Charles University of Prague
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Ph.D. Thesis

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Actuarial Mathematics in Non Life Insurance

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Děkuji mému školiteli, panu Prof. RNDr. Tomáši Ciprovi, DrSc. za ochotné vedení dizertační práce, cenné rady a připomínky a vůbec za všechno, co mě nejen během doktorského studia naučil. Děkuji mému zaměstnavateli, České kanceláři pojistitelů, za umožnění skloubit pracovní povinnosti s doktorandským studiem. Mým nejbližším patří dík za všestrannou podporu po celou dobu studia a za trpělivost.

Prohlašuji, že jsem svou dizertační práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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Abstrakt: Dizertační práce sleduje nové směry matematických metod při výpočtu technických rezerv v neživotním pojištění. Vychází ze stávajících zobecnění metody Chain Ladder, které dále rozšiřuje a zobecňuje. Pro metodu Mnichovský Chain Ladder (MCL) se představuje využití robustní regrese, byla odvozena studie citlivosti celkového odhadu na hodnotě parametru modelu a také se zavádí kalkulace celkové variability a mnohorozměrný MCL.

Dále se navrhuje modelování závislosti plnění, celkového závazku a rezervy v kontextu teorie vektorových autoregresních modelů, Grangerovy kauzality a dalších nástrojů současné ekonometrie. Speciálně, v závěru práce je ilustrována aplikace soustavy simultánních rovnic na problematiku odhadu celkového závazku v pojištění motorových vozidel.
Klíčová slova: Chain Ladder, Mnohorozměrné metody, Simultánní rovnice
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Abstract: The thesis suggests new directions of appliaction of mathematical method in non life insurance and is based on recently published versions of chain ladder method that are extended and generalised in many ways. Munich Chain ladder (MCL) is extended by robust regression, sensitivity study of ultimates depending on parameters value, calculation of variability and multivariate version.

Paid, Incurred and reserve amounts are later suggested to be modelled using econometrical techniques including vector autoregression models, Granger causality, what is compared with MCL. At the end a concept of simultaneous equation for estimating of the liability in motor insurance is implemented.
Keywords: Chain Ladder, Multivariate methods, Simultaneous equations

Chapter 1

Introduction

This Ph.D. thesis is dealing with classical actuarial non life techniques, it discusses their limitation and suggests their generalisation and alternative approaches using the feasible statistical and econometrical methods.

The goal of the work is to suggest new approaches for computation of technical reserves and later implement the proposed method in practical situation as well. The thesis is organised as follows.

In the first chapter, there the basic background for technical reserves in non life insurance is given. In subsequent second chapter we gave overview of standard actuarial techniques. Quite large attention is given there to Bühlmann's model of claims reserving that is very detailed and enables a lot of stochastic outputs. However from practical purposes chain ladder is used more in practice. This method is described later on.

Third chapter deals with recent development of Chain ladder regarding Munich and Multivariate models what is continued by author's views on generalisation of Munich Chain Ladder that were summarised in the fourth Chapter.

Fifth Chapter presents author's work on multivariate generalisation of Munich Chain ladder. Econometrical generalisation of technical reserves computation with alternative ways of modelling relation between paid and incurred data are given in sixth chapter. Finally seventh chapter presents application of simultaneous equation models for estimation of the claims volume.

1.1 Fundamental aspects of technical reserves

Technical reserves play significant role in insurance sector. Their value is important for overall economic results of the company and adequate level of technical reserve is reviewed by supervisory authority, auditors and other subjects in order to be sure that the insurance company will cover its liability.

Technical reserves are thus regulated in law and secondary legislative acts. Basic division of technical reserve could be as follows.

1. Unearned premium reserve (UPR) is set up in order to divide the collected premium into two parts. The first part is related with risk that might occur in the same year as the premium is paid. However if the policy contract remains valid in the following year(s) as well and the premium is paid prospectively for the whole time of insurance cover then the part of premium related to successive year(s) has to be given into the UPR. The usual method for its calculation is pro rata temporis method. For example if we have motor third part liability (also MTPL) contract written on 1st April 2008 with yearly premium 6000 CZK, written premium in 2008 is 6000 CZK, however only 4500 CZK consists for earned premium in 2008. The rest 1500 CZK will be given into UPR.

2. Claims reserve are divided onto RBNS and IBNR reserve. The whole thesis deals with this type of technical reserves

3. Equalisation Reserve is set up in that lines of business where loss ratios differs quite a lot across the years. In the year where loss ratio is low the reserve is set up and in the adverse years the amount is used in order to improve the results of that years. However the importance of this reserve is decreasing now what is connected with the fact that according to International Reporting Standards (IFRS) the equalisation reserve is not indeed reserve and the fluctuation of claims amount should be assumed in capital requirements.

4. Reserves covering CKP's liability

   Special type of reserves that are set up by insurance companies writing MTPL policies in order to cover standard not yet paid liabilities arising from already occurred claims of Czech Insurers' Bureau (CK-P) that deals with MTPL losses caused by uninsured and unknown drivers and also covers deficit of run-off MTPL business before 2000. It might be interesting that this reserve is from the point of view of CKP seen as assets that cover standard claims reserve of CKP arising from liabilities to damage parties. For evaluating its liabilities standard or new actuarial methods could be used and also it was found useful to use some statistical methods for evaluating and detection of uninsured cars, people, etc. These results could be seen in Jedlička (2007). From the point of view of CKP member companies Reserve covering CKP's liability is indeed reserve.

5. Other Reserves are set up in order to cover other specific liability and their setting has to be usually allowed by Czech National Bank (CNB) which play the role of supervisory institution for Czech insurance sector.

Non life Insurance companies are obliged to set up claims technical reserves for not yet unpaid claims which occurred in the past calendar years. The respective delay until the claim is paid is caused by the time between the date of accident and the date of reports to the insurer and moreover it will take another more time to settle the claims. In order to give realistic financial picture of the overall volume of the claims two types of technical reserves are set up.

1. RBNS (Reported but not Settled)
2. IBNR (Incurred but not Reported)

IBNER reserve is related to Incurred but not enough reserved claims and usually it is calculated together with "pure" IBNR.

1.2 Historical background for IBNR

Overview of historical development connected with application of IBNR worldwide and in Czech Republic was given in the paper of prof. Mandl (2005).

It states that one of the first publications regarding the Incurred but not reported reserves dates to 1933 (see T. F. Tarbell (1933). The basic principles written there are valid so far. First of all it is the fact that the estimation of IBNR is mathematical or statistical task. The method suggested there is related to recording the claims that were reported after the end of years of occurrence. IBNR is estimated according to the past development of the financial amount of that claims. In this article it is also stated that the estimate of IBNR might be quite easy for lines of business with fast compensations as property insurance where method as some percentage of RBNS might be even used. However the IBNR for lines of liability insurance with long time until the end of claims handling is rightly seen as more difficult task also in this very first paper.

In the paper of Tarbell, there is not worked with run-off triangle schemes only "recording" of IBNR claims is applied. This paper also takes into account the possibility of changes in frequency or severity of IBNR claims. The term of run-off triangle that is crucial for IBNR computation now was firstly introduced in the work of H.G. Verbeek (1972) as is again said in the work of Mandl (2005).

Moreover this article also describes development of technical reserves in Czech Republic for non life lines of business. After 1945 insurance sector was nationalised and only one insurance company remained after 1952. Technical reserves were practically limited to reserve fund that was similar to present equalisation reserves. In the good year when state insurance company made a profit, the part of the profit was taken to state budget and the rest was given into this reserve fund. In adverse years, if the collected premium was insufficient to pay compensation, the part of reserve fund was used to cover the liabilities. No claims reserves were applied and state insurance company worked on calendar year basis. However if the estimate of future liabilities was necessary (for example in case of handling a foreign claims arising e.g. from international motor liability insurance), the estimate was performed and the result might be used as a source for technical reserves of foreign partner.

The reserve funds interpretable as equalisation reserve remained some time after Velvet revolution when insurance sector was demonopolisated. In 1994 the law introduced obligation for insurance companies to create RBNS and IBNR types of reserves. Overview of present situation including also liability adequacy test for non life lines of business could be seen in the paper of Šváb (2005).

1.3 Claims reserves

RBNS reserve is set up for Reported But Not Settled claims and IBNR reserve deals with the problem of Incurred But Not Reported claims. The first one may be determined by individual estimates for each known not paid claim regarding the experiences and expert opinion of future compensation that is usually made by employee of claims department. The latter reserve could be determined only via mathematical methods using the known development of paid compensation and RBNS reserve. If RBNS reserve is not set up individually as estimate of future paid compensation for each and every claim, an actuary can use only data describing the development of paid compensation and estimate the sum of RBNS and IBNR reserves together.

1.4 Run off triangle schemes

We will mark $Y_{i,j}$, $i = 0, \dots, n$, $j = 0, \dots, n-i$ for data of paid claim or incurred where n notifies the dimension of the data sets. It is assumed that there is no development if n periods after accident pass. If we want to distinguish type of triangle we will add upper indices $Y^P_{i,j}$ for data of paid compensation or $Y^I_{i,j}$ for incurred data (sum of paid compensation and corresponding value of RBNS reserve). These data are usually analysed in the so called run-off triangles which could be seen as a matrix where only data in the upper left triangle are known and our aim is to estimate the future development in the lower right triangle. Each row is interpreted as one accident period and each column as a single development period (i.e. the variable $Y_{i,j}$ shows us overall paid or incurred value of all claims occurred in period i and paid or reported until j periods after the accident happened). Thus figures of each diagonal corresponds to one single calendar period.

Typical run-off triangle is defined as follow:

	0	1	2	3	4	5	6
0	978	2104	2134	2144	2174	2182	2174
1	1844	2552	2466	2480	2508	2454	
2	2904	4354	4698	4600	4644		
3	3502	5958	6070	6142			
4	2812	4882	4852				
5	2642	4406					
6	5022						

Figure 1.1: Example of Run-off triangle

Chapter 2

Traditional reserving methods

Reserving method could be classified in many ways. The basic division might be on stochastic and deterministic models. Some of the most known are briefly described below. Their description could be also find in standard actuarial textbooks as Cipra (1999) or Mandl (1999). Some of the presented methods are deterministic and the other are stochastic. Chain Ladder that will be generalised in many ways in the following chapters is described at the end of the chapter.

2.1 Loss Ratio Method

Loss Ratio method could be seen as the most straightforward reserving method. Let us denote l or more concretely l_i loss ratio that is assumed generally for all accident years together or different for various accident years. Then the overall claims reserve is computed as

$$R_i = EP_i \cdot l_i - Y^P_{i,n-i}$$

where EP_i stands for earned premium with respect to year i. It is crucial that the loss ratio must be known or assumed l_i so this method is rather cyclical one and is used in the cases where available data are not sufficient for performing another methods.

2.2 Bühlmann's model of claims reserving

We would like to remind this model to ensure that the roots for recent development in calculation of technical provision go back to the beginning of 1980's and that "only" the lack of data, changes of the methodologies etc. and other practical difficulties force us to implement more aggregate methods that are better useful in practical situations.

This model could be seen as one of the first stochastic approaches how to model claim reserves based on run-off triangles. Moreover it works with time between paying the compensation and claims occurrence as well as time between paying the compensation and claims report and requires individual data of each claims evolution.

To be able to perform individual development, notation $Z_{i,j}^{m,(k)}$ interpreted as amount paid (or incurred) on behalf kth claim occurred in time i and reported to insurer after following m periods will be introduced. Aim of the model is to predict the claim process for each claim since its report until finalisation of claim settlement.

Apart from that modelling, estimate of number of all claims that occurred in accident year i is important together with evolution of their reporting time to insurer. Model assumes that number of claims occurred in i is a random variable N_i and also we will mark T_i^k as calendar period when the k th claim is reported.

After that we have

$$Y_{i,j} = \sum_{m=0}^{j} \sum_{k=1}^{N_i} I[T_i^{(k)} = m].Z_{i,j}^{m,(k)}.$$

Paid compensation on behalf of reported claims occurred in i and paid until j following periods (i.e. until calendar periods $t = i+j$) can be rewritten as a sum of paid compensation on behalf of the claims occurred in i and reported until time $i + j$. We will also mark

$$Y_{i,j}^{m} = \sum_{k=1}^{N_i} I[T_i^{(k)} = m] \cdot Z_{i,j}^{m,(k)}$$

the overall amount paid after j periods on behalf of claims occurred in i and reported after m periods only.

Similarly as above final value for each claim is also defined:

$$Z_i^{(k)} = \lim_{j \to \infty} Z_{i,j}^{(k)} \qquad 1 \leq i \leq n,$$

that is to be estimated. Based on this definition it holds true that

$$Y_i = \sum_{m=0}^{\infty} \sum_{k=1}^{N_i} I[T_i^{(k)} = m] Z_i^{(k)} \cong \sum_{m=0}^{\infty} Y_i^m$$

where Y_i^m corresponds to whole amount of liability (regardless if it has been already paid) for accident period i and reported after m periods only. Also we can write

$$Y_i^m = \sum_{k=1}^{N_i} I[T_i^{(k)} = m] Z_i^{m,(k)}.$$

In that model run-off related to claim reports is considered as well. So

$$N_{i,j} = \sum_{k=1}^{N_i} I[T_i^{(k)} \leq j]$$

is interpreted as number of claims occurred in i and reported j periods after occurrence.

As we know, estimate of reserve $R_{i,j} = Y_i - Y_{i,j}$ is a general task. If Y are interpreted as Paid data we obtain the estimate of overall claims reserves (sum of RBNS and IBNR). If Y are interpreted as incurred data we obtain estimate of IBNR only. In that situation however we are sometimes interested how to separate IBNR onto pure IBNR (estimating the value of really unreported claims) and IBNER (Incurred But Not Enough Reserved). IBNER reflects fact that some claims could be hardly reserved in the whole amount soon after claims reporting. It holds for example for bodily injury claims in motor insurance where the scope of liability is known after longer period. Amount of lump sum compensation (e.g. pain and suffering) could be evaluated after end of medical treatment and the annuity compensation could be paid even for dozen of years.

Estimate of IBNR $R_{i,j} = Y_i - Y_{i,j}$ (if Y are Incurred data) could be rewritten

$$R_{i,j} = Y_i - Y_{i,j} = \sum_{m=0}^{j} Y_i^m - Y_{i,j}^m + \sum_{m=j+1}^{\infty} \sum_{k=1}^{N_i} I[T_i^{(k)} = m].Y_{i,j}^{(k),m}.$$

The first summand corresponds to IBNER (mark $\Gamma_{i,j}$) and the latter one to pure IBNR ($\Delta_{i,j}$).

It is also important to have some assumptions for above defined random variables or processes describing claim settlement if one has to implement the model.

In the article it is assumed the claim number distribution N_I is Poisson with parameter $V_i.v$ interpreted as volume of risk in underwriting period i and v is unknown parameter interpretable as loss frequency. For example in homogeneous portfolio one could use number of insured and loss frequency for v parameter if we do not assume that more than one claim from one contract could arise.

Other important assumptions are independence of number of claim occurrences N and respective times of accident T and also independence of random process of claims reporting and claims settlement among different accident periods.

Moreover random sequences describing claims development

$$Y_{i,j}^{m,(k)}, \qquad m \leq j$$

are for various claims independent and identically distributed and so we do not have to work with index k

In addition time of claims report $T_i^{(k)}$ are for various k i.i.d. with distribution function marked

$$F(m) = P(T_i^{(k)} \leq m)$$

so probability that the claim is reported just after m periods $p(m)$ is

$$p(m) = F(m) - F(m-1).$$

It is thus assumed that claims reporting is not dependent on time of occurrence.

Last assumption of basic model is connected with expected amount of paid compensation through so far reported claims

$$E[Z_{i,j+1}^{m,(k)} | Z_{i,d}^{m,(k)}, m \leq d \leq j] = \lambda_j^m . Z_{i,j}^{m,(k)}$$

and its variability

$$\mathrm{Var}[Z_{i,j+1}^{m,(k)}|Z_{i,d}^{m,(k)}, m \leq d \leq j] = \left(\sigma_j^m\right)^2 .f(Z_{i,j}^{m,(k)}),$$

where $f > 0$.

In extensions of the method leading to application some more assumptions are made as well. Probabilistic distribution of incurred amount of each claim is assumed to be logarithmic-normal so it holds for amount put in reserve or paid immediately after report that

$$\ln(Z_{i,m}^m) \sim N(\mu_m + (i-1).\ln(1+\delta), \sigma_0^2)$$

If we re apply this we will get conditional distribution under knowledge of pattern since report to time of reserve calculation $(Z_{i,m}^m, Z_{i,m+1}^m, \ldots, Z_{i,j}^m)$ as

$$\ln(Z_{i,j+1}) \sim N(\gamma_j + \ln(Y_{i,j}^m), \gamma_j\sigma^2)$$

After transformation we get for expectation

$$E(Z_{i,j+1}^m|Z_{i,j}^m, \ldots, Z_{i,m}^m) = Z_{i,j}^m.\exp(\gamma_j(1+\frac{\sigma^2}{2})) \cong Z_{i,j}\lambda_j$$

and for conditional variance

$$\mathrm{Var}(Z_{i,j+1}^m|Z_{i,j}^m, \ldots, Z_{i,m}^m) = Z_{i,j}^m.\sigma_j^2$$

It is seen as important simplification if

$$\lambda_j^m \cong \lambda_j$$

and also

$$\left(\sigma_j^m\right)^2 \cong \sigma_j^2$$

However this means that claim payment pattern reported after m periods depends only on delay after claims occurrence and not on delay after time of report. That does not need to be held in practical implications.

Initial values in the time of setting up the reserve could be formulated as

$$E(Z_{i,m}^m) = \exp(\mu_u + \frac{\sigma_0^2}{2})(1+\delta)^{(i-1)}$$

where claim inflation is considered as well.

2.2.1 Parameters estimates

As stated above, our aim is to estimate "separately" pure IBNR and IBNER based on information of $Y_{i,j}$ where $j \leq n-i$. We have to estimate both components of the sum

$$R_{i,n-i} = \Gamma_{i,n-i} + \Delta_{i,n-i} \qquad i = 1, \ldots, n.$$

The estimate of IBNER is possible to write as

$$\sum_{m=1}^{n-i} \Big(\prod_{j \geq n-i}^{\infty} \lambda_j - 1 \Big) . Y^m_{i,n-i} \equiv \sum_{m=1}^{\tilde{n}} (H^m_{\tilde{n}} - 1) Y_{i,\tilde{n}}$$

Pure IBNR $\Delta_{i,n-i}$ might be estimated as

$$\sum_{m=\tilde{n}+1}^{\infty} p(m).E(Z^m_i).V_i.v$$

Estimates of parameters v, $p(m)$ are straightforward.

$$\widehat{p(m).v} = \frac{\sum_{i=1}^{n+1-m} U_{i,m}}{\sum_{i=1}^{n+1-m} V_i}$$

and also

$$\widehat{v} = \sum_{m=1}^{\infty} \widehat{p(m).v}$$

Estimate of λ^m_j is proposed as

$$\widehat{\lambda^m_j} = \frac{\sum_{i=1}^{n-j+1} \frac{Y^m_{i+1,j} . Y^m_{i,j}}{U_{i,m}}}{\sum_{i=1}^{n-j+1} \frac{(Y^m_{i,j})^2}{U_{i,m}}} \tag{2.2.1}$$

where $U_{i,m}$ states for number of claims occurred in i and reported just after m periods after occurrence that is

$$U_{i,m} = N_{i,m} - N_{i,m-1}$$

and we define $N_{i,0} = 0$.

This estimate of λ^m_i is according to previous assumption BLUE. Obviously expected value of $\frac{Y^m_{i,j+1}}{Y_{i,jm}}$ equals λ^m_j for all i a its conditional variance if one does not know function f could be rewritten as

$$\sigma^m_j \sum_{k=1}^{U_{i,m}} f(Z^{(k)}_{i,j}) \cong K(\sigma^m_j)^2.U_{i,m}$$

Presented estimate 2.2.1 could be seen as special case of parameters estimate in linear model theory $X_i \sim (\mu, \sigma_i)$, that is

$$\hat{\mu} = \frac{\sum_{i=1}^{n} X_i \cdot \sigma_i^{-2}}{\sum_{i=1}^{n} \sigma_i^{-2}}$$

So we can determine estimate of overall paid amount for each claim occurred in i and reported after m period as $E(Z_i^m) = E\left(Z_{i,m}^m \prod_{j=m}^{\infty} \lambda_j^m\right)$. As we know that $E(Z_{i,m}^m) = c_m(1+\delta)^{i-1}$ it is sufficient to estimate parameters δ and c_m

These estimates $\hat{\delta}, \hat{c_m}$ will be derived if one minimises following error function

$$Q\left(\hat{\delta}, \hat{c_m}\right) = \sum_{i,mi+m \le n} \left(\frac{X_i, m^m}{U_{i,m}} - \hat{c_m}(1+\hat{\delta})^{i-1}\right)^2 U_{i,m}$$

Estimates for fixed δ are solution of that problem:

$$c_m(\delta) = \frac{\sum_{i=1}^{n-m} X_{i,m}^m (1+\delta)^{i-1}}{\sum_{i=1}^{n-m} U_{i,m}[(1+\delta)^{i-1}]^2}$$

Optimal estimate $\hat{\delta}$ is determined via minimising of function $Q(\delta, \widehat{c_m(\delta)})$. This solution could be obtained numerically.

2.2.2 Conclusion of the model

This model formulated more than 25 years ago is really detailed and gives realistic description of whole process of claims settlement and justifies the importance of setting up the all types of loss reserves and its interpretation.

However the structure of data and its complexity could be seen as drawback of the model. At least we have to use n run-off triangles where each of them considers known development of claims settlement according to period of claims report $m, m = 1, \ldots, n$. For large values of m not much different from n the problem of lack of data and related influence or result could occur.

2.3 Standard Chain Ladder Method

That is the most widely used method in loss reserving used for each single run-off triangle. It originates from intuitive deterministic assumptions which were later generalised to obtain stochastic model of chain ladder.

2.3.1 Standard Chain Ladder - deterministic approach

This method is described in the actuarial textbooks, e.g. Cipra (1999) or Mandl (1999) and is based on the assumption that ratios of following values in one raw are approximately constant independently on accident period i (but dependent on development period j). That is

$$Y_{i,j+1} \cong Y_{i,j}.f_j, \ i = 0, \ldots, n \ j = 0, \ldots, n-1 \tag{2.3.1}$$

Regarding the fact that we know only data $Y_{i,j}$ if $i+j \leq n$ estimate $\widehat{f_j}$ could be based on values $Y_{i,j+1}$, $i = 0, \ldots, n-j-1$ and $Y_{i,j}$, $i = 1, \ldots, n-j$ only.

Individual development factors are defined as $F_{i,j} = \frac{Y_{i,j+1}}{Y_{i,j}}$, $i = 0, \ldots n,$, $j = 0, \ldots, n-1$. Intuitive estimate $\widehat{f_j}$ could be formulated as arithmetic mean

$$\widehat{f_j} = \frac{1}{n-j} \sum_{i=0}^{n-j-1} F_{i,j}, \ j = 0, \ldots, n-1.$$

However the most popular estimate is different

$$\widehat{f_j} = \frac{\sum_{i=0}^{n-j-1} Y_{i,j+1}}{\sum_{i=0}^{n-j-1} Y_{i,j}}. \tag{2.3.2}$$

Its mathematical interpretation could be seen later, based on article Mack (1993). Our aim is to estimate ultimate values of paid or incurred data which is done according to following formulae:

$$\widehat{Y_{i,n}} = Y_{i,n-i} \prod_{j=n-i}^{n-1} \widehat{f_j}, \ i = 1, \ldots, n$$

and corresponding IBNR or sum of IBNR and RBNS reserves could be gained by subtracting the ultimate and diagonal figures:

$$R_i = \widehat{Y_{i,n}} - Y_{i,n-i}, \ i = 1, \ldots, n.$$

No reserve for accident year 0 is made since we assume that the claim handling is finished after n periods after accident.

Example

Let us assume following run-off triangle schemes representing Paid compensation development

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	45	372	1 514	2 728	4 058	5 002	5 795	7 055	7 322	7 818	7 926	9 074	9 481	9 691	9 715	9 736	9 861	10 082	10 082	10 152
1	9	419	1 188	2 643	3 318	4 439	5 332	5 669	5 847	5 982	6 406	6 702	6 802	6 808	6 808	6 853	6 874	6 874	6 910	
2	96	865	2 084	3 235	4 457	7 395	8 102	8 710	9 197	10 039	10 733	10 964	11 223	11 572	11 773	12 056	12 118	12 141		
3	16	170	1 171	2 510	6 455	7 222	8 501	9 056	10 480	11 608	12 393	12 916	13 124	13 694	13 870	13 930	14 163			
4	71	925	2 525	7 158	8 387	11 054	12 719	15 531	17 882	18 761	19 080	19 634	20 160	20 274	20 364	20 493				
5	253	952	2 854	3 744	5 185	6 234	9 792	11 737	12 893	14 309	14 609	14 857	15 458	15 523	15 595					
6	199	1 535	2 861	5 391	7 834	12 963	15 401	17 711	18 385	19 310	19 616	20 112	20 386	20 716						
7	246	1 348	3 369	6 350	14 823	18 154	20 897	22 260	23 445	24 272	25 820	26 382	26 906							
8	234	1 721	3 487	13 224	19 009	24 506	26 826	28 897	29 531	30 479	30 937	31 486								
9	142	603	5 793	9 092	13 430	15 514	17 344	17 941	19 396	19 801	20 223									
10	197	2 113	5 926	10 524	14 225	16 707	17 956	19 997	21 627	22 779										
11	43	1 591	9 316	14 940	20 551	24 373	26 698	28 573	30 371											
12	21	1 993	9 061	14 352	17 208	21 149	24 183	26 220												
13	74	2 305	8 577	12 837	16 611	19 966	23 652													
14	44	4 673	12 149	17 678	23 857	27 485														
15	196	5 171	16 437	23 153	28 565															
16	100	4 322	13 855	21 426																
17	262	4 941	13 131																	
18	335	4 135																		
19	228																			

Figure 2.1: Run - off triangle representing cumulative Paid data

The development of this process is relatively "smooth" that can be true for short tail non life business what is not the case of Motor Third part liability where separate analyses of property damage (short tail) and bodily injury claims (long tail) may be appropriate. Corresponding incurred portfolio is as follows.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	2502	8482	11 123	11 385	12 159	12 010	12 013	12 034	12 294	12 778	11 872	11 741	11 755	11 714	11 966	11 982	11 689	11 628	11 558	11 583
1	2826	5221	7 214	7 991	8 626	8 404	8 594	8 732	8 656	8 483	8 434	8 425	8 370	8 299	8 220	7 426	7 479	7 368	7 373	
2	5029	11012	13 759	14 674	14 672	14 403	14 721	14 505	14 350	13 743	13 850	13 778	14 155	14 104	13 759	13 933	14 038	13 965		
3	4538	13162	16 550	17 004	17 478	17 909	18 072	17 824	17 193	17 340	17 797	17 279	17 338	16 465	17 247	17 054	16 731			
4	5928	20480	23 324	23 743	25 237	24 954	25 770	24 865	25 794	25 520	25 342	25 281	24 654	24 714	24 486	24 358				
5	7132	16422	16 930	18 945	19 318	19 144	18 786	20 018	20 192	20 242	20 440	20 397	19 601	19 207	19 174					
6	10151	19 562	23 509	26 270	26 033	25 461	26 797	26 802	26 964	26 668	26 751	26 499	25 606	25 078						
7	10716	28 678	33 866	34 162	33 739	34 885	34 473	35 342	35 375	35 345	35 275	34 328	34 613							
8	15932	37 487	43 265	45 338	47 395	47 361	46 778	46 812	46 820	46 513	45 706	45 383								
9	9792	24304	27 571	29 165	29 238	30 083	29 647	29 257	29 025	28 846	28 677									
10	11147	26 238	29 848	31 610	32 201	32 240	32 167	32 194	32 192	32 083										
11	15210	35 030	39 254	40 282	40 789	41 534	41 565	41 032	41 347											
12	10384	30 454	35 567	36 615	38 244	38 667	38 628	38 864												
13	8994	25 783	26 424	29 888	32 357	33 370	33 311													
14	15919	34 124	37 478	37 382	37 315	37 251														
15	18245	40 305	45 693	45 361	44 838															
16	18742	39 769	42 078	41 155																
17	18387	34 187	36 493																	
18	20540	36 890																		

Figure 2.2: Run - off triangle representing cumulative Incurred data

Underlying triangle of numbers of claims is presented below:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	122	359	450	486	518	530	548	562	576	583	584	584	600	601	602	602	604	604	604	604
1	117	270	322	355	373	376	391	402	405	410	412	416	420	420	420	420	422	423	423	
2	207	436	537	571	611	649	675	684	695	704	704	707	722	724	724	725	725	725		
3	213	538	632	678	730	770	786	815	822	830	833	833	860	861	863	865	865			
4	259	708	861	941	985	1 035	1 074	1 084	1 112	1 128	1 133	1 134	1 146	1 150	1 150	1 151				
5	265	609	687	734	781	823	843	867	871	877	879	882	902	902	902					
6	404	806	961	1 024	1 081	1 104	1 119	1 135	1 143	1 149	1 152	1 154	1 179	1 180						
7	448	1 097	1 318	1 419	1 485	1 527	1 549	1 568	1 578	1 582	1 583	1 585	1 625							
8	532	1 274	1 549	1 612	1 657	1 707	1 741	1 766	1 777	1 787	1 792	1 804								
9	413	951	1 079	1 136	1 167	1 198	1 215	1 226	1 230	1 233	1 238									
10	469	1 017	1 192	1 276	1 324	1 346	1 387	1 404	1 427	1 434										
11	571	1 223	1 435	1 530	1 570	1 602	1 634	1 647	1 657											
12	371	1 042	1 222	1 293	1 335	1 359	1 383	1 390												
13	330	876	974	1 040	1 076	1 118	1 124													
14	640	1 192	1 320	1 394	1 435	1 463														
15	727	1 403	1 562	1 638	1 675															
16	569	1 232	1 373	1 437																
17	585	1 081	1 179																	
18	701	1 226																		
19	781																			

Figure 2.3: Run - off triangle representing cumulative numbers of claims

This run-off schemes will be used for basic illustration of chain ladder and its possible drawbacks. Firstly development factors for Paid triangle were computed:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
15,54	3,20	1,67	1,39	1,24	1,15	1,10	1,07	1,05	1,03	1,03	1,02	1,02	1,01	1,01	1,01	1,01	1,00	1,01

Figure 2.4: Estimates of development factors for paid triangle

We apply these standard estimates to complete available triangle into square what gives us following estimates of paid compensations for the future:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	45	372	1 514	2 728	4 058	5 002	6 795	7 065	7 322	7 818	7 928	9 074	9 481	9 691	9 715	9 736	9 861	10 082	10 082	10 152
1	9	419	1 188	2 643	3 318	4 439	5 332	5 869	5 947	5 982	6 406	6 702	6 802	6 808	6 808	6 853	6 874	6 874	6 910	6 958
2	96	865	2 084	3 235	4 457	7 395	8 102	8 710	9 197	10 039	10 733	10 964	11 223	11 572	11 773	12 056	12 118	12 141	12 187	12 251
3	16	170	1 171	2 510	6 465	7 222	8 501	9 056	10 480	11 608	12 393	12 916	13 124	13 594	13 870	13 930	14 183	14 282	14 313	14 412
4	71	925	2 525	7 158	8 367	11 054	12 719	15 531	17 882	18 761	19 080	19 634	20 160	20 274	20 364	20 493	20 705	20 880	20 925	21 069
5	253	952	2 864	3 744	5 185	6 234	9 792	11 737	12 893	14 309	14 609	14 857	15 458	15 523	15 596	15 729	15 892	16 026	16 061	16 172
6	199	1 535	2 861	5 391	7 834	12 963	15 401	17 711	18 385	19 310	19 618	20 112	20 386	20 718	20 894	21 074	21 292	21 472	21 518	21 687
7	246	1 348	3 369	6 350	14 823	18 154	20 897	22 260	23 445	24 272	25 920	26 362	26 906	27 335	27 570	27 807	28 095	28 332	28 393	28 589
8	234	1 721	3 487	13 224	19 009	24 506	26 826	28 897	29 531	30 479	30 937	31 486	32 248	32 763	33 043	33 328	33 672	33 957	34 030	34 265
9	142	603	5 793	9 092	13 430	16 514	17 344	17 941	19 396	19 801	20 223	20 851	21 356	21 697	21 883	22 072	22 300	22 488	22 536	22 692
10	197	2 113	5 926	10 524	14 225	16 707	17 955	19 997	21 627	22 779	23 531	24 263	24 850	25 247	25 463	25 683	25 948	26 167	26 224	26 405
11	43	1 591	9 316	14 940	20 551	24 373	26 698	28 573	30 371	31 951	33 006	34 032	34 856	35 412	35 716	36 024	36 396	36 703	36 782	37 037
12	21	1 993	9 061	14 352	17 208	21 149	24 183	26 220	28 017	29 474	30 448	31 394	32 154	32 667	32 947	33 231	33 575	33 858	33 931	34 186
13	74	2 306	8 577	12 837	16 611	19 966	23 662	26 000	27 782	29 227	30 193	31 131	31 885	32 394	32 672	32 953	33 293	33 575	33 647	33 880
14	44	4 673	12 149	17 678	23 857	27 485	31 512	34 641	37 015	38 940	40 226	41 477	42 481	43 159	43 529	43 904	44 357	44 732	44 828	45 139
15	196	5 171	16 437	23 153	28 665	35 375	40 557	44 584	47 640	50 118	51 773	53 383	54 675	55 548	56 024	56 507	57 090	57 573	57 697	58 096
16	100	4 322	13 855	21 426	29 794	36 896	42 301	46 502	49 669	52 273	53 999	55 679	57 026	57 936	58 433	58 937	59 545	60 049	60 178	60 594
17	282	4 941	13 131	21 974	30 556	37 840	43 383	47 690	50 959	53 609	55 380	57 102	58 484	59 417	59 927	60 443	61 068	61 584	61 716	62 143
18	335	4 135	13 238	22 159	30 804	38 147	43 736	48 078	51 373	54 046	55 829	57 666	58 959	59 900	60 413	60 934	61 563	62 084	62 217	62 648
19	228	3 538	11 326	18 953	26 355	32 637	37 419	41 134	43 953	46 239	47 768	49 251	50 444	51 249	51 688	52 133	52 672	53 117	53 231	53 600

Figure 2.5: Projection of paid triangle

We can learn even from this portfolio that Chain ladder is generally not suitable method due to large variability and large emphasis onto first observation for later accident periods

Better results could be obtained for triangle of numbers of claims where following estimates of development factors were computed:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2,21	1,16	1,06	1,04	1,03	1,02	1,02	1,01	1,01	1,00	1,00	1,02	1,00	1,00	1,00	1,00	1,00	1,00	1,00

Figure 2.6: Estimated of development factors for triangle of numbers of claim-
s

2.3.2 Standard Chain Ladder - stochastic model

In addition to previous approach we can obtain not only the point estimates of reserves but the variability and mean square error which will imply under normality assumption knowledge of overall distribution. Adequacy of normality assumption should be tested but it does not seem to contradict a reality due to Central limit theorem as run-off development is a sum of individual figures for each claim or policy contract.

The stochastic model was firstly presented in the article Mack (1993) and is based on 3 probabilistic assumptions regarding expectation, variability and inter row independencies.

It is assumed that for random vector Y holds

$$E(Y_{i,j+1}|Y_{i,j}, Y_{i,j-1}, \ldots, Y_{i,0}) = Y_{i,j} \cdot f_j, \; i = 0, \ldots, n, \; j = 0, \ldots n-1$$

and for its variability holds that

$$\mathrm{Var}(Y_{i,j+1}|Y_{i,j}, Y_{i,j-1}, \ldots, Y_{i,0}) = \sigma_j^2 . Y_{i,j}, \; i = 0, \ldots, n, \; j = 0, \ldots n-1.$$

To simplify the notation we will define $\mathbf{Y}_i(j) \equiv (Y_{i,0}, \ldots, Y_{i,j})$.

We can rewrite this into a linear model for each development period

$$Y_{i,j+1} = Y_{i,j} . f_j + \varepsilon_{i,j}, \; i = 0, \ldots, n \tag{2.3.3}$$

with notation $\mathrm{E}(\varepsilon_{i,j}|Y_i(j)) = 0$ and $\mathrm{Var}(\varepsilon_{i,j}|Y_i(j)) = \sigma_j^2 . Y_{i,j}$. Moreover it is assumed in Mack (1993) that loss development between different accident years are uncorrelated, that is $\mathrm{Covr}(Y_{i_1,j}, Y_{i_2,j}) = 0, \; i_1 \neq i_2$.

Using Aitken estimate for model (2.3.3) we obtain $\widehat{f}_j$ as

$$\widehat{f}_j = \frac{\sum_{i=0}^{n-j-1} Y_{i,j+1}}{\sum_{i=0}^{n-j-1} Y_{i,j}} \tag{2.3.4}$$

since from the theory of linear models is derived

$$\widehat{f}_j = (Y_{.,j}' V^{-1} Y_{.,j})^{-1} (Y_{.,j}' V^{-1} Y_{.,j+1}) =$$

$$= \sum_{i=0}^{n-j-1} (\sigma_j^{-2} Y_{i,j} . Y_{i,j}^{-1} . Y_{i,j})^{-1} . \sum_{i=0}^{n-j-1} \sigma_j^2 . Y_{i,j} Y_{i,j}^{-1} Y_{i,j+1}.$$

It is now easy to obtain (2.3.4), using notation $Y_{.,j} = (Y_{0,j}, \ldots, Y_{n-j-1,j})$ $Y_{.,j+1} = (Y_{0,j+1}, \ldots, Y_{n-j-1,j+1})$ and $V = \mathrm{Var}(\varepsilon_{.,j}) = \sigma_j^2 (Y_{.,j})$ for covariance matrix.

In this univariate case there is no need of σ_j^2 estimate for computation of $\widehat{f}_j$. However it is used for computing mean square error of the reserve. Mack (1993) suggested following straight forward estimate of variability of development factors

$$\widehat{\sigma_j^2} = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left(Y_{i,j} \left(\frac{Y_{i,j+1}}{Y_{i,j}} - \widehat{f}_j \right)^2 \right). \tag{2.3.5}$$

We can apply this formula to compute mean square error of the overall reserve what is again performed in the same article Mack (1993)

$$\mathrm{mse}(\hat{R}_i) = (\widehat{Y_{i,n}})^2 \sum_{k=n-i}^{N} \frac{\widehat{\sigma_k^2}}{\widehat{f_k}^2} \left(\frac{1}{\hat{Y}_{i,k}} + \frac{1}{\sum_{j=1}^{n-k} Y_{i,j}} \right). \tag{2.3.6}$$

Example

We can continue to work with data sets from previous example dealing with deterministic approach to Chain ladder. Its intuitive result that paid development factor could not be assumed as stable is verified through computation of standard deviation. We got following results:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_j	15,54	3,20	1,67	1,39	1,24	1,15	1,10	1,07	1,05	1,03	1,03	1,02	1,02	1,01	1,01	1,01	1,01	1,00	1,01
s_j	6501	1763	1195	856	447	333	218	164	116	98,1	122	41,4	48,1	29	30	21	38,2	10,7	

Figure 2.7: Standard deviation of development factors for Paid data

Moreover the results for Incurred scheme are not much better:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_j	2,30	1,13	1,04	1,02	1,01	1,00	1,00	1,00	1,00	0,99	0,99	0,99	0,98	1,00	0,99	0,99	0,99	1,00	1,00
s_j	1356	352	240	164	97,8	101	114	79,4	72,2	99,1	50,3	102	75,4	103	133	58,5	16,3	14,1	

Figure 2.8: Standard deviation of development factors for Incurred data

However variability of development factors coming from claims number triangle is quite low and useful for practical purposes:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_j	2,21	1,16	1,06	1,04	1,03	1,02	1,02	1,01	1,01	1,00	1,00	1,02	1,00	1,00	1,00	1,00	1,00	1,00	1,00
s_j	6,3	1,38	0,47	0,51	0,47	0,31	0,31	0,24	0,14	0,05	0,09	0,22	0,04	0,03	0,03	0,06	0,03	0	

Figure 2.9: Standard deviation of development factors for count data

Chapter 3

Recent Development in claims reserving

3.1 Munich Chain Ladder

3.1.1 General description

We could apply now stochastic model of Chain Ladder separately to Paid and Incurred data and we would expect that the ultimates of both triangles should be comparable since after sufficiently long development all claims are paid and almost no RBNS reserve should be booked. Since it does not hold in practice, paper Quarg (2004) introduced method analysing both triangles and their interdependencies simultaneously. Bachelor thesis describing this method with some illustrative examples based on data of MTPL sector was defended in 2005, see Fikar (2005).

We remind that we use upper right indices to distinguish values and parameters of each type of triangle, e.g. $Y_{i,j}^P, f_j^I, \sigma_j^P$ etc. Inter triangular dependencies are modelled via ratios of paid and incurred values

$$Q_{i,j} = (P/I)_{i,j} = \frac{Y_{i,j}^P}{Y_{i,j}^I}.$$

Average ratio for development period j is later defined as

$$q_j = (P/I)_j = \frac{\sum_{i=0}^n Y_{i,j}^P}{\sum_{i=0}^n Y_{i,j}^I}.$$

If Standard Chain Ladder method (SCL) is used instead of Munich Chain Ladder method (MCL) the problem with inconsistence exists for known data as well as for prediction. More accurately, it can be proved that if paid incurred ratio is under average in the time of estimate it will persist in the prediction for this accident year and vice versa what is claimed in the following theorem taken from Quarg (2004).

Theorem .1. *For every accident period i hold that ratio of observed and average paid to incurred ratio for last known development period and respective ratio for following prediction remain constant, or*

$$\frac{P/I_{i,j}}{P/I_j} = \frac{P/I_{i,a(i)}}{P/I_a(i)}$$

Proof. It is quite obvious that for $j > a(i)$ holds true

$$\widehat{P/I}_{i,j} = \frac{Y^P_{i,j}}{Y^I_{i,j}} = \frac{Y^P_{i,a(i)} \cdot \widehat{f^P_{a(i)}} \cdot \ldots \cdot \widehat{f^P_{j-1}}}{Y^I_{i,a(i)} \cdot \widehat{f^I_{a(i)}} \cdot \ldots \cdot \widehat{f^I_{j-1}}}$$

Moreover also

$$\widehat{f^P_s} \cdot \sum_{i=0}^{n} Y^P_{i,s} = \widehat{f^P_s} \cdot \left(\sum_{i=0}^{n-s-1} Y^P_{i,s} + \sum_{i=n-s}^{n} Y^P_{i,s}\right) \tag{3.1.1}$$

$$= \frac{\sum_{i=0}^{n-s-1} Y^P_{i,s+1}}{\sum_{i=0}^{n-s-1} Y_{i,s+1}} \cdot \sum_{i=0}^{n-s-1} Y^P_{i,s} + \sum_{i=n-s}^{n} \widehat{f^P_s} Y^P_{i,s} \tag{3.1.2}$$

$$= \sum_{i=0}^{n-s-1} Y^P_{i,s+1} + \sum_{i=n-s}^{n} Y^P_{i,s+1} \tag{3.1.3}$$

Thus it is possible to extend the sum that is used for estimates of development factors onto

$$f^P_s = \frac{\sum_{i=0}^{n} Y^P_{i,s+1}}{\sum_{i=0}^{n} Y^P_{i,s}} \qquad f^I_s = \frac{\sum_{i=0}^{n} Y^I_{i,s+1}}{\sum_{i=0}^{n} Y^I_{i,s}}$$

For simplicity of notation we do not distinguish between estimates and observed data. The right sort of data is implied by the value of calendar period.

If we apply results presented above onto (3.1.1) we will get

$$(P/I)_{i,j} = \frac{Y^P_{i,a(i)} \cdot \frac{\sum_{i=0}^n Y^P_{i,j}}{\sum_{i=0}^n Y^P_{i,a(i)}}}{Y^I_{i,a(i)} \cdot \frac{\sum_{i=0}^n Y^I_{i,j}}{\sum_{i=0}^n Y^I_{i,a(i)}}}$$

This result is equivalent with following

$$\frac{P/I_{i,j}}{P/I_j} = \frac{P/I_{i,a(i)}}{P/I_a(i)}$$

what proves the theorem.

□

MCL solves this problem very elegantly adjusting the developments factors. This adjustment is based on thought that if current paid to incurred ratio is low (i.e. below average) it means that it is not paid enough or is reserved more than usually comparing to another accident years. So it is expected that the amount of payments will be increased in future period which implies that the corresponding paid development factor should be increased and corresponding incurred factor should be lower than usual. If oppositely paid and incurred ratio is above average it may be interpreted that the future payment will be lower or increase of incurred will be substantially higher.

These types of dependencies are modelled for all development period after standardisation. Thus we use residual values with mean 0 and standard deviation 1 since

$$\mathrm{Res}(X|C) = \frac{X - \mathrm{E}(X|C)}{\sigma(X|C)}.$$

We formulate two regression models which finally produce following estimates of development factors.

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y^P_i(s)\right)|\mathbf{B}_i(s)\right) = \lambda^P \cdot \mathrm{Res}(Q^{-1}_{i,s}|Y^P_i(j))$$

and

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|Y^I_i(s)\right)|\mathbf{B}_i(s)\right) = \lambda^I \cdot \mathrm{Res}(Q_{i,s}|Y^I_i(j)).$$

It was switched from paid incurred ratio $Q_{i,s}$ to incurred paid ratio $Q^{-1}_{i,s}$ in order to obtain positive correlation in both cases. $\mathbf{B}_i(s)$ notifies two dimensional process $(Y_i(s)^P, Y_i(s)^I)$ of both data types in the time of reserve estimates.

$$\mathrm{E}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|\mathbf{B}_i(s)\right) = f^P_s + \lambda^P \frac{\sigma\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y_i(s)^P\right)}{\sigma(Q^{-1}_{i,s}|Y_i(s)^P)} \cdot (Q^{-1}_{i,s} - E(Q^{-1}_{i,s}|Y_i(s)^P)) \tag{3.1.4}$$

resp.

$$\mathrm{E}\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|\mathbf{B}_i(s)\right) = f^I_s + \lambda^I \frac{\sigma\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|Y_i(s)^I\right)}{\sigma(Q^{-1}_{i,s}|Y_i(s)^I)} \cdot (Q_{i,s} - E(Q_{i,s}|Y_i(s)^I)).$$

Moreover we assume that vectors $\mathbf{B}_{i_1}(s)$ and $\mathbf{B}_{i_2}(s)$ are stochastically independent if $i_1 \neq i_2$. Let us assume that $Q_{i,j}$ is defined as $\frac{Y^P_{i,j}}{Y^I_{i,j}}$. Parameters λ^P and λ^I determine then the adjustment of SCL development factors.

For practical implementation we have to obtain further estimates of $\sigma(Q^{-1}_{i,s}|Y_i(s)^I)$, $\sigma(Q_{i,s}|Y_i(s)^I)$ and $\sigma(Q^{-1}_{i,s}|Y_i(s)^P)$. Estimate of $E(Q_{i,s}|Y_i(s)^I)$ is formulated as

$$\widehat{q_s} = \sum_{i=0}^{n-s} Y^P_{i,s} / \sum_{i=0}^{n-s} Y^I_{i,s}$$

Estimate of variability of paid incurred ratio $\sigma(Q_{i,s}|Y_i(s)^I)$ is suggested as follows

$$\widehat{\rho^I_s} / \sqrt{Y^I_{i,s}}$$

using

$$\left(\widehat{\rho^I_s}\right)^2 = \frac{1}{n-s} \sum_{i=0}^{n-s} Y^I_{i,s} \cdot (Q_{i,s} - \widehat{q_s})^2.$$

In the same way we can obtain that

$$\widehat{q_s}^{-1} = \sum_{i=0}^{n-s} Y^I_{i,s} / \sum_{i=0}^{n-s} Y^P_{i,s}$$

estimates $E(Q^{-1}_{i,s}|Y_i(s)^P)$ and also

$$\widehat{\rho^P_s} / \sqrt{Y^P_{i,s}}$$

is estimate of $\sigma(Q^{-1}_{i,s}|Y^P_{i,s})$ using

$$\left(\widehat{\rho^P_s}\right)^2 = \frac{1}{n-s} \sum_{i=0}^{n-s} Y^P_{i,s} \cdot (Q^{-1}_{i,s} - \widehat{q_s}^{-1})^2.$$

3.1.2 Remarks to MCL

Estimate of regression parameters λ^P and λ^I was originally in the article Quarg (2004) obtained by ordinary least square method (OLS). If one changes theoretical values by above presented estimates the final projection could be easily obtained.

Despite the undoubtable benefits of MCL there are some open questions in that field. Some of them will be suggested to solve later in that thesis.

1. The underlying regression models for Paid (see formula 3.1.4) and Incurred data are regarded in practice as rather volatile. It could imply the question if the OLS method is appropriate for the data or even formulated model based on the Paid to Incurred ratios is the most proper one.

2. From practical point of view the information regarding the known value of reserves is useful for amount of payments in future periods but it does not have to be valid that so far paid amounts are useful to predict future development of incurred. That idea was mentioned by Verdier and Klinger (2005). Moreover it could be more more appropriate to use the value of reserve only as relevant information for Paid projection instead of whole incurred since in fact already paid amount, that is part of incurred amount, gives us no more information beyond standard chain lader model.

3. The consequences of the problem if the run-off is not ended after n period after claims' occurrence was mentioned in Quarg (2004). If we assume that outstanding reserve is set up adequately after n periods of development one could increase Paid value in upper right cell of triangle to match the paid and incurred data in that position and transformed value of $Y^P_{0,n}$ is to be interpreted as final payment for accident year 0. However in some examples of data with significant reserve development the run-off reserve model should be also mentioned in order to implement tail as well.

3.1.3 Example

If we continue with data from previous examples regarding the chain ladder we get following results for dependency of residuals in the case of paid scheme:

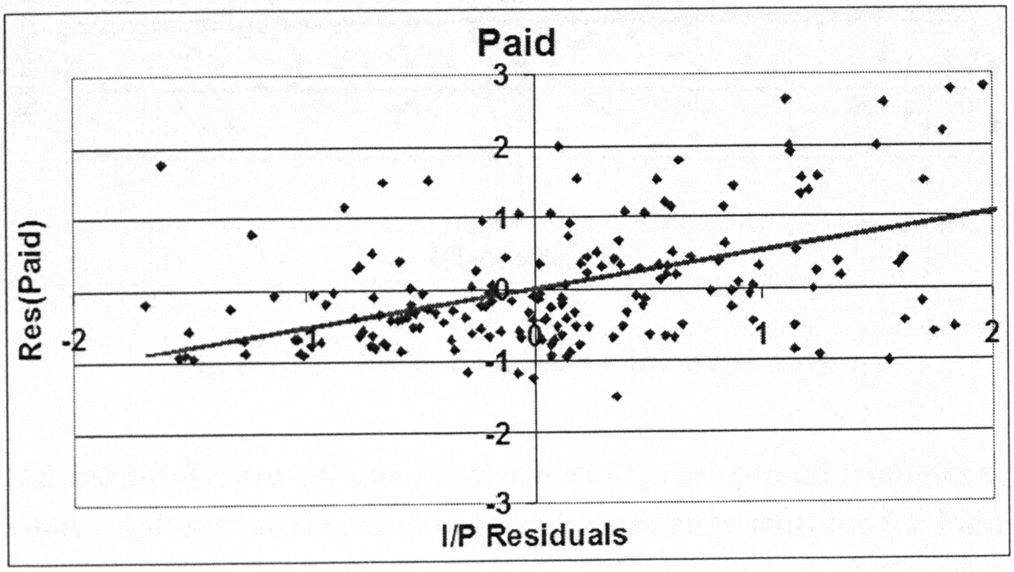

Figure 3.1: MCL Residuals - Paid data

In that situation sample correlation among standardised paid compensation and ratio I/P is approximately 0.5. On the other hand situation with correlation of incurred residuals and ratio P/I is much weaker, sample correlation is 0.08 only with following graphical demonstration:

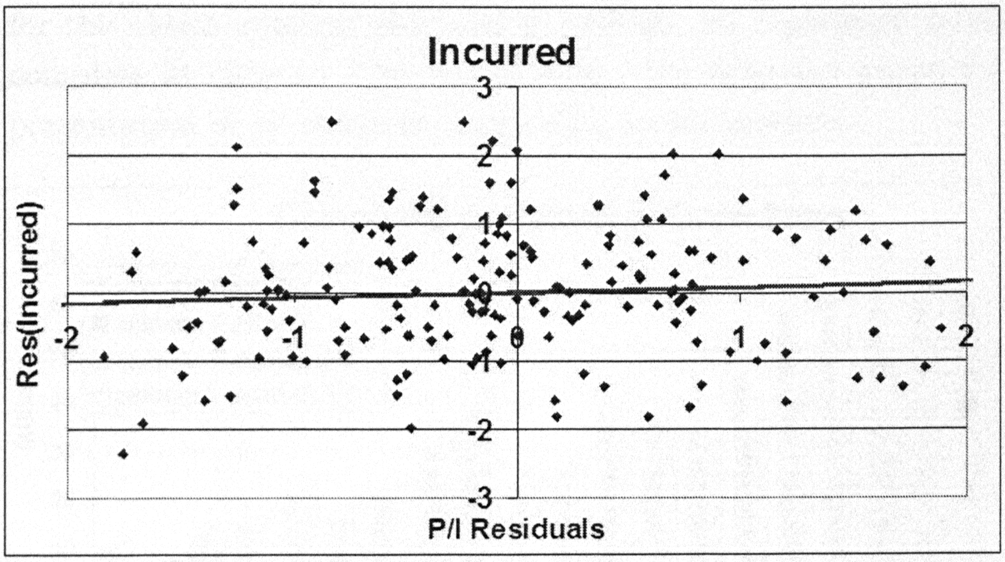

Figure 3.2: MCL Residuals - Incurred data

Similar results are obtained in the most of other run-off triangles schemes what implies that this kind of modelling is more suitable for Paid compensation than for Incurred process. Following graph shows the difference in the ratios of final prediction based on Paid or Incurred schemes depending on the type of method (SCL or MCL).

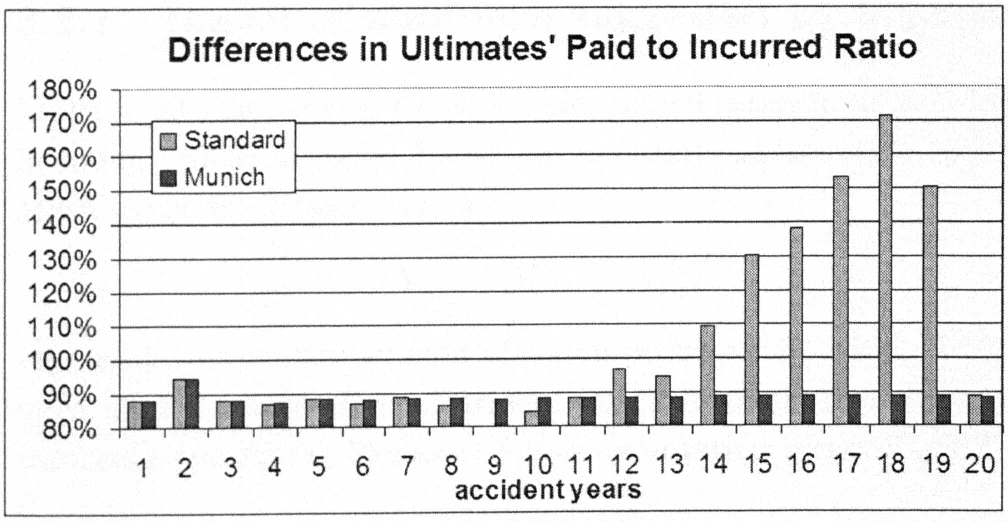

Figure 3.3: Paid to Incurred Ratios

The respective projection are thus much closer if we use MCL. However the fit is not completely done since the paid process is not finished even for the oldest accident year and so we have no experience to observe complete fit close to 1 for known data. See following graph for final presentation of all estimates depending on the method.

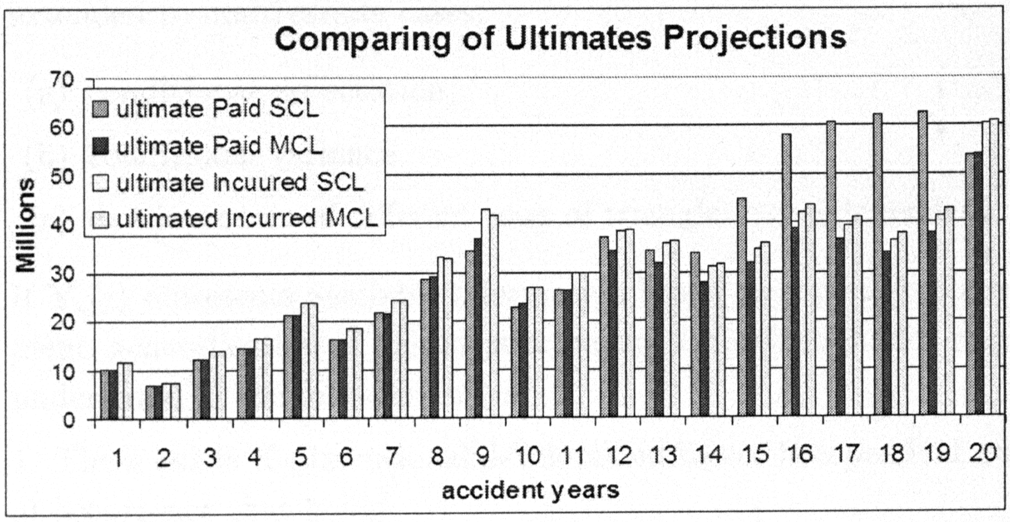

Figure 3.4: Comparing of ultimates

3.2 Multivariate Chain Ladder

3.2.1 Recall of approach suggested by Schmidt

Multivariate analogy of Chain Ladder model introduced in Prohl and Schmidt (2005) is again based on stochastic assumption of original Mack's model. Column vector

$$\mathbf{Y}_{i,j} = (Y_{i,j}^1, \ldots, Y_{i,j}^K)'$$

represents cumulative amount of claims occurred in period i and developed after j period after occurrence for all K simultaneously analysed insurance portfolios. Moreover following notation was also used

$$\Upsilon_{i,j} = \mathrm{diag}(\mathbf{Y_{i,j}})$$

Obviously $\mathbf{Y_{i,j}} = \Upsilon_{i,j}\mathbf{1}$, where $\mathbf{1}$ marks union vector of dimension K. Generalisation of one-dimensional formula $Y_{i,j+1} = Y_{i,j} \cdot F_{i,j}$ is then obviously

$$\mathbf{Y}_{i,j+1} = \Upsilon_{i,j} \cdot \mathbf{F}_{i,j}$$

where $\mathbf{F}_{i,j} = (F^1_{i,j}, \ldots, F^K_{i,j})'$ represents multivariate version of individual development factor.

3 basic stochastic assumptions proposed by Mack (1993) had to be also extended to multivariate cases:

(a) conditional expectation

(b) conditional variance

(c) developments of different rows of triangles are independent

If $\mathbf{Y}_i(j)$ represents available information based on j period of development, generalisation of the assumption suggested by Schmidt might be understood in the following ways.

1. There exists K-dimensional development factor independent on year of occurrence that holds

$$\mathrm{E}\left(\mathbf{Y}_{i,j+1}|\mathbf{Y}_i(j)\right) = \Upsilon_{i,j} \cdot \mathbf{f}_j$$

2. There exists matrix Σ_j so that

$$\mathrm{Cov}(\mathbf{Y}_{i1,j+1}, \mathbf{Y}_{i2,j+1}|\mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = \Upsilon^{1/2}_{i,j}\Sigma_j\Upsilon^{1/2}_{i,j}$$

if $i = i_1 = i_2$ and also

$$\mathrm{Cov}(\mathbf{Y}_{i1,j+1}, \mathbf{Y}_{i2,j+1}|\mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = 0$$

otherwise.

These assumption imply that

$$\mathrm{E}\left(\mathbf{F}_{i,j}|\mathbf{Y}_i(j)\right) = \mathbf{f}_j$$

and

$$\mathrm{Cov}(\mathbf{F}_{i1,j+1}, \mathbf{F}_{i2,j+1}|\mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = \Upsilon^{-1/2}_{i,j}\Sigma_j\Upsilon^{-1/2}_{i,j},$$

that is obvious analogy of one-dimensional formulae

$$\mathrm{E}(F_{i,j}|\mathbf{Y}_i(j)) = f_j$$

and

$$\mathrm{Var}(F_{i,j}|\mathbf{Y}_i(j)) = \sigma^2_j/Y_{i,j} \qquad i = 0, \ldots, n \qquad j = 0, \ldots n-1$$

We recall that in one-dimensional case of Mack's model estimate of f_j is to be found as

$$\widehat{f}_j = \sum_{i=0}^{n-j-1} w_i F_{i,j}$$

This estimate is unbiased if $\sum_{i=0}^{n-j-1} w_i = 1$. Linear model theory implies that OLS estimate is achieved if

$$w_i = \frac{Y_{i,j}}{\sum_{i=0}^{n-j-1} Y_{i,j}}$$

That gives us univariate Chain ladder estimator.

In multivariate case Schmidt suggested estimator $\mathbf{f}_j$ as

$$\widehat{\mathbf{f}}_j = \sum_{i=0}^{n-j-1} W_i \widehat{F}_{i,j}$$

Conditionally unbiased estimate is achieved if $\sum_{i=0}^{n-j-1} W_i = I$

Estimator that minimises mean square error is derived form linear model theory as

$$\widehat{\mathbf{f}}_j = \left(\sum_{i=0}^{n-j-1} \Upsilon_{i,j}^{1/2} \Sigma_j^{-1} \Upsilon_{i,j}^{1/2} \right) \sum_{i=0}^{n-j-1} \Upsilon_{i,j}^{1/2} \Sigma_j^{-1} \Upsilon_{i,j}^{1/2} \mathbf{F}_{i,j}$$

We suppose that estimator of Σ_j is important for practical purposes as well. However its specification is not included in the mentioned paper of Schmidt and Prohl (2005).

We could use classical estimator as

$$\widehat{\Sigma_j} = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left(\Upsilon_{i,j}^{1/2} \left(\widehat{\mathbf{F}_{i,j}} - \widehat{\mathbf{f}}_j \right) \right) \cdot \left(\Upsilon_{i,j}^{1/2} \left(\widehat{\mathbf{F}_{i,j}} - \widehat{\mathbf{f}}_j \right) \right)'$$

Drawback of that approach might be seen that $\widehat{\Sigma_j}$ is not well defined if $j \geq n-k$ what implies limited benefit of that method.

3.2.2 Recall of approach suggested by Kremer

Multivariate model in the paper of Kremer (2005) is suggested as follows

$$Y_{i,j+1} = Y_{i,j}.f_j + \varepsilon_{i,j} \qquad i = 0,\dots,n$$
$$\mathrm{E}(\varepsilon_{i,j}|\cdot) = 0 \qquad \mathrm{Var}(\varepsilon_{i,j}|\cdot) = \sigma_j^2.Y_{i,j}.$$

Thus it is assumed that $\forall j$ holds

$$Y_{i,j+1}^k = Y_{i,j}^k.f_j^k + \varepsilon_{i,j}^k \qquad i = 0,\dots,n \qquad k = 1,\dots,K$$

So original linear model is assumed for all of K analysed run-off triangles. Moreover it is assumed

$$\mathrm{Covr}(\varepsilon_{i,j}^{k1}, \varepsilon_{i,j}^{k2}|\cdot) = C_i^{k1,k2} \cdot \sqrt{Y_{i,j}^{k1}} \cdot \sqrt{Y_{i,j}^{k2}}$$

and

$$\mathrm{Var}(\varepsilon_{i,j}^k|\cdot) = \sigma_j^{k,2}.$$

If $i_1 \neq i_2$ or $j_1 \neq j_2$ then residuals are assumed to be uncorrelated, that is

$$\mathrm{Covr}(\varepsilon_{i1,j1}^{k1}, \varepsilon_{i2,j2}^{k2}|\cdot) = 0$$

Not only the estimate of development factor but also the estimator of variance is stressed in that approach. Estimate of $\mathbf{f_j}$ is suggested as Aitken's estimator since it corresponds to regression estimate with non-constant variance of residuals. However as is stated in Schmidt (2006) this approach could be seen as not effective enough since computation of large-dimensional inverse matrix $\widehat{\Psi}^{-1}$ might be time consuming.

In the proposed model, estimators of f_j^k are firstly calculated for each triangle separately. These estimators would be the optimal ones if $C_{i,j}^{k1,k2} = 0 \forall i, j, k_1, k_2$ For each run-off triangle $k = 1,\dots,K$ variability estimator corresponding above mentioned estimates of development factor is derived through following formulae

$$\widehat{\sigma_j^{2,k}} = \frac{\sum_{i=1}^{n-j-1}(Y_{i,j+1}^k - \widehat{f}_j^k Y_{i,j}^k)^2}{\sum_{i=1}^{n-j-1} Y_{i,j}} \qquad (3.2.1)$$

and also covariance estimator as

$$\widehat{C}_i^{k1,k2} = \frac{\sum_{i=1}^{n-j-1}(Y^{k1}_{i,j+1} - \widehat{f}^{k1}_j Y^{k1}_{i,j})(Y^{k2}_{i,j+1} - \widehat{f}^{k2}_j Y^{k2}_{i,j})}{\sum_{i=1}^{n-j-1}\sqrt{Y^{k1}_{i,j}}\sqrt{Y^{k2}_{i,j}}}$$

Note that the formula 3.2.1 is different from 2.3.5 suggested in original stochastic model for SCL.

In lth step the calculated estimators are used for updating a correlation structure that implies new estimator of development factors $\mathbf{f_j}^{l+1}$ based on inverse matrix $\widehat{\sigma_j^{2,k}}^{l}$ and $\widehat{C}_i^{k1,k2}{}_l$. This **iterative procedure** is repeated until the parameters estimates converge.

3.2.3 Example

Now we will illustrate the concept of iteration suggested by Kremer for situation of two triangle describing run-off of paid compensations:

	1	2	3	4	5	6	7	8
1	1 203 103	2 158 108	2 318 157	2 396 985	2 435 242	2 456 989	2 477 963	2 499 354
2	1 591 765	2 402 618	2 594 197	2 676 422	2 698 553	2 743 589	2 771 520	
3	1 538 127	2 352 495	2 558 737	2 657 903	2 718 632	2 772 554		
4	1 406 971	2 103 387	2 260 689	2 344 572	2 388 100			
5	1 422 361	2 179 732	2 365 049	2 434 525				
6	1 504 316	2 256 844	2 396 263					
7	1 655 792	2 326 276						
8	1 605 873							

Figure 3.5: Paid triangle portfolio 1

and also

	1	2	3	4	5	6	7	8
1	1 203 103	2 158 108	2 318 157	2 396 985	2 435 242	2 456 989	2 477 963	2 499 354
2	1 591 765	2 402 618	2 594 197	2 676 422	2 698 553	2 743 589	2 771 520	
3	1 538 127	2 352 495	2 558 737	2 657 903	2 718 632	2 772 554		
4	1 406 971	2 103 387	2 260 689	2 344 572	2 388 100			
5	1 422 361	2 179 732	2 365 049	2 434 525				
6	1 504 316	2 256 844	2 396 263					
7	1 655 792	2 326 276						
8	1 605 873							

Figure 3.6: Paid triangle portfolio 2

If we apply two separated univariate SCL computations to these triangles we would obtain for example $f_0^{(1)} = 1.52866$ and comparing of standard deviation depending on method (3.2.1 or 2.3.5) is presented in the tables and graph below

Differences in standard deviation estimates - Paid data k =1						
development period	0	1	2	3	4	5
Mack	138,95054	13,9695441	5,99102295	9,98160553	8,9406618	1,871301
Kremer	119,970602	12,8265038	5,3910063	8,87672808	7,18964907	1,321199

Figure 3.7: Differences in variance estimates 1

Differences in standard deviation estimates - Paid data k=2						
development period	0	1	2	3	4	5
Mack	62,78267	17,22121	4,75921	3,681599	5,286392	6,816385
Kremer	57,34894	16,59576	4,13301	3,072739	4,258274	4,79914

Figure 3.8: Differences in variance estimates 2

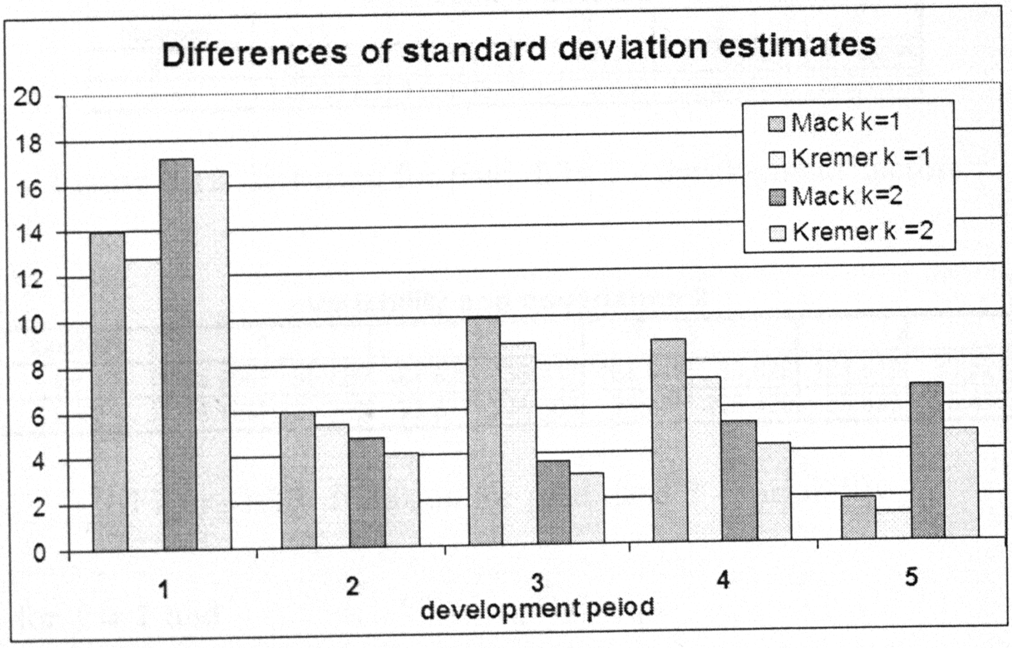

Figure 3.9: Differences in variance estimates 3

Application of iteration procedure gives following sequence of estimates for $\mathbf{f}_0$

development factors 1				
iteration:	1	2	3	4
k =1	1,52865678	1,51307772	1,51307672	1,51307671
k =2	1,50441857	1,4888684	1,48887377	1,48887379

Figure 3.10: Iteration for paid data 1 - development factors

and respective variability structure

variability and covariance 1						
iteration:	1		2		3	
k= 1	14 410,971	174,538	14 392,923	174,858	14 392,945	174,859
k =2	174,538	3 302,180	174,858	3 288,967	174,859	3 288,901

Figure 3.11: Iteration for paid data 1 - variability

Similar iterations were performed for other development periods resulting in

development factors 2					
iteration:	1	2	3	4	5
k =1	1,07729843	1,07744324	1,077435	1,07743495	1,07743495
k =2	1,07731527	1,07497313	1,07500419	1,07500439	1,07500439

Figure 3.12: Iteration for paid data 2 - development factors

variability and covariance 2								
iteration:	1		2		3		4	
k= 1	164,696	35,725	164,520	33,817	164,519	33,828	164,519	33,829
k =2	35,725	325,329	33,817	275,505	33,828	275,420	33,829	275,419

Figure 3.13: Iteration for paid data 2 - variability

for $j = 1$ and

development factors 3			
iteration:	1	2	3
k =1	1,034189	1,0341917	1,0341917
k =2	1,0312292	1,0311109	1,0311109

Figure 3.14: Iteration for paid data 3 - development factors

variability and covariance 3				
iteration:	1		2	
k= 1	29,063	-0,183	29,063	-0,188
k =2	-0,183	17,084	-0,188	17,082

Figure 3.15: Iteration for paid data 3 - variability

if $j = 2$. Iterations were not performed if $j \geq 3$ due to lack of observation.

The presented example quite well demonstrated that the iteration in that situation makes sense and that the computation is not so demanding since the requested number of iteration is very small.

Chapter 4

Univariate Munich Chain Ladder

Based on the evaluation in previous chapters we present some proposals how to solve possible drawbacks of MCL.

4.1 Methods how to estimate the slope parameters λ in MCL

In our opinion the proposed OLS method for estimating slope parameters λ^P and λ^P for all data is not the most suitable as was mentioned previously in Verdier and Klinger (2005) who suggested implementation of different mean and slope parameters of the model depending on development periods what on the other hand contradict the parsimony of the model stressed by Quarg and Mack (2004). In our approach we will try not to change the general construction of the model 3.1.4 but we will adjust the value of the slope parameters by omitting the outliers which may occur in this kind of situation generally across all development periods, see also Jedlička (2006).

We try to compare original ordinary least squares estimates of λ parameters with estimates obtained by some robust methods. We decided to use Huber's robust regression approach, bi square methods and Least trimmed squares (LTS) methods. Generally speaking the first two methods evaluate each observation and the outliers "receive" lower weight. Apart from this approach LTS method directly cuts off the outlying observation which does not correspond with probabilistic model. Differences between LTS1 and LTS2 are based on numbers of observations that are assumed not to contradict the model. It is about 60% in first situation and 75% approximately in the latter case.

LTS estimator or regression model parameters (see Cizek (2001) for more details) is generally defined as

$$\hat{\beta}^{LTS} = \arg \min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r^2_{[i]}(\beta),$$

where $r^2_{[i]}(\beta)$ represents i-th smallest value among $r^2_1(\beta), \ldots, r^2_n(\beta)$ and $r_i(\beta) = y_i - x_i^{'}\beta$, represents thus OLS residuals. It is important to specify how to select the value of trimming constant h. Generally holds $\frac{n}{2} < h \leq n$ that agrees with our assumption that 75% and 60% data does not contradict the model.

Even in the motivation example presented in Quarg (2004) could be seen significant difference between regression projection using OLS and LTS method in case of Paid data, see following graphs.

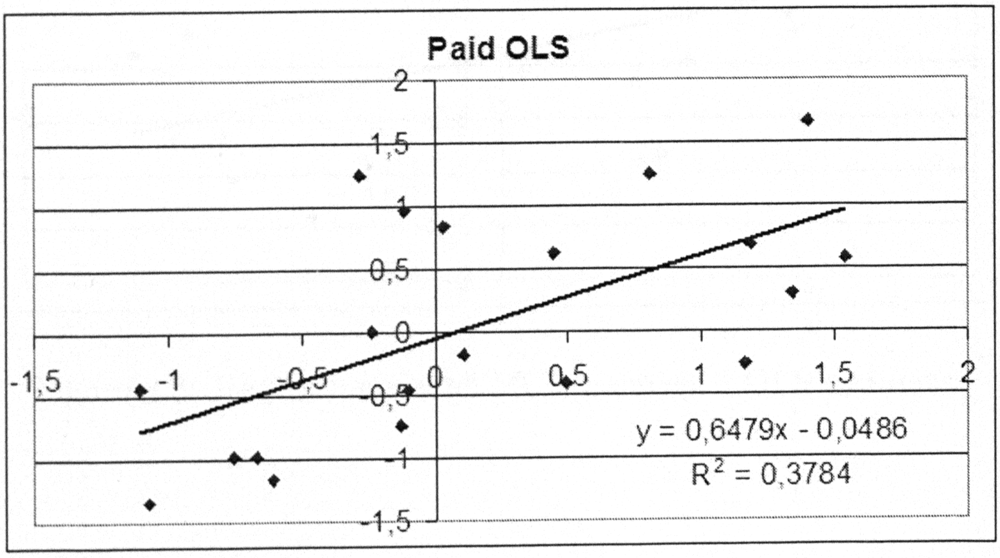

Figure 4.1: Regression model MCL Paid - OLS estimates

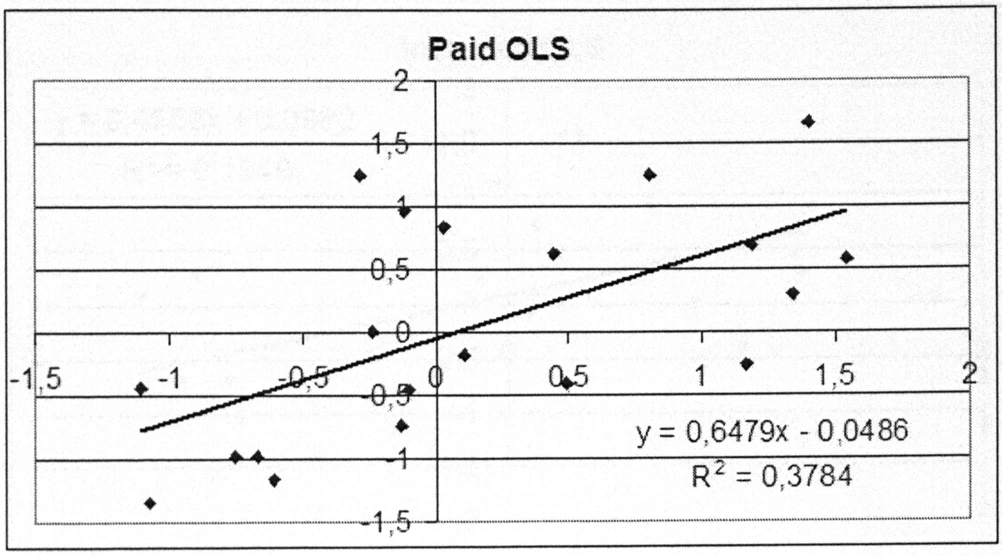

Figure 4.2: Regression model MCL Paid - LTS estimates

However as stated in the graph below no significant difference is presented in case of Incurred data in this specific situation:

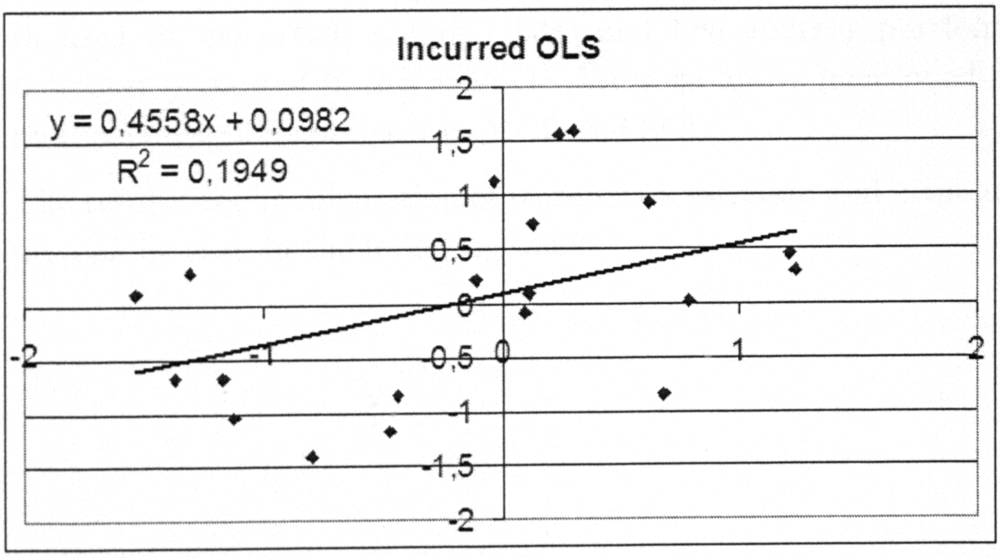

Figure 4.3: Regression model MCL Incurred - OLS estimates

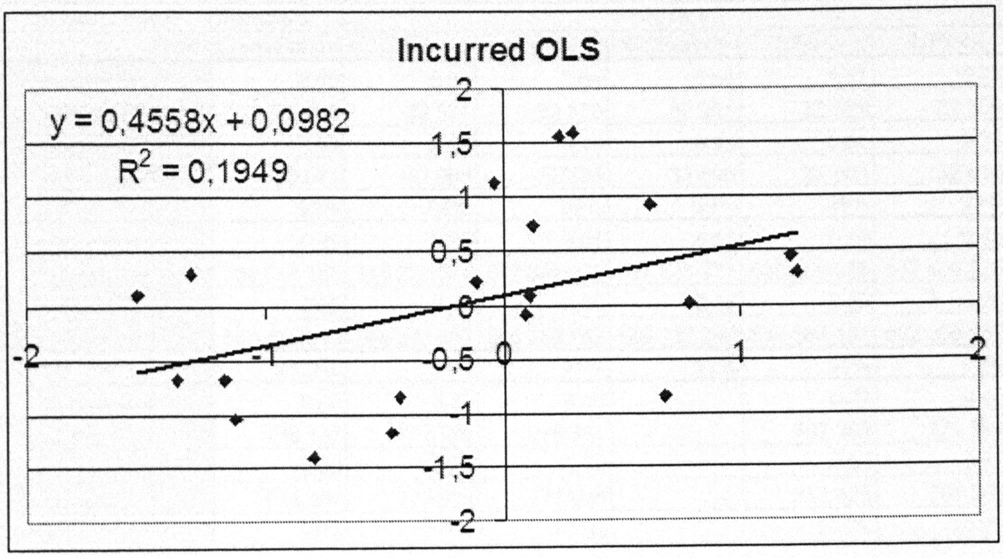

Figure 4.4: Regression model MCL Incurred - LTS estimates

Numerical Example

Parameter estimates of three different portfolio including original data used in the article Quarg (2004) and two another portfolios are moreover presented in this example. We used above mentioned robust methods, original approach to MCL and SCL.

The results of our calculation of parameters estimate and ultimate values can be seen in the following table.

	Chain Ladder	Standard	Munich				
	Type of regression		OLS	Hueber	Bi Square	LTS 1	LTS 2
portfolio 1	parameters P	0,00	0,64	0,64	0,64	1,17	0,77
	ultimates P	31 463	32 371	32 378	32 381	32 329	32 509
	parameters I	0,00	0,44	0,43	0,43	0,55	0,32
	ultimates I	33 071	32 688	32 693	32 696	32 901	32 816
	P/I	95%	99%	99%	99%	98%	99%
portfolio 2	parameters P	0,00	0,59	0,56	0,56	0,44	0,45
	ultimates P	680 614 181	516 229 574	517 841 075	517 821 077	530 909 088	532 450 325
	parameters I	0,00	0,14	0,16	0,15	0,25	0,28
	ultimates I	553 855 313	564 472 802	566 233 742	566 237 793	580 567 131	582 096 967
	P/I	123%	91%	91%	91%	91%	91%
portfolio 3	parameters P	0,00	0,39	0,40		0,22	0,40
	ultimates P	746 137	779 799	778 303		767 383	771 394
	parameters I	0,00	0,47	0,52		0,55	0,71
	ultimates I	816 194	773 813	771 926		762 426	764 263
	P/I	91%	101%	101%		101%	101%

Figure 4.5: Results of MCL based on robust regression

4.2 Elasticity of reserve

The question of interpretation of differences in the ultimate projection depending on applied regression estimate (as shown in the previous example) leads us to further sensitivity study of relationship between final projection and parameter estimate values. The derivation will be performed only for Paid data as the principles for Incurred are analogous.

We started from formula (3.1.4) to define estimate of development factor used in reserve calculation as

$$\widehat{f^P_{i,k}} = \widehat{f^P_k} + \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_k}}{\widehat{\rho^P_k}} \left(\widehat{Q_{i,k}}^{-1} - \widehat{q_k}^{-1} \right).$$

It is straightforward that ultimate value of paid amount due to claims occurred in accident period i is calculated as $\widehat{Y^P_{i,n}} = Y^P_{i,a(i)} \cdot \prod_{j=a(i)}^{n-1} \widehat{f_{i,j}}^P$ using notation $a(i) = n - i$.

If we inspect the value of paid ultimate estimate $\widehat{Y_{i,n}}$ as a function of $\widehat{\lambda^P}$ we can derive how strongly the ultimate values (and thus also reserve since reserve differs only by a known diagonal value) are affected by the choice of appropriate estimate of λ. We can write (all derivative are understood with respect to $\widehat{\lambda^P}$):

$$(\widehat{Y_{i,n}^P})' = \sum_{j=a(i)}^{n-1} \frac{Y_{i,a(i)}^P}{\widehat{f_{i,j}^P}} \cdot (\widehat{f_{i,j}^P})' \cdot \widehat{f_{i,a(i)}^P} \cdots \widehat{f_{i,n-1}^P} = \widehat{Y_{i,n}^P} \sum_{j=a(i)}^{n-1} \frac{\widehat{f_{i,j}^P}'}{\widehat{f_{i,j}^P}}.$$

Using formula $\widehat{f_{i,k}^P} = \widehat{f_k^P} + \widehat{\lambda^P} \cdot (\widehat{f_{i,k}^P})'$ we can make final adjustment of the above mentioned formula

$$\frac{(\widehat{Y_{i,n}^P})'}{\widehat{Y_{i,n}^P}} = \frac{1}{\widehat{\lambda^P}} \cdot \left[\sum_{j=a(i)}^{n-1} (1 - \frac{\widehat{f_j^P}}{\widehat{f_{i,j}^P}}) \right].$$

We further derived rather surprising result that $\mathrm{E}\left(\frac{(\widehat{Y_{i,n}^P})'}{\widehat{Y_{i,n}^P}} | \mathbf{B}_i(a(i)) \right) = 0$ if the expectation exists. That could be interpreted that there is no systematical influence of varying the regression estimates onto the ultimates values. It is rational that we do not see regression estimates as random variable since we are interested in the sensitivity only. It is easy to prove that $E((\widehat{f_{i,s}^P})' | \mathbf{B}_i(s), \widehat{\lambda^P}) = 0$ since the model assumptions imply that $E(Q_{i,s} | \mathbf{B}_i(s), \widehat{\lambda^P}) = q_s$ independently on accident period i. Using again formula $\widehat{f_{i,k}^P} = \widehat{f_k^P} + \widehat{\lambda^P} \cdot (\widehat{f_{i,k}^P})'$ we get $\mathrm{E}(\widehat{f_{i,k}^P} | \mathbf{B}_i(k), \widehat{\lambda^P}) = E(\widehat{f_k^P})$ Provided that both expectations exist we later obtain

$$\mathrm{E}\left(\frac{\widehat{f_{i,k}^P}}{\widehat{f_k^P}} | \mathbf{B}_i(k), \widehat{\lambda^P} \right) = \mathrm{E}\left(\frac{\widehat{f_k^P} + \widehat{\lambda^P} \widehat{(f_{i,k}^P)'}}{\widehat{f_k^P}} | \mathbf{B}_i(k), \widehat{\lambda^P} \right) =$$

$$= 1 + \widehat{\lambda^P} \mathrm{E}\left(\frac{\widehat{(f_{i,k}^P)'}}{\widehat{f_k^P}} | \mathbf{B}_i(k), \widehat{\lambda^P} \right) = 1.$$

This proves the formula $\mathrm{E}\left(\frac{(\widehat{Y_{i,n}^P})'}{\widehat{Y_{i,n}^P}} | \mathbf{B}_i(a(i)) \right) = 0.$

4.3 Variability and MSE calculation

Munich Chain Ladder gave us so far only formula for $\mathrm{E}\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P} | \mathbf{B}_i(s) \right)$ or $\mathrm{E}\left(\frac{Y_{i,s+1}^I}{Y_{i,s}^I} | \mathbf{B}_i(s) \right)$ and no information about the variability of development factors. We will derive this starting from regression model of residual data. It is again sufficient to perform the derivation for paid triangle only.

The standard linear model theory implies that

$$\mathrm{Var}\left(\mathrm{Res}\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|Y_i^P(s)\right)|\mathbf{B}_i(s)\right) = \frac{\sigma_R^2\cdot \mathrm{Res}^2\left(\frac{Y_{i,s}^I}{Y_{i,s}^P}|Y_i^P(s)\right)}{\sum_i\sum_{j,i+j\leq n}\mathrm{Res}^2\left(\frac{Y_{i,j}^I}{Y_{i,j}^P}|Y_i^P(s)\right)}$$

$$= \mathrm{Var}(\widehat{\lambda^P})\cdot \mathrm{Res}^2\left(\frac{Y_{i,s}^I}{Y_{i,s}^P}|Y_i^P(s)\right).$$

It is only special case of fact that in standard regression model $Y = X\beta+\varepsilon$ holds $\mathrm{Var}(\widehat{Y}) = \sigma^2 X(X'X)^{-1}X'$.

Rearranging this formula we obtain

$$\mathrm{Var}\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|\mathbf{B}_i(s)\right) = \mathrm{Var}(\widehat{\lambda^P})\cdot\sigma^2\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|Y_i^P(s)\right)\cdot\mathrm{Res}^2(Y_{i,s}^I/Y_{i,s}^P|Y_i(s)).$$

This may be made in very similar way as the shift between formula

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|Y_i^P(s)\right)|\mathbf{B}_i(s)\right) = \lambda^P\cdot\mathrm{Res}(Q_{i,s}^{-1}|Y_i^P(j))$$

and the consecutive one

$$\mathrm{E}\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|\mathbf{B}_i(s)\right) = f_s^P + \lambda^P\frac{\sigma\left(\frac{Y_{i,s+1}^P}{Y_{i,s}^P}|Y_i(s)^P\right)}{\sigma(Q_{i,s}^{-1}|Y_i(s)^P)}\cdot(Q_{i,s}^{-1} - E(Q_{i,s}^{-1}|Y_i(s)^P))$$

in case of conditional expectation. Actually in both cases one uses only fact that $\mathrm{Res}(\cdot|\mathrm{Y_I^P(S)})|B_i(s) = \mathrm{Res}(\cdot|\mathrm{B_i(s)})$.

It is straightforward to substitute the theoretical parameters by their estimates similarly as in formula for expectation and achieving that

$$\widehat{\sigma_{i,s}^{P,MCL_2}} = \mathrm{Var}(\widehat{\lambda^P})\cdot\widehat{\sigma_s^{P,SCL_2}}\cdot\mathrm{Res}^2\left(\widehat{\frac{Y_{i,s}^I}{Y_{i,s}^P}|Y_i(s))}\right)$$

This potentially enables us to calculate the mean square error for Munich Chain Ladder similarly as for Standard Chain Ladder where holds, see Mack (1993).

$$\text{mse}(\hat{R}_i) = \text{E}(R_i - \widehat{R_i}|\mathbf{Y}_i(j))^2 = \widehat{Y_{i,n}^2} \sum_{k=n-i}^{n} \frac{\widehat{\sigma_k^2}}{\widehat{f_k}^2} \left(\frac{1}{\hat{Y}_{i,k}} + \frac{1}{\sum_{j=1}^{n-k} Y_{i,j}} \right)$$

if we substitute factors of SCL by corresponding factors of MCL we will obtain following formula for mean square error of Paid data

$$\text{mse}(\hat{R}_i) = \text{E}(R_i - \widehat{R_i}|\mathbf{B}_i(j))^2 = \widehat{Y_{i,n}^{P\,2}} \sum_{k=n-i}^{n} \frac{\widehat{\sigma_{i,k}^{P\,2}}}{\widehat{f_{i,k}^P}^2} \left(\frac{1}{\hat{Y}_{i,k}^P} + \frac{1}{\sum_{j=1}^{n-k} Y_{i,j}^P} \right)$$

4.3.1 Example

In this example we will apply Munich Chain Ladder method on this set of Paid and Incurred schemes:

5 839	12 289	16 343	19 622	22 616	24 891	27 482	30 138	33 775	34 902	36 986	47 457
6 721	15 461	20 071	24 408	28 027	31 321	34 920	38 515	41 202	43 373	45 781	
7 067	15 449	20 300	23 864	27 674	30 676	37 419	41 497	44 058	47 227		
7 673	17 099	22 673	27 484	31 377	35 654	38 565	42 784	45 861			
7 006	15 019	20 674	25 019	29 424	33 857	37 984	42 950				
7 002	16 253	21 886	26 197	30 425	35 691	40 063					
7 135	14 873	19 176	23 712	27 571	31 858						
6 985	15 076	20 734	24 855	29 371							
6 625	14 370	18 612	22 504								
6 635	15 242	20 263									
7 506	15 673										
7 421											

Figure 4.6: MSE calculation - Paid data

5 839	12 289	16 343	19 622	22 616	24 891	27 482	30 138	33 775	34 902	36 986	47 457
6 721	15 461	20 071	24 408	28 027	31 321	34 920	38 515	41 202	43 373	45 781	
7 067	15 449	20 300	23 864	27 674	30 676	37 419	41 497	44 058	47 227		
7 673	17 099	22 673	27 484	31 377	35 654	38 565	42 784	45 861			
7 006	15 019	20 674	25 019	29 424	33 857	37 984	42 950				
7 002	16 253	21 886	26 197	30 425	35 691	40 063					
7 135	14 873	19 176	23 712	27 571	31 858						
6 985	15 076	20 734	24 855	29 371							
6 625	14 370	18 612	22 504								
6 635	15 242	20 263									
7 506	15 673										
7 421											

Figure 4.7: MSE calculation - Incurred data

The results comparing Mean Square Error based on MCL with that obtained from SCL shows us table and graph below.

		SCL				MCL			
year of origin	diagonal values	ultimate projection	value of reserve	MSE^0,5	MSE %	ultimate projection	value of reserve	MSE^0,5	MSE %
1	47 457	47 457	0	0		47 457	0	0	
2	45 781	58 742	12 961	947	7,3%	58 736	12 955	88	0,7%
3	47 227	64 075	16 848	1 034	6,1%	64 068	16 841	2	0,0%
4	45 861	65 602	19 741	1 668	8,4%	66 219	20 358	202	1,0%
5	42 950	66 243	23 293	2 334	10,0%	66 159	23 209	24	0,1%
6	40 063	68 627	28 564	2 526	8,8%	67 175	27 112	115	0,4%
7	31 858	61 488	29 630	3 649	12,3%	60 936	29 078	56	0,2%
8	29 371	64 405	35 034	4 067	11,6%	67 810	38 439	401	1,0%
9	22 504	57 267	34 763	3 865	11,1%	61 105	38 601	136	0,4%
10	20 263	62 124	41 861	4 155	9,9%	68 299	48 036	222	0,5%
11	15 673	63 885	48 212	4 471	9,3%	73 443	57 770	427	0,7%
12	7 421	66 221	58 800	5 275	9,0%	76 686	69 265	795	1,1%

Figure 4.8: MSE calculation - table of results

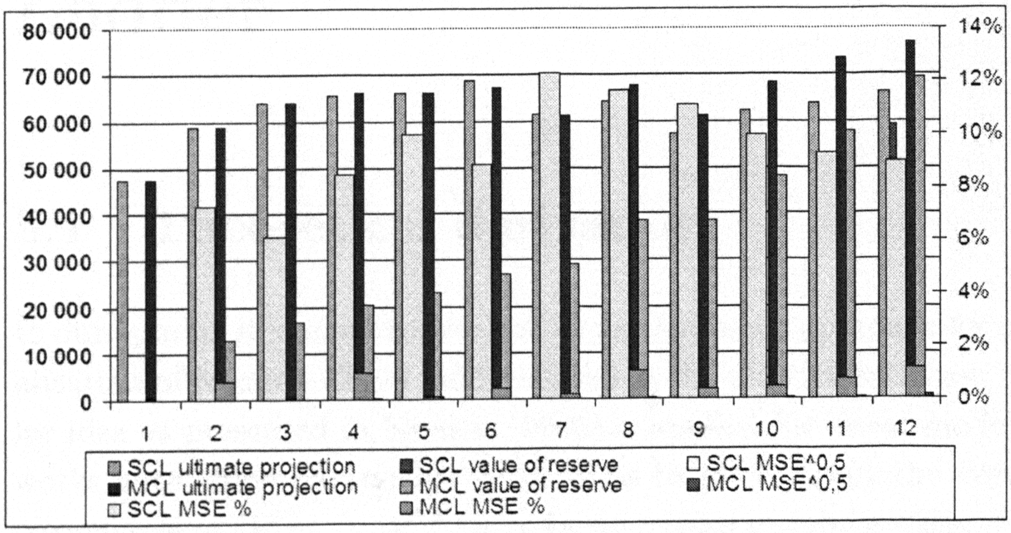

Figure 4.9: MSE calculation - chart of results

So extended information from both schemes lead to significant decrease of variability.

Chapter 5

Multivariate Munich Chain Ladder

5.1 Theoretical derivation

In our opinion it is more convenient to use Kremer's approach for generalisation of Munich Chain ladder model in the multivariate case. Similar idea as presented in Kremer (2005) is applied for linear model that works with slope parameters λ^P a λ^I as in MCL. Thus the vector of parameters of $(\lambda^{P,1}, \ldots, \lambda^{P,K})$ is to be estimated simultaneously if MCL model assumption holds for all triangles $k = 1, \ldots, K$

$$\text{Res}\left(\frac{Y_{i,s+1}^{P,k}}{Y_{i,s}^{P,k}}|Y_i(s)^{P,k}\right)|B_i(s)^k = \lambda^{P,k}\cdot\text{Res}((Q_{i,s}^k)^{-1}|Y_i(s)^P)+(\varepsilon_{i,j}^k|Y_i(s)^{P,k})$$

In univariate case it is assumed

$$\text{E}(\varepsilon_{i,j}|\cdot) = 0$$

and

$$\text{Var}(\varepsilon_{i,j}|\cdot) = \sigma^2$$

This could be extended into multivariate model as follows

$$\text{Covr}(\varepsilon_{i1,j1}^{k1}, \varepsilon_{i2,j2}^{k2}|\cdot) = 0$$

if $i_1 \neq i_2$ and

$$\mathrm{Covr}(\varepsilon^{k1}_{i,j1}, \varepsilon^{k2}_{i,j2}|\cdot) = 0$$

if $j_1 \neq j_2$ and for equal occurrence and development periods

$$\mathrm{Covr}(\varepsilon^{k1}_{i,j}, \varepsilon^{k2}_{i,j}|\cdot) = \sigma_{k1,k2}$$

and moreover we will mark

$$\sigma_{k,k} = \sigma^2_k$$

In more details we could specify multivariate version of MCL via following linear model of regression equations.

$$\begin{pmatrix} \mathbf{Y}^{P,1} \\ \mathbf{Y}^{P,2} \\ \vdots \\ \mathbf{Y}^{P,K} \end{pmatrix} = \begin{pmatrix} \mathbf{X}^{P,1} & & & \\ & \mathbf{X}^{P,2} & & \\ & & \ddots & \\ & & & \mathbf{X}^{P,K} \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} + \begin{pmatrix} \varepsilon^{P,1} \\ \varepsilon^{P,2} \\ \vdots \\ \varepsilon^{P,K} \end{pmatrix}$$

we use obvious notation

$$\mathbf{Y}^{P,k} = \begin{pmatrix} \mathrm{Res}\left(\frac{Y^{P,k}_{0,1}}{Y^{I,k}_{0,0}}|\cdot\right) \\ \mathrm{Res}\left(\frac{Y^{P,k}_{0,2}}{Y^{I,k}_{0,0}}|\cdot\right) \\ \vdots \\ \mathrm{Res}\left(\frac{Y^{P,k}_{n-1,1}}{Y^{I,k}_{n-1,0}}|\cdot\right) \end{pmatrix}$$

for response variable of the k-th model of development factors MCL of Paid data where corresponding explanatory variable is

$$\mathbf{X}^{P,k} = \begin{pmatrix} \mathrm{Res}\left(\frac{Y^{I,k}_{0,0}}{Y^{P,k}_{0,0}}|\cdot\right) \\ \mathrm{Res}\left(\frac{Y^{I,k}_{0,1}}{Y^{P,k}_{0,1}}|\cdot\right) \\ \vdots \\ \mathrm{Res}\left(\frac{Y^{I,k}_{n-1,0}}{Y^{P,k}_{n-1,0}}|\cdot\right) \end{pmatrix}$$

and also $\beta_k = \lambda^{P,k}$.

Based on above mentioned assumption of uncorrelated residuals in different periods we get

$$\mathrm{Var}\begin{pmatrix} \varepsilon^{P,1} \\ \varepsilon^{P,2} \\ \vdots \\ \varepsilon^{P,K} \end{pmatrix} = \Sigma \bigotimes I$$

Multivariate model is thus specified via set of linear regression equations and proposed procedure for practical implementation is then as follows

1. We get standard OLS estimator likewise in univariate case

$$\widehat{\lambda^{P,k}} = b_k = (\mathbf{X}^{P,k'} \cdot \mathbf{X}^{P,k})^{-1} \mathbf{X}^{P,k'} \mathbf{Y}^{P,k}$$

2. Matrix Σ is estimated using following formula

$$\widehat{\sigma_{k1,k2}} = \frac{\widehat{\varepsilon_{.,k1}} \widehat{\varepsilon_{.,k2}}}{n \cdot (n-1)/2}$$

where $\widehat{\varepsilon_{.,k1}}$ represents the vector of OLS calculated residuals of $k1$th model.

3. Estimator with non constant variance $\beta = \lambda^{\mathbf{P}}$ is derived as

$$\beta = (Z' \widehat{\Psi}^{-1} Z)^{-1} Z' \widehat{\Psi}^{-1} \mathbf{Y}^P$$

where $\widehat{\Psi} = \widehat{\Sigma} \bigotimes I$ a Z is block-diagonal matrix $\mathbf{X}^{P,k}$, thus

$$Z = \mathrm{diag}(\mathbf{X}^{P,1}, \ldots, \mathbf{X}^{P,K}).$$

This process could be performed repeatedly similarly as in Kremer (2005) if initial estimator is replaced by that one calculated in the 3th step. This is repeated until the estimated do not converge

However this straightforward generalisation does not work in practice as could be seen in the following example.

Example

Let us assume two different portfolios. The development of first one could be described by following paid and incurred run off schemes

70 667 658	106 235 024	113 790 024	116 865 272	118 966 938	119 483 611	120 232 235	121 575 748
84 887 287	128 779 906	137 573 849	141 563 543	143 563 773	145 441 033	146 913 953	
93 903 488	132 417 913	144 691 088	150 477 507	154 695 867	156 464 575		
79 621 124	118 210 520	127 184 148	131 385 163	134 039 374			
85 282 458	128 295 871	138 386 846	141 973 697				
86 071 140	131 556 922	141 114 004					
73 721 887	108 299 438						
65 546 609							

Figure 5.1: Multivariate MCL - Paid portfolio 1

118 506 251	140 996 652	141 482 399	141 929 883	137 522 272	133 710 514	132 341 628	135 181 646
141 696 100	161 891 554	165 259 658	171 807 922	176 732 226	165 977 422	164 715 724	
154 935 075	176 102 870	188 247 844	197 597 612	195 427 288	185 583 049		
141 633 570	174 807 935	181 270 807	189 366 804	184 790 868			
167 310 931	193 109 637	201 103 054	199 637 679				
162 574 602	193 335 020	193 869 181					
132 964 303	151 293 073						
125 634 620							

Figure 5.2: Multivariate MCL - Incurred portfolio 1

The past pattern of the second portfolio is in the sense of paid and incurred data shown below.

10 434 215	19 437 589	21 365 232	24 920 077	25 148 923	25 354 662	25 570 800	26 212 712
16 428 921	28 460 952	31 546 437	32 516 184	32 847 411	32 927 280	32 965 677	
23 055 067	45 880 533	51 748 640	55 212 771	55 702 358	58 012 681		
39 709 307	69 101 871	74 352 701	75 976 034	76 863 092			
54 707 836	83 756 779	91 237 581	95 509 093				
58 397 796	88 726 332	96 131 260					
61 364 456	90 923 720						
67 731 526							

Figure 5.3: Multivariate MCL - Paid portfolio 2

19 641 603	24 277 317	28 087 428	26 735 956	26 926 157	27 020 634	27 217 356	27 318 171
25 828 304	31 522 222	33 108 850	33 540 185	33 584 823	33 609 135	33 567 454	
39 339 979	59 316 077	62 225 886	65 392 469	66 297 526	64 630 096		
61 742 695	80 477 323	82 909 859	83 509 311	87 281 540			
87 313 132	108 577 637	115 885 844	121 691 590				
93 424 781	121 717 606	131 966 861					
89 802 273	110 374 569						
106 050 126							

Figure 5.4: Multivariate MCL - Incurred portfolio 2

If we apply separately original MCL on this portfolios we obtain $\lambda^{P,1} = 0.2506$ and also $\lambda^{P,2} = 0.3766$ with following residual dependencies:

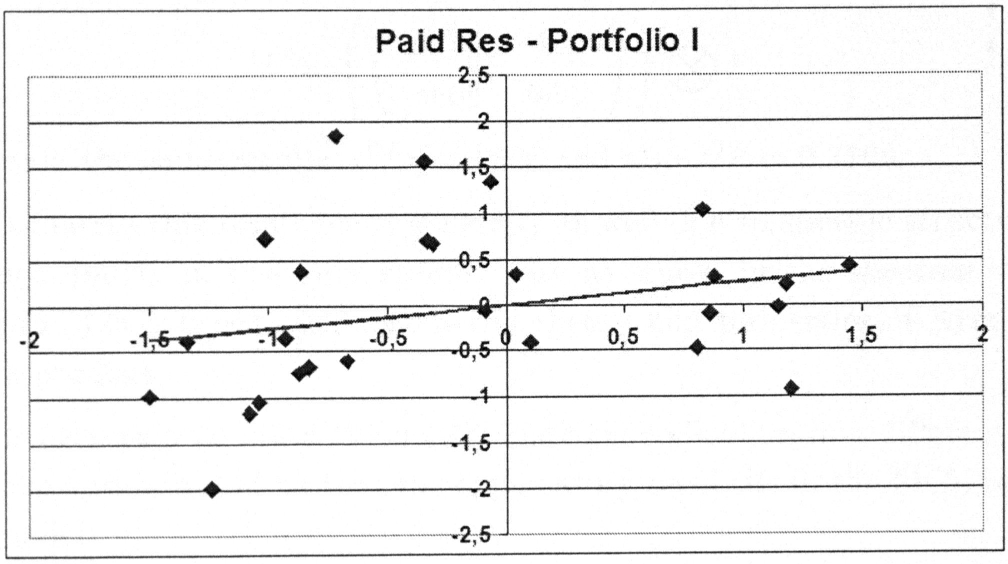

Figure 5.5: Multivariate MCL - initial dependency 1

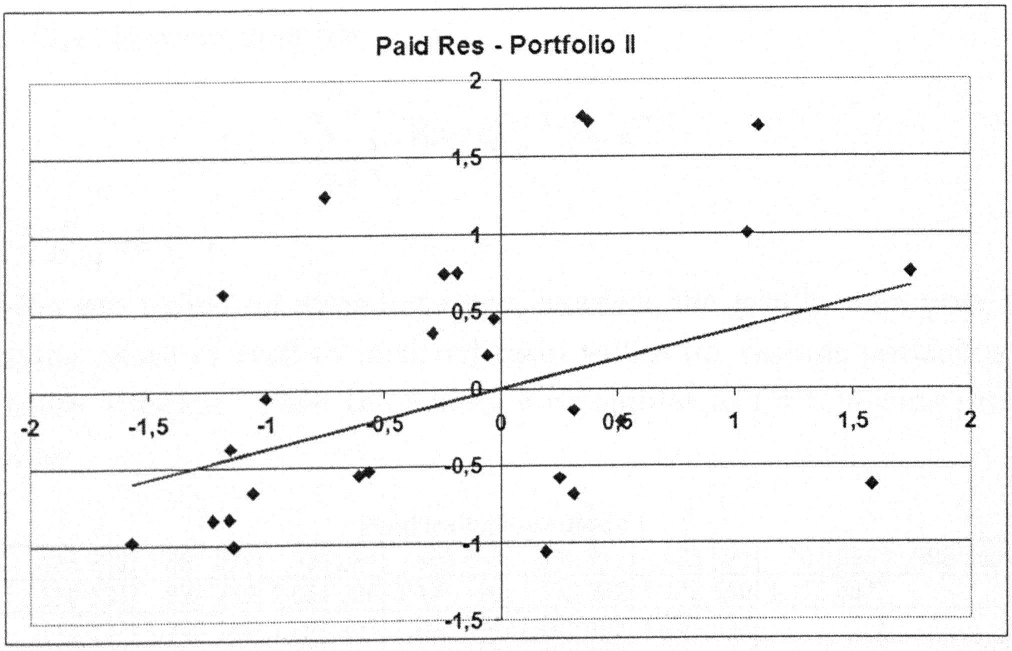

Figure 5.6: Multivariate MCL - initial dependency 2

If we moreover calculate the first step correlation among the residual of two portfolios we obtain value $\sigma_{1,2} = -0.04$ that implies rather weak correlation when we have $\sigma_{11} = 0.73$ and $\sigma_2 2 = 0.66$. After formulation of weight matrix in the form of

$$V = \left(\begin{pmatrix} 0.73 & -0.04 \\ -0.04 & 0.66 \end{pmatrix}\right) \bigotimes I$$

we obtain the same results $\lambda^{P,1} = 0.2506$ and also $\lambda^{P,2} = 0.3766$.

Unfortunately this result holds generally as well due to specific structure of variance matrix in this very specific case as stated in the theorem stated in Cipra (1984) based on Econometric theory and properties of Kronecker matrix product.

So it is necessary to make the multivariate generalisation in a different way. Intuitive option would be that the explanatory residuals in the MCL regression model

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y_i(s)^P\right)|B_i(s)\right) = \lambda^P \cdot \mathrm{Res}(Q^{-1}_{i,s}|Y_i(s)^P)$$

would be based on the values of Paid to Incurred ratios of all analysed portfolios. That is something like

$$\sum_{k=1}^{K} w_k \mathrm{Res}(Q^{-1,k}_{i,s}|Y_i(s)^{P,k})$$

where $\sum_{k=1}^{K} w_k = 1$.

This idea was tested on some data sets, however the results were poor if the underlying cause of paid to incurred ratio values for various portfolios is in some sense different. Then the results were similar as for following sets of 2 portfolios:

Paid traingle portfolio I							
585 810	867 277	926 051	950 953	966 917	973 521	987 856	995 360
639 531	998 282	1 084 328	1 125 190	1 153 382	1 172 550	1 178 857	
719 557	1 103 149	1 207 451	1 242 294	1 269 409	1 281 502		
741 477	1 204 523	1 302 308	1 346 757	1 372 547			
920 817	1 379 273	1 497 516	1 540 386				
1 039 810	1 523 180	1 605 089					
1 031 155	1 466 639						
939 083							

Figure 5.7: Multivariate MCL - Paid portfolio 1

Paid traingle portfolio I							
585 810	867 277	926 051	950 953	966 917	973 521	987 856	995 360
639 531	998 282	1 084 328	1 125 190	1 153 382	1 172 550	1 178 857	
719 557	1 103 149	1 207 451	1 242 294	1 269 409	1 281 502		
741 477	1 204 523	1 302 308	1 346 757	1 372 547			
920 817	1 379 273	1 497 516	1 540 386				
1 039 810	1 523 180	1 605 089					
1 031 155	1 466 639						
939 083							

Figure 5.8: Multivariate MCL - Incurred portfolio 1

Paid traingle portfolio II							
1 203 103	2 158 108	2 318 157	2 396 985	2 435 242	2 456 989	2 477 963	2 499 354
1 591 765	2 402 618	2 594 197	2 676 422	2 698 553	2 743 589	2 771 520	
1 538 127	2 352 495	2 558 737	2 657 903	2 718 632	2 772 554		
1 406 971	2 103 387	2 260 689	2 344 572	2 388 100			
1 422 361	2 179 732	2 365 049	2 434 525				
1 504 316	2 256 844	2 396 263					
1 655 792	2 326 276						
1 605 873							

Figure 5.9: Multivariate MCL - Paid portfolio 2

Incurred traingle portfolio II							
1 789 595	2 558 375	2 668 191	2 633 489	2 686 619	2 690 721	2 690 409	2 707 479
2 235 600	2 742 948	2 850 667	2 969 659	3 010 307	2 988 958	3 000 101	
2 177 712	2 758 224	3 003 116	3 189 306	3 196 665	3 137 857		
2 093 792	2 657 813	2 835 674	2 982 455	2 967 333			
2 276 596	2 897 413	3 059 559	3 145 055				
2 407 860	2 976 998	3 102 769					
2 424 122	3 008 002						
2 471 813							

Figure 5.10: Multivariate MCL - Incurred portfolio 2

It was not so bad in case of univariate MCL when we obtained

$$(\lambda_P^1, \lambda_I^1, \lambda_P^2, \lambda_I^2) = (0.24, 0.42, 0.02, 0.19)$$

However if we tried to formulate the average value of Paid to Incurred ratio we obtained worse result for first portfolio $\lambda_P^1 = -0.04$ even with wrong signature but slightly better results for the Paid data of second portfolio $\lambda_P^2 = 0.07$ where the original result was weak.

It is difficult to make any other statement apart from that this can be used if there exists some aggregate behaviour explaining development of paid and incurred values. It can be the case in the examination of reinsurance layers on the same portfolio in the multivariate way. However we have not reliable portfolio to test this since lack of data in the upper layers is crucial problem that can be better managed by large reinsurers only.

The final method of multivariate MCL possibly useful for different portfolios is based on the estimates of development factors from Kremer's method and its implementation onto MCL in a "classical" way.

That could be defined more formally if we assume the result of iteration in Multivariate SCL as $\mathbf{f}_j^{P,\infty}$ and $\mathbf{f}_j^{I,\infty}$ and the results for variance as $\sigma_j^{2,P,\infty}$ and $\sigma_j^{2,I,\infty}$.

Adjustment of MCL model 3.1.4 is then quite easy, see following formula where expectation and deviations of individual development factors change in appropriate way for k th portfolio:

$$\mathrm{E}\left(\frac{Y_{i,s+1}^{P,k}}{Y_{i,s}^{P,k}}|\mathbf{B}_i(s)\right) = \mathbf{f_s}^{P,k,\infty} + \lambda^{P,k}\frac{\sigma\left(\frac{Y_{i,s+1}^{P,k}}{Y_{i,s}^{P,k}}|Y_i(s)^{P,k}\right)}{\sigma(Q_{i,s}^{-1}|Y_i(s)^P)}\cdot(Q_{i,s}^{-1} - E(Q_{i,s}^{-1}|Y_i(s)^{P,k}))$$

This approach might be used in cases where multivariate structure comes from SCL development factors rather than from similar values for paid to incurred ratio that might be seen rather as a hypothetical task in some cases where there is not found any direct correlation for paid to incurred ratios in the same times.

5.2 Practical implementation

Let us assume the same data as in the example for Multivariate SCL (illustration of Kremer's approach for iterations). We will add corresponding incurred portfolios as well:

	1	2	3	4	5	6	7	8
1	1 789 595	2 558 375	2 668 191	2 633 489	2 686 619	2 690 721	2 690 409	2 707 479
2	2 235 600	2 742 948	2 850 667	2 969 659	3 010 307	2 988 958	3 000 101	
3	2 177 712	2 758 224	3 003 116	3 189 306	3 196 665	3 137 857		
4	2 093 792	2 657 813	2 835 674	2 982 455	2 967 333			
5	2 276 596	2 897 413	3 059 559	3 145 055				
6	2 407 860	2 976 998	3 102 769					
7	2 424 122	3 008 002						
8	2 471 813							

Figure 5.11: Multivariate SCL - Incurred 1

	1	2	3	4	5	6	7	8
1	812 282	1 034 568	1 064 829	1 042 494	1 054 725	1 086 409	1 088 613	1 091 268
2	951 213	1 223 191	1 252 336	1 329 148	1 376 532	1 393 769	1 386 340	
3	1 081 970	1 355 115	1 407 916	1 499 065	1 512 563	1 477 926		
4	1 312 629	1 486 078	1 612 666	1 636 749	1 607 076			
5	1 413 018	1 735 316	1 865 698	1 844 866				
6	1 627 386	1 833 106	1 895 400					
7	1 676 947	1 943 251						
8	1 628 922							

Figure 5.12: Multivariate SCL - Incurred 2

The iteration of development factors for this multivariate incurred scheme is as follows

development factors 1				
iteration:	1	2	3	4
k =1	1,272277	1,263685	1,263683	1,263682
k =2	1,195504	1,176306	1,17631	1,17631

Figure 5.13: Process of iteration, Incurred development factors 1

variability and covariance 1						
iteration:	1		2		3	
k= 1	6 685,447	287,726	6 678,371	285,979	6 678,417	285,989
k =2	287,726	4 584,962	285,979	4 516,283	285,989	4 516,190

Figure 5.14: Process of iteration, Incurred variability 1

for $j = 0$ and later on

development factors 2					
iteration:	1	2	3	4	5
k =1	1,055944	1,055772	1,055772	1,055772	1,055772
k =2	1,049781	1,052481	1,052481	1,052481	1,052481

Figure 5.15: Process of iteration, Incurred development factors 2

variability and covariance 2								
iteration:	1		2		3		4	
k= 1	838.985	49.580	838.974	49.381	838.974	49.381	838.974	49.381
k =2	49.580	832.540	49.381	830.520	49.381	830.520	49.381	830.520

Figure 5.16: Process of iteration, Incurred variability 2

for $j = 1$ and finally

development factors 3			
iteration:	1	2	3
k =1	1,0348722	1,0361997	1,0361997
k =2	1,0206671	1,0157373	1,0157373

Figure 5.17: Process of iteration, Incurred development factors 3

variability and covariance 3				
iteration:	1		2	
k= 1	1 809.140	247.194	1 808.600	248.331
k =2	247.194	1 669.075	248.331	1 675.571

Figure 5.18: Process of iteration, Incurred variability 3

if $j = 2$. Following iteration were not again computed for lack of data.

If we have this iteration completed we can compare the underlying information for MCL in both cases (univariate approach or suggested multivariate one). Results are for second portfolio (that is more suitable for MCL) as follows:

Differences in underlying parameters for MCL				
k=2	Paid	0	1	2
development factors	univariate	1,5044186	1,0773153	1,0312292
	multivariate	1,4888738	1,0750044	1,0311109
standard deviation	univariate	60,934266	17,000378	4,7569959
	multivariate	57,34894	16,595758	4,1330096

Figure 5.19: Impact on MCL 1

Differences in underlying parameters for MCL				
k=2	Incurred	0	1	2
development factors	univariate	1,1955036	1,049781	1,0206671
	multivariate	1,1763099	1,0750044	1,0157373
standard deviation	univariate	74,213888	31,15765	46,357444
	multivariate	67,202601	28,818747	40,933743

Figure 5.20: Impact on MCL 2

After all that computation we can formulate the linear models underlying MCL with that adjustment what gives us following model results

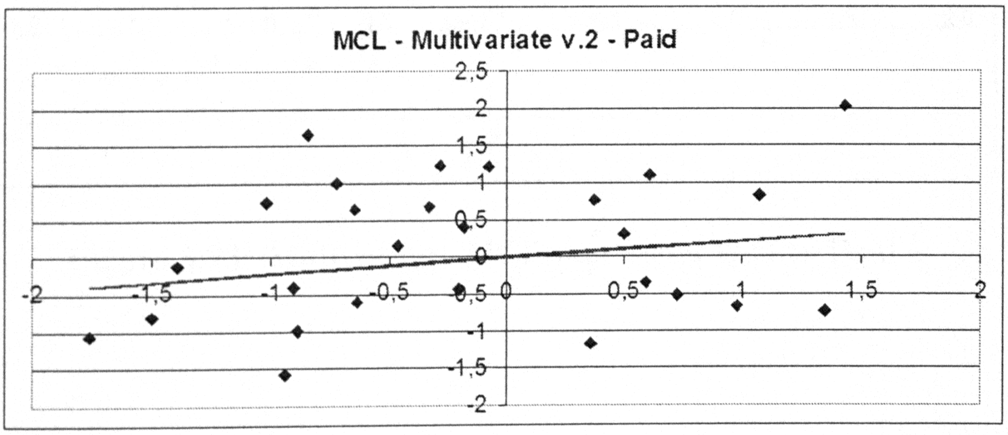

Figure 5.21: Multivariate MCL regression results Paid 1

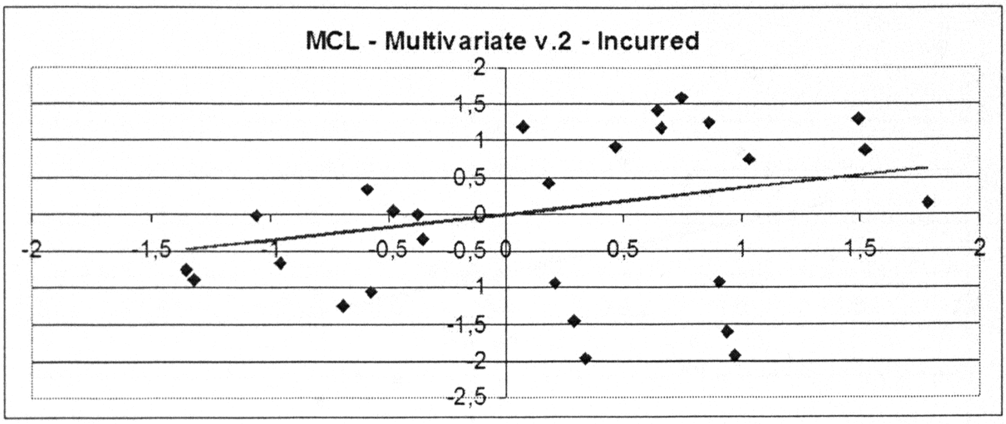

Figure 5.22: Multivariate MCL regression results Incurred 1

what is much better result than the result of "naive" multivariate generalisation, see following graphs:

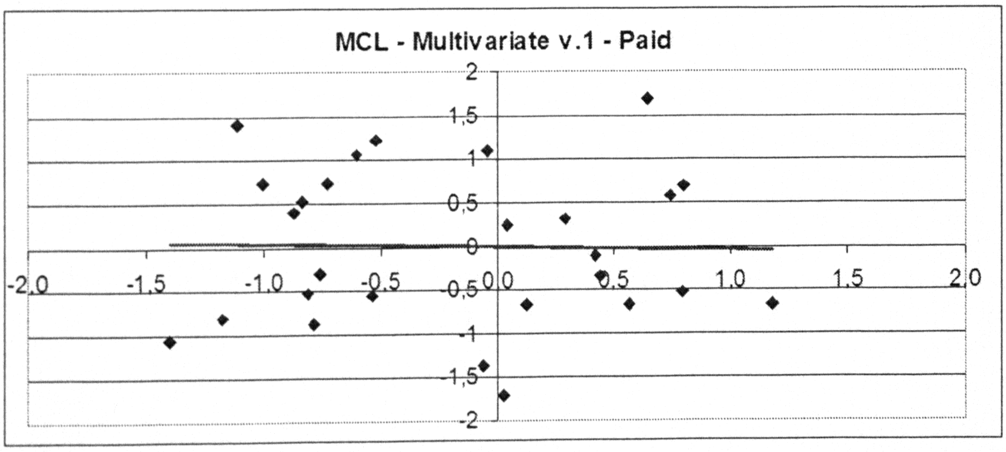

Figure 5.23: Multivariate MCL regression results Paid 2

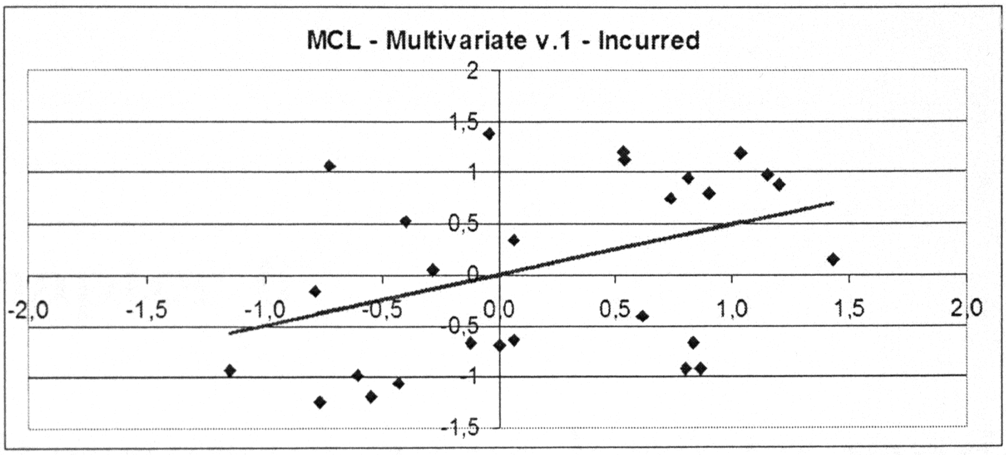

Figure 5.24: Multivariate MCL regression results Incurred 2

Chapter 6

Alternative ways how to model Paid and Incurred data

In addition to previously suggested generalisation of reserving methods we would like to continue now with further generalisation based on some econometrical methods that are suitable for application in claims reserving.

6.1 Bivariate time series

The further derivation is based on theory of vector auto regression as is described by Hamilton, see Hamilton (1994). Of course model of Chain Ladder is usually not stationary one and the claims evolution could not be seen as autoregressive process. However some analogy could be seen using the fact that development factor for the same level of delay is for various accident years the same. We will try to incorporate this onto multivariate process as well.

Firstly we have to review some basic facts concerning the vector autoregression. Hamilton defines p th order vector auto regression as

$$Y_t = c + \Phi_1 y_{t-1} + \ldots \Phi_p y_{t-p} + \varepsilon_t$$

where y_t corresponds to n th dimensional vector and Φ are interpreted as square matrices of auto regressive coefficients. It is assumed that the distribution of ε_t is multivariate normal, $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{Var}(\varepsilon_t) = \Omega$. We denote

$$\Pi^{'} = (c, \Phi_1 \ldots \Phi_p)$$

for simplification of further derivation.

Our goal is to achieve maximum likelihood estimates of Π and Σ which is based on assumption of multivariate conditional normal distribution of y_t

$$y_t | y_{t-1} \ldots y_{-p+1} \tilde{N}(\Pi^{'} x_t, \Omega)$$

After that maximum likelihood estimate of Π is derived as follows:

$$\widehat{\Pi}^{'} = \left[\sum_{t=1}^{T} y_t x_t^{'} \right] \left[\sum_{t=1}^{T} x_t x_t^{'} \right]^{-1}$$

In order to check the statement we can start from the last element of log likelihood

$$\sum_{t=1}^{T} \left[(y_t - \Pi^{'} x_t)^{'} \Omega^{-1} (y_t - \Pi^{'} x_t) \right]$$

$$= \sum_{t=1}^{T} \left[(y_t - \widehat{\Pi}^{'} x_t + \widehat{\Pi}^{'} x_t - \Pi^{'} x_t)^{'} \Omega^{-1} (y_t - \widehat{\Pi}^{'} x_t + \widehat{\Pi}^{'} x_t - \Pi^{'} x_t) \right]$$

$$= \sum_{t=1}^{T} \left[[\widehat{\varepsilon}_t + (\widehat{\Pi} - \Pi)^{'} x_t]^{'} \Omega^{-1} [\widehat{\varepsilon}_t + (\widehat{\Pi} - \Pi)^{'} x_t] \right]$$

Obviously OLS computed residuals are defined as $\widehat{\varepsilon}_t = y_t - \widehat{\Pi}^{'} x_t$. The above mentioned expression can be adjusted via multiplication onto

$$\sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} \widehat{\varepsilon}_t + 2 \sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} x_t + \sum_{t=1}^{T} x_t^{'} (\widehat{\Pi} - \Pi) \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} x_t$$

Using properties of trace operator we can get for first part of the formula

$$\sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} x_t = \mathrm{trace} \left[\sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} x_t \right] = \tag{6.1.1}$$

$$= \mathrm{trace} \left[\sum_{t=1}^{T} \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} x_t \widehat{\varepsilon}_t^{'} \right] = \tag{6.1.2}$$

$$= \mathrm{trace} \left[\Omega^{-1} (\widehat{\Pi} - \Pi)^{'} \sum_{t=1}^{T} x_t \widehat{\varepsilon}_t^{'} \right] \tag{6.1.3}$$

Due to the fact that $x_t^{'}\widehat{\varepsilon} = 0$ we can simplify the formula to

$$\sum_{t=1}^{T} \left[(y_t - \Pi^{'} x_t)^{'} \Omega^{-1} (y_t - \Pi^{'} x_t) \right] = \tag{6.1.4}$$

$$\sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} \widehat{\varepsilon}_t + \sum_{t=1}^{T} x_t^{'} (\widehat{\Pi} - \Pi) \Omega^{-1} (\widehat{\Pi} - \Pi)^{'} \tag{6.1.5}$$

If we define $x_t^* = (\widehat{\Pi} - \Pi)^{'} x_t$ we can then use the fact that $x_t^{*'} \Omega^{-1} x_t^* \geq 0$ and equals to 0 if $\widehat{\Pi} = \Pi$ which concludes the proof.

If we want to obtain maximum likelihood estimate of Ω we have to review likelihood function jointly after obtaining the estimate of Π, that is

$$L(\Omega, \widehat{\Pi}) = -(Tn)/2\log(2\pi) + T/2\log|\Omega^{-1}| - (1/2) \sum_{t=1}^{T} \widehat{\varepsilon}_t^{'} \Omega^{-1} \widehat{\varepsilon}_t$$

We can perform

$$\frac{\partial L(\Omega, \widehat{\Pi})}{\Omega^{-1}} = T/2 \frac{\partial \log|\Omega^{-1}|}{\partial \Omega^{-1}} - (1/2) \sum_{t=1}^{T} \frac{\partial \widehat{\varepsilon}_t^{'} \Omega^{-1} \widehat{\varepsilon}_t}{\partial \Omega^{-1}}$$

$$= T/2\Omega^{'} - (1/2) \sum_{t=1}^{T} \widehat{\varepsilon}_t \widehat{\varepsilon}_t{}^{'}$$

Maximum of likelihood is then achieved if

$$\Omega = 1/T \sum_{t=1}^{T} \widehat{\varepsilon}_t \widehat{\varepsilon}_t^{'}$$

6.2 Proposal of bivariate model in claims reserving

As was previously said, classical Chain Ladder model is NOT autoregressive model of order 1

$$X_t = c \cdot X_{t-1} +_t$$

Certain analogy however might be seen in the fact that we have the same development factor for various accident years

$$Xi, j = c_j \cdot X_{i,j-1} + \varepsilon_{i,j}$$

for that we used OLS estimate taking attention to non constat variance. Our aim here will be to construct such a kind of the model for bivariate series representing claims process as well.

For that purpose we start form the reserving model of Schnieper (1991) that was reviewed and extended by Huijuan Liu in 2007. We will work with basic evolution for incurred data $I_{i,j} \equiv Y^I_{i,j}$

$$I_{i,j} = I_{i,j-1} - D_{i,j} + N_{i,j}$$

where $N_{i,j}$ stands for amount of newly detected claims which from accounting point of view could be seen as the amount of expenses for setting up the reserve $R^T_{i,j}$. On the other hand $D_{i,j}$ is interpreted as positive development of claims from the point of view of the insurer. From the accounting perspective it can be seen as a surplus achieved due to reducing the reserve without respective payment $R^R_{i,j}$

Schnieper suggested following assumption for that model for expectations:

$$\mathrm{E}(N_{i,j}|I_{i,j-1}) = E_i\lambda_j$$

$$\mathrm{E}(D_{i,j}|I_{i,j-1}) = I_{i,j-1}\delta_j$$

It means that relative amount of new claims is proportional to adequately selected volume of risks that is assumed to be known and positive. Development evolution reminds chain ladder one with different lagged explaining variable.

Assumptions about model variance are also similar to chain ladder type of model, that is proportionality to volume of explaining variable:

$$\mathrm{Var}(N_{i,j}|I_{i,j-1}) = E_i\sigma_j^2$$

$$\mathrm{Var}(D_{i,j}|I_{i,j-1}) = Y_{i,j-1}\tau_j^2$$

and no other distribution properties are presented expect from the fact that the random sequences $(N_{i,j}, D_{i,j})$ are assumed to be independent for various accident years.

From that assumption it is quite rational that suggested estimators are also quite similar to chain ladder since the process of derivation would be the same under application of Aitken estimator as in Chapter describing the classical Chain Ladder:

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n+1-j} N_{i,j}}{\sum_{i=1}^{n+1-j} E_i} \quad \forall j$$

$$\widehat{\delta}_j = \frac{\sum_{i=1}^{n+1-j} D_{i,j}}{\sum_{i=1}^{n+1-j} I_{i,j-1}} \quad \forall j$$

$$\widehat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{E_i} \left(N_{i,j} - \widehat{\lambda}_j E_i\right)^2 \forall j$$

$$\widehat{\tau}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{I_{i,j-1}} \left(D_{i,j} - \widehat{\delta}_j I_{i,j-1}\right)^2 \forall j$$

Estimates of the projection could be then derived naturally using the basic formula of the model $I_{i,j} = I_{i,j-1} - D_{i,j} + N_{i,j}$ which implies one step ahead prediction as

$$\widehat{I_{2,n}} = \mathrm{E}(I_{2,n}|I_{2,n-1}) = \mathrm{E}(I_{2,n-1} - D_{2,n} + N_{2,n}|I_{2,n-1})$$

$$= I_{2,n-1} + \lambda_n E_2 = I_{2,n-1}(1-\delta_n) + \lambda_n E_2$$

Two step prediction could be written as

$$\widehat{X_{3,n}} = X_{3,n-2}(1-\delta_{n-1})(1-\delta_n) + E_3(1-\delta_n)\lambda_{n-1} + E_3\lambda_n$$

and further generalisation is quite straightforward.

In Huijuan Liu (2007) distribution assumptions were added to the model in the following way

$$\left(\frac{N_{i,j}}{E_i}|I_{i,j-1}\right) \sim N\left(\lambda_j, \frac{\sigma_j^2}{E_i}\right)$$

and also

$$\left(\frac{D_{i,j}}{I_{i,j-1}}|I_{i,j-1}\right) \sim N\left(\delta_j, \frac{\tau_j^2}{I_{i,j-1}}\right)$$

Under that assumptions MLE and OLS estimators will be the same and variance of the process was derived as

$$\mathrm{Var}(I_{i,j+t}|I_{i,j}) = (1-\delta_{j+t}^2)\mathrm{Var}(I_{i,j+t-1}|I_{i,j}) + \tau_{j+t}^2 \mathrm{E}(I_{j+t-1}|I_{i,j}) + E_i\sigma_{j+t}^2$$

and then error of estimates is formulated as

$$\text{Var}(I_{i,j+t}) = \widehat{I}^2_{i,j+t-1}\text{Var}(\widehat{\delta}_{j+t}) + \quad (6.2.1)$$
$$+(1-\delta_{j+t})^2\text{Var}(\widehat{I}_{i,j+t-1}) + \text{Var}(\widehat{\delta}_{j+t})\text{Var}(\widehat{I}_{i,j+t-1}) + E_i^2\text{Var}(\widehat{\lambda}_{j+t}) \quad (6.2.2)$$

and respective mean square error of the overall reserve

$$MSE(\widehat{R}|\cdot) = \text{Var}(R|\cdot) + \text{Var}(\widehat{R}|\cdot) = \sum_{i=1}^{n}\text{Var}(I_{i,n}|I_{i,j}) + \quad (6.2.3)$$
$$+\sum_{i=1}^{n}\text{Var}(\widehat{I}_{i,n}) + 2\sum_{t=1}^{n-1}\sum_{s=t}^{n}\text{Covr}(\widehat{I}_{t,n},\widehat{I}_{s,n}) \quad (6.2.4)$$

See Huijuan Liu (2007) for more details and alternative approach using the Monte Carlo simulation techniques.

Our work regarding the approach for setting the reserve evolution was done independently on work of Huijuan Liu as both contribution were presented in the same time, see Huijuan Liu (2007) and Jedlicka (2007). We started from the basic equation similarly as Huijuan Liu

$$R_{i,j+1} = R_{i,j} - P^d_{i,j+1} + R^T_{i,j+1} - R^R_{i,j+1}$$

and model for reserve development is then seen as

$$R^T_{i,j} - R^R_{i,j} = \gamma_j R_{i,j} + \varepsilon^C_{i,j}, \qquad \text{Var}(\varepsilon^C_{i,j}) = \sigma^2_C R_{i,j}$$

which is derived via

$$R_{i,j+1} = R_{i,j} - P^d_{i,j+1} + R^T_{i,j+1} - R^R_{i,j+1} = \quad (6.2.5)$$
$$R_{i,j} - \alpha_j R_{i,j} + R^T_{i,j+1} - R^R_{i,j+1} + \varepsilon^A_{i,j} = \quad (6.2.6)$$
$$\beta_j R_{i,j} + \varepsilon^B_{i,j} \quad (6.2.7)$$

Our approach is thus more general than in the paper of Schnieper and Huijuan Liu (2007) since we split incurred identity equation into actual both component (paid and reserve process) and we moreover modelled their dependencies as is shown later on.

Moreover quite natural assumption about payment development as a proportion of reserve was used

$$P^d_{i,j+1} = \alpha_j R_{i,j} + \varepsilon^A_{i,j}$$

Using the simplest model

$$R_{i,j+1} = \beta_j R_{i,j} + \varepsilon^B_{i,j}, \qquad \mathrm{Var}(\varepsilon^B_{i,j}) = \sigma^2_B R_{i,j}$$

reminding Chain Ladder for reserving process with following restrictions

$$\beta_j + \alpha_j - 1 = \gamma_j$$

and also

$$\varepsilon^C_{i,j} = \varepsilon^A_{i,j} + \varepsilon^B_{i,j}$$

6.2.1 Numerical illustration

Following example shows us how useful might be suggested two alternative generalisations in case we have unfinished schemes for the oldest accident years that are not properly fitted by any of MCL alternatives:

Paid	0	1	2	3	4	5	6	7	8	9	10	11
0	5 839	12 289	16 343	19 622	22 616	24 891	27 482	30 138	33 775	34 902	36 986	38 100
1	6 721	15 461	20 071	24 408	28 027	31 321	34 920	38 515	41 202	43 373	45 781	
2	7 067	15 449	20 300	23 864	27 674	30 676	37 419	41 497	44 058	47 227		
3	7 673	17 099	22 673	27 484	31 377	35 654	38 565	42 784	45 861			
4	7 006	15 019	20 674	25 019	29 424	33 857	37 984	42 950				
5	7 002	16 253	21 886	26 197	30 425	35 691	40 063					
6	7 135	14 873	19 176	23 712	27 571	31 858						
7	6 985	15 076	20 734	24 855	29 371							
8	6 625	14 370	18 812	22 504								
9	6 635	15 242	20 263									
10	7 506	15 673										
11	7 421											

Figure 6.1: Paid run-off triangle

Incurred	0	1	2	3	4	5	6	7	8	9	10	11
0	30 995	39 325	41 933	42 208	44 498	44 808	45 928	46 271	47 546	47 569	47 886	47 886
1	37 713	46 778	47 860	49 939	51 897	52 108	54 219	55 464	56 029	56 045	57 336	
2	39 214	47 350	50 974	53 669	55 342	60 270	60 551	59 961	61 258	62 298		
3	40 880	51 485	55 328	57 270	61 742	62 456	64 928	64 913	66 130			
4	44 025	54 152	57 151	61 659	61 489	61 876	63 014	63 387				
5	41 741	51 666	57 737	60 880	61 908	62 291	62 647					
6	40 841	51 836	55 032	56 139	56 048	56 492						
7	48 770	57 404	62 267	64 767	65 237							
8	50 687	58 297	58 190	58 595								
9	54 184	64 170	65 467									
10	58 829	70 328										
11	60 587											

Figure 6.2: Incurred run-off triangle

The ultimates values may be determined by using of the so far presented method starting with SCL to this alternative approaches. Results for Paid triangle are for accident years as follows:

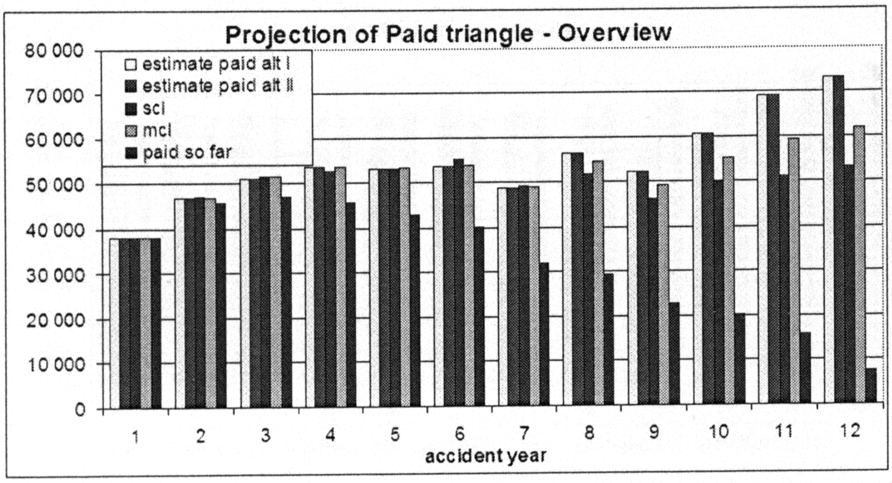

Figure 6.3: Projection Paid run-off triangle

Triangle of incurred values give us in addition following projections:

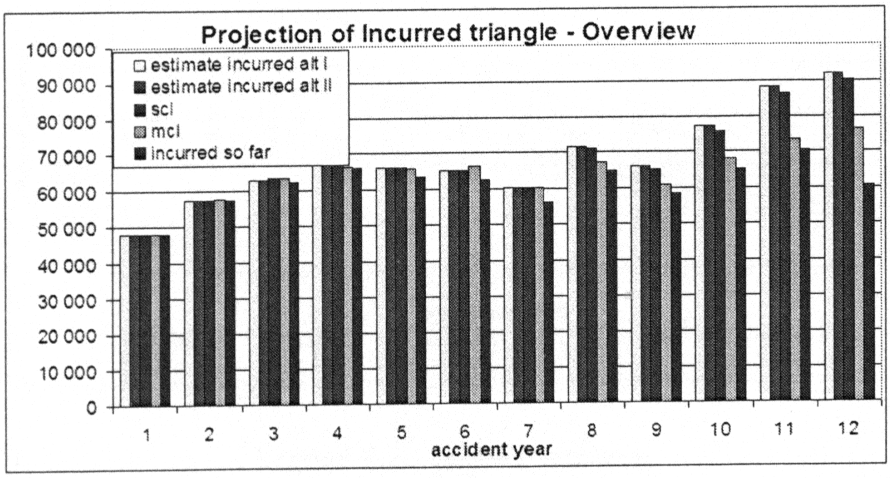

Figure 6.4: Projection of Incurred run-off triangle

We can see that the value of ultimates projection differ quite a lot. Overall fit is to be evaluated by the standard ratio. This fit of projection in alternative model (as expressed by Paid to incurred ratio) is better since we can well model the further development of RBNS

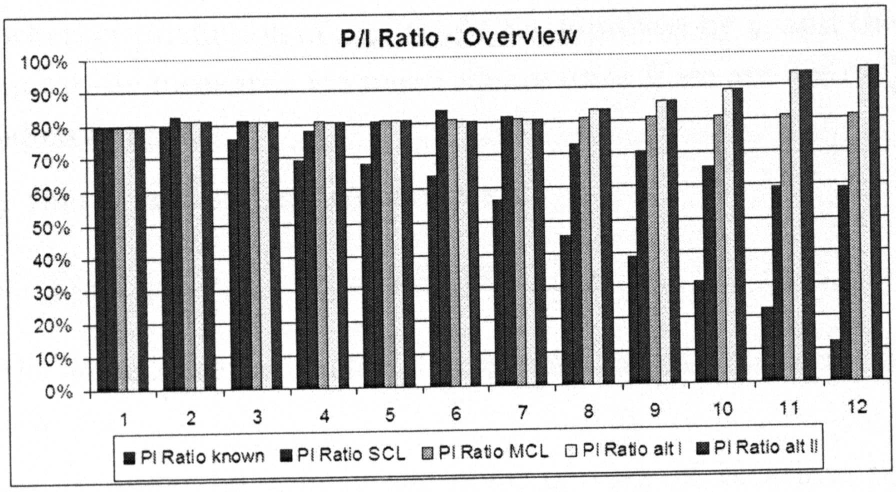

Figure 6.5: Paid to Incurred Ratio Alternative results

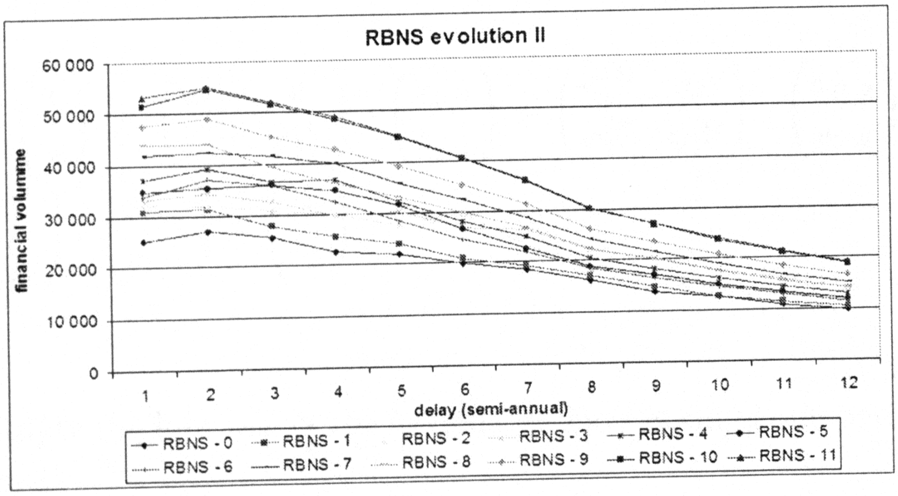

Figure 6.6: RBNS pattern

6.3 Concept of Granger Causality

In that subsection we will briefly remind the concept of Granger causality as stated in Hamilton (1994). We will restrict presented results on bivariate case only and so we are interested if one variable helps to predict the another one. Let us assume two time series

$$x_t, x_{t-1}, x_{t-2}, \dots$$

and also

$$y_t, y_{t-1}, y_{t-2}, \dots$$

The fact whether prediction of x_{t+s} might be improved by y_t and their lagged values is naturally measured via mean square error if we are restricted to the linear predictors only.

So we say that y fails to Granger cause x if

$$\text{MSE}\left[\text{E}(x_{t+s}|x_t, x_{t-1}\ldots)\right] = \text{MSE}\left[\text{E}(x_{t+s}|x_t, x_{t-1}\ldots, y_t, y_{t-1}, \ldots)\right]$$

which is the same as to say that x is exogenous in the time series sense with respect to y.

This situation might be seen in the point of view of bivariate time series theory as follows

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \psi_{11}^{(1)} & 0 \\ \psi_{21}^{(1)} & \psi_{22}^{(1)} \end{pmatrix} \cdot \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \psi_{11}^{(2)} & 0 \\ \psi_{21}^{(2)} & \psi_{22}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} x_{t-2} \\ y_{t-2} \end{pmatrix} + \cdots +$$
$$\begin{pmatrix} \psi_{11}^{(p)} & 0 \\ \psi_{21}^{(p)} & \psi_{22}^{(p)} \end{pmatrix} \cdot \begin{pmatrix} x_{t-p} \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

If we multiply the first row of the formula we have optimal one period ahead prediction as

$$\text{E}(x_{t+1}|x_t, x_{t-1}, \ldots y_t, y_{t-1}\ldots) = c_1 + \psi_{11}^{(1)} x_t + \psi_{11}^{(1)} x_{t-1} + \ldots + \psi_{11}^{(p)} x_{t-p+1}$$

Similarly also s ahead forecast depends on x only. So y does not Granger-cause x if Ψ is lower triangular $\forall j$.

The probably mostly used statistical or econometric test for significance of Granger causality is based on classical F test of null hypothesis in the model

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \ldots + \alpha_p x_{t-p} + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \ldots + \beta_p y_{t-p} + u_t$$

The F test is then based on null hypothesis

$$H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0$$

We can calculate the sum of squares in the full model

$$RSS_1 = \sum_{t=1}^{T} \widehat{u}_t^2$$

and also in the sub model without variables y which is

$$RSS_0 = \sum_{t=1}^{T} \widehat{e}_t^2$$

Test statistics is then constructed as

$$S_1 = \frac{(RSS_0 - RSS_1)/p}{RSS_1/(T - 2p - 1)}$$

which fulfills under the validity of H_0 hypothesis F distribution with p and $T - 2p - 1$ degrees of freedom.

6.4 Inspection of causality in bivariate claims models

So far we have presented two basic model for Paid claims development. The first one is the classical Chain Ladder

$$P_{i,j+1} = f_j \cdot P_{i,j} + \varepsilon_{i,j}$$

and the second one is based on alternative reserve development

$$P_{i,j+1} = P_{i,j} + \alpha_j R_{i,j} + \varepsilon_{i,j} \cdot P_{i,j}$$

If we combine both these two univariate approaches to generalised bivariate one, we can formulate

$$\begin{pmatrix} P_{i,j+1} \\ R_{i,j+1} \end{pmatrix} = \begin{pmatrix} f_j & \alpha_j \\ \delta_j & \beta_j \end{pmatrix} \cdot \begin{pmatrix} P_{i,j} \\ R_{i,j} \end{pmatrix} + \begin{pmatrix} \varepsilon_{i,j}^P \\ \varepsilon_{i,j}^R \end{pmatrix}$$

The mentioned two simple models could be seen as special cases if $\alpha_j = 0$ (then we obtain Chain Ladder model) or if $f_j = 1$ (since then we obtain alternative model for reserve development only).

We are particularly interested (from the point of view of concept of Granger causality) whether $\delta_j = 0$. If it holds true then so far paid compensation does not contain any predictive information for future reserving in the Granger's sense.

We can formulate the parameters estimate obtained as an application of vector auto regression

$$\widehat{\Pi_j} = \left[\sum_{i=1}^{n-j} \mathbf{Y}_i \mathbf{X}_i^{'}\right] \left[\sum_{i=1}^{n-j} \mathbf{X}_i \mathbf{X}_i^{'}\right]^{-1}$$

under notation that

$$\mathbf{Y}_i \equiv \begin{pmatrix} P_{i,j+1} \\ R_{i,j+1} \end{pmatrix}, \Pi_j \equiv \begin{pmatrix} f_j & \alpha_j \\ \delta_j & \beta_j \end{pmatrix}, \mathbf{X}_i \equiv \begin{pmatrix} P_{i,j} \\ R_{i,j} \end{pmatrix}, \Sigma \equiv \mathrm{Var} \begin{pmatrix} \varepsilon_{i,j}^P \\ \varepsilon_{i,j}^R \end{pmatrix}$$

The problem of variance matrix estimate is a bit more complicated since in fact homoscedasticity is not the case in the claims development processes. So estimator is suggested as

$$\widehat{\Sigma} = \frac{1}{n-j-1} \sum \widehat{\varepsilon_i} . \widehat{\varepsilon_i}^{'}$$

using notation

$$\widehat{\varepsilon_i} = \mathbf{Y}_i - \widehat{\Pi}^{'} \mathbf{X}_i$$

6.5 Numerical results and their interpretation

In this section we will not work with data of paid, reserve or incurred represented by triangular schemes. Rather we will see on data as time series and we will try to illustrate the causality relation. All of the results in this part were obtained using software R where procedure for Granger test is implemented in the package lmtest.

Let us work with following time series of Paid process P_t

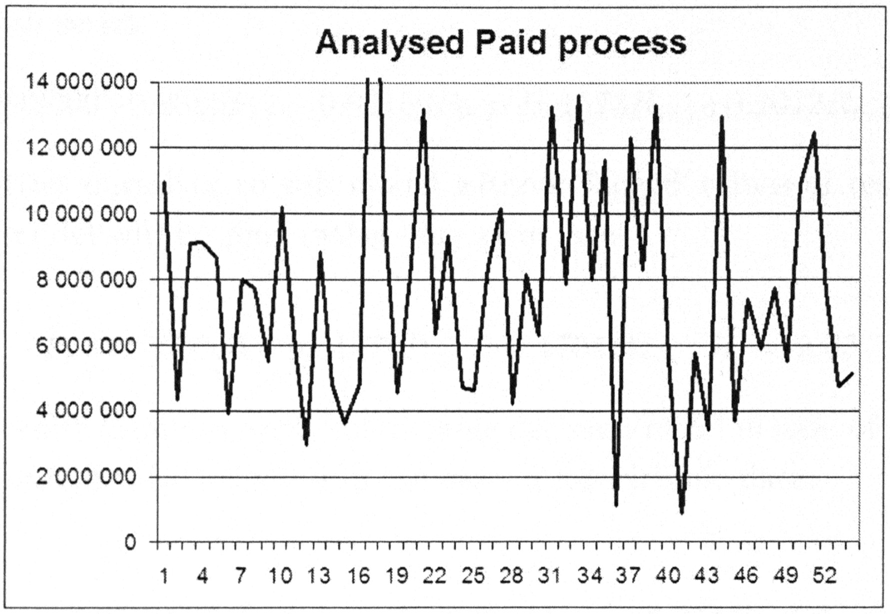

Figure 6.7: Paid process as time series

and reserve process R_t

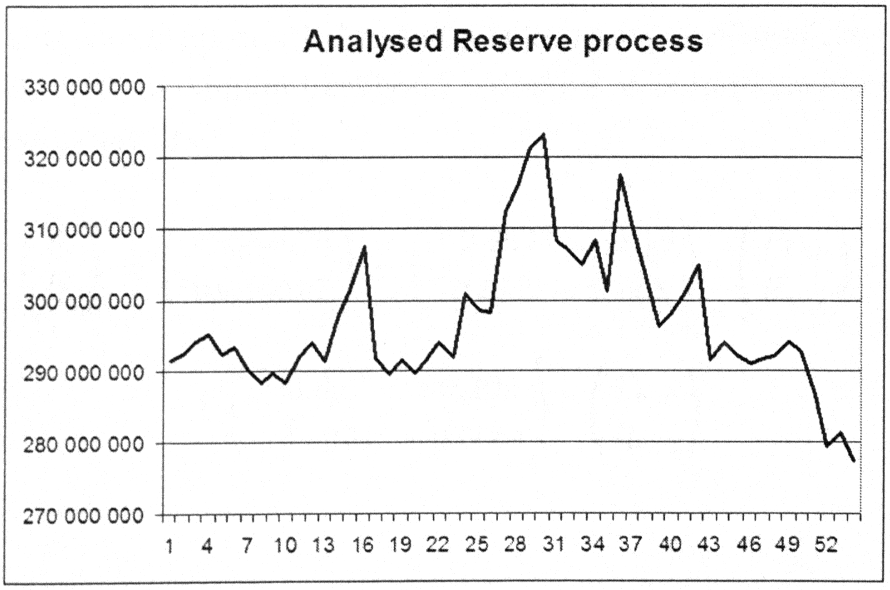

Figure 6.8: Reserve process as time series

which both correspond to one certain portfolio. We will work with the original series as well as with their logarithmic transformation.

Firstly we obtained following numerical results based on autoregression models for both series:

$$\widehat{P_t} = -2948000 - 0.2886P_{t-1} - 0.02159P_{t-2} - 0,1573R_{t-1} + 0.2012R_{t-2}, \quad R^2 = 0.2281$$

If we restrict ourselves to sub model without lagged values of reserves, remaining model will become rather very poor, see

$$\widehat{P_t} = 5265000 + 0.01772P_{t-1} + 0.1704P_{t-2} \quad R^2 = 0.03$$

Similar results (however with not so large decrease of R^2 in case of reduction to sub model) is also achieved in the case of logarithmic data:

$$\log(\widehat{P_t}) = -43.29 + 0.531\log(P_{t-1}) - 0.272\log(P_{t-2}) + 6.6299\log(R_{t-1}) - 3.816\log(R_{t-2})$$

with $R^2 = 0.28$

$$\log(\widehat{P_t}) = 10.34 + 0.5524\log(P_{t-1}) - 0.232\log(P_{t-2}) \quad R^2 = 0.244$$

However this simple approach does not give consistent estimate, so it is better to apply directly maximum likelihood estimates as is implemented e.g. in R with following results

$$\begin{pmatrix} \widehat{P_t} \\ \widehat{R_t} \end{pmatrix} = \begin{pmatrix} -2384976 \\ 10915217 \end{pmatrix} + \begin{pmatrix} -0.2941 & -0.16 \\ 0.052 & 0.968 \end{pmatrix} \cdot \begin{pmatrix} P_{t-1} \\ R_{t-1} \end{pmatrix} +$$

$$+ \begin{pmatrix} -0.027 & -0.202 \\ -0.0074 & -0.0064 \end{pmatrix} \cdot \begin{pmatrix} P_{t-2} \\ R_{t-2} \end{pmatrix}$$

which is however not so different from straightforward but not consistent approach.

Similarly we got for logarithmic data

$$\begin{pmatrix} \log(\widehat{P_t}) \\ \log(\widehat{R_t}) \end{pmatrix} = \begin{pmatrix} -45.61 \\ 0.852 \end{pmatrix} + \begin{pmatrix} 0.533 & 6.64 \\ -0.00011 & 0.9997 \end{pmatrix} \cdot \begin{pmatrix} \log(P_{t-1}) \\ \log(R_{t-1}) \end{pmatrix} +$$

$$\begin{pmatrix} -0.275 & -3.704 \\ -0.001 & -0.0418 \end{pmatrix} \cdot \begin{pmatrix} \log(P_{t-2}) \\ \log(R_{t-2}) \end{pmatrix}$$

In both situations lower left cell of square matrices is close to zero, which implies that paid process is not indeed informative for future development of reserving.

This fact was tested via Granger test in both logarithmic and original scales and this fact could be supported by following charts representing that additional information helps to improve the prediction in the paid process only.

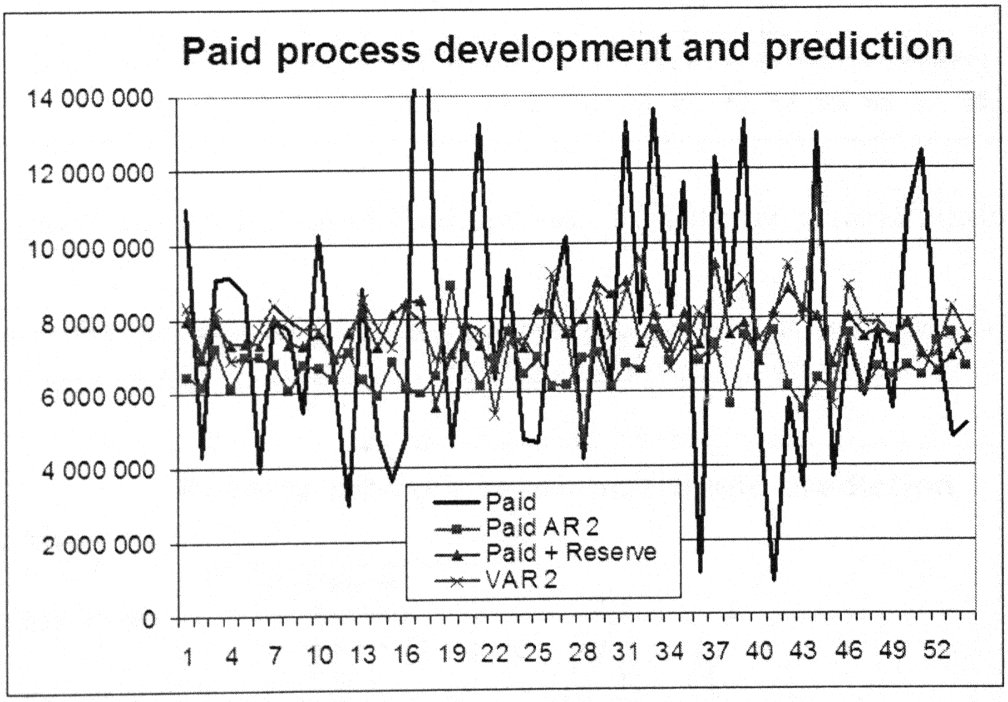

Figure 6.9: Projection of Paid process

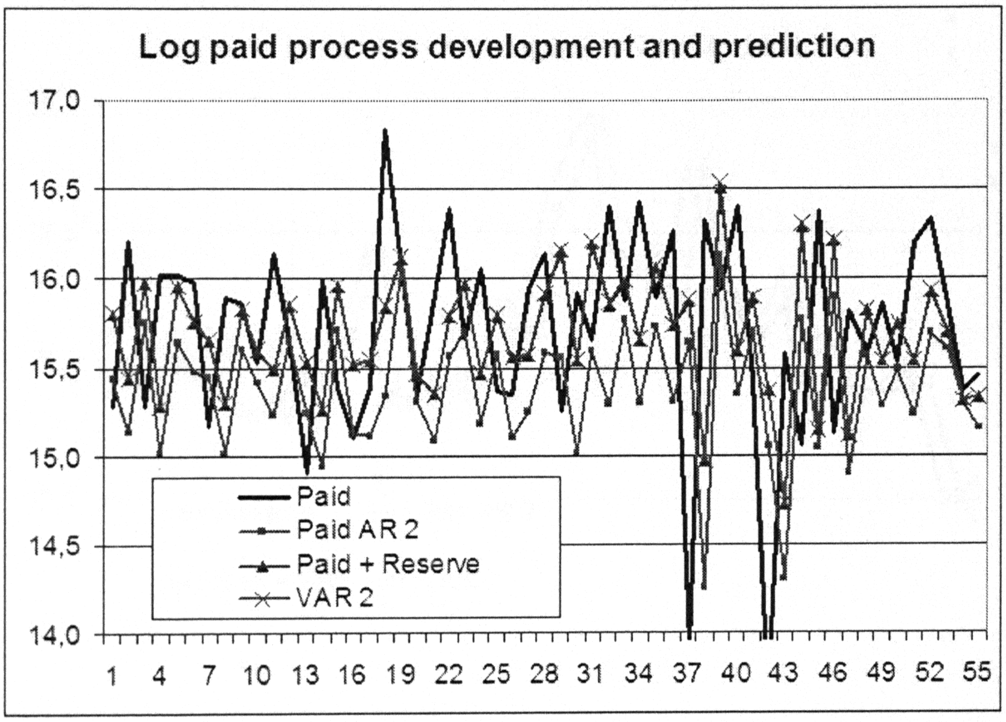

Figure 6.10: Projection of Paid process - logarithmic transformation

No improvement was achieved for reserve process and it holds for original data as well as for the data after logarithmical transformation.

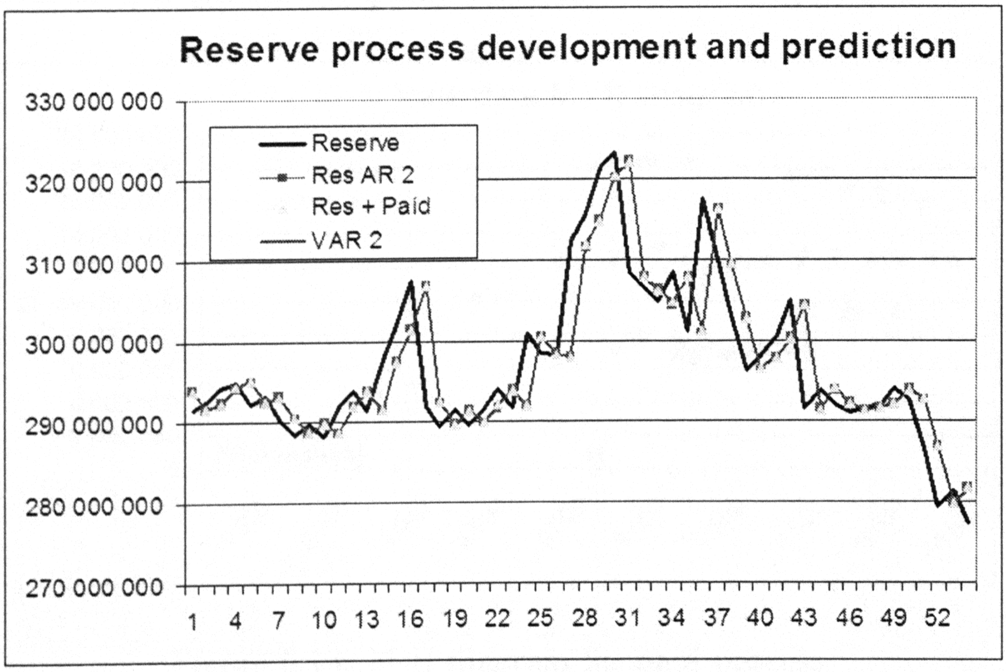

Figure 6.11: Projection of Reserve process

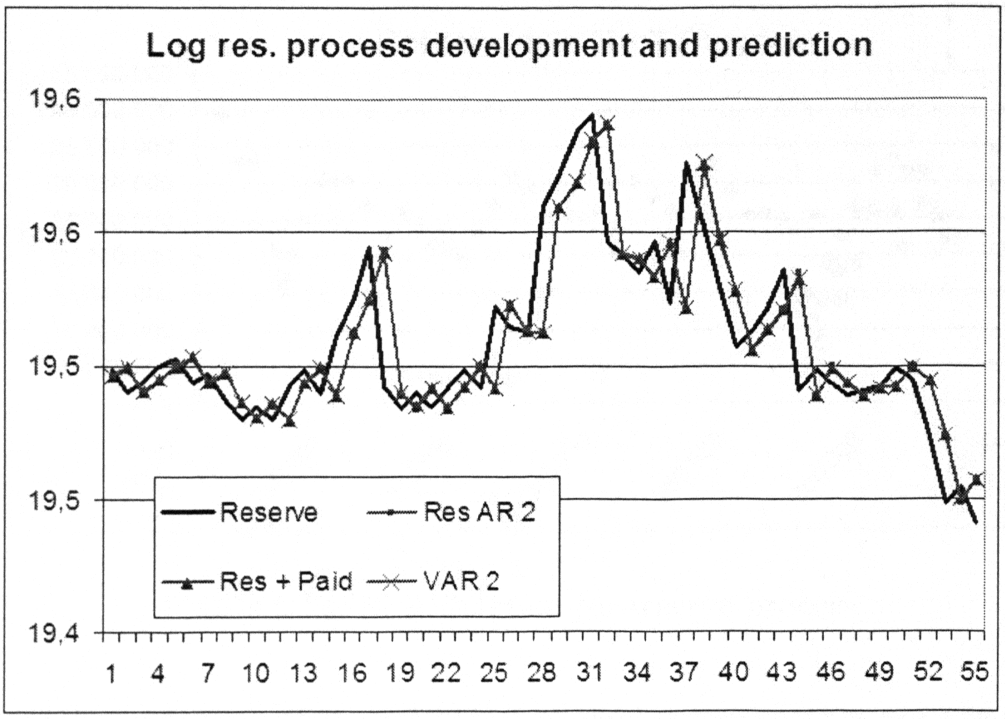

Figure 6.12: Projection of Reserve process - logarithmic transformation

Approach when using of logarithmic data is in this context more adequate since we can inspect it in the terms of so called m-r diagram which recommends us to use logarithmic transformation, see following graphs.

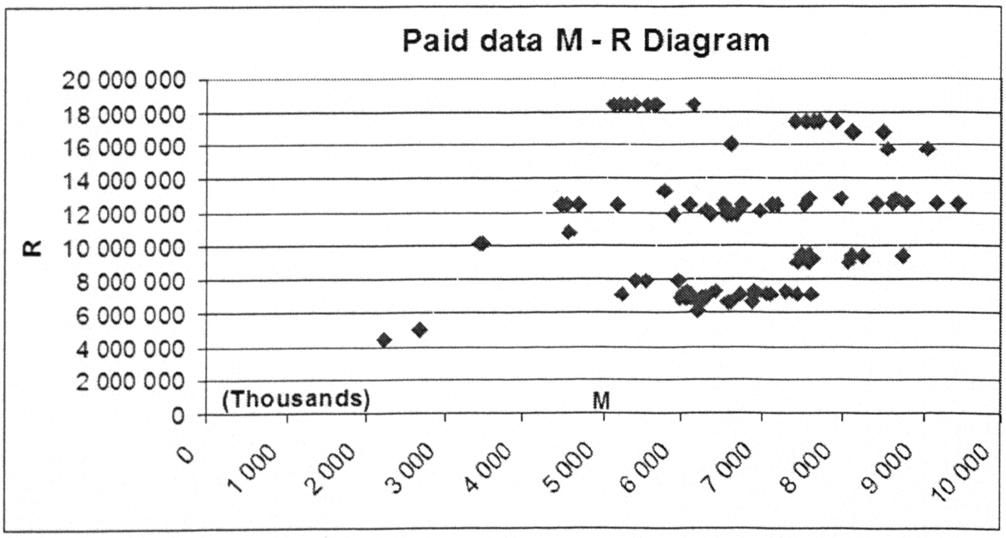

Figure 6.13: M-R diagram for paid process

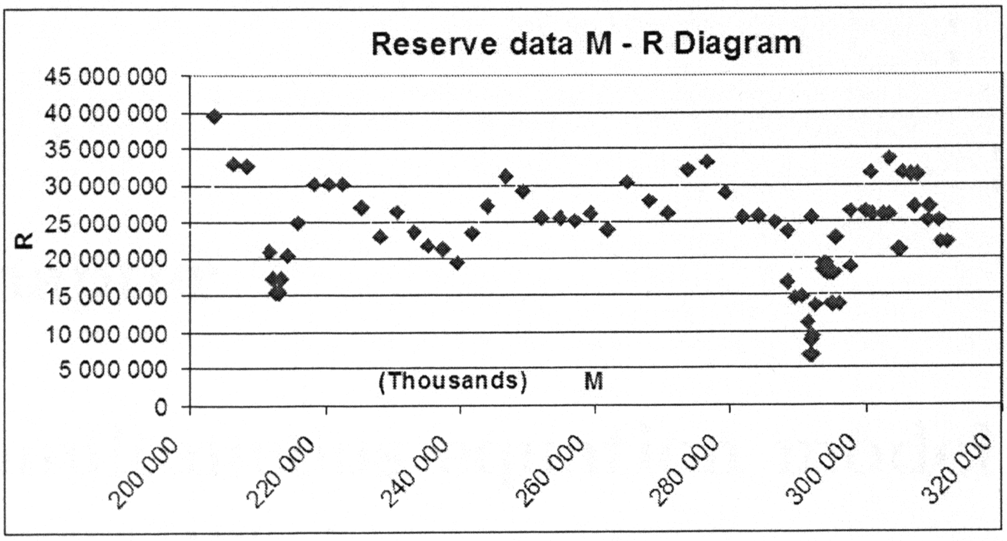

Figure 6.14: M-R diagram for reserve process

This result confirming that past values of paid compensation are not useful for prediction of future reserving (after interpretation for accident years) are giving an interpretation to the bivariate model based on paid and reserve values and their dependency in the one direction only rather than the MCL model for Paid and Incurred data with both side dependency of cumulative data.

Chapter 7

Simultaneous equation model in non life insurance

7.1 Motivation

The generalisation of reserving methods working more or less in the run off schemes is later extended to incorporate other information that is not explicitly readable from the data of schemes. A lot of papers was written devoted to the themes of using Bayesian statistical techniques and incorporating prior information or expert opinion into the model, see for example England, Verall (2005).

Our aim is not to use Bayesian approach again but we would like to formulate some econometrical model for non life insurance that could be used for prediction of overall liability volume and thus for reserving as well. The advantage of this approach is identification of the key factors influencing the value of the claims.

These exogenous factors affecting claims amount are (e.g. in MTPL) numbers of injured and killed in the accidents and special judical influence (compensation of social status, surviving dependants and other factors not explainable by development factors only).

One possibility is to use some regression approach like Probabilistic Trend Family (PTF) enabling quite flexible structure of time intervals related to explanatory variables, see for example Barnett, Zehnwirth (1999). This model incorporated the exogenous influences indirectly and its practical application were presented in the paper of Jedlička, Kočvara, Strnad (2005). Second proposed alternative is to extend claims model using directly the respective exogenous variables as appropriate for analysed lines of business.

Similar approach of incorporation econometrics into actuarial mathematics was made by Cipra (1998) for life insurance using respective relation among data of balance sheets in order to estimate future development of cash flow of life insurer.

7.2 Formulation of the non life model

We will deal with illustrative MTPL data and so that the choice of available variables is quite natural. It is based on records of MTPL claims that are suggested to be explained by data of transport accidents investigated by police.

We have performed the models for two different level of data aggregation. The first one is based on yearly aggregated data and the second one comes from quarterly aggregated values. The advantage of first approach is the fact that the model is not affected by seasonal effects but the lack of observation is the drawback of that level of aggregation and vice versa holds for quarterly data.

Firstly we would like to deal with the explanation of numbers of claims. It is quite natural for bodily injury claims that it will be based on numbers of seriously injured victims and fatalities. Equation suggested for explanation of number of seriously injures is as follows

$$S_t = \alpha_{1,0} + \alpha_{1,1} A_t + \alpha_{1,2} time + \varepsilon_{1,t}$$

Number of bodily injury is connected (since we analyse MTPL line of business) with number of injured victims

$$BI_t = \alpha_{2,0} + \alpha_{2,1} S_t + \alpha_{2,2} time + \varepsilon_{2,t}$$

In both model it is useful to add time as exogenous variable in order to analyse trend patterns. Crucial equation is the following one that explains financial volume of bodily injury claims

$$BIV_t = \alpha_{3,0} + \alpha_{3,1}LOS_t + \alpha_{3,2}FIS_t + \alpha_{4,2}time + \alpha_{5,2}BI_t + \varepsilon_{5,t}$$

The explanatory variable is not only the number of bodily-injury claims but also the indicators of times where significant changes of bodily injury compensation started. It is especially time of jump valorisation of point value for loss of social status in 2002 LOS_t and also beginning of survivors compensation in 2004 FIS_t.

We can solve similarly the model for property damage as well. We started from equation

$$LO_t = \alpha_{4,0} + \alpha_{4,1}A_t + \varepsilon_{4,t}$$

Motivation of this equation comes from the fact that number of claims without injuries are based somehow on the numbers of accidents and volume of the "routine" claims is affected by time inflation and numbers of reported loss occurrences

$$PDV_t = \alpha_{5,0} + \alpha_{5,1}LO_t + \alpha_{5,2}time + \varepsilon_{5,t}$$

Final "real" equation explains us the value of earned premium in the sense of numbers of insured vehicles and time indicator as well

$$EP_t = \alpha_{6,0} + \alpha_{6,1}V_t + \alpha_{6,2}TID_t + \varepsilon_{6,t}$$

and identity equation for overall liability is then formulated trivially as the sum of property and bodily-injury volume of claims

$$INC_t = PD_t + BIV_t$$

7.3 Numerical results

We work with following illustrative data coming from MTPL line of business. It could be divided into two parts. The first one is connected more or less with exogenous variables that affecting endogenous variables. The interesting connection is also that the data in the first table are widely available, the second table shows illustration of more specific and technical data after some transformations:

time	FIS_t	LOS_t	TID_t	V_t	A_t	F_t	S_t	L_t
1	0	0	0	4 082	52.925	0.265	1.085	5.083
2	0	0	0	4 082	51.332	0.342	1.402	7.468
3	0	0	0	4 082	50.914	0.368	1.647	7.865
4	0	0	0	4 082	56.345	0.361	1.391	6.647
5	0	0	0	4 130	41.343	0.237	1.011	5.191
6	0	0	0	4 220	44.178	0.279	1.472	7.532
7	0	0	0	4 216	46.846	0.371	1.667	8.429
8	0	0	0	4 181	53.297	0.332	1.343	7.145
9	0	1	0	4 230	42.140	0.235	0.971	5.341
10	0	1	0	4 263	45.377	0.33	1.479	7.651
11	0	1	0	4 364	48.933	0.384	1.646	8.777
12	0	1	0	4 374	54.268	0.365	1.396	7.244
13	0	1	1	4 326	44.594	0.254	0.955	5.401
14	0	1	1	4 331	48.503	0.35	1.427	8.408
15	0	1	1	4 334	50.584	0.398	1.7	9.493
16	0	1	1	4 190	52.170	0.317	1.171	7.01
17	1	1	1	4 124	48.070	0.202	0.839	5.411
18	1	1	1	4 166	48.430	0.31	1.291	7.891
19	1	1	1	4 188	48.442	0.338	1.534	8.867
20	1	1	1	4 242	51.542	0.365	1.214	7.374
21	1	1	1	4 194	48.863	0.195	0.785	5.339
22	1	1	1	4 319	47.702	0.273	1.218	7.672
23	1	1	1	4 346	49.278	0.326	1.299	8.207
24	1	1	1	4 312	53.419	0.333	1.094	6.756

Figure 7.1: Quarterly data for simultaneous equation 1

time	EP_t	LO_t	BI_t	PD_t	BIV_t	PDV_t	INC_t
2 000,00	3 109 130	67,30027	2,42675	64,87352	461 797	1 910 478	2 372 275
2 000,25	2 144 326	71,03004	2,42675	68,60329	461 797	1 910 478	2 372 275
2 000,50	2 419 776	65,61141	2,42675	63,18466	461 797	1 910 478	2 372 275
2 000,75	2 152 141	66,38528	2,42675	63,95853	461 797	1 910 478	2 372 275
2 001,00	2 908 283	73,90715	2,7675	71,13965	613 620	2 034 807	2 648 427
2 001,25	3 246 010	79,10649	2,7675	76,33899	613 620	2 034 807	2 648 427
2 001,50	3 012 290	77,56566	2,7675	74,79816	613 620	2 034 807	2 648 427
2 001,75	2 990 929	80,9477	2,7675	78,1802	613 620	2 034 807	2 648 427
2 002,00	3 245 449	79,67694	2,866	76,81094	813 632	2 073 283	2 886 915
2 002,25	3 443 872	82,23952	2,866	79,37352	813 632	2 073 283	2 886 915
2 002,50	3 397 827	79,52577	2,866	76,65977	813 632	2 073 283	2 886 915
2 002,75	3 500 487	76,92976	2,866	74,06376	813 632	2 073 283	2 886 915
2 003,00	3 665 062	75,97197	2,91325	73,05872	860 949	2 199 751	3 060 700
2 003,25	3 743 313	76,49488	2,91325	73,58163	860 949	2 199 751	3 060 700
2 003,50	3 858 146	76,00492	2,91325	73,09167	860 949	2 199 751	3 060 700
2 003,75	3 906 031	70,59523	2,91325	67,68198	860 949	2 199 751	3 060 700
2 004,00	3 988 745	75,04736	3,12675	71,92061	1 055 206	2 443 292	3 498 498
2 004,25	4 043 632	74,66841	3,12675	71,54166	1 055 206	2 443 292	3 498 498
2 004,50	4 154 573	70,39401	3,12675	67,26726	1 055 206	2 443 292	3 498 498
2 004,75	4 192 699	76,51222	3,12675	73,38547	1 055 206	2 443 292	3 498 498
2 005,00	4 183 960	76,50014	2,80175	73,69839	1 144 043	2 502 601	3 646 644
2 005,25	4 104 428	77,17912	2,80175	74,37737	1 144 043	2 502 601	3 646 644
2 005,50	4 314 332	73,9362	2,80175	71,13445	1 144 043	2 502 601	3 646 644
2 005,75	4 322 365	75,68453	2,80175	72,88278	1 144 043	2 502 601	3 646 644

Figure 7.2: Quarterly data for simultaneous equation 2

Limitation of the data is the fact that the division onto property damage and bodily injury is not available for sub year resolution. So we have to deal with trends only.

Alternatively we can work with data aggregated onto year's basis despite some drawbacks mentioned above.

time	LOS_t	FIS_t	TID_t	V_t	A_t	F_t	S_t	L_t
2000	0	0	0	4 082	211 516	1 336	5 525	27 063
2001	0	0	0	4 187	185 664	1 219	5 493	28 297
2002	1	0	0	4 308	190 718	1 314	5 492	29 013
2003	1	0	1	4 295	195 851	1 319	5 253	30 312
2004	1	1	1	4 180	196 484	1 215	4 878	29 543
2005	1	1	1	4 292	199 262	1 127	4 396	27 974

Figure 7.3: Yearly data for simultaneous equation 1

time	EP_t	LO_t	BI_t	BIV_t	PDV_t	INC_t
2000	9 825 373	270 327	9 707	1 847 187	7 641 912	9 489 100
2001	12 157 512	311 527	11 070	2 454 481	8 139 229	10 593 710
2002	13 587 634	318 372	11 464	3 254 529	8 293 130	11 547 659
2003	15 172 551	299 067	11 653	3 443 796	8 799 005	12 242 800
2004	16 379 649	296 622	12 507	4 220 825	9 773 166	13 993 991
2005	16 925 084	303 300	11 207	4 576 171	10 010 403	14 586 575

Figure 7.4: Yearly data for simultaneous equation 2

If we perform for example two stage least square methods onto quarterly data we will get following consistent estimates of the equations. This estimates were computed using R software using library systemfit:

$$\widehat{S}_t = 86.056 + 0.017A_t - 42.7time$$

$$\widehat{BI}_t = -13.11 - 0.00039S_t + 0.0671time$$

There is not so large problem that the explanatory power of that two equation is rather weak. The numbers of claims are known in rather short time after its occurrence. It is more important that fitted value compare very well to observed ones in the following estimated equation:

$$\widehat{BIV}_t = -137,800,000 + 89,570LOS_t + 1,055,600FIS_t + 69,212time + 244BI_t$$

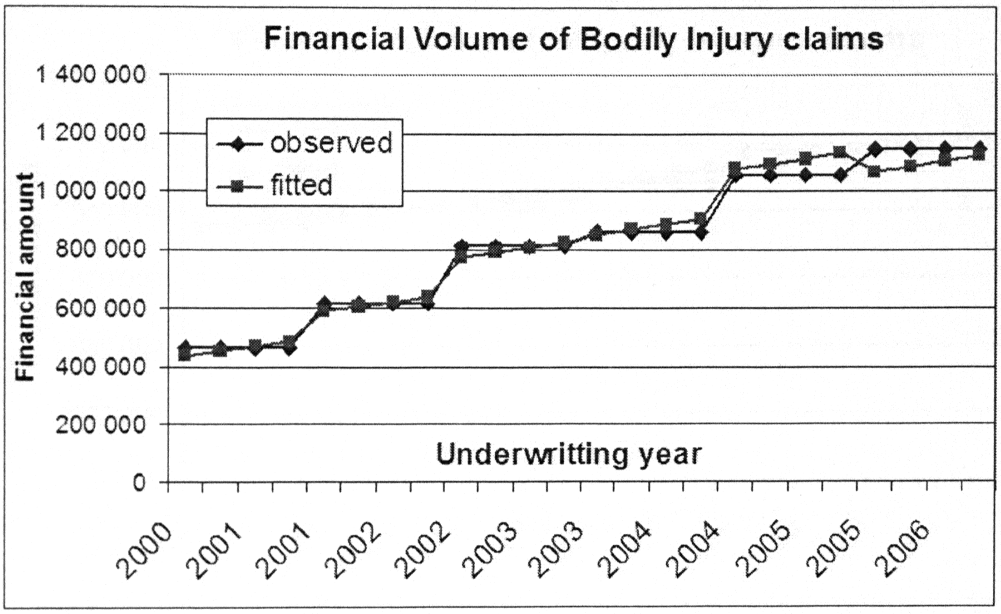

Figure 7.5: Model for amount of Bodily injury claims

The equation for loss occurrences is estimated as follows:

$$\widehat{LO}_t = 147893 - 1.485A_t$$

Corresponding financial amount arising from this type of claim could be estimated even with better interpretation:

$$\widehat{PDV}_t = -241,718,100 + 2.834LO_t + 121886.7time$$

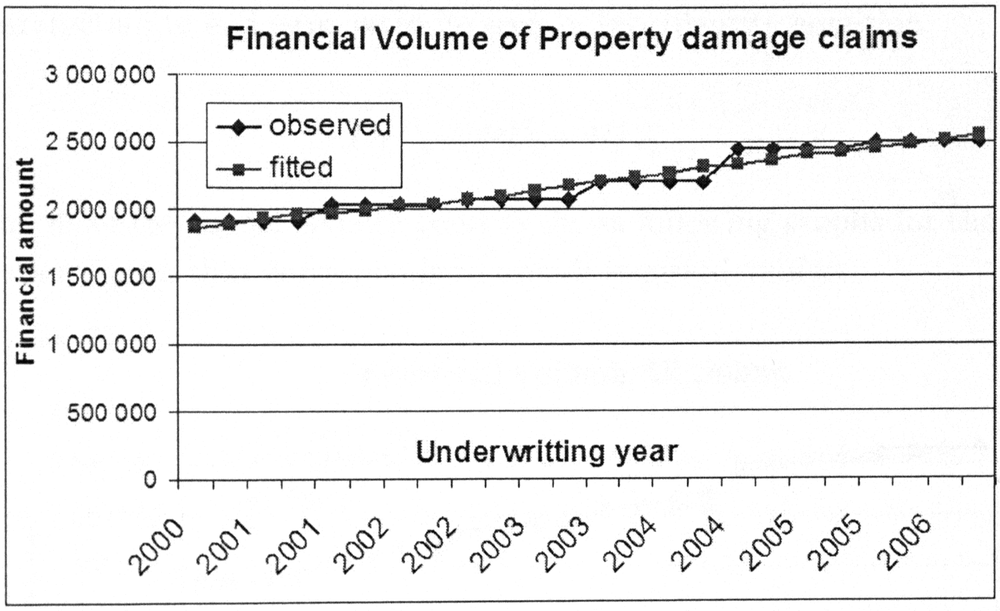

Figure 7.6: Model for amount of Property damage claims

Finally amount of earned premium depends on vehicle numbers and some time indicator in quite reasonable way as well.

$$\widehat{EP}_t = -6391200 + 1,7852V_t + 932800TID_t$$

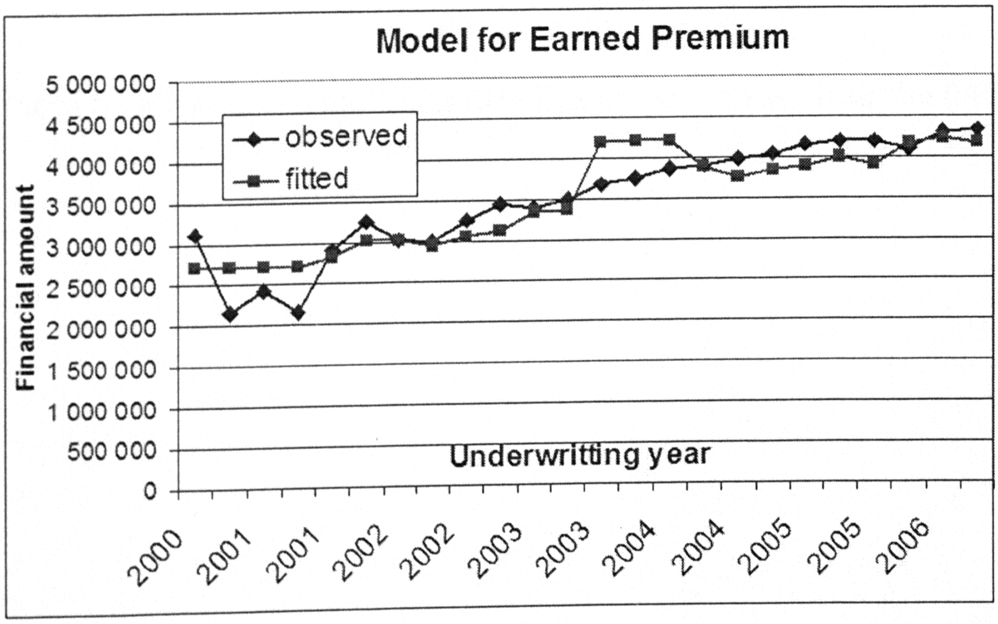

Figure 7.7: Model for amount of Earned premium

No parameters to estimate could be seen in last identity equation

$$INC_t = PD_t + BIV_t$$

Overall fit of the model is quite good as shows following graphs for the most important figure that corresponds to overall incurred values:

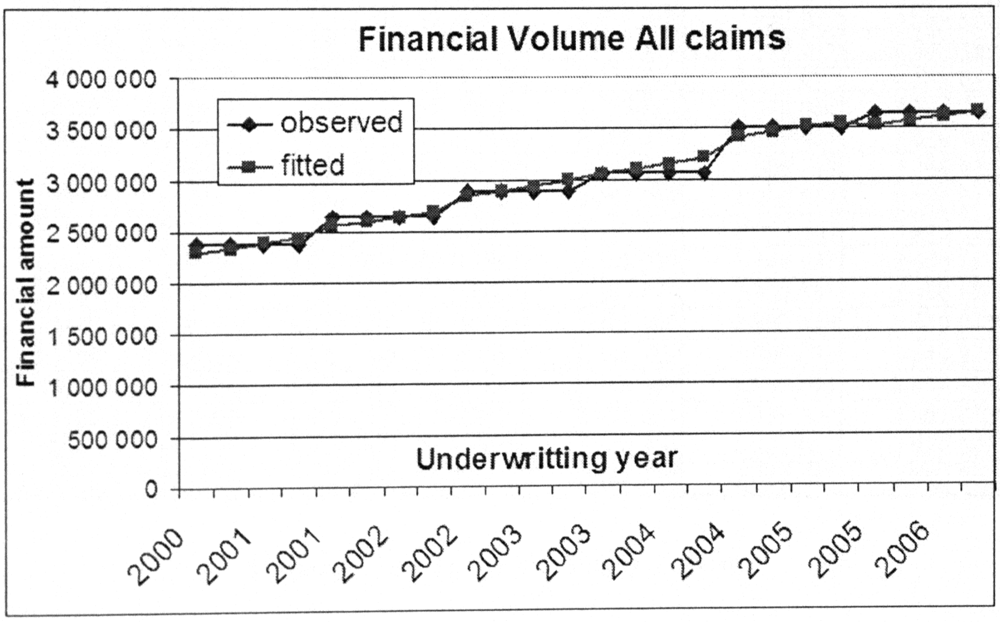

Figure 7.8: Chart of overall fit for Incurred value

If we focus on all equation and their fitted values we can see it in the following table:

fitted values of simultaneous equation model						
1) S	2) BI	3] BIV	4) LO	5) PDV	6) EP	7) INC
1,480706	2,701775	435 471	69,174235	1 865 399	2 718 131	2 300 870
1,442842	2,593412	452 777	71,541186	1 885 270	2 718 131	2 338 047
1,425025	2,513474	470 082	72,162269	1 931 143	2 718 131	2 401 225
1,507004	2,631315	487 388	64,092645	1 959 415	2 718 131	2 446 803
1,240355	2,798108	588 055	86,383292	1 968 509	2 824 810	2 556 564
1,278041	2,632899	605 361	82,170921	1 984 203	3 025 519	2 589 564
1,312877	2,572698	622 666	78,206687	2 019 054	3 018 380	2 641 721
1,412259	2,717384	639 972	68,621501	2 039 914	2 939 576	2 679 886
1,211214	2,881018	770 954	85,199073	2 073 998	3 049 010	2 844 952
1,255759	2,697255	788 260	80,389393	2 097 186	3 123 516	2 885 446
1,305746	2,648109	805 565	75,105728	2 135 371	3 347 097	2 940 936
1,386087	2,763581	822 871	67,178745	2 173 221	3 370 086	2 996 092
1,210345	2,954455	851 736	81,907738	2 206 415	4 196 484	3 058 151
1,266355	2,784903	869 042	76,099570	2 235 401	4 206 716	3 104 442
1,291176	2,693911	886 347	73,007526	2 267 265	4 213 271	3 153 613
1,307552	2,919524	903 653	70,650976	2 313 112	3 893 549	3 216 765
1,226913	3,067368	1 078 822	76,742941	2 330 931	3 744 987	3 409 752
1,222370	2,905712	1 096 127	76,208037	2 362 479	3 839 807	3 458 607
1,211890	2,826563	1 113 433	76,190206	2 405 100	3 888 489	3 518 533
1,254097	2,969669	1 130 739	71,584087	2 418 183	4 009 232	3 548 921
1,197704	3,155806	1 068 536	75,564666	2 448 689	3 900 504	3 517 225
1,167210	3,001650	1 085 842	77,289732	2 477 231	4 179 769	3 563 073
1,183415	2,986454	1 103 147	74,948040	2 516 920	4 240 047	3 620 067
1,243383	3,084162	1 120 453	68,795156	2 542 423	4 164 423	3 662 876

Figure 7.9: Table of fitted values

This approach is also very useful for estimating of the liability for subsequent accident years using the expected values of exogenous variables and to analyse time trends that affect the claims amount (typically loss inflation). The crucial task for application is that we can estimate the overall liability (that usually developed a few years) based on some exogenous factors that are known quite fast after the end of respective calendar year.

Limitation of one part of this model into future is the discussed problem of changing the limit when the accident is requested to be investigated by police. However the part of the model for bodily injury claims should be useful after any change in this limit as well since it is assumed that the accidents where some injury happens will have to be investigated anyway.

Chapter 8

Conclusion

This thesis firstly reviewed the current situation with application of the most popular chain ladder techniques and illustrated its limitation. After that the recently made generalisation (including Munich Chain Ladder and Multivariate Chain Ladder) were presented and later on discussed using author's experience from practical point of view and judgement.

After this first reviewing part of the thesis, generalisation of Munich Chain Ladder was suggested including application of robust regression, problems of elasticity of reserves and computation of mean square error what was illustrated in practical situation as well.

Moreover we introduced concepts how to perform multivariate generalisation of Munich Chain Ladder what helps to fit together paid and incurred inter dependencies with correlation among differen lines of business what can help with better evaluation of technical reserves in some cases where these dependencies are significant.

However there still remained some open problems for specific portfolios that can not be properly handled using this sort of methods and so we tried to incorporate deeper econometrical insight onto feasible data what was reflected in bivariate process and respective model dealing with long tailed lines, introducing a concept of Granger causality with relation of paid and reserves data and finally proposal of simultaneous equation model incorporating exogenous information onto development of claims value.

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