}(t)}+\rho_s\left\langle f\circ\eta^{(h)},\partial_t\eta^{(h)}\right\rangle_Q dt,\end{aligned}$$

and moreover there exist $C,c>0$ independent of h such that

$$\begin{aligned}&E_h(\eta^{(h)}(t))+c\fint_{t-h}^{t}\left\|u^{(h)}\right\|^2_{\Omega^{(h)}(t)}+\left\|\partial_t\eta^{(h)}\right\|^2_Q dt\\&+\int_0^t R_h(\nabla\eta^{(h)},\partial_t\eta^{(h)})+\nu\left\|\varepsilon u^{(h)}\right\|^2_{\Omega^{(h)}(t)}+h\left\|\nabla^{k_0}u^{(h)}\right\|^2_{\Omega^{(h)}(t)}dt\le C+Ct^2\end{aligned}$$

In both these estimates take $u^{(h)}$ and $\partial_t\eta^{(h)}$ to be continued by their initial values for $t<0$.

Proof. Theorem 7.2.8 translates for any $l\in\mathbb{N}_0$ such that $lh<T$ to

$$\begin{aligned}&E_h(\eta_{l+1})+\fint_0^s\frac{\rho_f}{2}\|v_{l+1}\|^2_{\tilde\Omega_l(t)}+\frac{\rho_s}{2}\|\partial_t\tilde\eta_{l+1}\|^2_Q dt\\&+\int_0^s R_h(\nabla\tilde\eta_{l+1},\partial_t\nabla\tilde\eta_{l+1})+\nu\|\varepsilon v_{l+1}\|^2_{\tilde\Omega_l(t)}+h\left\|\nabla^{k_0}v_{l+1}\right\|^2_{\Omega^{(h)}(t)}dt\\&\le E_h(\eta_l)+\fint_0^s\frac{\rho_f}{2}\|w_l\|^2_{\Omega_l}+\frac{\rho_s}{2}\|w_l\circ\eta_l\|_Q dt+\int_0^s\langle f,v_{l+1}\rangle_{\Omega_l}+\rho_s\langle f\circ\tilde\eta_l,\partial_t\tilde\eta_l\rangle_Q dt\end{aligned}$$

for $s\in[0,h]$. Now per construction $\|w_l(t,.)\|_{\Omega_l}=\|v_l(t,.)\|_{\tilde\Omega_{l-1}(t)}$ and $\|w_l\circ\eta_l\|_Q=\|\partial_t\tilde\eta_l\|_Q$ thus we can use a telescope argument to get the first energy inequality as we did in Theorem 6.1.4.

Next we use Young's inequality for the two force terms to obtain

$$\begin{aligned}&\int_0^t\rho_f\left\langle f,v^{(h)}\right\rangle_{\Omega^{(h)}(t)}+\rho_s\left\langle f\circ\eta^{(h)},\partial_t\eta^{(h)}\right\rangle_Q dt\\&\le\int_0^t\frac{1}{2\delta}\left(\rho_f\|f\|^2_{\Omega^{(h)}(t)}+\rho_s\left\|f\circ\eta^{(h)}\right\|^2_Q\right)+\frac{\delta}{2}\left(\rho_f\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)}+\rho_s\left\|\partial_t\eta^{(h)}\right\|^2_Q\right)dt\\&\le\frac{C}{\delta}t\|f\|^2_\infty+\int_0^t\frac{\delta\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)}+\frac{\delta\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt.\end{aligned}$$

Now dropping all non-negative terms on the left-hand side except those stemming from inertia and extending the interval to $[0,T]$, we have

$$\fint_{t-h}^{t} \frac{\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)} + \frac{\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt$$
$$\le E(\eta_0) - E_{\min} + \frac{1}{2}\|v_0\|^2_{\Omega_0} + \frac{C}{\delta}T\|f\|^2_\infty + \int_0^T \frac{\delta\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)} + \frac{\delta\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt$$

and summing over intervals for $T = hN$ thus gives

$$\int_0^T \frac{\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)} + \frac{\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt \le \sum_{l=1}^N h \fint_{(l-1)h}^{lh} \frac{\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)} + \frac{\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt$$
$$\le hN\left(C + \frac{C}{\delta}T\|f\|^2_\infty + \int_0^T \frac{\delta\rho_f}{2}\left\|v^{(h)}\right\|^2_{\Omega^{(h)}(t)} + \frac{\delta\rho_s}{2}\left\|\partial_t\eta^{(h)}\right\|^2_Q dt\right)$$

from which as before in Theorem 6.0.2 choosing $\delta = \frac{1}{2T}$ yields the desired estimate. □

Corollary 7.3.4 (Minimal no collision time)**.** *Assume that $\eta_0 \notin \partial\mathcal{E}$. Then there is a $T > 0$ depending only on η_0, v_0 and f such that $\eta^{(h)}(s)$ is injective on $\overline{Q}$ for all $t \in [0,T]$, h small enough, i.e. we have $\eta^{(h)}(t) \notin \partial\mathcal{E}$ and thus there is no collision.*

Proof. From the final estimate in the proof of Theorem 7.3.3 we get

$$\left\|\eta^{(h)} - \eta_0\right\|^2_Q = \left\|\int_0^T \partial_t\eta^{(h)}dt\right\|^2_Q \le \int_0^T \left\|\partial_t\eta^{(h)}\right\|^2_Q dt \le TC(1+T^2).$$

Using this bound for small enough T then allows us to apply the short-distance injectivity result Theorem 5.2.4. □

As a direct consequence of the uniform bounds of $\det(\nabla\eta^{(h)})$, the definitions of E_h and R_h as well as Theorem 5.2.8, we find that

Corollary 7.3.5 (Korn-type estimate)**.** *There is a constant just depending on the energy estimate in Theorem 7.3.3, such that*

$$\sup_{t\in[0,T-h]} \fint_t^{t+h} \left\|u^{(h)}(s)\right\|^2_\Omega ds + \int_0^T \left\|\partial_t\eta^{(h)}\right\|^2_{W^{1,2}(Q)} dt + \int_0^T \left\|u^{(h)}\right\|^2_{W^{1,2}(\Omega)} dt \le C,$$
$$\sup_{t\in[0,T]} h^{a_0}\left\|\eta^{(h)}\right\|^2_{W^{k_0,2}(Q)} + h\int_0^T \left\|\partial_t\eta^{(h)}\right\|^2_{W^{k_0,2}(Q)} dt + \int_0^T \left\|u^{(h)}\right\|^2_{W^{k_0,2}(\Omega)} dt \le C.$$

7.3.2 Proof of Theorem 7.1.2, step 2: The weak time-derivative

In the following, we may understand $\partial_t\eta^{(h)}$ and $u^{(h)}$ to be extended by their initial values for $t \in [-h, 0]$.

Lemma 7.3.6 (Length h bounds (fluid))**.** *Fix $T > 0$. Then there exists a constant C depending only on the initial data, such that the following holds:*

1. $\int_0^T \left\|\frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h}\right\|^2_{W^{-k_0,2}(Q)} dt \le C$
2. $\left\|\xi(t) - \xi(t-s_0)\circ\Phi^{(h)}_{-s_0}(t)\right\|_\Omega \le ChLip_{t,y}(\xi)$ *for all* $\xi \in C_0^\infty([0,T]\times\Omega)$, $s_0 \in [-h,h]$
3. $\left\|\xi - \xi\circ\Phi^{(h)}_{s_0}(t)\right\|_\Omega \le ChLip_y(\xi)$ *for all* $\xi \in C_0^\infty(\Omega)$, $s_0 \in [-h,h]$, $t \in [0,T]$

Here we use Lip_y and $Lip_{t,y}$ to distinguish the Lipschitz-constants with respect to space and space-time respectively.

Proof. The first estimate is shown in almost the same way as Theorem 6.2.1. Indeed, as we only test by functions that vanish on the boundary, we can afford to set ξ to 0 on the fluid-domain.

For the second estimate, let $\xi \in C_0^\infty([0,T] \times \Omega; \mathbb{R}^n)$ and calculate

$$\begin{aligned}
&\int_\Omega |\xi(t) - \xi(t-s_0) \circ \Phi_{-s_0}^{(h)}(t)|^2 dy \\
&= \int_\Omega |\int_{-s_0}^0 \partial_s \left(\xi(t+s) \circ \Phi_s^{(h)}(t)\right) ds|^2 dy \\
&= \int_\Omega |\int_{-s_0}^0 \left[\nabla\xi(t+s) \cdot u^{(h)}(t+s) + \partial_t \xi(t+s)\right] \circ \Phi_s^{(h)}(t) ds|^2 dy \\
&\leq s_0 \int_{-s_0}^0 \int_\Omega |\left[\nabla\xi(t+s) \cdot u^{(h)}(t+s) + \partial_t \xi(t+s)\right] \circ \Phi_s^{(h)}(t)|^2 dy ds \\
&\leq h^2 Lip_{t,y}(\xi)^2 \fint_{t-h}^{t} \int_\Omega \det(\nabla \Phi_{-s}^{(h)}(t+s)) \left(|u^{(h)}(s)|^2 + 1\right) ds \\
&\leq C h^2 Lip_{t,y}(\xi)^2 \fint_{t-h}^{t} \left(\left\|u^{(h)}(s)\right\|_\Omega + 1\right)^2 ds
\end{aligned}$$

using the uniform bounds of $\det(\nabla \Phi_{-s}^{(h)}(t+s))$ (Theorem 7.3.2) and the velocity Theorem 7.3.5. This implies (2). The third assertion follows by the very same arguments. ☐

The next proposition estimates the weak time-derivative of the global momentum.

Proposition 7.3.7. *There is a $m \geq k_0$ and a constant independent of h, such that for all $\xi \in C^0([0,T]; W_0^{m,2}(\Omega;\mathbb{R}^n))$ with $\operatorname{div}\xi = 0$ on $\Omega(t)$*

$$\int_0^T |\left\langle \frac{(\rho^{(h)} u^{(h)})(t) - (\rho^{(h)} u^{(h)})(t-h)}{h}, \xi(t) \right\rangle_\Omega | dt \leq C \|\xi\|_{L^2([0,T]; W^{m,2}(\Omega))}.$$

Proof. Let $\xi \in C^0([0,T] \times \Omega; \mathbb{R}^n)$ with $\operatorname{div}\xi(t) = 0$ on $\Omega^{(h)}(t)$ for all $t \in [0,T]$ and define $\phi := \xi \circ \eta^{(h)}$. Let us first split the integrand into two along the flow map.

$$\begin{aligned}
&\left\langle \frac{(\rho^{(h)} u^{(h)})(t) - (\rho^{(h)} u^{(h)})(t-h)}{h}, \xi(t) \right\rangle_\Omega \\
&= \left\langle \frac{(\rho^{(h)} u^{(h)})(t) - (\rho^{(h)} u^{(h)})(t-h) \circ \Phi_{-h}^{(h)}(t)}{h}, \xi(t) \right\rangle_\Omega \\
&- \left\langle \frac{(\rho^{(h)} u^{(h)})(t-h) - (\rho^{(h)} u^{(h)})(t-h) \circ \Phi_{-h}^{(h)}(t)}{h}, \xi(t) \right\rangle_\Omega =: J_1(t) - J_2(t)
\end{aligned}$$

Now we estimate $J_1(t)$ by changing variables on the fluid domain and using (7.3.2)

$$\begin{aligned}
\int_0^T |J_1(t)|dt &= \int_0^T \Bigg| \rho_s \left\langle \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h}, \phi \right\rangle_Q \\
&\quad + \rho_f \left\langle \frac{u^{(h)}(t) \circ \Phi_h^{(h)}(t-h) - u^{(h)}(t-h)}{h}, \xi(t) \circ \Phi_h^{(h)}(t-h) \right\rangle_{\Omega^{(h)}(t-h)} \Bigg| dt \\
&\le \int_0^T |\langle DE(\eta^{(h)}), \phi \rangle| + h^{a_0} |\langle \nabla^{k_0} \eta^{(h)}, \nabla^{k_0} \phi \rangle_Q| + |\langle D_2 R_h(\eta^{(h)}, \partial_t \eta^{(h)}), \phi \rangle| \\
&\quad + h |\langle \nabla^{k_0} \partial_t \eta^{(h)}, \nabla^{k_0} \phi \rangle_Q| + \nu |\langle \varepsilon u^{(h)}, \varepsilon \xi \rangle_{\Omega(t)}| + h |\langle \nabla^{k_0} u^{(h)}, \nabla^{k_0} \xi \rangle_{\Omega(t)}| \\
&\quad + \rho_f |\langle f, \xi \rangle_{\Omega(t)}| + \rho_s |\langle f \circ \eta, \phi \rangle_Q| dt \\
&\le c \int_0^T \left\| DE(\eta^{(h)}) \right\|_{W^{-2,q}(Q)} \|\xi(t)\|_{W^{2,q}(\Omega)} \left\| \eta^{(h)}(t) \right\|_{W^{2,q}(Q)} \\
&\quad + \left\| D_2 R(\eta^{(h)}, \partial_t \eta^{(h)}) \right\|_{W^{-1,2}(Q)} \|\xi(t)\|_{W^{2,q}(\Omega)} \left\| \eta^{(h)}(t) \right\|_{W^{2,q}(Q)} \\
&\quad + \left\| \eta^{(h)}(t) \right\|_{W^{k_0,2}(Q)} \left[h^{a_0} \left\| \nabla^{k_0} \eta^{(h)} \right\| \|\xi(t)\|_{C^{k_0}(\Omega)} + h \left\| \partial_t \eta^{(h)}(t) \right\|_{W^{k_0,2}(Q)} \|\xi(t)\|_{C^{k_0}(\Omega)} \right] \\
&\quad + h \left\| u^{(h)}(t) \right\|_{W^{k_0,2}(Q)} \|\xi\|_{W^{k_0,2}(Q)} + \left\| \varepsilon u^{(h)} \right\|_{\Omega(t)} \|\varepsilon \xi\|_{\Omega(t)} + \|f\|_\infty \|\xi\|_{\Omega(t)} \, dt
\end{aligned}$$

where we used that by Theorem 5.5.4 we have $\|\phi(t)\|_{W^{2,q}(Q)} \le c \|\xi(t)\|_{W^{2,q}(\Omega)} \left\|\eta^{(h)}(t)\right\|_{W^{2,q}(Q)}$ and $\|\phi(t)\|_{W^{k_0,2}(Q)} \le c \|\xi(t)\|_{C^{k_0}(\Omega)} \left\|\eta^{(h)}(t)\right\|_{W^{k_0,2}(Q)}$. From the energy estimate in Lemma 7.3.3 we know that $\left\|\eta^{(h)}(t)\right\|_{W^{2,q}(Q)}$ and $h^{a_0/2} \left\|\eta^{(h)}(t)\right\|_{W^{k_0,2}(Q)}$ are uniformly bounded in h and t. Thus every term is a product of a quantity which has (at least) a uniform $L^2([0,T])$-bound using the energy estimate and a term which can be estimated against $\|\xi(t)\|_{C^{k_0}(\Omega)}$. Choosing m such that $W^{m,2}(\Omega;\mathbb{R}^n)$ embeds into $C^{k_0}(\Omega;\mathbb{R}^n)$ then gives us

$$\int_0^T |J_1(t)|dt \le C \|\xi\|_{L^2([0,T];W^{m,2}(\Omega))}.$$

For $J_2(t)$ we first note that by the density preserving nature of Φ (see Theorem 7.3.2) we can obtain by a change of variables

$$\left\langle (\rho^{(h)} u^{(h)})(t-h) \circ \Phi_{-h}^{(h)}(t), \xi(t) \right\rangle_\Omega = \left\langle (\rho^{(h)} u^{(h)})(t-h), \xi(t) \circ \Phi_h^{(h)}(t-h) \right\rangle_\Omega,$$

and thus

$$J_2(t) = \left\langle (\rho^{(h)} u^{(h)})(t-h), \frac{\xi(t) - \xi(t) \circ \Phi_h^{(h)}(t-h)}{h} \right\rangle_\Omega \le \left\| (\rho^{(h)} u^{(h)})(t-h) \right\|_\Omega C Lip_y(\xi(t))$$

using Theorem 7.3.6 (3) as well as the uniform L^∞ bounds of $\rho^{(h)}$ and Theorem 7.3.5.

□

7.3.3 Proof of Theorem 7.1.2, step 3: Convergence to the limit

Now we again use our uniform bounds in h to find a converging sub-sequence and a suitable limit functions to the approximating sequences:

Lemma 7.3.8 (Weak compactness). *There exists a subsequence of $h \to 0$ (not relabeled) and limit functions $\eta \in C_w([0,T];W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$, $u \in L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$ such that*

$$\begin{aligned}
\eta^{(h)} &\rightharpoonup \eta && \text{in } C_w([0,T];W^{2,q}(Q;\mathbb{R}^n)) \\
\partial_t \eta^{(h)} &\rightharpoonup \partial_t \eta && \text{in } L^2([0,T];W^{1,2}(Q;\mathbb{R}^n)) \\
u^{(h)} &\rightharpoonup u && \text{in } L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))
\end{aligned}$$

Proof. Using the estimates from Theorem 7.3.3, we know that $E(\eta^{(h)}(t))$ is bounded independently of t and h and that $\int_0^T R(\eta^{(h)}, \partial_t\eta^{(h)})dt$ is also uniformly bounded in h. Thus by Assumptions 5.1.1 and 5.1.4, the sequence $(\eta^{(h)})_h$ is uniformly bounded in $L^\infty([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ which allows us to pick a subsequence and a limit η in the same space such that the first two assertions hold.

Finally we use the global Korn-inequality Theorem 5.2.8 to show that $\int_0^T \|u^{(h)}\|^2_{W^{1,2}(\Omega)} dt$ is uniformly bounded and extract a limit u (after possibly choosing another subsequence) such that the last assertion is true. □

Exactly by the same argument by which we obtained Theorem 6.2.2 and Theorem 6.2.3 we get:

Corollary 7.3.9 (Aubin-Lions & Minty (coupled solid)). *Let $b^{(h)} : t \mapsto \fint_{t-h}^t \partial_t\eta^{(h)} ds$. Then for a subsequence of h's (not relabeled) we have that*

$$b^{(h)} \to \partial_t\eta \text{ in } L^2([0,T]; L^2(Q;\mathbb{R}^n)) \text{ and } \eta^{(h)} \to \eta \text{ in } L^q([0,T]; W^{2,q}(Q;\mathbb{R}^n)).$$

In particular for almost all $t \in [0,T]$ we have

$$DE(\eta^{(h)}(t)) \to DE(\eta(t)) \text{ in } W^{-2,q}(Q;\mathbb{R}^n).$$

We now want to prove a similar result for the Eulerian velocity $u^{(h)}$. While we have an estimate on the time derivative of $\fint_{-h}^0 \rho^{(h)} u^{(h)}(t+s)ds$ in the form of Theorem 7.3.7, this estimate is in a dual space of functions which are divergence free on the fluid domain and thus in a time and h-dependent space. As a consequence, we are no longer in the realm of classic Aubin-Lions type theorems and instead need to prove a similar result directly.

Lemma 7.3.10 (Aubin-Lions (fluid)). *For each $t \in [0,T]$, $h > 0$ we define $\widetilde{m}^{(h)}(t) \in L^2(\Omega;\mathbb{R}^n)$ by*

$$\widetilde{m}^{(h)}(t) := \fint_{-h}^{0} (\rho^{(h)} u^{(h)})(t+s)ds.$$

For all (sufficiently small) $\delta > 0$ there exists a subsequence of h's (not relabeled) such that for all $A \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^{n\times n})$

$$\int_0^T \left\langle (u^{(h)})_\delta, A\widetilde{m}^h \right\rangle_\Omega dt \to \int_0^T \langle (u)_\delta, A\rho u\rangle_\Omega dt,$$

where $(\cdot)_\delta$ is the regularization operator defined in Theorem 5.3.11, as $u^{(h)}$ plays the role of a test function here.

Proof. We begin the proof with a couple of observations. First, as the operator $(\cdot)_\delta$ introduced in Theorem 5.3.11 is bounded and linear, we find that (for a non-relabeled subsequence) $(u^{(h)})_\delta \rightharpoonup (u)_\delta$ with $h \to 0$ in $L^2([0,T]; W^{1,2}(\Omega))$; cf. also Lemma 7.3.8.

Next, as $\widetilde{m}^{(h)}$ is uniformly bounded in $L^\infty([0,T]; L^2(\Omega))$ (see Theorem 7.3.5), we find that (after possibly choosing another subsequence) there exists a $\widetilde{m} \in L^\infty([0,T]; L^2(\Omega))$ such that $\widetilde{m}^{(h)} \rightharpoonup^* \widetilde{m}$ in that space. Since for $\xi \in C_0^\infty([0,T]\times\Omega)$ we have

$$\begin{aligned}\int_0^T \left\langle \widetilde{m}^{(h)}, \xi\right\rangle_\Omega dt &= \fint_{-h}^{0} \int_0^T \left\langle (\rho^{(h)} u^{(h)})(t+s), \xi(t)\right\rangle_\Omega dt\, ds \\ &= \int_0^T \left\langle (\rho^{(h)} u^{(h)})(t), \fint_{-h}^{0} \xi(t-s)\, ds\right\rangle_\Omega dt \to \int_0^T \langle \rho u, \xi\rangle_\Omega dt,\end{aligned}$$

we also know that $\widetilde{m} = \rho u$ almost everywhere.

Take a sequence $(h_i)_i$ with $h_i \to 0$ chosen such that all convergences outlined above, including the one in Theorem 7.3.9, hold true. Next fix $\epsilon > 0$. We aim to show that there is a N_ϵ, such that for another (non-relabled) subsequence and all $j > i > N_\epsilon$

$$|\int_0^T \Big\langle (u^{(h_i)}(t))_\delta, A(\widetilde{m}^{(h_i)}(t) - \widetilde{m}^{(h_j)}(t))\Big\rangle_\Omega \, dt| \leq c\epsilon, \tag{7.3.4}$$

which implies the result. Our strategy is based on the approach introduced in [152, Theorem 5.1]. Thus, we will split the time-interval $[0,T]$ into a finite number of sub-intervals of length σ and depending on ϵ we will first choose the regularizing parameter δ and then the length-parameter σ which will finally yield the sought number N_ϵ. Due to the changing fluid-domain, we first need to assure that $(u^{(h_i)}(t))_\delta$ are all divergence free on a fixed domain in which, for a given t, all $\Omega^{(h_i)}$ are included. For this, we use the uniform convergence of $\eta^{(h_i)} \to \eta$ that allows for any given $\delta > 0$ to take h_i small enough (N_ϵ large enough), such that $\hat{\Omega}_\delta(t) = \bigcap_{i\geq N_\epsilon} \Omega^{(h_i)}(t)$ and $\check{\Omega}_\delta(t) = \bigcup_{i\geq N_\epsilon} \Omega^{(h_i)}(t)$ satisfy a small Hausdorff-distance condition

$$\sup_{t\in[0,T]} \sup_{i\geq N_\epsilon} \Big(\sup_{y\in\Omega^{(h_i)}(t)} dist(y, \hat{\Omega}_\delta(t)) + \sup_{y\in\check{\Omega}_\delta(t)} dist(y, \Omega^{(h_i)}(t)) \Big) \leq \delta. \tag{7.3.5}$$

Next we may use the approximation introduced in Theorem 5.3.11 for $u^{(h_i)}$. The regularity of the domain allows to assume that

$$(\mathrm{div}((u^{(h_i)})_\delta(t)) = 0 \text{ in } \check{\Omega}_\delta(t).$$

Moreover Theorem 5.3.11 implies that for almost every t and every $m \in \mathbb{N}$

$$\begin{aligned} \left\|(u^{(h_i)}(t))_\delta\right\|_{W^{m,2}(\Omega)} &\leq c(\delta, m) \left\|u^{(h_i)}(t)\right\|_{W^{1,2}(\Omega)} \\ \left\|(u^{(h_i)}(t))_\delta\right\|_{L^2([0,T];W^{1,2}(\Omega))} &\leq c \left\|u^{(h_i)}(t)\right\|_{L^2([0,T];W^{1,2}(\Omega))} \\ \left\|(u^{(h_i)}(t))_\delta - u^{(h_i)}(t)\right\|_{L^2([0,T];L^2(\Omega))} &\leq c\delta^{\frac{2}{2+n}} \left\|u^{(h_i)}(t)\right\|_{L^2([0,T];W^{1,2}(\Omega))}. \end{aligned} \tag{7.3.6}$$

The parameter δ will be chosen later depending on ϵ. Furthermore we choose (in dependence of δ) a $\sigma > 0$, $N \in \mathbb{N}$ such that $T = N\sigma$.

Now for any $k \in \{0, ..., N\}$

$$\left\|\widetilde{m}^{(h_i)}(\sigma k)\right\|_\Omega^2 \leq \fint_{k\sigma-h}^{k\sigma} \left\|(\rho^{(h)} u^{(h)})(t)\right\|_\Omega^2 dt \leq C \left\|\rho^{(h)}\right\|_\infty \leq C\rho_{\max}$$

by the volume density-preserving nature of Φ and by Theorem 7.3.3. Here, $\rho_{\max}$ is a uniform upper bound on the density in the fluid and the solid, the latter of which can be easily derived from the energy bounds. As usual we continue v and $\partial_t \eta$ to negative times by their initial data. We can thus use compact embeddings to find a sub-sequence of $h_i \to 0$ such that $\widetilde{m}^{(h_i)}(\sigma k)$ converges strongly in $\left(W^{1,2}(\Omega; \mathbb{R}^n) \cap \{\mathrm{div} v|_{\check{\Omega}_\delta(t)} = 0\}\right)^*$ for all $k \in \{0, ..., N-1\}$. In particular we can choose the N_ϵ in such a way that for all $i, j \geq N_\epsilon$

$$\left\|\widetilde{m}^{(h_i)}(\sigma k) - \widetilde{m}^{(h_j)}(\sigma k)\right\|_{\left(W^{1,2}(\Omega;\mathbb{R}^n)\cap\{\mathrm{div} v|_{\check{\Omega}_\delta(t)}=0\}\right)^*} \leq \epsilon. \tag{7.3.7}$$

Now we rewrite for $t \in [\sigma k, \sigma(k+1))$

$$\begin{aligned} &\Big\langle (u^{(h_i)}(t))_\delta, A(\widetilde{m}^{(h_i)}(t) - \widetilde{m}^{(h_j)}(t))\Big\rangle_\Omega \\ &= \Big\langle (u^{(h_i)}(t))_\delta, A\widetilde{m}^{(h_i)}(t) - A\widetilde{m}^{(h_i)}(\sigma k)\Big\rangle_\Omega + \Big\langle (u^{(h_i)}(t))_\delta, A(\widetilde{m}^{(h_i)}(\sigma k) - \widetilde{m}^{(h_j)}(\sigma k))\Big\rangle_\Omega \\ &\quad + \Big\langle (u^{(h_i)}(t))_\delta, A\widetilde{m}^{(h_j)}(\sigma k) - A\widetilde{m}^{(h_j)}(t)\Big\rangle_\Omega =: \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t). \end{aligned}$$

For $i,j \geq N_\epsilon$ we find using (7.3.7) as well as (7.3.6)

$$\int_0^T \mathrm{II}(t)\,dt \leq C\epsilon.$$

The other two terms are estimated using the continuity in time of $\widetilde{m}^{(h_i)}$. Indeed we find that

$$\begin{aligned}\partial_\theta \widetilde{m}^{(h_i)}(\theta,y) &= \partial_\theta \left(\fint_{-h_i}^{0} (\rho^{(h_i)}u^{(h_i)})(\theta+s)ds \right)\\ &= \frac{1}{h_i}\partial_\theta\left(\int_{\theta-h_i}^{\theta} (\rho^{(h_i)}u^{(h_i)})(s)ds\right) = \frac{(\rho^{(h_i)}u^{(h_i)})(\theta) - (\rho^{(h_i)}u^{(h_i)})(\theta-h_i)}{h_i}\end{aligned}$$

and thus

$$\begin{aligned}|\int_0^T \mathrm{I}(t)\,dt| &= |\int_0^T \left\langle (u^{(h_i)}(t))_\delta, \int_{\sigma k}^t A\partial_\theta \widetilde{m}^{(h_i)}(\theta)d\theta \right\rangle_\Omega dt|\\ &\leq \sum_k \int_{\sigma k}^{(\sigma+1)k} \int_{\sigma k}^t |\left\langle (u^{(h_i)}(t))_\delta, A\frac{(\rho^{(h_i)}u^{(h_i)})(\theta) - (\rho^{(h_i)}u^{(h_i)})(\theta-h_i)}{h_i}\right\rangle_\Omega |d\theta dt\\ &= \sum_k \int_{\sigma k}^{\sigma(k+1)} \int_{\theta}^{\sigma(k+1)} |\left\langle (u^{(h_i)}(t))_\delta, A\frac{(\rho^{(h_i)}u^{(h_i)})(\theta) - (\rho^{(h_i)}u^{(h_i)})(\theta-h_i)}{h_i}\right\rangle_\Omega |dt\,d\theta\\ &\leq \sum_k \int_{\sigma k}^{\sigma(k+1)} \int_0^{\sigma} |\left\langle (u^{(h_i)}(\theta+s))_\delta, A\frac{(\rho^{(h_i)}u^{(h_i)})(\theta) - (\rho^{(h_i)}u^{(h_i)})(\theta-h_i)}{h_i}\right\rangle_\Omega |\,ds\,d\theta\\ &\leq \|A\|_\infty \int_0^\sigma \int_0^T |\left\langle (u^{(h_i)}(\theta+s))_\delta, \frac{(\rho^{(h_i)}u^{(h_i)})(\theta) - (\rho^{(h_i)}u^{(h_i)})(\theta-h_i)}{h_i}\right\rangle_\Omega |\,dt\,ds\\ &\leq \|A\|_\infty \int_0^\sigma \left\|(u^{(h_i)}(\cdot+s))_\delta\right\|_{L^2([0,T];W^{m,2}(\Omega))} ds \leq \|A\|_\infty C_\delta \sigma \left\|u^{(h_i)}\right\|_{L^2([0,T];W^{1,2}(\Omega))}\end{aligned}$$

using Theorem 7.3.7.

Using an analogous estimate on $\mathrm{III}(t)$, we find (7.3.4) by choosing σ small enough. □

Observe that due to the strong convergence of $\partial_t\eta^{(h)}$ (and consequently u on $\eta^h(t,Q)$) we find

$$\int_0^T \left\langle (u^{(h)})_\delta, A\widetilde{m}^h \right\rangle_{\Omega^h(t)} dt \to \rho_f \int_0^T \langle (u)_\delta, Au\rangle_{\Omega(t)}\,dt, \tag{7.3.8}$$

for all $A \in C_0^\infty(\Omega)$.

7.3.4 Proof of Theorem 7.1.2, Step 3a: Passing to the limit with the coupled PDEs

In the following we assume that T is small enough, such that a sequence of approximate solutions $(\eta^{(h)}, u^{(h)})$ exist on the interval $[0, T+h]$. Later it will be discussed how to prolong the solution up to the point of contact.

As before in the proof of Theorem 5.3.9 and Theorem 7.2.2, we use Theorem 5.3.11 to restrict ourselves to test functions $\xi \in C_0^\infty(\Omega;\mathbb{R}^n)$ with $\mathrm{div}\xi = 0$ in a neighborhood of $\Omega(t)$. We then construct $\phi^{(h)} := \xi \circ \eta^{(h)}$ and pass to the limit $h \to 0$. We proceed as in the proof of Theorem 6.0.2 and transfer the difference quotient to the test function to get

$$\begin{aligned}&\int_0^T \left\langle \frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h}, \phi^{(h)}(t)\right\rangle dt\\ &= -\int_0^T \left\langle \partial_t\eta^{(h)}(t), \frac{\phi^{(h)}(t+h) - \phi^{(h)}(t)}{h}\right\rangle dt \to -\int_0^T \langle \partial_t\eta(t), \partial_t\phi(t)\rangle\,dt.\end{aligned}$$

By the same arguments as in the last sections we find

$$\int_0^T \left\langle DE_h(\eta^{(h)}), \phi^{(h)} \right\rangle + \left\langle DR_h(\eta^{(h)}, \partial_t \eta^{(h)}), \phi^{(h)} \right\rangle dt$$
$$\to \int_0^T \langle DE(\eta), \phi \rangle + \langle DR(\eta, \partial_t \eta), \phi \rangle \, dt,$$

as well as

$$\int_0^T \rho_s \left\langle f \circ \eta^{(h)}, \phi^{(h)} \right\rangle_Q + \rho_f \langle f, \xi \rangle_{\Omega^{(h)}(t)} \, dt \to \int_0^T \rho_s \langle f \circ \eta, \phi \rangle_Q + \langle f, \xi \rangle_{\Omega(t)} \, dt,$$

and

$$\int_0^T \nu \left\langle \varepsilon u^{(h)}, \varepsilon \xi \right\rangle_{\Omega^{(h)}(t)} dt \to \int_0^T \nu \langle \varepsilon u, \varepsilon \xi \rangle_{\Omega(t)} \, dt.$$

What is left are the inertial term of the fluid. Again we transfer the difference quotient to the test function. For that we have to take into account the flow map $\Phi_h^{(h)}$.

$$\int_0^T \left\langle \frac{u^{(h)}(t) \circ \Phi_h^{(h)}(t-h) - u^{(h)}(t-h)}{h}, \xi(t) \circ \Phi_h^{(h)}(t-h) \right\rangle_{\Omega^{(h)}(t-h)} dt$$
$$= -\int_0^T \left\langle u^{(h)}(t), \frac{\xi(t+h) \circ \Phi_h^{(h)}(t) - \xi(t)}{h} \right\rangle_{\Omega^{(h)}(t)} dt$$
$$= -\int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{\xi(t+h) \circ \Phi_h^{(h)}(t) - \xi(t)}{h} \right\rangle_{\Omega^{(h)}(t)} dt$$
$$+ \int_0^T \left\langle (u^{(h)}(t))_\delta - u^{(h)}(t), \frac{\xi(t+h) \circ \Phi_h^{(h)}(t) - \xi(t)}{h} \right\rangle_{\Omega^{(h)}(t)} dt =: -\mathrm{I}^{\delta,h} + \mathrm{II}^{\delta,h}$$

where $(u^{(h)}(t))_\delta$ is a regularization in space, as defined in Theorem 5.3.11. Since the right-hand side in the scalar product of $\mathrm{II}^{\delta,h}$ is uniformly bounded in $L^\infty([0,T]; L^2(\Omega(t); \mathbb{R}^n))$, using (7.3.6) we know that $\mathrm{II}^{\delta,h}$ vanishes as $\delta \to 0$ (uniformly in h). For the first term we expand

$$\mathrm{I}^{\delta,h} = \int_0^T \left\langle (u^{(h)}(t))_\delta, \fint_0^h \partial_s \left(\xi(t+s) \circ \Phi_s^{(h)}(t) \right) ds \right\rangle_{\Omega^{(h)}(t)} dt$$
$$= \int_0^T \left\langle (u^{(h)}(t))_\delta, \fint_0^h \left(\partial_t \xi(t+s) - u^{(h)}(t+s) \cdot \nabla \xi(t+s) \right) \circ \Phi_s^{(h)}(t) ds \right\rangle_{\Omega^{(h)}(t)} dt$$
$$= \int_0^T \fint_0^h \left\langle (u^{(h)}(t))_\delta \circ \Phi_{-s}^{(h)}(t+s), \partial_t \xi(t+s) - u^{(h)}(t+s) \cdot \nabla \xi(t+s) \right\rangle_{\Omega^{(h)}(t+s)} ds \, dt$$
$$= \int_0^T \fint_0^h \left\langle (u^{(h)}(t))_\delta, \partial_t \xi(t+s) - u^{(h)}(t+s) \cdot \nabla \xi(t) \right\rangle_{\Omega^{(h)}(t+s)} ds \, dt$$
$$+ \int_0^T \fint_0^h \left\langle (u^{(h)}(t))_\delta, u^{(h)}(t+s) \cdot \nabla (\xi(t) - \xi(t+s)) \right\rangle_{\Omega^{(h)}(t+s)} ds \, dt$$
$$+ \int_0^T \fint_0^h \left\langle (u^{(h)}(t))_\delta \circ \Phi_{-s}^{(h)}(t+s) - (u^{(h)})_\delta(t), \partial_t \xi(t+s) - u^{(h)}(t+s) \cdot \nabla \xi(t+s) \right\rangle_{\Omega^{(h)}(t+s)} ds \, dt.$$

Since $\|\nabla(\xi(t)-\xi(t+s))\|_{L^\infty(\Omega)} \le Ch\|\partial_t\nabla\xi\|_{L^\infty([0,T]\times\Omega)}$ the second term in the last sum converges to zero as $h\to 0$. For the third term in the same sum, we may use Theorem 7.3.6 to see that the L^2-norm of the left-hand side in the scalar product is bounded by $ChLip_y((u^{(h)}(t))_\delta)$ which is in turn bounded by $hC_\delta\left\|u^{(h)}(t)\right\|_{W^{1,2}(\Omega)}$ so that this term vanishes for $h\to 0$. For the first term we aim to apply Theorem 7.3.10. To do so, we take $A_\delta \in C^0([0,T];C_0^\infty(\hat{\Omega}_\delta))$, such that $A_\delta(t)\to\chi_{\Omega(t)}$ almost everywhere in Ω. Hence we find by Theorem 7.3.10 in the form of (7.3.8) that

$$\begin{aligned}\lim_{h\to0}\mathrm{I}^{\delta,h} &= \lim_{h\to 0}\int_0^T \fint_0^h \left\langle (u^{(h)}(t))_\delta, \partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t)\right\rangle_{\Omega^{(h)}(t+s)} ds\,dt \\ &= \int_0^T \langle (u(t))_\delta, \partial_t\xi - u\cdot\nabla\xi A_\delta(t)\rangle_{\Omega(t)}\,dt \\ &\quad + \lim_{h\to0}\int_0^T\fint_0^h \left\langle (u^{(h)}(t))_\delta, u^{(h)}(t+s)\cdot\nabla\xi(t)(A_\delta(t)-\chi_{\Omega^{(h)}}(t+s))\right\rangle_\Omega ds\,dt.\end{aligned}$$

The last term is estimated by Hölder's inequality and Sobolev embedding. Indeed, for $a<\frac{n}{n-2}$ we find by (7.3.6)

$$\begin{aligned}&\left|\int_0^T\fint_0^h \left\langle (u^{(h)}(t))_\delta, u^{(h)}(t+s)\cdot\nabla\xi(t)(A_\delta(t)-\chi_{\Omega^{(h)}}(t+s))\right\rangle_\Omega ds\,dt\right| \\ &\le \int_0^T \left\|(u^{(h)}(t))_\delta\right\|_{L^{2a}(\Omega)} \fint_0^h \left\|u^{(h)}(t+s)\right\|_{L^a(\Omega)} \|(A_\delta(t)-\chi_{\Omega^{(h)}}(t+s))\|_{L^{2a'}(\Omega)}\,ds\,dt \\ &\le c\left\|(u^{(h)}(t))_\delta\right\|_{L^2([0,T];W^{1,2}(\Omega))}\sup_{t\in T}\left(\fint_0^h \left\|u^{(h)}(t+s)\right\|^2 ds\right)^{\frac12} \\ &\quad\times\left(\fint_0^h \|(A_\delta(t)-\chi_{\Omega^{(h)}}(t+s))\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{\frac12} \\ &\le c\left(\fint_0^h \|(A_\delta(\cdot)-\chi_{\Omega^{(h)}}(\cdot+s))\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{\frac12}.\end{aligned}$$

By the uniform convergence of $\eta^{(h)}\to\eta$, we find that

$$\lim_{h\to0}\left(\fint_0^h \|(A_\delta(\cdot)-\chi_{\Omega^{(h)}}(\cdot+s))\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{\frac12} = \|(A_\delta-\chi_\Omega))\|_{L^2([0,T];L^{2a'}(\Omega))}.$$

Finally, by passing to the limit $\delta\to0$ we have that

$$\lim_{\delta\to0}\lim_{h\to0}(-\mathrm{I}^{\delta,h}+\mathrm{II}^{\delta,h}) = -\int_0^T\langle u,\partial_t\xi - u\cdot\nabla\xi\rangle_{\Omega(t)}\,dt.$$

Thus we have shown that we obtain the right equation in the limit:

$$\begin{aligned}&\int_0^T -\rho_s\langle\partial_t\eta,\partial_t\phi\rangle_Q - \rho_s\langle v,\partial_t\xi - v\cdot\nabla\xi\rangle_{\Omega(t)} + \langle DE(\eta),\phi\rangle + \langle D_2R(\eta,\partial_t\eta),\phi\rangle + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt \\ &= \int_0^T \rho_s\langle f\circ\eta,\phi\rangle_Q + \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt - \rho_s\langle\eta_*,\phi(0)\rangle_Q - \rho_f\langle v_0,\xi(0)\rangle_{\Omega_0}.\end{aligned} \tag{7.3.9}$$

7.3.5 Proof of Theorem 7.1.2, Step 3b: Reconstruction of the pressure

As we do not want to consider the time-derivatives of the operator $\mathcal{B}_t$ we cannot go along the same lines as in the proof of Theorem 5.1.6. Instead we have to proceed in a global manner. We construct the pressure as a distribution.

Let $\psi \in C_0^\infty([0,T] \times \Omega)$. Take $\mathcal{B}$ to be the operator from Theorem 5.3.10 with respect to the domain Ω. To apply this operator to ψ, we need to normalize its mean by picking a $\tilde{\psi} \in C_0^\infty([0,T] \times \Omega)$ with $supp(\tilde{\psi}(t)) \cap \Omega(t) = \varnothing$ and $\int_\Omega \tilde{\psi}(t)dy = -\int_\Omega \psi(t)dy$ for all $t \in [0,T]$.

Now let $\xi(t) := \mathcal{B}(\psi(t) + \tilde{\psi}(t))$, $\phi(t,x) := \xi(t,\eta(t,x))$ and define a linear operator by

$$\begin{aligned} P(\psi) := \int_0^T & \langle DE(\eta(t)), \phi\rangle + \langle D_2R(\eta(t), \partial_t\eta(t)), \phi\rangle + \nu \langle \varepsilon u, \varepsilon\xi\rangle_{\Omega(t)} \\ & - \rho_f \langle f, \xi\rangle_{\Omega(t)} - \rho_s \langle f \circ \eta, \phi\rangle_Q - \rho_s \langle \partial_t\eta, \partial_t\phi\rangle_Q - \rho_f \langle u, \partial_t\xi - u\cdot\nabla\xi\rangle_{\Omega(t)}\, dt. \end{aligned}$$

Note that $P(\psi)$ is independent of the choice of $\tilde{\psi}$: Assume that $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are two such choices with corresponding ξ_1 and ξ_2. Then $\xi_1 - \xi_2 = \mathcal{B}(\tilde{\psi}_1 - \tilde{\psi}_2)$ has divergence 0 on $\Omega(t)$ and thus the above integral is the same because of (7.3.9). In particular if $supp(\psi(t)) \subset \eta(t,Q)$ (for all $t \in [0,T]$), we may choose $\tilde{\psi} \equiv \psi$ which implies (by the linearity of $\mathcal{B}$) that $P(\psi) = 0$. Hence $supp(P) \subset [0,T] \times \overline{\Omega(t)}$.

Furthermore it can be estimated that

$$\begin{aligned} &\int_0^T \langle DE(\eta(t)), \phi\rangle + \langle D_2R(\eta(t), \partial_t\eta(t)), \phi\rangle + \nu \langle \varepsilon u, \varepsilon\xi\rangle_{\Omega(t)} - \rho_f \langle f, \xi\rangle_{\Omega(t)} - \rho_s \langle f\circ\eta, \phi\rangle_Q\, dt \\ &\le T \sup_{t\in[0,T]} \|DE(\eta(t))\|_{W^{-2,q}(Q)} \|\phi(t)\|_{L^1([0,T],W^{2,q}(Q))} \\ &+ \int_0^T \|D_2R(\eta(t), \partial_t\eta(t))\|_{W^{-1,2}(Q)} \|\phi\|_{W^{1,2}(Q)} + \|\varepsilon u\|_{\Omega(t)} \|\varepsilon\xi\|_{\Omega(t)} + c\|f\|_\infty (\|\phi\|_Q + \|\xi\|_{\Omega(t)})dt \\ &\le C\|\phi\|_{L^1([0,T],W^{2,q}(Q))} + \|\xi\|_{L^2([0,T];W^{1,2}(\Omega))} \end{aligned}$$

via the known bounds on the terms in the weak equation. Finally using Theorem 5.5.4 we know that $\|\phi\|_{W^{2,q}(Q)} \le C\|\xi\|_{W^{2,q}(\Omega)}$. Consequently by the properties of the Bogovskiĭ-operator we find

$$\begin{aligned} C\|\phi\|_{L^1([0,T],W^{2,q}(Q))} + \|\xi\|_{L^2([0,T];W^{1,2}(\Omega))} &\le C\left\|\psi + \tilde{\psi}\right\|_{L^1([0,T],W^{2,q}(Q))} + C\left\|\psi + \tilde{\psi}\right\|_{L^2([0,T];L^2(\Omega))} \\ &\le C\|\psi\|_{L^1([0,T],W^{2,q}(Q))} + C\|\psi\|_{L^2([0,T];L^2(\Omega))} \end{aligned}$$

where for the last inequality we note that $\tilde{\psi}(t)$ can be chosen as a multiple of a fixed C_0^∞-function and thus its norm only needs to depend on $|\int_\Omega \psi(t)dy| \le c\|\psi(t)\|_\Omega$. Additionally for the other remaining terms we have

$$\begin{aligned} |\int_0^T \langle \partial_t\eta, \partial_t\phi\rangle_Q| &\le \|\partial_t\eta\|_{L^2([0,T]\times Q)} \|\phi\|_{W^{1,2}([0,T];L^2(Q))} \\ |\int_0^T \langle u, \partial_t\xi\rangle_{\Omega(t)}| &\le \|u\|_{L^2([0,T]\times\Omega)} \|\xi\|_{W^{1,2}([0,T];L^2(\Omega))} \\ |\int_0^T \langle u, u\cdot\nabla\xi\rangle_{\Omega(t)}| &\le \left\|u^2\right\|_{L^a([0,T];L^b(\Omega))} \|\xi\|_{L^{a'}([0,T];W^{1,b'}(\Omega))} \end{aligned}$$

where $a, b \in (1,\infty)$ are chosen in such a way that $|u|^2 \in L^a([0,T], L^b(\Omega))$, which is possible since $|u|^2 \in L^\infty([0,T], L^1(\Omega)) \cap L^1([0,T], L^p(\Omega))$ (with $p = \frac{n}{n-2}$ for $n > 2$ or p arbitrarily large for $n = 2$). Now bounding the norms of ξ and ϕ in terms of ψ as before proves that $P \in \mathcal{D}'([0,T] \times \Omega)$. Thus p is well defined via that operator and expanding

$$\int_0^T \langle \nabla p, \xi\rangle\, dt = P(\mathrm{div}\xi)$$

proves that it fulfills the right equations for $\xi \in C^\infty([0,T] \times \Omega)$. Moreover, it can be decomposed into

$$p \in L^\infty([0,T], W^{-1,q}(\Omega)) + L^2([0,T] \times \Omega) + W^{-1,2}([0,T], W^{-1,2}(\Omega)) \cap L^{a'}([0,T], W^{-1,b'}(\Omega)). \tag{7.3.10}$$

7.3.6 Proof of Theorem 7.1.2, Step 4: Energy inequality & maximal interval of existence

Above, we have shown existence of coupled weak solutions u, η on $[0, T]$ for some $T > 0$. As before we can now pick a maximal interval $[0, T_{\max})$ and use the energy bounds to conclude that either $T_{\max} = \infty$ or there exists a limit $\eta(T_{\max}) \in \partial\mathcal{E}$.

Finally, we observe that (7.1.8) follows by Theorem 7.3.3.

Remark 7.3.11 (On Lagrangian and Eulerian formulation)**.** *In the preceding proofs, we switched between the Lagrangian and the Eulerian point of view several times. While in the end, for the final equation we have to treat the solid as Lagrangian and fluid as Eulerian, we are free to change from one to the other as long as $h > 0$ during the proof.*

We used this prominently in defining the global Eulerian velocity u, as doing so made it easier to talk about convergence. But it should be noted that the same can be equally done in reverse. In the proof of the time-delayed problem Theorem 7.2.2, every time we used v, we could have similarly considered $\partial_t \Phi$ or its difference quotient respectively. In this way, the whole proof could be rewritten in terms of Φ, eliminating the need for v and u completely.

As long as $h > 0$ all the above considerations hold. It is only at the very last moment, where we take the limit $h \to 0$ that we are no longer guaranteed existence of a flow map Φ and have to introduce an Eulerian velocity to conclude the proof.

Chapter 8

Contactless rebound of elastic bodies in viscous incompressible fluids

8.1 Introduction

The chapter is organized as follows. In Section 8.2 we start by introducing the fluid-structure interaction model in the classical Eulerian-Lagrangian framework, which we then reformulate fully in the Eulerian setting, as needed for our numerical experiments. This is followed by the introduction of our reduced model of ODEs. The section is closed by the derivation of drag-formulas for the family of deformations that we consider for our numerical experiments and which form the model case for the analysis. Section 8.3 is dedicated to the main mathematical results of this paper and their proofs. In Subsection 8.3.1 we introduce our general assumptions and state the main theorems. In particular, we provide conditions that allow to prove or disprove rebound in the vanishing viscosity limit. Subsection 8.3.2 is dedicated to the proofs of these results. In Section 8.4, we first provide numerical experiments for the reduced model of ODEs. In the following subsection we introduce the numerical set up that allows to capture the bouncing behavior of elastic solids for small viscosities and provide some numerical experiments. We conclude the section with the comparison from a numerical standpoint of the ODE and PDE solutions (see Figure 8.8). Finally, in Section 8.5 we summarize and discuss our results.

8.2 Modeling of particle-wall approach and rebound in viscous fluids

In this section, we first recall the standard Eulerian-Lagrangian formulation of the fluid-structure interaction problem, which we then reformulate in a purely Eulerian setting. This is followed by presenting a class of reduced ODE models for the FSI problem.

8.2.1 The viscous fluid – elastic structure Eulerian-Lagrangian formulation

Consider an incompressible Newtonian fluid filling the region $\mathcal{F}(t)$, which surrounds an elastic particle whose position, at time t, will be denoted by $\mathcal{B}(t)$. We assume that the system composed by the fluid and the solid body occupies the region $\Omega = \mathcal{F}(t) \cup \overline{\mathcal{B}(t)}$, where Ω is an open subset of $\mathbb{R}^N$, $N = 2$ or $N = 3$. As it is customary in fluid mechanics, the equations expressing the conservation of mass and the balance of linear momentum for the fluid are given in the Eulerian reference frame and read as follows:

$$\begin{aligned} \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} &= 0, \\ \rho_f \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \right) &= \operatorname{div}_{\boldsymbol{x}} \sigma_f + \rho_f \boldsymbol{b}, \end{aligned} \qquad \text{in } \mathcal{F}(t), \tag{8.2.1}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity, ρ_f is the constant fluid density, and $\boldsymbol{b}(\boldsymbol{x}, t)$ represents (Eulerian) external bulk forces. Here the variable $\boldsymbol{x}$ denotes a position in the current (Eulerian) configuration, that is, $\boldsymbol{x} \in \mathcal{F}(t)$.

We recall that for Newtonian fluids the Cauchy stress tensor $\sigma_f(\boldsymbol{x},t)$ takes the form

$$\sigma_f = -p_f \mathbb{I}_N + 2\mu_f \mathbb{D}(\boldsymbol{v}), \tag{8.2.2}$$

where $p_f(\boldsymbol{x},t)$ denotes the fluid pressure, $\mathbb{I}_N$ is the N-dimensional identity matrix, μ_f is the constant dynamic viscosity, and $\mathbb{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla_{\boldsymbol{x}}\boldsymbol{v} + (\nabla_{\boldsymbol{x}}\boldsymbol{v})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{v}$. On the other hand, the balance equations for the elastic solid are given in the Lagrangian setting and can be written as

$$\begin{aligned} \rho_s \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} &= \operatorname{div}_{\boldsymbol{X}} \mathbb{P} + \rho_s \boldsymbol{B}, \\ J\rho_s &= \rho_s^0 \end{aligned} \qquad \text{in } \mathcal{B}_0, \tag{8.2.3}$$

where ρ_s and ρ_s^0 denote the density of the elastic solid at time t and in the reference configuration, respectively, $\boldsymbol{\eta}(\boldsymbol{X},t)$ is the displacement, $\mathbb{P}(\boldsymbol{X},t)$ is the first Piola–Kirchhoff stress, $\boldsymbol{B}(\boldsymbol{X},t)$ are the (Lagrangian) external forces acting on the solid, and $\mathcal{B}_0$ is the reference configuration of the solid. The variable $\boldsymbol{X}$ denotes a position in the reference (Lagrangian) configuration, that is, $\boldsymbol{X} \in \mathcal{B}_0$.

We assume that the structure is an incompressible hyperelastic solid, i.e.,

$$\mathbb{P} = \frac{\partial \mathcal{L}}{\partial \mathbb{F}}, \qquad \text{where} \qquad \mathcal{L}(\mathcal{F}, P) := \mathcal{W}(\mathcal{F}) - P(J-1).$$

Here $\mathcal{L}$ is the Lagrange function corresponding to the strain energy function $\mathcal{W}$ under the incompressibility restriction $J = 1$ and P is the associated Lagrange multiplier. The symbol $\mathbb{F}$ denotes the deformation gradient, i.e., if we define the deformation mapping $\boldsymbol{y}(\boldsymbol{X},t) := \boldsymbol{X} + \boldsymbol{\eta}(\boldsymbol{X},t)$, then

$$\mathcal{F}(\boldsymbol{X},t) := \nabla_{\boldsymbol{X}} \boldsymbol{y}(\boldsymbol{X},t) = \mathbb{I}_N + \nabla_{\boldsymbol{X}} \boldsymbol{\eta}(\boldsymbol{X},t).$$

Finally, J denotes the deformation gradient Jacobian, i.e. $J := \det \mathbb{F}$. Let us also mention here that throughout the following we assume that the deformation mapping $\boldsymbol{y}(\cdot,t)\colon \mathcal{B}_0 \to \mathcal{B}(t)$ is a sufficiently smooth bijection for all t.

The choice of the elastic material is determined by specifying the strain energy function $\mathcal{W}$. As a particular example used later in the numerical computations (see Section 8.4), we consider an incompressible neo-Hookean solid, for which the elastic strain energy is given by

$$\mathcal{W} := \frac{G_s}{2}(|\mathcal{F}|^2 - N), \tag{8.2.4}$$

where the constant G_s denotes the shear modulus. The Cauchy stress in the solid can be expressed in terms of the first Piola–Kirchhoff stress $\mathbb{P}$ as follows:

$$\sigma_s(\boldsymbol{x},t) = \frac{1}{J(\boldsymbol{X},t)} \mathbb{P}(\boldsymbol{X},t)\mathcal{F}^{\mathrm{T}}(\boldsymbol{X},t)\Big|_{\boldsymbol{X}=\boldsymbol{y}^{-1}(\boldsymbol{x},t)}.$$

As one can readily check, for the strain energy (8.2.4) the Cauchy stress in the solid takes the form

$$\sigma_s(\boldsymbol{x},t) = -\tilde{p}(\boldsymbol{x},t)\mathbb{I}_N + G_s \mathbf{B}(\boldsymbol{x},t) = -p(\boldsymbol{x},t)\mathbb{I}_N + G_s \mathbf{B}^d(\boldsymbol{x},t), \tag{8.2.5}$$

where $\tilde{p}(\boldsymbol{x},t) := P(\boldsymbol{y}^{-1}(\boldsymbol{x},t),t)$, $\mathbf{B}(\boldsymbol{x},t)$ is the left Cauchy–Green tensor, which is classically defined via

$$\mathbf{B}(\boldsymbol{x},t) := \mathcal{F}(\boldsymbol{X},t)\mathcal{F}^{\mathrm{T}}(\boldsymbol{X},t)\big|_{\boldsymbol{X}=\boldsymbol{y}^{-1}(\boldsymbol{x},t)}, \tag{8.2.6}$$

the symbol $\mathbf{B}^d := \mathbf{B} - (1/N)(tr\mathbf{B})\mathbb{I}_N$ denotes its deviatoric part, and finally $p := \tilde{p} - (1/N)tr\mathbf{B}$.

The conditions describing the interaction between the fluid and the solid comprise the continuity of the velocities and of the tractions:

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &= \frac{\partial \boldsymbol{\eta}(\boldsymbol{X},t)}{\partial t}\Big|_{\boldsymbol{X}=\boldsymbol{y}^{-1}(\boldsymbol{x},t)}, \\ \sigma_f \boldsymbol{n} &= \sigma_s \boldsymbol{n}, \end{aligned} \qquad \text{on } \partial\mathcal{B}(t), \tag{8.2.7}$$

where $\boldsymbol{n}$ is the unit normal to the fluid-solid interface. Finally, we prescribe no-slip boundary conditions on the boundary of the cavity, that is,

$$\boldsymbol{v} = \mathbf{0} \qquad \text{on } \partial\Omega.$$

Note that the above system is a well posed problem for the unknowns $\boldsymbol{v}, p$ describing the fluid in their variable-in-time Eulerian domain of definition and the deformation η and the pressure of the solid P given in their steady Lagrangian coordinates. Moreover, it admits a formal energy equality (derived for the reader's convenience in Subsection 8.6.2) of the following form:

$$\begin{aligned}\mathcal{K}(t) + \int_{\mathcal{B}_0} \mathcal{W}(\boldsymbol{X},t)\,d\boldsymbol{X} + \int_0^t \int_{\mathcal{F}(s)} 2\mu_f |\mathbb{D}(\boldsymbol{v}(\boldsymbol{x},s))|^2 d\boldsymbol{x}ds = \mathcal{K}(0) + \int_{\mathcal{B}_0} \mathcal{W}(\boldsymbol{X},0)\,d\boldsymbol{X} \\ + \int_0^t \int_{\mathcal{F}(s)} \rho_f \boldsymbol{b}(\boldsymbol{x},s)\cdot\boldsymbol{v}(\boldsymbol{x},s)\,d\boldsymbol{x}ds + \int_0^t \int_{\mathcal{B}_0} \rho_0^s \boldsymbol{B}(\boldsymbol{X},s)\cdot\frac{\partial\boldsymbol{\eta}}{\partial t}(\boldsymbol{X},s)\,d\boldsymbol{X}ds,\end{aligned} \tag{8.2.8}$$

where $\mathcal{K}(t)$ denotes the kinetic energy of the system at time t, that is

$$\mathcal{K}(t) := \int_{\mathcal{F}(t)} \frac{\rho_f}{2}|\boldsymbol{v}(\boldsymbol{x},t)|^2\,d\boldsymbol{x} + \int_{\mathcal{B}_0} \frac{\rho_0^s}{2}\left|\frac{\partial\boldsymbol{\eta}}{\partial t}(\boldsymbol{X},t)\right|^2\,d\boldsymbol{X}.$$

Let us also mention that under the assumption of small deformations, the elastic contribution to the energy given by (8.2.4) can be approximated as follows:

$$\int_{\mathcal{B}_0} \mathcal{W}(\boldsymbol{X},t)\,d\boldsymbol{X} \approx \int_{\mathcal{B}_0} \frac{G_s}{2}|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}(\boldsymbol{X},t)|^2\,d\boldsymbol{X}.$$

This approximation is also formally derived in Subsection 8.6.2.

Remark 8.2.1. *For simplicity, in this paper we restrict our attention to the case of a homogenous solid interacting with a homogenous fluid, that is, both ρ_f and ρ_0^s are assumed to be constant. However, the governing system of equations can be readily generalized to include the situation in which ρ_f (transformed into its Lagrangian counterpart) and ρ_0^s depend only on the space variable $\boldsymbol{X}$. Notice that if one defines the (Eulerian) global density ρ, these conditions correspond to requiring that the material parameter ρ is advected with velocity $\boldsymbol{v}$, that is, $\dot{\rho} = 0$. Similar considerations apply also to the fluid viscosity μ_f and to the shear modulus G_s. It is worth noting that also in this case one can derive a formal energy equality. However, while the theory of bouncing without contact introduced here relates well to compressible solid materials (see Subsection 8.6.1) considering compressible fluids might have a more dramatic impact, as topological contact is expected to happen in this case [58].*

In the next subsection we reformulate the mixed Lagrangian-Eulerian problem fully in the Eulerian frame. This allows for an efficient numerical implementation by finite element methods using a level-set function approach as described in Section 8.4 (see also [131] and the references therein).

8.2.2 The viscous fluid – elastic structure purely Eulerian formulation

The standard form of the fluid-structure interaction problem, as given in Section 8.2, consists of two sets of equations — one for the fluid and one for the solid — which are formulated in different configurations. While the fluid component is described in the physical Eulerian configuration, the equations for the solid are formulated in the reference (Lagrangian) configuration.

For our purposes, we find it more convenient to solve the whole problem in the Eulerian setting, where the interaction conditions (8.2.7) are satisfied automatically. In particular the unknowns become the global velocity in Eulerian coordinates $\boldsymbol{v}: \Omega \times [0,T] \to \mathbb{R}^N$, the global pressure $p: \Omega \times [0,T] \to \mathbb{R}$ and the left Cauchy–Green tensor $\mathbf{B}: \Omega \times [0,T] \to \mathbb{R}^{N\times N}$. We transfer the problem for the solid accordingly and rewrite the Eulerian form of the momentum balance for the solid as

$$\rho_s\left(\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{v}\right) = \operatorname{div}_{\boldsymbol{x}}\sigma_s + \rho_s\boldsymbol{b},$$

where the Cauchy stress σ_s is given by (8.2.5). The evolution equation for the Cauchy–Green tensor $\mathbf{B}$ can be derived directly from the kinematics. Indeed, (8.2.6) can be reformulated as

$$\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t) = \mathcal{F}(\boldsymbol{X},t)\mathcal{F}(\boldsymbol{X},t)^{\mathrm{T}}, \tag{8.2.9}$$

and differentiating both sides of (8.2.9) with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t)\right) = \frac{\partial\mathcal{F}}{\partial t}(\boldsymbol{X},t)\mathcal{F}(\boldsymbol{X},t)^{\mathrm{T}} + \mathcal{F}(\boldsymbol{X},t)\frac{\partial\mathcal{F}}{\partial t}(\boldsymbol{X},t)^{\mathrm{T}}. \tag{8.2.10}$$

By an application of the chain rule, the left-hand of (8.2.10) can be readily rewritten as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t)\right) = \frac{\partial\mathbf{B}}{\partial t}(\boldsymbol{y}(\boldsymbol{X},t),t) + (\boldsymbol{v}(\boldsymbol{y}(\boldsymbol{X},t),t)\cdot\nabla_{\boldsymbol{x}})\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t). \tag{8.2.11}$$

For clarity of exposition, we remark that (8.2.11) is merely the computation of the material time derivative of $\mathbf{B}$. On the other hand, in order to rewrite the right-hand side of (8.2.10), we observe that

$$\begin{aligned}\dot{\mathcal{F}}(\boldsymbol{X},t) = \frac{\mathrm{d}}{\mathrm{dt}}\nabla_{\boldsymbol{X}}\boldsymbol{y}(\boldsymbol{X},t) = \nabla_{\boldsymbol{X}}\frac{\partial\boldsymbol{\eta}}{\partial t}(\boldsymbol{X},t) &= \nabla_{\boldsymbol{X}}\boldsymbol{v}(\boldsymbol{y}(\boldsymbol{X},t),t)\\ &= (\nabla_{\boldsymbol{x}}\boldsymbol{v})(\boldsymbol{y}(\boldsymbol{X},t),t)\nabla_{\boldsymbol{X}}\boldsymbol{y}(\boldsymbol{X},t)\\ &= (\nabla_{\boldsymbol{x}}\boldsymbol{v})(\boldsymbol{y}(\boldsymbol{X},t),t)\mathcal{F}(\boldsymbol{X},t).\end{aligned} \tag{8.2.12}$$

In turn, (8.2.9) and (8.2.12) imply that

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial t}(\boldsymbol{X},t)\mathcal{F}(\boldsymbol{X},t)^{\mathrm{T}} + \mathcal{F}(\boldsymbol{X},t)\frac{\partial\mathcal{F}}{\partial t}(\boldsymbol{X},t)^{\mathrm{T}} &= (\nabla_{\boldsymbol{x}}\boldsymbol{v})(\boldsymbol{y}(\boldsymbol{X},t),t)\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t)\\ &\quad+\mathbf{B}(\boldsymbol{y}(\boldsymbol{X},t),t)(\nabla_{\boldsymbol{x}}\boldsymbol{v})(\boldsymbol{y}(\boldsymbol{X},t),t)^{\mathrm{T}}.\end{aligned} \tag{8.2.13}$$

Combining (8.2.10), (8.2.11), and (8.2.13), and passing to an Eulerian description of the motion (that is, substituting $\boldsymbol{x}$ for $\boldsymbol{y}(\boldsymbol{X},t)$) finally yields

$$\frac{\partial\mathbf{B}}{\partial t}(\boldsymbol{x},t) + (\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}})\mathbf{B}(\boldsymbol{x},t) = \nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x},t)\mathbf{B}(\boldsymbol{x},t) + \mathbf{B}(\boldsymbol{x},t)(\nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x},t))^{\mathrm{T}},$$

which enables to close the system of equations. Therefore, the governing equations for the incompressible neo-Hookean solid in the Eulerian setting read

$$\begin{aligned}&\mathrm{div}_{\boldsymbol{x}}\boldsymbol{v} = 0,\\ &\rho_s\left(\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{v}\right) = \mathrm{div}_{\boldsymbol{x}}\sigma_s + \rho_s\boldsymbol{b}, \quad \sigma_s = -p\mathbb{I}_N + G\mathbf{B}^d,\\ &\frac{\partial\mathbf{B}}{\partial t} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\mathbf{B} - (\nabla_{\boldsymbol{x}}\boldsymbol{v})\mathbf{B} - \mathbf{B}(\nabla_{\boldsymbol{x}}\boldsymbol{v})^{\mathrm{T}} = \mathbb{O}.\end{aligned} \tag{8.2.14}$$

Here and in the following $\mathbb{O}$ denotes the zero matrix; furthermore, we recall that $\mathbf{B}^d$ denotes the deviatoric part of the left Cauchy–Green tensor. Now, since both the fluid and the solid are described in the Eulerian frame of reference, we distinguish between the two simply by rheology. The formula for the Cauchy stress can be written in a unifying (essentially visco-elastic) manner as

$$\sigma \coloneqq -p\mathbb{I}_N + 2\mu\mathbb{D}(\boldsymbol{v}) + G\mathbf{B}^d, \tag{8.2.15}$$

where

$$\mu(\boldsymbol{x},t) \coloneqq \begin{cases}0 & \text{if } \boldsymbol{x}\in\mathcal{B}(t),\\ \mu_f & \text{if } \boldsymbol{x}\in\mathcal{F}(t),\end{cases} \qquad \text{and} \qquad G(\boldsymbol{x},t) \coloneqq \begin{cases}G_s & \text{if } \boldsymbol{x}\in\mathcal{B}(t),\\ 0 & \text{if } \boldsymbol{x}\in\mathcal{F}(t).\end{cases}$$

Moreover, we let

$$\mathbf{B} = \mathbb{I}_N \text{ in } \mathcal{F}(t) \qquad \text{and} \qquad \frac{\partial\mathbf{B}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\mathbf{B} - (\nabla\boldsymbol{v})\mathbf{B} - \mathbf{B}(\nabla\boldsymbol{v})^{\mathrm{T}} = \mathbb{O} \text{ in } \mathcal{B}(t).$$

Thus, the Cauchy stress σ is equal to σ_f in the fluid and to σ_s in the solid. Similarly, we define

$$\rho(\boldsymbol{x},t) := \begin{cases} \rho_s & \text{if } \boldsymbol{x} \in \mathcal{B}(t), \\ \rho_f & \text{if } \boldsymbol{x} \in \mathcal{F}(t). \end{cases}$$

Note that the domains $\mathcal{F}(t)$ and $\mathcal{B}(t)$ are advected with the velocity $\boldsymbol{v}$ and so do the material parameters ρ, μ, and G, i.e. their material time derivatives are equal to zero

$$\dot{\rho} = \dot{\mu} = \dot{G} = 0. \tag{8.2.16}$$

To summarize, the fully Eulerian FSI model is described by the following set of equations:

$$\begin{aligned} \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} &= 0, \\ \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}\right) &= \operatorname{div}_{\boldsymbol{x}} \sigma + \rho \boldsymbol{b}, \quad \sigma = -p\mathbb{I}_N + 2\mu \mathbb{D}(\boldsymbol{v}) + G\mathbf{B}^d, \\ \frac{\partial \mathbf{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\mathbf{B} - (\nabla_{\boldsymbol{x}} \boldsymbol{v})\mathbf{B} - \mathbf{B}(\nabla_{\boldsymbol{x}} \boldsymbol{v})^{\mathrm{T}} &= \mathbb{O}. \end{aligned} \tag{8.2.17}$$

It is worth noting that the fully Eulerian model admits the following formal energy equality:

$$\begin{aligned} &\int_\Omega \left(\frac{\rho(\boldsymbol{x},t)}{2}|\boldsymbol{v}(\boldsymbol{x},t)|^2 + \frac{G(\boldsymbol{x},t)}{2}(tr\mathbf{B}(\boldsymbol{x},t) - N)\right) d\boldsymbol{x} + \int_0^t \int_{\mathcal{F}(s)} 2\mu_f |\mathbb{D}(\boldsymbol{v}(\boldsymbol{x},s))|^2 \, d\boldsymbol{x} ds \\ &\quad = \int_\Omega \left(\frac{\rho(\boldsymbol{x},0)}{2}|\boldsymbol{v}(\boldsymbol{x},0)|^2 + \frac{G(\boldsymbol{x},0)}{2}(tr\mathbf{B}(\boldsymbol{x},0) - N)\right) d\boldsymbol{x} + \int_0^t \int_\Omega \rho(\boldsymbol{x},s)\boldsymbol{b}(\boldsymbol{x},s) \cdot \boldsymbol{v}(\boldsymbol{x},s) \, d\boldsymbol{x} ds. \end{aligned}$$

For the reader's convenience, a derivation of this formula is presented in the appendix to this paper (see Subsection 8.6.2).

Finally, let us note that for the sake of notational simplicity, we will from now on use the symbols ∇ and div for the Eulerian operators $\nabla_{\boldsymbol{x}}$ and $\operatorname{div}_{\boldsymbol{x}}$, respectively, while keeping the notation $\nabla_{\boldsymbol{X}}$ and $\operatorname{div}_{\boldsymbol{X}}$ for the differential operators in the reference configuration unchanged.

8.2.3 Reduced models

In view of the analytical challenges posed by the full FSI system described in Section 8.2, in this paper we propose a simplified model which we believe to adequately capture the essential features of the FSI phenomena under consideration, with special emphasis on the questions of contact and rebound. This is achieved via a two-step procedure. First, we consider a completely rigid particle and show that, under certain simplifying assumptions, its dynamics can be replaced by a single ODE. As a next step, we enrich the model by taking into account possible elastic deformations of the particle, which we approximate by a single scalar internal degree of freedom. In our simplified framework, this internal variable will be used to parameterize not only the change in shape of the particle (which will be reflected in the expression for the drag force, see Subsection 8.2.4 below), but also its elastic response. The final reduced model takes the form of two coupled ODEs with a highly non-linear damping term.

8.2.3.1 Dynamics of a rigid body as a second order ODE with non-linear damping

In this section we show that, under certain assumptions, the dynamics of a rigid body in a viscous incompressible fluid can be reformulated as a second order non-linear ODE, which takes the the form

$$\ddot{h} = -d(h)\dot{h}.$$

To be precise, following the presentation of Hillairet (see Section 3 in [101]), we assume that the system composed by the fluid and the rigid body occupies the entire half-space $\mathbb{R}^N_+$, $N = 2, 3$, and that the fluid adapts instantaneously to the solid, so that it can be effectively modeled by the quasi-static Stokes equations. Furthermore, if we suppose that the range of possible motions of the body consists only of translations in the direction $\boldsymbol{e}_N$, its position is uniquely determined by its distance from the set $\{x_N = 0\}$, denoted here and in the following with h. Let $\mathcal{B} \subset \mathbb{R}^N_+$ denote the bounded region occupied by the rigid body when $h = 0$ and define

$$\mathcal{B}_h \coloneqq \mathcal{B} + h\boldsymbol{e}_N, \qquad \mathcal{F}_h \coloneqq \mathbb{R}^N_+ \setminus \mathcal{B}_h. \tag{8.2.18}$$

With these notations at hand, and under the assumption that the fluid is homogeneous with density $\rho_f = 1$, our fluid-structure interaction problem is described by the system of equations

$$\begin{cases} -\mu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{0} & \text{in } \mathcal{F}_h, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \mathcal{F}_h, \\ \boldsymbol{v} = \dot{h}\boldsymbol{e}_N & \text{on } \partial\mathcal{B}_h, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \{x_N = 0\}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{at } \infty, \end{cases} \tag{8.2.19}$$

coupled with the continuity of the stresses across the fluid-solid surface, which in the present framework can be expressed via

$$M\ddot{h} = -\int_{\partial\mathcal{B}_h} (2\mu\mathbb{D}(\boldsymbol{v}) - p\mathbb{I}_N)\, \boldsymbol{n}\, d\mathcal{H}^{N-1} \cdot \boldsymbol{e}_N. \tag{8.2.20}$$

We recall that, as in the previous subsection, we use $\boldsymbol{v}$ and p to denote the velocity field and the pressure of the fluid, respectively. Moreover, the positive constants μ and M represent the viscosity of the fluid and the mass of the body, respectively. Finally, throughout the section $\boldsymbol{n}$ is always used to denote the outer unit normal vector to the fluid domain. The system (8.2.19)–(8.2.20) is further complemented with initial conditions of the form

$$h(0) = h_0 > 0, \quad \dot{h}(0) = \dot{h}_0.$$

It can be noted that in (8.2.19) the steady version of (8.2.1) and (8.2.2) is given, while the PDE for the solid (8.2.3) is reduced dramatically to (8.2.20).

The next result combines Lemma 4 and Lemma 5 in [101].

Lemma 8.2.2. *Let $h > 0$ be given and assume that $\partial\mathcal{B}$ is Lipschitz continuous. Then there exist a unique velocity field $\boldsymbol{s}_h$ and a pressure field π_h such that*

$$\begin{cases} -\Delta\boldsymbol{s}_h + \nabla\pi_h = \boldsymbol{0} & \textit{in } \mathcal{F}_h, \\ \operatorname{div}\boldsymbol{s}_h = 0 & \textit{in } \mathcal{F}_h, \\ \boldsymbol{s}_h = \boldsymbol{e}_N & \textit{on } \partial\mathcal{B}_h, \\ \boldsymbol{s}_h = \boldsymbol{0} & \textit{on } \{x_N = 0\}, \\ \boldsymbol{s}_h = \boldsymbol{0} & \textit{at } \infty. \end{cases} \tag{8.2.21}$$

Moreover, the following statements hold:

(i) *$\boldsymbol{s}_h$ is the unique global minimizer for the functional*

$$\mathcal{J}(\boldsymbol{u}; \mathcal{F}_h) \coloneqq 2\int_{\mathcal{F}_h} |\mathbb{D}(\boldsymbol{u})|^2\, d\boldsymbol{x}, \tag{8.2.22}$$

defined over the class

$$V_h \coloneqq \left\{\boldsymbol{u} \in H^1_0(\mathbb{R}^N_+; \mathbb{R}^N) : \operatorname{div}\boldsymbol{u} = 0, \textit{ and } \boldsymbol{u} = \boldsymbol{e}_N \textit{ on } \partial\mathcal{B}_h\right\}.$$

In particular, the pressure function π_h can be understood as the Lagrange multiplier associated to the divergence-free constraint in V_h.

(*ii*) *For every* $\tilde{\varphi} \in V_h$ *and* $z \in \mathbb{R}$, *if we let* $\varphi \coloneqq z\tilde{\varphi}$ *we have*

$$2\int_{\mathcal{F}_h} \mathbb{D}(\boldsymbol{s}_h) : \mathbb{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} = \int_{\partial\mathcal{B}_h} (2\mathbb{D}(\boldsymbol{s}_h) - \pi_h \mathbb{I}_N)\, \boldsymbol{n}\, d\mathcal{H}^{N-1} \cdot z\boldsymbol{e}_N. \tag{8.2.23}$$

(*iii*) *The function* $\boldsymbol{s}_h$ *depends smoothly on the parameter* h, *for all* $h \in (0,\infty)$.

As a consequence of Theorem 8.2.2 we see that the dynamics of the system are fully characterized by an initial value problem for a second order ODE with a non-linear damping term.

Lemma 8.2.3. *Assume that* $\partial\mathcal{B}$ *is Lipschitz continuous. Then, for every* $h_0 > 0$ *and* $\dot{h}_0 \in \mathbb{R}$, *the solvability of the fluid-structure interaction problem* (8.2.19)–(8.2.20) *reduces to that of the initial value problem*

$$\begin{cases} M\ddot{h} = -\mu\mathcal{J}(\boldsymbol{s}_h; \mathcal{F}_h)\dot{h}, \\ h(0) = h_0,\ \dot{h}(0) = \dot{h}_0. \end{cases} \tag{8.2.24}$$

Proof. Notice that for any given $h > 0$ and $\dot{h} \in \mathbb{R}$, letting $\boldsymbol{v} \coloneqq \dot{h}\boldsymbol{s}_h$ and $p \coloneqq \mu\dot{h}\pi_h$ yields a solution to (8.2.19). Moreover, using $\boldsymbol{\varphi} \coloneqq \mu\dot{h}\boldsymbol{s}_h$ as a test function in (8.2.23), we obtain

$$\begin{aligned} 2\mu\dot{h}\int_{\mathcal{F}_h} |\mathbb{D}(\boldsymbol{s}_h)|^2\, d\boldsymbol{x} &= \int_{\partial\mathcal{B}_h} (2\mathbb{D}(\boldsymbol{s}_h) - \pi_h\mathbb{I}_N)\, \boldsymbol{n}\, d\mathcal{H}^{N-1} \cdot \mu\dot{h}\boldsymbol{e}_N \\ &= \int_{\partial\mathcal{B}_h} (2\mu\mathbb{D}(\boldsymbol{v}) - p\mathbb{I}_N)\, \boldsymbol{n}\, d\mathcal{H}^{N-1} \cdot \boldsymbol{e}_N. \end{aligned} \tag{8.2.25}$$

In view of (8.2.25), we can then rewrite (8.2.20) as

$$M\ddot{h} = -2\mu\dot{h}\int_{\mathcal{F}_h} |\mathbb{D}(\boldsymbol{s}_h)|^2\, d\boldsymbol{x} = -\mu\mathcal{J}(\boldsymbol{s}_h; \mathcal{F}_h)\dot{h}.$$

This concludes the proof. □

8.2.3.2 Spring-mass model

In this subsection, we enrich the model described in (8.2.19)–(8.2.20) by considering also elastic deformations of the particle. It is well known that in the regime of small deformations, the dynamics of an elastic body reduces essentially to a (vectorial) wave equation (see Subsection 8.6.1). Consequently, as a first approximation, we will assume that the deformation of the particle can be described by a single scalar parameter ξ, which we can think of as the deformation of an internal spring with stiffness k carrying internal mass m, enclosed in a shell of mass M which is rigid with respect to the flow of surrounding fluid, but whose shape may change according to the value of the internal parameter ξ (the relevant notation is summarized in Figure 8.1; see Figure 8.2 for a schematic illustration of contactless rebound for the case of a deformable particle).

To be precise, let $\mathcal{P}$ denote the class of all admissible particle configurations, that is, $\mathcal{P}$ is the family of all bounded open subsets of $\mathbb{R}^N$ with Lipschitz continuous boundary and such that the intersection of their respective closures with the hyperplane $\{x_N = 0\}$ consists of only the origin. Given $\mathcal{B} \in \mathcal{P}$, we consider a one parameter family of diffeomorphisms $\{G_\xi : \mathcal{B} \to G_\xi(\mathcal{B}) : \xi \in \mathbb{R}\}$ such that $G_\xi(\mathcal{B}) \in \mathcal{P}$ for every $\xi \in \mathbb{R}$. Moreover, for every $h > 0$ and every $\xi \in \mathbb{R}$, we let

$$\mathcal{F}_{h,\xi} \coloneqq \mathbb{R}^N_+ \setminus (G_\xi(\mathcal{B}) + h\boldsymbol{e}_N)$$

and consider the energy functional

$$\mathcal{J}(\boldsymbol{u}; \mathcal{F}_{h,\xi}) \coloneqq 2\int_{\mathcal{F}_{h,\xi}} |\mathbb{D}(\boldsymbol{u})|^2\, d\boldsymbol{x}.$$

Compare these definitions with their counterparts in the previous subsection, i.e. (8.2.18) and (8.2.22), respectively. In particular, by an application of Theorem 8.2.2, we obtain that for each $h > 0$ and each $\xi \in \mathbb{R}$ there exists a vector field $\boldsymbol{s}_{h,\xi}$ that minimizes $\mathcal{J}(\cdot; \mathcal{F}_{h,\xi})$ over the class

$$V_{h,\xi} \coloneqq \left\{\boldsymbol{u} \in H^1_0(\mathbb{R}^N_+; \mathbb{R}^N) : \operatorname{div}\boldsymbol{u} = 0,\ \text{and } \boldsymbol{u} = \boldsymbol{e}_N \text{ on } G_\xi(\partial\mathcal{B}) + h\boldsymbol{e}_N\right\}.$$

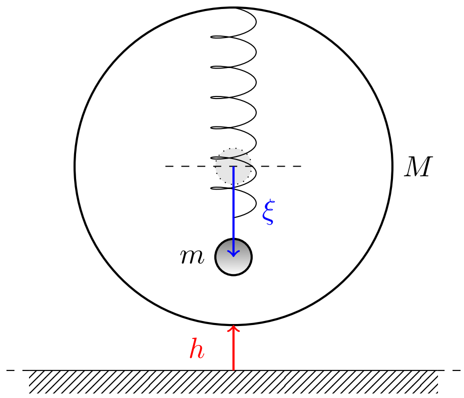


Figure 8.1: A spherical shell with an inner mass-spring system is surrounded by a viscous incompressible fluid.

Under the assumption that the range of possible motions of the deformable shell consists only of translations in the direction $\boldsymbol{e}_N$, reasoning as in Theorem 8.2.3 we see that its dynamics can be formulated as a second order ODE, with the exception that now at each time level t, the shape of the shell may change (depending on the value of ξ). Consequently, the mechanical force balance for such a system takes the form of the following system of two coupled ODEs:

$$M\ddot{h} = -k\xi - \mu\mathcal{J}(\boldsymbol{s}_{h,\xi}, \mathcal{F}_{h,\xi})\dot{h}, \tag{8.2.26}$$

$$m(\ddot{h} - \ddot{\xi}) = k\xi \tag{8.2.27}$$

with initial conditions

$$h(0) = h_0, \qquad \dot{h}(0) = \dot{h}_0,$$
$$\xi(0) = \xi_0, \qquad \dot{\xi}(0) = \dot{\xi}_0.$$

We remark that equation (8.2.26) is the analogue of (8.2.24), where the additional "internal" force is acting on the outer shell and with a more general drag force term which depends not only h, but also on the internal deformation ξ, while the second equation (8.2.27) expresses the dynamics of the internal mass-spring system in the frame accelerating with the outer shell.

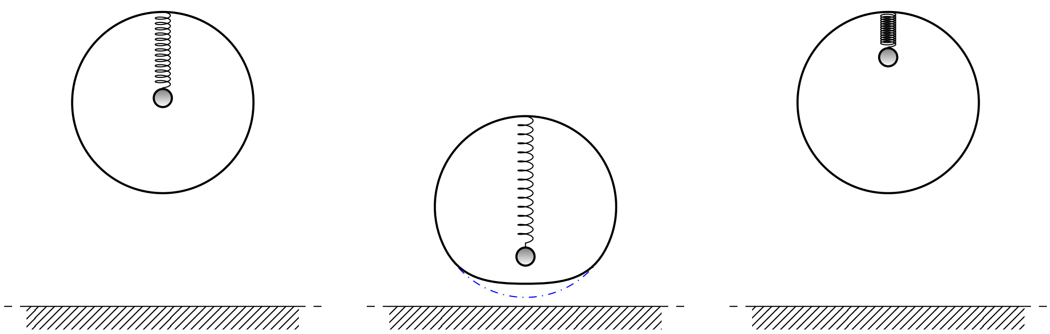


Figure 8.2: Schematic representation of contactless rebound for a deformable shell with an inner energy storing mechanism. The dash-dotted line represents the undeformed surface.

8.2.4 The drag force

As a consequence of the ODE reformulation of the FSI problem provided in (8.2.24) (resp. (8.2.27)), we see that the drag force exerted by the fluid on the solid body, i.e. the term $-\mu\mathcal{J}(\boldsymbol{s}_h;\mathcal{F}_h)\dot{h}$ (resp. $-\mu\mathcal{J}(\boldsymbol{s}_{h,\xi};\mathcal{F}_{h,\xi})\dot{h}$), can significantly influence the behavior of the system. Thus, in this section we collect some well known approximations of this force. In order to obtain a precise understanding of the near-to-contact dynamics, the focus of the section is on the dependence of $\mathcal{J}(\boldsymbol{s}_h;\mathcal{F}_h)$ on the parameter h, with special emphasis on the case $h \to 0^+$. We recall indeed that $h = 0$ corresponds to a collision of the body with the boundary of the container.

To be precise, in the following we present estimates of the drag formulas for both the two and three dimensional case. Furthermore, we compare them also with those resulting from the standard lubrication (Reynolds') approximation. For the purpose of this section, it is not restrictive to consider rigid particles. Additionally, in all cases we shall assume the particle is axi-symmetric with respect to the axis x_N ($N = 2, 3$) and that the part of the boundary $\partial\mathcal{B}$ that is closer to the wall can be described in a neighborhood of the origin by a graph of the form

$$\psi(x_1) = \gamma|x_1|^{1+\alpha} \quad \text{if } N = 2, \qquad \psi(x_1, x_2) = \gamma(x_1^2 + x_2^2)^{\frac{1+\alpha}{2}} \quad \text{if } N = 3. \tag{8.2.28}$$

8.2.4.1 Drag force estimates based on the variational formulation

We begin by noticing that, depending on the smoothness of the immersed particle, the drag force exerted by the viscous fluid can develop a singularity when the distance between the body and the boundary of the cavity tends to zero. This is made precise in the next result, which is due to Starovoitov (see Theorem 3.1 in [176]). A proof of the theorem is included in Subsection 8.6.3 for the reader's convenience.

Theorem 8.2.4. *Let $\mathcal{B}$ be an open bounded subset of $\mathbb{R}^N_+$ with Lipschitz continuous boundary and such that $\partial\mathcal{B} \cap \{x_N = 0\}$ consists of only the origin. For $\mathcal{J}$ and $\boldsymbol{s}_h$ given as in Theorem 8.2.2, let $D\colon (0,\infty) \to (0,\infty)$ be defined via*

$$D(h) \coloneqq \mathcal{J}(\boldsymbol{s}_h;\mathcal{F}_h).$$

Then D is locally Lipschitz continuous. Furthermore, the following statements hold:

(i) *if $N = 2$ and there are $\alpha, \gamma, r > 0$ such that in a neighborhood of the origin $\partial\mathcal{B}$ coincides with the graph of $\psi(x_1) \coloneqq \gamma|x_1|^{1+\alpha}$, for $|x_1| < r$, then there exists a positive constant c_1 such that for all $0 < h \le r^{1+\alpha}$*

$$D(h) \ge c_1 h^{\frac{-3\alpha}{1+\alpha}};$$

(ii) *if $N = 3$ and there are $\alpha, \gamma, r > 0$ such that in a neighborhood of the origin $\partial\mathcal{B}$ coincides with the graph of $\psi(x_1, x_2) \coloneqq \gamma(x_1^2 + x_2^2)^{\frac{1+\alpha}{2}}$, for $x_1^2 + x_2^2 < r^2$, then there exists a positive constant c_2 such that for all $0 < h \le r^{1+\alpha}$*

$$D(h) \ge c_2 h^{\frac{1-3\alpha}{1+\alpha}}.$$

Roughly speaking, Theorem 8.2.4 presents us with the crucial observation that the asymptotic behavior of D is deeply connected to the regularity of $\partial\mathcal{B}$ in a neighborhood of the nearest point to the fixed boundary of the container. It is worth noting that since for every $t \in (-1, 1)$ one has that

$$\frac{t^2}{2} \le 1 - \sqrt{1 - t^2} \le t^2,$$

an application of Theorem 8.2.4 with $\alpha = 1$ yields that if $N = 2$ and $\mathcal{B}$ is a disk then $D(h) \gtrsim h^{-3/2}$, while if $N = 3$ and $\mathcal{B}$ is a sphere then $D(h) \gtrsim h^{-1}$. In particular, as illustrated in Theorem 3.2 in [176] (see also Theorem 3 in [101]), one can then transform the differential equation obtained in Theorem 8.2.3 into a differential inequality; this, in turn, can be integrated to show that the rigid body cannot collide with the boundary of the container in finite time.

It is worth noting that the proof of the no-collision result in the papers [78, 101, 102], where the fluid is modeled by the Navier–Stokes equations, relies on the construction of a good (localized) approximation of the solution to the associated Stokes problem. A particularly interesting corollary of these constructions is that the asymptotic lower bounds provided by Theorem 8.2.4 are, in most cases, optimal. To be precise, we have the following theorem (for more information, see also the discussion at the end of Subsection 8.6.3).

Theorem 8.2.5. *Under the assumptions of Theorem 8.2.4, there exist two positive constants C_1, C_2 such that for all h sufficiently small*

$$D(h) \le \begin{cases} C_1 h^{\frac{-3\alpha}{1+\alpha}} & \text{if } N = 2, \\ C_2 h^{\frac{1-3\alpha}{1+\alpha}} & \text{if } N = 3 \text{ and } \alpha > 1/3, \\ C_2 |\log h| & \text{if } N = 3 \text{ and } \alpha = 1/3, \\ C_2 & \text{if } N = 3 \text{ and } \alpha < 1/3. \end{cases} \tag{8.2.29}$$

We conclude the section by observing that, in the present framework, if $\partial\mathcal{B}$ is sufficiently regular so that the body is prevented from colliding in finite time with the boundary of the container, then the system cannot produce a rebound.

Corollary 8.2.6. *Let h be a solution to* (8.2.24) *with initial conditions $h_0 > 0$ and $\dot{h}_0 < 0$, and assume that $h(t) > 0$ for every $t > 0$. Then h is a monotone function.*

Proof. Arguing by contradiction, assume that there are $\tau_1 < \tau_2$ such that $\dot{h}(\tau_1) = 0$ and $h(\tau_2) > \tilde{h} := h(\tau_1)$. Since $\min\{h(t) : t \in [0, \tau_2]\} > 0$ and by recalling that D is locally Lipschitz continuous in $(0, \infty)$, we see that the initial value problem (8.2.24) admits a unique solution in $[0, \tau_2]$, which must therefore agree with h. Notice, however, that h is also the unique solution to the initial value problem satisfying (8.2.24) on $[\tau_1, \tau_2]$ with initial conditions $h(\tau_1) = \tilde{h}$ and $\dot{h}(\tau_1) = 0$. Consequently $h \equiv \tilde{h}$ on $[\tau_1, \tau_2]$, which contradicts $h(\tau_2) > \tilde{h}$. □

8.2.4.2 Drag force estimates based on Reynolds' approximation

Similarly to above, throughout the subsection we consider an axi-symmetric particle $\mathcal{B}$. In particular, if $\partial\mathcal{B}$ satisfies (8.2.28), then in a neighborhood of the nearest-to-contact point $\partial\mathcal{B}_h$ (see (8.2.18)) can be conveniently described as the graph of

$$g(r) = h + \gamma r^{1+\alpha}, \tag{8.2.30}$$

where r denotes the distance from the symmetry axis. With this notation at hand and in view of the lubrication (Reynolds') approximation (see Subsection 8.6.3), we obtain that the vertical component of the drag force exerted on the particle can be effectively estimated by

$$F_{\text{lub}} := -12\mu\dot{h} \begin{cases} 2\displaystyle\int_0^\infty \int_r^\infty \frac{r'}{g(r')^3}\, dr' dr & \text{if } N = 2, \\ \pi\displaystyle\int_0^\infty \int_r^\infty \frac{rr'}{g(r')^3}\, dr' dr & \text{if } N = 3. \end{cases} \tag{8.2.31}$$

An exact comparison of the drag formulas in (8.2.31) with the resulting expressions derived in subsubsection 8.2.4.1 is only possible for particular values of α, for which the Reynolds based expression can be integrated analytically. In particular, assuming circular (when $N = 2$) or spherical (when $N = 3$) shape of the solid ball with radius R, we get

$$g(r) := h + R - \sqrt{R^2 - r^2} \sim h + \frac{r^2}{2R^2}.$$

Substituting $\alpha = 1$ and $\gamma = 1/(2R)$ into (8.2.31) allows to analytically resolve the integrals, which ultimately yields $F_{\text{lub}} = -\mu D_{\text{lub}}(h)\dot{h}$, where

$$D_{\text{lub}}(h) \coloneqq \begin{cases} 3\sqrt{2}\pi \left(\dfrac{R}{h}\right)^{\frac{3}{2}} & \text{if } N = 2, \\ 6\pi \dfrac{R^2}{h} & \text{if } N = 3; \end{cases} \tag{8.2.32}$$

see also eq. (7-270) in [124], eq. (2.18) in [25], and and eq. (1.1) in [45].

On the other hand, in order to compare the expressions for the drag force for other values of α, we compute numerically the lubrication theory shape factor D_{lub} from (8.2.31) and compare it with the analytical estimates in (8.2.29) in Figure 8.3. Note that the match is very good for the case $N = 2$ and reasonable for the case $N = 3$ at least in the vicinity of $\alpha = 1$, corresponding to the sphere.

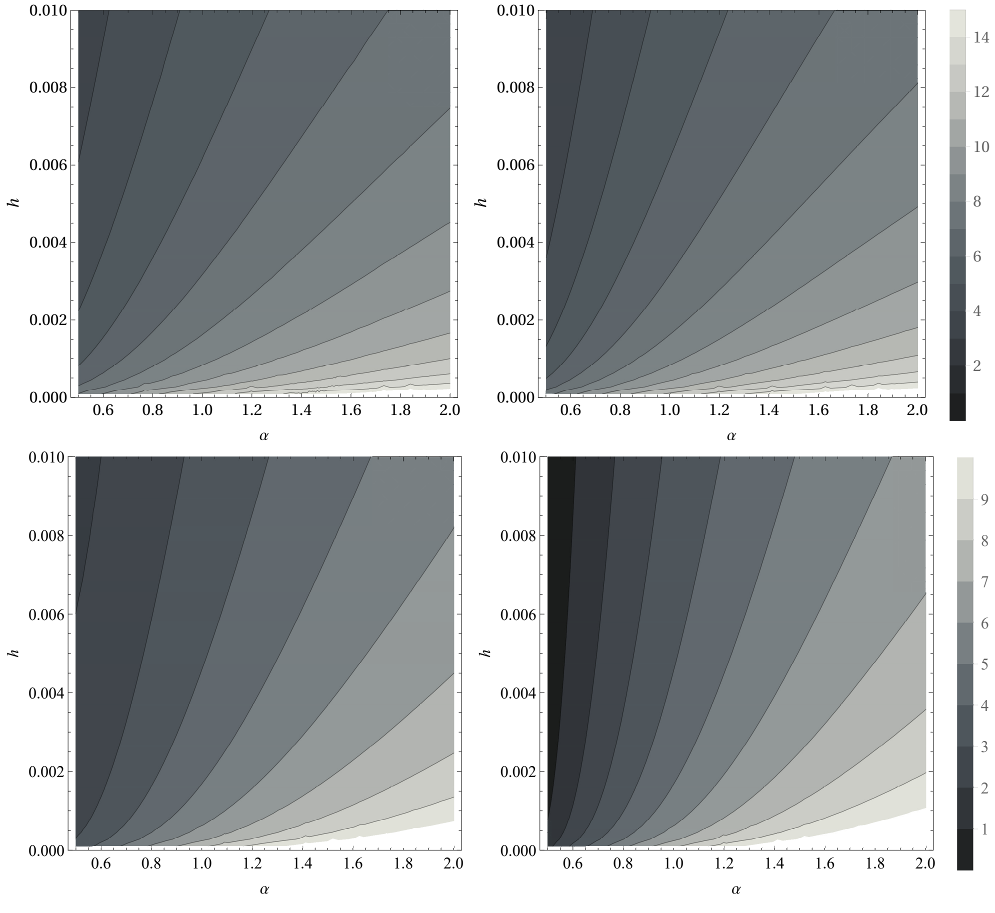


Figure 8.3: Logarithm of the drag force shape factor based on the Reynolds approximation (left) and on the analytical estimate (right) for $N = 2$ (top row) and $N = 3$ (bottom row).

8.3 Global well-posedness and qualitative behavior of solutions to the reduced model

In this is section, we undertake a rigorous analytical study of the reduced model that was previously introduced in subsubsection 8.2.3.2. We begin by addressing the question of global well-posedness and we then proceed to investigate qualitative properties of solutions as we vary the viscosity parameter μ. In this direction, we present two results which highlight very different behaviors with regard to particle rebound. For clarity of exposition, we postpone the proofs to Subsection 8.3.2. In addition, we refer the reader to Subsection 8.4.1 for some numerical experiments on the model considered in this section.

8.3.1 Statement of the main results

Throughout the section we consider the system of ODEs

$$\begin{cases} \ddot{h} - \ddot{\xi} = ab(\xi), \\ \ddot{h} = -b(\xi) - \mu\mathcal{D}(h,\xi)\dot{h}, \\ h(0) = h_0,\ \dot{h}(0) = \dot{h}_0, \\ \xi(0) = \xi_0,\ \dot{\xi}(0) = \dot{\xi}_0. \end{cases} \tag{8.3.1}$$

Here a and μ are positive constants, while the functions b and $\mathcal{D}$ serve as proxies for the elastic response of the solid and the drag force, respectively. Notice indeed that the system given by (8.2.26)–(8.2.27) is a particular case of (8.3.1), corresponding to the choices

$$b(\xi) \coloneqq \frac{k\xi}{M}, \qquad a \coloneqq \frac{M}{m}, \qquad \mathcal{D}(h,\xi) \coloneqq \frac{\mathcal{J}(s_{h,\xi}, \mathcal{F}_{h,\xi})}{M}. \tag{8.3.2}$$

In our first result, the aim is to identify conditions for which the body is prevented from colliding with the boundary of the container in finite time. Our analysis is in spirit very close to that of [101] (see also Theorem 8.2.4 and the subsequent discussion). To this end, we define

$$B(y) \coloneqq \int_0^y b(w)\, dw,$$

and make the following assumptions:

(B.1) $b\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous;

(B.2) B is coercive, that is, $B(y) \to \infty$ as $|y| \to \infty$.

Additionally, on $\mathcal{D}\colon (0,\infty)\times\mathbb{R} \to (0,\infty)$ we require an analogous regularity condition and a singular asymptotic lower bound that is uniform with respect to the variable ξ. To be precise, throughout the following we always work under the following set of assumptions:

(D.1) the map $(h,\xi) \mapsto \mathcal{D}(h,\xi)$ is locally Lipschitz continuous in $(0,\infty)\times\mathbb{R}$;

(D.2) there exist a constant $c > 0$ and $\alpha \in [1,\infty)$ such that for all $h > 0$ and $\xi \in \mathbb{R}$

$$\mathcal{D}(h,\xi) \geq ch^{-\alpha}.$$

It is worth noting that the assumptions above are satisfied, for example, by the drag force exerted on a circular or spherical structure (see in particular (8.2.32)).

We are now ready to state a no-contact result.

Proposition 8.3.1. *Let b and $\mathcal{D}$ be given in such a way that* (B.1), (B.2), (D.1), *and* (D.2) *are satisfied. Then, for every a, μ, h_0, $\dot{h}_0$, ξ_0, $\dot{\xi}_0 \in \mathbb{R}$ with a, μ, $h_0 > 0$ there exists a unique global solution to* (8.3.1)*, denoted by (h_μ, ξ_μ). In particular, $h_\mu(t) > 0$ for all $t > 0$.*

Having established existence of global solutions, the remainder of the section is dedicated to characterizing the different qualitative behaviors of h_μ, as we let $\mu \to 0^+$. For our next result, in addition to the assumptions of Theorem 8.3.1, we require that

(B.3) $B \geq 0$,

and furthermore, we restrict our attention to the case where the function $\mathcal{D}$ does not depend on the variable ξ and obeys a power law in h. To be precise, we assume the following:

(D.3) there exist three constants $C_1, C_2 > 0$ and $\alpha \in [1, \infty)$ and a locally Lipschitz continuous function $g\colon (0,\infty) \to [C_1, C_2]$ such that for all $h > 0$ and all $\xi \in \mathbb{R}$ we have

$$\mathcal{D}(h, \xi) = g(h) h^{-\alpha}.$$

We are now in position to state the first of our main results.

Theorem 8.3.2. *Under the assumptions of Theorem 8.3.1, set $\xi_0 = \dot{\xi}_0 = 0$ and let $H\colon [0,\infty) \to [0,\infty)$ be defined via*

$$H(t) \coloneqq \max\{0, h_0 + \dot{h}_0 t\}.$$

Then the following statements hold:

(i) *Assume that $b(0) = 0$ and that $\dot{h}_0 < 0$. Then, as $\mu \to 0^+$, we have that $h_\mu \to H$ and $\xi_\mu \to 0$ uniformly in $[0, t_0]$, where $t_0 \coloneqq -h_0/\dot{h}_0$.*

(ii) *Assume that $b(0) = 0$ and that $\dot{h}_0 \geq 0$. Then, as $\mu \to 0^+$, we have that $h_\mu \to H$ and $\xi_\mu \to 0$ uniformly on compact subsets of $[0, \infty)$.*

(iii) *Assume that $\dot{h}_0 < 0$, that b satisfies* (B.3)*, and that $\mathcal{D}$ is given as in* (D.3)*. Let $\xi\colon [0,\infty) \to \mathbb{R}$ be defined via $\xi(t) = 0$ if $t \leq t_0$, while if $t > t_0$ we let ξ be the unique solution to the initial value problem*

$$\begin{cases} \ddot{\xi} + ab(\xi) = 0, \\ \xi(t_0) = 0, \ \dot{\xi}(t_0) = -\dot{h}_0. \end{cases}$$

Then, as $\mu \to 0^+$, we have that $h_\mu(t) \to H$ and $\xi_\mu \to \xi$ uniformly on compact subsets of $[0, \infty)$.

A few comments are in order. First, let us mention that we are primarily interested in the case $\dot{h}_0 < 0$; the case of a non-negative initial velocity is mainly stated for comparison. Next, observe that (D.3) can be interpreted as a rigidity condition on the solid body (see Theorem 8.2.4 and Theorem 8.2.5). It is also worth noting that the function H given in the theorem is monotone. In particular, the conclusions of statement (iii) in Theorem 8.3.2 can be summarized as follows: in the vanishing viscosity limit, a "rigid" solid (in the sense of condition (D.3)) moving towards the wall will impact the boundary of the container in finite time (to be precise, at $t = t_0$) and it won't separate from the container's wall thereafter. Thus, rather surprisingly, the reduced model predicts that, as we let the viscosity parameter go to zero, the system composed of a smooth rigid shell with an inner mass-spring mechanism (as described in subsubsection 8.2.3.2) approaches a state where the motion of the shell and that of the spring are perfectly decoupled and the shell cannot move away from the wall after collision. This trapping effect is readily explained by observing that if we could instantaneously invert the direction of the velocity, the shell would experience a drag force of equal intensity. More specifically, the resistance of the fluid to the movement of the body does not distinguish on whether the shell is approaching or receding from the wall. Notably, for positive (but small) values of the viscosity parameter, the very same phenomenon that prevents from collision is also the primary obstruction to rebound.

Next, we show that the nearly paradoxical situation described by Theorem 8.3.2 can be partly resolved by allowing for qualitative changes in the shape of the solid. This has the effect of introducing an asymmetry in the problem which can potentially prevent the trapping phenomenon illustrated above. These changes, however, need to be significant enough to be reflected in the asymptotic behavior of $\mathcal{D}$ (which we recall should be understood as an approximation to the drag force exerted on the body by the surrounding fluid environment) as h approaches zero.

The running assumptions for the last result of the section are the following:

(B.4) $b(y)y > 0$ for all $y \neq 0$;

(D.4) $\mathcal{D}$ is non-decreasing as a function of ξ, that is,

$$\mathcal{D}(h,\xi_1) \leq \mathcal{D}(h,\xi_2)$$

for every $h > 0$ and every $\xi_1 \leq \xi_2$;

(D.5) there exist three constants $\delta_1, c_1 > 0$ and $\gamma_1 \in [\alpha,\infty)$ such that for every $h > 0$ we have

$$\mathcal{D}(h,-\delta_1) \geq c_1 h^{-\gamma_1};$$

(D.6) there exist a constant $\delta_2 > 0$ and a function $\gamma\colon (0,\infty) \to [0,\infty)$ such that

$$\int_0^h \gamma(y)y^{-1}\,dy \to 0$$

as $h \to 0^+$ and with the property that for every $h > 0$ we have

$$\mathcal{D}(h,-\delta_2) \leq \gamma(h)h^{-\gamma_1}.$$

Theorem 8.3.3. *Under the assumptions of Theorem 8.3.1, let $\xi_0 = \dot{\xi}_0 = 0$ and assume that $\mathcal{D}$ satisfies* (D.4), (D.5)*, and* (D.6)*. Furthermore, let b be given satisfying* (B.4) *so that there exists a unique $y^- < 0$ with the property that $2aB(y^-) = \dot{h}_0^2$. Assume that $y^- < -\delta_2$, where δ_2 is given as in* (D.6)*. Then, for every subsequence $\{h_n\}_n \subset \{h_\mu\}_\mu$ there exists $T > t_0 := -h_0/\dot{h}_0$ such that*

$$\lim_{n\to\infty} h_n(T) > 0.$$

We remark that although Theorem 8.3.3 holds for every choice of the initial velocity $\dot{h}_0$, the result is of particular interest in the case where $\dot{h}_0 < 0$. Indeed, since in this case we have that $h_n(t_0) \to 0$ (see statement (i) in Theorem 8.3.2), the theorem implies that $\{h_n\}_n$ converges to a function which is not monotone.

Remark 8.3.4. *Notice that* (B.4) *implies* (B.3)*. Therefore, the main difference between Theorem 8.3.2 and Theorem 8.3.3 is that condition* (D.3) *is replaced by* (D.4)–(D.6)*. We mention here that our prototypical examples for the drag shape factor $\mathcal{D}$ are motivated by the drag force estimates obtained in Subsection 8.2.4 (see in particular* (8.2.29) *and* (8.2.32)*) and are given by*

$$\mathcal{D}_1(h,\xi) := h^{-c\xi-3/2}, \qquad \mathcal{D}_2(h,\xi) := h^{-\max\{\xi,0\}-1}, \tag{8.3.3}$$

where c is a positive constant. Notice that in view of (8.2.29) *and* (8.2.32)*, $\mathcal{D}_1$ is modeled in such a way that positive values of ξ induce a flattening of an (initially) spherical particle in two space dimension, while $\mathcal{D}_2$ does the same in three space dimensions. Moreover, both allow for adequate choices of δ_2 and y^-. In particular, they satisfy* (D.1), (D.5)*, and* (D.6)*. Notice that the monotonicity requirement in* (D.4) *holds for all $h \leq 1$; thus both examples can be suitably modified to satisfy* (D.4)*. Additionally, as it becomes apparent from the proof of Theorem 8.3.3 (see in particular* (8.3.24)*), it is enough to assume that* (D.4) *holds for $h \leq h_0 + \varepsilon$, where ε can be any positive number. Finally, notice that* (D.2) *is automatically satisfies for d_2 and holds for d_1 provided that c is chosen opportunely.*

8.3.2 Proofs of the main results

In this section we collect the proofs of the results stated above. Due to the technical nature of the arguments presented below, we give first a schematic outline of the general principles behind the results. In particular, let us mention that the proof of Theorem 8.3.2 is based on the key observation that the falling object reaches a minimum distance from the wall at which escape is no longer possible. Recalling that the leading term in the drag coefficient does not distinguish on whether the solid is approaching or receding from the wall, in this configuration the elastic response of the inner spring is not sufficient to overcome the singular force that resists motion in the fluid. Conversely, in the proof of Theorem 8.3.3 we exploit the fact that, as the object is approaching the wall, the change of shape effectively stops the particle at a greater distance than the one that would be reached by the undeformed configuration. As the deformation parameter reverts these changes, the symmetry is broken and the elastic response is now sufficient to generate a vertical motion away from the wall. This effective rebound is physical in the sense that it withstands the vanishing viscosity limit.

Proof of Theorem 8.3.1. In view of the regularity assumptions (B.1) and (D.1), the existence of local solutions to (8.3.1) follows directly from Peano's theorem. Let (h,ξ) be a maximal solution defined on the interval $(0,T)$ and assume by contradiction that $T<\infty$. We divide the proof into two steps.

Step 1: Multiplying the first equation in (8.3.1) by $(\dot h-\dot\xi)$, the second one by $a\dot h$, and adding together the resulting expressions, we arrive at

$$(\ddot h-\ddot\xi)(\dot h-\dot\xi)+a\ddot h\dot h=-ab(\xi)\dot\xi-a\mu\mathcal{D}(h,\xi)\dot h^2. \tag{8.3.4}$$

Define the auxiliary function

$$F(t):=(\dot h(t)-\dot\xi(t))^2+a\dot h(t)^2+2aB(\xi(t))$$

and notice that integrating (8.3.4) yields the following energy equality

$$F(t)+2a\mu\int_0^t\mathcal{D}(h(s),\xi(s))\dot h(s)^2\,ds=F(0). \tag{8.3.5}$$

This *energy equality* relates well to the one formally derived for the full fluid-structure interaction (8.2.8). Indeed, $(\dot h(t)-\dot\xi(t))^2+a\dot h(t)^2$ resembles the kinetic energy $\mathcal{K}(t)$, while $2aB(\xi(t))$ represents the elastic contribution given by the density $\mathcal{W}$ and the dissipation (stemming from the fluid) is the same in both energy equalities.

Since the integral on the left-hand side is non-negative, in view of (B.2) we conclude that $\xi,\dot h$, and $\dot\xi$ are bounded. Consequently, since by assumption $T<\infty$, we obtain that h is also bounded in $[0,T]$. This implies that necessarily $h(T)=0$, since otherwise the solution would admit an extension, hence contradicting the maximality of the solution (h,ξ).

Step 2: Next, we take T_1 to be the smallest time instance for which $h(T_1)=0$. In view of the previous step we have that $T_1\le T<\infty$. We notice that by multiplying the second equation in (8.3.1) by $\chi_{\{\dot h<0\}}$ we get

$$\begin{aligned}\left(\ddot h+b(\xi)\right)\chi_{\{\dot h<0\}}&=-\mu\mathcal{D}(h,\xi)\dot h\chi_{\{\dot h<0\}}\\&\ge-\mu ch^{-\alpha}\dot h\chi_{\{\dot h<0\}}\\&\ge-\mu ch^{-\alpha}\dot h\chi_{\{\dot h<0\}}-\mu ch^{-\alpha}\dot h\chi_{\{\dot h>0\}}\\&=-\mu ch^{-\alpha}\dot h,\end{aligned}$$

where in the first inequality we have used the lower bound given by (D.2). Integrating both sides in the previous inequality yields

$$\int_0^t\left(\ddot h(s)+b(\xi(s))\right)\chi_{\{\dot h<0\}}(s)\,ds\ge-\mu c\int_0^t h^{-\alpha}(s)\dot h(s)\,ds=-\mu c\int_{h_0}^{h(t)}y^{-\alpha}\,dy. \tag{8.3.6}$$

Notice that since by assumption $\alpha\ge1$, the right-hand side of (8.3.6) tends to infinity as $t\to T_1^-$. Set $U:=\{s\in(0,t):\dot h(s)<0\}$ and observe that if $U=(0,t)$ then

$$\int_0^t\ddot h(s)\chi_{\{\dot h<0\}}(s)\,ds=\dot h(t)-\dot h_0, \tag{8.3.7}$$

while if this is not the case then we write U as the union of at most countably many disjoint open intervals, i.e.

$$U = \bigcup_{i=1}^{\infty}(s_i, t_i).$$

Without loss of generality we assume that $s_i \leq s_j$ if $i \leq j$; furthermore, we notice that $\dot{h}(t_1) = 0$, $\dot{h}(s_i) = 0$ for all $i \geq 2$, and $\dot{h}(t_i) = 0$ for all $i \geq 2$ provided that $t_i \neq t$. Consequently, we have

$$\int_0^t \ddot{h}(s)\chi_{\{\dot{h}<0\}}(s)\, ds = \sum_{i=1}^{\infty}\int_{s_i}^{t_i} \ddot{h}(s)\, ds = \min\{\dot{h}(t), 0\} - \min\{\dot{h}_0, 0\}. \tag{8.3.8}$$

Combining (8.3.7) and (8.3.8) with the bounds obtained in the previous step shows that the left-hand side of (8.3.6) remains bounded as $t \to T_1^-$, thus yielding a contradiction.

In turn, we obtain that $h > 0$ in $[0, T]$ and the existence of a global solution follows by Step 1. Moreover, the uniqueness of solutions is now direct consequence of the Picard–Lindelöf theorem and (D.1). □

In the remainder of this section, we study the asymptotic behavior of solutions as the viscosity parameter μ approaches zero. To be precise, in the following we fix a sequence $\mu_n \to 0^+$ and denote with (h_n, ξ_n) the solution to (8.3.1) (given by Theorem 8.3.1) relative to the choice $\mu = \mu_n$.

Lemma 8.3.5. *Under the assumptions of Theorem 8.3.1, let (h_n, ξ_n) be solutions as above. Then there exist two Lipschitz continuous functions $h, \xi\colon \mathbb{R} \to \mathbb{R}$, with h non-negative, such that (up to the extraction of a subsequence, which we do not relabel) $h_n \to h$ and $\xi_n \to \xi$ uniformly on compact subsets of $[0, \infty)$.*

Proof. As a consequence of the energy estimate (8.3.5), we see that

$$\sup\left\{\|\xi_n\|_{L^\infty([0,\infty))} + \|\dot{h}_n\|_{L^\infty([0,\infty))} + \|\dot{\xi}_n\|_{L^\infty([0,\infty))} : n \in \mathbb{N}\right\} < \infty. \tag{8.3.9}$$

Since the sequence $\{h_n\}_n$ is equi-Lipschitz continuous and $h_n(0) = h_0$ for every n, it is also equi-bounded in $[0, T]$ for every $T > 0$. The desired result then follows by the Arzelà–Ascoli theorem. □

Lemma 8.3.6. *Assume that $b(0) = 0$, $\xi_0 = 0$, $\dot{\xi}_0 = 0$, and let (h, ξ) be given as in Theorem 8.3.5. Then, the following hold:*

(i) *if $\dot{h}_0 \geq 0$ we have that $h(t) = h_0 + \dot{h}_0 t$ and $\xi(t) = 0$ for every $t \geq 0$;*

(ii) *if $\dot{h}_0 < 0$ we have that $h(t) = h_0 + \dot{h}_0 t$ and $\xi(t) = 0$ for every $t \leq t_0 \coloneqq -h_0/\dot{h}_0$.*

Proof. Since by assumption $h(0) = h_0 > 0$, there exists $t_1 > 0$ such that $h(t) > 0$ in $[0, t_1)$. For any $t_2 < t_1$, let $\varepsilon \coloneqq \min\{h(t) : t \in [0, t_2]\}$. Then, for $t \in (0, t_2)$ we have

$$|\ddot{h}_n(t)| \leq \|b(\xi_n)\|_{L^\infty} + \mu_n\|\dot{h}_n\|_{L^\infty} \max\left\{\mathcal{D}(y, \xi_n) : y \in [\varepsilon, \|h_n\|_{L^\infty}]\right\}.$$

Thus, $\{h_n\}_n$ is bounded in $C^{1,1}((0, t_2))$ and by the Arzelà–Ascoli theorem we find for that for a subsequence $\dot{h}_n \to \dot{h}$ uniformly.

Next, notice that by integrating the second equation in (8.3.1) we arrive at

$$\dot{h}_n(t) - \dot{h}_0 = -\int_0^t b(\xi_n(s))\, ds - \mu_n\int_0^t \mathcal{D}(h_n(s), \xi_n(s))\dot{h}_n(s)\, ds.$$

Letting $n \to \infty$ in the previous identity yields

$$\dot{h}(t) - \dot{h}_0 = -\int_0^t b(\xi(s))\, ds. \tag{8.3.10}$$

Subtracting the second equation in (8.3.1) to the first one we obtain

$$\ddot{\xi}_n = -(1 + a)b(\xi_n) - \mu_n\mathcal{D}(h_n, \xi_n)\dot{h}_n. \tag{8.3.11}$$

Therefore, reasoning as above, we conclude that $\xi \in C^1((0,t_2))$ and that eventually extracting a subsequence we also have $\dot{\xi}_n \to \dot{\xi}$. Integrating the equation in (8.3.11) and passing to the limit with respect to n we see that

$$\dot{\xi}(t) = -(1+a)\int_0^t b(\xi(s))\,ds.$$

In turn, ξ is of class C^2 in $(0,t_2)$ and solves the initial value problem

$$\begin{cases} \ddot{\xi} + (1+a)b(\xi) = 0, \\ \xi(0) = \dot{\xi}(0) = 0. \end{cases}$$

Since by assumption $b(0) = 0$, we readily deduce that ξ is identically equal to zero in $[0,t_2]$. This, together with (8.3.10), implies that $\dot{h}(t) = \dot{h}_0$ and therefore that $h(t) = h_0 + \dot{h}_0 t$ for every $t \in [0,t_2]$.

Finally, assuming first that $\dot{h}_0 < 0$, we notice that if we can choose $t_1 \geq t_0$ then there is nothing else to do. If this is not the case, then we can assume without loss of generality that $t_1 < t_0$ is such that $h(t_1) = 0$. In this case, letting $t_2 \to t_1^-$ would then imply that $h(t_1) = h_0 + \dot{h}_0 t_1 > 0$, thus leading to a contradiction. On the other hand, if $\dot{h}_0 \geq 0$ the proof is similar, but simpler; thus we omit the details. This completes the proof. □

In the following proposition we address the more delicate case in which $\dot{h}_0 < 0$ and $t \geq t_0$.

Proposition 8.3.7. *Under the assumptions of Theorem 8.3.2, let h_n, ξ_n, h, and ξ be given as in Theorem 8.3.5. Then, if $\dot{h}_0 < 0$ we have that $h(t) = 0$ in $[t_0,\infty)$.*

Proof. Assume first that $\alpha > 1$ and fix $\varepsilon > 0$. We claim that there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n \geq N(\varepsilon)$ then

$$h_n(t)^{\alpha-1} \leq \varepsilon \tag{8.3.12}$$

for every $t \geq t_0$. The desired result then follows by the arbitrariness of ε. We begin by observing that adding the first equation in (8.3.1) to a multiple of the second equation yields

$$(1+a)\ddot{h}_n - \ddot{\xi}_n = -a\mu_n g(h_n) h_n^{-\alpha}\dot{h}_n.$$

Let $t \geq t_0$ be such that $h_n(t) < h_0$. Then, integrating the previous identity and by means of a change of variables we obtain

$$\begin{aligned} (1+a)\dot{h}_n(t) - (1+a)\dot{h}_0 - \dot{\xi}_n(t) &= -a\mu_n \int_0^t g(h_n(s))h_n(s)^{-\alpha}\dot{h}_n(s)\,ds \\ &= -a\mu_n \int_{h_0}^{h_n(t)} g(y)y^{-\alpha}\,dy \\ &\leq -\frac{C_2 a\mu_n}{1-\alpha}\left(h_n(t)^{1-\alpha} - h_0^{1-\alpha}\right). \end{aligned}$$

Rearranging the terms in the previous inequality yields

$$h_n(t)^{\alpha-1} \leq \frac{C_2 a\mu_n}{\alpha-1}\left((1+a)\dot{h}_n(t) - (1+a)\dot{h}_0 - \dot{\xi}_n(t) + \frac{C_2 a\mu_n}{\alpha-1}h_0^{1-\alpha}\right)^{-1}.$$

Therefore, to prove (8.3.12) it is enough to show that

$$\frac{C_2 a\mu_n}{\alpha-1} \leq \varepsilon\left((1+a)\dot{h}_n(t) - (1+a)\dot{h}_0 - \dot{\xi}_n(t) + \frac{C_2 a\mu_n}{\alpha-1}h_0^{1-\alpha}\right);$$

in the following it will be convenient to rewrite this condition as

$$(1+a)\dot{h}_0 + \frac{C_2 a\mu_n}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right) \leq (1+a)\dot{h}_n(t) - \dot{\xi}_n(t). \tag{8.3.13}$$

Let us remark here that since by assumption $\dot h_0 < 0$, it is possible to choose n large enough so that the left-hand side in (8.3.13) is negative. Consequently, if arguing by contradiction we assume that (8.3.13) does not hold, we obtain that

$$\left|(1+a)\dot h_0 + \frac{C_2 a\mu_n}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right)\right| < |(1+a)\dot h_n(t) - \dot\xi_n(t)|. \tag{8.3.14}$$

Squaring both sides in (8.3.14) and by Young's inequality we see that

$$\begin{aligned}\left[(1+a)\dot h_0 + \frac{C_2 a\mu_n}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right)\right]^2 &< \left((1+a)\dot h_n(t) - \dot\xi_n(t)\right)^2 \\ &\leq \left(1+\frac{1}{\delta}\right)(\dot h_n(t) - \dot\xi_n(t))^2 + (1+\delta)a^2\dot h_n(t)^2 \end{aligned}\tag{8.3.15}$$

holds for every $\delta > 0$. In particular, if we let $\delta = 1/a$, the right-hand side in (8.3.15) can be rewritten as

$$\left(1+\frac{1}{\delta}\right)(\dot h_n(t) - \dot\xi_n(t))^2 + (1+\delta)a^2\dot h_n(t)^2 = (1+a)\left[(\dot h_n(t) - \dot\xi_n(t))^2 + a\dot h_n(t)^2\right],$$

and therefore, from the energy equality (8.3.5), we see that

$$\begin{aligned}(1+a)\left[(\dot h_n(t) - \dot\xi_n(t))^2 + a\dot h_n(t)^2\right] &= (1+a)^2\dot h_0^2 - 2a(1+a)B(\xi_n(t)) \\ &\quad - 2a(1+a)\mu_n\int_0^t g(h_n(s))h_n(s)^{-\alpha}\dot h_n(s)^2\,ds. \end{aligned}\tag{8.3.16}$$

Further, expanding the square on the left-hand side of (8.3.15) we obtain the quantity

$$(1+a)^2\dot h_0^2 + \frac{2C_2 a(1+a)\mu_n}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right)\dot h_0 + \mathcal{O}(\mu_n^2). \tag{8.3.17}$$

Thus, combining (8.3.16) and (8.3.17) with (8.3.15) for $\delta = 1/a$ and rearranging the terms in the result inequality, we arrive at

$$B(\xi_n(t)) + \mu_n\int_0^t g(h_n(s))h_n(s)^{-\alpha}\dot h_n(s)^2\,ds \leq \frac{C_2\mu_n}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right)(-\dot h_0) + \mathcal{O}(\mu_n^2).$$

Let $t_1 < t_0$ be such that

$$h_0^{1-\alpha} + \frac{C_2}{C_1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right) < (h_0 + \dot h_0 t_1)^{1-\alpha}. \tag{8.3.18}$$

Then, since by assumption $t \geq t_0$, we have

$$\begin{aligned}C_1\int_0^{t_1} h_n(s)^{-\alpha}\dot h_n(s)^2\,ds &\leq \frac{B(\xi_n(t))}{\mu_n} + \int_0^t g(h_n(s))h_n(s)^{-\alpha}\dot h_n(s)^2\,ds \\ &\leq \frac{C_2(-\dot h_0)}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right) + \mathcal{O}(\mu_n).\end{aligned}$$

We claim that letting $n \to \infty$ in the previous inequality leads to a contradiction to the definition of t_1. Indeed, since in $[0, t_1]$ we have that h_n and $\dot h_n$ converge uniformly to h and $\dot h$, respectively. Moreover, since (B.3) implies that $b(0) = 0$, we are in a position to apply Theorem 8.3.6 and conclude that

$$\begin{aligned}\frac{C_1(-\dot h_0)}{\alpha-1}\left[(h_0 + \dot h_0 t_1)^{1-\alpha} - h_0^{1-\alpha}\right] &= C_1\int_0^{t_1} h(s)^{-\alpha}\dot h(s)^2\,ds \\ &\leq \frac{C_2(-\dot h_0)}{\alpha-1}\left(\varepsilon^{-1} - h_0^{1-\alpha}\right). \end{aligned}\tag{8.3.19}$$

As one can readily check, (8.3.19) is in contradiction with (8.3.18) and the claim is proved. Thus, we have shown that if $t \geq t_0$ and $h_n(t) < h_0$ for every n sufficiently large then $h_n(t)^{\alpha-1} \leq \varepsilon$. Assume for the sake of contradiction that there exists $t > t_0$ such that $h_n(t) \to h(t) \geq h_0$. Since $h(t_0) = 0$, there must be a point $\tau \in (t_0, t)$ such that $h(\tau) = h_0/2$. Let $N \in \mathbb{N}$ be such that

$$\left| h_n(\tau) - \frac{h_0}{2} \right| < \frac{h_0}{4}$$

for all $n \geq N$. Notice that for every such n we have that $h_n(\tau) < h_0$ and therefore $h_n(\tau) \leq \varepsilon$, provided n is large enough. This implies that $0 < h_0/2 = h(\tau) \leq \varepsilon$. Letting $\varepsilon \to 0$ leads to a contradiction.

If $\alpha = 1$, the argument presented above can be suitably modified to prove that

$$\frac{1}{|\log h_n(t) - \log h_0|} \leq \varepsilon$$

rather than (8.3.12). Since the proof requires only minimal changes, we omit the details. □

Lemma 8.3.8. *Under the assumptions of Theorem 8.3.1, let h_n, ξ_n, h, and ξ be given as in Theorem 8.3.5. Then, if $\dot{h}_0 < 0$, $b(0) = 0$, and $h(t) = 0$ for $t \geq t_0$ we have that ξ is the unique solution to the initial value problem*

$$\begin{cases} \ddot{\xi} + ab(\xi) = 0, \\ \xi(t_0) = 0, \ \dot{\xi}(t_0) = -\dot{h}_0. \end{cases}$$

Proof. We begin by noticing that if we let $z_n \coloneqq h_n - \xi_n$, we have that $\ddot{z}_n = ab(\xi_n)$. Therefore, the sequence $\{z_n\}_n$ is bounded in $C^{1,1}$. In turn, by Arzelá-Ascoli, we see that there exists a function z such that, up to the extraction of a subsequence (which we do not relabel), $z_n \to z$ and $\dot{z}_n \to \dot{z}$ uniformly on compact subsets of $[0, \infty)$. Integrating the equation for z_n and passing to the limit in n yields

$$\dot{z}(t) - \dot{z}(0) = \int_0^t ab(\xi(s)) \, ds.$$

Therefore $z \in C^2(0, \infty)$ and satisfies $\ddot{z} = ab(\xi)$. Since by assumption we have that $h(t) = 0$ for every $t \geq t_0$, in view of Theorem 8.3.6 we are left to show that $\dot{\xi}(t_0) = -\dot{h}_0$. This follows by observing that

$$\dot{h}_0 = \lim_{t \nearrow t_0} \dot{h}(t) - \dot{\xi}(t) = \dot{z}(t_0) = \lim_{t \searrow t_0} \dot{z}(t) = \lim_{t \searrow t_0} -\dot{\xi}(t).$$

This concludes the proof. □

Proof of Theorem 8.3.2. Combining the results of Theorem 8.3.6, Theorem 8.3.7 and Theorem 8.3.8, we obtain that for every sequence $\mu_n \to 0^+$, the corresponding sequences of solutions, i.e. $\{h_n\}_n$ and $\{\xi_n\}_n$, admit a subsequence with the desired convergence properties. As one can readily check with a standard argument by contradiction, this implies that the convergence holds for the entire family. Hence, the proof is complete. □

We conclude the section with the proof of Theorem 8.3.3.

Proof of Theorem 8.3.3. We present the proof for the case that $\gamma_1 > 1$, where γ_1 is the constant given in (D.5). The case that $\gamma_1 = \alpha = 1$ follows by the same arguments (replacing powers by logarithms at the relevant places).

We divide the proof into several steps.

Step 1: Arguing by contradiction, assume that $h_n(t) \to \max\{h_0 + \dot{h}_0 t, 0\}$ for all $t \geq 0$. Then, an application of Theorem 8.3.8 yields that eventually extracting a subsequence we have that $\xi_n \to \xi$, where ξ is the solution to

$$\ddot{\xi} + ab(\xi) = 0, \tag{8.3.20}$$

with initial conditions $\xi(t_0) = 0$ and $\dot{\xi}(t_0) = -\dot{h}_0$. Furthermore, from (B.2) and (B.4) we see that there are exactly two points y^-, y^+, with $y^- < 0 < y^+$, such that $2aB(y^\pm) = \dot{h}_0^2$. Let

$$t^\pm \coloneqq 2 \left| \int_0^{y^\pm} \left(\dot{h}_0^2 - 2aB(y) \right)^{-1/2} dy \right|.$$

Observe that $t^{\pm}$ are finite by the positivity assumption of (B.4), since (by Taylor expansion) $\dot{h}_0^2 - 2aB(y) = 2ab(y^{\pm})(y^{\pm}-y)+\mathcal{O}((y^{\pm}-y)^2)$. Further notice that the points $y^{\pm}$ are turning points for the non-linear oscillator (8.3.20), whose period is given by $t^+ + t^-$. We then define $t_1 \coloneqq t_0 + t^+$ and $t_2 \coloneqq t_0 + t^+ + t^-$. With this notation at hand, we have that

$$\begin{aligned} y^- \le \xi(t) \le y^+ \qquad & \text{for } t \ge 0, \\ \xi(t) > 0 \qquad & \text{for } t \in (t_0, t_1), \\ \xi(t) < 0 \qquad & \text{for } t \in (t_1, t_2). \end{aligned}$$

Step 2: In this step we prove that for every $\varepsilon > 0$ with $6\varepsilon < t_2 - t_1$ there exists $N(\varepsilon)$ such that if $n \ge N(\varepsilon)$ then $\dot{h}_n(t) \ge 0$ in $(t_1 + 3\varepsilon, t_2 - 3\varepsilon)$. To this end, observe that by the uniform convergence of ξ_n to ξ, there exists a positive δ such that

$$\xi_n(t) \le -\delta \tag{8.3.21}$$

for all $t \in (t_1 + \varepsilon, t_2 - \varepsilon)$ and all n sufficiently large. Arguing by contradiction, suppose that for a subsequence of $\{h_n\}_n$ (which we do not relabel) we can find points $\tau_n \in (t_1 + 3\varepsilon, t_2 - 3\varepsilon)$ with the property that $\dot{h}_n(\tau_n) < 0$. Observe that necessarily $\dot{h}_n(t) \le 0$ in $(t_1 + \varepsilon, \tau_n)$. Indeed, if this was not the case then h_n would admit a local maximum at a point σ_n in this interval. This leads to a contradiction since (8.3.21), together with (B.4), implies that $\ddot{h}_n(\sigma_n) = -b(\xi_n(\sigma_n)) > 0$. Let $t_{\varepsilon,n} \in (t_1 + \varepsilon, t_1 + 2\varepsilon)$ be such that

$$h_n(t_1 + \varepsilon) - h_n(t_1 + 2\varepsilon) = \dot{h}_n(t_{\varepsilon,n})\varepsilon.$$

Letting $n \to \infty$ we see that $\dot{h}_n(t_{\varepsilon,n}) \to 0$. Integrating the second equation in (8.3.1) between $t_{\varepsilon,n}$ and $t \in (t_{\varepsilon,n}, t_1 + 3\varepsilon)$, using the fact that h_n is non-increasing in this interval, and (8.3.21) we arrive at

$$\dot{h}_n(t) - \dot{h}_n(t_{\varepsilon,n}) \ge -\int_{t_{\varepsilon,n}}^{t} b(\xi_n(s))\, ds \ge \beta(t - t_{\varepsilon,n}),$$

where $\beta \coloneqq \min\{-b(y) : y \in [y^-, -\delta]\}$. Integrating the previous inequality between $t_1 + 2\varepsilon$ and $t_1 + 3\varepsilon$ we then conclude that

$$h_n(t_1 + 3\varepsilon) - h_n(t_1 + 2\varepsilon) - \dot{h}_n(t_{\varepsilon,n})\varepsilon \ge \frac{\beta\varepsilon^2}{2},$$

which in turn implies

$$h_n(t_1 + 3\varepsilon) - h_n(t_1 + \varepsilon) \ge \frac{\beta\varepsilon^2}{2}.$$

Letting $n \to \infty$ leads to a contradiction, since the left hand side was assumed to converge to zero.
Step 3: Let t_n be such that

$$h_n(t_n) = \min\left\{h_n(t) : t \in \left[0, \frac{t_1 + t_2}{2}\right]\right\}.$$

The purpose of this step is to prove that there exists a constant $K > 0$ such that for every n sufficiently large we have

$$K h_n(t_n)^{\gamma_1 - 1} \ge \mu_n. \tag{8.3.22}$$

To see this, fix $\varepsilon > 0$ such that $\xi(t) > -\delta_1/2$ in $[0, t_1 + 3\varepsilon]$, where δ_1 is given as in (D.5). Using the fact that $\xi_n \to \xi$ uniformly in $[0, t_2]$ and the result of the previous step it is possible to find a number $N(\varepsilon)$ such that if $n \ge N(\varepsilon)$ then the following properties are satisfied:

$$\begin{aligned} \xi_n(t) \ge -\delta_1 \quad & \text{in } [0, t_1 + 3\varepsilon], \\ \dot{h}_n(t) \ge 0 \quad & \text{in } [t_1 + 3\varepsilon, t_2 - 3\varepsilon]. \end{aligned} \tag{8.3.23}$$

Notice that by (8.3.23) it follows that $t_n \in [0, t_1 + 3\varepsilon]$. Moreover, in view of (D.4), (D.5), and (8.3.23), for every $t \in [0, t_1 + 3\varepsilon]$ we have

$$\begin{aligned}\left(\ddot{h}_n(t) + b(\xi_n(t))\right)\chi_{\{\dot{h}_n \le 0\}}(t) &= -\mu_n \mathcal{D}(h_n(t), \xi_n(t))\dot{h}_n(t)\chi_{\{\dot{h}_n \le 0\}}(t)\\ &\ge -\mu_n \mathcal{D}(h_n(t), -\delta_1)\dot{h}_n(t)\chi_{\{\dot{h}_n \le 0\}}(t)\\ &\ge -\mu_n c_1 h_n(t)^{-\gamma_1}\dot{h}_n(t)\chi_{\{\dot{h}_n \le 0\}}(t)\\ &\ge -\mu_n c_1 h_n(t)^{-\gamma_1}\dot{h}_n(t). \end{aligned} \tag{8.3.24}$$

Reasoning as in (8.3.8) (see also (8.3.9)), we conclude that there exists a constant $k > 0$ such that

$$k \ge \int_0^{t_n} \left(\ddot{h}_n(t) + b(\xi_n(t))\right)\chi_{\{\dot{h}_n \le 0\}}(t)\,dt \ge \frac{\mu_n c_1}{\gamma_1 - 1}\left(h_n(t_n)^{1-\gamma_1} - h_0^{1-\gamma_1}\right), \tag{8.3.25}$$

where the second inequality is obtained by integrating the estimate in (8.3.24). Notice that (8.3.25) can be rewritten as

$$\frac{(\gamma_1 - 1)k}{c_1} h_n(t_n)^{\gamma_1 - 1} \ge \mu_n\left(1 - h_0^{1-\gamma_1} h_n(t_n)^{\gamma_1 - 1}\right),$$

and that the right-hand side can be further estimated from below by $\mu_n/2$, provided n is large enough. In particular, we have show that (8.3.22) holds for $K = 2(\gamma_1 - 1)k/c_1$.

Step 4: With this estimate at hand can proceed as follows. Since by assumption $y^- < -\delta_2$, where δ_2 is the constant given as in (D.6), eventually replacing ε with a smaller number, we can find T_1, T_2 such that $T_1 < T_2$, $4\varepsilon < T_2 - T_1$, $(T_1, T_2) \subset (t_1 + 3\varepsilon, t_2 - 3\varepsilon)$, and with the property that $\xi_n(t) \le -\delta_2$ for every $t \in (T_1, T_2)$. Reasoning as in Step 2 of the proof, for every n we can find a point $\tau_{\varepsilon,n} \in (T_1, T_1 + \varepsilon)$ in such a way that

$$h_n(T_1) - h_n(T_1 + \varepsilon) = \dot{h}_n(\tau_{\varepsilon,n})\varepsilon.$$

Notice that for every $t \in (T_1, T_2)$, (8.3.22) and (D.5) imply that

$$\begin{aligned}\ddot{h}_n(t) &= -b(\xi_n(t)) - \mu_n \mathcal{D}(h_n(t), \xi_n(t))\dot{h}_n(t)\\ &\ge -b(\xi_n(t)) - K h_n(t_n)^{\gamma_1 - 1}\gamma(h_n(t))h_n(t)^{-\gamma_1}\dot{h}_n(t)\\ &\ge -b(\xi_n(t)) - K\gamma(h_n(t))h_n(t)^{-1}\dot{h}_n(t).\end{aligned}$$

Integrating the previous inequality between $\tau_{\varepsilon,n}$ and $t \in (\tau_{\varepsilon,n}, T_2)$ we obtain

$$\begin{aligned}\dot{h}_n(t) - \dot{h}_n(\tau_{\varepsilon,n}) &\ge -\int_{\tau_{\varepsilon,n}}^t b(\xi_n(s))\,ds - K\int_{\tau_{\varepsilon,n}}^t \gamma(h_n(s))h_n(s)^{-1}\dot{h}_n(s)\,ds\\ &= -\int_{\tau_{\varepsilon,n}}^t b(\xi_n(s))\,ds - K\int_{h_n(\tau_{\varepsilon,n})}^{h_n(t)} \gamma(y)y^{-1}\,dy\\ &\ge -\int_{\tau_{\varepsilon,n}}^t b(\xi_n(s))\,ds - K\int_0^{h_n(T_2)} \gamma(y)y^{-1}\,dy,\end{aligned} \tag{8.3.26}$$

where in the last inequality we have used the fact that h_n is non-decreasing in (T_1, T_2). Integrating (8.3.26) from $\tau_{\varepsilon,n}$ to T_2 yields

$$\begin{aligned}h_n(T_2) - h_n(\tau_{\varepsilon,n}) - \dot{h}_n(\tau_{\varepsilon,n})(T_2 - \tau_{\varepsilon,n}) \ge &-\int_{\tau_{\varepsilon,n}}^{T_2}\int_{\tau_{\varepsilon,n}}^t b(\xi_n(s))\,dsdt\\ &- K(T_2 - \tau_{\varepsilon,n})\int_0^{h_n(T_2)} \gamma(y)y^{-1}\,dy.\end{aligned}$$

In view of (D.6), by letting $n \to \infty$ in the previous inequality we obtain

$$0 = \lim_{n\to\infty} h_n(T_2) - h_n(\tau_{\varepsilon,n}) - \dot{h}_n(\tau_{\varepsilon,n})(T_2 - \tau_{\varepsilon,n}) \ge \lim_{n\to\infty} -\int_{\tau_{\varepsilon,n}}^{T_2}\int_{\tau_{\varepsilon,n}}^t b(\xi_n(s))\,dsdt. \tag{8.3.27}$$

To conclude, it is enough to notice that the right-hand side of (8.3.27) is positive. Indeed, if we set $\tilde{\beta} \coloneqq \min\{-b(y) : y \in [y^-, -\delta_2]\} > 0$, we get

$$-\int_{\tau_{\varepsilon,n}}^{T_2} \int_{\tau_{\varepsilon,n}}^{t} b(\xi_n(s))\, ds dt \geq \frac{1}{2}(T_2 - T_1 - \varepsilon)^2 \tilde{\beta} > 0.$$

We have thus arrived at a contradiction and the proof is complete. □

8.4 Numerical results

In this section, we present some numerical experiments in order to further strengthen our main conjecture, which was previously formulated in the introduction. We begin by illustrating that the "effectively deformable" reduced model, for which the internal spring deformation is coupled with the damping term representing the drag force, does indeed produce a physical rebound. We conclude the section with the comparison from a numerical standpoint of the ODE and PDE solutions. The striking similarities that we observe suggest the relevance of the reduced model for the description of the rebound phenomenon.

8.4.1 Reduced model

In the numerical simulations we shall consider a particular variant of the reduced model (8.3.1). To be precise, we take

$$\mathcal{D}(h, \xi) \coloneqq \frac{c_1 h^{-c_2\xi - 3/2} + c_3}{M},$$

where the first term on the right-hand side is in accordance with (8.3.3) and reflects the change of flatness parameterized by ξ, and the second constant term describes the standard Stokes drag in the absence of geometrical constraints. Furthermore, if a and b are given as in (8.3.2), then the system of governing equations for h and ξ can be written as

$$M\ddot{h} = -k\xi - \mu\left(c_1 h^{-c_2\xi - 3/2} + c_3\right)\dot{h},$$
$$m(\ddot{h} - \ddot{\xi}) = k\xi,$$

with initial conditions

$$h(0) = h_0, \qquad \dot{h}(0) = \dot{h}_0,$$
$$\xi(0) = \xi_0, \qquad \dot{\xi}(0) = \dot{\xi}_0.$$

8.4.1.1 Numerical results

In order to demonstrate the critical effect of the change of flatness for the reduced model, we compare the two situations in which $c_2 = 0$ and $c_2 \neq 0$. In both cases an internal energy storage mechanism is present in the form of a mass-spring element. In the first case the elongation of the spring does not affect the drag force (rigid shell model). For this model we proved that a physical rebound is not possible, see Theorem 8.3.1 and Theorem 8.3.2. In the other case, the elongation of the spring does affect the drag force (effectively deformable model); in this setting a physical rebound can be expected in view of Theorem 8.3.3.

The qualitative different behaviors that the two settings can exhibit are summarized in Figure 8.4, where we plot the evolution of h, that is, the distance to the wall, as a function of time t for several values of the fluid viscosity μ for the rigid shell model (left column) and for the effectively deformable model (right column). The top row shows the larger time interval $(0, 2)$ s, while on the bottom row we zoom into the vicinity of the supposed rebound instant. The figure clearly demonstrates the severe effect of the inclusion of a coupling between the internal deformation parameter ξ and the drag force on the dynamics of the system. For the rigid shell model, the response converges with decreasing viscosity to the "hit-and-stick" solution, i.e., to the piecewise affine function $H(t) = \max\{0, h_0 + \dot{h}_0 t\}$ (see Theorem 8.3.2). Note that as a result of the presence of the internal spring, the solutions for the rigid shell model are non-monotone, but as the amplitude of these oscillations diminishes with decreasing fluid viscosity, this bouncing does not correspond to the physical rebound as we defined it (that is, it doesn't withstand the vanishing viscosity limit).

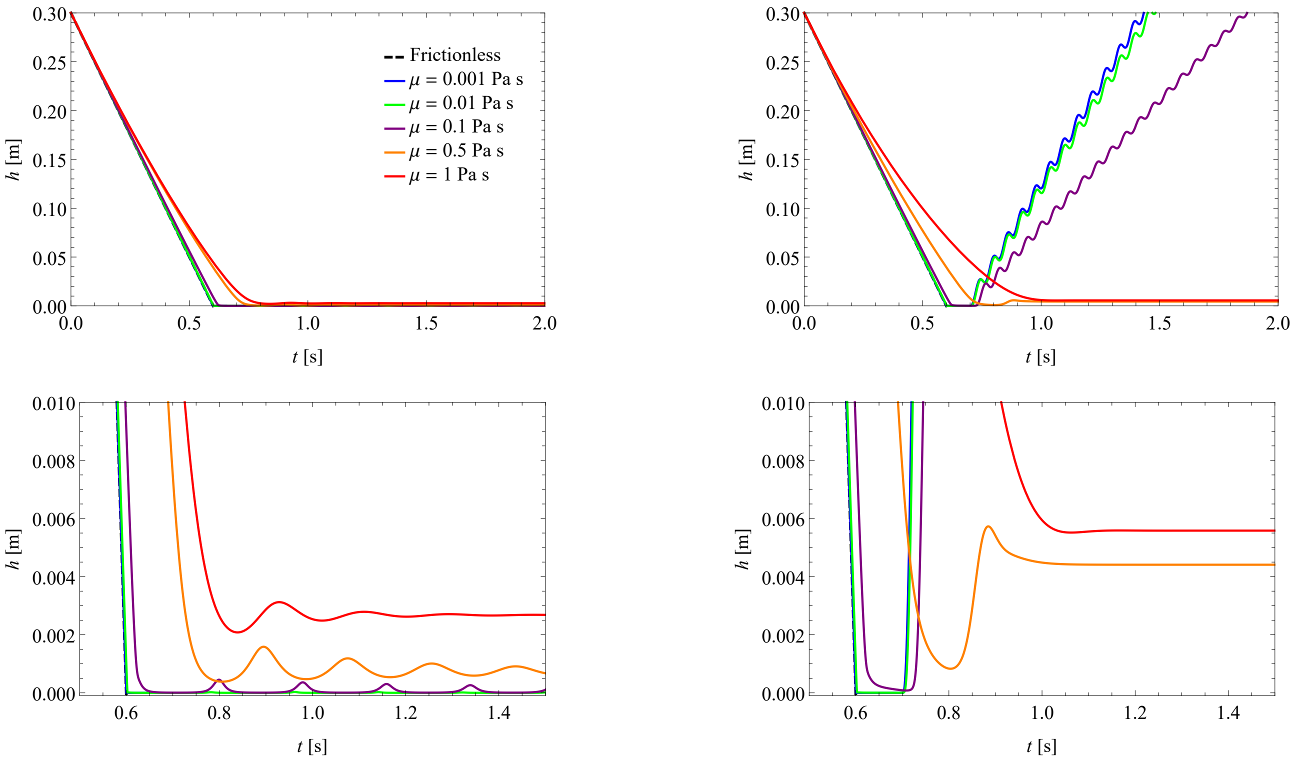


Figure 8.4: Physical rebound is not possible for the rigid shell model (left). The effectively deformable model can produce a rebound (right). The graphs in the bottom row are close-ups in the vicinity of the supposed rebound instant.

The situation is very different for the effectively deformable model. For the highest values of viscosity (red line), the body bounces off very mildly and its motion is rather quickly slowed down due to friction in the fluid. But with decreasing viscosity, the rebound becomes more and more pronounced and the solutions appear to be converging to an expected frictionless limit. Note how oscillations of the internal spring manifest themselves in the motion of h, becoming less and less damped as μ goes to zero. Interestingly, our simulations indicate that the kinetic energy corresponding to the outer shell, i.e. the fraction $M/(M+m)$ of the total kinetic energy of the system, is lost during the rebound in the vanishing viscosity limit. This suggests that a proper physical rebound (i.e. a perfectly elastic vacuum situation), would correspond in our reduced model to the case $M \to 0^+$, that is, to the situation in which the entire mass of the body is carried by the internal mass and the outer shell is massless.

The values of the parameters used in the depicted simulations are as follows: $k = 10000$, $c_1 = 0.1$, $c_2 = 20$ for the effectively deformable model (and it is set equal to zero in the rigid case), $c_3 = 7.4$, $M = 1$, and finally $m = 8.2$. It is worth noting that this particular choice of the parameters is in accordance with the assumptions of Theorem 8.3.3 (see also Theorem 8.3.4). Indeed, since the energy estimate (8.3.5) implies that $\sup_t |\xi(t)| \le |\dot{h}_0|/\sqrt{k} = 1/200$, we have that $\xi(t) < 1/(2c_2) = 1/40$ for all $t > 0$. Moreover, the choice is motivated by our effort to match the solutions to the reduced model with the finite element solutions to the full FSI problem described in Subsection 8.4.3. Finally, for the initial conditions we considered the following values: $h_0 = 0.3$, $\dot{h}_0 = 0.5$, and $\xi_0 = \dot{\xi}_0 = 0$.

8.4.2 Full FSI model

The standard form of the fluid-structure interaction problem, as presented in Section 8.2, is typically treated by the so-called arbitrary Lagrangian–Eulerian (ALE) method (see, for example, [57, 170]) where the solid part is Lagrangian, but the fluid problem is transformed into a certain special configuration which reflects the changes of the shape of the fluid domain but is not disrupted by the (possibly vigorous) motion of the fluid within the domain. The ALE method can be used to tackle the problem of contact in fluid-structure interactions, but often requires the use of sophisticated adaptive remeshing techniques to keep the fluid domain in the contact region well resolved.

For our specific problem, however, we find it more convenient to pass to a fully Eulerian description. This formulation was derived in Subsection 8.2.2, and revolves around the system of equations

$$\begin{aligned} &\mathrm{div}\boldsymbol{v} = 0, \\ \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) &= \mathrm{div}\sigma, \quad \sigma = -p\mathbb{I}_N + 2\mu\mathbb{D}(\boldsymbol{v}) + G\mathbf{B}^d, \end{aligned}$$

where viscosity μ is positive in the fluid and zero in the solid, the elastic modulus G is positive in the solid and zero in the fluid, the density ρ is equal to ρ_s in the solid and ρ_f in the fluid, and

$$\mathbf{B} = \mathbb{I}_N \text{ in } \mathcal{F}(t) \qquad \text{and} \qquad \frac{\partial \mathbf{B}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\mathbf{B} - (\nabla\boldsymbol{v})\mathbf{B} - \mathbf{B}(\nabla\boldsymbol{v})^{\mathrm{T}} = \mathbb{O} \text{ in } \mathcal{B}(t).$$

For the numerical implementation of the above model, we employ the conservative level-set method with reinitialization, which facilitates the tracking of the boundary between the fluid and the solid domain. In particular, we add a new scalar unknown χ, defined via

$$\chi(\boldsymbol{x}, t) \coloneqq \begin{cases} 1 & \text{if } \boldsymbol{x} \in \mathcal{B}(t), \\ 0 & \text{if } \boldsymbol{x} \in \mathcal{F}(t), \end{cases}$$

and which is smeared out so that it changes smoothly across an interfacial zone with characteristic thickness ε. The newly obtained regularized level set function is denoted here by χ_ε. As the elastic solid moves in the fluid, the level set function χ_ε is advected by the fluid velocity, i.e.

$$\frac{\partial \chi_\varepsilon}{\partial t} + \boldsymbol{v}\cdot\nabla\chi_\varepsilon = 0, \tag{8.4.1}$$

and, in order to ensure stability of the method and a good resolution of the interfacial zone, the level set must also be reinitialized during the simulations (see for example [156]). This is done by solving the following equation:

$$\mathrm{div}\left[\bar{\chi}_\varepsilon(1-\bar{\chi}_\varepsilon)\frac{\nabla\chi_\varepsilon}{|\nabla\chi_\varepsilon|}\right] - \varepsilon\Delta\bar{\chi}_\varepsilon = 0,$$

where χ_ε is the solution of the advection equation (8.4.1) at the reinitialization time. The solution $\bar{\chi}_\varepsilon$ is then assigned to χ_ε and the evolution of χ_ε is then continued according to (8.4.1). Note that the reinitialization smears out the level set function to the required diffuse profile. In one dimension, this reads

$$\chi_\varepsilon(x) = \frac{1}{2}\left(1 + \tanh\frac{x}{2\varepsilon}\right), \tag{8.4.2}$$

so indeed the parameter ε controls the thickness of the diffuse interface. Similarly, the material parameters ρ, μ, and G are prescribed to change smoothly across the interface by setting

$$\rho(\chi_\varepsilon) \coloneqq \chi_\varepsilon\rho_s + (1-\chi_\varepsilon)\rho_f, \quad \mu(\chi_\varepsilon) \coloneqq \chi_\varepsilon\mu_s + (1-\chi_\varepsilon)\mu_f, \quad G(\chi_\varepsilon) \coloneqq \chi_\varepsilon G_s + (1-\chi_\varepsilon)G_f.$$

In order to reduce the complexity of the problem, which is further enhanced by the necessity to solve for the evolution of the tensor $\mathbf{B}$, we simplified the model by assuming that both the elastic deformations and the velocities are small. Consequently, we omit all convective terms, including the one in the evolution equation for $\mathbf{B}$. In the same spirit, we also assume that $(\nabla \boldsymbol{v})\mathbf{B} \approx \nabla \boldsymbol{v}$ in the solid, while in the fluid the equation is regularized in such a way that the evolution equation for $\mathbf{B}$ can be solved easily.[1] In particular, due to the regularized level set function we introduce a global left Cauchy-Green tensor $\mathbf{B}\colon \Omega \times [0,T] \to \mathbb{R}^{N\times N}$, where $\mathbf{B} = (1-\chi_\varepsilon)\mathbf{B}_f + \chi_\varepsilon \mathbf{B}_s$, with $\mathbf{B}_f$ being a canonical approximation of the identity. All together we arrive at the following set of equations that we solved numerically

$$\begin{aligned}
\operatorname{div}\boldsymbol{v} &= 0,\\
\frac{\partial \chi_\varepsilon}{\partial t} + \boldsymbol{v}\cdot\nabla\chi_\varepsilon &= 0, \qquad \operatorname{div}\left[\bar{\chi}_\varepsilon(1-\bar{\chi}_\varepsilon)\frac{\nabla\chi_\varepsilon}{|\nabla\chi_\varepsilon|}\right] - \varepsilon\Delta\bar{\chi}_\varepsilon = 0,\\
\rho(\chi_\varepsilon)\frac{\partial \boldsymbol{v}}{\partial t} &= \operatorname{div}\sigma, \qquad \sigma = -p\mathbb{I}_N + 2\mu(\chi_\varepsilon)\mathbb{D} + G(\chi_\varepsilon)\mathbb{B}^d,\\
\frac{\partial \mathbf{B}_s}{\partial t} &= 2\mathbb{D}, \qquad \frac{\partial \mathbf{B}_f}{\partial t} + \mathbf{B}_f - \mathbb{I}_N = \mathbb{O}, \qquad \mathbf{B} = (1-\chi_\varepsilon)\mathbf{B}_f + \chi_\varepsilon\mathbf{B}_s,
\end{aligned}$$

which is optimized for the performance of the numerical simulations. The problem is implemented with the finite element method in the open source finite element library FEniCS (see [4]) and discretized on a regular triangular mesh. Finally, the equations for $\mathbf{B}_s$ and $\mathbf{B}_f$ are solved locally and their solutions then inserted immediately into the balance equation of linear momentum. In the numerical implementation, the time derivatives are approximated with the backward Euler time scheme, that is,

$$\frac{\partial \mathbf{B}}{\partial t} \approx \frac{\mathbf{B}-\mathbf{B}_0}{\Delta t},$$

where $\mathbf{B}_0$ represents the value of $\mathbf{B}$ at the previous time step. The time step Δt, in turn, is chosen adaptively according to the speed of the fluid from the previous time step in such a way that the CFL condition holds, i.e.

$$\frac{v_{\max}\Delta t}{h_{\min}} = \frac{1}{2}.$$

Here $v_{\max}$ denotes the maximum value of the velocity magnitude and $h_{\min}$ is the minimum size of the element. The local integration of $\mathbf{B}$ gives

$$\mathbf{B}_s = \mathbf{B}_0 + 2\mathbb{D}\Delta t, \quad \mathbf{B}_f = \frac{\mathbb{I}_N + \mathbf{B}_0\Delta t}{1+\Delta t}, \quad \mathbf{B} = (1-\chi_\varepsilon)\mathbf{B}_f + \chi_\varepsilon\mathbf{B}_s.$$

Thus, the only global unknowns are the velocity $\boldsymbol{v}$, the pressure p, and the level-set function χ_ε. While velocity and pressure are approximated by the classical P2/P1 Taylor–Hood element, the level-set function is approximated with the P2 element. The non-linearities are treated with the exact Newton method and the resulting set of linear equations is then solved with the direct solver MUMPS.

8.4.2.1 Numerical results

We have numerically investigated the rebound of an elastic ball in a viscous fluid environment. The radius of the ball considered is 0.2 m and its center is initially located 0.5 m from the bottom wall in a square container of size 0.8 m. At the boundary of the container, no-slip boundary conditions are prescribed. As mollification parameter we used $\varepsilon = 0.0011$.

The initial velocity of the ball is 0.5 m/s (downwards) and the fluid is initially at rest. Throughout the simulation, body forces have been switched off. The following material parameters have been prescribed:

$$\begin{aligned}
\rho_f &= 1.0\ \text{kg/m}^3, & \rho_s &= 1001.0\ \text{kg/m}^3,\\
\mu_f &= 0.1\ \text{Pa s}, & \mu_s &= 0.0\ \text{Pa s},\\
G_f &= 0.0\ \text{Pa}, & G_s &= 50\,000.0\ \text{Pa}.
\end{aligned}$$

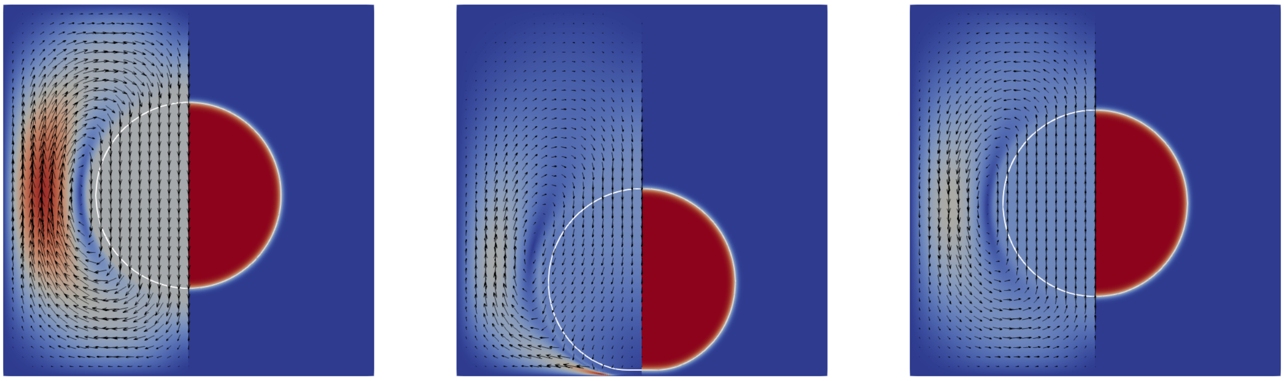


Figure 8.5: Velocity (left side) and level-set (right side) at (a) moving down (b) rebound (c) moving up. The white contour depicts the interface where the value of the level-set function is equal to 0.5.

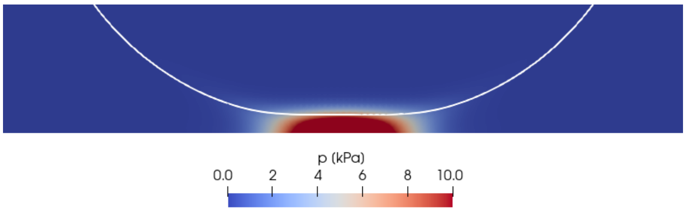


Figure 8.6: Pressure in the fluid at the time of rebound.

Figure 8.5 shows the snapshots of the velocity and level-set fields at three time instances: moving down (panel a), during the rebound (panel b) and moving up (panel c). Since the viscosity considered is relatively high, the process is dissipative and the velocity magnitude is gradually decreasing with time. We remark that contact between the elastic ball and the wall never takes place – it is indeed prevented by the development of a high-pressure region around the point where contact would normally be expected, see Figure 8.6. The formation of this hydrodynamic pressure spike then facilitates the rebound.

It is also worth noting how, in Figure 8.6, the ball gets deformed, with the "impacting" face becoming very flat during the rebound phase. We fitted the shape of the interface at the bottom of the ball with the function

$$y = d_1 + d_2|x|^a,$$

where, for simplicity, we fixed $d_2 = 1/(2R) = 2.5$. This choice of d_2 is optimal for a circle of radius R. The dependence of the exponent a on h is shown in Figure 8.7, which demonstrates the significant flattening during the rebound, that is, we observe larger values of a as the distance h approaches its minimal value.

8.4.3 A comparison of the two models

Let us now compare in detail the numerical simulations performed for the reduced (effectively deformable) ODE model with the finite element solutions obtained for the full FSI problem. Throughout the section, we refer in particular to Figure 8.8, where we display the distance to the wall h as a function of time for both models and for several values of the viscosity parameter. It is worth noting that for the FSI model h is defined as the distance between the wall and 0.5-level set. Observe that with the choice of parameters made in subsubsection 8.4.1.1,

[1]Note that the set of equations obtained through these reductions (that is, in the regime where deformations and velocities are small) was compared to the full (large strain, large velocities) model. The difference turned out to be negligible, which implies that the simplifications are admissible.

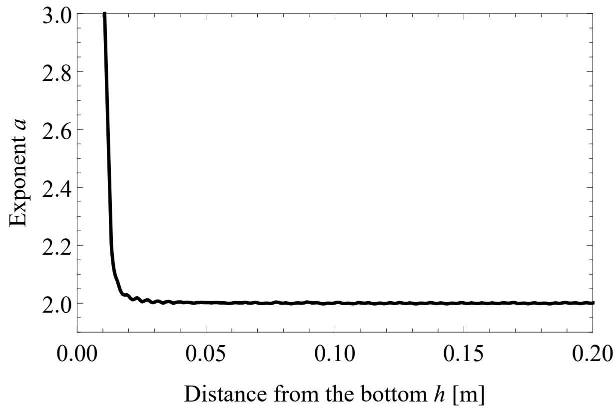
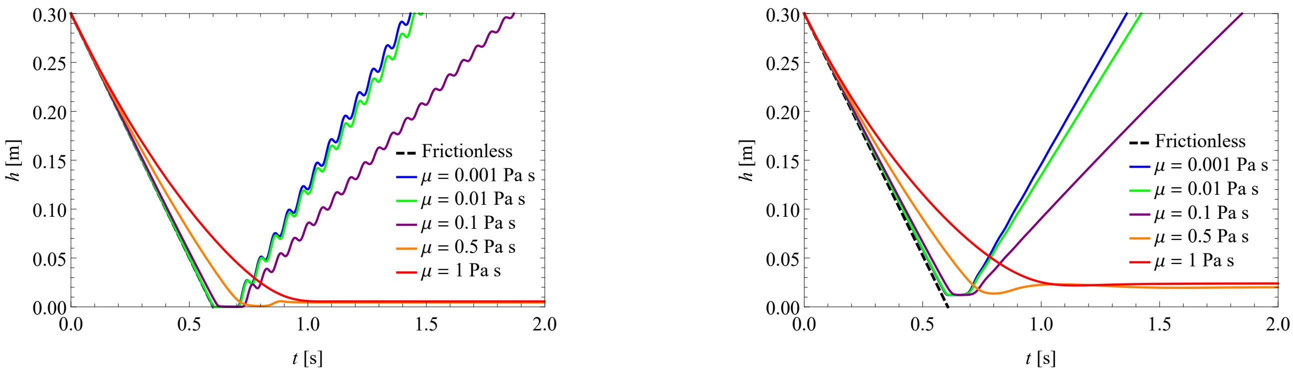


Figure 8.7: The dependence of the exponent a on h.

the match between the two sets of solutions is satisfactory in terms of the duration of the rebound phase and also regarding the mean body velocity after the rebound.



(a) (b)

Figure 8.8: Comparison of numerically obtained solutions to the reduced model of ODEs (a) and FEM solutions (b).

A comment on the oscillatory behavior of h after rebound in the ODE solutions is in order. While the oscillations are unmistakably due to the internal spring-mass system, which is effectively undamped for low viscosity values, it is worth mentioning that somewhat similar "free oscillations" were also observed in the simulations performed for the full FSI model for sufficiently small values of the shear modulus G (not included in this paper). This can be seen as an indication that this particular feature of the ODE model should not be a-priori regarded as completely non-physical.

Next, we note that the vertical offset of the solutions from the x-axis suggests that the finite element solutions bounce off at greater distances from the wall when compared to solutions of the ODE model. To some extent, this can be attributed to the effect of the level set approximation, i.e., to the diffuse interface between the ball and the fluid in the FSI model. Indeed, the fact that the material parameters are "smeared" over the diffuse interface of thickness ε (see (8.4.2)) poses certain limitations on the minimal distance that (the 0.5-level set of) the deformable structure may reach. Despite this issue, we are confident that for our choices of ε, spatial resolution, and high-enough viscosity μ, the rebound due to the pressure singularity is not a mere artifact of the diffuse interface approach.

8.5 Summary and concluding remarks

In this paper, we investigate how serious and physically relevant is the so called no-contact paradox, that is, the absence of body-body and body-wall topological contact for elastic particles in an incompressible Stokes fluid with no-slip boundary conditions imposed on all boundaries. We were driven partially by the question whether the no-slip boundary condition in fluid-structure interaction problems must be avoided and branded as non-physical, or whether it can be redeemed somehow. We believe that we have provided an affirmative answer to the latter question, as we have shown that even in the absence of topological contact between an elastic body in motion towards a rigid wall, an effective rebound can be achieved, which is physical in the sense that it withstands the vanishing viscosity limit.

It is known and it has been proved rigorously that neither topological contact nor rebound are possible for perfectly rigid bodies (for a demonstration of this phenomenon see, for example, Figure 1.2; see also subsubsection 8.2.3.1 and Theorem 8.2.6). On the other hand, the inclusion of elastic deformations of the solid bodies has been hypothesized as a promising ingredient towards obtaining a physical rebound. We tried to follow this path, yet, to simplify the notoriously difficult fluid-structure interaction problem, we devised a simplified ODE model (see subsubsection 8.2.3.2) which captures the features that we believe to be essential.

The model comprises a ball (immersed in an incompressible Stokes fluid with no-slip boundary conditions prescribed both on the boundary of the container and on the fluid-solid interface) which is moving towards a rigid wall. As a simplified model of elasticity, we introduced a single scalar internal parameter ξ, which can be visualized as the elongation of a spring attached to a certain mass within the ball (see Figure 8.1). In this setting, when the ball is subjected to the drag and internal push-and-pull from the spring, we have proved that contact cannot happen in finite time for any value of the viscosity parameter, and moreover that there is no rebound in the vanishing viscosity limit (see subsubsection 8.2.3.1; see also Theorem 8.3.2 and Theorem 1.3.1). In view of this fact, we conjectured that the internal mechanical energy storage alone is not a sufficient mechanism to ensure particle rebound.

As a next step in our analysis, we have investigated how allowing for deformations of the solid body changes the picture. This was achieved by coupling the internal deformation parameter with the drag formula. As a model case, we considered a one-parameter family of graphs describing the near-to-contact shape of the solid body by a general power function of the form

$$y = h + c|x|^{\alpha},$$

where h is the distance of the closest point to the wall, while c and α are parameters possibly depending on the elastic deformation, i.e., we take $c = c(\xi)$ and $\alpha = \alpha(\xi)$. We have derived the corresponding parameterization of the drag force exerted by the fluid on the ball for such “deformed” configurations as they approach the wall (see Subsection 8.6.3). These formulas are consistent with the standard lubrication (Reynolds’) theory (see subsubsection 8.2.4.2 and Subsection 8.6.3) and read

$$\mathcal{D}(h,\xi) \sim h^{-\frac{3\alpha(\xi)}{1+\alpha(\xi)}} \quad \text{if } N = 2, \qquad \mathcal{D}(h,\xi) \sim h^{\frac{1-3\alpha(\xi)}{1+\alpha(\xi)}} \quad \text{if } N = 3.$$

It is worth noting that in the context of the standard Hertz theory of contact, the shape of the solid body does not change dramatically in the sense that “spheres deform to ellipses”, so that the shape exponent α remains unaltered. Inspired however by real-world observations, where much more dramatic changes in the “flatness” of an impacting body are often observed, we relaxed the assumption of the Hertz theory that α is constant and allowed it to change according to the elastic deformation described by ξ.

Surprisingly, this appears to be the key missing ingredient – the feature that allows to reproduce a physically meaningful rebound. Indeed, in Section 8.3 we have proved the possibility of a rebound that withstands the vanishing viscosity limit. Furthermore, our proofs are supplemented with numerical simulations of the ODE system (see Figure 8.4). It is worth noting that the reduced model can predict rebound while incorporating at the same time the defining feature of our problem, that is, the lack of topological contacts. This is a direct consequence of the fact that, in view of the imposed no-slip boundary conditions, the drag force exerted by the fluid blows up as the distance of the body from the wall approaches zero.

Not only the ODE model admits a rigorous analysis of the effective rebound process, but, despite its apparent simplicity, it also shows a striking capability to reproduce qualitative characteristics of the rebound process when compared to finite element simulations of the full fluid-structure interaction problem (see Figure 8.8). This gives us the confidence to consider the rigorously proved result for the ODE model as a reliable proof-of-concept for the general fluid-structure interaction problem outlined in Section 8.2 and to strengthen our main conjecture from the introduction, i.e., the claim that *a qualitative change in the flatness of the solid body as it approaches the wall, together with some elastic energy storage mechanism within the body, allows for a physically meaningful rebound even in the absence of topological contact.*

8.6 Appendix of the chapter

8.6.1 Reduction of the full elastic dynamics to the wave equation

In this appendix, we motivate the design of the reduced model from subsubsection 8.2.3.2; in particular, we formally justify the reduction of the elastic structure to a spring-mass model. To simplify the derivation, let us for a moment replace the incompressible neo-Hookean material with a compressible one, for which the elastic strain energy is given by

$$\mathcal{W}_{\mathrm{comp}} := \frac{G}{2}(|\mathcal{F}|^2 - N - 2\ln J).$$

Then the resulting first Piola–Kirchhoff stress tensor reads

$$\mathbb{P} = \frac{\partial \mathcal{W}_{\mathrm{comp}}}{\partial \mathbb{F}} = G\left(\mathcal{F} - \mathcal{F}^{-\mathrm{T}}\right). \tag{8.6.1}$$

Let us assume that the deformations are small, i.e., $|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}| \ll 1$. By performing Taylor expansions around the identity readily obtain

$$\mathcal{F}^{-\mathrm{T}} = \left(\mathbb{I}_N + (\nabla_{\boldsymbol{X}}\boldsymbol{\eta})^{\mathrm{T}}\right)^{-1} = \mathbb{I}_N - (\nabla_{\boldsymbol{X}}\boldsymbol{\eta})^{\mathrm{T}} + \mathcal{O}\left(|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2\right), \tag{8.6.2}$$

$$J = \det(\mathbb{I}_N + \nabla_{\boldsymbol{X}}\boldsymbol{\eta}) = 1 + tr(\nabla_{\boldsymbol{X}}\boldsymbol{\eta}) + \mathcal{O}\left(|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2\right) = 1 + \mathrm{div}_{\boldsymbol{X}}\boldsymbol{\eta} + \mathcal{O}\left(|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2\right). \tag{8.6.3}$$

Notice that upon inserting (8.6.2) into (8.6.1) and neglecting all terms of order $\mathcal{O}\left(|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2\right)$ we recover the well-known formula of the stress for small strains, that is

$$\mathbb{P} \approx G\left(\nabla_{\boldsymbol{X}}\boldsymbol{\eta} + (\nabla_{\boldsymbol{X}}\boldsymbol{\eta})^{\mathrm{T}}\right) =: 2G\mathbb{D}_{\boldsymbol{X}}(\boldsymbol{\eta}). \tag{8.6.4}$$

Under the assumption that the material is nearly incompressible, that is $\rho_s \approx \rho_s^0$, combining the balance of mass $(8.2.3)_2$ with (8.6.3) and neglecting all terms of order $\mathcal{O}\left(|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2\right)$, we obtain

$$\rho_s^0 \approx J\rho_s^0 \Rightarrow J \approx 1 \Rightarrow \mathrm{div}_{\boldsymbol{X}}\boldsymbol{\eta} \approx 0.$$

Moreover, inserting (8.6.4) in the balance of linear momentum $(8.2.3)_1$ yields the classical wave equation for $\boldsymbol{\eta}$, that is

$$\rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \approx G\Delta_{\boldsymbol{X}}\boldsymbol{\eta} + \rho_s^0 \boldsymbol{B}.$$

8.6.2 Formal energy equality for the FSI problem

We begin this appendix with the formal derivation of an energy equality for the fully Eulerian model (8.2.17). To this end, we multiply the balance of linear momentum by the velocity $\boldsymbol{v}$ and integrate the resulting relation over Ω. Then, by the Gauss theorem and (8.2.16) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\boldsymbol{v}|^2\,d\boldsymbol{x} &= -\int_\Omega \left(-p\mathbb{I}_N + 2\mu\mathbb{D}(\boldsymbol{v}) + G\mathbf{B}^d\right) : \mathbb{D}(\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,d\boldsymbol{x}\\ &= -\int_\Omega \left(2\mu|\mathbb{D}(\boldsymbol{v})|^2 + G\mathbf{B} : \mathbb{D}(\boldsymbol{v})\right)d\boldsymbol{x} + \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\,d\boldsymbol{x},\end{aligned} \tag{8.6.5}$$

where in the last equality we have used the incompressibility condition. Next, we take the trace of the transport equation for $\mathbf{B}$ (see (8.2.14)), we multiply it by $G/2$, integrate it over Ω, and use again (8.2.16) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{G}{2} tr\mathbf{B}\, d\boldsymbol{x} = \int_\Omega G\mathbf{B} : \mathbb{D}(\boldsymbol{v})\, d\boldsymbol{x}. \tag{8.6.6}$$

Upon adding together (8.6.5) and (8.6.6) we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\boldsymbol{v}|^2\, d\boldsymbol{x} + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{G}{2} tr\mathbf{B}\, d\boldsymbol{x} + \int_\Omega 2\mu|\mathbb{D}(\boldsymbol{v})|^2\, d\boldsymbol{x} = \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\, d\boldsymbol{x}.$$

In view of (8.2.16), the second term in the previous equality can be modified in such a way that the energy balance takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\boldsymbol{v}|^2\, d\boldsymbol{x} + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{G}{2}(tr\mathbf{B} - N)\, d\boldsymbol{x} + \int_\Omega 2\mu|\mathbb{D}(\boldsymbol{v})|^2\, d\boldsymbol{x} = \int_\Omega \rho\boldsymbol{b}\cdot\boldsymbol{v}\, d\boldsymbol{x}. \tag{8.6.7}$$

Upon integrating (8.6.7) over the time interval $(0,t)$, and recalling that $\mu = 0$ in $\mathcal{B}(t)$, finally yields

$$\begin{aligned}\int_\Omega \left(\frac{\rho(\boldsymbol{x},t)}{2}|\boldsymbol{v}(\boldsymbol{x},t)|^2 + \frac{G(\boldsymbol{x},t)}{2}(tr\mathbf{B}(\boldsymbol{x},t) - N)\right) d\boldsymbol{x} + \int_0^t \int_{\mathcal{F}(s)} 2\mu_f|\mathbb{D}(\boldsymbol{v}(\boldsymbol{x},s))|^2\, d\boldsymbol{x}ds \\ = \int_\Omega \left(\frac{\rho(\boldsymbol{x},0)}{2}|\boldsymbol{v}(\boldsymbol{x},0)|^2 + \frac{G(\boldsymbol{x},0)}{2}(tr\mathbf{B}(\boldsymbol{x},0) - N)\right) d\boldsymbol{x} + \int_0^t \int_\Omega \rho(\boldsymbol{x},s)\boldsymbol{b}(\boldsymbol{x},s)\cdot\boldsymbol{v}(\boldsymbol{x},s)\, d\boldsymbol{x}ds.\end{aligned}$$

One can observe now that the first integral consists of two terms: the kinetic energy, with density $E_k = \rho|\boldsymbol{v}|^2/2$, and the elastic strain energy, with density given by $\mathcal{W}$ expressed in the Eulerian framework (see (8.2.4)). Notice also that the second integral respresents the standard Newtonian dissipation $2\mu|\mathbb{D}|^2$. Upon transforming the integrals over $\mathcal{B}(t)$ to the reference configuration $\mathcal{B}_0$, recalling that $G = 0$ in $\mathcal{F}(t)$, that $J = 1$, and by employing the identity $tr\mathbf{B} = tr\left(\mathcal{F}\mathcal{F}^{\mathrm{T}}\right) = |\mathcal{F}|^2$, the energy equality can be recast to a form which is perhaps more standard in the FSI setting (cf. the energy equality for the reduced model (8.3.5)):

$$\begin{aligned}\mathcal{K}(t) + \int_{\mathcal{B}_0}\mathcal{W}(\boldsymbol{X},t)\, d\boldsymbol{X} + \int_0^t\int_{\mathcal{F}(s)} 2\mu_f|\mathbb{D}(\boldsymbol{v}(\boldsymbol{x},s))|^2 d\boldsymbol{x}ds = \mathcal{K}(0) + \int_{\mathcal{B}_0}\mathcal{W}(\boldsymbol{X},0)\, d\boldsymbol{X} \\ + \int_0^t\int_{\mathcal{F}(s)} \rho_f\boldsymbol{b}(\boldsymbol{x},s)\cdot\boldsymbol{v}(\boldsymbol{x},s)\, d\boldsymbol{x}ds + \int_0^t\int_{\mathcal{B}_0}\rho_0^s\boldsymbol{B}(\boldsymbol{X},s)\cdot\frac{\partial\boldsymbol{\eta}}{\partial t}(\boldsymbol{X},s)\, d\boldsymbol{X}ds,\end{aligned}$$

where $\mathcal{K}(t)$ denotes the kinetic energy of the system at time t, that is

$$\mathcal{K}(t) := \int_{\mathcal{F}(t)}\frac{\rho_f}{2}|\boldsymbol{v}(\boldsymbol{x},t)|^2\, d\boldsymbol{x} + \int_{\mathcal{B}_0}\frac{\rho_0^s}{2}\left|\frac{\partial\boldsymbol{\eta}}{\partial t}(\boldsymbol{X},t)\right|^2\, d\boldsymbol{X}.$$

Finally, if we now restrict our attention to the regime of small deformations (see Subsection 8.6.1) and therefore neglect all terms of the second and higher order in $|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|$, we get

$$|\mathcal{F}|^2 = |\mathbb{I}_N + \nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2 = N + 2\mathrm{div}_{\boldsymbol{X}}(\boldsymbol{\eta}) + |\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2 \approx N + |\nabla_{\boldsymbol{X}}\boldsymbol{\eta}|^2.$$

This readily implies that

$$\int_{\mathcal{B}_0}\mathcal{W}(\boldsymbol{X},t)\, d\boldsymbol{X} \approx \int_{\mathcal{B}_0}\frac{G_s}{2}|\nabla_{\boldsymbol{X}}\boldsymbol{\eta}(\boldsymbol{X},t)|^2\, d\boldsymbol{X}.$$

8.6.3 Drag force estimates

The material in this appendix is meant to complement our treatment of the drag force in Subsection 8.2.4 by providing a proof of the analytical estimates and a derivation of the drag force predicted by the lubrication approximation theory.

The drag force based on the variational formulation

We begin this first part of the appendix with the proof of Theorem 8.2.4. The argument we present here is directly adapted from the proof of Lemma 3 in [101].

Proof of Theorem 8.2.4. We divide the proof into two steps.
Step 1: Assume first that $N = 2$. Then, by a density argument, it is enough to show that for every $\boldsymbol{v} \in C_c^\infty(\mathbb{R}^2_+;\mathbb{R}^2) \cap V_h$

$$c_1 \le h^{\frac{3\alpha}{1+\alpha}} \|\nabla \boldsymbol{v}\|^2_{L^2},$$

where c_1 is a positive constant independent of $\boldsymbol{v}$. To see this, using the notation introduced in (8.2.30), we define

$$\mathcal{F}_h(\delta) \coloneqq \{(x_1, x_2) : |x_1| < \delta, 0 < x_2 < g(|x_1|)\} \subset \mathcal{F}_h$$

and integrate $\operatorname{div}\boldsymbol{v} = 0$ in $\mathcal{F}_h(\delta)$ to obtain

$$\int_{\partial\mathcal{F}_h(\delta)\cap\partial\mathcal{B}_h} \boldsymbol{v}\cdot\boldsymbol{n}\, d\mathcal{H}^1 = -\int_{\mathcal{F}_h(\delta)\cap\{|x_1|=\delta\}} \boldsymbol{v}\cdot\boldsymbol{n}\, d\mathcal{H}^1.$$

Since $\boldsymbol{v} = \boldsymbol{e}_2$ on $\partial\mathcal{B}_h$, we see that

$$\mathcal{L} \coloneqq \int_{\partial\mathcal{F}_h(\delta)\cap\partial\mathcal{B}_h} \boldsymbol{v}\cdot\boldsymbol{n}\, d\mathcal{H}^1 = \int_{\partial\mathcal{F}_h(\delta)\cap\partial\mathcal{B}_h} n_2\, d\mathcal{H}^1 = 2\delta,$$

where the last equality is obtained via a direct computation, parameterizing the domain of integration. Similarly, but also using the fact that $g(\delta) = g(-\delta)$, we obtain that

$$\mathcal{R} \coloneqq \int_{\mathcal{F}_h(\delta)\cap\{|x_1|=\delta\}} \boldsymbol{v}\cdot\boldsymbol{n}\, d\mathcal{H}^1 = \int_0^{g(\delta)} \left(v_1(\delta, x_2) - v_1(-\delta, x_2)\right)\, dx_2,$$

and an application of Hölder's and Poincaré's inequalities yields

$$\begin{aligned}\int_0^{g(\delta)} |v_1(\delta, x_2) - v_1(-\delta, x_2)|\, dx_1 &\le g(\delta)^{1/2}\|v_1(\delta,\cdot) - v_1(-\delta,\cdot)\|_{L^2((0,g(\delta));\mathbb{R}^2)}\\ &\le g(\delta)^{3/2}\left\|\frac{\partial v_1}{\partial x_2}(\delta,\cdot) - \frac{\partial v_1}{\partial x_2}(-\delta,\cdot)\right\|_{L^2((0,g(\delta));\mathbb{R}^2)}.\end{aligned}$$

In turn,

$$2\delta = \mathcal{L} \le |\mathcal{R}| \le \sqrt{2} g(\delta)^{3/2}\left(\int_0^{g(\delta)} \left(|\nabla v_1(\delta, x_2)|^2 + |\nabla v_1(-\delta, x_2)|^2\right)\, dx_2\right)^{1/2}.$$

Integrating the previous inequality over $\delta \in (0, r)$ yields

$$\begin{aligned} r^2 &\le \sqrt{2}\sup_{\delta\in(0,r)}\left\{g(\delta)^{3/2}\right\}\int_0^r\left(\int_0^{g(\delta)}\left(|\nabla v_1(\delta,x_2)|^2 + |\nabla v_1(-\delta,x_2)|^2\right)\, dx_2\right)^{1/2} d\delta\\ &\le \sqrt{2}\sup_{\delta\in(0,r)}\left\{g(\delta)^{3/2}\right\} r^{1/2}\left(\int_0^r\int_0^{g(\delta)}\left(|\nabla v_1(\delta,x_2)|^2 + |\nabla v_1(-\delta,x_2)|^2\right)\, dx_2 d\delta\right)^{1/2}\\ &= \sqrt{2}\sup_{\delta\in(0,r)}\left\{g(\delta)^{3/2}\right\} r^{1/2}\|\nabla\boldsymbol{v}\|_{L^2},\end{aligned}$$

where in the second to last step we have used Hölder's inequality. Consequently, recalling that g is given as in (8.2.30), if we let $r = h^{1/(1+\alpha)}$ we obtain

$$\frac{1}{\sqrt{2}} \le \left(\frac{\sup\left\{h + \gamma\delta^{1+\alpha} : \delta\in(0,r)\right\}}{r}\right)^{3/2}\|\nabla\boldsymbol{v}\|_{L^2} = (1+\gamma)^{3/2} h^{\frac{3\alpha}{2(1+\alpha)}}\|\nabla\boldsymbol{v}\|_{L^2}.$$

The desired result readily follows.
Step 2: Now, assume that $N = 3$. Reasoning as in the previous step, but with the aid of cylindrical coordinates (δ, θ, z), we readily deduce that

$$\pi\delta^2 \le g(\delta)^{3/2}\delta\int_0^{2\pi}\left(\int_0^{g(\delta)}|\nabla\boldsymbol{v}|(\delta,\theta,z)|^2\, dz\right)^{1/2} d\theta$$

holds for every $\boldsymbol{v} \in C_c^\infty(\mathbb{R}^3_+;\mathbb{R}^3) \cap V_h$. Thus, integrating the previous inequality over $\delta \in (0,r)$ and by means of Hölder's inequality, we get

$$\frac{\pi r^3}{3} \le \sup_{\delta\in(0,r)} \left\{ g(\delta)^{3/2}\delta^{1/2} \right\} \int_0^r \delta^{1/2} \int_0^{2\pi} \left(\int_0^{g(\delta)} |\nabla \boldsymbol{v}|(\delta,\theta,z)|^2 \, dz \right)^{1/2} d\theta d\delta$$
$$\le \sqrt{2\pi} \sup_{\delta\in(0,r)} \left\{ g(\delta)^{3/2}\delta^{1/2} \right\} r^{1/2} \|\nabla \boldsymbol{v}\|_{L^2}.$$

Therefore, setting once again $r = h^{1/(1+\alpha)}$, we get

$$\frac{1}{3}\sqrt{\frac{\pi}{2}} \le \sup_{\delta\in(0,r)} \left\{ g(\delta)^{3/2} \right\} r^{-2} \|\nabla \boldsymbol{v}\|_{L^2} \le (1+\gamma)^{3/2} h^{\frac{3\alpha-1}{2(1+\alpha)}} \|\nabla \boldsymbol{v}\|_{L^2}.$$

This concludes the proof. □

Next, we turn our attention to the proof of Theorem 8.2.5, which we only sketch here. Recalling that by definition $D(h) = \min\{\mathcal{J}(\boldsymbol{u};\mathcal{F}_h) : \boldsymbol{u} \in V_h\}$, the conclusions of Theorem 8.2.5 follow if we can exhibit a competitor, namely $\boldsymbol{w}_h \in V_h$, for which $\mathcal{J}(\boldsymbol{w}_h;\mathcal{F}_h)$ is bounded from above by the right-hand side of (8.2.29). To achieve this, one has to construct a velocity field which allows for the fluid to escape the aperture in between the solid body and the boundary of the container in a nearly optimal way. For $N = 2$, such a construction was carried out by Gérard-Varet and Hillairet (see Section 4.1 and Proposition 8 in [78]). Their argument is adapted from the analogous construction for a two-dimensional disk, due to Hillairet (see Section 4 in [101]). The construction for $N = 3$ is due to Hillairet and Takahashi (see Section 3.1 in [102]) for a sphere, and can be suitably modified for the more general shapes that we consider in this paper.

The drag force based on the Reynolds approximation In this second part of the appendix, we are interested in approximating the drag force exerted on a particle immersed in a Newtonian fluid, which is moving towards a rigid wall, when both the wall and the fluid-solid interface are subjected to no-slip boundary conditions. Considering an axi-symmetric situation (as in Figure 8.5), a good approximation can be found by calculating and integrating the pressure under the solid body, which indeed represents the major contribution to the drag force [124]. The pressure profile can be estimated using the so-called lubrication, or Reynolds', approximation. This yields the following ODE (see eq. (7-256) in [124] for $N = 2$; see eq. (4.22) in [11] for $N = 3$):

$$\frac{d}{dr}\left(r^{N-2} g^3 \frac{dp}{dr} \right) = 12\mu r^{N-2}\dot{h} \qquad N = 2,3,$$

where r is the distance from the symmetry axis, and g is defined as in (8.2.30). Integrating the previous equation from 0 to r', and recalling that by assumption the particle is axi-symmetric, yields

$$\frac{dp}{dr}(r') = 12\mu\dot{h}\frac{r'}{g(r')^3(N-1)} \qquad N = 2,3.$$

Integrating now between r and R we obtain

$$p(r) - p(R) = -\frac{12\mu\dot{h}}{N-1}\int_r^R \frac{r'}{g(r')^3}\, dr'. \tag{8.6.8}$$

Assuming $R \gg 1$ then the pressure difference on the left-hand side corresponds to the actual dynamic pressure, which constitutes the main contribution to the drag force. Integrating the pressure difference over the surface of the solid body yields the (pressure contribution) to the drag force

$$\tilde{\boldsymbol{F}}_{\text{lub}} := -\int_\Gamma (p(r) - p(R))\, \boldsymbol{n}\, d\mathcal{H}^{N-1},$$

where $\boldsymbol{n}$ is the outer unit normal to Γ (pointing inside the fluid). By symmetry, we only need to evaluate the vertical component of the force, since all other components are zero:

$$\tilde{F}_{\text{lub}} := \tilde{\boldsymbol{F}}_{\text{lub}} \cdot \boldsymbol{e}_N = -\int_\Gamma (p(r) - p(R))\, \boldsymbol{n}\, d\mathcal{H}^{N-1} \cdot \boldsymbol{e}_N.$$

In view of (8.6.8), letting $R \to \infty$ in the expression above yields

$$F_{\text{lub}} \coloneqq \lim_{R\to\infty} \tilde{F}_{\text{lub}} = -12\mu\dot{h} \begin{cases} 2\displaystyle\int_0^\infty \int_r^\infty \frac{r'}{g(r')^3}\,dr'dr & \text{if } N = 2, \\ \pi\displaystyle\int_0^\infty \int_r^\infty \frac{rr'}{g(r')^3}\,dr'dr & \text{if } N = 3, \end{cases}$$

which is the desired approximation.

Chapter 9

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