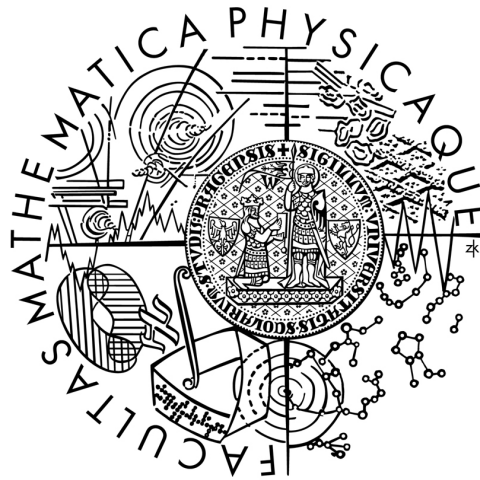


Charles University in Prague
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MASTER THESIS



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Dimensional transmutation in quantum theory

Institute of Theoretical Physics

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Study program: Theoretical physics

2008

I would like to thank Prof. Jiří Hořejší for his patience with my delayed progress, valuable discussions on the subject and many suggestions and corrections to the text, Prof. Pavel Exner, for his advices on the quantum-mechanical part, and Mgr. Ondřej Maršálek and Dr. Michael Prouza, for their help in obtaining the literature; the latter also for the corrections of the final manuscript – and to my parents, who supported me financially even in the extended length of the study. Nevertheless, the most grateful I am to my wife, Ivaana, for her love and support. Typeset in LyX, an open source document processor.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 8. 8. 2008

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Název práce: Dimenzionální transmutace v kvantové teorii

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Katedra (ústav): Ústav teoretické fyziky

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Abstrakt: Práce se zabývá dvěma vybranými modely – z kvantové teorie pole nehmotnou skalární elektrodynamikou (tzv. Colemanův-Weinbergův model) a z kvantové mechaniky kontaktním (δ -funkčním) potenciálem (ve dvou dimenzích) – které jsou zdánlivě invariantní vůči nějakému druhu škálových transformací, a tak při vhodné volbě jednotek obsahují pouze bezrozměrné parametry. Ukazuje se, že i kvantově-mechanickém případě je potřeba formální definici modelu doplnit další procedurou a že užití různých fyzikálních regularizací vede ke stejným výsledkům, které se navíc shodují s předpověďmi matematicky rigoróznější metody samosdružených rozšíření operátorů. V této práci prezentujeme podrobné výpočty podporující tento závěr; na rozdíl od běžné literatury tak ovšem činíme přímočarými metodami, které umožňují krok za krokem vidět, proč tomu tak je – veškeré potřebné znalosti z pokročilejší funkcionální analýzy jsou navíc shrnuty v dodatku. V části věnované kvantové teorii pole aplikujeme podobný přístup, kde výsledky získané pomocí abstraktních funkcionálních metod “znovuobjevujeme” pomocí obvyklé poruchové teorie. V jejím rámci navíc ukazujeme, jak z teorie obdržet předpovědi i pro jiné veličiny, než samotné hmoty zúčastněných částic.

Klíčová slova: dimenzionální transmutace, skalární elektrodynamika, kontaktní interakce, samosdružená rozšíření

Title: Dimensional transmutation in quantum theory

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Abstract: This work deals with two models – from the quantum field theory it is the massless scalar electrodynamics (the so-called Coleman-Weinberg model) and from quantum mechanics it is the contact (δ -function) potential (in two dimensions) – that are apparently invariant under some sort of scale transformations and thus they, in suitably chosen units, contain only dimensionless parameters. It turns out that even in the quantum-mechanical case one has to add an additional procedure to the formal definition of the model and that the use of different physical regulators leads to the same results, that furthermore agree with the predictions of the mathematically rigorous method of self-adjoint operator extensions. In this work, we present detailed calculations supporting this result. Contrary to the common literature, we do so in a straightforward manner, which can be followed step by step (with all the necessary elements of functional analysis summarised in the Appendix). In quantum field theory we apply a similar approach, when we “rediscover” the results of the abstract functional methods in the ordinary perturbation theory. In its framework, we further show how to obtain predictions also for other quantities than particle masses.

Keywords: dimensional transmutation, scalar electrodynamics, contact interactions, self-adjoint extensions

Preface

The dimensional transmutation is a phenomenon, that (in fact quite frequently) occurs in quantum theory, when there is a result that cannot be reconstructed from the original parameters due to a lack of quantities with the right dimension, e.g. the appearance of a state of a definite mass or energy in a theory, where all the constants are, in some appropriate sense, dimensionless. At first sight, this can be intriguing, but looking deeper and considering the different faces of the phenomenon, the reason appears to be always the same. It is simply not the right way to do physics, if we try to write down a formal expression, then to explore its consequences and finally we wonder that they did not turn out to be uniquely determined.

We are all used, and often even forced, to employ the “physical” treatment of many models that is often mathematically nonrigorous, particularly in the quantum field theory. But even though the usual approach based on a local Lagrangian and the perturbation theory is plagued with obstacles, by carefully circumventing them we can extract sensible physical predictions. This circumvention often requires additional information that have not been included in the original formal expression, such as the definition of a renormalisation scale, a choice of a symmetry that we demand to be maintained after quantisation, or anything that helps us to remedy the frivolity with which we wrote a too formal expression at the beginning. Only when these additional pieces are included can we speak about a specific physical theory.

The “dimensionally transmuted” theories are the most prominent manifestations of this general rule. The massless scalar electrodynamics, an apparently scale invariant theory, indeed contains a dimensionful parameter, the renormalisation scale. It is often referred to as an “arbitrary” parameter, but in fact it is no more arbitrary than the coupling constants themselves, it just only serves to pick a particular version from the parameter space of the theory. The need to carefully define the delta-function pseudopotential may be discomfoting as we are not used to such complications in quantum mechanical systems, but we are just unlucky that the notion of the delta-function gives us a concise notation to write down a practically meaningless expression which we are later forced to correct.

These considerations may seem enough to dismiss the topic of the work as completely uninteresting. Nevertheless, the transmuted systems are still interesting in many ways – mainly because the formality of the usual means to handle quantum theories is too severe for them and thus they need a special care to be treated with. Those theories then not only teach us new methods that we have to use to handle them, but they also point out the weakest parts of our typical chain of thinking.

The existing literature relevant to the quantum-mechanical part¹ is very rich at the first sight (given that the two-dimensional delta-function model does not have a groundbreaking practical application), but it can be quickly realised that many articles partly or fully overlap in their content. In fact, there are even monographs and review articles; yet we hope we have something to add: we try to present the arguments in such a way which is as simple as is feasible, yet detailed enough – and, when possible, self-contained – to convince the reader that the conclusion

¹From which we present only a very small sample in the References.

that the transmutation occurs is indeed the right one to make. To this end, we focus almost solely on the delta-function potential, which, though seemingly more artificial, is technically much less obscured than its inverse square cousin, yet it shows an unexpected wealth of intricacies.

However, is the dimensional transmutation only a theoretical nicety, or does it have some relevance for the real world? The problem with scale invariant systems is that they are often indiscernible from similar systems with a small amount of explicit symmetry breaking, although it is possible to conjecture (cf. [2]) that the fact that, in certain cases, the latter is not present is somehow fundamental. Nevertheless, a very striking evidence that the unusual features of scale invariant systems in fact survive the transition to a more realistic physical setting is provided by the aforementioned inverse square potential, which is actually applicable to molecular physics ([5]).

A wide survey of practical applications of the δ -function potentials is presented in the introduction to the monograph [28]. Naturally, most of such applications use the three-dimensional case, which does not exhibit dimensional transmutation in the strict sense, but it is interesting to note an article by Thorn ([3]) who uses the two-dimensional δ -function as an extremely simplified model for quark confinement in the infinite momentum frame, where only two transverse coordinates remain as independent variables. Actually, this remark points to an extreme broad field (which we do not discuss any further), where the dimensional transmutation may even take a firm part of the processes that lead to the existence of the world around us – namely the fact that a part of the mass of the hadrons is expected to originate in this way from the QCD interactions of massless gluons with the approximately massless quarks ([13]). Also, the quantum mechanical δ -function potential, considered as a crude nonrelativistic limit of a ϕ^4 quantum field theory ([16]), can be used as a tool to discuss some of the features of the former.

Finally, let us note that the Coleman-Weinberg model is actually not critically dependent on the masslessness of the theory, as already discovered by the original authors. The conclusions drawn from the loop-corrected effective potential can be even used to put a lower bound to the Higgs mass ([13]), but it shall be noted that it is very weak, in the sense that the current experimental bounds are far above it.

Part I

The Coleman-Weinberg model

1 Effective potential

Although we do not claim any historical accuracy, it seems likely that the term “dimensional transmutation” first appeared exactly 35 years ago in the work of Sidney Coleman and Erick Weinberg ([29]). Their treatment of massless scalar electrodynamics using a (at that time quite pioneering) functional approach has since become a rather standard – even textbook – material and is sometimes referred to as the Coleman-Weinberg (hereafter C-W) model. In this first section, we review a (smaller) part of the original paper that deals directly with the massless scalar electrodynamics in detail.

1.1 The method

Since the times of the C-W paper, functional tools in quantum field theory have become a part of the common knowledge and are often taught in undergraduate courses – we will thus present only a brief summary (see [18], Ch. 9 and 11, [13], Ch. 7, etc.).

The *Green functions* of a scalar field

$$G_n(x_1, \dots, x_n) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle \quad (1)$$

can be calculated taking functional derivatives of the *generating functional* $Z[J]$, which is actually the vacuum-vacuum amplitude in the presence of an external source J , that is

$$Z[J] = \langle 0 | e^{-i2HT} | 0 \rangle_J = \int \mathcal{D}\phi \exp \left(i \int_{-T}^T d^4x (L[\phi] + J\phi) \right) \quad (2)$$

(where it is implied that $T \rightarrow \infty(1 - i\epsilon)$ and the functional integration indicated by $\mathcal{D}\phi$ is eventually performed over any other dynamical field in the Lagrangian) and

$$\begin{aligned} G_n(x_1, \dots, x_n) &= \frac{\int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) \exp(i \int d^4x (L[\phi] + J\phi))}{\int \mathcal{D}\phi \exp(i \int d^4x (L[\phi]))} = \\ &= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)} \right) \dots \left(-i \frac{\delta}{\delta J(x_n)} \right) Z[J] \Big|_{J=0} \end{aligned} \quad (3)$$

where the proper time-to-infinity limit is again understood (note that it is crucial to keep the track of the various imaginary units and that our straightforward conventions for the Green functions are slightly different from those of C-W). As a next step, the disconnected Green functions are exponentiated using the Cluster theorem and the *connected generating functional* $W[J]$ is defined by

$$Z[J] = e^{iW[J]}. \quad (4)$$

A generalisation of the Legendre transform then allows one to define the *effective action* Γ . First, we must define the conjugate variable to J , which is the *classical field*

$$\phi_c(x) = \frac{\delta W}{\delta J(x)}. \quad (5)$$

The transformation then reads

$$\Gamma[\phi_c] = W[J] - \int d^4x J(x) \phi_c(x) \quad (6)$$

where, as with the ordinary Legendre transform, the dependence on J through W drops out. Also, the conjugation of variables is reflexive, that is

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = -J(x). \quad (7)$$

This apparently innocent equation is actually at the core of the whole machinery, when applied to (possibly) spontaneously broken theories. We must always take into account that the external source J has been added only as a computational tool – the real physical model is reached only when J equals zero everywhere. Imagine a model in which there is a nonzero value ϕ_0 of the classical field such that $\delta \Gamma[\phi_c]/\delta \phi_c = 0$. By (5) this value is the connected 1-point Green function (remember that it was obtained for $J = 0$, which is the point where $\delta W/\delta J$ is evaluated to extract the Green functions), that is, the vacuum expectation value of the field. By Lorentz invariance, this value must be constant throughout the spacetime (otherwise there would be a preferred direction), allowing a great level of simplification. For a general classical field, Γ is a complicated functional, but restricted to constant fields, it must be of the form

$$\Gamma[\phi_c(x) = \phi] = -cV(\phi), \quad (8)$$

where the *effective potential* $V(\phi)$ is an ordinary function, not a functional (there is no other information than the single value of ϕ that Γ could depend on), the minus sign is just convenient, c is an (actually infinite) normalisation constant to be specified shortly and the condition $\delta \Gamma[\phi_c]/\delta \phi_c = 0$ becomes just $dV/d\phi = 0$.

The last property suggests that the name “effective potential” has not been chosen randomly. Clearly, when V has an extreme at ϕ_0 , it is sensible to approach the theory in terms of a new field $\phi - \phi_0$ (which has a zero vacuum expectation value and thus is readily given a particle interpretation²) that is, to build a perturbation theory centred around this value of the field. On the other hand, we do not know yet, what to do if there is more than one such extreme, nor is it evident that there is a difference between a maximum and a minimum (as it should be for a potential).

This question can be settled by the formulae of the previous few paragraphs. By (6), $V(\phi_0)$ is proportional to $-W[0]$ for $J = 0$ (as the second term drops out)

²Because ϕ creates, among other things, 1-particle states from the vacuum, the non-vanishing value of $\langle 0|\phi|0\rangle$ would imply an undesirable non-orthogonality of the vacuum and 1-particle states. (This simple but interesting and often overlooked point is taken from [8]).

and by (4) and (2) this is in turn proportional to the mean value of energy in the vacuum (now we see the reason for the minus in the definition of V and the imaginary unit in the definition of W). However, the vacuum is, by definition, the state with the *lowest* energy (and its mean value) – thus, when we ask, which of the apparently many (often contradictory) answers to the same question “what value does the effective potential attain when $J = 0$ ” is correct, we must answer that it is the one that gives the lowest value, that is, the absolute minimum of V , what finishes the “potential” interpretation of V . Consequently, it is the value of ϕ where this minimum is attained, that gives the actual vacuum expectation value of the field.³

The last point to clarify is how to actually compute the effective potential. This is in fact straightforward, taking into account that the effective action is the generating functional of the one-particle-irreducible (1PI) Green functions. If we choose to call $\Gamma^n(x_1, \dots, x_n)$ just the sum of 1PI graphs with n external legs (without any further factors), then

$$i\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \Gamma^n(x_1, \dots, x_n) \phi_c(x_1) \dots \phi_c(x_n) \quad (9)$$

In momentum space, we can always factor out the overall delta function by translation symmetry (4-momentum conservation) and thus we define, as usual,

$$\Gamma^n(k_1, \dots, k_n) (2\pi)^4 \delta^4(k_1 + \dots + k_n) = \int d^4x_1 \dots d^4x_n \Gamma^n(x_1, \dots, x_n) e^{-i(k_1x_1 + \dots + k_nx_n)} \quad (10)$$

(with this factor, the Γ^n 's are correctly calculated using the common set of Feynman rules). Inverting the Fourier transform⁴, substituting to the previous equation and evaluating for $\phi_c(x) = \phi$ we get

$$\begin{aligned} i\Gamma[\phi_c] &= \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \int \frac{d^4x_1 d^4k_1}{(2\pi)^4} \dots \frac{d^4x_n d^4k_n}{(2\pi)^4} \Gamma^n(k_1, \dots, k_n) (2\pi)^4 \delta^4\left(\sum k_n\right) e^{i(k_1x_1 + \dots + k_nx_n)} = \\ &= \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \int d^4k_1 \dots d^4k_n \Gamma^n(k_1, \dots, k_n) (2\pi)^4 \delta^4\left(\sum k_n\right) \delta(k_1) \dots \delta(k_n) = \\ &= (2\pi)^4 \delta^4(0) \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \Gamma^n(0, \dots, 0) \end{aligned} \quad (11)$$

³In fact, the other extremes of V are illusory altogether. The restriction to constant classical fields leads to subtleties that arise when a lower value of Γ is achieved on a field with “islands” of different values in different parts of the spacetime, than on a constant field, parallel to the Maxwell construction in thermodynamics close to a phase transition, (as explained e.g. in [18], Ch. 11.3., where further parallels between those two subjects are exploited), that make the true effective potential convex. Aside of this remark, we will not need to deal with this issue, because it does not affect the absolute minimum.

⁴Note that the Fourier transforms in the two parts of this work are different, in compliance with the tradition of the particular field: in Part 1 we put the whole $(2\pi)^{-d}$ into the inverse transform, in Part 2 we use the “symmetric” version with $(2\pi)^{-d/2}$ in both formulae.

If we hide the awkward constant $(2\pi)^4 \delta^4(0)$ into the insofar arbitrary c from (8), then the effective potential generates the 1PI functions at zero external momenta, namely

$$\Gamma^n(0, \dots, 0) = -i \frac{d^n V(\phi)}{d\phi^n} \Big|_{\phi=0} \quad (12)$$

Conversely, the effective potential can be obtained by summing i times the 1PI functions with zero external momenta – the $n!$ will be cancelled (barring some symmetry factors) by the permutations of the external lines and the ϕ^n can be conveniently accounted for by assigning the weight ϕ to every such an external line.

The utility of the effective potential is twofold. First of all, it serves as a tool to look for spontaneous symmetry breaking, which may be not evident from the Lagrangian. Clearly, if we calculate V by summing some diagrams with external ϕ 's, as sketched before, we will always get the value $V(0) = 0$. Thus, when we somehow find a minimum away from the origin and V is negative there, then the value of ϕ_0 where this happens is the vacuum expectation value of the field. Then we should define the particle contents of the theory in terms of the shifted field $\phi' = \phi - \phi_0$ and this is where V turns out to be useful the second time: considered as a function of ϕ' it generates its 1PI functions at $\phi' = 0$, that is, $\phi = \phi_0$ (this is consistent, since $J = 0$ for $\phi = \phi'$), allowing us in principle to get some information about the broken phase of the theory directly.⁵

1.2 Massless scalar electrodynamics

The method, chosen by C-W to evaluate the effective potential, is to expand it in the number of loops – that is the first correction to the tree-level effective potential (which is just the classical potential in the Lagrangian) is calculated by summing all possible graphs with one loop and an arbitrary number of external legs. This expansion is certainly not worse, than any perturbation scheme in powers of coupling constants, but has many advantages, as thoroughly discussed in the C-W paper. From the practical point of view, it is clear that a possible vacuum expectation value of the scalar field in any $\phi \leftrightarrow -\phi$ symmetric theory can not be seen in any finite order of the perturbation theory (all the relevant graphs simply vanish) and thus some kind of infinite summation is necessary.

At the end of the day, the effective potential is found to be a power series in the coupling constant(s). However, this means that in any theory with a single coupling

⁵In the presence of other fields (it will be the case in the rest of Part I), the effective action (and thus the effective potential) can be generalised to a generating functional of 1PI functions of all the fields, using the same method as we just did, only adding more sources for the additional fields and more classical fields as their conjugate variables. It seems logical that the zero-momentum 1PI functions of the theory in the broken phase (hereafter “the broken theory”) appear as its derivatives at the point where $\phi = \phi_0$ and the other classical fields (that do not develop vacuum expectation values) are zero. Nevertheless, we have not found a simple but feasible explanation, why it should be the case for the functions of the other fields alone – yet it repeatedly turns out to be the case, as will be shown in the next sections. For this reason (which is more likely due to our incapability, than a flaw in the theory), we will avoid making conclusions about the other fields based solely on the information from the effective potential.

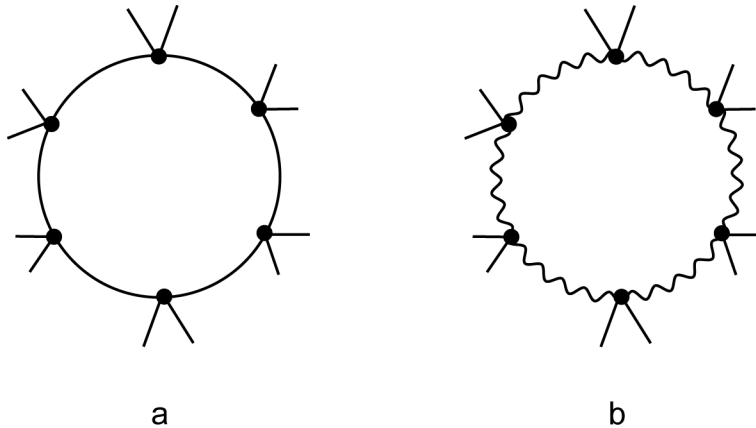


Figure 1: The types of diagrams that contribute to the one-loop effective potential in Landau gauge. The a) type allows both ϕ_1 and ϕ_2 orbiting in the loop (but without a change of type in one diagram).

constant, any possible minimum comes from the interplay of terms of different order in the coupling constant, putting its reality in doubt. This is where the scalar electrodynamics kicks in: due to renormalisation requirements, there is not only the e from the covariant derivative, but also another independent constant λ , the ϕ^4 coupling, which must be added to renormalise the scalar-scalar scattering.

It should be stressed that, for the result to be derived in this section, the choice of *massless* scalar electrodynamics is not crucial. On the other hand, this is not only the simplest but mainly the right one to fit in the topic of this work – as a massless theory with both couplings dimensionless, it is naively scale invariant, that is, there is no way for it to give any prediction of a definite mass, energy, length or timescale. Yet we will be quite soon forced by its infrared divergences to introduce a dimensionful quantity in it, just to get any meaningful results. This step will inevitably break the scale invariance of the theory at the end, in a process that deserves the title “dimensional transmutation”. Let us now see, without further protraction, how this happens, in the one-loop calculation of C-W.

The Lagrangian of scalar electrodynamics is

$$\frac{1}{2} (\partial_\mu \phi_1 - eA_\mu \phi_2)^2 + \frac{1}{2} (\partial_\mu \phi_2 + eA_\mu \phi_1)^2 - \frac{\lambda}{4!} (\phi_1^2 + \phi_2^2)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \text{counterterms.} \quad (13)$$

We have used two real fields instead of a single complex one to make the calculation easier: due to the global⁶ $U(1)$ symmetry, the potential can depend only on the magnitude ϕ of the two-component vector (ϕ_1, ϕ_2) and we can thus calculate only the diagrams with, e.g. all ϕ_1 's as external lines. Then the particle orbiting in the loop cannot change from ϕ_1 to ϕ_2 or back (this would require an $\phi_1^3 \phi_2^2$ vertex which is not present). As to the “photon” field, we will work in the Landau gauge, with the propagator

⁶Here we do *not* claim the independence of the potential on *local* gauge transformation – we will get to this point later.

$$-i \frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{k^2 + i\epsilon} \quad (14)$$

as it further eliminates any graphs, where the loop particle changes from the scalar to the photon – the derivatives in the $A_\mu(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1)$ vertex become the multiplications by the respective momenta, but the external momentum is zero and the momentum of the internal ϕ_2 is the same as that of the photon by momentum conservation, which gives zero when contracted with the transverse propagator in the Landau gauge. Thus, there are only three types of graphs (Fig. 1) and, moreover, they are nearly identical: they differ only in the different couplings and numerical constants because all the Landau-gauge propagator numerators contracted together give just a factor of 3 (the transverse projector is idempotent and the final contraction gives its trace)⁷.

Let us first evaluate the graphs with ϕ_1 in the loop. To every vertex there is a propagator and the contribution of such a pair is $(-i\lambda)i/(k^2 + i\epsilon)$; the external legs of the vertex contribute ϕ^2 times 1/2 for the symmetry in their exchange. Any such graph is symmetric with respect to rotations and reflections, which gives the symmetry factor $1/2n$ for the graph with n vertices. Finally, there is the overall i from (12) and the sum over all possible numbers of vertices, thus

$$\begin{aligned} V_{\phi_1} &= \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\frac{1}{2}\lambda\phi^2}{k^2 + i\epsilon} \right)^n = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(1 - \frac{\frac{1}{2}\lambda\phi^2}{k^2 + i\epsilon} \right) = \\ &= \frac{1}{2} \int \frac{d^4k_E}{(2\pi)^4} \ln \left(1 + \frac{\lambda\phi^2}{2k_E^2} \right) \end{aligned} \quad (15)$$

where we have first summed the infinite sum and then Wick-rotated (the Jacobian has eaten the imaginary unit in front of the integral). The utility of summing all the 1-loop graphs is now clear: individually, the graphs are more and more infrared-divergent, because we evaluate them at the worst possible external momenta, but after the summation, the integral is, in fact, infrared-finite.

Returning to k for the integration variable for notational simplicity, cutting off at $k^2 = \Lambda$, introducing the shortcut $a = \lambda\phi^2/2$ and going to spherical coordinates, we have

$$\begin{aligned} V_{\phi_1} &= \frac{2\pi^2}{2(2\pi)^4} \int_0^\Lambda \ln \left(1 + \frac{a}{k^2} \right) k^3 dk = \\ &= \frac{1}{64\pi^2} \left(\ln \left(1 + \frac{a}{\Lambda^2} \right) \Lambda^4 + a\Lambda^2 - a^2 \ln \left(1 + \frac{\Lambda^2}{a} \right) \right). \end{aligned} \quad (16)$$

The terrible quartic divergence is only apparent, because for large Λ we can expand the logarithm, dropping terms that vanish in the $\Lambda \rightarrow \infty$ limit (we can also drop the 1 in the last logarithm):

⁷The Landau-gauge formulation has its advantages and drawbacks – we will encounter both of them in Ch. 3

$$\begin{aligned}
V_{\phi_1} &\rightarrow \frac{1}{64\pi^2} \left(\left(\frac{a}{\Lambda^2} - \frac{a^2}{2\Lambda^4} \right) \Lambda^4 + a\Lambda^2 - a^2 \ln \frac{\Lambda^2}{a} \right) = \\
&= \frac{1}{64\pi^2} \left(2a\Lambda^2 - a^2 \ln \frac{\Lambda^2}{a} - \frac{a^2}{2} \right) = \frac{\lambda\phi^2}{128\pi^2} \left(2\Lambda^2 + \frac{\lambda\phi^2}{2} \left(\ln \frac{\lambda\phi^2}{2\Lambda^2} - \frac{1}{2} \right) \right). \quad (17)
\end{aligned}$$

The full unrenormalised 1-loop effective potential can now be easily written down: the loops with ϕ_2 differ only in the incomplete cancellation of $1/4!$ in the rule for the vertex, thus they are obtained by the substitution $\lambda \rightarrow \lambda/3$ and the photon-loop is given by a similar substitution $\lambda \rightarrow 2e^2$ for the different vertices (note that the different sign of the photon vertex cancels against the different sign in the propagator) and introducing an overall factor of 3, as already discussed. We must also add the tree-level contribution⁸ $\lambda/4!$ and the contribution of the ϕ -related counterterms $\frac{B}{2} (\phi_{1_0}^2 + \phi_{2_0}^2) + \frac{C}{4!} (\phi_{1_0}^2 + \phi_{2_0}^2)^2$:

$$\begin{aligned}
V &= \left(\frac{\Lambda^2}{32\pi^2} \left(\frac{2}{3}\lambda + 3e^2 \right) - \frac{B}{2} \right) \phi^2 + \\
&+ \left(\frac{1}{256\pi^2} \left(\lambda^2 \ln \frac{\lambda\phi^2}{2\Lambda^2} + \frac{\lambda^2}{9} \ln \frac{\lambda\phi^2}{6\Lambda^2} + 12e^4 \ln \frac{e^2\phi^2}{\Lambda^2} - \left(\frac{5\lambda^2}{9} + 6e^4 \right) \right) + \frac{\lambda - C}{4!} \right) \phi^4. \quad (18)
\end{aligned}$$

What will be the renormalisation conditions? First of all, we are working with *massless* scalar electrodynamics and thus we want the mass of ϕ to vanish. In the terms of the effective potential⁹, we demand that $d^2V/d\phi^2 = 0$ for $\phi = 0$, that is, $B = \frac{\Lambda^2}{16\pi^2} \left(\frac{2}{3}\lambda + 3e^2 \right)$. To fix C we could try a similar approach, demanding $d^4V/d\phi^4 = \lambda$ for $\phi = 0$, but by the Leibniz rule $d^4V/d\phi^4$ necessarily contains the term where ϕ^4 is derived four times (to a constant) and $\ln\phi^2$ is kept, which is divergent as $\phi \rightarrow 0$. In fact, we can not be surprised with this result, because it is the same infrared divergence we would encounter in the ordinary perturbation theory for massless electrodynamics. The way out, suggested by C-W is to demand

$$\left. \frac{d^4V}{d\phi^4} \right|_{\phi=M} = \lambda \quad (19)$$

with M arbitrary. This condition may seem quite artificial, but is perfectly admissible as a renormalisation scheme, because it is consistent with the tree level. Moreover, it will be given a physical (though not on-shell) meaning in the next section. Using the elementary, but convenient formulae $d^4\phi^4/d\phi^4 = 4!$ and $d^4(\phi^4 \ln c\phi^4)/d\phi^4$, we see that

⁸The $4!$ is stripped away when formulating the Feynman rule, but restored again as a symmetry factor.

⁹If we were to impose a condition with a general mass, the correct way is, of course, to demand that $\Gamma^2(m^2) = 0$, not of $\Gamma^2(0) = -m^2$ and thus the on-shell renormalisation is not directly possible in terms of the effective potential. However, for the vanishing mass, those two conditions coincide – and actually we will use the later even for massive particles for simplicity, resting assured by C-W that in the 1-loop order the difference actually does not matter.

$$\begin{aligned} \left. \frac{d^4 V}{d\phi^4} \right|_{\phi=M} &= \frac{4!}{256\pi^2} \left(\lambda^2 \ln \frac{\lambda M^2}{2\Lambda^2} + \frac{\lambda^2}{9} \ln \frac{\lambda M^2}{6\Lambda^2} + 12e^4 \ln \frac{e^2 M^2}{\Lambda^2} - \left(\frac{5\lambda^2}{9} + 6e^4 \right) \right) + \\ &+ \frac{100}{256\pi^2} \left(\frac{10}{9} \lambda + 12e^2 \right) + \lambda - C \end{aligned} \quad (20)$$

and to fulfil the renormalisation condition, C must cancel the rest of the expression, except for the λ . The renormalised form of the potential is now easy to obtain. When we put this C in (18), the logarithms combine, dropping the dependencies on the cutoff and the couplings inside them and the $\frac{5\lambda^2}{9} + 6e^4$ term cancels altogether. Finally, the potential is

$$\begin{aligned} V &= \left(\frac{\lambda}{4!} + \frac{\frac{10}{9}\lambda^2 + 12e^4}{256\pi^2} \left(\ln \frac{\phi^2}{M^2} - \frac{100}{24} \right) \right) \phi^4 = \\ &= \left(\frac{\lambda}{4!} + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} \right) \left(\ln \frac{\phi^2}{M^2} - \frac{25}{6} \right) \right) \phi^4. \end{aligned} \quad (21)$$

To look for its extremes, we write its first derivative

$$\frac{dV}{d\phi} = \left(\frac{\lambda}{6} + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} \right) \left(4 \ln \frac{\phi^2}{M^2} - \frac{88}{6} \right) \right) \phi^3. \quad (22)$$

If we choose a fixed renormalisation point M , the form of V can be a good approximation only for those ϕ such that $\ln \frac{\phi^2}{M^2} \ll 1$, because higher powers of such logarithms are expected to appear in higher orders of the perturbation theory and would make the expansion questionable. In such a case, the second term is negative and an extreme is indeed possible. It cannot arise from the cancellation of the terms proportional to λ and λ^2 that are of different order. However λ and e^4 are so far independent and there is no reason why they could not be comparable. If they indeed are, we must drop the λ^2 term for consistency, because we have not calculated the term up to e^8 that would be of a similar magnitude. To make the things even easier, we can choose the renormalisation scale to be just the location of the extreme (let us call it ϕ_0). Then

$$0 = \frac{dV}{d\phi}(\phi_0) = \left(\frac{\lambda}{6} - \frac{11e^4}{6\pi^2} \right) \phi_0^3 \quad (23)$$

and consistency requires that if ϕ_0 is nonzero, then

$$\lambda = \frac{33e^4}{8\pi^2}. \quad (24)$$

After such formal manipulations, particularly within the quantum field theory, it is always good to step back and try to interpret the result – we have made a lot of assumptions and even the choice of the renormalisation point was done assuming that there is an extreme. The way out of the labyrinth is provided by the renormalisation group approach¹⁰. If we are talking about one particular theory

¹⁰It has become customary to cite the original article of Gell-Mann and Low ([19]) in this context, but often an actually more useful account of this method can be found in any QFT textbook.

(i.e., a theory with a given set of amplitudes), we cannot choose M and λ in (19) arbitrarily – once we have specified the renormalisation condition at one particular value of M , the conditions for other values are firmly determined from the RG equations. Thus, what (23) really says is that *if there is a nonzero renormalisation point ϕ_0 such that (24) holds, then this point is an extreme of the effective potential*. The important point, as the somewhat lengthy RG analysis of C-W, which we will not reproduce here, shows, is that if we are given with e and λ small but arbitrary at some renormalisation point, we can *always* find a ϕ_0 such that (24) holds at this point. Moreover, using the coupling-constant relation, we can easily see that

$$V = \frac{3e^4}{64\pi^2} \left(\ln \frac{\phi^2}{\phi_0^2} - \frac{1}{2} \right) \phi^4 \quad (25)$$

and

$$V(\phi_0) = -\frac{3e^4}{128\pi^2} \phi_0^4, \quad (26)$$

thus it is an absolute minimum, lower than the zero value at origin. We see that, regardless of the actual initial values of coupling constants, the scalar field always develops an expectation value.

From here, the discussion of the spontaneous symmetry breaking goes as usual, even though it was not apparent from the Lagrangian. The requirement that $\phi = \phi_0$ for the minimum of V is fulfilled at infinitely many points that differ only by the rotation in the $\phi_1 - \phi_2$ plane – the choice among them is the spontaneous symmetry breaking. However arbitrary this choice is, if the actual vacuum expectation value does not fall into the ϕ_1 or ϕ_2 direction, we must rotate the fields so it does, otherwise the mass matrix¹¹ would not be diagonal, because

$$\frac{\partial^2 V}{\partial \phi_1 \partial \phi_2} = \frac{3e^4}{8\pi^2} \left(\ln \frac{\phi^2}{\phi_0^2} + 1 \right) \phi_1 \phi_2 \quad (27)$$

which indeed vanishes for $\phi = \phi_0$ only if one of its components is zero.

Let us choose $\phi_0 = (\phi_0, 0)$ in the ϕ_1 -direction. Then the masses of the scalars are

$$\begin{aligned} \frac{\partial^2 V}{\partial \phi_1 \partial \phi_1}(\phi_0) &= \frac{3e^4}{8\pi^2} \phi_0^2 \\ \frac{\partial^2 V}{\partial \phi_2 \partial \phi_2}(\phi_0) &= 0 \end{aligned} \quad (28)$$

and there indeed is a massless scalar in agreement with the Goldstone theorem (which is nevertheless expected not to appear as a physical state, as in the usual Higgs model¹²). The mass of the “photon” after the symmetry breaking is just stated in the C-W paper and for the time being we cannot actually do much more, unless we use the (unsettled) argument of footnote 5. If we do so, we can express the lowest

¹¹Cf. footnote 9; we will not stress this point from now on.

¹²This is one of the point on which we will not elaborate anymore, as explained in the last paragraph of this section.

order (tree-level) contribution to the effective potential as the function of A_μ simply as

$$V_A = -\frac{e^2}{2} A_\mu A^\mu \phi^2 \quad (29)$$

what gives the same lowest-order contribution to the photon mass as C-W stated, namely $e^2 \phi_0^2$.

A final remark is in order here. Soon after the original C-W paper appeared, it has been pointed out by Jackiw (see [25], where he also introduced a functional method that allows the evaluation of V in any gauge) that the effective functional is not gauge invariant¹³ and thus it is not clear, whether the C-W calculation, confined to the particular choice of the Landau gauge, should be trusted. Jackiw even constructed a gauge, in which the 1-loop contribution vanishes altogether, but he acknowledges that in this gauge, higher-order corrections become extremely important. The problem has been thoroughly investigated since then and already in the same year, Jackiw and Dolan ([15]) confirmed that the spontaneous symmetry breaking is most likely a gauge-invariant phenomenon, even though the concrete form of the potential depends on the choice of gauge. We will not go much deeper in proving this conclusion – nevertheless we will derive the coupling-constant relation in the next section using a very different gauge, which can be considered as a small piece of evidence for it.

In many respects, the C-W model is parallel to the well-known Abelian Higgs model, which has received a great deal of attention over the past decades and there are in fact numerous complications to it, as often happens in the quantum field theory – to deal with all of them is far beyond our reach. We have thus chosen a particular topic that is usually not stressed in the literature – in the following two sections, we present several hints on how one could handle the C-W model in the ordinary perturbation theory. We hope that the reader will forgive us the purported ignorance of many obstacles that could stand in our way (e.g., that we do not prove the unitarity, finiteness to higher orders nor gauge-invariance of the calculations).

2 Ordinary perturbation theory I: scalar particles

2.1 Lagrangian and diagrams

Having studied C-W potential, we are left in a somehow unsatisfactory state: the abstract functional tools have revealed the likely particle contents of the theory, but so far we know little about the interactions of these particles. One could boldly go on and compute the full effective action in the original theory (i.e. in the one-loop approximation) and then extract the 1PI functions of the spontaneously broken theory by calculating higher functional derivatives of the effective action at the new minimum, but nobody expects this task to be simple. For our purpose we choose a different approach which is technically simpler, albeit not free of subtleties, allowing us to calculate desired quantities in conventional perturbation theory. At the end,

¹³It is meant that it is not invariant to *local* gauge transformations. The argument we used to limit external legs to ϕ_1 's is based on global symmetry and is not affected by this issue.

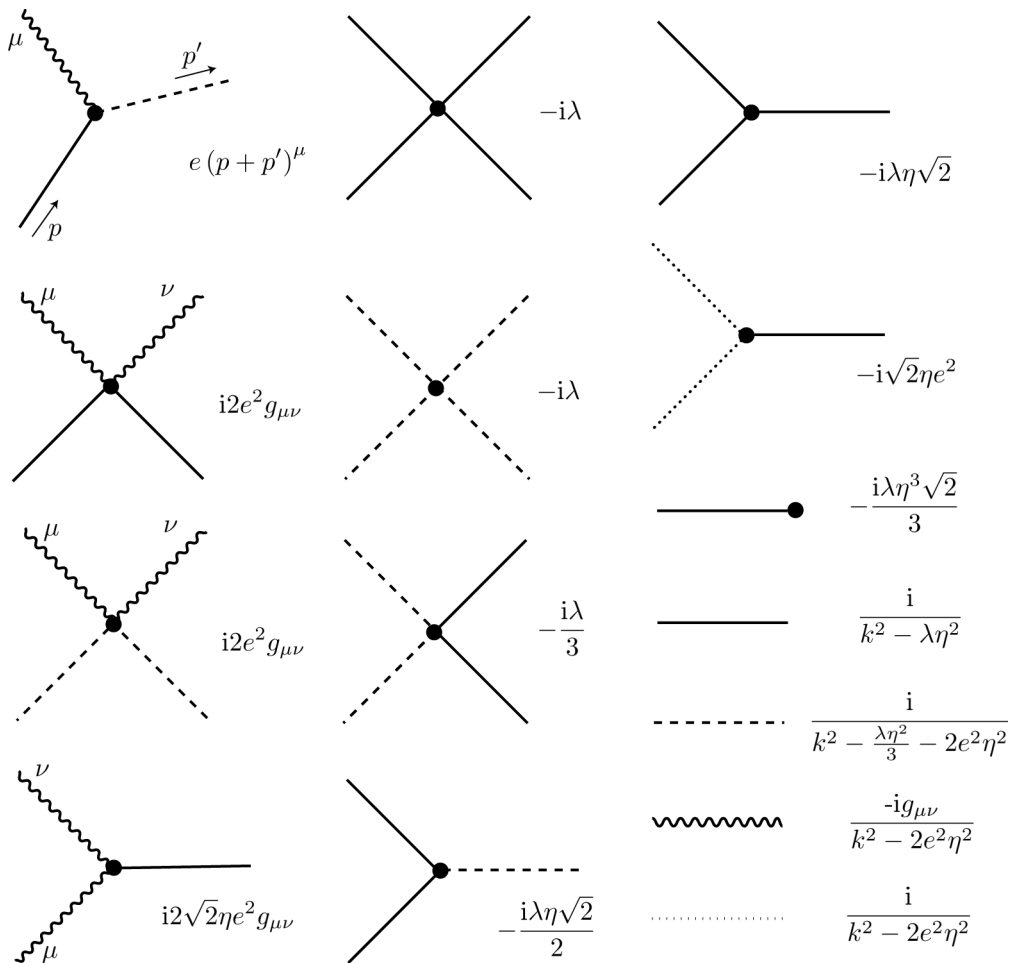


Figure 2: Diagrams and propagators in the broken theory

we hope to be equipped not only with a definite algorithm to predict observables, but also with a slightly better understanding of what happened to the theory.

As we are basically still asking the theory “how should we treat you?”, we try to be as open-minded as possible. We thus borrow only two results of the C-W model as (qualified) assumptions for our computations and for both we are ready to throw them overboard on the first signal of contradiction. The first one will be that the scalar field develops some vacuum expectation value (and so must be shifted to start the perturbation theory) and the second that, at least to the lowest relevant order, $\lambda \sim e^4$. The second assumption is not inevitable, but dramatically simplifies ordering the diagrams and this relations at the end proves necessary, yet fully justifiable through the aforementioned renormalisation group analysis.

As for the open-mindedness, we start with what we feel is the less suggestive formulation of scalar electrodynamics, using one complex scalar field. The Lagrangian then reads

$$(\partial_\mu \phi_0 + ieA_\mu \phi_0) (\partial^\mu \phi_0 + ieA^\mu \phi_0)^+ - \frac{\lambda}{6} (\phi_0^+ \phi_0)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (30)$$

where by ϕ_0 we mean the field before there are any shifts carried on it (not the bare field). The gauge field kinetic term will never be affected by our calculations

and is by itself gauge invariant, we thus omit it in the following. Using the first assumption, we perform the shift on the scalar field and write $\phi_0 = \phi + \eta$, assuming for simplicity η to be real. The terms quadratic in ϕ turn out to be

$$\eta^2 (\phi^2 + 4\phi^+ \phi + \phi^{+2}) \quad (31)$$

what is of course readily diagonalised via a transformation to

$$\begin{aligned} \phi_1 &= \frac{1}{\sqrt{2}} (\phi + \phi^+) \\ \phi_2 &= \frac{1}{\sqrt{2}i} (\phi - \phi^+) \end{aligned} \quad (32)$$

where the numerical factors preserve the normalisations of the fields and the factor of i fixes the sign of the kinetic term. Then, systematically neglecting constant terms, we arrive at the Lagrangian of the form

$$\begin{aligned} &\frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 - \frac{\lambda \eta^2}{2} \left(\phi_1^2 + \frac{\phi_2^2}{3} \right) + e^2 \eta^2 A_\mu A^\mu + \\ &+ e A^\mu (\phi_2 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2) + \frac{e^2}{2} A_\mu A^\mu (\phi_1^2 + \phi_2^2) - \\ &- \sqrt{2} e \eta A^\mu \partial_\mu \phi_2 + \sqrt{2} e^2 \eta A_\mu A^\mu \phi_1 - \\ &- \frac{\lambda}{4!} (\phi_1^4 + \phi_2^4) - \frac{\lambda}{12} \phi_1^2 \phi_2^2 - \frac{\lambda}{6} \sqrt{2} \eta \phi_1 \phi_2^2 - \frac{\lambda}{6} \sqrt{2} \eta \phi_1^3 - \frac{\lambda}{3} \sqrt{2} \eta^3 \phi_1 \end{aligned} \quad (33)$$

We already see the notational disaster related to the innocent introduction of η instead of $\eta/\sqrt{2}$ as the shift of the complex field, but it always helps to stay alert, if we make well-known equations look a bit different, so we stick to it. This Lagrangian is truly reminiscent of the Abelian Higgs model, only here we have some extra vertices, lacking the cancellations that come naturally when the shift is in accordance with the classical Lagrangian. Having our open-mindedness program in mind, we do not resort to the U-gauge formulation, as it would already presume our attitude to the theory. Instead we exploit the techniques already developed for the conventional Higgs models and use an R_ξ -gauge formulation, taking the first line of (33) as the free field Lagrangian and the gauge-fixing term to be

$$-\frac{1}{2} \left[\frac{1}{\sqrt{\xi}} \left(\partial_\mu A^\mu - \xi e \eta \sqrt{2} \phi_2 \right) \right]^2 \quad (34)$$

where the cross term in the binomial expansion exactly cancels the unpleasant $\sqrt{2} e \eta A^\mu \partial_\mu \phi_2$ term in the Lagrangian.

This operation seems to reduce the load of possible vertices of the theory by one, but it actually evokes the ghosts (that would otherwise form a free field without any influence on the theory) with their vertex¹⁴, so we are still left with a fair share of eleven. Nevertheless, thanks to our $\lambda \sim e^4$ assumption their values vary greatly

¹⁴There is no interaction of the ghosts with the gauge field in the Abelian theory, but their interaction with the ‘‘Higgs field’’ is governed by the form of the gauge fixing function ([18], Ch. 21.1).

in order – and they are further distinguished by the powers of η as it is the only dimensionful parameter of our theory and thus contributions to quantities of given dimension are strongly restricted by its power-counting. Fig. 2 shows all of them together with their Feynman weights and the propagators of all the particles in the Feynman gauge, $\xi = 1$.

2.2 Counterterms

Having the fields shifted will help us a lot in taming the infrared divergences of the theory (as for now we have no massless propagators), but not with the ultraviolet ones. Thus to formulate renormalised perturbation theory, we still need a set of counterterms. The crucial (if obvious) observation is that we already have them. Had we have forgotten that we are only reformulating the *massless scalar electrodynamics*, we could set up a list of renormalisation conditions, specifying some features of the symmetry broken theory, but then we would not have learnt much about what happened with the former. Indeed, we would end up having to *specify* the masses of the respective particles, instead of having *computed* their ratio.¹⁵

Thus we have to formulate our conditions in the original, unbroken theory, and calculate the counterterms according to them – actually, for the purely scalar sector of the theory, we have already done so in the previous section! We could in principle obtain the counterterms without any reference to the effective potential, but the condition (19) (with $M = \sqrt{2}\eta$) has the great advantage, that it is applicable in the unbroken theory yet interpretable in the broken-phase theory, namely as the 1PI function of four ϕ_1 's at zero four-momenta (nevertheless, we will have to repeat the calculation because we use a different regulator).

The counterterms depending on A_μ of the theory will be discussed in the next section and for the moment, let us set also the wave-function renormalisation aside, as the sample calculation will deal only with $p = 0$ terms. The shift of the counterterms then reads (already in the terms of two real fields):

$$\begin{aligned}
& \frac{B}{2} (\phi_{1_0}^2 + \phi_{2_0}^2) + \frac{C}{4!} (\phi_{1_0}^2 + \phi_{2_0}^2)^2 + DA^\mu (\phi_{2_0} \partial_\mu \phi_{1_0} - \phi_{1_0} \partial_\mu \phi_{2_0}) + \\
& + EA_\mu A^\mu (\phi_{1_0}^2 + \phi_{2_0}^2) = \frac{B}{2} (\phi_1^2 + 2\phi_1 \eta + \phi_2^2) + \\
& + \frac{C}{4!} (\phi_1^4 + 4\phi_1^3 \sqrt{2}\eta + 12\eta^2 \phi_1^2 + 8\sqrt{2}\eta^3 \phi_1) + \frac{C\phi_2^2}{12} (\phi_1^2 + 2\sqrt{2}\eta\phi_1 + 2\eta^2) + \\
& \frac{C}{4!} \phi_2^4 + DA^\mu (\phi_2 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2) + \sqrt{2}\eta DA^\mu \partial_\mu \phi_2 + \\
& EA_\mu A^\mu (\phi_1^2 + 2\sqrt{2}\eta\phi_1 + 2\eta^2 + \phi_2^2) + \text{const. terms} \tag{35}
\end{aligned}$$

¹⁵We have not actually stressed this fact, but it is the *ratio* of the scalar and photon masses that is considered the main result of the C-W paper. Indeed, the expectation value of the field is not a directly measurable quantity and thus the absolute values of the masses are not really predicted, whereas the expectation value drops when the ratio is calculated.

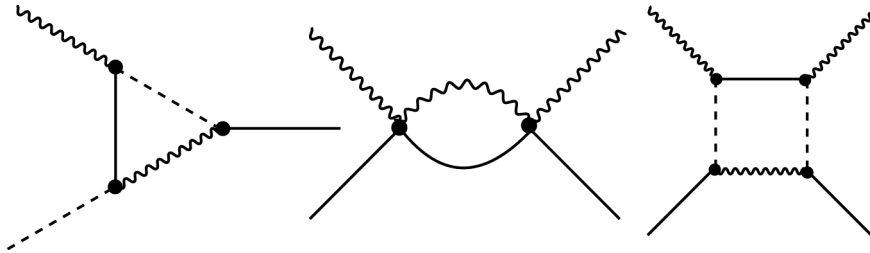


Figure 3: Leading contributions to D and E counterterms

To choose the right diagrams to consider, it is useful to know what order of e are the individual quantities. We already know that $C \sim e^4$, whereas with the use of dimensional regularisation that is to follow, B will turn out to be exactly zero, at least to one-loop order. Fig. 3 shows the leading order contribution to the remaining two counterterms, showing that $D \sim e^3$ and $E \sim e^4$ to the leading order.

2.3 Effective potential in dimensional regularisation

As already noted, we will use a different regulator, namely the dimensional regularisation through the calculation, for its ultimate computational simplicity. The price to pay is the need to rederive the C-W effective potential in this language. The final result is expected to be unchanged, but the valuable information is the specific form of the counterterms.

We will work again in Landau gauge for simplicity. An astute reader may now well object against the use of two different gauges in a single calculation – actually we should be using the same gauge for the potential (that gives us the counterterm) and the rest of diagrams, but in this case, to this order, there is no difference. To higher orders, it is possible to calculate the effective potential in any gauge using the aforementioned method of Jackiw – or we could just use the (plain, not just R_0) Landau gauge consistently: in fact, we would be awarded with various simplifications (due to the transverse vector propagator and the ghostlessness of the theory) in the following two sample calculations, but at the expense of missing the interesting interplay between many types of diagrams.

Recall from the previous section that the only contribution that was actually important came from the photon loops and that the sum of such 1-loop diagrams to V was (in the Euclidean space)

$$V_{loop} = \frac{3}{2} \int \frac{d^4 k}{(2\pi^4)} \ln \left(1 + \frac{e^2 \phi^2}{k^2} \right) \quad (36)$$

in d dimensions, after performing the angular integration, we get

$$V_{loop} = \frac{2\pi^{(d/2)} (d-1)}{\Gamma(d/2)} \int_0^\infty \ln \left(1 + \frac{a}{k^2} \right) k^{(d-1)} dk \quad (37)$$

where the $d-1$ factor reflects the fact that the 3 before the original integral is in fact $\text{Tr } g_{\mu\nu} - 1$ and thus dimension dependent and a has been introduced as shorthand for $e^2 \phi^2$. We do not introduce the customary dimensionful rescaling factor, as

there is always an appropriate scale introduced by renormalisation conditions; the dimensionful logarithms we get may seem slightly ugly, but they do appear only in intermediate steps. Integrating the previous expression by parts, we get

$$\int_0^\infty \ln\left(1 + \frac{a}{k^2}\right) k^{(d-1)} dk = \left[\ln\left(1 + \frac{a}{k^2}\right) \frac{k^d}{d} \right]_0^\infty + \int_0^\infty \frac{2a}{(k^3 + ka)} \frac{k^d}{d} dk \quad (38)$$

At least for $d \in (1, 2)$ the surface term vanishes and the remaining integral is convergent and evaluates to

$$\frac{a^{d/2} \pi}{\sin\left(\frac{\pi d}{2}\right) d} \quad (39)$$

where we already see the poles in even values of d . Putting back the multiplicative factors and expanding in $\epsilon = d - 4$ and adding the tree and counterterm contributions, we have

$$V = \frac{(\lambda + C) \phi^4}{4!} + \frac{e^4 \phi^4}{128\pi^2} \left(\frac{12}{\epsilon} - 5 + 6 \ln\left(\frac{e^2 \phi^2}{4\pi}\right) + 6\gamma \right) \quad (40)$$

There is no term with less than four powers of ϕ and thus the requirement of masslessness is satisfied automatically. (It is sometimes customary to present a counterterm for the divergence in $d = 2$ in the 2-point amplitude, but it is not in fact necessary, as a single point in the complex plane does not stand in our way to the physical dimension.) Enforcing (19) for $M = \sqrt{2}\eta$ we have

$$C = \frac{3e^4}{8\pi^2} \left(-\frac{6}{\epsilon} - 10 - 3 \ln\left(\frac{e^2 \eta^2}{2\pi}\right) - 3\gamma \right) \quad (41)$$

and the effective potential turns exactly equal to (21), as expected.

2.4 1-point amplitude: justifying the shift

Finally, we are in the position to compute some quantity in the broken theory. Obviously, the first one shall be the 1-point amplitude. Even if we, in accordance with our open-minded attitude, disregard the arguments of the previous section that led us to the search for minima of the effective potential, the requirement of the vanishing vacuum expectation value of the field is still, as already stressed, essential for the particle interpretation of the field. The vacuum expectation value $\langle 0 | \phi_1(x) | 0 \rangle$ is just the full 1-point Green function, but as we are looking for its zero value, we may in fact calculate its 1PI version only, as the initial propagator with all possible corrections, enters as a multiplicative factor. We could use the Lorentz invariance to argue that any nonzero value must be constant and thus when working in momentum space, the external momentum can be set to zero, but it will not enter the expressions anyway, and furthermore it is naturally guaranteed by the conservation delta function in the vertices.

It may look a little conspicuous at the first sight, that the two lowest order contributions (Fig. 4b and c) are apparently of order e^2 and thus with a little chance

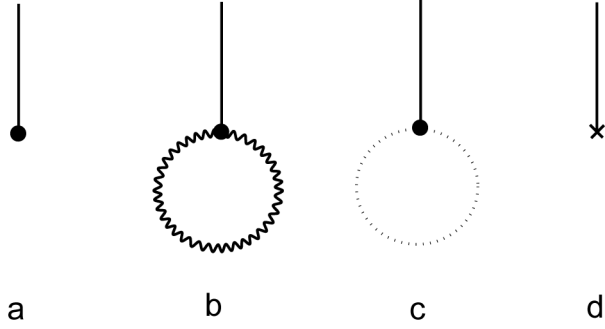


Figure 4: The relevant 1-point diagrams

to cancel against any λ -related term. This puzzle is easily resolved after evaluating them, because they turn out to be proportional to the mass terms in the corresponding propagators, that are to lowest order $\sim e^2$. Actually, we can foresee this even before the evaluation, using dimensional analysis. The vacuum expectation value has obviously the dimension of mass, but since we have stripped of one propagator from it, which would have contributed a factor of $(mass)^{-2}$, our 1PI amplitude is proportional to $(mass)^3$. But the order e^2 diagrams are proportional to η^1 and thus could not contribute, had they not been rescued by the mass terms in the propagators that are of course $\sim \eta^2$, but also $\sim e^2$.

The fact that η is always accompanied by some power of a coupling constant in a vertex or a propagator restricts any possible contributions to the leading (as we now know $\sim e^4\eta^3$) terms in the amplitude – it is simply quite difficult to collect three powers of η from the vertices without exceeding the fourth power of e and the propagators are of little help, bringing two powers of η but also at least two powers of e . From at least fourteen diagrams of the order $\sim e^4$ only four of them have the right dimension (see Fig. 4): the simple λ vertex (a), the gauge boson (b) and ghost (c) loops and the counterterm diagram (d). (Note that, unlike the scalar boson, the gauge boson does not receive any corrections to the mass in the first order as there is no 2-point gauge boson diagram $\sim e^2\eta^2$.)

The contribution of the gauge boson diagram is equal to¹⁶

$$\Gamma_{\text{gauge}}^1(0) = \frac{i2\sqrt{2}}{2}e^2\eta g_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{-ig^{\mu\nu}}{k^2 - 2e^2\eta^2} = \sqrt{2}e^2\eta \frac{-id}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{2e^2\eta^2}\right)^{\left(1 - \frac{d}{2}\right)} \quad (42)$$

where we have explicitly shown the symmetry factor coming from the line beginning and ending at the same point, note also that $g_{\mu\nu}g^{\mu\nu} = d$. Expanding in $\epsilon = d - 4$ we get finally:

$$\Gamma_{\text{gauge}}^1(0) = -\frac{i\sqrt{2}\eta^3 e^4}{4\pi^2} \left(\frac{4}{\epsilon} + 2 \ln\left(\frac{e^2\eta^2}{2\pi}\right) - 1 + 2\gamma\right) \quad (43)$$

¹⁶From now on, we drop the $i\epsilon$ terms in the denominator for brevity. The evaluations of the d -dimensional integrals are based on the Minkowski-space formulae in Appendix A to [18], but, contrary to their conventions, we put $\epsilon = d - 4$.

In a similar fashion, the ghost loop is (note that the minus sign from the fermion loop cancels the minus sign in the vertex):

$$\begin{aligned}\Gamma_{\text{ghost}}^1(0) &= i\sqrt{2}e^2\eta \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - 2e^2\eta^2} = -\sqrt{2}e^2\eta \frac{-i}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{2e^2\eta^2}\right)^{\left(1 - \frac{d}{2}\right)} = \\ &= \frac{i\sqrt{2}\eta^3 e^4}{4\pi^2} \left(\frac{1}{\epsilon} + \frac{\ln\left(\frac{e^2\eta^2}{2\pi}\right) - 1 + \gamma}{2}\right)\end{aligned}\quad (44)$$

The tree contribution (that forces us to work so hard in a first place to remove it) is simply:

$$\Gamma_{\text{tree}}^1(0) = -\frac{1}{3}i\sqrt{2}\lambda\eta^3 \quad (45)$$

and the counterterm contribution is equally simple to obtain, yet a little lengthy:

$$\Gamma_{\text{CT}}^1(0) = \frac{i\sqrt{2}\eta^3 e^4}{4\pi^2} \left(\frac{3}{\epsilon} + \frac{3}{2} \ln\left(\frac{e^2\eta^2}{2\pi}\right) + 5 + \frac{3}{2}\gamma\right) \quad (46)$$

Putting it all together we have:

$$\Gamma^1(0) = \Gamma_{\text{gauge}}^1(0) + \Gamma_{\text{ghost}}^1(0) + \Gamma_{\text{tree}}^1(0) + \Gamma_{\text{CT}}^1(0) = \frac{i\sqrt{2}\eta^3}{24\pi^2} (33e^4 - 8\lambda\pi^2) \quad (47)$$

and there are (to our little surprise) two possible solutions

$$\Gamma^1(0) = 0, \text{ when } \begin{cases} \eta = 0 \\ \text{or} \\ \lambda = \frac{33e^4}{8\pi^2} \end{cases} \quad (48)$$

We now see in an explicit way how the loop corrections reset the vacuum expectation value of the ‘‘Higgs field’’ to zero, given the right relation among the coupling constants. It is also interesting to note that although we have chosen to order the perturbation expansion according to the powers of e , allowing in principle many 2-loop diagrams to appear, it were only the 1-loop corrections that survived after the dimensional analysis. For the moment we have only re-derived the already known relation between the coupling constant, but this procedure can be clearly carried out to higher orders. Note also that the counterterm from the original theory was indeed sufficient to remove the divergence of the amplitude.

2.5 Scalar mass

In the discussion of the effective potential we saw the mass of the scalar to come out easily, but we expressed some qualification on the assignment of the mass to the gauge boson. Now we are in a fully complementary situation. The mass of the gauge boson can be, in the lowest order, read easily from the Lagrangian and equals

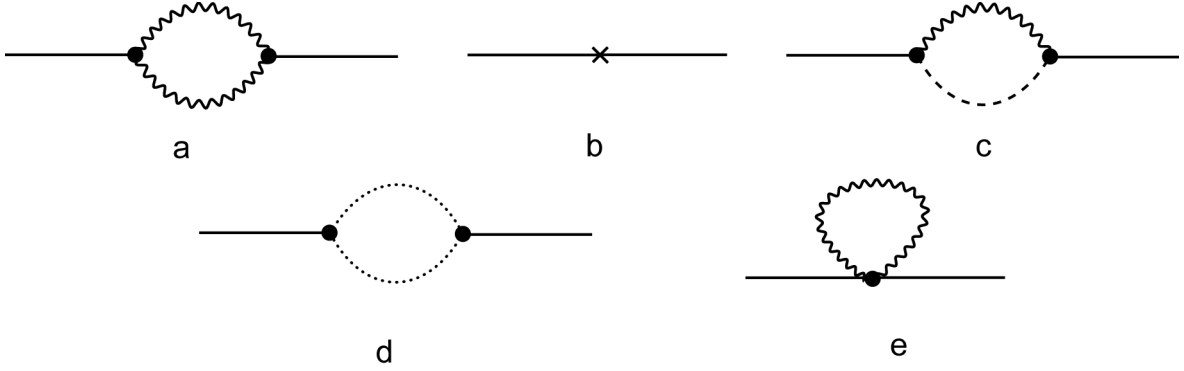


Figure 5: The relevant 2-point diagrams

to $\sqrt{2}e\eta$, in an agreement with our previous “estimate”, and it is also quickly seen that to this order it does not receive any further corrections. On the other hand, the tree mass of the scalar disagrees completely. We will now show that this is again remedied by the properly ordered radiative corrections. To make the comparison with the previous results easier, we again calculate the squared mass as the 2-point 1PI amplitude at zero momentum.

Again for dimensional reasons, the amplitude shall be proportional to η^2 . Concerning the vertices, the best “ η to e ratio” is offered by the 1-scalar-2-gauge bosons and ghost vertices, both allowing η^2 to be reached at the e^4 . There are two further e^2 diagrams (already present in the unbroken theory), whose propagators also offer terms $\sim \eta^2 e^4$, and we should not forget the $\sim C\eta^2$ counterterm, which is of the same order and will serve to control the divergences. On the other hand, from now on we will disregard the $\sim \lambda$ terms in the denominators of scalar propagators, and treat the respective terms in the Lagrangian as interactions. Indeed, those “tree” terms are of the same order as the 1-loop corrections we are just about to calculate, at least concerning the factors outside the logarithms.¹⁷ All these graphs are summarised in Fig. 5, with the lettering corresponding to the subscripts used in the calculation below. Their contributions are:

$$\begin{aligned} \Gamma_a^2(0) &= \frac{1}{2} \left(2\sqrt{2}\right)^2 e^4 \eta^2 \int \frac{d^d k}{(2\pi)^d} \frac{d}{(k^2 - 2e^2 \eta^2)^2} = \\ &= 4e^4 \eta^2 \frac{id}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{2e^2 \eta^2}\right)^{\left(2 - \frac{d}{2}\right)} = -\frac{i\eta^2 e^4}{4\pi^2} \left(\frac{8}{\epsilon} + 4 \ln\left(\frac{e^2 \eta^2}{2\pi}\right) + 2 + 4\gamma\right) \end{aligned} \quad (49)$$

(note the symmetry factor and the lack of a need for Feynman parametrisation when restricted to zero external momentum)

$$\Gamma_b^2 = i\frac{C}{4!} \cdot 12 \cdot 2 \cdot \eta^2 = iC\eta^2 = -\frac{i\eta^2 e^4}{4\pi^2} \left(-\frac{9}{\epsilon} - \frac{9}{2} \ln\left(\frac{e^2 \eta^2}{2\pi}\right) - 15 - \frac{9}{2}\gamma\right) \quad (50)$$

¹⁷The appearance of logarithms of coupling constants may cast some doubts on the validity of the perturbation theory. Nevertheless, they are so prominent only due to our notational honesty (think of any standard-model calculation with logarithms of m_W for example).

$$\begin{aligned}
\Gamma_c^2(0) &= e^2 \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - 2e^2\eta^2)^2} = e^2 \frac{id}{2(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{2e^2\eta^2}\right)^{(1-\frac{d}{2})} = \\
&= -\frac{i\eta^2 e^4}{4\pi^2} \left(-\frac{2}{\epsilon} - \ln\left(\frac{e^2\eta^2}{2\pi}\right) + \frac{1}{2} - \gamma\right) \tag{51}
\end{aligned}$$

$$\begin{aligned}
\Gamma_d^2(0) &= -2e^4\eta^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - 2e^2\eta^2)^2} = -2e^4\eta^2 \frac{id}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{2e^2\eta^2}\right)^{(2-\frac{d}{2})} = \\
&= -\frac{i\eta^2 e^4}{4\pi^2} \left(-\frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{e^2\eta^2}{2\pi}\right) - \frac{1}{2}\gamma\right) \tag{52}
\end{aligned}$$

$$\begin{aligned}
\Gamma_e^2(0) &= \frac{2}{2} e^2 \int \frac{d^d k}{(2\pi)^d} \frac{d}{k^2 - 2e^2\eta^2} = e^2 \frac{-id}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{2e^2\eta^2}\right)^{(1-\frac{d}{2})} = \\
&= -\frac{i\eta^2 e^4}{4\pi^2} \left(\frac{4}{\epsilon} + 2 \ln\left(\frac{e^2\eta^2}{2\pi}\right) + 2\gamma - 1\right) \tag{53}
\end{aligned}$$

Adding all the terms together with the tree-level inverse propagator, we finally get:

$$\Gamma^2(0) = -i\lambda\eta^2 + \frac{27ie^4\eta^2}{8\pi^2} = -\frac{3ie^4\eta^2}{4\pi^2} \tag{54}$$

where in the last step we have used the already established relation between the coupling constants from (48). The result is again in agreement with the previous section, as for the notation $\phi_0 = \sqrt{2}\eta$, which corrects the extra 1/2 in our formula.

Looking at Eq. (35), we can see several more new counterterms in the broken phase, that must correspond to new divergent loop diagrams, otherwise they would spoil the alleged renormalisability of the theory. Aside from the ϕ_2 -related terms that we have already promised to ignore there is, concerning the scalar particles only, just the 3-point vertex of ϕ_1 , which however contributes to physical process only in orders higher, than our current e^4 level (there is of course no kinematically available process with three identical external legs). It is interesting to note that not only there is no change in the ϕ_1^4 counterterm, but there are actually no new graphs for this process in the broken theory up to the e^4 order. Thus, the calculation of the scalar-scalar scattering to this order would proceed exactly as in the unbroken phase, except that the infrared divergences are now regulated by the masses of the particles.

3 Ordinary perturbation theory II: photons

3.1 Inverse propagator of the photon

It is about time to stop cheating and to return to the original Landau gauge for the calculations that use its counterterms. This is in fact a slightly delicate manner,

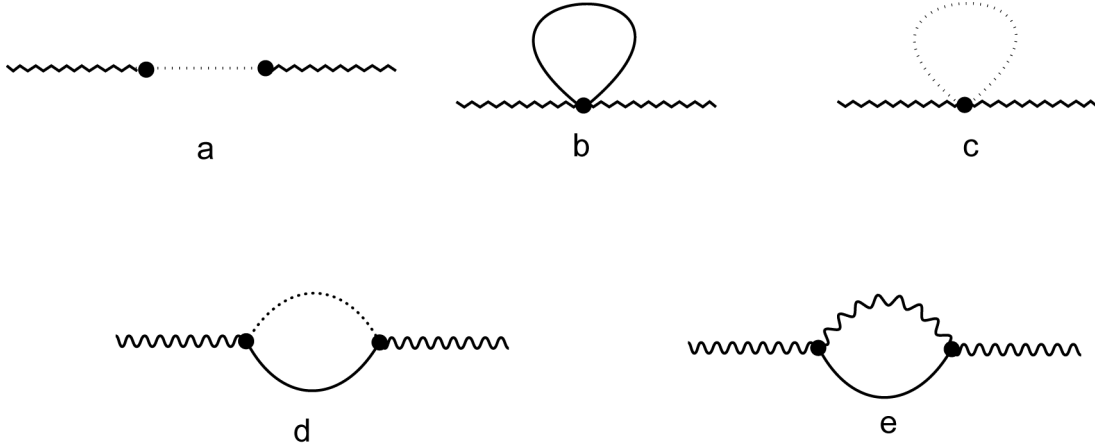


Figure 6: The various contributions to the photon propagator

particularly when working with 1PI functions – indeed, one may wonder what the inverse propagator is, when the propagator does not *have* an inverse. An unambiguous way to deal with it is the first-order formalism of Nakanishi, using an auxiliary field, but similar results can be used considering the Landau gauge as a $\xi \rightarrow 0$ limit of the covariant gauges, when necessary (cf. e.g. [24] and references cited therein).

As the gauge-fixing term remains unchanged with respect to the unbroken theory, we are obviously not introducing any ghosts and also there is no mass term for the alleged Goldstone boson ϕ_2 . On the other hand, we have not gotten rid of the $A_\mu \partial^\mu \phi_2$ vertex that has the potential to make the Feynman rules rather messy. Nevertheless, when it appears inside a graph, it inevitably gives zero when the momentum (corresponding to the derivative) is contracted with the transverse Landau-gauge propagator. The only questionable contributions are those, where this vertex is connected to an external photon leg, when calculating a quantity for which the external photon leg does not contribute a propagator.

Again, a rigorous discussion is possible (and again, see [24], according to which there is an intrinsic ambiguity in the longitudinal part of the Green functions in Landau gauge), but for the two-point function, we can easily present a formal $\xi \rightarrow 0$ argument. In a general covariant gauge (after the symmetry breaking that induces the mass m_V for the photon), the propagator

$$-i \frac{g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 - \xi m_V^2}}{k^2 - m_V^2} = \frac{-i}{k^2 - m_V^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{-i\xi}{k^2 - \xi m_V^2} \frac{k_\mu k_\nu}{k^2} \quad (55)$$

can be obtained from the Fourier transform of the quadratic part of the Lagrangian

$$g_{\mu\nu} k^2 - k_\mu k_\nu + \frac{1}{\xi} k_\mu k_\nu - m_V^2 g_{\mu\nu} = (k^2 - m_V^2) \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{k^2 - \xi m_V^2}{\xi} \frac{k_\mu k_\nu}{k^2} \quad (56)$$

by the well-known trick of inverting the coefficient of the transverse and longitudinal parts separately (see [11]). Consider now for example the graph in Fig. 6a. Its contribution is proportional to $k_\mu k_\nu / k^2$, that is, longitudinal. In the $\xi \rightarrow 0$ limit,

adding this contribution to (56) has no effect, as this term diverges anyway – thus we can ignore it.

As for the inverse propagator, there are three more $\sim e^4$ graphs (Fig. 6b–d), contribution of which is to this order the same as in the unbroken phase (barring the masses in the propagators that nevertheless add powers of coupling constants) and thus they do not give rise to any photon mass and also their divergences are cancelled by the wave-function renormalisation counterterm. The only new graph in the broken phase is that of Fig. 6e. It may (actually has to) be divergent, for there is the $2\eta^2 EA_\mu A^\mu$ counterterm (cf. (35)) to cancel it. However, the divergence must be momentum independent, otherwise the counterterm would be useless.

Let us now check this assumption by explicitly evaluating the diagram (the counterterm will be derived in the next subsection). For an arbitrary external momentum q , its contribution to the inverse propagator is (ignoring the $\sim e^4$ mass of ϕ_1)

$$\Gamma_e^{2A}(q)_{\mu\nu} = 8e^4\eta^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{-g_{\mu\nu}}{(k^2 - 2e^2\eta^2)(k^2 + q^2)} + \frac{k_\mu k_\nu}{(k^2 - 2e^2\eta^2)(k^2 + q^2)k^2} \right) \quad (57)$$

and it can be manipulated using Feynman parametrisation into

$$\begin{aligned} \Gamma_e^{2A}(q)_{\mu\nu} &= -8e^4\eta^2 \int \frac{d^d l}{(2\pi)^d} dx \frac{g_{\mu\nu}}{(l^2 - (2e^2\eta^2 - xq^2)(x-1))^2} + \\ &+ 16e^4\eta^2 \int \frac{d^d l}{(2\pi)^d} dx dy \frac{l_\mu l_\nu + y^2 q_\mu q_\nu}{(l^2 - (yq^2(y-1) + 2xe^2\eta^2))^3} \end{aligned} \quad (58)$$

(we have to split the integral in two, because “ l ” has different meanings in each of the terms and the limits on the x and y are understood to be $(0; 1)$).

The $q_\mu q_\nu$ term is indeed convergent and thus from now on, we will restrict ourselves to $q = 0$. In this case, the first and the second term are very similar, particularly when we switch from x to $1 - x$ in the first of them:

$$\Gamma_e^{2A}(0)_{\mu\nu} = 8e^4\eta^2 \left(\int \frac{d^d l}{(2\pi)^d} dx \frac{-g_{\mu\nu}}{(l^2 - 2e^2\eta^2 x)^2} + \int \frac{d^d l}{(2\pi)^d} dx \frac{2l_\mu l_\nu}{(l^2 - 2e^2\eta^2 x)^3} \right) \quad (59)$$

and this similarity is even more prominent after the integration – actually the second term is minus one half the first, that is

$$\begin{aligned} \Gamma_e^{2A}(0)_{\mu\nu} &= -8e^2\eta \frac{ig_{\mu\nu}}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \int \left(\frac{1}{2e^2\eta^2 x}\right)^{(2-\frac{d}{2})} dx = \\ &= \frac{i\eta^2 e^4 g_{\mu\nu}}{4\pi^2} \left(\frac{2}{\epsilon} + \ln\left(\frac{e^2\eta^2}{2\pi}\right) + \gamma - 1\right). \end{aligned} \quad (60)$$

It only remains to be found out whether this divergence is cancelled by the shifted counterterm and whether the resulting (hopefully finite) shift of the photon “mass” (the inverse propagator at origin) will agree with the value obtained by shifting the effective potential. To this end, we need to sum just another infinite sequence of graphs.

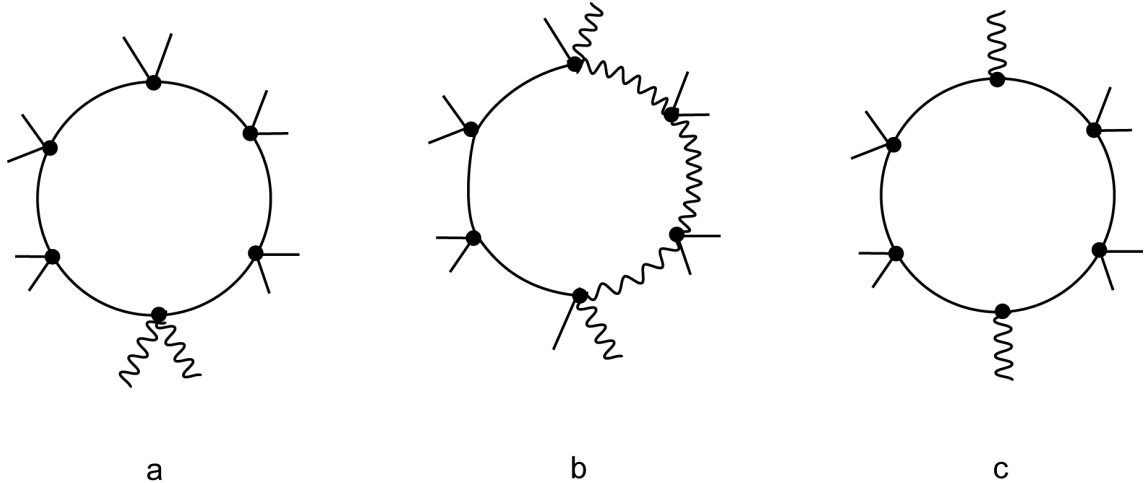


Figure 7: Three types of graphs that contribute to the two-photon effective potential

3.2 Two-external-photon-legs contributions to the effective potential

The calculation of the full effective potential in 1-loop approximation is in principle possible in the same manner as in the purely scalar case, but contains an overwhelming amount of combinatorics. As we have already noted, the task is simplified greatly thanks to the vanishing vacuum expectation value of the gauge field: even in the broken theory, we are still taking derivatives at its zero and thus we need to know only the relevant terms in the Taylor expansion of V . Working again in the Landau gauge, we immediately see that all diagrams with one external A^μ vanish. It corresponds to the fact that the 1-vector-2-scalar vertex is (necessarily from the Lorentz invariance) derivative and thus, even at the tree level, enters only the momentum-dependent terms in the expansion of the effective action. The two- A^μ term does not vanish and will present us (among other things) with the counterterm we need to finish the calculation of the photon inverse propagator.

In Landau gauge, there are three types of relevant graphs with two external gauge bosons, as shown in Fig. 7, all derived from the basic graphs of Fig. 1 (we again consider only external ϕ_1 thanks to gauge invariance of V): a) the $\phi^2 A_\mu A^\mu$ vertex can be simply inserted anywhere on the loop with the two external gauge boson legs, with ϕ_1 as well as ϕ_2 orbiting around; b) the $\phi^2 A_\mu A^\mu$ can be inserted on the loop with only one external A^μ , changing the orbiting particle from the scalar to the gauge boson; again, a second insertion anywhere else will allow the loop to close; c) the $A^\mu (\phi_2 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2)$ vertex can be inserted anywhere in the loop, resulting in a change of the orbiting field, which has to be compensated somewhere else by the same vertex, to allow the loop to close.

The contribution of the first class of diagrams is very similar to that of the scalar-only case (15) except for three important differences, namely some overall factors (from which the most important is $A^2 e^2$), one extra propagator relative to the power of $\lambda \phi^2$ and the lack of the rotation symmetry. Thus,

$$V_a \sim iA^2 e^2 \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left(\frac{\lambda \phi^2}{2k^2} \right)^n \frac{1}{k^2} = A^2 e^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \frac{\lambda \phi^2}{2}} \quad (61)$$

and it is well-known that such “tadpole” integrals are proportional to the “mass” in the propagator in the dimensional regularisation, as they vanish for massless particles in the loop – but it means that the contribution is proportional to $e^2 \lambda$ and thus can be disregarded to our order.

In a similar fashion we get the contribution of all graphs of the type “c”, because the relative position of the two extra vertices does not influence the value of the graph when the external momenta are all zero. The expression then is quite similar to the previous one, we only have one extra propagator, two loop momenta in the numerator (from the extra vertices) and the factor $(n + 1)$ that counts all the relative positions of the extra vertices, thus

$$V_c \sim A^\mu A^\nu e^2 \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left(\frac{\lambda \phi^2}{2k^2} \right)^n \frac{k^\mu k^\nu}{k^4} (n + 1) = A^\mu A^\nu e^2 \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{\left(k^2 - \frac{\lambda \phi^2}{2} \right)^2} \quad (62)$$

and it can be ignored by the same argument as the before (or by explicit calculation, if the former is not convincing enough).

The “b” class of diagrams needs a little more effort, but it is the one that actually contributes to the e^4 order¹⁸. In the Landau gauge, the numerator of photon propagator is idempotent and thus the contribution from the photon lines in the loops does not depend on their number. But the vertices with external scalar legs carry different coupling constants when they are attached to photon line than on a scalar line, therefore the contributions of different graphs with the same number of external legs differ. We will thus sum over all possible splittings between vertices on photon and scalar lines for each fixed total number of vertices. In this case we also need to get all the multiplicative constants right.

Similarly to the case of Sec. 1.2, the difference between the signs in vertices and propagators compensates, if we pair them together. Then there remains one scalar and one photon propagator – together with the two extra vertices, they supply four imaginary units and one minus sign (and four inverse powers of the loop momentum). Taking into account the imaginary unit in the definition of V , the factor 4 from the extra vertices and the reflection symmetry (around the horizontal axis in the orientation of Fig. 7), we arrive at

$$V_b = -i2A^\mu A^\nu e^4 \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left(\frac{\phi^2}{2k^2} \right)^n \frac{\phi^2 \left(g_{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)}{k^4} \sum_{i=0}^n \lambda^i (2e^2)^{n-i} \quad (63)$$

The finite sum is easily worked out, and so is then the infinite one:

¹⁸Note that the smallest diagram of this kind, without any further ϕ_1^2 insertions, is similar to the diagram of Fig. 6f, when the external scalar lines are replaced by $\sqrt{2}\eta$.

$$\begin{aligned}
V_b &= -i2A^\mu A^\nu e^4 \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left(\frac{\phi^2}{2k^2} \right)^n \frac{\phi^2 (g_{\mu\nu} - \frac{k^\mu k^\nu}{k^2})}{k^4} \left(\frac{\lambda^{n+1} - (2e^2)^{n+1}}{\lambda - 2e^2} \right) = \\
&= -\frac{i2A^\mu A^\nu \phi^2 e^4}{\lambda - 2e^2} \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \frac{(g_{\mu\nu} - \frac{k^\mu k^\nu}{k^2})}{k^4} \left(\lambda \left(\frac{\lambda \phi^2}{2k^2} \right)^n - 2e^2 \left(\frac{e^2 \phi^2}{k^2} \right) \right) = \\
&= -\frac{i2A^\mu A^\nu \phi^2 e^4}{\lambda - 2e^2} \int \frac{d^d k}{(2\pi)^d} \frac{(g_{\mu\nu} - \frac{k^\mu k^\nu}{k^2})}{k^2} \left(\frac{\lambda}{k^2 - \frac{\lambda \phi^2}{2}} - \frac{2e^2}{k^2 - e^2 \phi^2} \right) \quad (64)
\end{aligned}$$

The integral is in fact the same as the one from the previous subsection, Eq. (57), only with different constants around and thus can be obtained from it just by the replacement of the constants. Most terms in the final $\epsilon \rightarrow 0$ expansion of (57) are actually independent of the “mass” in the propagator and thus they can be factored out from both term in the last bracket in (64), cancelling the unpleasant factor $\lambda - 2e^2$. After some manipulations, we get

$$V_{2A} = -\left(\frac{e^2}{2} + E \right) A^2 \phi^2 - \frac{A^2 \phi^2 e^4}{16\pi^2} \left(\frac{2}{\epsilon} + \ln \left(\frac{\phi^2}{4\pi} \right) + \gamma - 1 + \frac{\lambda \ln \lambda - 2e^2 \ln 2e^2}{\lambda - 2e^2} \right) \quad (65)$$

(where we have added the tree-level and counterterm two-photon potential) and we may hope that the somewhat ugly last term will be improved by the renormalisation, as was the logarithm of λ in sec. (1.2).

Now we impose the condition¹⁹:

$$\left. \frac{\partial^4 V}{\partial \phi^2 \partial A^2} \right|_{\phi=\sqrt{2}\eta, A=0} = -2e^2 \quad (66)$$

because the condition at $\phi = 0$ is again inapplicable (this time we take only two derivatives with respect to ϕ , but we also have only two powers of ϕ to beat the logarithm) and solve for E . The calculations are trivial, but it is interesting to see how they work. The derivatives with respect to A amount just to the multiplication by 2 and so do the derivatives with respect to ϕ , except for the logarithm, from which another +6 arises. Four times $-e^2/2$ is exactly what we want the expression to be and so the counterterm must cancel the rest; what survives after putting it back into (65) is just the +6 from the derivative of the logarithm and the obligatory $\ln \phi^2 / (2\eta^2)$, that is

$$V_{2A} = -\frac{e^2}{2} A^2 \phi^2 - \frac{A^2 \phi^2 e^4}{16\pi^2} \left(\ln \frac{\phi^2}{2\eta^2} - 6 \right) \quad (67)$$

and the counterterm is

¹⁹By $\phi = \sqrt{2}\eta$ we mean, of course, that $\phi_1 = \sqrt{2}\eta$ and $\phi_2 = 0$, according to our previous discussion.

$$E = -\frac{e^4}{16\pi^2} \left(\frac{2}{\epsilon} + \ln \frac{\eta^2}{4\pi} + \gamma + 5 + \frac{\lambda \ln \lambda - 2e^2 \ln 2e^2}{\lambda - 2e^2} \right). \quad (68)$$

The contribution of the counterterm to the inverse propagator in the shifted-field perturbation theory is $4iEg_{\mu\nu}$, as can be read from (35). Comparing with (60), we conclude that the e^4 contribution to the inverse propagator at zero momenta is

$$\Gamma^{2A}(0)_{\mu\nu} = \frac{i\eta^2 e^4 g_{\mu\nu}}{4\pi^2} \left(\ln 2e^2 - 6 - \frac{\lambda \ln \lambda - 2e^2 \ln 2e^2}{\lambda - 2e^2} \right) \approx -\frac{3i\eta^2 e^4 g_{\mu\nu}}{2\pi^2} \quad (69)$$

where the last equality is obtained simply by assuming that λ is of a higher order than $2e^2$ (one even does not need to use the actual coupling constant relation). Moreover, noting that

$$\left. \frac{\partial^4 V}{\partial A^2} \right|_{\phi=\sqrt{2}\eta, A=0} = -2\eta^2 e^2 + \frac{3\eta^2 e^4}{2\pi^2} \quad (70)$$

(one factor of two is from the derivative, the second from the $\sqrt{2}$ in front of η), we see that the value obtained in this way is again the same.

We could of course continue with evaluating particular quantities in the C-W model, but we hope that we have convinced the reader that this theory is as tractable as the standard Abelian Higgs model, provided that one carefully takes into account the relative orders of the relevant quantities. A final remark is in order here. The $\phi^2 A^2$ counterterm does not actually seem like a lot of information, but we can, following [9], expect the counterterms to respect the gauge invariance of the unbroken theory, thus they should take the form

$$(Z_2 - 1) (\partial_\mu \phi_0 + ieA_\mu \phi_0) (\partial^\mu \phi_0 + ieA^\mu \phi_0)^+ - \frac{B}{2} \phi_0^+ \phi_0 - \frac{C}{6} (\phi_0^+ \phi_0)^2 - \frac{1}{4} (Z_3 - 1) F^{\mu\nu} F_{\mu\nu}. \quad (71)$$

It means that e.g. the D , in the notation of (35), is just $2E/e$ and so on and the remaining photon wave-function renormalisation counterterm can (or better said, has to) be calculated directly without resorting to the effective potential formalism.

Part II

Two-dimensional delta-function

4 Physicist's approach

4.1 Preliminaries

In the first part we have examined the dimensional transmutation using a classic and feasible example, which nevertheless turned out to be technically rather involved and partly obscure, unsurprisingly due to its field-theory nature. In such a case, one can usually ask whether there is some, preferably non-relativistic, quantum mechanical analogue. There we could possibly hope for an exact solution to the theory, which can be much more illuminating than the perturbative treatment we had to use in the field theory. The answer is a qualified yes: quantum mechanical systems that in some sense exhibit dimensional transmutation do exist, but they are not quite a direct generalisation of the Coleman-Weinberg model. In this section we will examine this statement from a rather informal point of view – the informality will be remedied in the rest of this part.

The notion of a massless particle is relativistic and thus it is not immediately clear how to deal with masslessness in non-relativistic quantum mechanics and whether it is even possible. Some insight can be obtained from the group-theoretical point of view (for more details see [12] and references cited therein): it turns out that quantum mechanical systems correspond to projective representations of the Galilei group distinguished by the value of the mass, whereas setting $m = 0$ corresponds to a true (not projective) representation which cannot be consistently interpreted as a dynamical system.

On the other hand, this is by no means an obstacle to finding scale invariant models in quantum mechanics: in a non-relativistic theory, we do not have enough fundamental physical constants (in fact, we have only \hbar) to express all the dimensions of physical quantities in terms of powers of length or mass. Instead, these two remain independent and the mass does not enter the scale transformations at all. In one-particle models we can even think of the particle mass as another fundamental parameter of the theory and then express all dimensions in powers of length, as in [4], but as the nature of the problem is quite simple, we will adopt a more transparent approach. Let us examine the spectrum of the one-particle Hamiltonian

$$H_* = -\frac{\hbar^2 \Delta}{2m} + \lambda_* V(\mathbf{x}) = \frac{\hbar^2}{2m} \left(-\Delta + \frac{2m\lambda_*}{\hbar^2} V(\mathbf{x}) \right) \quad (72)$$

where we have explicitly extracted the coupling constant from the potential, assuming that $V(\mathbf{x})$ is of a simple form that allows such a splitting naturally (such as a power law). Obviously we can focus on the operator in the bracket (which we will simply call H in the following), whose eigenvalues, if any, must have the dimension of $length^{-2}$ in order to give the spectrum of H_* the dimension of energy. But if $V(\mathbf{x})$ has itself the dimension of $length^{-2}$ then the “coupling constant” $\lambda = \frac{2m\lambda_*}{\hbar^2}$ is dimensionless and we recover the same line of argument as when introducing the

Coleman-Weinberg model: there is no way to construct a quantity with the right dimension from the parameters of the theory at hand.

The statement that $V(x)$ is of the form “coupling times function” is essentially vague, so let us specify the most obvious such case, the potentials that are homogeneous in the coordinate. If the degree of homogeneity is -2 , then we have perfectly the situation of the previous paragraph. Let us now consider a function that satisfies

$$H(\mathbf{x})\Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \quad (73)$$

Because the Laplacian is also homogeneous of the degree -2 , the Hamiltonian as a whole has exceptionally simple scaling properties and we have for any ξ (using the trivial fact that $\xi\mathbf{x}$ is equally good as an independent variable as \mathbf{x} is):

$$H(\mathbf{x})\Psi(\xi\mathbf{x}) = H(\xi^{-1}\xi\mathbf{x})\Psi(\xi\mathbf{x}) = \xi^2 H(\xi\mathbf{x})\Psi(\xi\mathbf{x}) = \xi^2 E\Psi(\xi\mathbf{x}) \quad (74)$$

The same function Ψ evaluated at scaled points is a solution of the former equation with an “eigenvalue” (in the usual physical sense) of $\xi^2 E$. There is no possibility of a discrete spectrum when a single solution to (73) generates solutions with arbitrary energies with the same sign (as ξ^2 is positive). Taking into account that any negatively-homogeneous function tends to zero at infinity, we can have a physically sensible continuous spectrum covering $(0, \infty)$, but if even a single bound state appears, the Hamiltonian fails to be bounded from below. That again is not surprising, because any such a bound state would necessarily introduce a dimensionful scale that is not present in the original form of H .

Saying that the potential is minus-2-homogeneous appears to be only a fancy expression for $V \sim f(\Omega)/r^2$ but there is a notable correction to this statement. Consider the δ -function in d dimensions for which

$$\int \delta(\xi\mathbf{x}) f(\mathbf{x}) d^d\mathbf{x} = \frac{1}{\xi^d} \int \delta(\mathbf{y}) f(\mathbf{y}/\xi) d^d\mathbf{y}. \quad (75)$$

The right hand side is evaluated at $\mathbf{y}/\xi = 0 \iff \mathbf{y} = 0$ and thus the only effect of the dilatation on the delta is the rescaling by the $-d$ th power, that is, homogeneity. In a general space dimension, the δ -function is -2 -homogeneous only when multiplied by r^{d-2} , but for $d = 2$ we get a particularly simple scale-invariant Hamiltonian

$$H = -\Delta + \lambda\delta(\mathbf{x}) \quad (76)$$

which will serve us as a perfect sandbox for our discussion, whereas we postpone the comments on its physical relevance to the very end of Part II. From the mathematical point of view, the expression (76) obviously needs some care to make it meaningful, especially when it is to act on the possibly wild functions from L^2 and this is why it is often referred to as a “pseudopotential”.

4.2 Formal scattering solution and the need for regularisation

Nevertheless, we have promised an informal approach in this section and thus we will boldly go on for a moment, following mainly the treatment of [26] but with slightly

different conventions and a more detailed commentary. We have already dismissed the chance of finding a bound state for a scale-invariant Hamiltonian, so we shall rather investigate the scattering. The potential is so friendly that we can dare to try to directly solve the Lippmann-Schwinger equation ([10], Ch. 3.7) for the scattering state

$$|\Psi_{\mathbf{p}}\rangle = |\mathbf{p}\rangle + \frac{1}{E - H_0 + i\epsilon} H_I |\Psi_{\mathbf{p}}\rangle \quad (77)$$

where of course $H = H_0 + H_I$ and $E = p^2$. (Working with H instead of H_* amounts to using units where $\hbar = 1$ and $m = \frac{1}{2}$.) In the momentum representation

$$\begin{aligned} \tilde{\psi}_{\mathbf{p}}(\mathbf{k}) &= \langle \mathbf{k} | \mathbf{p} \rangle + \int d\mathbf{k}' d\mathbf{x} d\mathbf{x}' \langle \mathbf{k} | \frac{1}{E - H_0 + i\epsilon} | \mathbf{k}' \rangle \langle \mathbf{k}' | \mathbf{x} \rangle \langle \mathbf{x} | H_I | \mathbf{x}' \rangle \psi_p(\mathbf{x}') = \\ &= \delta(\mathbf{p} - \mathbf{k}) + \int d\mathbf{x} \frac{1}{p^2 - k^2 + i\epsilon} \frac{1}{(2\pi)^{d/2}} e^{i\mathbf{k}\mathbf{x}} \lambda \delta(\mathbf{x}) \psi_p(\mathbf{x}) = \\ &= \delta(\mathbf{p} - \mathbf{k}) + \frac{\lambda}{(2\pi)^{d/2} (p^2 - k^2 + i\epsilon)} \psi_p(\mathbf{0}) \end{aligned} \quad (78)$$

with $\psi(\mathbf{x}) = \langle \mathbf{x} | \Psi \rangle$ and keeping the spatial dimension arbitrary (for a good reason to be explained later). $\psi_{\mathbf{p}}(\mathbf{0})$ can now be determined self-consistently by the Fourier transform from the equation

$$\begin{aligned} (2\pi)^{d/2} \psi_{\mathbf{p}}(\mathbf{0}) &= \int \tilde{\psi}_{\mathbf{p}}(\mathbf{k}) d^d \mathbf{k} = 1 + \frac{\lambda \psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} \int \frac{d^d \mathbf{k}}{(p^2 - k^2 + i\epsilon)} = \\ &= 1 + \frac{\lambda \psi_{\mathbf{p}}(\mathbf{0}) S_{d-1}}{(2\pi)^{d/2}} \int_0^\infty \frac{k^{d-1} dk}{p^2 - k^2 + i\epsilon} \end{aligned} \quad (79)$$

(S_n is the surface of n -dimensional sphere, that is, the surface of a unit ball in $n + 1$ dimensions). Similarly to the usual $d = 3$ scattering, for a sufficiently short-range potential (which we indeed have) the scattering solution tends asymptotically to

$$\psi_{\mathbf{p}}(\mathbf{r}) \sim e^{i\mathbf{p}\mathbf{r}} + f(\mathbf{p}', \mathbf{p}) \frac{e^{ipr}}{r^{(d-1)/2}} \quad (80)$$

for $r \rightarrow \infty$, where $f(\mathbf{p}', \mathbf{p})$ is the scattering amplitude²⁰ (barring an overall phase factor) whose absolute value squared gives the differential cross section. (This result is almost obvious, taking into account that the intensity of a spherical wave in d dimensions has to fall like $r^{(1-d)}$ to carry finite energy to spatial infinity, the nontrivial fact is the absence of numerical factors, see Appendix C of [4] for a derivation.) When (79) is solved, the $\psi_{\mathbf{p}}(\mathbf{0})$ is just a known constant and Fourier transforming (78) we get

$$\psi_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\mathbf{r}} + \frac{\lambda \psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} \int \frac{e^{i\mathbf{k}\mathbf{r}} d^d \mathbf{k}}{(p^2 - k^2 + i\epsilon)} \rightarrow e^{i\mathbf{p}\mathbf{r}} - \frac{\lambda \psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} \frac{(2\pi)^{(1+d)/2}}{2p^{(3-d)/2}} \frac{e^{i(pr + \pi(3-d)/4)}}{r^{(d-1)/2}} \quad (81)$$

²⁰In the usual (though slightly misleading) notation where $\mathbf{p}' = p \frac{\mathbf{r}}{r}$

for $r \rightarrow \infty$ (by Eq. C5 in [4]). Regardless of the specific numerical factors, we see from the comparison with (80) that the scattering amplitude is in any dimension proportional to $\psi_{\mathbf{p}}(0)$ times a power of p . We shall thus first evaluate $\psi_{\mathbf{p}}(0)$ (or rather $\lambda\psi_{\mathbf{p}}(0)$, which turns out to be wiser) to see whether it is finite but nonzero – otherwise we are guaranteed to get no scattering or a nonsense, independently of the details of the rest of the expression.

So we formally solve (79) for

$$\frac{\lambda\psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} = \frac{1}{\frac{(2\pi)^d}{\lambda} + S_{d-1} \int_0^\infty \frac{k^{d-1} dk}{(k^2 - p^2 - i\epsilon)}} \quad (82)$$

and as the integral is divergent in two or more dimensions (the one-dimensional case is in some respects very different and we do not treat it here) at large momenta, $\psi_{\mathbf{p}}(0)$ indeed vanishes and there is no scattering for any chosen value of λ .

This situation is quite reminiscent to the UV divergences of the field theory, where holding the formal “bare” coupling in the Lagrangian fixed at any value does not usually lead to a sensible interacting theory (for a more elaborate discussion of these matters, see [6]). It is thus tempting to regulate the integral by introducing a cut-off, let λ depend on the cut-off and hope for a successful renormalisation with nontrivial results. It is indeed how we will proceed now, but we do not claim it to be more than a mere heuristic (for a similar approach, based on the dimensional regularisation, the reader is referred to [4]). On the one hand, the situation is quite similar to the one in the field theory, as we are trying to circumvent the troubles caused by the introduction of a strictly local interaction, on the other hand we are lacking the whole machinery that renders renormalisation in the field theory physically understandable.

In the first few dimensions, the regulated integral is (for such a large Λ for which we have not only $\Lambda^2 > p^2$ to work out the “ $i\epsilon$ ” continuations, but we can also replace $\operatorname{arctanh} x\Lambda$ with its proper limit at infinity):

$$\int_0^\Lambda \frac{k^{d-1} dk}{(k^2 - p^2 - i\epsilon)} =$$

$$d = 2: \quad \frac{1}{2} \ln \left(\frac{\Lambda^2}{-p^2 - i\epsilon} + 1 \right) = \frac{1}{2} \ln \left(1 - \frac{\Lambda^2}{p^2} + i\epsilon \right) = \frac{1}{2} \ln \left(\frac{\Lambda^2}{p^2} - 1 \right) + \frac{i\pi}{2}$$

$$d = 3: \quad \Lambda - \sqrt{-p^2 - i\epsilon} \operatorname{arctan} \frac{\Lambda}{\sqrt{-p^2 - i\epsilon}} = \Lambda - p \operatorname{arctanh} \frac{\Lambda}{p} = \Lambda + \frac{i\pi}{2} p$$

$$d = 4: \quad \frac{\Lambda^2}{2} + p^2 \frac{1}{2} \ln \left(\frac{\Lambda^2}{-p^2 - i\epsilon} + 1 \right) = \frac{\Lambda^2}{2} + p^2 \frac{1}{2} \ln \left(\frac{\Lambda^2}{p^2} - 1 \right) + \frac{i\pi}{2} p^2$$

$$d = 5: \quad \frac{\Lambda^3}{3} + \Lambda p^2 - p^2 \sqrt{-p^2 - i\epsilon} \operatorname{arctan} \frac{\Lambda}{\sqrt{-p^2 - i\epsilon}} = \frac{\Lambda^3}{3} + \Lambda p^2 + \frac{i\pi}{2} p^3 \quad (83)$$

and so on, following a very clear pattern. In two dimensions, the coupling constant must be taken in the form

$$\lambda = 1/\left(-\frac{1}{2\pi}\ln \Lambda + c\right) \quad (84)$$

where c stands for a cut-off independent, finite (possibly zero) term, for the denominator to be finite when $\Lambda \rightarrow \infty$. Similarly, in three dimensions we will require

$$\lambda = 1/\left(-\frac{1}{2\pi^2}\Lambda + c\right). \quad (85)$$

The advertised finite results then read (after removing the cutoff)

$$\frac{\lambda\psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} = \frac{1}{\pi(4\pi c + i\pi - \ln p^2)} \quad (d=2) \quad (86)$$

$$\frac{\lambda\psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^3 c + 2i\pi^2 p} \quad (d=3) \quad (87)$$

where, as usual, we do not introduce any artificial scale to remedy the dimensionful arguments of logarithms – instead we wait for an on-shell renormalisation scheme to do it for us to explicitly see its inevitability.

On the contrary, in higher dimensions the situation is hopeless. There are always divergent terms proportional to p that cannot be incorporated into the coupling constant, thus for $d > 3$ the denominator is always infinite and the scattering amplitude vanishes identically as noted e.g. in [27].²¹

4.3 Bound states and renormalisation

With the cut-off in place, the Hamiltonian (76) is no longer scale invariant and so we can re-open the question of possible bound states. The eigenvalue equation for this operator

$$(\Delta + E)\psi(\mathbf{x}) = \lambda\delta(\mathbf{x})\psi(\mathbf{x}) \quad (88)$$

is also readily put into momentum representation

$$(p^2 - E)\tilde{\psi}(\mathbf{p}) = -\frac{\lambda}{(2\pi)^{d/2}}\psi(\mathbf{0}) \quad (89)$$

(note that E is now negative and thus we do not have to deal with complex continuations; we will write $E = -B$ to stress it). Again, we find a consistency relation

$$\psi(\mathbf{0}) = -\frac{\lambda S_{d-1}\psi(\mathbf{0})}{(2\pi)^d} \int_0^\infty \frac{p^{d-1} dp}{p^2 + B} \quad (90)$$

²¹This result is not directly related to the main topic, but is interesting for its own and we will see it emerging naturally from the machinery of the next section.

from which $\psi(\mathbf{0})$ readily drops out and the rest can be solved for B after cutting-off the integral and using the already established cut-off dependent coupling. In two dimensions we get

$$-\frac{1}{4\pi}\ln \Lambda^2 + c = -\frac{1}{4\pi}\ln \left(\frac{\Lambda^2}{B} + 1 \right) \Rightarrow c = \frac{1}{4\pi}\ln B \quad (91)$$

when $\Lambda \rightarrow \infty$. This has a unique solution for B regardless of the value of c , thus a suitably regularised δ -function potential possesses exactly one bound state independently of the “finite part” of the coupling constant. As we are used from the field theory, the value of c is dependent on the details of the “renormalisation” procedure and the natural parametrisation of the theory is in terms of the (in principle observable) bound state energy. From (86) we have

$$\frac{\lambda\psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} = -\frac{1}{\pi \left(\ln \frac{p^2}{B} - i\pi \right)} \quad (92)$$

and it is now easy to extract the scattering amplitude comparing Eqs. (80) and (81) for $d = 2$ as

$$f(\mathbf{p}', \mathbf{p}) = \sqrt{\frac{2\pi i}{p}} \frac{1}{\ln \frac{p^2}{B} - i\pi}. \quad (93)$$

The amplitude is completely isotropic, or, phrased in the language of partial-wave decomposition, only the s-wave is affected, and the appearance of B in the denominator clearly violates the apparent scale-invariance of the Hamiltonian. It is often stated that this energy is a new parameter that is not calculable from the “original” parameters of the theory. This is not completely untrue, as the Hamiltonian (76) makes no reference to the cut-off and so “ λ ” in it refers to the coupling evaluated at $\Lambda \rightarrow \infty$, that is, *zero*, from which we hardly compute anything. But in such a case the theory is obviously not complete yet. Nevertheless, when we consider the cut-off theory at a definite value of Λ , then B can be “calculated” from the actual coupling constant, but at the end this amounts to changing one way to parametrise the theory to another, not a physical prediction. The mechanism for scale-invariance breaking is exactly the same as in the field theory: we define the theory solely in the means of dimensionless parameters, but some of them have to be given at *some* value of the dimensionful cut-off, that is, in a scale non-invariant way. The presence of Λ in (91) allows B to consistently appear without violating any dimensional arguments, hence allowing c to be related to B . The most wonderful thing, as always, is that the dependence on Λ can be chosen such that it completely disappears at the end (the independence of the result on the choice of the cut-off procedure is illustrated in [31]).

To make the comparison complete, we repeat the same procedure also for the three-dimensional case. Eq. (90) is solved to

$$-\frac{1}{2\pi^2}\Lambda + c = -\frac{1}{2\pi^2} \left(\Lambda - \frac{\pi}{2}\sqrt{B} \right) \Rightarrow c = \frac{\sqrt{B}}{4\pi}. \quad (94)$$

Again, there is a unique solution, but only for positive values of c – for negative values there is no bound state and thus another way of renormalisation must be used. Eq. (87) then reads

$$\frac{\lambda\psi_{\mathbf{p}}(\mathbf{0})}{(2\pi)^{d/2}} = \frac{1}{8\pi^2\sqrt{B} + 2i\pi^2p} \quad (d = 3) \quad (95)$$

and the scattering amplitude becomes

$$f(\mathbf{p}', \mathbf{p}) = -\frac{1}{\sqrt{B} + ip}, \quad (96)$$

i.e. again an isotropic (s-wave) form. Even though there is no dimensional transmutation, because λ is not dimensionless, this is just an aesthetic difference: again we have to put new information into the theory by hand, no matter if it is the function $\lambda(\Lambda)$ (that is, c), or the specific value of B . On the other hand we see that the presence of Λ in (94) was in this case not necessary to allow a relation between c and B .

4.4 Square-well approximation

Some insight into the regularisation procedure can be gained when we consider the δ -function potential to be a limit of some more regular potentials. This method, using a square well, is commented on in [13] for the bound state, but we will again present its application to the scattering (focusing on the 2-dimensional case). The reason is simply that having a potential without a bound state is a perfectly acceptable situation, whereas a potential without any scattering is not – thus the argument is more convincing.²²

First we need a few more notions regarding the two-dimensional scattering ([7]). We already know the asymptotics of the scattering solution (cf. Eq. 80):

$$\psi_{\mathbf{p}}(\mathbf{r}) \sim e^{i\mathbf{p}\mathbf{r}} + f(\mathbf{p}', \mathbf{p}) \frac{e^{ipr}}{\sqrt{r}}. \quad (97)$$

In two dimensions, the plane wave can also be expanded in “spherical” (we should probably say “circular” in 2D, but we keep the usual terminology) waves according to the formula

$$e^{i\mathbf{k}\mathbf{r}} = e^{ikr \cos \theta} = \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\theta} = \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos(m\theta) \quad (98)$$

where we have introduced the convenient shorthand $\epsilon_0 = 1$, $\epsilon_{m>0} = 2$ and J_m are the Bessel functions. The functions $\frac{1}{\sqrt{2\pi}} e^{im\theta}$, $m \in \mathbb{Z}$ constitute a basis of the unit circle and play the role of spherical harmonics in two dimensions. They allow the angular

²²A similar calculation, which nevertheless *assumes* (106) from the beginning, can be found in [1]. It is also interesting to note that the authors interpret the same result from a completely different viewpoint, namely that the renormalisation is a deliberate interference with the model and it is its unrenormalised (thereby trivial) version that shall be considered more natural (even though they admit, that the renormalised version may well be more appropriate to describe some actual physical phenomena).

separation of variables for a spherically symmetric Schroedinger equation. For a finite-range potential, the general radial solution for the m -th wave (it is enough to consider $m \geq 0$, as the radial equation depends only on m^2) at infinity is

$$R_m(r) = A_m J_m(kr) + B_m N_m(kr), \quad (99)$$

where N_m is the Neumann function. Because N_m is singular at the origin, B_m is exactly zero when a scattering centre is not present. Thus it is not surprising that the phase shifts turn out to be related to the fraction of the Neumann function in the solution, namely

$$\tan \delta_m = -B_m/A_m. \quad (100)$$

This can be seen more clearly considering the asymptotic of (99) for $r \rightarrow \infty$:

$$\begin{aligned} R_m &\sim A_m \cos \left(kr - \frac{\pi}{2} \left(m + \frac{1}{2} \right) \right) + B_m \sin \left(kr - \frac{\pi}{2} \left(m + \frac{1}{2} \right) \right) \sim \\ &\sim \cos \left(kr - \frac{\pi}{2} \left(m + \frac{1}{2} \right) + \delta_m \right) \end{aligned} \quad (101)$$

if the phase shift δ_m obeys (100).

Consider now a spherical (circular) potential well, whose diameter is a and depth $\lambda/(\pi a^2)$ so that its “volume” is always λ . The corresponding potential then tends to $-\lambda$ times the δ -function in the limit $a \rightarrow 0$. The scattering solution can be obtained simply by matching the free solution outside the well (99) (where in our units simply $k = \sqrt{E}$) with the inside solution

$$R_m^{well}(r) = C_m J_m \left(\sqrt{E + \frac{\lambda}{\pi a^2}} r \right) \equiv C_m J_m(k_\lambda r) \quad (102)$$

so that the wave function and its derivative are continuous on the well boundary. The result is somewhat lengthy

$$\tan \delta_m = \frac{\sqrt{k_\lambda} J_{m+1}(k_\lambda a) J_m(ka) - \sqrt{k} J_m(k_\lambda a) J_{m+1}(ka)}{\sqrt{k_\lambda} J_{m+1}(k_\lambda a) N_m(ka) - \sqrt{k} J_m(k_\lambda a) N_{m+1}(ka)}, \quad (103)$$

but it can be easily checked that it goes to zero as $a \rightarrow 0$ for any m .

This result is not surprising, as we have already seen that the δ -function potential gives no scattering, unless specially treated. Because the discussion for an arbitrary m would be rather cumbersome, let us focus on the s-wave, which is the most promising one, as it was the only contribution we found in the previous section. For $m = 0$ the leading behaviour of the phase shift as $a \rightarrow 0$ is found to be, after some manipulations

$$\tan \delta_0 \sim \frac{\pi}{2} \frac{1}{\ln \frac{\sqrt{E}}{2} + \ln a + \gamma + \frac{J_0(\sqrt{\frac{\lambda}{\pi}})}{J_1(\sqrt{\frac{\lambda}{\pi}}) \sqrt{\frac{\lambda}{\pi}}}} \quad (104)$$

where γ is the Euler-Mascheroni constant. The trouble is in the $\ln a$ term in the denominator that forces the whole expression to vanish. As a remedy, we can again let the coupling constant depend on a . The exact way is not clear yet, but the form of the last term suggests that λ will have to tend to zero to produce a divergent quantity. With this assumption, we again use the asymptotic of the Bessel functions and we find that

$$\frac{J_0\left(\sqrt{\frac{\lambda}{\pi}}\right)}{J_1\left(\sqrt{\frac{\lambda}{\pi}}\right)\sqrt{\frac{\lambda}{\pi}}} \sim \frac{2\pi}{\lambda} \quad (105)$$

as $\lambda \rightarrow 0$. Now it is clear that, in order to tame the $\ln a$ term while not introducing other divergent quantities, the coupling constant must be of the form

$$\lambda = \frac{1}{\frac{1}{2\pi}\ln\frac{d}{a} - c} \quad (106)$$

where c is finite and d is a so far arbitrary constant introduced to make the following discussion easier. (Note that this is consistent with the preceding assumption, because λ really does tend to zero for small a .)

This formula is reminiscent of the regularised coupling constant (84), but to make a direct comparison we must note that the square-well formulation has naturally led us to reverse the sign in the definition of the coupling constant (that is why we have written the finite part with the opposite sign). With this qualification in mind, we can ask how to match the two procedures. Using (91) we get (already after the $a \rightarrow 0$ limit and writing $E = p^2$ for easier comparison)

$$\tan \delta_0 = \frac{\pi}{\ln\frac{p^2 d^2}{4B} + 2\gamma} = \frac{\pi}{\ln\frac{p^2 d^2 e^{2\gamma}}{4B}} \quad (107)$$

The scattering amplitude is

$$f(\mathbf{p}) = \frac{1}{\sqrt{2\pi i p}} (e^{2i\delta_0} - 1) = -\frac{1}{\sqrt{2\pi i p}} \frac{2 \tan \delta_0}{\tan \delta_0 + i} = \sqrt{\frac{2\pi i}{p}} \frac{1}{\ln\frac{p^2 d^2 e^{2\gamma}}{4B} - i\pi} \quad (108)$$

which is identical with (93) if we put

$$d = \sqrt{4e^{-2\gamma}} \quad \text{and} \quad \Lambda = \frac{d}{a}. \quad (109)$$

Thus we see how the cut-off in momentum space corresponds to the shrinking of the well in coordinate space.

4.5 Scale anomaly

To conclude this section, let us briefly investigate the preceding calculation from the point of view of the scale symmetry. In fact, this term is a little misleading, since, as a dynamical system, our model is *not* invariant with respect to the transformation

$r \rightarrow r' = \xi r$. Indeed, the equation of motion²³ for a classical system of a particle under the influence of a potential, after such a transformation (the $1/2$ is just m in our units) reads

$$\frac{\xi}{2} \frac{d^2 \mathbf{x}}{dt^2} = \frac{1}{\xi} \frac{dV(\xi \mathbf{x})}{d\mathbf{x}} \quad (110)$$

and is actually invariant when V is *plus-2-homogeneous*, which is the case of the harmonic oscillator (which is very well-known to be devoid of intricacies and its coupling constant is not dimensionless)! For a *minus-2-homogeneous* potential, the RHS of (110) contains a factor ξ^{-3} that can be obtained on the LHS by the simultaneous change of $r \rightarrow r' = \xi r$ and $t \rightarrow t' = \xi^2 t$, which is usually called the scale transformation²⁴ – it is not surprising after all, because in the units where m and \hbar are held fixed, the dimension of time is *length*².

The quantum-mechanical generator of scale transformation can be guessed from the transformation of the wave function for $\xi = 1 + \epsilon$ with ϵ small

$$\psi((1 + \epsilon) \mathbf{x}, (1 + 2\epsilon) t) = e^{-i2\epsilon t H} \psi((1 + \epsilon) \mathbf{x}, t) \approx$$

$$\approx (1 - i2\epsilon t H) (\psi(\mathbf{x}, t) + \epsilon \mathbf{x} \nabla \psi(\mathbf{x}, t)) \approx (1 - i2\epsilon t H + i\epsilon \mathbf{x} \mathbf{p}) \psi(\mathbf{x}, t) \quad (111)$$

The operator²⁵ $2tH - \mathbf{x} \mathbf{p}$ is not even formally self-adjoint (because $(\mathbf{x} \mathbf{p})^+ = \mathbf{p} \mathbf{x} = \mathbf{x} \mathbf{p} - i$) but the operator

$$D = tH - \frac{1}{4} (\mathbf{x} \mathbf{p} + \mathbf{p} \mathbf{x}) \quad (112)$$

is (we rescaled it according to the usual convention). Moreover (assuming, as always, $H = p^2 + V(\mathbf{x})$) the time derivative

$$\begin{aligned} \frac{dD}{dt} &= \frac{\partial D}{\partial t} + i[H, D] = H - \frac{i}{4} [H, \mathbf{x} \mathbf{p} + \mathbf{p} \mathbf{x}] = H - \frac{i}{4} (\{[H, \mathbf{x}], \mathbf{p}\} + \{[H, \mathbf{p}], \mathbf{x}\}) = \\ &= H - \frac{i}{4} (-4ip^2 + i2\mathbf{x} \nabla V(\mathbf{x})) = V + \frac{1}{2} \mathbf{x} \nabla V(\mathbf{x}) \end{aligned} \quad (113)$$

²³A reader may note that we avoid using the variational principle in this chapter. The reason is that invariance properties of the action become particularly unclear under the transformations that scale time (what will be the case here).

²⁴Even though there was no time in (74), there is a clear correspondence of those two transformations taking into account either that the quantum time evolution only depends on Et , or simply that for the free particle, $E' = (dr'/dt')^2 = \xi^{-2} E$ (the only thing that makes the comparison odd is that in (74) we scaled the wave function *not* the parameters of the Hamiltonian, thus the inverse power of ξ).

²⁵Here we are slightly punished for not discerning between c-numbers and operators in the notation. The next-to-last expression in (111) is valid in the coordinate representation. Turning the multiplication by the coordinate and the derivation into appropriate operators, we get an operator expression independent of a representation. Note that (in the usual formulation of quantum mechanics that we use) there is no “time operator” and t is an external c-number parameter, which is thus independent of any representation chosen on the Hilbert space.

vanishes indeed just if the potential is minus-2-homogeneous, what further justifies D as the scale operator ([26]).

For a square well from the previous chapter, we get

$$\frac{dD}{dt} = V + \frac{\lambda}{2\pi a} \delta(r - a). \quad (114)$$

To avoid mistakes, let us integrate over a C_0^∞ function (for simplicity, we take one that is rotationally invariant) and then let $a \rightarrow 0$:

$$\begin{aligned} \int \frac{dD}{dt} f(\mathbf{x}) d\mathbf{x} &= 2\pi \int \frac{dD}{dt} f(r) r dr = -\frac{2\lambda}{a^2} \int_0^a f(r) r dr + \frac{\lambda}{a} \int_0^\infty f(r) \delta(r - a) r dr \rightarrow \\ &\rightarrow \lambda f(0) \left(-\frac{2}{a^2} \left[\frac{r^2}{2} \right]_0^a + 1 \right) = 0 \end{aligned} \quad (115)$$

by continuity of f . It is pleasing that the formal scale invariance of the δ -function is preserved in its representation as the limit of the shrinking square wells. On the other hand, it does not seem that letting $\lambda \rightarrow 0$ could ever break this invariance, what seems to be in straight contradiction with the scattering amplitude we calculated!

Actually, we are here at a rather slippery slope, because of the formal manipulations involved, but, in the spirit of this section, we can be bold for a moment. In the expression $\frac{2\lambda(a)}{a^2} \int_0^a f(r) r dr$, λ is not a function of r and thus can be put under the integral sign. As $a \rightarrow 0$ it does not matter if we write $\lambda(a)$ or $\lambda(r)$ in the integral, so the limiting “distribution” is $-\frac{2\pi}{\ln r + c'} \delta(\mathbf{x})$, where $c' = 2\pi c + \ln d$ to avoid cluttering the notation. The quotes remind us that this expression does not make much sense as a functional on C_0^∞ – however, there is a natural way to think of it.

For the point interaction, the scattering solution acquires its asymptotic form in the whole space, as we have seen in practice during the calculation of the phase shift. For the phase shift to be nonzero, the solution must contain the Neumann function, which diverges at the origin and thus the action of the ordinary δ on it is not defined. The fact that such a function can hardly be in any generalised sense the eigenvalue of the Hamiltonian is what forces the scattering to be trivial in the first place. On the other hand, integrating $-\frac{2\pi}{\ln r + c'} \delta(\mathbf{x})$ with the Neumann function, we get a finite result, because the $\ln r$ term in the denominator compensates for its logarithmic singularity.

Still not having fully clarified its meaning, we can investigate the behaviour of this “potential” under the scale symmetry. Firrst, it is instructive to see how the homogeneity of the ordinary δ appears in spherical coordinates, namely that²⁶

$$r \frac{\partial}{\partial r} \delta(\mathbf{x}) = r \frac{\partial}{\partial r} \frac{\delta(r)}{\pi r} = \frac{1}{\pi} \left(-\frac{1}{r} \delta(r) - \delta'(r) \right) = -\frac{2}{\pi r} \delta(r) = -2\delta(\mathbf{x}). \quad (116)$$

²⁶The coordinate changes on δ are a little tricky, particularly because in the spherical coordinates, its support resides at the boundary of the integration interval for r , which is why there are two different notions of a “radial δ ” in use, which differ by a factor of 2 ([30]). Nevertheless, when one is in doubt, it is always possible to check the results after integration with a smooth function.

By the same method we get

$$\begin{aligned}
& r \frac{\partial}{\partial r} \frac{2\pi}{\ln r + c'} \delta(\mathbf{x}) = 2r \frac{\partial}{\partial r} \frac{\delta(r)}{r(\ln r + c')} = \\
& = -2 \left(\frac{\delta(r)}{r(\ln r + c')^2} + \frac{\delta(r)}{r(\ln r + c')} + \frac{\delta'(r)}{\ln r + c'} \right) = -\frac{4\pi}{\ln r + c'} \delta(\mathbf{x}) - \frac{2\pi\delta(\mathbf{x})}{(\ln r + c')^2}
\end{aligned} \tag{117}$$

and thus, formally, $dD/dt \neq 0$. “Formally” means that the extra term in the last expression is even more difficult to make sense of than $-\frac{2\pi}{\ln r + c'} \delta(\mathbf{x})$ was – nevertheless, this calculation may be considered as a hint that the conservation of D is somehow broken.

Actually, it can be easily shown, that the conservation of D is basically impossible. Consider an eigenstate ψ of a nonzero energy E . Then, using the simple time-dependence of an energy eigenstate,

$$\frac{d}{dt} \langle \psi(t) | D | \psi(t) \rangle = \frac{d}{dt} \left(tE \|\psi(0)\|^2 - \frac{1}{4} \langle \psi(0) | \mathbf{x}\mathbf{p} + \mathbf{p}\mathbf{x} | \psi(0) \rangle \right) = E \|\psi(0)\|^2 \neq 0. \tag{118}$$

and thus it cannot be that $dD/dt = 0$ identically. On this formal level, it is questionable what could we make of this manipulation for a “non-normalisable” state from the continuous spectrum, but we have seen that the renormalised two-dimensional δ always possesses a bound state (which is a blatant violation of the scale symmetry anyway).

From the practical point of view, Eq. (118) can be understood as a warning sign that quantum mechanics and scale symmetry do not fit together too well – or that our approach to the problem was too careless of some mathematical subtleties. In the next section, we will address the later issue, incidentally shedding some light also on the former one.

5 Mathematician’s approach

5.1 Motivation

In the previous section, we did not care much about the formal details of our calculations: for example, when we wrote down (76) we did not specify on which functions it shall act, nor have we shown that a reasonable choice exists. In fact, many interesting and correct conclusions about quantum-mechanical systems can be drawn without much care about these matters. Nevertheless, in the case of singular potentials, methods of functional analysis can be extremely helpful. At the end we will see that *all* the dynamics of a “ δ -function” potential can be found just by carefully examining the set of functions on which a certain simple operator is prescribed to act.

The key observation is particularly simple: the δ -interaction is so localised that the particle is free everywhere except the origin. As the origin is the boundary of the rest of the space, we could try to describe the system by the free Hamiltonian (that is, in our units, just minus the Laplacian) supplied with an appropriate

boundary condition here. To do this, we will first specify, what we really mean by the “free Hamiltonian”, because the action of a differential operator on the very general functions from $L^2(\mathbb{R}^n)$ is not obvious – the implementation of the boundary condition will then turn out quite naturally. The resulting operator will be not self-adjoint and thus not acceptable as a physical Hamiltonian (see sec. A.4). It turns out that the remedy is to soften the boundary condition, a process in which a free dimensionful parameter inevitably appears, allowing the same kind of dimensional transmutation in two dimensions that we have already seen.

This section is devoted to making this idea precise. This task could have been accomplished in one sentence: all the relevant results can be found in the comprehensive monograph [28]. On the other hand, their rigorous treatment requires that the reader has a considerable mathematical background to understand even the basic points and many references at hand to explore the details. Being aware of this we try to present here a self-contained account of the topic, from what we believe to be a slightly more straightforward point of view. The necessary elements of functional analysis, centred around von Neumann’s theory of self-adjoint extensions, are presented in the Appendix.

5.2 The free Hamiltonian and a boundary condition

To investigate the free Hamiltonian from the formal point of view, we should find a convenient space of functions smooth enough to be directly differentiated by the Laplacian. In fact, this choice is to some extent a matter of taste, but a very convenient (and conservative in the sense that it is rather small) choice is the space C_0^∞ of infinitely differentiable functions with compact support, which is indeed dense in L^2 (in any dimension). The natural question whether the Laplacian on this domain is a closed operator becomes easier when phrased differently: is C_0^∞ closed in the (Δ -graph) norm $\|\phi\|_\Delta^2 = \|\phi\|^2 + \|\Delta\phi\|^2$? The negative answer is well known ([23], Ch. XIV.) and the closure of C_0^∞ in this norm (or in one of the various equivalent norms that are commonly used) even has a name – it is the 2nd **Sobolev space**, $H^2(\mathbb{R}^d)$ ²⁷, which turns out to be exactly the space of functions from $L^2(\mathbb{R}^d)$ for that $\Delta\phi \in L^2$ in the sense of distributions, e.g. there is a vector $\psi \in L^2$ such that $(\psi, \eta) = (\phi, \Delta\eta)$ for all $\eta \in C_0^\infty$. This is in turn equivalent to the characterisation (160) of the domain of the adjoint operator to the Laplacian on C_0^∞ and so (cf. footnote 41 in sec. A.2) the Laplacian on H^2 (we will call it H_0) is self-adjoint.

In order to understand better, what kind of functions is the H^2 made of, we employ the Fourier transform²⁸ ([20], Ch. IX.6–7). For a C_0^∞ function, the Laplacian

²⁷The terminology and notation concerning the Sobolev spaces is slightly variable. In general, the Sobolev space $W^{s,p}(\Omega)$ is a space of L^p -integrable functions on a set Ω , with L^p -integrable derivatives up to the s -th order, endowed with the norm that is the sum of L^p -norms of these derivatives (where, as already mentioned, equivalent norms exist), but only the spaces $H^s = W^{s,2}$ are Hilbert spaces and thus of particular interest in the operator theory – note that e.g. Reed and Simon ([20]) use simply the notation W_s for them.

²⁸The Fourier transform on L^2 , sometimes called the Fourier-Plancherel operator, is a unitary map (denoted \mathcal{F}) on the whole L^2 which is the unique continuous extension of the Fourier transform on L^1 . Whereas the latter is given simply in the usual integral form, the integral is not necessarily meaningful for an L^2 function and a limiting procedure is used instead.

is via \mathcal{F} unitarily equivalent with the operator of multiplication by p^2 , because we have $-\widehat{\Delta\phi}(\mathbf{x}) = p^2\hat{\phi}(\mathbf{p})$ if $\phi \in C_0^\infty$. (From now on we will omit the $\hat{\cdot}$ if the confusion is unlikely.) It is easily seen that the operator of multiplication by p^2 is self-adjoint on the set $D(p^2)$ of functions $\phi \in L^2$ such that $p^2\phi \in L^2$ – the vector η , such that $(\eta, \psi) = \int \overline{\phi(\mathbf{p})}p^2\psi(\mathbf{p})$ is clearly $p^2\phi$ and by definition, it must be in L^2 for ϕ to be in the domain of the adjoint, which thus equals to $D(p^2)$. It can be proved directly that this set closes the operator p^2 (see [32], Ch. 5.5) but because, as we already know, the closure of the Laplacian on C_0^∞ is self-adjoint, its self-adjoint extension is unique and thus $D(p^2)$ is the Fourier transform of H^2 (we will simply identify such pairs of spaces, if the confusion is unlikely).

The characterisation of H^2 by the Fourier transform will help us to understand the following important result, the **Sobolev’s lemma**: If $l < n - d/2$, then any element of $H^n(\mathbb{R}^d)$ is a C^l function. (In our case, $n = 2$ and thus for $d = 1$, we deal with once continuously differentiable function, for $d = 2, 3$ the functions are continuous and for $d \geq 4$ no “good” behaviour is guaranteed.) The reasoning is roughly that if $\phi \in H^2$, then, neglecting the angular dependence, $\int p^{2n} |\phi(\mathbf{p})|^2 p^{d-1} dp < \infty$, thus the integrand decays at infinity at least as $p^{-1-\epsilon}$, and therefore at least $\phi \sim p^{-d/2-n-\epsilon}$, and so $p^l\phi \sim p^{l-d/2-n-\epsilon}$. The Riemann-Lebesgue lemma guarantees that the Fourier transform of a L^1 function is continuous and $p^l\phi$ is integrable in d dimensions, if it decays at least as $p^{-d-\epsilon}$, that is if $l - d/2 - n < -d \iff l < n - d/2$. But $p^l\phi$ is the transform of the l -th derivative²⁹ and if it is continuous, then $\phi \in C^l$.

The utility of the Sobolev’s lemma is clear. For a general L^2 function it is not sensible to speak about its “value at a point” and thus the possibility of imposing a condition at origin is doubtful, whereas we can easily do so for a continuous function³⁰: to implement the point interaction in $d = 2, 3$ we will simply restrict the domain of the H_0 to the H^2 functions that vanish at origin, as this is where the formally infinitely strong interaction takes place (in 1D we could even restrict the derivative to vanish, but we will not deal with this case in this work); we will call the $-\Delta$ with this domain just H . Because the functions from H^2 are not guaranteed to be continuous for $d \geq 4$, it is not surprising that no such point interaction can be constructed – in fact, it can be proved ([20], Ch. X.I) that the closure of the Laplacian on the set of C_0^∞ functions that vanish in a open neighbourhood of origin (which is indeed a very generous definition of vanishing) is self-adjoint for $d \geq 4$ and thus equal to the free Hamiltonian. This result is reminiscent of the problems we encountered at the end of Sec. 4.2 and will be illustrated nicely when we identify the deficiency subspaces in lower dimensions.

5.3 Deficiency subspaces of H

First, we need to find the domain of H^* . Obviously, $H_0 \supset H$ which implies that $H_0 \subset H^*$, i.e. $D(H^*)$ contains any H^2 function. Consider now the set D of functions $\phi \in L^2$ for those there is a $c \in \mathbb{C}$ such that $p^2\phi - c \in L^2$. Certainly, $H^2 \subset D$ (for

²⁹This rather sloppy point can be easily refined using induction, starting from $l = 0$.

³⁰Using a similar argument as in the explanation of the Sobolev’s lemma, it can be shown that for $d = 2, 3$ the H^2 functions are even slightly smoother, than just continuous, that is they are *Hölder continuous*, with the Hölder exponent $\alpha < 1$ for $d = 2$ and $\alpha < 1/2$ for $d = 3$ ([32]).

$c = 0$) and for a given ϕ , the constant c is uniquely determined, because a constant cannot be the difference of two L^2 functions (they form a vector space, after all). For $\phi \in D$ and any $\psi \in D(H)$, we have

$$(T\psi, \phi) = \int \overline{\psi(\mathbf{p})} p^2 \phi(\mathbf{p}) d^d \mathbf{p} = c \int \overline{\psi(\mathbf{p})} d^d \mathbf{p} + (\psi, p^2 \phi - c). \quad (119)$$

But the integral is equal to $\psi(\mathbf{x} = \mathbf{0}) = 0$ and because $p^2 \phi - c \in L^2$, we have $\phi \in D(H^*)$ and $H^* \phi = p^2 \phi - c$ (as $c = 0$ for H^2 functions, this result is consistent with the fact that H_0 is an extension of H).

What are the functions in D that are not in H^2 , if there are any? Without any loss of generality, we may assume that $c = 1$ (we can obtain any c just by rescaling the function). Then, $\phi = 1/p^2 + \eta/p^2$, where $\eta \in L^2$ and thus it decays at least as $p^{d/2}$ in infinity. But this means that for a sufficiently large p , the first term is much larger and thus $\phi \sim 1/p^2$ which cannot be in L^2 for $d > 3$. Thus, in this case $D = H^2$. On the contrary, for $d = 2, 3$, the only problem for ϕ to be in L^2 is the potential singularity at zero, which is easily cured by a suitable choice of η . Nevertheless, the slow $1/p^2$ decay forbids it to be in H^2 as expected, thus we have potentially a great wealth of functions in $D - H^2$. As an additional remark we note that the Laplacian (considered in the sense of distributions) on a function, whose Fourier transform is $1/p^2$, is proportional to $\delta(x)$ which is exactly the kind of singular behaviour we can expect for H^* in the coordinate space, because this term gives zero in the scalar product with a function that vanishes at origin. The action of H^* is then “do the minus Laplacian boldly and drop any δ if it occurs”.

Insofar without any claim that D is the whole $D(H^*)$, we can still ask whether there are some portions of the deficiency subspaces of H in D . It is well-known that the Laplacian has no eigenvalues on H^2 . But for the functions in $D - H^2$, the eigenfunction equation is much richer, namely

$$p^2 \phi + c = a \phi \quad (120)$$

and is readily solved by

$$\phi = \frac{c}{p^2 - a}. \quad (121)$$

It remains to check whether this is in D , but it is elementary, because

$$p^2 \phi = \frac{cp^2}{p^2 - a} = c + \frac{ac}{p^2 - a}. \quad (122)$$

The constant term is exactly c , as required, and the second term is in L^2 (for $d = 2, 3$) if and only if $p^2 - a$ has no real roots, that is for $a \in \mathbb{C} - [0; \infty)$ – particularly there is one solution in \mathcal{K}_+ and one in \mathcal{K}_- . In fact, multiplies of the vectors $\psi(\mathbf{p})_{\pm} = \frac{1}{p^2 \mp i}$ do indeed exhaust the deficiency subspaces of H ([14]) and they can be easily written in the coordinate space, noting that they are the Fourier transform of the free Green functions $G_k(0, x)$ for the “unphysical” momentum $k^2 = \pm i$ (which is just the integral kernel of the resolvent of H_0 ; we keep using k^2 as the independent variable to make the comparison with well-known formulae easier, whereas k is always taken with $\text{Im } k > 0$ where the ambiguity arises). Explicitly, they are

$$\psi(\mathbf{x})_{\pm} = \frac{i\pi}{2} H_0^{(1)} \left(\frac{\sqrt{2}}{2} (i \mp 1) r \right) \quad \text{for } d = 2$$

$$\psi(\mathbf{x})_{\pm} = \frac{\sqrt{2\pi}}{2} \frac{e^{-\frac{\sqrt{2}}{2}(1 \mp i)r}}{r} \quad \text{for } d = 3 \quad (123)$$

(cf. [28], note that our normalisation different – it is chosen so that the expressions are the Fourier transform of $\psi(\mathbf{p})_{\pm}$, as they are defined above, what is useful in further calculations). Here $H_0^{(1)}$ is the Hankel function of the first kind and order zero³¹, in a slight notational collision.

5.4 Self-adjoint extensions of H and bound states

Since we explicitly know its deficiency subspaces, it is easy to list all possible self-adjoint extensions of H , using the theorem of sec. A.3. Both deficiency subspaces are one-dimensional: given a chosen vector from each, both of the same norm, the only unitary maps are those that map one of them on the other modulo a phase factor. Thus all the self-adjoint extensions of H are members of the 1-parametric family of operators

$$H_{\theta} : D(H_{\theta}) = \{ \phi + c\psi_+ + ce^{i\theta}\psi_-; \phi \in D(H), c \in \mathbb{C} \}, \theta \in \left[0; \frac{\pi}{2} \right)$$

$$H_{\theta}(\phi + c\psi_+ + ce^{i\theta}\psi_-) = H\phi + ic\psi_+ - ice^{i\theta}\psi_-; \quad (124)$$

First we note, that for $\theta = \pi$, we have (in Fourier image) $\psi_+ - \psi_- = \frac{2i}{p^4+1}$, which is in H^2 . Moreover,

$$H_{\pi}(\psi_+ - \psi_-) = i\psi_+ + i\psi_- = \frac{2ip^2}{p^4+1} = p^2(\psi_+ - \psi_-) = H_0(\psi_+ - \psi_-) \quad (125)$$

and as H and H_0 agree on $D(H)$, we see that, as promised, one of the self-adjoint extensions of H , namely H_{π} is identical with the free Hamiltonian H_0 . Yet, there is (for $d = 2, 3$) a plethora of other ones that are not.

We have already seen that H^* has many eigenvalues, so a natural question is whether some of them are inherited by some of the H_{π} . It would be the case, if some of the eigenvectors $\frac{c}{p^2-a}$ were in $D(H_{\theta})$, that is if

$$\phi_H(\mathbf{p}) = \frac{c}{p^2-a} - \left(\frac{1}{p^2-i} + \frac{e^{i\theta}}{p^2+i} \right) =$$

$$= \frac{(c-1-e^{i\theta})p^4 + (e^{i\theta}(a+i) + a-i)p^2 + c + ai(1-e^{i\theta})}{(p^2-a)(p^4+1)} \in D(H) \quad (126)$$

³¹Note that the Hankel function, which is a linear combination of a Bessel and a Neumann function, carries the $\ln r$ singularity of the Neumann function at origin, which is weaker than the pole found in three dimensions. Additional properties of these functions can be found in the book [17], whose notation we (along with almost all references we cite) follow.

for some $a \in (-\infty, 0)$ (the H_θ were constructed to be self-adjoint and thus their spectrum is real, but non-negative real eigenvalues were already excluded in (122)) and some θ . Because $D(H) \subset H^2$, the p^4 term in the denominator must vanish, what is easily accomplished because c is so far arbitrary. The requirement that $\int \phi_H(\mathbf{p}) d^d \mathbf{p} = 0$ is somewhat harder to tackle and must be considered in both relevant dimensions separately.

For $d = 3$ we get (already with $c = 1 + e^{i\theta}$ and after some simplifications),

$$\begin{aligned} \frac{\sqrt{-a}}{\pi^2} \int \phi_H(\mathbf{p}) d^d \mathbf{p} &= 2a(\cos \theta + 1) + \sqrt{-2a}(\cos \theta - \sin \theta + 1) + \\ &+ i(2a \sin \theta + \sqrt{-2a}(\sin \theta + \cos \theta - 1)). \end{aligned} \quad (127)$$

The condition for this expression to be zero may be viewed as a system of two homogeneous equations for two unknown variables, $2a$ and $\sqrt{-2a}$. The determinant

$$\begin{aligned} (\cos \theta + 1)(\sin \theta + \cos \theta - 1) - \sin \theta(\cos \theta - \sin \theta + 1) &= \\ \cos^2 \theta - 1 + \sin^2 \theta &= 0 \end{aligned} \quad (128)$$

vanishes identically, so the solution always exists and implies that (keeping in mind that a is negative)

$$-2a = \frac{\cos \theta + \sin \theta - 1}{\sin \theta} \sqrt{-2a} \Rightarrow \sqrt{-2a} = \frac{\cos \theta + \sin \theta - 1}{\sin \theta}. \quad (129)$$

The last expression is positive for $\theta \in [0; \frac{\pi}{2}] \cup (\pi; 2\pi)$ and then there is, for every such H_θ exactly one eigenvalue (bound state)

$$a = \frac{\sin \theta - 1}{\cos \theta + 1}, \quad (130)$$

whereas for $\theta \in [\frac{\pi}{2}; \pi]$ there are no bound states.

For $d = 2$, we get

$$-\frac{1}{\pi} \int \phi_H(\mathbf{p}) d^d \mathbf{p} = \ln(-a)(1 + \cos \theta) + \frac{\pi}{2} \sin \theta + i \left(\ln(-a) \sin \theta + \frac{\pi}{2} (1 - \cos \theta) \right). \quad (131)$$

Again, demanding that real and imaginary parts vanish separately provides us with two equations

$$\ln(-a) = -\frac{\pi \sin \theta}{2(1 + \cos \theta)} \quad \text{and} \quad \ln(-a) = -\frac{\pi(1 - \cos \theta)}{2 \sin \theta} \quad (132)$$

and a simple calculation shows that they are identical (if their singularities are properly treated). Contrary to the three-dimensional case, Eq. (132) has always a solution for a , except for the free case $\theta = \pi$ and so there is exactly one bound state for every H_θ with the eigenvalue

$$a = -\exp\left(-\frac{\pi \sin \theta}{2(1 + \cos \theta)}\right). \quad (133)$$

5.5 Scattering

The mathematically rigorous scattering theory is a huge and very involved field into which we will not plunge deeper than it is necessary to derive the correct formulae for the scattering amplitudes. Nevertheless, it shall be clear that we still cannot safely proceed in the usual physical manner – when the very difference between the H_θ 's and the free Hamiltonian is in the domain, what would the “non-normalisable” scattering solutions (that are not even in the Hilbert space) be for them? The trouble with the H_θ 's is that they cannot be written on the form $-\Delta + V$ and thus the whole machinery developed for potential scattering (i.e. the Lippmann-Schwinger equation) is not directly applicable. As it often happens, a trouble can be easily made an advantage: it will turn out quickly that we do not need such machinery at all. To this end, we must first understand, how are the “non-normalisable eigenfunctions” properly handled (see [21], Ch. XI.6).

A self-adjoint operator A with a purely discrete spectrum can always be diagonalised (in a generalised sense) by its *eigenfunction expansion*. That is if we choose a set of eigenfunctions ϕ_n such that $A\phi_n = a_n\phi_n$ and for an arbitrary $\psi \in L^2(\mathbb{R}^n)$ we put

$$\psi_n = \int \overline{\phi_n(\mathbf{x})} \psi(\mathbf{x}) d^d \mathbf{x} \quad (134)$$

then we have $(A\psi)_n = a_n\psi_n$. If we now replace the discrete index with a continuous one, we can obtain a similar “diagonalised” expression for the free Hamiltonian using the Fourier transformation (where the role of the ϕ_n is played by the functions $e^{-i\mathbf{k}\mathbf{x}}$, labelled by different values of \mathbf{k}), because clearly

$$\widehat{(H_0\psi)}(\mathbf{k}) = k^2 \widehat{\psi}(k). \quad (135)$$

The natural question is then, whether a similar transformation can be found, for which (132) holds with H_θ instead of H_0 and the corresponding “eigenfunctions” can be interpreted as the scattering solutions, similar to the solutions of the Lippmann-Schwinger equation. That is, if there are some functions $g(\mathbf{x}, \mathbf{k})$ such that

$$\int g(\mathbf{x}, \mathbf{k}) (H_\theta\psi)(\mathbf{x}) d^d \mathbf{x} = k^2 \int g(\mathbf{x}, \mathbf{k}) \psi(\mathbf{x}) d^d \mathbf{x}. \quad (136)$$

for every $\psi \in D(H_\theta)$.

We will first examine the case $d = 3$ – the advantage of this choice will be evident. The solution of (136) does not seem easy unless we make some assumptions on the form of $g(\mathbf{x}, \mathbf{k})$ (and look if they can or cannot be satisfied). First of all we want $g(\mathbf{x}, \mathbf{k})$ to have the asymptotics of a scattering solution, that is (barring an inessential overall factor)

$$g(\mathbf{x}, \mathbf{k}) \rightarrow e^{i\mathbf{k}\mathbf{x}} + f(k', k) \frac{e^{ikr}}{r} \text{ as } r \rightarrow \infty. \quad (137)$$

From the analysis of the previous section, we expect only the isotropic s-wave to be affected, that is $f(k', k) = f(k)$. Finally, because the interaction is concentrated at

the origin, one may be led to assume that the asymptotic form is satisfied throughout the whole space (except for the origin, where it diverges)³². Eq. (136) then becomes

$$\int \left(e^{i\mathbf{k}\mathbf{x}} + f(k) \frac{e^{ikr}}{r} \right) (H_\theta \psi)(\mathbf{x}) d\mathbf{x} = k^2 \int \left(e^{i\mathbf{k}\mathbf{x}} + f(k) \frac{e^{ikr}}{r} \right) \psi(\mathbf{x}) d\mathbf{x}. \quad (138)$$

If we write $\psi = \phi + c(\psi_+ + e^{i\theta}\psi_-)$ with $\phi \in D(H)$ then we can easily infer (e.g. from (124)) that

$$\int e^{i\mathbf{k}\mathbf{x}} (H_\theta \psi)(\mathbf{x}) d\mathbf{x} = k^2 \int e^{i\mathbf{k}\mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} - (2\pi)^{\frac{3}{2}} c (1 + e^{i\theta}) \quad (139)$$

(the factor $(2\pi)^{\frac{3}{2}}$ accounts for the unnormalised Fourier transform used) and subtracting this from the previous equation, we get

$$f(k) \int \frac{e^{ikr}}{r} (H_\theta \psi)(\mathbf{x}) d\mathbf{x} = k^2 f(k) \int \frac{e^{ikr}}{r} \psi(\mathbf{x}) d\mathbf{x} + (2\pi)^{\frac{3}{2}} c (1 + e^{i\theta}). \quad (140)$$

Because $H_\theta \psi = -\Delta\phi + ic(\psi_+ - e^{i\theta}\psi_-)$, the integral on the LHS can be split into three. In the first term, we integrate per partes: the Laplacian of e^{ikr}/r is $-k^2 e^{ikr}/r$ plus a δ -function term³³ that vanishes when integrated with $\phi \in D(H)$, the remaining two terms are dealt with by introducing a notation $I_\pm = \int \frac{e^{ikr}}{r} \psi_\pm d\mathbf{x}$. If we similarly split the integral in the RHS, we find that the first term cancels against the $k^2 \frac{e^{ikr}}{r} \psi$ term we have just found on the LHS and the remaining two are again conveniently expressed using I_\pm – altogether we get:

$$f(k) (icI_+ - e^{i\theta}icI_-) = k^2 f(k) (cI_+ + e^{i\theta}cI_-) + (2\pi)^{\frac{3}{2}} c (1 + e^{i\theta}) \quad (141)$$

We observe that c readily drops out and we can solve for $f(k)$:

$$f(k) = (2\pi)^{\frac{3}{2}} \frac{1 + e^{i\theta}}{I_+ (i - k^2) - e^{i\theta} I_- (i + k^2)} \quad (142)$$

It only remains to evaluate the easy integrals

$$I_\pm = \frac{\sqrt{2\pi}}{2} \int \frac{e^{ikr}}{r} \frac{e^{i\sqrt{\pm i}r}}{r} d\mathbf{x} = (2\pi)^{\frac{3}{2}} \int_0^\infty e^{i(\sqrt{\pm i}+k)r} dr = \frac{(2\pi)^{\frac{3}{2}} i}{(\sqrt{\pm i} + k)} \quad (143)$$

and use the result

³²A reader that does not find this argument particularly compelling may consider Eq. (78) in the previous section as an alternative source of inspiration – or think of this as just an ansatz that has yet to prove its utility.

³³In other words, $\frac{e^{ikr}}{r}$ is the Green function of the differential operator $\Delta + k^2$.

$$f(k) = -\frac{i(1+e^{i\theta})}{\frac{i-k^2}{\sqrt{i+k}} - \frac{k^2-(-i)}{\sqrt{-i+k}} e^{i\theta}} = \frac{1}{i\frac{\sqrt{i-e^{i\theta}}\sqrt{-i}}{1+e^{i\theta}} - ik} = \left(\frac{\sin\theta - \cos\theta - 1}{\sqrt{2}(1+\cos\theta)} - ik\right)^{-1}. \quad (144)$$

Note that the first term of the last expression diverges as $\theta \rightarrow \pi/2$ (the amplitude vanishes in the free case) and its square equals minus the eigenvalue of H_θ that we found for $\theta \in [0; \frac{\pi}{2}) \cup (\pi; 2\pi)$ (see (130); the expression for the square holds for all θ , but outside the given interval, there is no eigenvalue) – reproducing the result (96) of the previous section (the apparently different sign is correct – the term itself is negative for $\theta \in [0; \frac{\pi}{2}) \cup (\pi; 2\pi)$, so the negative root must be taken).

As for the case $d = 2$ we can adopt a similar, yet slightly modified approach. Using the argument preceding Eq. (138) and the general asymptotics of the scattering function (80) we may be led to try the ansatz

$$g(\mathbf{x}, \mathbf{k}) = e^{i\mathbf{k}\mathbf{x}} + f(k) \frac{e^{ikr}}{\sqrt{r}}. \quad (145)$$

But, contrary to the three-dimensional case, the chosen spherical wave is *not* the Green function of $\Delta + k^2$ in two dimensions and thus the trick that we used to proceed from (140) to (141) will fail, the terms involving ϕ will not drop from the equation and it will not be possible to solve for $f(k)$. In fact, we were rather lucky when we picked the form (138) for the scattering function.

Having learnt where the ansatz fails, it is easy to improve it using the true Green function of $\Delta + k^2$ in two dimensions, that is, assuming

$$g(\mathbf{x}, \mathbf{k}) = e^{i\mathbf{k}\mathbf{x}} + \frac{i\sqrt{-2\pi ik}}{2} f(k) H_0^{(1)}(kr) \quad (146)$$

where the factors were chosen according to the asymptotic of the Hankel function, so that $f(k)$ indeed is the scattering amplitude³⁴. The steps that led us to Eq. (142) can now be almost exactly repeated, that is

$$f(k) = 2\pi \frac{1+e^{i\theta}}{I_+(i-k^2) - e^{i\theta} I_-(i+k^2)} \quad (148)$$

with the difference only in the numerical factor that depends on the dimension and of course in the definition of the auxiliary integrals,

$$I_\pm = \frac{i\sqrt{-2\pi ik}}{2} \int H_0^{(1)}(kr) \psi_\pm d\mathbf{x}. \quad (149)$$

³⁴The asymptotics is

$$H_0^{(1)}(kr) \rightarrow \frac{\sqrt{2}}{i\sqrt{\pi kr}} e^{i(kr+\pi/4)} \quad (147)$$

(cf. Eq. 3.18 in [26]). The conventions for the scattering amplitude in two dimension differ by phase among the authors, depending on the inclusion of the phase factors from the asymptotics of the Hankel function in the amplitude. Throughout this work we use the convention that guarantees (145).

An integral of a product of two Hankel functions is not always easily evaluated, but in this case, we can easily proceed, resorting back to the Fourier representation of the functions (from which we have obtained them in the first place), that is

$$\begin{aligned}
I_{\pm} &= \frac{\sqrt{-2\pi i k}}{4\pi^3} \int \left(\int \frac{e^{i\mathbf{p}\mathbf{x}}}{p^2 - k^2 - i\epsilon} d\mathbf{p} \right) \left(\int \frac{e^{i\mathbf{q}\mathbf{x}}}{q^2 \mp i} d\mathbf{q} \right) d\mathbf{x} = \\
&= \frac{\sqrt{-2\pi i k}}{4\pi^3} \iiint \frac{e^{i(\mathbf{p}-\mathbf{q})\mathbf{x}}}{(p^2 - k^2 - i\epsilon)(q^2 \mp i)} d\mathbf{p} d\mathbf{q} d\mathbf{x} = \\
&= \sqrt{\frac{2k}{i\pi}} \iint \frac{\delta(\mathbf{p} - \mathbf{q})}{(p^2 - k^2 - i\epsilon)(q^2 \mp i)} d\mathbf{p} d\mathbf{q} = \sqrt{\frac{2k}{i\pi}} \int \frac{1}{(p^2 - k^2 - i\epsilon)(p^2 \mp i)} d\mathbf{p} = \\
&= -\sqrt{\frac{\pi k}{2i}} \frac{2 \ln(-k - i\epsilon) + 2 \ln k \pm i\pi}{k^2 \mp i} = 2\sqrt{\frac{\pi k}{2i}} \frac{\ln k^2 - \frac{i\pi}{2}(2 \mp 1)}{-k^2 \pm i}. \quad (150)
\end{aligned}$$

The longer the calculation was, the easier is now to use the result:

$$f(k) = \sqrt{\frac{2\pi i}{k}} \frac{1 + e^{i\theta}}{\ln k^2 - \frac{i\pi}{2} + e^{i\theta} \left(\ln k^2 - \frac{3i\pi}{2} \right)} = \sqrt{\frac{2\pi i}{k}} \frac{1}{\ln k^2 - \frac{i\pi}{2} \frac{3+e^{i\theta}}{1+e^{i\theta}}}. \quad (151)$$

The second term in the denominator is easily manipulated into

$$\frac{i\pi}{2} \frac{3 + e^{i\theta}}{1 + e^{i\theta}} = i\pi - \frac{\pi \sin \theta}{2(1 + \cos \theta)} = i\pi + \ln B \quad (152)$$

where B is minus the eigenvalue of H_{θ} and thus finally

$$f(k) = \sqrt{\frac{2\pi i}{k}} \frac{1}{\ln \frac{k^2}{B} - i\pi} \quad (153)$$

again in agreement with the result of the previous section (93).

5.6 Scale anomaly revisited

In the formalism of self-adjoint extensions, the dimensional transmutation looks even more strange than using the regulator. We started with a model with no intrinsic scale and the only parameter we introduced was an ‘‘angle’’ that moreover appeared only as phase of a number with a unit modulus, yet there are dimensional quantities derived from it.

Nevertheless, we can easily see how the scaling argument of Eq. (74) is broken. To this end, we take one of the self-adjoint extensions H_{θ} (from now on, we will work in two dimensions) and rescale one of the functions in its domain, namely we calculate

$$2\pi (\psi_+ + e^{i\theta} \psi_-) (\xi \mathbf{x}) = \int e^{i\xi \mathbf{p}\mathbf{x}} (\psi_+ + e^{i\theta} \psi_-) (\mathbf{p}) d\mathbf{p} =$$

$$= \int e^{i\mathbf{p}\mathbf{x}} \frac{\psi_+ + e^{i\theta}\psi_-}{\xi^2} \left(\frac{\mathbf{p}}{\xi} \right) d\mathbf{p} = \int e^{i\mathbf{p}\mathbf{x}} \left(\frac{1}{p^2 - i\xi^2} + \frac{e^{i\theta}}{p^2 + i\xi^2} \right) d\mathbf{p} \quad (154)$$

and ask whether the rescaled function is also in the domain of H_θ (the reason for this choice is obvious, as $D(H)$ is clearly invariant under rescaling). It would be so, if there was a c such that

$$\phi = \frac{1}{p^2 - i\xi^2} + \frac{e^{i\theta}}{p^2 + i\xi^2} - c \left(\frac{1}{p^2 - i} + \frac{e^{i\theta}}{p^2 + i} \right) \in D(H). \quad (155)$$

This problem is similar to that we have solved in Sec. 5.4 (when we searched for the bound states) and so it can be tackled by the same procedure. Putting all terms on the common denominator, we find that the coefficient of the leading power of p must vanish for $\phi \in H^2$ and it does so only if $c = 1$. But then it is found that

$$\frac{1}{2\pi} \int \phi(\mathbf{p}) d\mathbf{p} = -\ln \xi ((1 + \cos \theta) + i \sin \theta) \quad (156)$$

and it vanishes only for $\xi = 1$ (that is no scaling at all) and $\theta = \pi$ (that is the free case). This is why the occurrence of a single bound state is possible: its scaled wave function is no longer in the domain of H_θ and thus does not form another bound state.

An almost equally interesting question is, whether the scaled function is in the domain of some *other* extension of H , let us call it H_η . The condition is almost similar, that is

$$\phi = \frac{1}{p^2 - i\xi^2} + \frac{e^{i\theta}}{p^2 + i\xi^2} - c \left(\frac{1}{p^2 - i} + \frac{e^{i\eta}}{p^2 + i} \right) \in D(H), \quad (157)$$

but the calculation is much more cumbersome, because of the two different angles. However, after some manipulation, it can be found that

$$\begin{aligned} & \frac{1}{2\pi} \int \phi(\mathbf{p}) d\mathbf{p} = -\ln \xi (1 + \cos \theta) + \\ & + \frac{\pi}{4} \left(\sin \theta - \sin \eta \frac{1 + \cos \theta}{1 + \cos \eta} \right) - i \ln \xi \sin \theta + \frac{i\pi}{4} \left(1 - \cos \theta - \frac{\sin \theta \sin \eta}{1 + \cos \eta} \right). \end{aligned} \quad (158)$$

Assuming for simplicity $\theta \notin \{0, \pi\}$ we can safely solve for vanishing of both real and imaginary parts, getting two equations

$$\begin{aligned} & \frac{\pi}{4} \left(\frac{\sin \eta}{1 + \cos \eta} - \frac{\sin \theta}{1 + \cos \theta} \right) = \ln \xi \\ & \frac{\pi}{4} \left(\frac{\sin \eta}{1 + \cos \eta} - \frac{1 - \cos \theta}{\sin \theta} \right) = \ln \xi \end{aligned} \quad (159)$$

that in fact happen to be identical (after some manipulation with the θ -dependent term). The map $\eta \rightarrow \frac{\sin \eta}{1 + \cos \eta}$ is a one-to-one map between $[0; \pi) \cup (\pi; 2\pi)$ and \mathbb{R} and

thus there is always a (unique) solution. Moreover, as the scaling factor ξ varies through $(0; \infty)$, the image of a chosen $D(H_\theta)$ ranges through the domains of all other extensions of H , except for the free Hamiltonian. That is, the set of all non-free H_θ is still scale invariant; the invariance is broken at the instant we choose *one* of them to become the actual Hamiltonian of the system. We also see that for a non-free H_θ , the scaling does not operate within its domain, thus it is not possible, e.g., to satisfy $[D, H] = iH$ in any sensible way as a relation between unbounded operators.

Conclusions

There are actually no conclusions in the strict sense, that is, we did not find any results that were previously unknown. Yet, to our best knowledge, a reasonable part of the content of this work has been published never before.

The Coleman-Weinberg potential, reviewed in Ch. 1, is well known, and the relatively simple derivation we have given has been significantly improved since then. Nevertheless, we are not aware of any work that shows how to perform actual perturbative calculations in this theory – a gap we were trying to remedy in Ch. 2–3, where examples of such calculations are given. In our opinion, the most interesting point is that the "tree level" quantities of the broken phase are changed by one-loop corrections, because they are actually of the same order in the coupling constant.

In Ch. 4, we review the various "physical" viewpoints of the delta-function potential found in the literature, with the main goal to put the relevant formulae into the same context. The results of Ch. 5 are also already known, but they are usually derived using the angular decomposition. We think that the momentum-space approach is much easier to comprehend and more straightforward; we also present the calculations in very high detail. Also, we have explicitly characterised the action of the adjoint operator to the original (non-self-adjoint) Hamiltonian, which is an important piece of information which we found not to be stated very clearly in the previous works.

Of particular interest (in our opinion) is that many questions in the theory, even apparently very distinct, can be handled using the same method, which we have shown on several occasions. Moreover, the calculations of Sec. 5.5 provide a notable example that one can find an eigenexpansion (in a mathematically well-defined sense) even without solving the Lippmann-Schwinger equation (actually without being able to write one down). Finally, we give an appendix which presents the mathematical subtleties of the subject in a way that we hope to be concise, yet fully understandable to anyone with a basic mathematical knowledge on undergraduate level.

Part III

Appendix

A Self-adjoint extensions of symmetric operators

In this Appendix we follow mainly the classic books [22, 20], but we omit most of the rigorous proofs (which can be found in the literature), whereas we try to present some, mostly informal, motivations behind the ideas.

A.1 Basic notions for unbounded operators

Let us first recall some elementary notions. A **Hilbert space** \mathcal{H} is a linear space (always infinite-dimensional for us) with a norm that stems naturally from an inner product $\|x\| = \sqrt{\langle x, x \rangle}$ (where $\langle x, y \rangle$ is conjugate-linear in the first argument³⁵) that is **complete** in the corresponding metric, i.e. every Cauchy sequence in \mathcal{H} (a sequence for which there is always some N such that for all $n, m > N$, the distance $\|x_n - x_m\|$ is arbitrarily small) converges to an element of \mathcal{H} ³⁶. An **operator** on \mathcal{H} is a linear map T from \mathcal{H} to some subspace of \mathcal{H} , its **norm** is the smallest C such that $\|Tx\| \leq C \|x\|$ for all $x \in \mathcal{H}$ and if $C < \infty$ then T is **bounded**³⁷. An **adjoint** of a bounded operator T is the operator T^* that satisfies $\langle T^*x, y \rangle = \langle x, Ty \rangle$ for $\forall x, y \in \mathcal{H}$ and if $T = T^*$ then T is called **self-adjoint**.

It is well known, yet worth a brief remark, that when dealing with infinite-dimensional spaces, one has to be careful with the geometric insights from the finite-dimensional case. This is nicely illustrated on the issue of linear subspaces: in the case of \mathbb{R}^n , it is clear that any proper subspace is a closed set, i.e. if a sequence in a hyperplane converges, it obviously converges to a point *in* the hyperplane. On the other hand, in the infinite-dimensional case, we can have both closed and non-closed subspaces. Moreover, two quite counter-intuitive situations do often appear: a closed proper subspace can be isomorphic to the whole space and a non-closed subspace can be **dense** in \mathcal{H} , i.e. any element of \mathcal{H} can be arbitrarily well approximated by the elements of the subspace.

A very useful tool, that will also give an additional insight into these matters, is the notion of a **graph** of an operator $\Gamma(T)$ which is the subset of $\mathcal{H} \times \mathcal{H}$ containing all elements of the form $\langle x, Tx \rangle$ ³⁸. $\Gamma(T)$ is always a linear subspace of $\mathcal{H} \times \mathcal{H}$ by linearity of T (e.g. $\alpha \langle x, Tx \rangle = \langle \alpha x, \alpha Tx \rangle = \langle \alpha x, T(\alpha x) \rangle \in \Gamma(T)$). The relation between the two oddities, non-closed subspaces and unbounded linear operators, is nicely clarified by the **closed graph theorem**: a linear map between Hilbert (even Banach) spaces is bounded if and only if its graph is closed (i.e. it is a closed subset

³⁵We avoid the bra-ket notation as it can be misleading when later dealing with operators that are not self-adjoint.

³⁶A complete linear space with a norm, but without an inner product is a **Banach** space. The requirement of completeness may seem a little artificial (even to a reader that is used to it) because it is not often used directly, nevertheless it is hidden in the various useful theorems about Banach spaces.

³⁷On a finite-dimensional space, *every* linear operator is bounded.

³⁸The notation shall be obvious and not confused with the notation for the scalar product!

of $\mathcal{H} \times \mathcal{H}$). An important consequence of it is the **Hellinger-Toeplitz theorem**: an operator T on a Hilbert space \mathcal{H} that satisfies $(Tx, y) = (x, Ty)$ for $\forall x, y \in \mathcal{H}$ is closed and thus bounded.

On the other hand, almost every operator we deal with in quantum theory is unbounded, starting from the simple case of the Laplacian. Yet we would like (at least for the Hamiltonian) to have a property similar to the bounded-case self-adjointness, which guarantees the reality of the spectrum (that is, the reality of observed quantities!), the ability to make functions of the operators and so on. It turns out that the loophole is that we work with operators that are not defined on the whole \mathcal{H} , but only on a subspace of \mathcal{H} , called the **domain** $D(T)$. The domain cannot be a closed subspace, because it would again be a Hilbert space and the Hellinger-Toeplitz theorem would hold – actually we will see shortly that the natural requirement for $D(T)$ is to be dense in \mathcal{H} , which is also intuitively pleasant as the least possible concession.

In many respects, unbounded operators require much more care in manipulation than the bounded ones, due to the limited domains where they make sense. That is also the case when we try to define self-adjointness for them. First, the **adjoint of an unbounded operator** will be also unbounded, so we need to specify its domain, which will be those $\phi \in \mathcal{H}$ for which there is a $\eta \in \mathcal{H}$ such that

$$(T\psi, \phi) = (\psi, \eta) \quad \text{for } \forall \psi \in D(T) \quad (160)$$

and on these vectors we may then define $T^*\phi = \eta$. (The requirement for $D(T)$ to be dense shall now be clear: if its closure $\overline{D(T)}$ was not the whole \mathcal{H} , then $\overline{D(T)}$ would be a closed proper subspace which is known to have a nonempty orthogonal complement. Then for any $v \in \overline{D(T)}^\perp$ we have $(\psi, \eta + v) = (\psi, \eta)$ for $\forall \psi \in D(T)$ and so the choice of η is not unique).

Now if $(T\psi, \phi) = (\psi, T\phi)$ for all $\psi, \phi \in D(T)$, the vector $T\phi$ satisfies (160), and thus in such a case $D(T^*) \supseteq D(T)$ and actions of T and T^* on vectors from $D(T)$ coincide – we write that $T \subset T^*$ (because also $\Gamma(T) \subset \Gamma(T^*)$) and we call T **symmetric**. The reason for not writing $T = T^*$ right away is that we are not guaranteed to have exhausted all vectors for which (160) holds – the domain of T^* can be larger than the domain of T . Only if those two domains *do* coincide then $T = T^*$ and we call T **self-adjoint**. As (160) is required to hold for $\forall \psi \in D(T)$, we can only make it harder to accomplish by enlarging the $D(T)$, thus this may only shrink $D(T^*)$. Sometimes (but not always!) the two domains can be brought together this way – in such a case we call the resulting self-adjoint operators (there can be a lot of them) **self-adjoint extensions** of T .

A.2 Deficiency subspaces and indices

Why do we ever make the difference between operators that act in the same way but differ only in their domains? It is not difficult to realise that by only slightly changing the domain of an operator in a suitable way, we can radically change its spectrum (for example if we remove some of its eigenvectors). For symmetric operators, the situation is even more dramatic, because it can be proved that a closed symmetric operator that is not self-adjoint has always at least one half-plane

of complex numbers in its spectrum³⁹, whereas the spectrum of a self-adjoint (even unbounded) operator is purely real! But maybe the most convincing argument that only the self-adjoint operators are the good candidates for Hamiltonians comes from a rather different perspective, in the form of the Stone's theorem, to which we will return at the end of the section with enough machinery at hand.

For a moment, consider the subspace A of $\mathcal{H} \times \mathcal{H}$, that is spanned by the vectors in the form $\langle -T\psi, \psi \rangle$ for $\forall \psi \in D(T)$. Any vector $\langle \phi, \eta \rangle$ for which

$$0 = (\langle -T\psi, \psi \rangle, \langle \phi, \eta \rangle) = (\psi, \eta) - (T\psi, \phi) \quad \text{for } \forall \psi \in D(T) \quad (161)$$

belongs to A^\perp . But in (161) we immediately recognise the condition (160), thus $A^\perp = \Gamma(T^*)$ while an elementary argument⁴⁰ shows that an orthogonal complement to a subspace is always closed. Thus T^* is closed, and so is then every self-adjoint operator – with no loss of generality, we can first take the **closure** of a non-closed symmetric operator (i.e. work with the operator whose graph is $\overline{\Gamma(T)}$), when seeking its self-adjoint extensions⁴¹.

As in the bounded case, for an unbounded self-adjoint operator we can easily see that if $T\phi = T^*\phi = \lambda\phi$, then

$$\bar{\lambda}(\phi, \phi) = (\lambda\phi, \phi) = (T\phi, \phi) = (\phi, T^*\phi) = (\phi, T\phi) = \lambda(\phi, \phi). \quad (162)$$

Thus any eigenvalue of a self-adjoint operator is real. In fact, the same argument goes through for a symmetric operator, but *not* for its adjoint, as in general $T^*\phi = \lambda\phi \not\Rightarrow T\phi = \lambda\phi$, because it may be that $\phi \in D(T^*) - D(T)$ and thus $T\phi$ makes no sense. Moreover, if $(T^* - \lambda)\phi = 0$ ⁴², then obviously for $\forall \psi \in D(T)$

$$0 = (\psi, (T^* - \lambda)\phi) = ((T - \bar{\lambda})\psi, \phi) \Rightarrow \phi \in \text{Ran } (T - \bar{\lambda})^\perp. \quad (163)$$

This argument is easily reverted and so

$$\text{Ran } (T - \bar{\lambda})^\perp = \text{Ker } (T^* - \lambda) \quad (164)$$

which means that if there is a complex eigenvalue λ of T^* then $T - \bar{\lambda}$ is not invertible on the whole \mathcal{H} and so $\bar{\lambda}$ is in the spectrum of T (even though it is not an eigenvalue).

The trouble with operators that are only symmetric is now clear – $D(T)$ is too small for $T - \lambda$ to cover the whole space, whereas $D(T^*)$ is too big because it includes the undesired eigenvectors of complex eigenvalues. Naturally, we would like to examine the difference between the two domains, but as $D(T)$ is dense, it cannot have an orthogonal complement, which makes the task harder. An elegant solution is to consider the orthogonal (in the $\mathcal{H} \times \mathcal{H}$ inner product) complement of the closed

³⁹The **spectrum** of T is the set of $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not a one-to-one map between $D(T)$ and \mathcal{H} ; this definition encompasses the eigenvalues, the continuous spectrum and some exotic cases that (luckily) never appear for a physical Hamiltonian.

⁴⁰ $x_n \rightarrow x$ and $(x_n, \psi) = 0 \Rightarrow (x, \psi) = 0$ by the continuity of the scalar product

⁴¹It is generally perfectly possible that $\overline{\Gamma(T)}$ is not a graph of a (single-valued) operator, but it is never the case when T is symmetric. Moreover, a rather technical argument shows that always $(\overline{\Gamma(T)})^* = T^*$, which further justifies the generality in always taking the closure. From now on we will assume that T is closed.

⁴²The set $\{\phi : A\phi = 0\} = \text{Ker } A$ is the **kernel** of A ; similarly the set $\{\phi : \phi = A\psi \text{ for some } \psi\} = \text{Ran } A$ is the **range** of A

subspace $\Gamma(T)$ in $\Gamma(T^*)$. It should be somehow related to the kernels of $T^* - \lambda$, so let us see if it contains some of them. The space $\Gamma(T)^\perp \cap \Gamma(T^*)$ is the set of such elements $\langle \psi, T^* \psi \rangle \in \Gamma(T^*)$ that

$$0 = (\langle \phi, T\phi \rangle, \langle \psi, T^* \psi \rangle) = (\phi, \psi) + (T\phi, T^* \psi) \text{ for } \forall \phi \in D(T). \quad (165)$$

For $\psi \in \text{Ker}(T^* - \lambda)$ this can be evaluated as

$$0 = (\phi, \psi) + \lambda(T\phi, \psi) = (\phi, \psi) + \lambda(\phi, T^* \psi) = (\phi, \psi)(1 + \lambda^2). \quad (166)$$

Because $D(T)$ has no orthogonal complement, the first term cannot vanish identically and thus $\lambda = \pm i$. In fact, it can be proved that the search is now over: the whole graph of T^* is the orthogonal sum of pairs of vectors from $D(T)$ and $\text{Ker}(T^* \pm i)$ and their images⁴³. Thus, $\Gamma(T) = \Gamma(T^*)$, i.e. T is self-adjoint, if and only if $\text{Ker}(T^* \pm i) = \emptyset$, which is known as **the basic criterion for self-adjointness**. The spaces $\mathcal{K}_\pm = \text{Ker}(T^* \mp i)$ are called the **deficiency subspaces** and their dimensions (n_+, n_-) the **deficiency indices** of T .

A.3 Healing the deficiency

For a given symmetric operator, the task is to get rid of the deficiency subspaces and the obstacle to it is that we cannot simply start adding vectors from, say, \mathcal{K}_+ to $D(T)$, because we must keep $T = T^*$ and this would immediately contradict (162). Fortunately, there is a way out: what if we added some vectors $\phi = \phi_+ + \phi_-$, where $\phi_\pm \in S_\pm \subset \mathcal{K}_\pm$, (eigenvectors of T^* for $\pm i$)? Then for every pair of such vectors, we must demand the symmetry of the operator, thus

$$\begin{aligned} 0 &= (T(\phi_+ + \phi_-), \psi_+ + \psi_-) - (\phi_+ + \phi_-, T(\psi_+ + \psi_-)) = \\ &= -i(\phi_+ - \phi_-, \psi_+ + \psi_-) - i(\phi_+ + \phi_-, \psi_+ - \psi_-) = -2i((\phi_+, \psi_+) - (\phi_-, \psi_-)). \end{aligned} \quad (167)$$

The requirement that $(\phi_+, \psi_+) = (\phi_-, \psi_-)$ simply means that the map from, say, S_+ to S_- (let us call it U), which tells us which ϕ_- to add to a chosen ϕ_+ , is *unitary*⁴⁴. Then (assuming the deficiency indices of T to be finite for simplicity) the two subspaces S_\pm from which the vectors are taken, must have the same dimension, because an unitary matrix is always regular. The action of T on the added vectors is determined by the already known action of T^* , so we have

$$T(\phi_+ + U\phi_+) = i\phi_+ - iU\phi_+ \quad (168)$$

$$(T + i)(\phi_+ + U\phi_+) = 2i\phi_+ \text{ and } (T - i)(\phi_+ + U\phi_+) = -2iU\phi_+. \quad (169)$$

⁴³It can be also shown that the spaces $\text{Ker}(T^* - \lambda)$ have the same dimension as $\text{Ker}(T^* \pm i)$ for λ in the lower (upper) open complex half-plane. For obvious reasons, they can be neither contained of $D(T)$, nor identical to $\text{Ker}(T^* \pm i)$, so they are in general made of some linear combinations of the former.

⁴⁴It can be shown that it is sufficient also for the symmetry of matrix elements of T between the ϕ 's and vectors from $D(T)$.

But it means that $\text{Ran } (T + i)$ has been enlarged by S_+ and vice versa and thus, by (164), the deficiency indices have been both lowered by the same value, the dimension of S_{\pm} . Have these indices not been equal, we would eventually end up with one of them zero and no way to get rid of the remainder of the other – such a symmetric operator would not have any self-adjoint extensions at all.

Even though these ideas are mostly vague, the reader shall not be surprised that they can be refined into a rigorous **theorem** (originally due to von Neumann): For a closed symmetric operator T with deficiency indices (n_+, n_-) , the following holds:

1. T is self-adjoint if and only if $n_+ = n_- = 0$
2. A has self-adjoint extensions if and only if $n_+ = n_-$ and there is a one-to-one correspondence between those extensions and unitary maps U from \mathcal{K}_+ to \mathcal{K}_- . The domain of an extension corresponding to U is $D(T_U) = \{\phi + \phi_+ + U\phi_+; \phi \in D(T), \phi_+ \in \mathcal{K}_+\}$ and $T_U(\phi + \phi_+ + U\phi_+) = T(\phi) + i\phi_+ - iU\phi_+$.

Note that a unitary matrix of dimension n depends on n^2 real parameters, thus there is an n^2 -parametric family of self-adjoint extensions for a symmetric operator with deficiency indices (n, n) .

A.4 Stone's theorem

To conclude this section, we introduce the **Stone's theorem**, as promised. The theorem says that, given a one-parameter unitary group $U(t)$ on a Hilbert space, which is in some sense continuous, there is always a self-adjoint operator A (the *infinitesimal generator* of the group) such that $U(t) = e^{itA}$. This statement is of great importance in quantum mechanics, because it is such a unitary group, generated by the Hamiltonian, that defines the dynamics of the system ([10], Ch. 3.1). Let us see (again, without any aspiration for rigour), why the generator of such a group (and so the Hamiltonian) has to be self-adjoint.

The correct definition of the formal expression e^{itA} requires much care (that is, the spectral theorem) and is outside our scope, but it is not hard to believe, that in some sense, $A \longleftrightarrow \frac{1}{it} \frac{dU(t)}{dt} \Big|_{t=0}$. Assume we have proved that there is a dense set of vectors, on which the following makes sense, then (note that $U(0) = I$ for a unitary group and $U^*(t) = U^{-1}(t) = U(-t)$)

$$(A\phi, \psi) = \lim_{t \rightarrow 0} \left(\left(\frac{U(t) - I}{it} \right) \phi, \psi \right) = \lim_{t \rightarrow 0} \left(\phi, \left(\frac{U(-t) - I}{-it} \right) \psi \right) = (\phi, A\psi) \quad (170)$$

so A is symmetric. Were it not self-adjoint, there would be some ψ in its, say, \mathcal{K}_+ . Then for any $\phi \in D(A)$, we have

$$\frac{d}{dt} (U(t)\phi, \psi) = (iAU(t)\phi, \psi) = -i(U(t)\phi, A^*\psi) = (U(t)\phi, \psi). \quad (171)$$

But that means, that $f(t) = (U(t)\phi, \psi)$ solves $f' = f$ and so it is a constant times e^t . But $U(t)$ is unitary, its matrix elements are bounded, and so f must be identically zero, particularly $0 = f(0) = (\phi, \psi)$. But if $D(A)$ is dense, this is a contradiction, thus \mathcal{K}_+ is empty (and so is \mathcal{K}_- by a similar argument).

The situation much resembles the simple case of complex numbers: $|e^{ix}| = 1$ only for x real. We can think of self-adjoint operators as just the operators that are “real enough” not to destroy the unitarity of e^{itA} . It is the basic criterion for self-adjointness that justifies this point of view.

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