PHD THESIS:

Easton's theorem and large cardinals

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Abstract

The continuum function $\alpha \mapsto 2^\alpha$ on regular cardinals is known to have great freedom. Say that F is an *Easton function* iff for regular cardinals α and β, $\mathrm{cf}(F(\alpha)) > \alpha$ and $\alpha < \beta \to F(\alpha) \leq F(\beta)$. The classic example of an Easton function is the *continuum function* $\alpha \mapsto 2^\alpha$ on regular cardinals. If GCH holds then any Easton function is the continuum function on regular cardinals of some cofinality-preserving extension $V[G]$; we say that F is *realised* in $V[G]$. We say that κ is $F(\kappa)$*-hypermeasurable* iff there is an elementary embedding $j : V \to M$ with critical point κ such that $H(F(\kappa))^V \subseteq M$; j will be called a *witnessing embedding.* Our original results concern two topics:

Compatibility of a continuum function given by F and of measurable cardinals. We will show that if GCH holds then for any Easton function F there is a cofinality-preserving generic extension $V[G_0]$ such that if κ, closed under F, is $F(\kappa)$-hypermeasurable in V and there is a witnessing embedding j such that $j(F)(\kappa) \geq F(\kappa)$, then κ will remain measurable in $V[G_0]$.

Compatibility of a continuum function given by F and of singular strong limit cardinals failing SCH.

- We will show that if GCH holds then for any Easton function F there is a cardinal-preserving generic extension $V[G_1]$ such that if κ, closed under F, is $F(\kappa)$-hypermeasurable in V and there is a witnessing embedding j such that $j(F)(\kappa) \geq F(\kappa)$, then κ has cofinality ω in $V[G_1]$ and $V[G_1] \models 2^\kappa = F(\kappa)$. $V[G_1]$ is a generic extension of $V[G_0]$ obtained by a forcing which adds an unbounded ω-sequence to each such κ.

- We will show that if GCH holds then for any Easton function F which satisfies some mild restrictions and preserves all Mahlo cardinals there is a cardinal-preserving generic extension $V[G_2]$ realising F such that if κ is $F(\kappa)$-hypermeasurable in V, then κ has cofinality ω in $V[G_2]$ and $V[G_2] \models 2^\kappa = F(\kappa)$. The mild restrictions mentioned above require that F preserves GCH at some places, which contrasts with the restriction placed on F in $V[G_1]$. The forcing used to obtain this model is the *extender based Prikry forcing.*

- We say that F is *toggle-like* if for all regular cardinals α, $F(\alpha)$ is either α^+ or α^{++} (F "toggles" GCH on and off). Let Σ be a subclass of κ^{++}-hypermeasurable cardinals. We will show that if GCH holds and F is toggle-like and $F(\kappa^+) = \kappa^{++}$ for every measurable cardinal κ, then there is a cardinal-preserving generic extension $V[G_3]$ realising F where SCH fails exactly on elements of Σ. It is our conviction that the restriction $F(\kappa^+) = \kappa^{++}$ for every measurable cardinal κ can be removed, thus obtaining an almost optimal result.

1 Introduction

This thesis deals with the following problem: Given an Easton function F defined on regular cardinals, what does F (and the universe V) have to satisfy so that there is a class generic extension V^* so that

1. V^* has the same cofinalities as V, F is realised in V^* *and* the large cardinal structure of V is preserved in V^* (we shall concentrate on measurable cardinals).

2. V^* has the same cardinals as V, F is realised in V^* *and* some global pattern of failures of SCH is achieved in V^* which corresponds to the large cardinal structure in V.

These questions are a generalization of the results of W.B.Easton in [4]. We are motivated by the conviction that large cardinals play an important role in set theory and so it is of great interest to find out how they interact with the continuum function. The same holds for the failure of SCH at singular cardinals, which is inherently a large cardinal problem.

The original results answering, at least partially, the above questions are given in Theorems 5.7, 5.17, 6.6, 6.21, and 6.28. The argument for Theorems 5.7 and 5.17 was accepted for publication [7].

The thesis is divided as follows: In Section 2 we review the original proof in [4] and some basic notions and concepts concerning extenders and hypermeasurable (strong) cardinals. In Section 3 we show that preservation of large cardinals introduces new and important considerations if we want to realise an Easton function F (we for instance show that the original forcing in [4] may actually kill many large cardinals). In Section 4 we review and generalize slightly a new way how to preserve measurable cardinals while failing GCH. This technique is based on [9]. In Section 5, we prove original results concerning the question 1. above. In Section 6, we prove original results concerning the question 2. above.

2 Preliminaries

In this section we review some background results and techniques which are referred to later in the text.

2.1 Notational conventions

Our notation is standard, following [18]. In particular, $\mathbb{P}_\kappa = \langle(\mathbb{P}_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa\rangle$ denotes a forcing iteration of length κ where $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$ for every $\alpha < \kappa$. If $\alpha < \kappa$ then $\mathbb{P}_\alpha$ denotes the restriction of $\mathbb{P}_\kappa$ to α. For each $\alpha < \kappa$, we can factor $\mathbb{P}_\kappa$ into the forcing $\mathbb{P}_\alpha$ and the tail iteration of length $\kappa \setminus \alpha$ defined in $V^{\mathbb{P}_\alpha}$; we write this as

$$\mathbb{P}_\kappa \cong \mathbb{P}_\alpha * \mathbb{P}_\kappa \setminus \mathbb{P}_\alpha, \tag{2.1}$$

where $\mathbb{P}_\kappa \setminus \mathbb{P}_\alpha$ is the tail iteration.

Similarly, if G_κ is a generic filter for $\mathbb{P}_\kappa$, we denote as G_α its restriction to $\mathbb{P}_\alpha$. Very often we use capital G to refer to a generic filter for an iteration, while small g refers to a generic filter for a successor step of an iteration: for instance a generic $G_{\alpha+1}$ for $\mathbb{P}_{\alpha+1}$ factors as $G_\alpha * g_\alpha$.

To avoid confusion, we explicitly state that we are going to use the notions κ-distributive, κ-closed and their variants for $<\kappa$-distributive and $<\kappa$-closed (just as κ-cc is in fact used for antichains of size $<\kappa$).

We denote as On the class of ordinal numbers, and sometimes we denote as Reg the class of regular cardinals.

2.2 Easton class forcing

We will shortly review an original result of Easton [4] who showed that there is very little ZFC can prove about the continuum function on regular cardinals. It should be emphasized that this result really holds just for ZFC, and fails badly for extensions of ZFC of the type ZFC + (there are large cardinals).[1] See discussion in Section 3 and following.

Definition 2.1 *A class function F defined on all regular cardinals is called an* Easton function *if it satisfies the following two conditions Let κ, μ be arbitrary regular cardinals:*

1. *If $\kappa < \mu$, then $F(\kappa) \leq F(\mu)$;*
2. *$\kappa < \mathrm{cf}(F(\kappa))$.*

[1]By the rather vague term "large cardinal" we will generally mean a cardinal which implies the existence of a non-trivial embedding from V to some transitive model M (hence we will start from a measurable cardinal).

Note that Cantor's theorem $\kappa < 2^\kappa = F(\kappa)$ is implied by (2) above.

We say that an Easton function F is *realised* in some cardinal-preserving extension $V^* \supseteq V$ of V if F is the continuum function $\alpha \mapsto 2^\alpha$ in V^*, i.e. for every α in the domain of F, $V^* \models 2^\alpha = F(\alpha)$.

Theorem 2.2 (Easton) *Assume GCH. If F is an Easton function then there is a cofinality-preserving forcing extension realising F. Hence the two conditions in Definition 2.1 above are the only conditions provable in ZFC about the continuum function.*

We will not give a detailed proof of the theorem (see for instance [22]) but will emphasize some points which are interesting for us with respect to the forcing which Easton used to prove the theorem.

Definition 2.3 *Let α be a regular cardinal and β an ordinal greater than 0. Then* $\mathrm{Add}(\alpha, \beta)$ *denotes the forcing for adding β-many Cohen subsets of α. For notational convenience we will construe* $\mathrm{Add}(\alpha, \beta)$ *as the $< \alpha$-supported product of* $\mathrm{Add}(\alpha, 1)$ *of length β, where conditions in* $\mathrm{Add}(\alpha, 1)$ *are functions from α to 2 with domain of size less than α.*

Easton used the product of $\mathrm{Add}(\lambda, F(\lambda)$ for regular λ's to achieve the desired result. He used a special kind of support, which is now called the *Easton support* (denoted by the superscript $\prod^E$):

$$\prod_{\lambda \in \mathrm{Reg}}^{E} \mathrm{Add}(\lambda, F(\lambda)) \tag{2.2}$$

Definition 2.4 *We say that a product $\prod_{\lambda \in \mathrm{Reg}} \mathbb{P}_\lambda$ for some forcing notions $\mathbb{P}_\lambda$ in V has the* Easton support *if the support of $p \in \prod_{\lambda \in \mathrm{Reg}} \mathbb{P}_\lambda$ is bounded below each $\lambda \in \mathrm{Reg}$.*

Notice that we could say equivalently that the support of p is bounded below each regular limit cardinal λ (conditions are trivially bounded if λ is a successor cardinal). Since Easton started with a ground model satisfying GCH, a regular limit cardinal is the same as a (strongly) inaccessible cardinal, and hence the Easton support is a non-trivial requirement only if inaccessible cardinals are present in V.

Remark 2.5 Every condition in the forcing (2.2) is a proper class and so the product (2.2) formally does not exist (as it would contain as elements proper classes). In order to rectify this situation, we will identify (2.2) with the union

$$\mathbb{P}^F =_{df} \bigcup_{\lambda \in \mathrm{Reg}} \mathbb{P}^F_{\leq \lambda}, \tag{2.3}$$

where $\mathbb{P}^F_{\leq\lambda}$ equals to $\prod^E_{\lambda\in\mathrm{Reg}\cap\lambda+1}\mathrm{Add}(\lambda, F(\lambda))$. To define ordering on $\mathbb{P}^F$, we identify $\mathbb{P}^F_{\leq\bar\lambda}$ with the pointwise image of the obvious complete embedding from $\mathbb{P}^F_{\leq\bar\lambda}$ to $\mathbb{P}^F_{\leq\lambda}$ (whenever $\bar\lambda < \lambda$ are regular). In practice, however, we will use the "naive" representation in (2.2). Even with the formally correct representation in (2.3), it is still not automatic that a generic extension via $\mathbb{P}^F$ satisfies all the axioms of ZF since $\mathbb{P}^F$ is a proper class. The forcing $\mathbb{P}^F$ is however very mild, in particular it factors at each regular λ into a λ^+-closed upper part (a proper class) and a λ^+-cc lower part (a set); see below (2.4). The closure of the upper part (in combination with Lemma 2.6 (1)) is enough to conclude that the axioms of ZF are satisfied in a generic extension via $\mathbb{P}^F$. A detailed account can be found in [18], p. 236, or in [6], p. 39.

The preservation of cofinalities in a generic extension via $\mathbb{P}^F$ follows from the following product analysis (this lemma is often called "Easton's lemma").

Lemma 2.6 *Assume $\mathbb{P}, \mathbb{Q} \in V$ are forcing notions, and $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is κ-closed. Then the following holds:*

1. $1_\mathbb{P} \Vdash \check{\mathbb{Q}}$ *is κ-distributive;*
2. $1_\mathbb{Q} \Vdash \check{\mathbb{P}}$ *is κ-cc;*
3. *As a corollary, if G is generic for $\mathbb{P}$ over V and H is generic for $\mathbb{Q}$ over V, then $G \times H$ is generic for $\mathbb{P} \times \mathbb{Q}$, i.e. G and H are mutually generic.*

Proof. Ad (1). Assume $\dot{f}$ is forced by $\langle \tilde{p}, \tilde{q}\rangle$ to be a $\mathbb{P} \times \mathbb{Q}$-name for a function from $\mu < \kappa$ into ordinal numbers. We will find a condition $q_\infty \leq \tilde{q}$ such that for every G generic for $\mathbb{P}$ containing $\tilde{p}$ and H generic for $\mathbb{Q}$ over $V[G]$ containing q_∞, the condition q_∞ defines the function $\dot{f}^{G\times H}$ in $V[G]$.

The proof is an analogue of the proof of the fact that a κ-closed forcing notion does not add a new μ-sequence for $\mu < \kappa$, except that instead of determining a single value of $\dot{f}(\alpha)$, $<\kappa$ different values – corresponding to an antichain in $\mathbb{P}$ – will be found.

In preparation for the argument define for each $\alpha < \mu < \kappa$ and some $q_\alpha \leq \tilde{q} \in \mathbb{Q}$ the following procedure. Simultaneously construct a decreasing sequence $\langle q^\xi_\alpha \,|\, \xi < \zeta_\alpha\rangle$ of conditions in $\mathbb{Q}$ and a sequence of pairwise incompatible conditions below $\tilde{p}$, $A_\alpha = \{p^\xi_\alpha \,|\, \xi < \zeta_\alpha\}$, and ordinals a^ξ_α for $\xi < \zeta_\alpha$ such that $\langle q^\xi_\alpha, p^\xi_\alpha\rangle \Vdash \dot{f}(\alpha) = a^\xi_\alpha$. As $\mathbb{P}$ is κ-cc, the construction will terminate – yielding a maximal antichain A_α – at some $\zeta_\alpha < \kappa$. Let $\tilde{q}_\alpha$ be the lower bound of q^ξ_α for $\xi < \zeta_\alpha$ (such bound exists as $\mathbb{Q}$ is κ-closed).

By induction carry out the above procedure for all $\alpha < \mu$, making sure that the resulting conditions $\tilde{q}_\alpha$ form a decreasing sequence $\langle \tilde{q}_\alpha \,|\, \alpha < \mu\rangle$. Let q_∞ be the lower bound of this sequence.

Let G be a generic for $\mathbb{P}$ containing $\tilde{p}$ and H generic for $\mathbb{Q}$ over $V[G]$ containing q_∞. It follows that $\dot{f}^{G\times H}(\alpha) = x$ iff x is identical to $a_\alpha^{\xi_0}$, where $p_\alpha^{\xi_0}$ is the unique element of A_α in G (A_α is a maximal antichain below $\tilde{p}$, and so is hit by all generics G containing $\tilde{p}$). This definition takes place in $V[G]$ and hence $\dot{f}^{G\times H} \in V[G]$.

Ad (2). Assume $\tilde{q} \in \mathbb{Q}$ forces that $\dot{A}$ is a sequence of length κ of elements of $\mathbb{P}$ in $V^{\mathbb{Q}}$. Construct by induction a decreasing sequence $\langle q_\xi \,|\, \xi < \kappa\rangle$ of conditions below $\tilde{q}$ such that q_ξ decides the value of $\dot{A}$ at ξ, i.e. $q_\xi \Vdash \dot{A}(\xi) = a_\xi$. At limit stage of the construction, we can take a lower bound as $\mathbb{Q}$ is κ-closed. It follows that $\tilde{A} = \{a_\xi \,|\, \xi < \kappa\}$ is an antichain, if $\tilde{q}$ forced $\dot{A}$ to be an antichain. This is a contradiction.

Ad (3). It is enough to show that G hits all maximal antichains of $\mathbb{P}$ in $V[H]$. As by (2) $\mathbb{P}$ is still κ-cc in $V[H]$, and $\mathbb{Q}$ does not add new $<\kappa$ sequences, it follows that the maximal antichains in $V[H]$ coincide with the the maximal antichains in V.

Note. The forcing notion $\mathbb{Q}$ may be a class forcing, providing that the forcing relation is definable so that the truth lemma for $\mathbb{Q}$ holds. Sufficient conditions are identified in the standard reference book for class forcing [6]. The forcings we will be dealing with will satisfy these sufficient conditions.

(Lemma 2.6) □

Lemma 2.6 has an immediate application to $\mathbb{P}^F$: if $\lambda \in \mathrm{Reg}$ then the product $\mathbb{P}^F$ can be factored as

$$\mathbb{P}^F \cong \prod^E_{\lambda'\in\mathrm{Reg}\cap\lambda+1} \mathrm{Add}(\lambda', F(\lambda')) \times \prod^E_{\lambda'\in\mathrm{Reg}\setminus\lambda+1} \mathrm{Add}(\lambda', F(\lambda')) \qquad (2.4)$$

Denote $\prod^E_{\lambda'\in\mathrm{Reg}\cap\lambda+1} \mathrm{Add}(\lambda', F(\lambda'))$ by $\mathbb{P}^F_{\leq\lambda}$ and $\prod^E_{\lambda'\in\mathrm{Reg}\setminus\lambda+1} \mathrm{Add}(\lambda', F(\lambda'))$ by $\mathbb{P}^F_{>\lambda}$.[2] $\mathbb{P}^F_{>\lambda}$ is obviously λ^+-closed. Assuming GCH, a Δ-lemma argument combined with the Easton support of $\mathbb{P}^F_{\leq\lambda}$ implies that $\mathbb{P}^F_{\leq\lambda}$ is λ^+-cc. The argument that $\mathbb{P}^F$ preserves cofinalities is now straightforward:

Sketch of the proof of Theorem 2.2. We will show that the forcing $\mathbb{P}^F$ preserves all cofinalities. Let G be a generic filter for $\mathbb{P}^F$ and assume for contradiction that some regular cardinal κ is singularized in $V[G]$ and has cofinality λ. Since λ is regular in $V[G]$, it must be regular in V as well. Factor $\mathbb{P}^F$ as $\mathbb{P}^F_{\leq\lambda} \times \mathbb{P}^F_{>\lambda}$ and write $G = G_{\leq\lambda} \times G_{>\lambda}$. There are two ways how to argue now, using either the item (1) or the item (2) in Lemma 2.6.

The proof usually refers to (2) and the argument is in the "non-reverse" order, i.e. we look at $\mathbb{P}^F$ as $\mathbb{P}^F_{>\lambda} \times \mathbb{P}^F_{\leq\lambda}$. $\mathbb{P}^F_{>\lambda}$ is λ^+-closed in V, and so it must be the forcing $\mathbb{P}^F_{\leq\lambda}$ which adds a cofinal subset to κ of length λ. However, by (2) of Lemma 2.6 the forcing $\mathbb{P}^F_{\leq\lambda}$ is λ^+-cc in $V[G_{>\lambda}]$ and hence regularity of all cardinals $\mu \geq \lambda^+$ is preserved. In particular $\kappa > \lambda$ cannot be singularized by forcing with $\mathbb{P}_{\leq\lambda}$ over $V[G_{>\lambda}]$. Contradiction.

[2]See Remark 2.5 for a formally correct way of representing $\mathbb{P}^F_{>\lambda}$.

Though less common, it is possible to view the product $\mathbb{P}^F$ in the "reverse order", i.e. we look at $\mathbb{P}^F$ as $\mathbb{P}^F_{\leq\lambda} \times \mathbb{P}^F_{>\lambda}$ and use item (1) of Lemma 2.6. Let $\lambda < \kappa$ be as in the previous paragraph. The forcing $\mathbb{P}^F_{\leq\lambda}$ preserves regularity of all cardinals $\mu \geq \lambda^+$, and in particular of κ. The forcing $\mathbb{P}^F_{>\lambda}$ is however still λ^+-distributive in $V[G_{\leq\lambda}]$ by (1), and so it cannot add a new cofinal subset to κ of size λ when forced over $V[G_{\leq\lambda}]$. Contradiction. (Theorem 2.2) □

Remark 2.7 We will see in Section 3 that $\mathbb{P}^F$ typically destroys large cardinals since it is a product of forcing notions existing in V. In order to preserve large cardinals, it is necessary to use iteration (see Sections 3 and 4).

We close this subsection with a definition of a notion which a strengthening of the κ-cc condition. We say that a forcing notion $\mathbb{P}$ is *κ-Knaster* if every subset $X \subseteq \mathbb{P}$ of size κ has a subfamily $Y \subseteq X$ of size κ such that all elements of Y are pairwise compatible.

Lemma 2.8 *If $\mathbb{P}$ is κ-Knaster and $\mathbb{Q}$ is κ-cc, then $\mathbb{P} \times \mathbb{Q}$ is κ-cc.*

Proof. It suffices to show that $1_P \Vdash \mathbb{Q}$ is κ-cc. Assume $\tilde{p} \Vdash \dot{A}$ is an antichain in $\mathbb{Q}$ of size κ. Define a set $X = \{p_\alpha \,|\, \alpha < \kappa\}$ of conditions below $\tilde{p}$ such that $p_\alpha \Vdash \dot{A}(\alpha) = q_\alpha$ for some q_α. By κ-Knasterness of $\mathbb{P}$, there is a subfamily $X' = \{p_{\alpha_\xi} \,|\, \xi < \kappa\}$ of X such that the conditions in X' are pairwise compatible. It follows that the set $A' = \{q_{\alpha_\xi} \,|\, p_{\alpha_\xi} \in X'\}$ is an antichain in V. This is a contradiction. □

2.3 Extenders and hypermeasurable cardinals

As the topic of this thesis is centered around preservation of large cardinals in forcing extensions, and in particular of measurable cardinals, we will review some facts concerning measurable and hypermeasurable (or strong) cardinals.

If $j : V \to M$ is a measure ultrapower via some $U \subseteq \mathscr{P}(\kappa)$ then U is not an element of M. It follows that measure ultrapowers capture only a "small" segment of the V_α-hierarchy of V – $V_{\kappa+1} \subseteq M$, but $V_{\kappa+2} \not\subseteq M$. *Extenders* enable us to capture an arbitrarily large initial segment of the V_α-hierarchy of V. Extenders are thus useful in dealing with *hypermeasurable*, or *strong*, cardinals.

Definition 2.9 *A cardinal κ is λ-hypermeasurable (or λ-strong), where λ is a cardinal number, if there is an elementary embedding j with a critical point κ from V into a transitive class M such that $\lambda < j(\kappa)$ and $H(\lambda)^V \subseteq M$.*

Remark 2.10 Note that Definition 2.9 is slightly different from the definition found in [18] or [20], where κ is called $\kappa + \alpha$ or just α-strong if $V_{\kappa+\alpha}$ is included in M. The main difference is that we use the $H_\alpha = H(\alpha)$ hierarchy instead of the V_α hierarchy to measure the strength of the embedding j. Note that under GCH, there is a straightforward correspondence between the measurement of the strength of an embedding using the structures $H(\kappa^{+\alpha})$ and $V_{\kappa+\alpha}$ for an inaccessible κ and ordinal number α. It holds that if M is an inner model of ZFC then $V_{\kappa+\alpha} \subseteq M$ holds iff $H(\kappa^{+\alpha}) \subseteq M$ holds. This correspondence is however lost if GCH fails.

Starting with some $j : V \to M$, we define a (κ, λ)-extender derived from j which will capture the desired initial segment of M (and if M contains some desired segment of V, we capture this segment of V as well). Our definition of extenders is standard, see for instance [20] or [1]. We stress some useful points which are not usually made explicit.

Assume that $j : V \to M$ (M transitive) is an elementary embedding with a critical point κ and $\lambda < j(\kappa)$ is a V-cardinal. We define the (κ, λ)-extender ultrapower M_E derived from j to be the transitive collapse of $X_E \subseteq M$, where[3]

$$X_E = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\lambda]^{<\omega}\} \tag{2.5}$$

Standard arguments show that X_E is a model of ZFC since it is the direct limit of an ω-directed system of models of ZFC

$$\langle (X_a, id_{a,b}) \mid a, b \in [\lambda]^{<\omega}, a \subseteq b \rangle, \tag{2.6}$$

where $X_a \subseteq M$ for each $a \in [\lambda]^{<\omega}$ and the embeddings are the identity, i.e. $X_a \prec X_b$ for $a \subseteq b$. Each X_a is an isomorphic copy the measure ultrapower

$$\bar{X}_a = \{[f]_{U_a} \mid f : \kappa \to V\}, \tag{2.7}$$

where $U_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}$ and $j_a : V \to \bar{X}_a$ is the respective elementary embedding. If we set

$$i_a([f]_{U_a}) = j(f)(a) \tag{2.8}$$

then

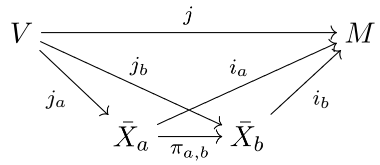
$$X_a = i_a[\bar{X}_a] \tag{2.9}$$

The models $\bar{X}_a$ themselves also form an ω-directed system together with the natural projections $\pi_{a,b}$ from $b \to a$ for $a \subseteq b$

$$\langle (\bar{X}_a, \pi_{a,b}) \mid a, b \in [\lambda]^{<\omega}, a \subseteq b \rangle \tag{2.10}$$

[3] We shall use the expression in (2.5) interchangibly with $X_E = \{j(f)(a) \mid a \in [\lambda]^{<\omega}, f : [\kappa]^{|a|} \to V\}$.

It follows that the following embeddings commute



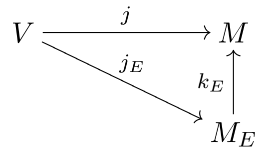
The direct limit of the system (2.6) is the class X_E defined in (2.5) above, and it holds that

$$j : V \to X_E \text{ is elementary, or equivalently } X_E \prec M \tag{2.11}$$

If k_E is the transitive collapse of X_E, then if we define $j_E : V \to M_E$ by

$$j_E = k_E^{-1} \circ j \tag{2.12}$$

then the following diagram commutes:



and moreover

$$M_E = \{j_E(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\lambda]^{<\omega}\} \tag{2.13}$$

As λ is obviously included in X_E (every $\alpha < \lambda$ can be expressed as $j(id)(\alpha)$), the following observation is immediate.

Observation 2.11 *If we assume GCH, then $H(\lambda)^M$ is included in M_E.*

Proof. It is immediate that $H(\lambda)^M$ is a subset of X_E (since $H(\lambda)^M$ has size λ in M by GCH), and consequently the transitive collapse k_E is the identity on $H(\lambda)^M$. (Observation 2.11) □

Corollary 2.12 *If M contains $H(\lambda)$ of V, then M_E also contains $H(\lambda)$ of V.*

We have as yet ignored the question what are the closure properties of M_E as regards sequences existing in V.

Observation 2.13 *Let $j : V \to M$ with critical point κ be as above. If the cofinality of λ is at least κ^+ and $H(\lambda)^V$ is included in M, then*

$$^{\kappa}M_E \cap V \subseteq M_E \tag{2.14}$$

In particular, if $\mathrm{cf}(\lambda) \geq \kappa^+$ and κ is λ-strong, then M_E is closed under κ sequences existing in V.

Proof. We use the following variant of X_E:

$$X_E^* = \{j(f)(a) \mid f : [\kappa]^{<\kappa} \to V, a \in ([\lambda]^{<\lambda})^M\} \tag{2.15}$$

The direct limit analysis of X_E^* still applies, hence X_E^* is a model of ZFC. However, the directed system

$$\langle (X_a, i_{a,b}) \mid a, b \in ([\lambda]^{<\lambda})^M, a \subseteq b \rangle \tag{2.16}$$

is now $\mathrm{cf}(\lambda)$-directed under the inclusion relation (as $[\lambda]^{<\lambda}$ of V is included in M). Assume now that $\langle x_\alpha \mid \alpha < \kappa \rangle \subseteq X_E^*$ is a sequence of elements in V and each x_α is in some X_{a_α} for some $a_\alpha \in ([\lambda]^{<\lambda})^M$. If the cofinality of λ is at least κ^+ then there is some a_0 such that $a_\alpha \subseteq a_0$ for each a_α. By the standard analysis of the closure of a measure ultrapower, X_{a_0} is closed under κ-sequences in V and contains $\langle x_\alpha \mid \alpha < \kappa \rangle$. (Observation 2.13) □

Note that if GCH holds below κ in V, then in fact

$$X_E = X_E^* \tag{2.17}$$

To argue for (2.17), let h be any bijection $h : [\kappa]^{<\kappa} \to \kappa$ such that for each cardinal $\bar{\kappa} < \kappa$ the restrictions of h to $\bar{\kappa}^{<\bar{\kappa}}$ is a bijection from $[\bar{\kappa}]^{<\bar{\kappa}}$ to $\bar{\kappa}$. Then there is in M a bijection $j(h) \restriction \lambda$ between $([\lambda]^{<\lambda})^M$ and λ, immediately yielding the identity $X_E^* = X_E$. If GCH does not hold below κ, X_E^* may be bigger than X_E.

The representation using the finite tuples of λ, i.e. $X_E = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\lambda]^{<\omega}\}$, is in this sense the least possible, and also canonical in the sense that $[\lambda]^{<\omega}$ of V is always included in M. The canonicity is also used in the abstract definition of an (κ, λ)-extender which does not refer to any $j : V \to M$ (see for instance [20]).

To conclude, as we will start with GCH, we can think of X_E (with respect to some fixed j) as a uniquely defined class closed under κ-sequences in V if the cofinality of λ is at least κ^+.

2.4 Lifting of embeddings

As regards preservation of measurability (and more) of a given cardinal κ in generic extensions, it turns out that very often we can argue for preservation by lifting an embedding existing in the ground model. This strategy is very useful for forcings which are sufficiently closed (see Section 5), but there are notable examples where it cannot be used: for instance iterations of Prikry-type forcing notions (see Section 6 for an argument for preservation of measurability in this case).

We will review some basic results in this section (this review is based on a similar review in [1]).

Definition 2.14 *Let $\mathbb{P} \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for $\mathbb{P}$, and H an N-generic filter for $j(\mathbb{P})$. We say that j^* lifts the embedding j if j^* is an elementary embedding $j^* : M[G] \to N[H]$ extending j.*

There is a simple sufficient condition which guarantees the existence of a lifting j^*. The following lemma is due to Silver.

Lemma 2.15 (Lifting lemma) *Let $\mathbb{P} \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for $\mathbb{P}$, and H an N-generic filter for $j(\mathbb{P})$. If $j[G] \subseteq H$, i.e. if the pointwise image of G under j is included in H, then*

1. *j lifts to $j^* : M[G] \to N[H]$, and*
2. *$j^*(G) = H$.*

Proof. Ad (1). Define $j^*(\dot{x}^G) = (j(\dot{x}))^H$. This definition is sound since if $\dot{x}^G = \dot{y}^G$, then there is $p \in G$ forcing $\dot{x} = \dot{y}$; by the assumption that $j[G] \subseteq H$, $j(p)$ is in H and forces $j(\dot{x}) = j(\dot{y})$, implying $(j(\dot{x}))^H = (j(\dot{y}))^H$. Now let $\varphi(x_0, \ldots)$ be an arbitrary formula: $M[G] \models \varphi(\dot{x}_0^G, \ldots)$ iff there is some $p \in G$ such that $p \Vdash \varphi(\dot{x}_0, \ldots)$ iff $j(p) \Vdash \varphi(j(\dot{x}_0), \ldots)$ then $N[H] \models \varphi(j^*(\dot{x}_0^G, \ldots)$, again by the inclusion $j[G] \subseteq H$. Conversely, if $M[G] \not\models \varphi(\dot{x}_0^G, \ldots)$, then $M[G] \models \neg\varphi(\dot{x}_0^G, \ldots)$ and we may conclude as above that $N[H] \not\models \varphi(j(\dot{x}_0), \ldots)$.

j^* extends j since if $x \in M$ then $j^*(x) = (j(\check{x}))^H$ and by elementarity $j(\check{x}) = \check{j(x)}$ and so $(\check{j(x)})^H = j(x)$.

Ad (2). If $\dot{G}$ is the canonical name for the generic filter G, $j(\dot{G})$ is by elementarity the canonical name for the generic filter H. (Lemma 2.15) □

As we will be dealing with extender embeddings, it is useful to notice that by using names for the elements of $M[G]$, we can argue that if j is an extender embedding, then so will be the lift j^*:

Lemma 2.16 *Let the assumptions of Lemma 2.15 hold. Assume further that $j : M \to N$ is an extender embedding, i.e. $N = \{j(F)(a) \mid F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of j. Then j^* is also an extender embedding, and moreover the parameters A and B of the extender embedding j remain the same, i.e.*

$$N[j^*(G)] = \{j^*(F)(a) \mid F \in M[G], F : A \to M[G], a \in B\}. \tag{2.18}$$

Proof. Let $x \in N[j^*(G)] = N[H]$ be given. We need to show that there is $\bar{F} : A \to M[G]$ in $M[G]$ and $a \in B$ such that $x = j^*(\bar{F})(a)$. As j is an extender embedding, the name $\dot{x} \in N$ for x can be expressed as $\dot{x} = j(F)(a)$ for some $F : A \to M$ and $a \in B$. W.l.o.g. we may assume that the range of F contains only names. Define in $M[G]$ a function $\bar{F}$ as follows: $\bar{F}(a) = (F(a))^G$ for each $a \in A$. By elementarity of j^*, $j^*(\bar{F})(a) = (j^*(F)(a))^H$ for every $a \in B$. It follows:

$$x = \dot{x}^H = (j(F)(a))^H = (j^*(F)(a))^H = j^*(\bar{F})(a) \tag{2.19}$$

(Lemma 2.16) □

Remark 2.17 If $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed measurability of κ in M (i.e. j is definable in M), then κ is still measurable in $M[G]$, *providing* that j^* is definable in $M[G]$.

Remark 2.18 Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed measurability of κ in M. Assume further that j^* is definable in $M[G]$. Let $U_j = \{X \subseteq \kappa \,|\, X \in M, \kappa \in j(X)\}$ be the normal ultrafilter derived from j. Then U_{j^*} extends the ultrafilter U_j: $U_j \subseteq U_{j^*}$. Note however that the extension U_{j^*} is in general very difficult to find[4] unless some powerful structural information such as the embedding j is available.

In view of the Lifting lemma 2.15, the crucial part of the arguments dealing with the preservation of measurability consists in finding an N-generic H containing the pointwise image of G. This may be rather difficult in some cases, but if the forcing notion $\mathbb{P}$ is sufficiently distributive and the embedding to be lifted is an extender embedding, the existence of such an H is straightforward.

Lemma 2.19 *Assume $j : M \to N$ is an extender embedding as in Lemma 2.16 and $N = \{j(F)(a) \,|\, F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Let G be M-generic for a forcing notion $\mathbb{P} \in M$. If M satisfies that $\mathbb{P}$ is $|A|^+$-distributive, then*

$$H = \{q \in j(\mathbb{P}) \,|\, \exists p \in G, j(p) \leq q\}$$

is N-generic for $j(\mathbb{P})$ and contains the pointwise image of G.

Proof. H is obviously a filter. We show it is a generic filter. Let $D = j(F)(d)$ be a dense open set. We may assume that the range of F consists of dense open sets in $\mathbb{P}$. Let $\{a_\xi \,|\, \xi < |A|\}$ be the enumeration of A. By distributivity, $X = \bigcap_{\xi<|A|} F(a_\xi)$ is dense. Let $p \in X$ be in G; then $M \models \forall a \in A, p \in F(a)$, and by elementarity it follows that $N \models \forall a \in j(A), j(p) \in j(F)(a)$. In particular, $j(p) \in j(F)(d) = D$. (Lemma 2.19) □

[4]With the notable exception when the forcing notion $\mathbb{P}$ is of size $<\kappa$; in this case any normal ultrafilter U in M generates a normal ultrafilter in the generic extension.

However, it is not true conversely that if $\mathbb{P}$ fails to be $|A|^+$-distributive, then H cannot be in some sense generated from the generic filter G. In fact, the construction in [9] shows that in the context of Sacks forcing, distributivity can be replaced by the weaker property of $|A|$-fusion (diverting from the notation in this case, $|A|$-fusion refers to sequences of length $|A|$ and not $< |A|$; see Lemma 4.3 below).

We close this preliminary section by a technical lemma which is useful in constructing generic filters and is tacitly used throughout the arguments.

Lemma 2.20 *Assume $N \subseteq M$ are inner models of ZFC and ${}^\lambda N \cap M \subseteq N$, i.e. N is closed under λ-sequences in M. If $\mathbb{P} \in N$ is λ^+-cc in M and G is $\mathbb{P}$-generic over M, then $N[G]$ is closed under λ-sequences in $M[G]$.*

Proof. Note that G is clearly a generic filter for $\mathbb{P}$ over N as well. To show that $N[G]$ is closed under λ-sequences in $M[G]$, we first show that it is enough to consider closure under sequences of ordinals. Assume $\bar{N} \subseteq \bar{M}$ are inner models of ZFC, then

$$\text{If } {}^\lambda\text{On} \cap \bar{M} \subseteq \bar{N} \text{ then } {}^\lambda\bar{N} \cap \bar{M} \subseteq \bar{N} \tag{2.20}$$

Given a sequence s in ${}^\lambda\bar{N}\cap\bar{M}$, choose in $\bar{N}$ some large enough $(H(\lambda'))^{\bar{N}}$ such that $s \subseteq (H(\lambda'))^{\bar{N}}$ and choose a wellordering $<$ of $(H(\lambda'))^{\bar{N}}$ also existing in $\bar{N}$. Considering a suitable $(\gamma, \in)$ which isomorphic (in $\bar{N}$) to $((H(\lambda'))^{\bar{N}}, <)$, it is obvious that the sequence s can be captured as some λ-sequence of ordinals in γ. Hence (2.20) follows.

Assume now that $\mathbb{P} \in N$ satisfies the assumptions of the lemma. It is enough to show that every λ-sequence s of ordinals existing in $M[G]$ has a $\mathbb{P}$-name existing N. Let $\dot{s}$ be a $\mathbb{P}$-name existing in M for s, and assume for simplicity that the empty condition forces that $\dot{s}$ is a λ-sequence of ordinals. For each $\alpha < \lambda$, let A_α be a maximal antichain in $\mathbb{P}$ containing conditions p in $\mathbb{P}$ such that there is some $\xi \in \text{On}$ and $p \Vdash \dot{s}(\alpha) = \xi$. Define a name $\dot{s}^*$ by

$$\dot{s}^* = \{(p, [\alpha, \xi]) \,|\, p \in A_\alpha, p \Vdash \dot{s}(\alpha) = \xi\}, \tag{2.21}$$

where $[\cdot, \cdot]$ is a canonical name for an ordered pair. It is immediate that $1_{\mathbb{P}} \Vdash \dot{s} = \dot{s}^*$. Since $\mathbb{P}$ is λ^+-cc, the size of $\dot{s}^*$ is at most λ and this implies that $\dot{s}^*$ lies in N, as desired. (Lemma 2.20) $\Box$

Note that by (2.20) the closure under sequences is "upwards absolute" because it is determined by λ-sequences of ordinal numbers: if $N \subseteq M$ and N is closed under λ- sequences in M, then every model $\bar{N}$ of ZFC between N and M, i.e. $N \subseteq \bar{N} \subseteq M$, is also closed under λ-sequences in M. Also, if N is closed under λ-sequences in M and $\mathbb{P} \in M$ is λ^+-distributive in M and G is a generic filter for $\mathbb{P}$ then N is still closed under λ-sequences in $M[G]$.

3 Forcing and preservation of large cardinals

As our main interest is the behaviour of the continuum function (on regular cardinals) with respect to large cardinals, we will focus on forcings which add a prescribed number of new subsets to regular cardinals α. The results in this section are folklore (although it is difficult to give exact references). Unlike the results in Section 2, however, they are more specific to our topic.

Recall the original forcing of Easton [4], reviewed in Section 2.2. We now show that this "product-style" forcing will typically destroy large cardinals.

First notice that preservation of a large cardinal κ when adding many new subsets of κ requires some sort of forcing preparation below the cardinal κ.

Observation 3.1 *Assume κ is a measurable cardinal and $\gamma < \kappa$ is a successor ordinal. If $2^\kappa = \kappa^{+\gamma}$ then there is a $X \subseteq \kappa$ of inaccessible cardinals such that $2^\alpha \geq \alpha^{+\gamma}$ for each $\alpha \in X$ and X is an element of some normal measure at κ.*

Proof. Let U be a normal measure at κ and $j : V \to M$ be the ultrapower embedding derived from U. As all subsets of κ are included in M, 2^κ in M must be at least $(2^\kappa)^V = \kappa^{+\gamma}$: $M \models \kappa^{+\gamma} \leq 2^\kappa < j(\kappa)$ by the inaccessibility of $j(\kappa)$ in M. By the Los theorem, $X = \{\xi < \kappa \,|\, \xi \text{ is inaccessible}, \xi^{+\gamma} \leq 2^\xi\}$ is in the measure U.

Notice that if $2^\kappa = \kappa^{+\gamma}$ for some γ which is "simply" describable, we may include γ's greater than κ as well (γ may also be a limit ordinal of cofinality at least κ^+). For instance if $2^\kappa = \kappa^{+(\kappa^+)}$, then the set $\{\xi < \kappa \,|\, \xi \text{ is inaccessible}, \xi^{+(\xi^+)} \leq 2^\xi\}$ must be big. This analysis in principle captures all possible values of 2^κ, but the formulation is less elegant as we need to refer to the specific measure U used in the argument: Assume $2^\kappa = \kappa^{+\gamma}$ for some ordinal γ (either a successor ordinal, or with cofinality at least κ^+). Fix a normal measure U and the corresponding ultrapower embedding $j : V \to M$. As before we obtain $\gamma \leq \kappa^{+\gamma} \leq (2^\kappa)^M < j(\kappa)$; in particular there is some $f_\gamma : \kappa \to \kappa$ representing γ in M, i.e. $[f_\gamma]_U = \gamma$. It follows that the set $\{\xi < \kappa \,|\, \xi \text{ is inaccessible}, \xi^{+f_\gamma(\xi)} \leq 2^\xi\}$ is in U. (Observation 3.1) $\Box$

It may consistently happen that even adding a single new subset of κ requires some forcing preparation (see [16]).

Observation 3.2 *Assume $V = L[U]$ where U is a normal measure at κ. If $\mathbb{P}$ is a κ-distributive forcing notion which adds a new subset of κ then κ is no longer measurable in $V^{\mathbb{P}}$.*

Proof. Let G be a generic filter for $\mathbb{P}$ and assume that κ is still measurable in $L[U][G]$ and let j be an embedding witnessing measurability of κ. By the definition of $L[U][G]$, j is an embedding from $L[U][G]$ to $L[j(U)][j(G)]$

and $j(G)$ is a generic filter over $L[j(U)]$ for $j(\mathbb{P})$. Note that $L[U][G]$ and $L[j(U)][j(G)]$ must contain the same subsets of κ. By Kunen's iteration arguments, we know that $L[j(U)]$ is an iterate of $L[U]$ and in particular $L[j(U)] \subseteq L[U]$ and $(\mathscr{P}(\kappa))^{L[U]} = (\mathscr{P}(\kappa))^{L[j(U)]}$. It follows that every new subset of κ added by G must be added by the forcing $j(\mathbb{P})$ over $L[j(U)]$. But by elementarity, $j(\mathbb{P})$ is $j(\kappa)$-distributive in $L[j(U)]$, contradiction. (Observation 3.2) □

However, even with preparation we may lose measurability in the final generic extension if we use product-style forcing. We first proof an observation which is interesting in its own right (see [1]).

Observation 3.3 *Assume κ is not measurable in some inner model M and there is some $\mathbb{P}$ in M such that $\mathbb{P}$ forces that κ is measurable in $M^{\mathbb{P}}$. Then $\mathbb{P} \times \mathbb{P}$ contains antichains of size κ in M, or equivalently $\mathbb{P}$ contains antichains of size κ in $M^{\mathbb{P}}$. In particular $\mathbb{P}$ must have size at least κ (in M).*

Proof. Let $G \subseteq \mathbb{P}$ be an M-generic filter and $\dot{U}$ a name forced by $1_{\mathbb{P}}$ to be a measure in $M[G]$. We will construct in $M[G]$ a κ-sequence of incompatible conditions in $\mathbb{P}$.

First notice that if p forces $\dot{X} \in \dot{U}$ and the membership in $\dot{U}$ of all $\dot{Y}$ such that p forces $\dot{Y} \subseteq \dot{X}$ is decided uniquely by all $q \leq p$, i.e. for all $q \leq p$, $q \Vdash \dot{Y} \in \dot{U}$ or for all $q \leq p$, $q \Vdash \dot{X} \setminus \dot{Y} \in \dot{U}$ then

$$W = \{A \subseteq \kappa \mid A \in M, p \Vdash A \cap \dot{X} \in \dot{U}\} \tag{3.1}$$

is a measure existing in M. We will use (3.1) in our construction.

To start the construction, choose $X_0 \subseteq \kappa$ (X_0 can be taken in M) and $p_0, \bar{p}_0$ with p_0 in G such that $p_0 \Vdash X_0 \in \dot{U}$ and $\bar{p}_0 \Vdash (\kappa \setminus X_0) \in \dot{U}$; this is possible otherwise by (3.1) there would be a measure in M. In the next step apply the above procedure to $X_0 \in (\dot{U})^G$, choosing some $X_1 \subseteq X_0$ (X_1 can again be taken in M) and conditions $p_1, \bar{p}_1$ with p_1 in G such that $p_1 \Vdash X_1 \in \dot{U}$ and $\bar{p}_1 \Vdash (X_0 \setminus X_1) \in \dot{U}$. Notice that $\bar{p}_0, \bar{p}_1$ must be incompatible since $(\kappa \setminus X_0)$ and $(X_0 \setminus X_1)$ are disjoint. Continue in this fashion for all $n \in \omega$ building an antichain $\{\bar{p}_n \mid n \in \omega\}$ (always assuming that the non-barred p_n's are chosen in G). At stage ω, we know that $X_\omega = \bigcap X_n$ is in $(\dot{U})^G$ by the κ-completeness of the measure $(\dot{U})^G$ in $M[G]$. Choose $p_\omega \in G$ forcing $\dot{X}_\omega \in \dot{U}$ (note that X_ω may exist only in $M[G]$). Using (3.1), there must be $X_{\omega+1}$ ($X_{\omega+1}$ can be taken in M) and $p_{\omega+1}, \bar{p}_{\omega+1}$ below p_ω with $p_{\omega+1}$ in G such that $p_{\omega+1} \Vdash X_{\omega+1} \cap \dot{X}_\omega \in \dot{U}$ and $\bar{p}_{\omega+1} \Vdash (\dot{X}_\omega \setminus X_{\omega+1}) \in \dot{U}$.

Due to the κ-completeness of the measure $(\dot{U})^G$ we can continue the construction for all $\alpha < \kappa$, obtaining an antichain $\{\bar{p}_\alpha \mid \alpha < \kappa\}$ as desired (always assuming that the non-barred p_α's are chosen in G). Note that the possible lack of closure of $\mathbb{P}$ which could otherwise block the construction

at limit stages is compensated by the closure properties of the measure $\dot{U}$.
(Observation 3.3) □

Remark 3.4 This result is known to be optimal. Kunen showed [21] that there is a κ-cc forcing notion S such that the product $S \times S$ is not κ-cc and S can "revive" measurability of κ in the sense that κ is measurable in V, is not measurable in some forcing extension V^*, but becomes measurable again after forcing with S in V^* (S is a κ-Souslin tree construed as a forcing notion). Also Cohen forcing at κ, i.e. a κ^+-cc forcing, can "revive" measurability of κ. We will use some results and definitions from Hamkins [17] concerning the so called *gap forcing* to show the latter claim. We say that a forcing notion $\mathbb{S}$ is a gap forcing if there is $\delta < \kappa$ such that $\mathbb{S}$ factors into $\mathbb{Q} * \mathbb{R}$ where $|\mathbb{Q}| < \delta$ is non-trivial and $\mathbb{R}$ is forced by $\mathbb{Q}$ to be δ^+-closed. Assuming GCH for simplicity, let κ be measurable in V and $\mathbb{P}$ be an Easton-supported iteration of $\mathrm{Add}(\alpha, 1)$ for every inaccessible $\alpha < \kappa$. Let G be $\mathbb{P}$-generic. We will show that κ is not measurable in $V[G]$ but becomes measurable again if we force with $\mathrm{Add}(\kappa, 1)$ (defined in $V[G]$) over $V[G]$. Assume for contradiction that κ is still measurable in $V[G]$. Clearly our $\mathbb{P}$ is a gap forcing – for instance for $\delta = \alpha_0^+$ where α_0 is the least inaccessible cardinal. The fact that $\mathbb{P}$ is a gap forcing implies the following: If $j : V[G] \to M[j(G)]$ witnesses measurability of κ in $V[G]$ then j is an extension of some j^* definable in V such that $j^* = j \restriction V : V \to M$ and M is closed under κ-sequences in V. In particular M contains all subsets of κ present in V. Notice that $\mathbb{P} = j^*(\mathbb{P})_\kappa$ and so $j(G)$ factors as $G * g$ where g is $M[G]$-generic for the iteration $j(\mathbb{P})$ in the interval $[\kappa, j(\kappa))$ and in particular adds a new subset of κ over $M[G]$. This is a contradiction since all subsets of κ existing in $V[G]$ are present in $M[G]$ (since M contains all nice $\mathbb{P}$-names for subsets of κ). However it is known by work of Silver that $\mathbb{P} * \mathrm{Add}(\kappa, 1)$ preserves measurability of κ, which proves the claim.

Now we can show that even with a preparation forcing below κ, we can destroy a large cardinal.

Observation 3.5 *Assume $V = L[U]$ where U is a normal measure at κ. Let $\mathbb{R} = \mathbb{P} \times \mathbb{Q}$ be a forcing notion such that $\mathbb{P}$ is κ-closed and adds a new subset of κ and $\mathbb{Q}$ is an arbitrary forcing notion such that $\mathbb{Q} \times \mathbb{Q}$ is κ-cc. Then κ is no longer measurable in $V^{\mathbb{R}}$.*

Proof. Notice that if $\mathbb{P}$ is a Cohen forcing adding a new subset of κ and $\mathbb{Q}$ is either a product or an iteration of Cohen forcings adding a new subset to each inaccessible $\alpha < \kappa$ then the conditions of the observation are satisfied, and it follows that κ fails to be measurable in $V^{\mathbb{R}}$.

It is enough to argue that forcing with $\mathbb{Q}$ over $V^{\mathbb{P}}$ cannot "revive" the measurability of κ which is killed in $V^{\mathbb{P}}$ by Observation 3.2. By Easton lemma 2.6, $\mathbb{Q} \times \mathbb{Q}$ is still κ-cc after forcing with $\mathbb{P}$. The proof is concluded by applying Observation 3.3. (Observation 3.5) $\square$

The above discussion implies that if we want to add new subsets to a large cardinal κ *and* we want to have a chance of preserving the largeness of κ in the generic extension, we should add new subsets to "many" regular cardinals below κ, and moreover we should do it by an iterated forcing.

We will fix the following notion for the rest of the section. Let F be an Easton function according to Definition 2.1 defined on all regular cardinals. Our aim is to define in general terms a forcing notion which will realise the Easton function F (making it the continuum function in the generic extension) and satisfy the conditions discussed above.

In as much as the desired forcing must be an iteration, the iteration will need to have some "space": The following observation shows that iteration on successive cardinals will tend to collapse cardinals.

Observation 3.6 *Assume $\kappa < \lambda$ are regular cardinals. If κ^* is a cardinal greater than λ, then forcing with $\mathrm{Add}(\kappa,\kappa^*) * \mathrm{Add}(\lambda,1)$ collapses κ^* to λ.*

Proof. Let $\langle x_\xi \,|\, \xi < \kappa^* \rangle$ be the enumeration of subsets of κ in $V^{\mathrm{Add}(\kappa,\kappa^*)}$; for each $\xi < \kappa^*$, the set $D_\xi = \{p \in \mathrm{Add}(\lambda,1) \,|\, \exists \alpha < \lambda, p \restriction [\alpha, \alpha+\kappa) = x_\xi\}$ is dense. Consequently, there is a surjection from λ onto κ^* in the generic extension of V by $\mathrm{Add}(\kappa,\kappa^*) * \mathrm{Add}(\lambda,1)$. (Observation 3.6) $\square$

The concept of the "space" mentioned above is technically captured by the closure points of the function F. We say that a cardinal κ is a closure point of F if $\mu < \kappa$ implies $F(\mu) < \kappa$. We will enumerate in the increasing order the closed unbounded class of closure points of F as $\langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$. Note that every i_α must be a limit cardinal and $i_{\beta+1}$ has cofinality ω for every β. If κ is a regular closure point, then κ equals i_κ.

We will now give a full definition of a forcing notion to realise an Easton function F. It will be a combination of (reverse) Easton iteration and of product-style Easton forcing. Later in the text we will define several variants of this forcing (for more examples of this type of forcing see [8]).

Definition 3.7 ***(General form of forcing)*** *Let an Easton function F satisfying the conditions (1), (2) of 2.1 be given. Let $\langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$ be an increasing enumeration of the closure points of F.*

Define an iteration $\mathbb{P}^F = \langle (\mathbb{P}_{i_\alpha}, \dot{\mathbb{Q}}_{i_\alpha}) \,|\, \alpha < \mathrm{On} \rangle$ indexed by $\langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$ as follows:

If $\alpha + 1$ is a successor ordinal we define

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{\mathbb{Q}}_{i_\alpha}, \tag{3.2}$$

where $\dot{\mathbb{Q}}_{i_\alpha}$ *is a name for the Easton-supported product*

$$\prod^{E}_{i_\alpha \leq \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda))$$

with λ *ranging over regular cardinals.*

If γ *is a limit ordinal, then* $\mathbb{P}_{i_\gamma}$ *is an inverse limit unless* i_γ *is a regular cardinal, in which case* $\mathbb{P}_{i_\gamma}$ *is a direct limit (the usual Easton support).*

Note that we use the same symbol, i.e. $\mathbb{P}^F$, for both the product forcing in (2.2) and the iteration defined above; we will always specify which forcing we have in mind. Also, we may later drop for notational reasons the superscript ^{E} from $\prod$ which denotes the Easton support as in (2.2). Notice that conditions in $\mathbb{P}^F$ are proper classes; to define $\mathbb{P}^F$ correctly, see Remark 2.5. As regards the definition of the forcing, recall that each i_α is a limit cardinal; the forcing at i_α is thus non-trivial only if i_α is a regular and limit cardinal. As we will assume GCH, such i_α's are (strongly) inaccessible cardinals.

We now show that $\mathbb{P}$ preserves all cofinalities.

Lemma 3.8 *Assuming GCH,* $\mathbb{P}^F$ *of Definition 3.7 preserves all cofinalities.*

Proof. Let G be a generic filter for $\mathbb{P}^F$ and suppose for contradiction that some some regular cardinal κ is singularized in $V[G]$ and has cofinality μ (μ is a regular cardinal both in $V[G]$ and V). We will distinguish two cases, mostly for notational reasons (the argument is otherwise the same in both cases):

Case (1). $\mu = i_\alpha$ for some inaccessible i_α. Factor $\mathbb{P}^F$ as

$$\big(\mathbb{P}^F_{i_\alpha} * \mathrm{Add}(i_\alpha, F(i_\alpha))\big) * \big(\prod^{E}_{\substack{i^+_\alpha \leq \lambda < i_{\alpha+1} \\ \lambda \in \mathrm{Reg}}} \mathrm{Add}(\lambda, F(\lambda)) * \mathbb{P}^F_{\mathrm{tail}}\big), \tag{3.3}$$

where the product $\prod^E_{i^+_\alpha \leq \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda))$, λ regular, is defined in $V^{\mathbb{P}^F_{i_\alpha}}$, and $\mathbb{P}^F_{\mathrm{tail}}$ is written for $\mathbb{P}^F \setminus \mathbb{P}^F_{i_\alpha+1}$. Write (3.3) as $\mathbb{P}_0 * \mathbb{P}_1$. We will show that $\mathbb{P}_0$ is i^+_α-cc and forces that $\mathbb{P}_1$ is i^+_α-distributive.

Since i_α is an inaccessible closure point of F, $\mathbb{P}^F_{i_\alpha}$ is included in V_{i_α}. Due to the Easton support of $\mathbb{P}^F$, all conditions in $\mathbb{P}^F_{i_\alpha}$ have their support bounded in i_α, and it follows that $\mathbb{P}^F_{i_\alpha}$ is i^+_α-cc by the Δ-system argument: the size of every antichain is bounded by the size of $^{<i_\alpha}V_{i_\alpha}$ which is i_α. Also, $\mathbb{P}^F_{i_\alpha}$ forces that $\mathrm{Add}(i_\alpha, F(i_\alpha))$ is i^+_α-cc, and thus $\mathbb{P}^F_{i_\alpha} * \mathrm{Add}(i_\alpha, F(i_\alpha))$ is i^+_α-cc.

The rest of the product $\prod^E_{i^+_\alpha \leq \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda))$, λ regular, is forced by $\mathbb{P}_0$ to be i^+_α-distributive, by Lemma 2.6. Furthermore, $\mathbb{P}^F_{i_{\alpha+1}}$ clearly forces that $\mathbb{P}^F_{\text{tail}}$ is $i^+_{\alpha+1}$-closed, where $i^+_\alpha < i_{\alpha+1}$. It follows $\mathbb{P}_1$ is forced to be i^+_α-distributive.

Now we reach contradiction just like in the proof of Theorem 2.2. $\mathbb{P}_0$ preserves regularity of all κ such that $i^+_\alpha \leq \kappa$ by its being i^+_α-cc. However, $\mathbb{P}_1$ does not add new i_α-sequences, and so κ cannot have cofinality i_α in $V[G]$. Contradiction.

Case (2). μ is a regular cardinals between i_α and $i_{\alpha+1}$, i.e. $i_\alpha < \mu < i_{\alpha+1}$, for some α, where i_α may be singular. As above, we factor $\mathbb{P}^F$ at μ:

$$\big(\mathbb{P}^F_{i_\alpha} * \prod^E_{\substack{i_\alpha \leq \lambda \leq \mu \\ \lambda \in \text{Reg}}} \text{Add}(\lambda, F(\lambda))\big) * \big(\prod^E_{\substack{\mu < \lambda < i_{\alpha+1} \\ \lambda \in \text{Reg}}} \text{Add}(\lambda, F(\lambda)) * \mathbb{P}^F_{\text{tail}}\big), \tag{3.4}$$

where both the products in (3.4) are defined in $V^{\mathbb{P}^F_{i_\alpha}}$. Write (3.4) as $\mathbb{P}_0 * \mathbb{P}_1$. As in Case (1), we show that $\mathbb{P}_0$ is μ^+-cc and $\mathbb{P}_1$ is forced to be μ^+-distributive. Even when i_α is singular, $\mathbb{P}^F_{i_\alpha}$ is i^{++}_α-cc by GCH. As by our assumption $i^+_\alpha \leq \mu$, we still have that $\mathbb{P}_0$ is μ^+-cc, arguing as in Case (1) (here we use the fact that $\prod^E_{i_\alpha \leq \lambda \leq \mu} \text{Add}(\lambda, F(\lambda))$, λ regular, is forced by $\mathbb{P}^F_{i_\alpha}$ to be $i^{++}_\alpha \leq \mu^+$-cc). Exactly as in Case (1), $\mathbb{P}_1$ is μ^+-distributive over $V^{\mathbb{P}_0}$. This yields a contradiction. (Lemma 3.8) $\square$

Remark 3.9 Referring to Remark 2.5, the fact that $\mathbb{P}^F$ factors into $\mathbb{P}_0 * \mathbb{P}_1$ at every regular cardinal, where the second forcing is sufficiently distributive over $V^{\mathbb{P}_0}$, is enough to argue that a generic extension by $\mathbb{P}^F$ preserves all axioms of ZF(C).

Before we move on, notice that if $\mathbb{P}^F$ should stand a chance of preserving a measurable cardinal κ, for instance, the Easton function F used in the definition of $\mathbb{P}^F$ must behave “reasonably” below κ: cf. Observation 3.1.

4 Failure of GCH at a measurable cardinal

It is known, see [14] (see Section 4.2 for a short discussion), that failure of GCH at a measurable cardinal is consistency-wise much stronger than measurability. As we aim to realise an arbitrary Easton function F and preserve large cardinals (for instance measurable cardinals), it is obvious that we have to take this restriction into account.

The first consistency proof by Silver used the supercompact cardinal to obtain a measurable cardinal where GCH fails (for a proof see for instance [18]). It was Woodin (unpublished) who found how to achieve the same result from a much weaker hypothesis: hypermeasurability. His idea included a technique of a modification of a generic to allow for lifting (for review, see [1], or [3]).

The concept of modification however seems rather restrictive and so we introduce another technique in this section for producing a measurable cardinal failing GCH which includes a Sacks forcing, developed in [9]. Our review will focus on issues which are relevant for our original results later in the text.

4.1 Sacks forcing at inaccessible cardinals

The original results in Section 5 are centered around the technique developed in [9] which uses the Sacks forcing instead of the Cohen forcing in the lifting arguments. We will give a brief review here in a slightly generalized setting.

Though the concept of a perfect tree can be formulated for an arbitrary regular cardinal, see also [19], we will use the forcing at inaccessible cardinals only and this introduces further simplifications.

Definition 4.1 *If α is an inaccessible cardinal, then $p \subseteq 2^{<\alpha}$ is a perfect α-tree if the following conditions hold:*

1. *If $s \in p, t \subseteq s$, then $t \in p$;*
2. *If $s_0 \subseteq s_1 \cdots$ is a sequence in p of length less than α, then the union of s_i's belongs to p;*
3. *For every $s \in p$ there is some $s \subseteq t$ such that t is a splitting node, i.e. both $t * 0$ and $t * 1$ belong to p;*
4. *Let $\mathrm{Split}(p)$ denote the set of s in p such that both $s * 0$ and $s * 1$ belong to p. Then for some (unique) closed unbounded set $C(p) \subseteq \alpha$, $\mathrm{Split}(p) = \{s \in p \,|\, \mathrm{length}(s) \in C(p)\}$.*

A perfect α-tree is an obvious generalization of the perfect tree at ω ordered by inclusion; there is only one non-trivial condition, and this concerns the limit levels of the tree: if $s \in p$ is an element at a limit level and the splitting nodes $t \subseteq s$ are unbounded in s, then s must be a splitting node as well (continuous splitting). As α is inaccessible, and consequently every level of p is of size $<\alpha$, the trees obeying (4) above are dense in the trees having continuous splitting.

Generalized perfect trees can be used to define a natural forcing notion.

Definition 4.2 *The forcing notion* $\mathrm{Sacks}(\alpha,1)$ *contains as conditions perfect* α*-trees, the ordering is by inclusion (not the reverse inclusion), i.e.* $p \leq q$ *iff* $p \subseteq q$*. Or generally, the forcing notion* $\mathrm{Sacks}(\alpha,\lambda)$*, where* $0<\lambda$ *is an ordinal number, is a product of length* λ *of the forcing* $\mathrm{Sacks}(\alpha,1)$ *with support of size at most* α*, i.e. a condition* p *in* $\mathrm{Sacks}(\alpha,\lambda)$ *is a function from* λ *to* $\mathrm{Sacks}(\alpha,1)$ *such that* $\{\xi<\lambda \,|\, p(\xi) \neq 1_{\mathrm{Sacks}(\alpha,1)}\}$ *has size at most* α*.*

For p a condition in $\mathrm{Sacks}(\alpha,1)$, let $\langle \alpha_i \,|\, i<\alpha\rangle$ be the increasing enumeration of $C(p)$ and let $\mathrm{Split}_i(p)$ be the set of s in p of length α_i. For $p,q \in \mathrm{Sacks}(\alpha,1)$ let us write $p \leq_\beta q$ iff $p \leq q$ and $\mathrm{Split}_i(p)=\mathrm{Split}_i(q)$ for $i<\beta$. In the generalization for the product $\mathrm{Sacks}(\alpha,\lambda)$ we write $p \leq_{\beta,X} q$ (where X is some subset of λ of size less than α) iff $p \leq q$ (i.e. for all $i<\lambda, p(i) \leq q(i)$) and moreover for each $i \in X$, $p(i) \leq_\beta q(i)$.

We will define several useful notions and state some properties.

Lemma 4.3 *The forcing* $\mathrm{Sacks}(\alpha,\lambda)$ *satisfies the following* α-fusion property*: Suppose* $p_0 \geq p_1 \geq \dots$ *is a descending sequence in* $\mathrm{Sacks}(\alpha,\lambda)$ *of length* α *and suppose in addition that* $p_{i+1} \leq_{i,X_i} p_i$ *for each* i *less than* α*, where* X_i *form an increasing sequence of subsets of* λ *of size less than* α *whose union is the union of the supports of* p_i*'s; such a sequence will be called a* fusion sequence*. Then the* p_i*'s have a lower bound in* $\mathrm{Sacks}(\alpha,\lambda)$ *(obtained by taking intersections at each component).*

Proof. Note the role of X_i's in the lemma. It is not required that in the step from p_i to p_{i+1} we keep all splitting levels $\leq i$ in the trees in the support of p_i: we are allowed to thin out more at some coordinates (in $\mathrm{supp}(p_i) \setminus X_i$),[5] providing we eventually "catch" all coordinates, i.e. $\bigcup_{i<\alpha} X_i = \bigcup_{i<\alpha} \mathrm{supp}(p_i)$.

The proof itself is obvious: if $\xi<\lambda$ is a coordinate in $\bigcup_{i<\alpha}\mathrm{supp}(p_i)$ there is some $j<\alpha$ and X_j such that $\xi \in X_j$. As the X_i's form an increasing chain, we have that $p_{i+1}(\xi) \leq_i p_i(\xi)$ for all $j \leq i < \alpha$. This is enough to conclude that the decreasing sequence at the coordinate ξ has a lower bound $\tilde{p}_\xi$. This works for every $\xi \in \bigcup_{i<\alpha}\mathrm{supp}(p_i)$, so the desired condition $\tilde{p}$ is defined to have the support $\bigcup_{i<\alpha}\mathrm{supp}(p_i)$ and for each ξ in the support we set $\tilde{p}(\xi)=\tilde{p}_\xi$. (Lemma 4.3) □

[5] As we require that each X_i has size less than α (for reasons which will be apparent in Lemma 4.6), this difference will have size α if the support of p_i has size α.

Definition 4.4 *Assume p is a condition in* Sacks(α, λ)*, X is a subset of λ of size less than α and β is less than α. Then an (X, β)-*thinning *of p is an extension of p obtained by thinning each $p(i)$ for $i \in X$ to a subtree consisting of all nodes compatible with some particular node on the β-th splitting level of $p(i)$.*

Definition 4.5 *Assume D is a dense open set in* Sacks(α, λ)*. We say that* $p \in$ Sacks(α, λ) reduces *D iff for some subset X of λ of size less than α and some $\beta < \alpha$ any (X, β)-thinning of p meets D.*

The following important property holds:

Lemma 4.6 *Let $\{D_i \mid i < \alpha\}$ be a collection of α-many dense open sets in* Sacks(α, λ)*. Then for each p there is a condition $q \leq p$, obtained as a lower bound of a fusion sequence, such that q reduces each D_i in the above sense.*

Proof. We will inductively build a decreasing sequence $q = p_0 \geq p_1 \geq \dots$ of length α and an increasing sequence $X_0 \subseteq X_1 \subseteq \dots$ of subsets of λ of size less than α such that $\bigcup_{i<\alpha} X_i = \bigcup_{i<\alpha} \mathrm{supp}(p_i)$. At stage $k < \alpha$, first take a lower bound of all $p_{k'}$ for $k' < k$ and denote this lower bound as r_k. According to some fixed suitable strategy fixed at the beginning choose some X_k of size less than α satisfying the properties above. Let $\langle f_\zeta \mid \zeta < \mu \rangle$ be an enumeration of $P = \prod_{\xi \in X_k} (\mathrm{Split}_k(r_k(\xi)) \times \{0, 1\})$, where $|P| = \mu < \alpha$. Build a decreasing sequence of conditions $r_k \geq r_k^{f_0} \geq \dots$ (taking lower bounds at limits) of length $\mu < \alpha$ such that each $r_k^{f_{\zeta+1}}$ extends $r_k^{f_\zeta}$ and satisfies (4.1):

$$\text{the restriction of } r_k^{f_{\zeta+1}} \text{ to nodes } f_\zeta \text{ meets } D_k, \tag{4.1}$$

and for $\xi \in X_k$, $r_k^{f_{\zeta+1}}(\xi)$ is identical to $r_k^{f_\zeta}(\xi)$ except for the subtrees at the nodes determined by f_ζ. Note that the supports of $r_i^{f_\zeta}$'s are increasing and may be bigger than X_k (at coordinates outside X_k we do not need to preserve the k-th splitting level so we keep thinning out as needed to satisfy (4.1)).

Since $\mu < \alpha$, there is a lower bound to the sequence $r_k \geq r_k^{f_0} \geq \dots$, and this will be the desired p_k. (Lemma 4.6) □

Note that Sacks(α, λ) is obviously α^{++}-cc (by the GCH at α) and α-closed. It follows that under GCH forcing with Sacks(α, λ) preserves all cardinal (and cofinalities), except perhaps α^+. We use Lemma 4.6 to argue the case for α^+.

Lemma 4.7 *Forcing with* Sacks(α, λ) *preserves α^+.*

Proof. Let $\dot{f}$ be forced to be a function from α to α^+. We argue that the empty condition forces that the range of $\dot{f}$ is bounded in α^+. This follows directly from the argument in Lemma 4.6: Let D_i for each $i < \alpha$ be the dense open set of conditions deciding the value of $\dot{f}(i)$. If G is a generic filter, by Lemma 4.6 there is a $q \in G$ reducing all D_i's. Clearly, for every $i < \alpha$ the reduction of D_i determines less than α many choices for the interpretation of $\dot{f}$ at i and this suffices. (Lemma 4.7) □

Notice that Sacks(α, λ) adds λ-many new subsets of α (the intersection of all trees in a generic filter at a given coordinate determines a unique subset of α; this subset in turn determines the whole generic at the given coordinate).

As discussed after Lemma 2.19, it is the α-fusion property which is strong enough to replace the restrictive condition of distributivity in Lemma 2.19. We will briefly review here the argument of [9] in a slightly more general setting (for details consult [9]).

Theorem 4.8 *(GCH) Let κ be a λ-hypermeasurable cardinal with λ greater than κ and of cofinality at least κ^+. Assume further that there is a witnessing embedding j and a function $f_\lambda : \kappa \to \kappa$ such that $j(f_\lambda)(\kappa) = \lambda$. Then there is a forcing iteration $\mathbb{S} = \langle S_\alpha \,|\, \alpha \leq \kappa+1\rangle$ of generalized Sacks forcings which preserves measurability of κ and forces $2^\kappa = \lambda$. Moreover, the generic for the $j(\kappa)$-th stage of the iteration $j(\mathbb{S})$ is in some sense "generated" from the generic at stage κ of $\mathbb{S}$.*

We will give a sketch of the proof. Fix a λ-hypermeasurable extender embedding $j : V \to M$ with critical point κ; we may still assume that $j(f_\lambda)(\kappa) = \lambda$. As the cofinality of λ is at least κ^+, M can be taken to be closed under κ-sequences. Also, by GCH we have that $\lambda < j(\kappa) < \lambda^+$. We define the iteration $\mathbb{S} = \langle S_\alpha \,|\, \alpha \leq \kappa + 1\rangle$ as an Easton-supported forcing iteration of length $\kappa + 1$ which at every inaccessible $\alpha < \kappa$ adds $f_\lambda(\alpha)$-many new subsets of α using the forcing Sacks$(\alpha, f_\lambda(\alpha))$ and at stage κ adds λ-many new subsets of κ using Sacks(κ, λ). Let us write the generic $G_{\kappa+1}$ for $\langle S_\alpha \,|\, \alpha \leq \kappa + 1\rangle$ as $G * g$, where G is $\mathbb{S}_\kappa$-generic over V and g is Sacks(κ, λ)-generic over $V[G]$.

Our aim is to lift the embedding j to $V[G * g]$. Using the fact that $j(f_\lambda)(\kappa) = \lambda$, we can proceed as in [9] to lift partially to $j : V[G] \to M[G * g * H]$, where H is a generic for the iteration in the interval $(\kappa, j(\kappa))$.

Now it remains to find a generic h for Sacks$(j(\kappa), j(\lambda))$ over $M[G*g*H]$ containing the pointwise image of g. Denote $j[g]$ by h^*. As g is a set of conditions in Sacks(κ, λ) of $V[G]$, h^* is a set of conditions in Sacks$(j(\kappa), j(\lambda))$ of $M[G * g * H]$. The following lemma describes the "intersection" of the conditions in h^*.

Lemma 4.9 *For $\alpha < j(\lambda)$ let t_α be the intersection of the trees $j(p)(\alpha), p \in g$. If α belongs to the range of of j, then t_α is a $(\kappa, j(\kappa))$-*tuning fork*, i.e. a subtree of $2^{<j(\kappa)}$ which is the union of two cofinal branches which split at κ. If α does not belong to the range of j, then t_α consists of exactly one cofinal branch through $2^{<j(\kappa)}$.*

Proof. First notice that the intersection of $\bigcap_{C \in V[G]} j(C)$ where C is a closed unbounded set in κ is equal to $\{\kappa\}$. The intersection obviously contains κ. Now let $C_{f_\lambda} \subseteq \kappa$ be the closed unbounded set of closure points of f_λ, i.e. for all $\xi \in C_{f_\lambda}$, $f_\lambda(x) < \xi$ for each $x \in [\xi]^{<\omega}$ (without loss of generality, all elements of C_{f_λ} are limit cardinals). As $j(f_\lambda)(\kappa) = \lambda$ and $\bigcap_{C \in V[G]} j(C) \subseteq j(C_{f_\lambda})$, it is obvious that any element ξ of the intersection in the interval $(\kappa, j(\kappa))$ must be a limit cardinal greater than λ. But any such hypothetical ξ can be expressed as $j(f)(x)$ for some $f : [\kappa]^{<\omega} \to \kappa$ and $x \in [\lambda]^{<\omega}$. If C_f is the closed unbounded set of closure points of f, it is immediate that $j(C_f)$ cannot contain $\xi = j(f)(x)$. Notice that not only $j(f)(x) = \xi$ is not in $j(C_f)$, but the whole interval $[\lambda, \xi]$ is disjoint from $j(C_f)$ (as $x \in [\lambda]^{<\omega}$, the least closure point of $j(f)$ above λ must be greater than ξ). It follows that there is for each $\xi < j(\kappa)$ a closed unbounded set $C_\xi = C_{f_\lambda} \cap C_f$ such that

$$(\kappa, \xi] \cap j(C_\xi) = \emptyset. \tag{4.2}$$

The analysis of the intersection of closed unbounded sets is important as the following fact holds. If C is a closed unbounded subset of κ in $V[G]$ and X is a subset of λ of size at most κ, then any condition $p \in \text{Sacks}(\kappa, \lambda)$ in $V[G]$ has an extension q such that for all $i \in X$, $C(q(i))$ (= the set of splitting levels of the tree $q(i)$) is a subset of C. As every $\alpha < j(\lambda)$ can be expressed as some $j(f)(a)$ where f is a function from $[\kappa]^{<\omega}$ to λ, by applying the above fact with $X = \text{rng}(f)$ we obtain that $C(j(q)(\alpha))$ is a subset of $j(C)$. It follows by (4.2) that there is for each $\xi < j(\kappa)$ a condition r_ξ in g such that the tree $j(r_\xi)(\alpha)$ does not split between κ and ξ (though it may split at κ).

As $M[G * g * H]$ contains all subsets of κ existing in $V[G * g]$, it follows that the intersection t_α of the $j(p)(\alpha)$, $p \in g$, is a subtree of $2^{<j(\kappa)}$ which is the union of at most two cofinal branches which can only differ at κ.

If α is in the range of j than it is obvious that all trees $j(p)(\alpha)$, $p \in g$, do branch at κ (by elementarity and by the "continuous splitting" of a perfect tree). If α is not in the range of j, then the intersection t_α does not split at κ (the proof can be found in [9]). (Lemma 4.9) □

Definition 4.10 *For $\alpha < j(\lambda)$ in the range of j, let $(x(\alpha)_0, x(\alpha)_1)$ be the branches that make up the $(\kappa, j(\kappa))$-tuning fork at α, where $x(\alpha)_0(\kappa) = 0$ and $x(\alpha)_1(\kappa) = 1$. For $\alpha < j(\lambda)$ not in the range of j let $x(\alpha)_0$ denote the unique branch constituting the intersection of the $j(p)(\alpha)$, $p \in g$.*

Lemma 4.11 *Let h consist of all conditions p in* Sacks$(j(\kappa), j(\lambda))$ *of* $M[G * g * H]$ *such that for each* $\alpha < j(\lambda)$, $x(\alpha)_0$ *is contained in* $p(\alpha)$*. Then h is generic for* Sacks$(j(\kappa), j(\lambda))$ *of* $M[G * g * H]$ *and contains* $j[g]$*.*

Proof. Let D be a dense open set in Sacks$(j(\kappa), j(\lambda))$ in $M[G * g * H]$. As an element of $M[G * g * H]$, it can be written as $j(f)(d)$ for some f and $d \in [\lambda]^{<\omega}$. Without loss of generality we may assume that $f(a)$ is a dense open set in Sacks(κ, λ) for every $a \in [\kappa]^{<\omega}$.

Using Lemma 4.6, there is a condition $q \in g$ such that q reduces all dense open sets $f(a)$. By elementarity, $j(q)$ reduces all dense open sets $j(f)(a)$ for $a \in [\lambda]^{<\omega}$ and in particular reduces $j(f)(d) = D$.

In $M[G * g * H]$ choose a subset X of $j(\lambda)$ of size less than $j(\kappa)$ and $\alpha < j(\kappa)$ such that any (X, α)-thinning of $j(q)$ meets D. Now for each $i \in X$ thin $j(q)$ by choosing an initial segment of $x(i)_0$ on the α-th splitting level of $j(q)(i)$. As this sequence of choices is in $M[G * g * H]$ (for proof see [9]), it follows that this thinned out condition belongs to h and meets D. So h is generic for Sacks$(j(\kappa), j(\lambda))$ of $M[G * g * H]$ over $M[G * g * H]$ as desired.[6] (Lemma 4.11) □

Amongst the main advantages of [9], apart from the fact that we avoid the "modification" argument as in the Woodin-style approach (see for instance [1] or a slightly different argument in [2]) is that we don't have to enlarge the universe $V[G * g]$ to complete the lifting. This adds a degree of uniformity which will be used later in this thesis.

4.2 The optimal strength for failure of GCH on a measurable cardinal

In [14], M.Gitik (using the ideas of Mitchell concerning coherent sequences of measures) identified the following optimal consistency strength of a measurable cardinal κ failing GCH: as regards consistency, the assumption $o(\kappa) = \kappa^{+\alpha}$ is necessary for a measurable cardinal κ such that $2^\kappa = \kappa^{+\alpha}, \alpha \leq 2$. This restrictions is optimal in the sense that using the hypothesis we for instance have that $o(\kappa) = \kappa^{++}$ is enough to build a generic extension where κ is measurable and $2^\kappa = \kappa^{++}$ (see [13]). Note that by [9], the forcing with the Sacks forcing can also be used to force $2^\kappa = \kappa^{++}$ with κ remaining measurable from the assumption $o(\kappa) = \kappa^{++}$.

Anticipating a little, in Theorem 5.7, we realise an Easton function F and preserve some measurable cardinals. Since we also realize F on successor cardinals, it seems that we need the full strength of hypermeasurability to lift the embedding. For instance if $F(\kappa) = \kappa^{++}$, $F(\kappa^+) = \kappa^{+3}$ and $F(\kappa^{++}) = \kappa^{+4}$, then we need an embedding $j : V \to M$ such that $H(\kappa^{++})^V \subseteq M$ to

[6] One must also verify that any two conditions in h are compatible with each other; for argument, see [9].

lift j to the generic extension (see argument in Lemma 5.9). Thus it is an open question for us whether in the above particular example it suffices to start with $o(\kappa) = \kappa^{++}$ and realise F on κ, κ^+ and κ^{++}.

5 Easton functions and large cardinals

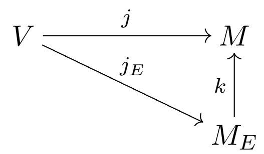
In this section we show some original results which concern the interaction between an Easton function F defined on regular cardinals and large cardinals in the universe V.

Before we start, we provide two observations on hypermeasurable cardinals which do not seem to fit elsewhere. For brief review of hypermeasurable cardinals, see Section 2.3.

Assume that F is an Easton function. The following observation can be shown easily, for instance using the arguments in [20].

Observation 5.1 *(GCH) If κ is $F(\kappa)$-hypermeasurable, where F is an Easton function, and $j : V \to M$ is a witnessing embedding, then j can be factored through some $j_E : V \to M_E$ and $k : M_E \to M$ such that j_E is an extender embedding with $A = [\kappa]^{<\omega}$ and $B = [F(\kappa)]^{<\omega}$ witnessing the $F(\kappa)$-hypermeasurability of κ. Moreover, if $j(F)(\kappa) \geq F(\kappa)$, then also $j_E(F)(\kappa) \geq F(\kappa)$.*

Proof. Consider the following commutative triangle:



By the construction of the extender, it follows that k is the identity on $F(\kappa)$. The following holds: $k(j_E(F)(\kappa)) = k(j_E(F))(k(\kappa)) = k(j_E(F))(\kappa) = j(F)(\kappa)$. If $\mu = j_E(F)(\kappa) < F(\kappa)$ were true, then k would be the identity at μ, implying that $j(F)(\kappa) = \mu$, which is a contradiction. (Observation 5.1) □

The above fact allows us to use only extender embeddings in our arguments and these will be used tacitly throughout.

The second observation concerns the properties of a singular λ in M, where M is the target model for λ-hypermeasurability. Perhaps surprisingly, λ can be regular (and more).

Observation 5.2 *(GCH) Let $j : V \to M$ for some transitive M be an embedding with critical point κ such that $H(\lambda)^V \subseteq M$, $\kappa < \lambda < j(\kappa)$ and λ is inaccessible in M (such an embedding exists for example if κ is λ-hypermeasurable for some V-inaccessible $\lambda > \kappa$). Then there exists $\bar{\lambda} \leq \lambda$ singular in V and an embedding $k : V \to N$ which witnesses that κ is $\bar{\lambda}$-hypermeasurable and $\bar{\lambda}$ is inaccessible in N.*

Proof. Without loss of generality assume that j is an extender embedding, that is $M = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\lambda]^{<\omega}\}$. If λ is singular in V, then we are done. So assume that λ is regular (and hence inaccessible) in V. For each $f : [\kappa]^{<\omega} \to \kappa$ define a function $\theta_f : [\lambda]^{<\omega} \to \lambda$ by setting $\theta_f(a) = j(f)(a)$ if $j(f)(a) < \lambda$ and $\theta_f(a) = 0$ otherwise. Working in V, let $C_f \subseteq \lambda$ by a closed unbounded set of limit cardinals closed under θ_f; as $\kappa^+ < \lambda$ and the number of all θ_f is κ^+, the intersection $C = \bigcap_f C_f$ is a closed unbounded set in λ. Let $\bar{\lambda}$ be some singular cardinal in C greater than κ and let $H = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\bar{\lambda}]^{<\omega}\}$. If $\pi : H \cong N$ is the transitive collapse map, we obtain that $\pi \circ j : V \to N$ witnesses that κ is $\bar{\lambda}$-hypermeasurable. We are done once we show that $\bar{\lambda}$ is regular in N. This will follow from the fact that $H \cap \lambda = \bar{\lambda}$. Let $\alpha \in H \cap \lambda$ be given; it is of the form $j(f)(a)$ for some $f : [\kappa]^{<\omega} \to \kappa$ and $a \in [\bar{\lambda}]^{<\omega}$. As $\alpha < \lambda$, $j(f)(a) = \theta_f(a) < \bar{\lambda}$ by the selection of $\bar{\lambda}$ in C_f. Conversely, if $\alpha < \bar{\lambda}$, then $\alpha = j(id)(\alpha)$. Finally, without loss of generality we may assume that there is some f_λ such that $j(f_\lambda)(\kappa) = \lambda$ and hence $\lambda \in H$. Since then $\pi(\lambda) = \bar{\lambda}$, the observation follows. (Observation 5.2) □

5.1 Preservation of measurable cardinals

Recall that a class function F defined on regular cardinals is called an *Easton function* if it satisfies the following two conditions which were shown by Easton to be the only conditions provable about the continuum function on regular cardinals in ZFC. Let κ, μ be arbitrary regular cardinals:

1. If $\kappa < \mu$, then $F(\kappa) \leq F(\mu)$;
2. $\kappa < \mathrm{cf}(F(\kappa))$.

As discussed above if a given large cardinal κ should remain measurable in a generic extension realizing a given Easton function F, the properties of the cardinal κ and the properties of the function F need to combine in a suitable way which requires more than the conditions given in Definition 2.1. See Section 3.

We capture a sufficient condition for preservation of measurability in the following definition.

Definition 5.3 *We say that a cardinal κ is* good *for F, or shortly F-*good*, if the following properties hold:*

1. *$F[\kappa] \subseteq \kappa$, i.e. κ is closed under F;*
2. *κ is $F(\kappa)$-hypermeasurable and this is witnessed by an embedding $j : V \to M$ such that $j(F)(\kappa) \geq F(\kappa)$.*

Our forcing to realise a given Easton function F will be a combination of the Sacks forcing $\mathrm{Sacks}(\bar{\alpha}, \bar{\beta})$ (see Definition 4.2) and of the Cohen forcing $\mathrm{Add}(\alpha, \beta)$, where $\bar{\alpha}, \alpha$ are regular cardinals and $\bar{\beta}, \beta$ are ordinal numbers. For notational convenience we will construe $\mathrm{Add}(\alpha, \beta)$ as the $< \alpha$-supported product of $\mathrm{Add}(\alpha, 1)$ of length β, where conditions in $\mathrm{Add}(\alpha, 1)$ are functions from α to 2 with domain of size less than α.

As our aim is the preservation of large cardinals, we cannot use the standard Easton product-style forcing, but we need to use some kind of (reverse Easton) iteration. We will define our forcing according to Defintion 3.7 with one modification: we will use the Sacks forcing $\mathrm{Sacks}(\alpha, F(\alpha))$ for every regular closure point α of F.

Definition 5.4 *Assume GCH. Let an Easton function F satisfying the conditions (1), (2) of 2.1 be given. Let $\langle i_\alpha \,|\, \alpha < \mathrm{On}\rangle$ be an increasing enumeration of the closure points of F.*

We will define an iteration $\mathbb{P}^F = \big\langle \langle \mathbb{P}_{i_\alpha} \,|\, \alpha < \mathrm{On}\rangle, \langle \dot{\mathbb{Q}}_{i_\alpha} \,|\, \alpha < \mathrm{On}\rangle \big\rangle$ indexed by $\langle i_\alpha \,|\, \alpha < \mathrm{On}\rangle$ such that:

- *If i_α is not an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{\mathbb{Q}}_{i_\alpha}, \tag{5.1}$$

  where $\dot{\mathbb{Q}}_{i_\alpha}$ is a name for

$$\prod_{i_\alpha < \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda)),$$

  where λ ranges over regular cardinals and the product has the Easton support.

- *If i_α is an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{\mathbb{Q}}_{i_\alpha}, \tag{5.2}$$

  where $\dot{\mathbb{Q}}_{i_\alpha}$ is a name for

$$\mathrm{Sacks}(i_\alpha, F(i_\alpha)) \times \prod_{i_\alpha < \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda)),$$

  where λ ranges over regular cardinals and the product has the Easton support.

- *If γ is a limit ordinal, then $\mathbb{P}_{i_\gamma}$ is an inverse limit unless i_γ is a regular cardinal, in which case $\mathbb{P}_{i_\gamma}$ is a direct limit (the usual Easton support).*

Lemma 5.5 *Under GCH, $\mathbb{P}^F$ preserves all cofinalities.*

Proof. See the proof of Lemma 3.8. The only new feature of $\mathbb{P}^F$ compared to Lemma 3.8 is the inclusion of the Sacks forcing. It is enough to show that $\mathrm{Sacks}(\kappa, F(\kappa)) \times \mathrm{Add}(\kappa^+, F(\kappa^+))$ preserves cofinalities. The product is κ-closed and κ^{++}-cc, hence just the cardinal κ^+ needs a special argument. For this, it suffices to use the usual fusion-style argument for the Sacks forcing in $V^{\mathrm{Add}(\kappa^+, F(\kappa^+))}$. (Lemma 5.5) □

In fact, working harder we can show that $\mathrm{Sacks}(\kappa, \alpha)$ forces that the Cohen forcing $\mathrm{Add}(\kappa^+, \beta)$ (for arbitrary ordinals α, β) remains κ^+-distributive. This will be useful in further arguments.

Lemma 5.6 *Let κ be an inaccessible cardinal and α an ordinal number. Let $\mathbb{P}$ be any κ^+-closed forcing notion.*

1. $\mathrm{Sacks}(\kappa, 1)$ *forces that $\check{\mathbb{P}}$ is κ^+-distributive.*
2. *Or more generally,* $\mathrm{Sacks}(\kappa, \alpha)$ *forces that $\check{\mathbb{P}}$ is κ^+-distributive.*

Proof. Ad (1). Denote $\mathbb{S} = \mathrm{Sacks}(\kappa, 1)$. The proof is a generalization of the usual argument which shows that a κ^+-closed forcing notion does not add new κ-sequences. The difference is in the treatment of the Sacks coordinates in $\mathbb{S} \times \mathbb{P}$ which are obviously not κ^+-closed; however, they are closed under fusion limits of length κ, and this will suffice to argue that new κ sequences cannot appear between $V^{\mathbb{S}}$ and $V^{\mathbb{S}\times\mathbb{P}}$.

Let $\langle s, p\rangle$ force that $\dot{f} : \kappa \to \mathrm{On}$. It is enough to find a condition $\langle \tilde{s}, \tilde{p}\rangle \leq \langle s, p\rangle$ such that if $\langle \tilde{s}, \tilde{p}\rangle \in G \times H$, where $G \times H$ is a generic for $\mathbb{S} \times \mathbb{P}$, then $\dot{f}^{G\times H} = f$ can be defined in $V[G]$.

We will define a decreasing sequence of conditions $\langle\langle s_\alpha, p_\alpha\rangle \,|\, \alpha < \kappa\rangle$ deciding the values of $\dot{f}(\alpha)$ for $\alpha < \kappa$ where $\tilde{s}$ will be the fusion limit of $\langle s_\alpha \,|\, \alpha < \kappa\rangle$ and $\tilde{p}$ will be the lower bound of $\langle p_\alpha \,|\, \alpha < \kappa\rangle$.

Set $\langle s_0, p_0\rangle = \langle s, p\rangle$. Assume $\langle s_{\alpha'}, p_{\alpha'}\rangle$ are constructed for $\alpha' < \alpha$ and let first $\langle \bar{s}_\alpha, \bar{p}_\alpha\rangle$ be a lower bound of $\langle s_{\alpha'}, p_{\alpha'}\rangle$'s; we show how to construct $\langle s_\alpha, p_\alpha\rangle$. Let S_α denote the set of splitting nodes of rank α in $\bar{s}_\alpha$ (the first splitting node has rank 0). Pick some $t \in S_\alpha$, and considering its immediate continuations $t{*}0$ and $t{*}1$, find conditions $\langle r_{t*0}, p_{t*0}\rangle$, $\langle r_{t*1}, p_{t*1}\rangle$ and ordinals α_{t*0}, α_{t*1} such that the following conditions hold:

1. $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1}$;
2. $r_{t*0} \leq \bar{s}_\alpha \restriction t * 0$, $r_{t*1} \leq \bar{s}_\alpha \restriction t * 1$;
3. $\langle r_{t*0}, p_{t*0}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*0}$ and $\langle r_{t*1}, p_{t*1}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*1}$.

Continue in this fashion considering successively all $t \in S_\alpha$, taking care to form a decreasing chain $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$, for $t, t' \in S_\alpha$ (where t' is considered after t) in the Cohen forcing. We define:

1. $p_\alpha =$ the lower bound of $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$;
2. $s_\alpha =$ the amalgamation of the subtrees r_{t*0}, r_{t*1} for all $t \in S_\alpha$.

Finally, define $\langle \tilde{s}, \tilde{p} \rangle$ as the fusion limit of s_α's at the first coordinate and as the lower bound at the second coordinate.

Let $G \times H$ be a generic for $\mathbb{S} \times \mathbb{P}$ containing $\langle \tilde{s}, \tilde{p} \rangle$. In $V[G]$ define a function $f' : \kappa \to \mathrm{On}$ as follows: $f'(\alpha) = \beta$ iff $\beta = \alpha_{t*i}$, for $i \in \{0, 1\}$, where t is a splitting node of rank α in $\tilde{s}$ and $t * i \subseteq \bigcup_{s \in G} \mathrm{stem}(s)$.

It is straightforward to verify that $f' = f = \dot{f}^{G \times H}$.

Ad (2). The proof proceeds in exactly the same way as (1) except that a generalized fusion is used for the Sacks(κ, α) forcing (it is essential here that the conditions in the Sacks forcing can have support of size κ). (Lemma 5.6) □

We can now state the main theorem.

Theorem 5.7 *Assume GCH and let F be an Easton function according to Definition 2.1. Then the generic extension by $\mathbb{P}^F$ preserves all cofinalities and realises F, i.e. $2^\kappa = F(\kappa)$ for every regular cardinal κ. Moreover, if a cardinal κ is good for F, then it will remain measurable.*

The proof will be given in a sequence of lemmas.

It is obvious that the Easton function F is realised in $V^{\mathbb{P}^F}$. It remains to prove that each F-good cardinal κ remains measurable in the generic extension. Let an F-good cardinal κ be fixed. Fix also a $j : V \to M$ an $F(\kappa)$-hypermeasurable extender embedding witnessing the F-goodness of κ.

$$V \xrightarrow{\quad j \quad} M$$

The properties of the Easton function F imply that $\mathrm{cf}(F(\kappa)) > \kappa$, so in particular M is closed under κ-sequences in V. It also holds that $F(\kappa) < j(\kappa) < F(\kappa)^+$, $j(F)(\kappa) \geq F(\kappa)$ (by goodness), $M = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [F(\kappa)]^{<\omega}\}$, and $H(F(\kappa))^V = H(F(\kappa))^M$. Note that M is not closed even under κ^+-sequences in V, but the correct capturing of $H(F(\kappa))$ implies that ${}^{<\mathrm{cf}(F(\kappa))}H(F(\kappa)) \subseteq M$, so M is closed under $< \mathrm{cf}(F(\kappa))$-sequences providing that they refer to objects in $H(F(\kappa))$.

We fix some notation first. Let G be a generic for $\mathbb{P}^F$. As usual, we will write G_α for the generic G restricted to $\mathbb{P}_\alpha$. The generic for $\dot{\mathbb{Q}}_\alpha$ taken in $V[G_\alpha]$ will be denoted as g_α; it follows that $G_{\alpha+1} = G_\alpha * g_\alpha$.

For reasons of notational simplicity, we write $\mathbb{P}^M$ for $j(\mathbb{P}^F)$. Recall that $\mathbb{P}^F$ is defined as an iteration along the closure points $\langle i_\alpha \mid \alpha < \mathrm{On} \rangle$ of F; by elementarity, $\mathbb{P}^M$ is defined using the closure points of $j(F)$, which we will denote as $\langle i^M_\alpha \mid \alpha < \mathrm{On} \rangle$. Since j is the identity on $H(\kappa)$, the closure points of F and $j(F)$ coincide up to and including κ, i.e. $\langle i_\alpha \mid \alpha \leq \kappa \rangle = \langle i^M_\alpha \mid \alpha \leq \kappa \rangle$.

Because κ is regular, we also have that $i_\kappa = \kappa$. By elementarity, $j(\kappa)$ is closed under $j(F)$, and as $j(\kappa)$ is regular in M, it follows that $j(\kappa) = i^M_{j(\kappa)}$ and so in particular $F(\kappa) \leq j(F)(\kappa) < i^M_{\kappa+1} < j(\kappa) < F(\kappa)^+ < i_{\kappa+1}$.

The general strategy of the proof is to lift the embedding j to $V[G]$. This amounts to finding a suitable generic for $\mathbb{P}^M$. As the cardinal structure between V and M is the same up to and including $F(\kappa)$, it follows that the generics for the V-regular cardinals $\leq F(\kappa)$ need to be "copied" from the $V[G]$-side. The forcing $\mathbb{P}^M$ at the M-cardinals in the interval $(F(\kappa), j(\kappa))$ (and at $F(\kappa)$ if $F(\kappa)$ is singular in V but regular in M) will be shown to be sufficiently well-behaved so that the corresponding generics can be constructed in $V[G]$. The next step is the forcing $\mathbb{P}^M$ at $j(\kappa)$ where the task is twofold: not only do we need to find a generic, but we need to find one which contains the pointwise image under j of g_κ. Precisely to resolve this difficult point, we have included the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$ at stage κ because by [9] the point-wise image of the generic g_κ (or rather of its Sacks part) will (almost) generate the correct generic for $j(\kappa)$. Finally, we lift to all of $V[G]$ using Lemma 2.19.

We will first lift the embedding j to $V[G_\kappa]$. As $H(\kappa)^V = H(\kappa)^M$, $\mathbb{P}_\kappa = \mathbb{P}^M_\kappa$ and it follows we can copy the generic G_κ.

Note: In order to keep track of where we are, we will use the following dotted arrow convention to indicate that we are in the process of lifting the embedding j to $V[G_\kappa]$, but we have not yet completed the lifting. Once we lift the embedding, the arrow will be printed in solid line.

$$V[G_\kappa] \overset{j}{\dashrightarrow} M[G_\kappa]$$

Recall by the definition of $\mathbb{P}^F$ that the next step of iteration $\mathbb{Q}_\kappa$ in $V[G_\kappa]$ is the product $\mathrm{Sacks}(\kappa, F(\kappa)) \times \prod_{\kappa<\lambda<i_{\kappa+1}} \mathrm{Add}(\lambda, F(\lambda))$, where λ ranges over regular cardinals in V and the product has the Easton support; the corresponding forcing in $M[G_\kappa]$, to be denoted $\mathbb{Q}^M_\kappa$, is $\mathrm{Sacks}(\kappa, j(F)(\kappa)) \times \prod_{\kappa<\lambda<i^M_{\kappa+1}} \mathrm{Add}(\lambda, j(F)(\lambda))$, where λ ranges over regular cardinals in M.

Remark 5.8 For typographical reasons, we employ the following notation for $\mathbb{Q}_\kappa$ and $\mathbb{Q}^M_\kappa$.

- We write $i(\kappa+1)$ for $i_{\kappa+1}$ and $i^M(\kappa+1)$ for $i^M_{\kappa+1}$;
- If λ is a regular cardinal in V in the interval $[\kappa, i(\kappa+1))$, then $\mathcal{Q}_\lambda$ stands for the forcing $\mathrm{Sacks}(\kappa, F(\kappa))$ if $\lambda = \kappa$, and for the forcing $\mathrm{Add}(\lambda, F(\lambda))$ if $\lambda \neq \kappa$;
- If $\mu < \mu'$ are cardinals in V (μ, μ' may be singular) in the interval $[\kappa, i(\kappa+1))$ then we write $\prod_{[\mu,\mu')} \mathcal{Q}_\lambda$ for the product $\mathbb{Q}_\kappa$ restricted to the interval $[\mu, \mu')$ (and similarly for other intervals (μ, μ') etc.). Thus for instance $\prod_{[\kappa,i(\kappa+1))} \mathcal{Q}_\lambda = \mathbb{Q}_\kappa$.

– Analogously, if $\bar{\lambda}$ is a regular cardinal in M in the interval $[\kappa, i^M(\kappa+1))$, then $\mathcal{Q}^M_{\bar{\lambda}}$ stands for the forcing $\mathrm{Sacks}(\kappa, j(F)(\kappa))$ in $M[G_\kappa]$ if $\bar{\lambda} = \kappa$, and for the forcing $\mathrm{Add}(\bar{\lambda}, j(F)(\bar{\lambda}))$ in $M[G_\kappa]$ if $\bar{\lambda} \neq \kappa$;

– If $\mu < \mu'$ are cardinals in M (μ, μ' may be singular in M) in the interval $[\kappa, i^M(\kappa+1))$ then we write $\prod^M_{[\mu,\mu')} \mathcal{Q}^M_{\bar{\lambda}}$ for the $M[G_\kappa]$-product $\mathbb{Q}^M_\kappa$ restricted to the interval $[\mu, \mu')$ (and similarly for other intervals (μ, μ') etc.);

– Generic filters for these forcings (once they are found in the case of the forcing in $M[G_\kappa]$) shall be denoted in the same fashion using the notation $g_{[\mu,\mu')}$ and $g^M_{[\mu,\mu')}$, respectively.

Now we return to the proof. We will proceed to show that $g_{[\kappa,F(\kappa)]}$ can be used to find in $V[G_\kappa]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

We will first correct the possible discrepancy between the values of $F(\lambda)$ and $j(F)(\lambda)$ for V-regular λ in the interval $[\kappa, F(\kappa)]$ (recall that $\lambda \leq F(\kappa)$ is a cardinal in V if and only if it is a cardinal in M, but $F(\kappa)$ may be regular M but singular in V, so we need to remember in which universe we are: V or M). By elementarity of j, $j(\kappa)$ is closed under $j(F)$ and by the F-goodness of κ, $F(\kappa) \leq j(F)(\kappa) < j(\kappa)$. Let λ_0 be the least regular cardinal greater than κ such that $F(\kappa) < F(\lambda_0)$ ($\lambda_0 \leq \mathrm{cf}((F(\kappa))$ as $F(\mathrm{cf}(F(\kappa)))$ has cofinality greater than $\mathrm{cf}(F(\kappa))$ and therefore cannot equal $F(\kappa)$). For a regular $\lambda \in [\kappa, \lambda_0)$, $F(\lambda) = F(\kappa) \leq j(F)(\kappa) \leq j(F)(\lambda)$. Also $\lambda < \lambda_0 \leq \mathrm{cf}(F(\kappa)) \leq F(\kappa) < j(\kappa)$ and $j(\kappa)$ is closed under $j(F)$; it follows that $j(F)(\lambda) < j(\kappa)$ and hence $F(\lambda)$, $j(F)(\lambda)$ both have V-cardinality $F(\kappa)$. Any bijection between $F(\lambda)$ and $j(F)(\lambda)$ for a given λ generates an isomorphism between the forcings $\mathrm{Sacks}(\kappa, F(\kappa))$ and $\mathrm{Sacks}(\kappa, j(F)(\kappa))$ if $\lambda = \kappa$ and between $\mathrm{Add}(\lambda, F(\lambda))$ and $\mathrm{Add}(\lambda, j(F)(\lambda))$ otherwise. Denote these isomorphic forcings as $\mathcal{Q}^*_\lambda$, i.e. $\mathcal{Q}_\lambda \cong \mathcal{Q}^*_\lambda$. If a V-regular λ lies in the interval $[\lambda_0, F(\kappa)]$, then $j(F)(\lambda) < j(\kappa) < F(\kappa)^+ \leq F(\lambda)$ and so $j(F)(\lambda) < F(\lambda)$. It follows we can truncate the product $\mathcal{Q}_\lambda$ at the ordinal $j(F)(\lambda)$; let $\mathcal{Q}^{**}_\lambda$ denote this truncation. It is immediate that

$$\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda =_{df} \prod_{[\kappa,\lambda_0)} \mathcal{Q}^*_\lambda \times \prod_{[\lambda_0,F(\kappa)]} \mathcal{Q}^{**}_\lambda \tag{5.3}$$

is completely embeddable into $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda$ and so there is a generic filter, to be denoted $g^+_{[\kappa,F(\kappa)]}$, existing in $V[G]$, which is $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$-generic over $V[G_\kappa]$. The generic $g^+_{[\kappa,F(\kappa)]}$ will be used to find a generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

The manipulation to obtain $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ ensures agreement for $\lambda \leq F(\kappa)$ between the lengths of the products $\mathcal{Q}_\lambda$ and $\mathcal{Q}^M_{\bar{\lambda}}$ in $V[G_\kappa]$ and $M[G_\kappa]$, respectively, but a word of caution is in order. For instance if $F(\kappa) > \kappa^+$ is regular, $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ is never identical with $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_{\bar{\lambda}}$: Already for $F(\kappa) = \kappa^{++}$ the forcing $\mathrm{Add}(\kappa^{++}, j(F)(\kappa^{++}))$ in $M[G_\kappa]$ fails to capture all conditions in $\mathrm{Add}(\kappa^{++}, j(F)(\kappa^{++}))$ in $V[G_\kappa]$ as the supports in this forcing are κ^+-sequences extending above κ^{++}, and some such sequences are missing in $M[G_\kappa]$ (for instance if $F(\kappa) = \kappa^{++}$, $(\kappa^{+3})^M$ has cofinality κ^+ in V). Accordingly, we only have (when $F(\kappa)$ is regular in V greater than κ^+) $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda \subseteq \prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$.

We will deal separately with the two cases: $F(\kappa)$ regular in V, and $F(\kappa)$ singular in V.

Lemma 5.9 *Assume $F(\kappa)$ is* regular *in V. There is in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$, which we will denote as $g^M_{[\kappa,i^M(\kappa+1))}$.*

Proof. As $F(\kappa)$ is regular in V, it is also regular in M. Consequently, the forcing $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ is $F(\kappa)^+$-cc in $M[G_\kappa]$ and as $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ is $F(\kappa)^+$-closed, the forcings are mutually generic in the sense of Lemma 2.6. It follows that we can deal with $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ and $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ separately.

A) The product $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$.

We will use $g^+_{[\kappa,F(\kappa)]}$ to obtain the required generic; in fact we will show that the intersection $g^M_{[\kappa,F(\kappa)]} = g^+_{[\kappa,F(\kappa)]} \cap \prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ is $M[G_\kappa]$-generic for $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$.

We will argue that a maximal antichain $A \in M[G_\kappa]$ in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ will stay maximal in $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$, and so will be hit by $g^+_{[\kappa,F(\kappa)]}$.

For $p \in \prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ write

$$\mathrm{supp}(p) = \{\langle \lambda, \alpha \rangle \mid p(\lambda)(\alpha) \neq 1\}, \tag{5.4}$$

where 1 stands for the empty condition in the relevant forcing, and analogously for $A \subseteq \prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$,

$$\mathrm{supp}(A) = \{\langle \lambda, \alpha \rangle \mid \exists p \in A, \langle \lambda, \alpha \rangle \in \mathrm{supp}(p)\}. \tag{5.5}$$

We will show that if $A \in M[G_\kappa]$ is a maximal antichain in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ and $p \in \prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ is arbitrary, then

$$X = \mathrm{supp}(p) \cap \mathrm{supp}(A) \in M[G_\kappa] \text{ and } p \restriction X \in M[G_\kappa]. \tag{5.6}$$

Providing we know (5.6), $p \restriction X$ must be compatible with some $a \in A$, and because p and a are compatible on the supports, they must be compatible everywhere. It follows that A stays maximal in $V[G_\kappa]$. To argue for (5.6), the $F(\kappa)^+$-cc of $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ in $M[G_\kappa]$ implies that the size of $\mathrm{supp}(A)$ in $M[G_\kappa]$ is at most $F(\kappa)$. Since the size of $\mathrm{supp}(p)$ is strictly less than $F(\kappa)$, (5.6) will follow from the following property (5.7).

$$\text{If a set } x \in M[G_\kappa] \text{ has size at most } F(\kappa) \text{ in } M[G_\kappa], \text{ then } {}^{<F(\kappa)}x \cap V[G_\kappa] \subseteq M[G_\kappa]. \tag{5.7}$$

Let $f : x \to F(\kappa)$ be a 1-1 function, $f \in M[G_\kappa]$, and let $\vec{s} \in {}^{<F(\kappa)}x \cap V[G_\kappa]$ be given. Working in $V[G_\kappa]$, it is obvious that $f[\vec{s}] \in H(F(\kappa))$. Since $H(F(\kappa))$ is the same in $V[G_\kappa]$ and $M[G_\kappa]$, $f[\vec{s}] \in M[G_\kappa]$. But as f is in $M[G_\kappa]$, so is $f^{-1}[f[\vec{s}]] = \vec{s}$.

B) The product $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

Notice that every dense open set of $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$ in $M[G_\kappa]$ is of the form $(j(f)(a))^{G_\kappa}$, $a \in [F(\kappa)]^{<\omega}$, where $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name, for some $f : [\kappa]^{<\omega} \to H(\kappa^+)$. Without loss of generality, we may assume that the range of all such f contains just names for dense open sets.[7] For each such f, the set $\{\langle j(f)(a), 1\rangle \mid a \in [F(\kappa)]^{<\omega}\}$ is a $\mathbb{P}^M_\kappa$-name in M, which interprets as a family $\{(j(f)(a))^{G_\kappa} \mid a \in [F(\kappa)]^{<\omega}\}$ of at most $F(\kappa)$ many dense open sets in $M[G_\kappa]$ – it follows the intersection $\mathscr{D}_f = \bigcap_{a\in[F(\kappa)]^{<\omega}} (j(f)(a))^{G_\kappa}$ is dense in $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$ since the forcing notion $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$ is $F(\kappa)^+$-distributive in $M[G_\kappa]$. As there are only $(\kappa^+)^\kappa = \kappa^+$ such f's, and $M[G_\kappa]$ is closed under κ-sequences in $V[G_\kappa]$, we can construct a generic in $V[G_\kappa]$ meeting all the dense sets $\mathscr{D}_f$ for all suitable f. Let us denote this generic as $g^M_{(F(\kappa),i^M(\kappa+1)}$.

We finish the proof by setting $g^M_{[\kappa,i^M(\kappa+1))} = g^M_{[\kappa,F(\kappa)]} \times g^M_{(F(\kappa),i^M(\kappa+1))}$.

(Lemma 5.9) $\Box$

Lemma 5.10 *Assume $F(\kappa)$ is* singular *in V with cofinality $\delta < F(\kappa)$ (recall that $\kappa^+ \le \delta$ by the definition of Easton function). There is in $V[G_\kappa * g_{[\kappa,F(\kappa))}]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$, which we will denote as $g^M_{[\kappa,i^M(\kappa+1))}$.*

The singularity of $F(\kappa)$ implies that $M[G_\kappa]$ may not be closed in $V[G_\kappa]$ under $<F(\kappa)$-sequences of elements of $F(\kappa)$, but just under $<\delta$-sequences. It follows that the argument given in Lemma 5.9, in particular (5.7), cannot be used as it stands. However, we will argue that the desired generic can be constructed via "approximations" by induction along some sequence of regular cardinals cofinal in $F(\kappa)$.

In preparation for the argument, we will define a certain procedure which will be used in the argument. Let $\langle \gamma_i \mid i < \delta\rangle$ be a sequence of regular cardinals cofinal in $F(\kappa)$, with $\delta < \gamma_0$ (we may assume that this sequence belongs to M if $F(\kappa)$ is singular in M, as in that case, $F(\kappa)$ has the same cofinality in M as it has in V). Generalizing our notation, if $\gamma_{i+1} < \mu$, where μ is an M-cardinal (μ will in fact be always either $F(\kappa)$ or $i^M(\kappa+1)$), and $p \in \prod^M_{[\kappa,\mu)} \mathcal{Q}^M_{\bar{\lambda}}$ is a condition, let p_{γ_i} denote p restricted to $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_{\bar{\lambda}}$ (the "lower part of p") and p^{γ_i} denote p restricted to $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_{\bar{\lambda}}$ (the "upper part

[7]Formally, $f(s)$ will be a name for a dense open set in the forcing $\mathbb{P}_\kappa$, and so $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be a name for a dense open set in $\mathbb{P}^M_{j(\kappa)}$. We will abuse notation and identify every $j(f)(a)$ with a $\mathbb{P}_\kappa$-name for a dense open set in $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

of p") (the parameter μ will be understood from the context). Note that for each γ_i, $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_{\lambda}$ is γ_i^+-closed and $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_{\lambda}$ is γ_i^+-cc in $M[G_\kappa]$.

Let γ_i, $f : [\kappa]^{<\omega} \to H(\kappa^+)$, and $a \in [\gamma_i]^{<\omega}$ be arbitrary and assume that $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name for a dense open set in $\prod^M_{[\kappa,\mu)} \mathcal{Q}^M_{\lambda}$, where μ is either $F(\kappa)+1$ or $i^M_{\kappa+1}$. Let us denote $(j(f)(a))^{G_\kappa}$ as D. Assume further that p is a condition in $\prod^M_{[\kappa,\mu)} \mathcal{Q}^M_{\lambda}$.

Definition 5.11 $\overline{q} \in \prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_{\lambda}$ *is said to* γ_i-reduce D below p *if the following holds:*

1. *$\overline{q}$ extends the upper part of p, i.e. $\overline{q} \leq p^{\gamma_i}$ in $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_{\lambda}$;*
2. *The set $\overline{D} = \{q \leq p_{\gamma_i} \in \prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_{\lambda} \,|\, q \cup \overline{q} \in D\}$ is dense open in $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_{\lambda}$ below the lower part of p.*

We will show how to construct such a reduction $\overline{q}$ (the argument is essentially the one used to prove the Easton lemma 2.6 (1)). Choose some (r_0, s_0) such that $r_0 \cup s_0 \in D$ and $r_0 \leq p_{\gamma_i}$ and $s_0 \leq p^{\gamma_i}$. At stage ξ of the construction, let r'_ξ be any condition which is incompatible with the set of all previous conditions $\{r_\zeta \,|\, \zeta < \xi\}$ (if there is such) and let s'_ξ be a lower bound of $\langle s_\zeta \,|\, \zeta < \xi\rangle$. Choose $r_\xi \leq r'_\xi$ and $s_\xi \leq s'_\xi$ such that $r_\xi \cup s_\xi \in D$. The construction is well-defined since $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_{\lambda}$ is γ_i^+-cc and consequently the process will stop at some $\rho < \gamma_i^+$. Set $\overline{q}$ to be the lower bound of all s_ζ for $\zeta < \rho$. We will show that $\overline{q}$ indeed γ_i-reduces D below p according to Definition 5.11. We only need to check the condition (2) as (1) is obvious. Let $q \leq p_{\gamma_i}$ be given. It follows from the construction that there is some r_ζ such that q and r_ζ are compatible with some lower bound $\tilde{r}$. Also, $r_\zeta \cup s_\zeta \in D$ and consequently $\tilde{r} \cup \overline{q} \in D$ by openness. Note that as $p \in M[G_\kappa]$ by assumption, the construction can be carried out in $M[G_\kappa]$ and consequently $\overline{q}$ will also be in $M[G_\kappa]$.

We will need to distinguish several cases which will be handled in a sequence of Sublemmas. Notice that by Observation 5.2, we cannot disregard the possibility that $F(\kappa)$ is singular in V while it is regular in M.

Sublemma 5.12 *If the cofinality of $F(\kappa)$ in V is κ^+ and $F(\kappa)$ is singular in V ($F(\kappa)$ can be either regular or singular in M) then there is in $V[G_\kappa * g_{[\kappa,F(\kappa))}]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\lambda}$.*

Proof. Fix the two following sequences:

1. Sequence $\langle \gamma_i \,|\, i < \kappa^+\rangle$ of regular cardinals cofinal in $F(\kappa)$, where $\kappa^+ < \gamma_0$;

2. Sequence $\langle j(f_\alpha) \mid \alpha < \kappa^+\rangle$, where $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ enumerates all $f : [\kappa]^{<\omega} \to H(\kappa^+)$ such that $f(s)$ is a name for a dense open set in $\mathbb{P}_\kappa$ for every $s \in [\kappa]^{<\omega}$; $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will thus range over names for dense open sets in $\mathbb{P}^M_{j(\kappa)}$ but we will abuse notation and identify every $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ with a name restricted to $\prod^M_{[\kappa, i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ in $M[G_\kappa]$.

By induction on $i < \kappa^+$, we will construct conditions $p_i \in M[G_\kappa]$ the tails of which will reduce all dense open sets in $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ according to Definition 5.11. We will also consider their limit – a "master condition" – p_∞ (possibly outside $M[G_\kappa]$) .

Fix in advance some wellordering $<_0 \in M[G_\kappa]$ of the pairs in $\kappa^+ \times [F(\kappa)]^{<\omega}$ such that the restriction of $<_0$ to $k \times [\gamma_k]^{<\omega}$ for each $k < \kappa^+$ has order type γ_k. Assume that p_i have been constructed for all $i < k$ and we need to construct p_k. First let r_k be a lower bound of p_i for $i < k$ and work below this condition. Carry out the following construction in $M[G_\kappa]$. By induction on $<_0$ restricted to $k \times [\gamma_k]^{<\omega}$ construct a decreasing chain of conditions $\overline{q}_{(\xi,a)}$ in $\prod^M_{(\gamma_k,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ as follows. At stage (ξ, a), let first $r_{(\xi,a)}$ be the lower bound of $\overline{q}_{(\xi',a')}$ for $(\xi', a') <_0 (\xi, a)$. Using the argument below Definition 5.11, set $\overline{q}_{(\xi,a)}$ to be a condition which γ_k-reduces the dense open set with the name $j(f_\xi)(a)$ below $(r_k)_{\gamma_k} \cup r_{(\xi,a)}$. Since the induction has length γ_k and we consider only the initial segment of order type k of the functions in the sequence $\langle j(f_\xi) \mid \xi < \kappa^+\rangle$ (which exists in $M[G_\kappa]$), the lower bound of all $\overline{q}_{(\xi,a)}$ exists in $M[G_\kappa]$. Denoting this lower bound $\overline{q}$, we set p_k to be equal to the union of the lower part of r_k and $\overline{q}$, i.e. $p_k = (r_k)_{\gamma_k} \cup \overline{q}$.

Set p_∞ to be a lower bound of $\langle p_i \mid i < \kappa^+\rangle$ (p_∞ may exist only in $V[G_\kappa]$). Let us write $p^{\leftarrow}_\infty$ for p_∞ restricted to the interval $[\kappa, F(\kappa))$ and $p^{\rightarrow}_\infty$ for the rest of p_∞ defined at the interval $[F(\kappa), i^M_{\kappa+1})$. Note that $p^{\leftarrow}_\infty$ is an element of the forcing $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$, while $p^{\rightarrow}_\infty$ is not an element of any of the forcings introduced so far (it is just a union of certain conditions which exists in $V[G_\kappa]$).

Define the desired generic $g^M_{[\kappa,i^M(\kappa+1))}$ as follows. Assume now that h is a $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$-generic filter over $V[G_\kappa]$ containing the condition $p^{\leftarrow}_\infty$, and set $h' = \{p^{\rightarrow}_\infty\} \cup \{q \in \prod^M_{[F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda \mid p^{\rightarrow}_\infty \leq q\}$. We claim that $g^M_{[\kappa,i^M(\kappa+1))} = (h \times h') \cap M[G_\kappa]$ is $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$.

Let $D = (j(f)(a))^{G_\kappa}$ dense open be given, where $a \in [\gamma_{k'}]^{<\omega}$ for some $k' < \kappa^+$. We will show that $g^M_{[\kappa,i^M(\kappa+1))}$ meets D. Assume that the set D was dealt with at substage (ξ, a) of the inductive construction of p_∞ at stage $k \geq k'$, where $j(f)$ is considered. Under this notation, recall that the set $\overline{D} = \{q \leq (r_k)_{\gamma_k} \mid q \cup \overline{q}_{(\xi,a)} \in D\}$ is dense in $M[G_\kappa]$ below $(r_k)_{\gamma_k}$ in $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_\lambda$. If A is a maximal antichain contained in $\overline{D}$, then

$$A \text{ remains maximal in } \prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda \text{ in } V[G_\kappa] \tag{5.8}$$

To see that (5.8) is true, we argue as in Lemma 5.9 (5.6): Since A is a maximal antichain contained in a dense set, it is a maximal antichain in the whole forcing $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_{\bar\lambda}$. As $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_{\bar\lambda}$ is γ_k^+-cc in $M[G_\kappa]$ and a support of a condition p in $\prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda$ has size $< \gamma_k$, the closure property of $M[G_\kappa]$

$$\text{If a set } x \in M[G_\kappa] \text{ has size at most } \gamma_k \text{ in } M[G_\kappa], \\ \text{then } {}^{<\gamma_k}x \cap V[G_\kappa] \subseteq M[G_\kappa] \tag{5.9}$$

ensures that (5.8) is true. It follows that h restricted to $\prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda$ must hit A; let a be an element of h such that $a_{\gamma_k} \in \overline{D}$. It follows that $a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$ meets D. As both a and $p^{\leftarrow}_\infty$ are in h, there is some $a' \in h$ below both of them. But then $a' \cup p^{\rightarrow}_\infty \in h \times h'$ and $a' \cup p^{\rightarrow}_\infty \leq a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$, and so $a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$ is in $g^M_{[\kappa,i^M(\kappa+1))}$ and meets D.

We finish the proof by arguing that $g^+_{[\kappa,F(\kappa))}$ can be used to find in $V[G]$ some such generic h containing $p^{\leftarrow}_\infty$. By the homogeneity[8] of the forcing $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ there is $r \in g^+_{[\kappa,F(\kappa))}$ and an automorphism $\pi : \prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda \cong \prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ such that $\pi(r) = p^{\leftarrow}_\infty$; it now follows that $h = \pi[g^+_{[\kappa,F(\kappa))}]$ is $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$-generic containing $p^{\leftarrow}_\infty$ as desired. (Sublemma 5.12) □

Sublemma 5.13 *If the cofinality of $F(\kappa)$, which we denote δ, is greater than κ^+ in V, and $F(\kappa)$ is singular in V, then there is in $V[G_\kappa * g_{[\kappa,F(\kappa))}]$ a $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar\lambda}$.*

Proof. We will need to distinguish two cases.

Case (1): $F(\kappa)$ is regular in M.

Recall the sequences $\langle \gamma_i \,|\, i < \delta \rangle$, where $\kappa^+ < \gamma_0$, and $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$ which we used in the inductive construction in Sublemma 5.12. Unlike in Sublemma 5.12, we do not make the assumption that $\delta = \kappa^+$. Thus the two inductions cannot be merged together as in Sublemma 5.12 and a more complicated argument is called for. We will construct the desired generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar\lambda}$ in two steps.

A) The forcing $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_{\bar\lambda}$.

Intuitively, we need to define a generic for $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_{\bar\lambda}$ by building a decreasing list of conditions using induction along $\langle \gamma_i \,|\, i < \delta \rangle$ and simultaneously along $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$.[9] As both inductions can lead the construction

[8] In fact, we need homogeneity only for the Cohen forcing part of the forcing $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ above γ_0 as the master condition $p^{\leftarrow}_\infty$ is trivial below γ_0. This implies that we can disregard the Sacks forcing here.

[9] This time, $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be identified with $\mathbb{P}_\kappa$-names for dense open sets in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_{\bar\lambda}$.

outside the model $M[G_\kappa]$, we need to find a way to compatibly extend conditions "locally" without leaving the class $M[G_\kappa]$. We shall do this by dividing the supports of the conditions into segments corresponding to some elementary substructures existing in $M[G_\kappa]$.

Let m_α for $\alpha < \kappa^+$ denote the following elementary substructure of some large enough $H(\theta)^{M[G_\kappa]}$ which is closed under $<F(\kappa)$-sequences existing in $M[G_\kappa]$:

$$m_\alpha = \mathrm{SkolemHull}^{H(\theta)^{M[G_\kappa]}}\big(\{\prod_{[\kappa,F(\kappa)]}^{M} \mathcal{Q}^M_{\bar\lambda}\}\cup F(\kappa)+1\cup\{j(f_\xi)\,|\,\xi\leq\alpha\}\big). \quad (5.10)$$

Notice that each m_α has size $F(\kappa)$ in $M[G_\kappa]$ and contains as elements all dense open sets of the form $(j(f_\xi)(a))^{G_\kappa}$ for $a \in [F(\kappa)]^{<\omega}$ and $\xi \leq \alpha$.

We will build a matrix of conditions $\{p_{i,\alpha}\,|\,i<\delta,\alpha<\kappa^+\}$ in $\prod^M_{[\kappa,F(\kappa)]}\mathcal{Q}^M_{\bar\lambda}$ with δ-many rows each of length κ^+ such that the conditions will be decreasing both in the rows and the columns. Moreover, for every $i<\delta$ and every $\alpha<\kappa^+$, the sequence of conditions in the α-th column up to i, i.e. $\langle p_{k,\alpha}\,|\,k<i\rangle$, will exist in m_α. We will construct the matrix in δ-many steps, each of length κ^+ (i.e. we will be completing rows first).

The first "square" of the matrix $p_{0,0}$ will be filled in as follows. By definition of m_0, all dense open sets in $\prod^M_{[\kappa,F(\kappa)]}\mathcal{Q}^M_{\bar\lambda}$ of the form $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ are in m_0; by elementarity, they are dense open in m_0. Working inside m_0, carry out the reduction argument described in Sublemma 5.12. In particular, $p_{0,0}$ will γ_0-reduce all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a\in[\gamma_0]^{<\omega}$ (below the trivial condition 1 as we are filling in the first square). The square $p_{0,1}$ will be filled in in exactly the same way (considering f_0 and f_1), but working below the condition $p_{0,0}$ which is present in m_1. In particular $p_{0,1}$ will γ_0-reduce below $p_{0,0}$ all dense open sets of the form $(j(f_1)(a))^{G_\kappa}$ for $a\in[\gamma_0]^{<\omega}$. Proceed this way at every successor ordinal, obtaining $p_{0,\alpha+1}$. At a limit ordinal $\lambda<\kappa^+$, first take a lower bound q of $\langle p_{0,\alpha}\,|\,\alpha<\lambda\rangle$ which by the closure properties of m_λ exists in m_λ, and then work below this lower bound; the resulting $p_{0,\lambda}$ will γ_0-reduce below q all dense open sets of the form $(j(f_\xi)(a))^{G_\kappa}$ for $a\in[\gamma_0]^{<\omega}$ and $\xi\leq\lambda$. After κ^+ steps we have completed the 0-th row of the matrix. Note that the limit of $\langle p_{0,\alpha}\,|\,\alpha<\kappa^+\rangle$ may not exist in $M[G_\kappa]$.

We now need to complete row 1. In order to complete the first square in row 1, we need to find $p_{1,0}$ compatible with all conditions in the 0-th row of the matrix. Though the lower bound of these conditions may not exist in $M[G_\kappa]$, we will argue that an intersection of the union of the conditions (or more precisely of their supports) in the 0-th row with m_0 is in $M[G_\kappa]$, and even in m_0, i.e.

$$m_0\cap\bigcup_{\alpha<\kappa^+}\mathrm{supp}(p_{0,\alpha})\in m_0 \quad (5.11)$$

To see that (5.11) is true, we argue similarly as in Lemma 5.9. Each $p_{0,\alpha}$ is obviously in $M[G_\kappa]$, and consequently $\mathrm{supp}(p_{0,\alpha}) \cap m_0$ is in $M[G_\kappa]$ and in particular in m_0. The intersection (5.11) can thus be viewed as the union of a κ^+-sequence of elements in m_0. But as m_0 has size $F(\kappa)$ in $M[G_\kappa]$, such a sequence exists in $M[G_\kappa]$ due to the following closure property

$$^{\kappa^+}F(\kappa) \in M[G_\kappa], \tag{5.12}$$

which is implied by the $F(\kappa)$-hypermeasurability of κ and the fact that κ^+ is smaller than the cofinality of $F(\kappa)$.

It follows there is $p_{1,0}$ which γ_1-reduces all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_1]^{<\omega}$ below the condition $\bigcup_{\alpha<\kappa^+} p_{0,\alpha}$ restricted to m_0. In general for $\alpha < \kappa^+$, the condition $p_{1,\alpha}$ will reduce the relevant dense open sets below the common lower bound of $\bigcup_{\alpha<\kappa^+} p_{0,\alpha}$ restricted to m_α and the union of previous $p_{1,\beta}$ for $\beta < \alpha$.

It is immediate that the above construction can be repeated for any successor ordinal $i+1$ below δ, i.e. if the matrix has been completed up to the stage i, we can fill in the $i+1$-th row by the above argument.

Assume now that $i < \delta$ is a limit ordinal. First consider the sequence $\langle p_{k,\alpha} \,|\, k < i\rangle$ for a single $\alpha < \kappa^+$. As the sequence is of length less then cofinality $F(\kappa)$ in $M[G_\kappa]$ and contains elements from m_α, which has size $F(\kappa)$ in $M[G_\kappa]$, we can infer from

$$^{i}F(\kappa) \in M[G_\kappa] \tag{5.13}$$

that the sequence exists in $M[G_\kappa]$, and in particular in m_α. Let $q_{i,\alpha} \in m_\alpha$ denote the lower bound of the sequence $\langle p_{k,\alpha} \,|\, k < i\rangle$ for each $\alpha < \kappa^+$. Now repeat the above argument for the successor step considering the restrictions of $\bigcup_{\alpha<\kappa^+} q_{i,\alpha}$ to m_β's for $\beta < \kappa^+$.

We finish the construction by taking the limit of the whole matrix $\{p_{i,\alpha} \,|\, i < \delta, \alpha < \kappa^+\}$, obtaining some set p_∞ existing in $V[G_\kappa]$ (for instance first taking limits of the rows and then the single limit of this sequence). Let $p_\infty^{\leftarrow}$ denote the restriction of p_∞ to the interval $[\kappa, F(\kappa))$ (note that $p_\infty^{\leftarrow}$ is a condition in $\prod_{[\kappa,F(\kappa))} \mathcal{Q}_\lambda^+$) and $p_\infty^{\rightarrow}$ the restriction of p_∞ to $\{F(\kappa)\}$ (note that $p_\infty^{\rightarrow}$ is a union of conditions in $(\mathrm{Add}(F(\kappa), j(F)(F(\kappa))))^{M[G_\kappa]}$ which exists in $V[G_\kappa]$). Arguing as at the end of Sublemma 5.12, we find a $\prod_{[\kappa,F(\kappa))} \mathcal{Q}_\lambda^+$-generic h, where $p_\infty^{\leftarrow}$ is in h, and define h' to be generated by $p_\infty^{\rightarrow}$, such that $h \times h' \cap M[G_\kappa]$ is $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda^M$-generic over $M[G_\kappa]$. Let us denote this generic as $g^M_{[\kappa,F(\kappa)]}$.

B) The forcing $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}_\lambda^M$ in $M[G_\kappa]$.

The regularity of the cardinal $F(\kappa)$ in M implies that $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda^M$ and $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}_\lambda^M$ are mutually generic. Working in $V[G_\kappa]$, we construct the generic $g^M_{(F(\kappa),i^M(\kappa+1))}$ for $\prod^M_{(F(\kappa),i^M(\kappa+1))}\mathcal{Q}^M_{\bar\lambda}$ exactly as in case B) of Lemma 5.9.

It follows that $g^M_{[\kappa,i^M(\kappa+1))} = g^M_{[\kappa,F(\kappa)]} \times g^M_{(F(\kappa),i^M(\kappa+1))}$ is the desired $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_{\bar\lambda}$.

Case (2): $F(\kappa)$ **is singular in** M**.**

Recall once again the sequence $\langle \gamma_i \,|\, i < \delta\rangle$, where $\kappa^+ < \gamma_0$, and the sequence $\langle j(f_\alpha) \,|\, \alpha < \kappa^+\rangle$ which we used in the inductive construction in Sublemma 5.12 and Case (1) of the present Sublemma.

The singularity of $F(\kappa)$ in $M[G_\kappa]$ introduces an important simplification into the construction: $\langle \gamma_i \,|\, i < \delta\rangle$ can be picked in $M[G_\kappa]$ this time. Just run the argument in Sublemma 5.12 with the following modification: Start with $j(f_0)$ and run the argument using just this one function $j(f_0)$, obtaining some master condition $p^{f_0}_\infty$. Since the sequence $\langle \gamma_i \,|\, i < \delta\rangle$ is in $M[G_\kappa]$, so is $p^{f_0}_\infty$. Now deal with $j(f_1)$ and so on by induction on $\alpha < \kappa^+$. At each $\alpha < \kappa^+$ we can take the lower bound of the conditions $p^{f_\beta}_\infty$ for $\beta < \alpha$ as we have closure under κ-sequences. Denote the constructed generic as $g^M_{[\kappa,i^M(\kappa+1))}$. (Sublemma 5.13) □

This also ends the proof of the whole lemma. (Lemma 5.10) □

It follows we have completed one more step in finding suitable generics for $\mathbb{P}^M$.

$$V[G_\kappa] \xrightarrow{\;j\;} M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$$

In order to construct another generic, we need to verify that we have preserved closure under κ-sequences of $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$ in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$.

Lemma 5.14 *$M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$ is closed under κ-sequences in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$.*

Proof. As mentioned above, $M[G_\kappa]$ remains closed under κ-sequences in $V[G_\kappa]$ as the forcing $\mathbb{P}_\kappa$ is κ-cc. Let us denote as g_S the projection of $g_{[\kappa,i(\kappa+1))}$ to the Sacks forcing. By Lemma 5.6, the forcing $\mathrm{Add}(\kappa^+, F(\kappa^+))$ is κ^+-distributive after the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$, and consequently it is enough to show closure just in $V[G_\kappa * g_S]$. Recall the "manipulation" argument just before Lemma 5.9 which removes the discrepancy between the values $F(\kappa)$ and $j(F)(\kappa)$; the modification of $\mathrm{Sacks}(\kappa, F(\kappa))$ changes this forcing to $S^* = \mathrm{Sacks}(\kappa, j(F)(\kappa))$. Due to closure of $M[G_\kappa]$ under κ-sequences, S^* is the same in $V[G_\kappa]$ and in $M[G_\kappa]$ and is the first step of the product $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_{\bar\lambda}$ in the iteration $\mathbb{P}^M$ at stage κ. We are going to work in $V[G_\kappa * g^*_S] = V[G_\kappa * g_S]$, where g^*_S is the generic for S^*, and as such is present in $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$.

Let X be a κ-sequence of ordinal numbers in $V[G_\kappa * g^*_S]$, and let this be forced by some $p_0 \in g^*_S$. By the fusion argument (carried out in $V[G_\kappa]$), there is for every $r \leq p_0$ some $p_X \leq r$ such that if p_X is in g^*_S, then X can be uniquely determined from p_X and g^*_S restricted to the support of p_X. Since such p_X are dense below p_0, some such p_X is in g^*_S, and as p_X and g^*_S are present in $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$, so is X. (Lemma 5.14) □

The preservation of closure allows us to prove:

Lemma 5.15 *We can construct in* $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$ *an* $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$*-generic for the stage* $\mathbb{P}^M_{[i^M_{\kappa+1},j(\kappa))}$.

Proof. As in Lemma 5.9, case B), work in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$ and construct by recursion a generic filter H hitting all dense sets. (Lemma 5.15) □

It follows we can lift partially to $V[G_\kappa]$:

$$V[G_\kappa] \xrightarrow{\quad j \quad} M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))} * H]$$

The next step is to lift to the $\prod_{[\kappa,i(\kappa+1))} \mathcal{Q}_\lambda$-generic $g_{[\kappa,i(\kappa+1))}$ over $V[G_\kappa]$.

Again due to Lemma 5.6, coupled with Lemma 2.19, the only non-trivial part of this step is to lift to the generic filter g_S for Sacks$(\kappa, F(\kappa))$. This follows directly from the technique in [9], which is reviewed (and sufficiently generalized) in Section 4.1 (note that the condition $j(f_\lambda)(\kappa) = \lambda$ in the proof of Theorem 4.8 can be easily replaced by our assumption that $j(F)(\kappa) \geq F(\kappa)$).

Let us denote the generic generated by $j[g_S]$ as h_0. It follows we can lift to $V[G_\kappa * g_S]$:

$$V[G_\kappa * g_S] \xrightarrow{\quad j \quad} M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))} * H * h_0]$$

We finish the lifting by an application of Lemma 2.19 in two stages, first to the rest of the product $\prod_{[\kappa^+,i(\kappa+1))} \mathcal{Q}_\lambda$ and then to the rest of the iteration $\mathbb{P}^F$:

$$V[G] \xrightarrow{\quad j \quad} M[j(G)]$$

This proves Theorem 5.7 and shows that κ remains measurable in the generic extension by $\mathbb{P}^F$.

5.2 Preservation of strong cardinals

In [24], Menas showed using a "master condition" argument that *locally definable* (see Definition 5.16 below) Easton functions F can be realised while preserving supercompact cardinals. We will show how to adapt his result to strong cardinals using the above arguments.

Definition 5.16 *An Easton function F, see Definition 2.1, is said to be* locally definable *if the following condition holds:*

There is a sentence ψ and a formula $\varphi(x,y)$ with two free variables such that ψ is true in V and for all cardinals γ, if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma (F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \tag{5.14}$$

Theorem 5.17 *(GCH) Assume F is locally definable in the sense of Definition 5.16. If $\mathbb{P}^F$ is the forcing notion as in Definition 5.4, then $V^{\mathbb{P}^F}$ realises F and preserves all strong cardinals.*

Proof. First note that since ψ is true in V, there exists a closed unbounded class of cardinals C_ψ such that if $\beta \in C_\psi$, then $H(\beta) \models \psi$. It also holds that the closed unbounded class C_ψ is included in the closed unbounded class C_F of closure points of F.

Assume κ is a strong cardinal. We first show that κ is closed under F. Choose some β greater than κ such that $H(\beta)$ satisfies ψ and let $j : V \to M$ be an embedding witnessing β-hypermeasurability of κ; in particular $H(\beta)^V \subseteq M$ and $\beta < j(\kappa)$. Notice that for every $\alpha < \kappa$, the following equivalence is true by elementarity of j:

$$\exists \xi \in (\alpha, \kappa), \xi \text{ closed under } F \text{ iff } \exists \xi \in (\alpha, j(\kappa)), \xi \text{ closed under } j(F) \tag{5.15}$$

Since β in the interval $(\alpha, j(\kappa))$ was chosen to satisfy ψ and thus it is closed under F (and $j(F)$), we conclude that the closure points of F are unbounded in κ, and consequently κ is closed under F.

Let G be a generic filter for $\mathbb{P}^F$. Assume that $\beta > \kappa$ is a singular cardinal such that $H(\beta)$ satisfies ψ (it follows that β is a closure point of F). We claim that every extender embedding $j : V \to M$ witnessing the β^{++}-hypermeasurability of κ can be lifted to a $j^* : V[G] \to M[j(G)]$ with $H(\beta^+)$ of $V[G]$ included in $M[j(G)]$, thereby witnessing that κ is still β^+-hypermeasurable in $V[G]$. As β can be arbitrarily large, this implies that κ is still strong in $V[G]$.

Let $\beta > \kappa$ singular such that $H(\beta) \models \psi$ be given. Let $j : V \to M$ be a β^{++}-hypermeasurable witnessing embedding; that is $\beta^{++} < j(\kappa) < \beta^{+++}$ and $M = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [\beta^{++}]^{<\omega}\}$. Since κ is closed under F, $j(\kappa)$ is closed under $j(F)$. Moreover, since $j(F)$ is locally definable in M via the formulas ψ and $\varphi(x,y)$ and $H(\beta)^M = H(\beta)^V$, it follows that $H(\beta)^M \models \psi$ and consequently F and $j(F)$ are identical on the interval $[\omega, \beta)$; in particular β is closed under $j(F)$. The fact that $H(\beta^{++})$ is correctly captured by M implies that $\mathbb{P}^F$ and $j(\mathbb{P}^F)$ coincide up to stage β, i.e. $\mathbb{P}^F_\beta = j(\mathbb{P}^F)_\beta$, and thus we may "copy" the generic G_β, i.e. G restricted to β, and use it as a generic for $j(\mathbb{P}^F)_\beta$. Moreover, as $\mathbb{P}^F_\beta$ is β^{++}-cc, all nice $\mathbb{P}^F_\beta$-names for subsets of β^+ in V are included in M, and consequently all subsets of β^+ existing in $V[G_\beta]$ are also present in $M[G_\beta]$. It follows that $H(\beta^{++})$ of $V[G_\beta]$ equals $H(\beta^{++})$ of $M[G_\beta]$.

Applying the notation of the previous section, we denote $\beta = i(\bar\beta) = i^M(\bar\beta)$, where i and i^M enumerate the closure points of F and $j(F)$, respectively, and $\bar\beta \le \beta$ is some ordinal. The singularity of β in M implies that the next step of the iteration, the product $\mathbb{Q}^M_\beta$ in $M[G_\beta]$, is trivial at β, and so

$$\mathbb{Q}^M_\beta = \prod\nolimits^M_{[\beta^+, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda \tag{5.16}$$

is the Easton-supported product of Cohen forcings in the interval $[\beta^+, i^M(\bar\beta+1))$, where $i^M(\bar\beta+1) < j(\kappa)$ is the next closure point of $j(F)$ after β.

Due to Lemma 2.6, we know that the forcing notion $\prod^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda$, i.e. $(\mathrm{Add}(\beta^+, j(F)(\beta^+)))^{M[G_\beta]} \times (\mathrm{Add}(\beta^{++}, j(F)(\beta^{++})))^{M[G_\beta]}$ and the forcing notion $\prod^M_{[\beta^{+3M}, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda$ are mutually generic, and hence we can deal with them separately.

As $\beta^{++} \le F(\beta^+) \le F(\beta^{++})$ and the size of $j(F)(\beta^+)$ and $j(F)(\beta^{++})$ is β^{++} in V (due to closure of $j(\kappa)$ under $j(F)$), we can "manipulate" the forcing $\mathrm{Add}(\beta^+, F(\beta^+)) \times \mathrm{Add}(\beta^{++}, F(\beta^{++}))$ of $V[G_\beta]$ just like in (5.3) before Lemma 5.9. We obtain a forcing notion $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$ and a $V[G_\beta]$-generic $g^+_{[\beta^+,\beta^{++}]}$ for $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$. Since $H(\beta^{++})$ of $V[G_\beta]$ is correctly captured in $M[G_\beta]$, we can argue as in Lemma 5.9, case A) that maximal antichains in $\prod^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda$ existing in $M[G_\beta]$ remain maximal in $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$. It follows that

$$g^+_{[\beta^+,\beta^{++}]} \cap M[G_\beta] \text{ is } M[G_\beta]\text{-generic for } \prod\nolimits^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda. \tag{5.17}$$

Arguing as in Lemma 5.14, $M[G_\beta]$ is easily seen to be still closed under κ-sequences in $V[G_\beta]$. Consequently, we may construct a $M[G_\beta]$-generic for $\prod^M_{[\beta^{+3M}, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda$ just like in Lemma 5.9, Case B). Similarly, we construct a generic for the iteration $j(\mathbb{P}^F)$ up to the closure point $j(\kappa)$ (see Lemma 5.15). We finish the proof by first lifting to the Sacks forcing at κ, using [9] and the generalization in Section 4.1 of this paper, and then to the rest of the forcing above κ (see the end of the proof for Theorem 5.7, just before this Section 5.2), finally obtaining

$$j^* : V[G] \to M[j^*(G)]. \tag{5.18}$$

Notice that $M[j^*(G)]$ captures all subsets of β in $V[G]$, and hence κ is still β^+-hypermeasurable in $V[G]$. (Theorem 5.17) □

Remark 5.18 As mentioned in Section 4.2, the proof of the Theorem 5.7 and 5.17 uses heavily the assumption that $H(F(\kappa))^V$ is included in M to argue that κ remains measurable (or strong). It is an open question for us if the theorem can be proved just from the assumption $o(\kappa) = F(\kappa)$.

6 Easton functions and global failure of SCH

In Theorem 5.7 we have shown that the final model $V^{\mathbb{P}^F}$ realises a given Easton function F and preserves measurability of some (the so called "F-good") large cardinals. It is natural to ask if it is possible to globally change the cofinality of all measurable cardinals in $V^{\mathbb{P}^F}$ while preserving all cardinals. Thus, if κ is measurable in $V^{\mathbb{P}^F}$ and GCH fails at κ, SCH will fail at κ if it remains a strong limit cardinal with cofinality ω. We show that this is indeed possible, by iterating a forcing developed by K.Prikry in [25], see Definition 6.1 here, along (some) measurable cardinals. In fact, in Section 6.1 we shall show two ways of doing it: (i) an application of the iteration with full support developed by M. Magidor ([23]) and (ii) an application of the Easton-supported iteration developed by M. Gitik (see for instance [11]).

However, the use of the forcing as in Definition 6.1 implies that the cardinal κ where we want to fail SCH needs to be first a measurable cardinal failing GCH. We have observed above in Observation 3.1 that this implies failure of GCH on a measure one set below this cardinal κ. This limits unnecessarily the eligible Easton functions F if we aim at obtaining cardinals failing SCH and not care to have them measurable first. There exists a more complicated Prikry-style forcing developed by M. Magidor and M. Gitik in [10] which achieves this task: it cofinalizes a sufficiently large κ to a cofinality ω *and simultaneously* blows up its powerset. We study the iteration of this type of forcing in Section 6.2 obtaining some original results in this area.

6.1 Iteration of the simple Prikry forcing

We first review the definition of the forcing which we will call "simple Prikry forcing", and denote as $\mathrm{Prk}(\kappa)$.

Definition 6.1 *A condition in* $\mathrm{Prk}(\kappa)$ *is of the form* (s, A) *where* s *is a finite sequence in* κ *and* A *is a subset of* κ *which lies in some fixed normal* κ*-complete ultrafilter* U *on* κ*. We assume that* $\max(s) < \min(A)$*. We say that* (s, A) *is stronger than* (t, B)*,* $(s, A) \leq (t, B)$*, if* s *end-extends* t*,* $A \subseteq B$ *and* $s \setminus t \subseteq B$*. We say that* (s, A) *directly extends* (t, B)*,* $(s, A) \leq^* (t, B)$*, if* (s, A) *extends* (t, B) *and moreover* $s = t$*.*

In the terminology of [11], $\mathrm{Prk}(\kappa)$ is the canonical example of a *Prikry type forcing notion*, that is P is a Prikry type forcing notion if there are orderings $\leq^* \subseteq \leq$ of P, where $\leq^*$ is called a direct extension and $\leq$ an extension and for every $p \in P$ and a sentence σ there is $q \leq^* p$ deciding σ. The ordering $\leq^*$ is typically more closed than $\leq$.

All antichains in $\mathrm{Prk}(\kappa)$ have size at most $\kappa^{<\omega} = \kappa$, and hence $\mathrm{Prk}(\kappa)$ is κ^+cc (this does not require GCH). The direct extension relation $\leq^*$ is

κ-closed which implies that $\mathrm{Prk}(\kappa)$ does not add new subsets of κ. If follows that $\mathrm{Prk}(\kappa)$ preserves all cardinals.

We will now describe how to iterate the forcing $\mathrm{Prk}(\kappa)$. Essentially, there are two options: the full support, and the Easton support. We first review the full support iteration:

Definition 6.2 (Full support iteration) *An iteration with full support* $\mathbb{R}^{\text{full}} = \mathbb{R} = \langle (\mathbb{R}_\alpha, \dot{R}_\alpha) \,|\, \alpha \in \mathrm{On} \rangle$ *is defined by recursion along* $\alpha < \mathrm{On}$. *We will suppress the superscript notation "full" in* $\mathbb{R}^{\text{full}}$ *if there is no risk of confusion.*

For every $\alpha < \mathrm{On}$ *let* $\mathbb{R}_\alpha$ *be a set of all elements* p *of the form* $\langle \dot{p}_\gamma \,|\, \gamma < \alpha \rangle$, *where for every* $\gamma < \alpha$,

$$p \restriction \gamma = \langle \dot{p}_\beta \,|\, \beta < \gamma \rangle \in \mathbb{R}_\gamma \tag{6.1}$$

and $p \restriction \gamma \Vdash$ *"*$\dot{p}_\gamma$ *is a condition in* $\dot{R}_\gamma$*," where* $\dot{R}_\gamma$ *is either* $\mathrm{Prk}(\gamma)$ *or a trivial forcing.*

Let $p = \langle \dot{p}_\gamma \,|\, \gamma < \alpha \rangle$ *and* $q = \langle \dot{q}_\gamma \,|\, \gamma < \alpha \rangle$ *be elements of* $\mathbb{R}_\alpha$. *Then* p *is stronger than* q, $p \leq q$, *iff*

1. *for every* $\gamma < \alpha$,
$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq \dot{q}_\gamma \text{ in } \dot{R}_\gamma; \tag{6.2}$$

2. *there exists a finite subset* $b \subseteq \alpha$ *so that for every* $\gamma \in \alpha \setminus b$,
$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq^* \dot{q}_\gamma \text{ in } \dot{R}_\gamma. \tag{6.3}$$

If the set in item (2) is empty, the we call p *a direct extension of* q *and denote it as* $p \leq^* q$.

Note that even if κ is a Mahlo cardinal, the forcing $\mathbb{R}^{\text{full}}_\kappa$ fails to be κ-cc. However, in certain applications (see [12]), it is useful to have κ-cc at the stage κ of an iteration. We may achieve this by requiring that the conditions have the Easton support; Definition 6.3 is due to M. Gitik.

Definition 6.3 (Easton support iteration) *An iteration* $\mathbb{R}^{\text{Easton}} = \mathbb{R} = \langle (\mathbb{R}_\alpha, \dot{R}_\alpha) \,|\, \alpha \in \mathrm{On} \rangle$ *is defined by recursion along* $\alpha < \mathrm{On}$. *We will suppress the superscript notation "Easton" in* $\mathbb{R}^{\text{Easton}}$ *if there is no risk of confusion.*

For every $\alpha < \mathrm{On}$ *let* $\mathbb{R}_\alpha$ *be a set of all elements* p *of the form* $\langle \dot{p}_\gamma \,|\, \gamma \in g \rangle$, *where*

1. $g \subseteq \alpha$;

2. g *has the Easton support, i.e. for every inaccessible* $\beta \leq \alpha$, $\beta > |g \cap \beta|$, *provided that for every* $\gamma < \beta$, $|\mathbb{R}_\gamma| < \beta$;

3. *for every* $\gamma \in g$,
$$p \restriction \gamma = \langle \dot{p}_\beta \,|\, \beta \in g \cap \gamma \rangle \in \mathbb{R}_\gamma \tag{6.4}$$
and $p \restriction \gamma \Vdash$ "$\dot{p}_\gamma$ *is a condition in* $\dot{R}_\gamma$," *where* $\dot{R}_\gamma$ *is either* $\mathrm{Prk}(\gamma)$ *or a trivial forcing.*

Let $p = \langle \dot{p}_\gamma \,|\, \gamma \in g \rangle$ *and* $q = \langle \dot{q}_\gamma \,|\, \gamma \in f \rangle$ *be elements of* $\mathbb{R}$. *Then* p *is stronger than* q, $p \leq q$, *iff*

1. $g \supseteq f$;

2. *for every* $\gamma \in f$,
$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq \dot{q}_\gamma \text{ in } \dot{R}_\gamma; \tag{6.5}$$

3. *there exists a finite subset* $b \subseteq f$ *so that for every* $\gamma \in f \setminus b$,
$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq^* \dot{q}_\gamma \text{ in } \dot{R}_\gamma. \tag{6.6}$$

If the set in item (3) is empty, the we call p *a direct extension of* q *and denote it as* $p \leq^* q$.

By results in [11], both iterations $\mathbb{R}^{\mathrm{full}}$ and $\mathbb{R}^{\mathrm{Easton}}$ are themselves Prikry-type, i.e. if p is a condition in either of the forcings and σ is a sentence then there is a direct extension $q \leq^* p$ deciding σ.

Lemma 6.4 *Both iterations preserve (under some mild cardinal arithmetic assumptions in case of* $\mathbb{R}^{\mathrm{Easton}}$*) all cardinals, and also all axioms of ZFC:*

1. *At each cardinal* κ, $\mathbb{R}^{\mathrm{full}} = \mathbb{R}$ *factors into* $\mathbb{R}_{\kappa+1} * \mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ *such that* $\mathbb{R}_{\kappa+1}$ *is* κ^+*-cc and* $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ *does not add new subsets of* κ^+. *In particular,* $\mathbb{R}$ *preserves all axioms of ZFC and all cardinals.*

2. *Assuming SCH, at each cardinal* κ, $\mathbb{R}^{\mathrm{Easton}} = \mathbb{R}$ *factors into* $\mathbb{R}_{\kappa+1} * \mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ *such that* $\mathbb{R}_{\kappa+1}$ *preserves cardinals* $\lambda \geq \kappa^+$ *and* $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ *does not add new subsets of* κ^+. *In particular* $\mathbb{R}$ *preserves all axioms of ZFC and all cardinals.*

Proof. **Ad (1)**. Let us denote $\mathbb{R} = \mathbb{R}^{\mathrm{full}}$, and let κ be a cardinal. The interesting case is when κ is a limit of non-trivial stages of the iteration $\mathbb{R}$, i.e. if there is a $\lambda \leq \kappa$ and an increasing sequence of cardinals $\langle \kappa_\alpha \,|\, \alpha < \lambda \rangle$ such that $\kappa = \sup(\langle \kappa_\alpha \,|\, \alpha < \lambda \rangle)$ and each $\dot{R}_\alpha$ is a name for the simple Prikry forcing. Since we are dealing with a full support iteration, we do not need to distinguish the cases when κ is regular, or singular. $\mathbb{R}_\kappa$ is κ^+-cc by the following argument: if $p \in \mathbb{R}_\kappa$ then there exists a finite subset $b \subseteq \kappa$ where the the first coordinate of the condition in the Prikry forcing is non-trivial (at coordinates outside b there are only direct extensions of the empty condition, and these are compatible), i.e. there is a finite sequence of names $\langle \dot{s}_\alpha \,|\, \alpha \in b\rangle$ with $\dot{s}_\alpha$ being a name for a non-empty finite sequence in κ_α. As there are only $\kappa^{<\omega} = \kappa$ many such sequences, it follows that there are at most κ many incompatible conditions, and hence $\mathbb{R}_\kappa$ is κ^+-cc. Since $\dot{R}_\kappa$ is either trivial or the simple Prikry forcing, we also have that $\mathbb{R}_{\kappa+1}$ is κ^+-cc. The fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ does not add new subset of κ^+ follows from the fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ satisfies the Prikry condition and the direct extension relation in $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ is κ^{++}-closed.

This is enough to argue that $\mathbb{R}$ preserves all cardinals: assume that some κ^+ is collapsed to κ and factor $\mathbb{R}$ into $\mathbb{R}_{\kappa+1}$ and $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$. Since $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ cannot collapse κ^+, it must be $\mathbb{R}_{\kappa+1}$, but this is impossible as $\mathbb{R}_{\kappa+1}$ is κ^+-cc. Preservation of axioms of ZFC follows similarly as in Remark 2.5 using the fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ does not add new subsets of κ^+ for every κ.

Ad (2). Let $\mathbb{R} = \mathbb{R}^{\mathrm{Easton}}$, and let κ be the interesting case as above in (1). Unlike in (1) we cannot argue that every p in $\mathbb{R}_\kappa$ is determined as regards compatibility by a finite sequence of names $\langle \dot{s}_\alpha \,|\, \alpha \in b\rangle$. We need to distinguish the cases when κ is regular and singular. Notice that in both cases, κ needs to be strong limit since it is the limit of a sequence $\langle \kappa_\alpha \,|\, \alpha < \lambda\rangle$, $\lambda \leq \kappa$, of inaccessible cardinals.

Case 1: κ is regular. In this case κ is strong limit and regular, and hence inaccessible. It follows that $\kappa^{<\kappa} = \kappa$ and by Easton support of $\mathbb{R}_\kappa$, this is enough to conclude that $\mathbb{R}_\kappa$ is κ^+-cc. In fact, if κ is Mahlo, a standard argument shows that $\mathbb{R}_\kappa$ is κ-cc. Since $\dot{R}_\kappa$ is either trivial or the simple Prikry forcing, also $\mathbb{R}_{\kappa+1}$ is κ^+-cc.

Case 2: κ is singular. In this case κ is a strong limit singular cardinal. Since $\mathbb{R}_\kappa$ has size 2^κ, it is obviously $(2^\kappa)^+$-cc. By SCH (this is the only place where we need an additional assumption), $2^\kappa = \kappa^+$ and so $\mathbb{R}_\kappa$ is κ^{++}-cc. It follows that $\mathbb{R}_\kappa$ preserves all cardinals $\lambda \geq \kappa^{++}$. We need a special argument to show that κ^+ is preserved as well. This is a standard argument based on the Prikry properties of $\mathbb{R}_\kappa$ using the fact that the cofinality of κ is some $\delta < \kappa$ and hence if $\mathbb{R}_\kappa$ collapsed κ^+, it would need to add a cofinal δ-sequence to κ^+.[10]

Preservation of axioms of ZFC and of cardinals follows exactly as in (1).
(Lemma 6.4) $\square$

Remark 6.5 Notice that $\mathbb{R}^{\mathrm{full}}$ has the following nice property: every two direct extensions p, q in $\mathbb{R}^{\mathrm{full}}$ of an empty condition $1_{\mathbb{R}^{\mathrm{full}}} = 1$ are compatible. This is very useful in showing that the initial segment $\mathbb{R}^{\mathrm{full}}_\kappa$ of the iteration

[10]In fact, it is known from the results in inner model theory that it is very hard to collapse successors of singular cardinals. Thus if we for instance assume that there is no inner model with a Woodin cardinal in our universe, κ^+ cannot be collapsed by a general inner model argument.

preserves measurability κ (see the first proof of Theorem 6.6). This contrasts with $\mathbb{R}^{\text{Easton}}$ which fails to have this property.

Now we can show:

Theorem 6.6 *Let F be an Easton function as in Definition 2.1 and $\mathbb{P}^F$ a forcing iteration as in Definition 5.4. Let Δ denote the class of F-good cardinals as in Definition 5.3. Assume that GCH holds in V. Then: There is a forcing iteration $\mathbb{R}$ of the simple Prikry forcing such that in the generic extension by $\mathbb{P}^F * \mathbb{R}$ all cardinals are preserved, the function F is realized and if κ is in Δ, then its cofinality is changed to ω.*

We will first review the properties of the generic extension $V[G]$ by $\mathbb{P}^F$:

1. $V[G]$ is a cofinality preserving extension of V realizing F.
2. $V[G]$ satisfies SCH.
3. All F-good cardinals of V, i.e. all $\kappa \in \Delta$, remain measurable in $V[G]$.
4. The measurability of $\kappa \in \Delta$ is witnessed in $V[G]$ by some extender embedding $j^* : V[G] \to M[j^*(G)]$, where j^* lifts some extender embedding $j : V \to M$ witnessing F-goodness of κ in V.

We will give two proofs of the theorem. The author first constructed a proof given as Proof 2 using an iteration with the Easton support. Then M.Magidor in personal communication suggested to the author that in the case of the simple Prikry forcing it is much more elegant to use an iteration with full support, as in [23] – this is the Proof 1.

Proof 1: Full support iteration

The first proof is be based on the full support iteration of the simple Prikry forcing $\text{Prk}(\kappa)$. Work in $V[G]$ and let $\mathbb{R}$ be defined as in Definition 6.2, with $\dot{R}_\alpha$ being a name for the forcing $\text{Prk}(\alpha)$ whenever $\mathbb{R}_\alpha$ forces that α is measurable. By Lemma 6.4, $\mathbb{R}$ preserves cardinals, and obviously does not change the continuum function in $V[G]$ – hence F is still realized in a generic extension by $\mathbb{R}$. It remains to verify that all elements of Δ will be cofinalized to a cofinality ω. In fact, we show that all measurable cardinals in $V[G]$ will be cofinalized.

Let us denote by $\mathscr{M}$ the class of measurable cardinals in $V[G]$. Note that in general $\Delta \subseteq \mathscr{M}$, but $\Delta = \mathscr{M}$ may not be true. Clearly, it is enough to show

$$\text{For every } \alpha \in \mathscr{M}, \mathbb{R}_\alpha \Vdash \alpha \text{ is measurable.} \tag{6.7}$$

The proof uses the following property of the full support iteration $\mathbb{R}$ of the simple Prikry forcing (where 1 denotes the greatest, or equivalently empty condition in the relevant forcing):

$$\text{For all } p, q \leq^* 1, p, q \text{ are compatible.} \tag{6.8}$$

Let κ in $\mathscr{M}$ be fixed. We shall show that $\mathbb{R}_\kappa$ forces that κ is measurable. Let H_κ be a generic for $\mathbb{R}_\kappa$. Define a measure U on κ in $V[G][H_\kappa]$ as follows:

$$X \in U \text{ iff } \exists p \in H_\kappa, \exists p' \leq^* 1 \in j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa, p^\frown p' \Vdash \kappa \in j^*(\dot{X}), \tag{6.9}$$

where $\dot{X}$ is a $\mathbb{R}_\kappa$-name for a subset of κ and $1 = 1_{j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa}$. We claim that U is a κ-complete ultrafilter in $V[G][H_\kappa]$. In the paragraphs below a primed condition (e.g. p') will refer to elements of $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$, while a non-primed condition (e.g. p) will refer to elements of $H_\kappa \subseteq \mathbb{R}_\kappa$ (unless stated otherwise).

We first state a simple fact:

Fact 6.7 *If σ is a sentence then there are r, r', such that $r \in H_\kappa$, $r' \leq^* 1_{j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa}$ and $r^\frown r'$ decides σ.*

Proof. By the Prikry property of $j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa$, the greatest condition $1_{\mathbb{R}_\kappa}$ in $\mathbb{R}_\kappa$ forces for some $r' \leq^* 1_{j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa}$ in $j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa$ the sentence "$r' \Vdash \sigma \vee r' \Vdash \neg\sigma$". Since H_κ is a generic filter, there is some $r \in H_\kappa$ forcing either $r' \Vdash \sigma$ or $r' \Vdash \neg\sigma$, or equivalently $r^\frown r'$ decides σ. (Fact 6.7) $\square$

We finish the first proof of Theorem 6.6 by the following lemma:

Lemma 6.8 *U defined in (6.9) is a κ-complete ultrafilter in $V[G][H_\kappa]$.*

Proof. U is correctly defined. Note that if $\dot{X}_0$ and $\dot{X}_1$ are two names and they interpret as the same subset of κ in $V[G][H_\kappa]$, i.e. $(\dot{X}_0)^{H_\kappa} = (\dot{X}_1)^{H_\kappa}$, then they are decided in the same way by conditions according to (6.9): Assume for contradiction that there are $r_0^\frown r_0'$ and $r_1^\frown r_1'$ such that $r_0^\frown r_0' \Vdash \kappa \in j^*(\dot{X}_0)$ and $r_1^\frown r_1' \Vdash \kappa \notin j^*(\dot{X}_1)$. Let $p \in H_\kappa$ force that $\dot{X}_0 = \dot{X}_1$; $j(p)$ is of the form $p^\frown p'$ where p' is a direct extension of 1 (because there is only finite number of coordinates with non-direct extensions in p and hence these coordinates are bounded in κ). It follows that all these three conditions $r_0^\frown r_0', r_1^\frown r_1', p^\frown p'$ are compatible which is a contradiction.

U is a filter. The empty condition in $j^*(\mathbb{R}_\kappa)$ forces that $\kappa \in j^*(\kappa)$, and so $\kappa \in U$. Let X, Y be in U and let $\dot{X}, \dot{Y}$ be their respective names. If $p^\frown p'$ forces that κ is in $j^*(\dot{X})$ and $r^\frown r'$ forces the same for $j^*(\dot{Y})$ then clearly the common lower bound forces that κ is in the intersection. Also trivially, if $X \subseteq Y$ are subsets of κ, then by the maximality principle we can choose names $\dot{X}, \dot{Y}$ such that the empty condition forces that $\dot{X} \subseteq \dot{Y}$. It follows that $X \in U$ implies $Y \in U$: if $p^\frown p' \Vdash \kappa \in j^*(\dot{X})$, then also $p^\frown p' \Vdash \kappa \in j^*(\dot{Y})$.

U is an ultrafilter. Let X be a subset of κ. Choose names $\dot{X}$ and $\dot{X}^c$ such that the empty condition forces that $\dot{X}^c$ is the complement of $\dot{X}$. Then $1_{j^*(\mathbb{R}_\kappa)} \Vdash \kappa \in j^*(\dot{X} \cup \dot{X}^c)$. As H_κ is generic and $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ satisfies the Prikry property, the are $p^\frown p'$ and $r^\frown r'$ as in Fact 6.7 deciding whether or not κ is in $j^*(\dot{X})$ or $j^*(\dot{X}^c)$, respectively. By the compatibility of $p^\frown p'$ and $r^\frown r'$, it must be that exactly one of these conditions decides its relevant sentence positively, otherwise we could consider a common lower bound and derive a contradiction.

U is a κ-complete ultrafilter. Let $\langle X_\alpha \mid \alpha < \delta\rangle$ be sets in U for some $\delta < \kappa$. By definition (6.9), there are $p_\alpha^\frown p'_\alpha$, $\alpha < \delta$, forcing that κ is in $j^*(\dot{X}_\alpha)$. Let $r^\frown r'$, $r \in H_\kappa$, $r' \leq^* 1_{j^*(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa}$ decide the sentence $\kappa \in \bigcap_{\alpha<\delta} j^*(\dot{X}_\alpha)$. We claim that $r^\frown r'$ must decide the sentence positively. Assume otherwise. Let $\bar{p}$ be the greatest lower bound of p'_α's and s' a condition $\leq^*$ below r' and $\bar{p}$. Then also $r^\frown s'$ decides $\kappa \in \bigcap_{\alpha<\delta} j^*(\dot{X}_\alpha)$ negatively. There must be some $r_0 \leq r$ in H_κ and $s'_0 \leq s'$ and α such that $r_0^\frown s'_0$ forces $\kappa \notin j^*(\dot{X}_\alpha)$. However, this is a contradiction since $r_0^\frown s'_0$ is compatible with $p_\alpha^\frown p'_\alpha$. (Lemma 6.8) $\square$

This ends the first proof of Theorem 6.6 (note that GCH or SCH was never used in the argument).

Proof 2: Easton support iteration

Now we will give an alternative proof of Theorem 6.6. Work in $V[G]$ and let $\mathbb{R}$ be defined as in Definition 6.3, with $\dot{R}_\alpha$ being a name for the forcing $\mathrm{Prk}(\alpha)$ whenever $\mathbb{R}_\alpha$ forces that α is measurable and an element of Δ. By Lemma 6.4, $\mathbb{R}$ preserves cardinals, and obviously does not change the continuum function in $V[G]$ – hence F is still realized in a generic extension by $\mathbb{R}$. It remains to verify that all elements of Δ will be cofinalized to a cofinality ω. Clearly, as in the first proof, it is enough to show

$$\text{For every } \alpha \in \Delta, \mathbb{R}_\alpha \Vdash \alpha \text{ is measurable.} \tag{6.10}$$

Let $\kappa \in \Delta$ be fixed. κ is a measurable cardinal in $V[G]$ and this is witnessed by an embedding $j^* : V[G] \to M[j^*(G)] =_{df} M^*$ which is a lift of an embedding $j : V \to M$ in V. Recall that the original j was an extender embedding, i.e.

$$M = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [F(\kappa)]^{<\omega}\} \tag{6.11}$$

The lifted j^* is also an extender embedding so that

$$M^* = \{j^*(f^*)(a) \mid f^* : [\kappa]^{<\omega} \to V[G], a \in [F(\kappa)]^{<\omega}\} \tag{6.12}$$

Note that each f^* is defined from some $f \in V$ with its domain containing only $\mathbb{P}^F$-names by setting

$$f^*(a) = (f(a))^G, \text{ for each } a \in \mathrm{dom}(f) \tag{6.13}$$

Preservation of measurability of κ by $\mathbb{R}_\kappa$ follows directly from [11], p.91, if $F(\kappa) = \kappa^+$ (or if the cofinality of $F(\kappa)$ is κ^+). We provide a general argument which works for arbitrary $F(\kappa)$ (assuming $\kappa \in \Delta$). Before we start the proof, recall that the Easton-supported iteration $\mathbb{R}^{\mathrm{Easton}}$ fails to satisfy the property that all direct extensions of a given condition are compatible. Thus we cannot proceed as in the first proof of Theorem 6.6.

In order to show that κ remains measurable in $\mathbb{R}_\kappa$ we have to define a measure at κ. Following the argument in [11] we will find a family of conditions in $j^*(\mathbb{R}_\kappa)$ which will answer compatibly the questions

$$\text{“is } \kappa \text{ in } j^*(\dot{X}),\text{”} \tag{6.14}$$

where $\dot{X}$'s are $\mathbb{R}_\kappa$-names for subsets of κ. If $F(\kappa) > \kappa^+$, then there are more than κ^+-many such names $\dot{X}$ and this prevents us from taking lower bounds when constructing the (to-be) compatible family of conditions (M^* is closed only under κ-sequences in $V[G]$). A standard way to circumvent this obstacle is to group the $\leq^*$-dense open sets (see Definition 6.9) corresponding to the relevant questions into κ^+-many segments such that each segment can be determined by a single condition (each segment will typically have size greater than κ^+).

The basic idea of the proof is to show that this grouping can be achieved by considering a family $\{f_\alpha \mid f_\alpha : \kappa \to H(\kappa)^V, \alpha < \kappa^+\}$ in V which determines a family $\{f^*_\alpha \mid f^*_\alpha : \kappa \to H(\kappa)^{V[G]}, \alpha < \kappa^+\}$ of functions in $V[G]$ which is universal in that the ranges of $j^*(f^*)$'s capture all $\leq^*$-dense opens sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_{\kappa+1}$ (and in particular the $\leq^*$-dense open sets corresponding to the questions (6.14)). Thus we will “borrow” some degree of GCH at κ from the original V.

Definition 6.9 *1. $D \subseteq \mathbb{R}_\kappa$ is $\leq^*$-dense open if D is open and for every $p \in \mathbb{R}_\kappa$ there is $d \in D$ and $d \leq^* p$.*

2. We say that p and q are $\leq^*$-compatible (or direct compatible*) if there is a direct extension below p and q. We say that p and q are $\leq^*$-incompatible, $p \perp^* q$, if there is no direct extension below p and q.*

3. $A \subseteq \mathbb{R}_\kappa$ is a $\leq^*$-antichain if all elements of A are $\leq^*$-incompatible. A is a maximal $\leq^*$-antichain if $a \notin A$ implies that there is some $\bar{a} \in A$ such that a and $\bar{a}$ are direct compatible.

4. We say that $\mathbb{R}_\kappa$ is κ^*-cc if all $\leq^*$-antichains are smaller than κ.

Note that a $\leq^*$-antichain may not be an antichain in the usual $\leq$-relation. However, every antichain is also a $\leq^*$-antichain. But a maximal antichain may not be a maximal $\leq^*$-antichain.

As regards the κ^*-cc chain condition, notice by way of example that $\mathrm{Prk}(\kappa)$ is still κ^{+*}-cc as conditions with the same first coordinate are direct compatible.

The usual correspondence between dense sets and antichains still holds:

Lemma 6.10 *Assume $G \subseteq \mathbb{R}_\kappa$ hits all $\leq^*$-maximal antichains. Then it hits all $\leq^*$-dense open sets (but may miss some usual dense open sets).*

Proof. It is enough to show that if D is $\leq^*$-dense open and $A \subseteq D$ is a maximal $\leq^*$-antichain in D, then it is maximal in $\mathbb{R}_\kappa$. Let $a \notin D$ be given; we want to show that there is $\bar{a} \in A$ such that a and $\bar{a}$ are direct compatible. By $\leq^*$-density of D, there is some $d \in D$, $d \leq^* a$. If d is in A, set $\bar{a} = d$. If d is not A, by the maximality of A in D there is some $\bar{a} \in A$ and r such that $a \geq^* d \geq^* r$ and $\bar{a} \geq^* r$, and hence $\bar{a}$ and a are direct compatible. (Lemma 6.10) □

The reason for introducing $\leq^*$-antichains is that there are generally smaller than $\leq^*$-dense open sets in $\mathbb{R}_\kappa$.

Lemma 6.11 *$\mathbb{R}_\kappa$ is κ^*-cc.*

Proof. Emulate the usual proof for Easton-supported iteration (see for instance [18], Theorem 16.9 and 16.30). The basic setup of the argument is that using the Fodor theorem one can thin out every κ-sequence of conditions in $\mathbb{R}_\kappa$ to a subsequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ such that there are $\gamma < \xi < \eta < \kappa$ with $\operatorname{supp}(p_\xi) \cap \xi \subseteq \gamma$, $\operatorname{supp}(p_\eta) \cap \eta \subseteq \gamma$ and moreover $\operatorname{supp}(p_\xi) \subseteq \eta$. By induction hypothesis $\mathbb{R}_\gamma$ is κ^*-cc and hence there is some $q \in \mathbb{R}_\gamma$ $\leq^*$-extending $p_\xi \cap \mathbb{R}_\gamma$ and $p_\eta \cap \mathbb{R}_\gamma$. Above γ the supports of p_ξ and p_η are disjoint and by the definition of the direct extension in Easton support (outside support everything is direct) there is a direct extension below p_ξ and p_η. (Lemma 6.11) □

Let H_κ be a generic filter for $\mathbb{R}_\kappa$. It is also a generic filter for $j^*(\mathbb{R})_\kappa$ over M^*. Let us assume that $j^*(\mathbb{R}_\kappa)$ is defined at κ, that is $\dot{R}_\kappa$ is $\operatorname{Prk}(\kappa)$. As $j(F)(\kappa) \geq F(\kappa)$,[11] the least measurable cardinal above κ is greater than $F(\kappa)$ and hence $\operatorname{Prk}(\kappa)$ forces over $M^*[H_\kappa]$ that $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_{\kappa+1}$ is $F(\kappa)^+$ $\leq^*$-closed.

Lemma 6.12 *Let σ in M^* be a $j^*(\mathbb{R})_{\kappa+1}$-name (where $j^*(\mathbb{R})_{\kappa+1} = \mathbb{R}_\kappa * \operatorname{Prk}(\kappa)$ in M^*) for a maximal $\leq^*$-antichain in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$. We claim that there is a name $\bar{\sigma}$ such that $j^*(\mathbb{R})_{\kappa+1}$ forces $\bar{\sigma} = \sigma$, and moreover for some f in V and $a \in [F(\kappa)]^{<\omega}$, $f : [\kappa]^{<\omega} \to H(\kappa)^V$, we have that $j^*(f^*)(a) = \bar{\sigma}$ (see (6.12) and (6.13) for the meaning of f^*). In particular there are only $(\kappa^\kappa)^V = \kappa^+$ functions f^* which enumerate (names for) maximal $\leq^*$-antichains in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$.*

Proof. By the definition of M^*, $\sigma = j^*(f^*)(a) = (j(f)(a))^{j(G)}$ for some f in V. It is enough to show that in fact f can be taken to be a mapping from $[\kappa]^{<\omega}$ to $H(\kappa)$.

We first argue that we can choose for σ an equivalent name $\bar{\sigma}$ which is an element of $H(j(\kappa))$ in M^*: By Lemma 6.11 applied to $j^*(\mathbb{R}_\kappa)$ we know

[11] Surprisingly, this condition seemed accidental due to $\mathbb{P}^F$ forcing, but in fact it seems essential even here.

that $j^*(\mathbb{R})_{\kappa+1}$ forces that σ is an antichain of size less than $j(\kappa)$, i.e. that it is an element of $H(j(\kappa))$ in the generic extension of M^* by $j^*(\mathbb{R})_{\kappa+1}$. W.l.o.g we can identify elements of $H(j(\kappa))$ with bounded subsets of $j(\kappa)$. Hence we know that $j^*(\mathbb{R})_{\kappa+1}$ forces in M^* that σ is a bounded subset of $j(\kappa)$. Moreover since $j^*(\mathbb{R})_{\kappa+1}$ is κ^+-cc in M^*, it forces a bound on σ; let $\alpha_\sigma < j(\kappa)$ be this bound:

$$M^* \models j^*(\mathbb{R})_{\kappa+1} \Vdash \sigma \subseteq \alpha_\sigma < j(\kappa) \tag{6.15}$$

Hence there is a nice $j^*(\mathbb{R})_{\kappa+1}$-name for σ, to be denoted as $\bar{\sigma}$, which is an element of $H(j(\kappa))$ of M^*. We again identify $\bar{\sigma}$ with some bounded subset of $j(\kappa)$ in M^*.

Going back to the original V, notice that because $\bar{\sigma}$ is a bounded subset of $j(\kappa)$, it must have been added by the iteration $j(\mathbb{P}^F)_{j(\kappa)}$ over M. By $j(\kappa)$-cc of the forcing $j(\mathbb{P}^F)_{j(\kappa)}$ in M, we can choose a nice $j(\mathbb{P}^F)_{j(\kappa)}$-name $\bar{\bar{\sigma}}$ for $\bar{\sigma}$ which itself can be identified with a bounded subset of $j(\kappa)$, this time in M.

As a bounded subset of $j(\kappa)$ in M, $\bar{\bar{\sigma}}$ is an element of $H(j(\kappa))$ of M. It follows we can write $\bar{\bar{\sigma}}$ as $j(f)(a)$ for some $f : \kappa \to H(\kappa)$, $f \in V$. By defining $f^*(a) = (f(a))^G$ for all a in the domain of f, we obtain

$$j^*(f^*)(a) = (j(f)(a))^{j(G)} = (\bar{\bar{\sigma}})^{j(G)} = \bar{\sigma}, \tag{6.16}$$

as desired. (Lemma 6.12) □

Work in $V[G][H_\kappa]$, where H_κ is a generic for $\mathbb{R}_\kappa$. To finish the proof of the Theorem 6.6, define the following construction: let $\langle f^*_\alpha \,|\, \alpha < \kappa^+\rangle$ be some enumeration of the relevant f^*'s as identified in Lemma 6.12. For each α, the family of names for $\leq^*$-antichains in the forcing $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$ in $M^*[H_\kappa]$ determined by $j^*(f^*_\alpha)$, i.e. $\{(j^*(f^*_\alpha)(a))^{H_\kappa} \,|\, a \in [F(\kappa)]^{<\omega}\} =_{df} \{A_{\alpha_\gamma} \,|\, \gamma < F(\kappa)\}$, exists in $M^*[H_\kappa]$ and has size less or equal $F(\kappa)$. We can assume that the empty condition in $\mathrm{Prk}(\kappa)$, $1_{\mathrm{Prk}(\kappa)}$, forces that each A_{α_γ} is a maximal $\leq^*$-antichain.

By induction construct for each $j^*(f^*_\alpha)$ a sequence of conditions $\langle q_{\alpha_\gamma} \in j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1} \,|\, \gamma < F(\kappa)\rangle$ such that q_{α_γ}'s are forced by $1_{\mathrm{Prk}(\kappa)}$ to form a $\leq^*$-decreasing chain in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$. Choose each q_{α_γ} so that it is forced by $1_{\mathrm{Prk}(\kappa)}$ to be a direct extension of some element in the maximal $\leq^*$-antichain A_{α_γ} (this can be arranged as each A_{α_γ} is (forced to be) a maximal $\leq^*$-antichain).

Let $\bar{q}_\alpha$ be the limit of q_{α_γ}'s. Arrange the construction so that $1_{\mathrm{Prk}(\kappa)}$ forces that for $\alpha < \kappa^+$, $\bar{q}_\alpha$'s form a $\leq^*$-decreasing chain. Set

$$Q = \{q \in j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1} \,|\, \exists \alpha < \kappa^+, 1_{\mathrm{Prk}(\kappa)} \Vdash \bar{q}_\alpha \leq^* q\} \tag{6.17}$$

The conditions in Q (compatibly) meet all maximal $\leq^*$-antichains in $j^*(\mathbb{R}) \setminus j^*(\mathbb{R})_{\kappa+1}$, and by Lemma 6.10 they meet all $\leq^*$-dense open sets in the same forcing.

Define a measure U as follows, where X is a subset of κ in $V[G][H_\kappa]$:

$$X \in U \text{ iff } \exists r \in H_\kappa, \exists p \leq^* 1_{\mathrm{Prk}(\kappa)}, \text{ and } \exists q \in Q \text{ such that } r^\frown p^\frown q \Vdash \kappa \in j^*(\dot{X}) \tag{6.18}$$

The argument that U is a measure is practically identical to the argument in the first proof for (6.9) providing we modify Fact 6.7 as follows:

Fact 6.13 *If σ is a sentence that there is $r \in H_\kappa$, $p \leq^* 1_{\mathrm{Prk}(\kappa)}$, and $q \in Q$ such that $r^\frown p^\frown q$ decides σ.*

Proof. By the Prikry property of $\mathrm{Prk}(\kappa)$ and $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$, the empty condition $1_{\mathbb{R}_\kappa}{}^\frown 1_{\mathrm{Prk}(\kappa)}$ forces that "the set of all conditions in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$ deciding σ is a $\leq^*$-dense open set". By the construction of Q, there is some $q \in Q$ capturing this dense open set, i.e. $1_{\mathbb{R}_\kappa}{}^\frown 1_{\mathrm{Prk}(\kappa)}$ forces "$q \Vdash \sigma \vee q \Vdash \neg\sigma$". As in Fact 6.7 this successively translates into the desired claim. (Fact 6.13) □

This ends the alternative proof of Theorem 6.6.

Remark 6.14 One can argue (see [11]) that the models obtained in Proof 1 and Proof 2 are different. We do not know so far whether one can find an interesting statement related to cofinalization which distinguishes these two models.

6.2 Iteration of the extender based Prikry forcing

In the previous section we have shown (in a global setting) how to obtain a strong limit cardinal κ (a former larger cardinal) which has cofinality ω and fails SCH. However, since κ was at one stage of the construction a measurable cardinal failing GCH, by reflection properties of measurable cardinals, this implies that GCH will fail on a big set of cardinals below κ.

In [10], M.Magidor and M.Gitik showed how to obtain such a cardinal κ without failing GCH below, using the so called *extender based Prikry forcing*. However, at least *prima facie* some GCH type assumptions on cardinals below κ are in fact necessary to show that this forcing behaves reasonably at κ (see Theorem 6.21), at least if arbitrary F's are considered.

We will use the results in Theorem 5.7 to prove some results concerning the realisation of an Easton function F in the context of global failure of SCH.

We first review the definition of the extender based Prikry forcing following [11], with some small corrections according to [5] in Definitions 6.17 and 6.18.

Let $\kappa < \lambda$ be cardinals, κ regular and λ of cofinality at least κ^{++}.

Definition 6.15 *A commutative system of embeddings is called a* nice system for (κ, λ) *if the following conditions hold:*

1. *$\langle \lambda, \leq_E \rangle$ is a κ^{++}-directed partial order, i.e. if $\{\alpha_\xi \mid \xi < \kappa^+\}$ is a subset of λ then there is some $\bar{\alpha} < \lambda$ such that $\alpha_\xi \leq_E \bar{\alpha}$ for all $\xi < \kappa^+$.*

2. *$\langle U_\alpha \mid \alpha < \lambda \rangle$ is a Rudin-Keisler commutative sequence of κ-complete ultrafilters over κ with projections $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \lambda, \alpha \geq_E \beta \rangle$.*

3. *For every $\alpha < \lambda$, $\pi_{\alpha\alpha}$ is the identity on a fixed set $\bar{X}$ which belongs to every U_β, $\beta < \lambda$.*

4. *(Commutativity) For every $\alpha, \beta, \gamma < \lambda$ such that $\alpha \geq_E \beta \geq_E \gamma$ there is $Y \in U_\alpha$ so that for every $\nu \in Y$*

$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)) \tag{6.19}$$

5. *For every $\alpha < \beta$, $\gamma < \lambda$ if $\gamma \geq_E \alpha, \beta$ then*

$$\{\nu \in \kappa \mid \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in U_\gamma \tag{6.20}$$

6. *U_κ is a normal ultrafilter.*

7. *$\kappa \leq_E \alpha$ when $\kappa \leq \alpha < \lambda$.*

8. *(Full commutativity at κ) For every $\alpha, \beta < \lambda$ and $\nu < \kappa$, if $\alpha \geq_E \beta$ then $\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\nu))$.*

9. *(Independence of the choice of projections to κ) For every $\alpha, \beta, \kappa \leq \alpha$, $\beta < \lambda$, $\nu < \kappa$*

$$\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\nu) \tag{6.21}$$

10. *Each U_α is a P-point ultrafilter, i.e. for every $f \in {}^\kappa\kappa$, if f is not constant mod U_α, then there is $Y \in U_\alpha$ such that for every $\nu < \kappa$, $|Y \cap f^{-1}(\{\nu\})| < \kappa$.*

The existence of a nice systems for (κ, λ) follows in a straightforward way if κ is a λ-hypermeasurable cardinal and GCH holds. Implicit in [10] is the following weakening of the hypermeasurability assumption (and of GCH) which also implies the existence of a nice system for (κ, λ).

Observation 6.16 *Let κ be a regular cardinal and $\lambda > \kappa$ a cardinal with cofinality at least κ^{++}. Assume that $2^\kappa = \kappa^+$. Assume further that there exists an embedding $j : V \to M$ with a critical point κ such that*

1. *M is closed under κ-sequences in V*

2. *$j(\kappa) > \lambda$*

3. *$([\lambda]^{\kappa^+})^V \subseteq M$*

4. $|[\lambda]^{\kappa^+}| = \lambda$ *in* M *(and hence also in* V*)*

5. *For some* $f_\lambda : \kappa \to \kappa$, $j(f_\lambda)(\kappa) = \lambda$

Then there exists a nice system for (κ, λ).

Proof. Define for $\kappa \leq \alpha \leq \beta < \lambda$ that $\alpha \leq_E \beta$ iff $j(f)(\beta) = \alpha$ for some $f : \kappa \to \kappa$. The single interesting property which may fail to hold in this context (when we use a weaker embedding than a λ-hypermeasurable embedding) is (1) in Definition 6.15, i.e. that $\langle \lambda, \leq_E \rangle$ is a κ^{++}-directed partial order. It is enough to verify that there exists in V an enumeration h such that $j(h)$ enumerates $[\lambda]^{\kappa^+}$ in V and M in λ-many steps so that each subset of λ of size at most κ^+ occurs cofinally often in the enumeration. To this effect define h with a domain κ to satisfy (where $\mu_\alpha = |[f_\lambda(\alpha)]^{\alpha^+}|$ in V): If α is a Mahlo cardinal, then h restricted do μ_α enumerates $[f_\lambda(\alpha)]^{\alpha^+}$ so that each subset of $f_\lambda(\alpha)$ of size at most α^+ occurs cofinally often in the enumeration. Then $j(h)$ restricted to $(\mu_\kappa)^M = |[j(f_\lambda)(\kappa)]^{\kappa^+}|^M = j(f_\lambda)(\kappa) = \lambda$ enumerates $[\lambda]^{\kappa^+}$ both in V and M in λ-many steps and with cofinal repetitions.

See the construction of a nice system in [11] or [10] for the other properties. (Observation 6.16) $\square$

Before we define the forcing notion, we first need some auxiliary definitions related to the nice system in Definition 6.15. Let us denote $\pi_{\alpha\kappa}(\nu)$ by ν^0, where $\kappa \leq \alpha < \lambda$ and $\nu < \kappa$ (this is independent of α). By ^{0}-increasing sequence of ordinals we mean a sequence $\langle \nu_1, \ldots, \nu_n \rangle$ of ordinals below κ so that

$$\nu_1^0 < \nu_2^0 < \ldots < \nu_n^0 \tag{6.22}$$

For every $\alpha < \lambda$ we shall always mean by writing $X \in U_\alpha$ that $X \subseteq \bar{X}$, in particular it will imply that for $\nu_1, \nu_2 \in X$ if $\nu_1^0 < \nu_2^0$ then the size of $\{\alpha \in X \mid \alpha = \nu_1^0\}$ is $< \nu_2^0$. The following weak version of normality holds since U_α is a P-point: if $X_i \in U_\alpha$ for $i < \kappa$ then also $X = \{\nu \mid \forall i < \nu^0, \nu \in X_i\} \in U_\alpha$.

Let $\nu < \kappa$ and $\langle \nu_1, \ldots, \nu_n \rangle$ be a finite sequence of ordinals below κ. Then ν is called *permitted* for $\langle \nu_1, \ldots, \nu_n \rangle$ if $\nu^0 > \max(\{\nu_i^0 \mid 1 \leq i \leq n\})$.

Now we are ready to define the extender based Prikry forcing notion.

Definition 6.17 *The extender based Prikry forcing* $\mathrm{Prk}_E(\kappa, \lambda)$ *is defined as follows. The set of forcing conditions consists of all the elements* p *of the form* $\{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \setminus \{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\}$, *where*

1. $g \subseteq \lambda$ *of cardinality* $\leq \kappa$ *which has a maximal element in* $\leq_E$*-ordering and* $0 \in g$. *Further let us denote* g *by* $\mathrm{supp}(p)$, $\max(g)$ *by* $\mathrm{mc}(p)$, T *by* T^p *and* $p^{\max(g)}$ *by* p^{mc} *("mc" for maximal coordinate).*

2. *For* $\gamma \in g$, p^γ *is a finite* ^{0}*-increasing sequence.*

3. *T is a tree with a trunk p^{mc} consisting of ^{0}-increasing sequences. All the splittings in T are required to be on sets in $U_{\mathrm{mc}(p)}$, i.e. for every $\eta \in T$, if $\eta \geq_T p^{\mathrm{mc}}$ then the set*

$$\mathrm{Suc}_T(\eta) = \{\nu < \kappa \mid \eta^\frown\langle\nu\rangle \in T\} \in U_{\mathrm{mc}(p)}. \tag{6.23}$$

We also require that for $\eta_1 \geq_T \eta_0 \geq_T p^{\mathrm{mc}}$

$$T_{\eta_1} \text{ is a subtree of } T_{\eta_0}, \tag{6.24}$$

where T_η denotes the set of σ such that $\eta^\frown\sigma$ belongs to T.

4. *For every $\gamma \in g$, $\pi_{\mathrm{mc}(p),\gamma}(\max(p^{\mathrm{mc}}))$ is not permitted for p^γ.*

5. *For every $\nu \in \mathrm{Suc}_T(p^{\mathrm{mc}})$*

$$|\{\gamma \in g \mid \nu \text{ is permitted for } p^\gamma\}| \leq \nu^0 \tag{6.25}$$

6. *$\pi_{\mathrm{mc}(p),0}$ project p^{mc} onto p^0, in particular p^{mc} and p^0 are of the same length.*

The ordering $\leq_{\mathrm{Prk}_E(\kappa,\lambda)} = \leq$ is defined as follows:

Definition 6.18 *We say that p extends q, $p \leq q$ if*

1. $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

2. *For every $\gamma \in \mathrm{supp}(q)$, p^γ is an end-extension of q^γ.*

3. $p^{\mathrm{mc}(q)} \in T^q$.

4. *For every $\gamma \in \mathrm{supp}(q)$: $p^\gamma \setminus q^\gamma = \pi_{\mathrm{mc}(q),\gamma}{}''((p^{\mathrm{mc}(q)} \setminus q^{\mathrm{mc}(q)} \restriction \mathrm{length}(p^{\mathrm{mc}}) \setminus (i+1))$, where $i \in \mathrm{dom}(p^{\mathrm{mc}(q)})$ is the largest such that $p^{\mathrm{mc}(q)}(i)$ is not permitted for q^γ.*

5. *$\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ maps T^p to a subtree of T^q.*

6. *For every $\gamma \in \mathrm{supp}(q)$, for every $\nu \in \mathrm{Suc}_{T^p}(p^{\mathrm{mc}})$ if ν is permitted for p^γ, then*

$$\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu)). \tag{6.26}$$

The ordering $\leq^*_{\mathrm{Prk}_E(\kappa,\lambda)} = \leq^*$ is defined as follows:

Definition 6.19 *Let $p, q \in \mathrm{Prk}_E(\kappa,\lambda)$. We say that p is a direct extension of q ($p \leq^* q$) if:*

1. $p \leq q$.

2. *For every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$.*

We state without a proof the following facts:

Fact 6.20 *Assume that the universe V satisfies the conditions set out in Observation 6.16. Then the following holds of* $\mathrm{Prk}_E(\kappa,\lambda)$*:*

1. $\mathrm{Prk}_E(\kappa,\lambda)$ *is* κ^{++}*-cc.*

2. $\leq^*$ *is* κ*-closed.*

3. $(\mathrm{Prk}_E(\kappa,\lambda), \leq, \leq^*)$ *satisfies the Prikry property.*

4. $\mathrm{Prk}_E(\kappa,\lambda)$ *preserves cardinals.*

5. $\mathrm{Prk}_E(\kappa,\lambda)$ *does not add new bounded subsets of* κ *and the cofinality of* κ *is* ω *in the generic extension.*

6. $\kappa^\omega = 2^\kappa = \lambda$ *in the generic extension.*

As regards (6), note that $\lambda \leq 2^\kappa$ is true by the way the forcing $\mathrm{Prk}_E(\kappa,\lambda)$ is set up. The other direction, i.e. $2^\kappa \leq \lambda$ follows from the number of nice names for subsets of κ: by κ^{++}-cc of the forcing, this is λ^{κ^+}. Thus a direct way to ensure that $2^\kappa = \lambda$ is to have $\lambda^{\kappa^+} = \lambda$. In Theorem 6.21 this is achieved by restricting F to be trivial at κ^+ (for κ a Mahlo cardinal). This restriction is also important to ensure that $\mathrm{Prk}_E(\kappa,\lambda)$ behaves correctly over some generic extensions (see Theorem 6.21).

The following Theorem is an application of the results in Theorem 5.7. In contrast to Theorem 5.7 and Theorem 6.6, however, Easton functions F allowed in Theorem 6.21 are less restrictive in one significant aspect: we no longer require that F should satisfy some reflection properties below a large cardinal in question – cf. the assumption in Theorem 5.7 that if κ should remain measurable then we need to have some $j : V \to M$ witnessing "correct behaviour" of F below κ. This less restrictive formulation is made possible by the inclusion of the extender based Prikry forcing into the argument. Note however that the inclusion of the extender based Prikry forcing brings with it restrictions of its own. We will discuss these restrictions, and possible ways of eliminating some of them, in Remark 6.26 after the proof of the theorem.

Let κ^+-hypermeasurable mean that κ is measurable. Also let us introduce the following notation: $\Theta = \{\kappa \,|\, \kappa \text{ is } F(\kappa)\text{-hypermeasurable}\}$ and $\mathrm{Lim}(\Theta) = \{\kappa \in \Theta \,|\, \kappa \text{ is a limit point of } \Theta\}$.

Theorem 6.21 *Assume GCH. Let F be an Easton function which satisfies the following properties:*

1. *F preserves Mahlo cardinals, i.e. for all α Mahlo, $F[\alpha] \subseteq \alpha$;*

2. *For all Mahlo cardinals* α, $F(\alpha^+) = \max(F(\alpha), \alpha^{++})$;

3. *For all* $\kappa \in \mathrm{Lim}(\Theta)$, $F(\kappa) = \kappa^+$.

If F *satisfies these properties then there is a cardinal-preserving extension* V^* *such that the continuum function in* V^* *satisfies:*

1. *If* κ *in* V *is a regular cardinal which is not Mahlo then* $2^\kappa = F(\kappa)$ *in* V^*.

2. *If* κ *in* V *is in* Θ *then* $2^\kappa = F(\kappa)$ *and* κ *is a singular strong limit cardinal with cofinality* ω *in* V^*.

3. *If* κ *in* V *is a Mahlo cardinal not in* Θ *then* $2^\kappa = \kappa^+$ *in* V^*.

The proof will be given in a sequence of lemmas. The general strategy is as follows: The desired forcing $\mathbb{P}$ will be of the form $\mathbb{P} * \mathbb{R}$, where $\mathbb{P}$ realises F everywhere except on Mahlo cardinals, and $\mathbb{R}$ realises F on elements of Θ by a combination of the simple Prikry forcing $\mathrm{Prk}(\kappa)$ if $F(\kappa) = \kappa^+$, or of the extender based Prikry forcing $\mathrm{Prk}_E(\kappa, F(\kappa))$ otherwise. The remaining Mahlo cardinals not handled by $\mathbb{R}$ will be left to satisfy GCH (see Remark 6.26).

Definition 6.22 *Let* $\mathbb{P}$ *be defined as* $\mathbb{P}^F$ *in Definition 5.4 with the following modifications:*

1. *The domain of* $\mathbb{P}$ *includes all regular cardinals except Mahlo cardinals and successors of Mahlo cardinals, i.e. the forcing is empty at all Mahlo cardinals and their successors.*

2. *The iteration points* i_α*'s (see Definition 5.4) will not depend on* F *and always be the Mahlo cardinals of* V, *i.e.* $\mathbb{P}$ *will be an iteration of products of the Cohen forcing at regular cardinals which are not Mahlo and the points of iteration will be the Mahlo cardinals in* V.

Lemma 6.23 *The forcing* $\mathbb{P}$ *preserves cofinalities and realises* F *on all regular cardinals except Mahlo cardinals (and their successors). Also, if* κ *is in* Θ *then* κ *remains measurable in* $V^{\mathbb{P}}$ *if* $F(\kappa) = \kappa^+$, *or if* $F(\kappa) > \kappa^+$ *then* κ *retains sufficient "largeness" so that the forcing* $\mathrm{Prk}_E(\kappa, F(\kappa))$ *can be defined at* κ *and satisfies properties in Fact 6.20.*

Proof. Let G denote the generic filter for $\mathbb{P}$. $\mathbb{P}$ obviously realises F everywhere except at Mahlo cardinals (and their successors) and preserves cofinalities, so we need to show that it preserves relevant properties of elements in Θ.

Case 1: $\kappa \in \Theta$ and $F(\kappa) = \kappa^+$.

The intended forcing $\mathbb{R}$ in the second stage will use just the simple Prikry forcing $\mathrm{Prk}(\kappa)$: thus it is enough to verify that κ remains measurable in $V[G]$. This is an easy application of the argument in Theorem 5.7: let $j : V \to M$ in V witness the measurability of κ, in particular $\kappa^+ < j(\kappa) < \kappa^{++}$. We will lift j to $j^* : V[G] \to M[j^*(G)]$. Since $\mathbb{P}_\kappa = j(\mathbb{P})_\kappa$, we can use the generic G_κ for $j(\mathbb{P})_\kappa$ in M. Since $\mathbb{P}$ is not defined at Mahlo cardinals, $j(\mathbb{P})$ is trivial at the Mahlo cardinal κ in M, and also at κ^+ since F is trivial at successors of Mahlo cardinals. It follows that $j(\mathbb{P}_\kappa) \setminus j(\mathbb{P})_\kappa$ is a κ^{++M}-closed iteration in $M[G_\kappa]$ which has $j(\kappa)$-cc in $M[G_\kappa]$. Since $j(\kappa)$ has size κ^+ in $V[G_\kappa]$ we can build in $V[G_\kappa]$ a $M[G_\kappa]$-generic H for $j(\mathbb{P}_\kappa) \setminus j(\mathbb{P})_\kappa$ in κ^+-many steps. The forcing $\mathbb{P} \setminus \mathbb{P}_\kappa$ is κ^+-distributive in $V[G_\kappa]$ and so we may lift j to the whole model $V[G]$ by applying Lemma 2.19.

Case 2: $\kappa \in \Theta$ and $F(\kappa) > \kappa^+$.

We want to show that κ retains sufficient "largeness" for a reasonable definition of $\mathrm{Prk}_E(\kappa, F(\kappa))$ in $V[G]$. We show that if we lift j witnessing $F(\kappa)$-hypermeasurability to j^* using the argument in Theorem 5.7 then j^* will satisfy all properties identified in Observation 6.16. This will imply that there exists a nice system for $(\kappa, F(\kappa))$ in a generic extension by $\mathbb{P}$.

We clearly have that $2^\kappa = \kappa^+$ in $V[G]$ since the forcing $\mathbb{P}$ avoids Mahlo cardinals (and Mahlo cardinals remain strongly inaccessible in $V^{\mathbb{P}}$).

We will analyze the forcing $j(\mathbb{P})$ in order to show that the lifting argument in Theorem 5.7 applies here. Recall the notational conventions employed in the proof of Theorem 5.7 in Remark 5.8 (with the understanding that we are avoiding Mahlo cardinals). In particular let for $\lambda \in [\kappa, F(\kappa)]$ denote $\mathcal{Q}_\lambda$ the forcing $\mathrm{Add}(\lambda, F(\lambda))$ of the pertinent model, and the same for $\mathcal{Q}^M_\lambda$. Also, let $i_{\kappa+1} = i(\kappa+1)$ denote the next Mahlo cardinal in V greater than κ, and similarly for $i^M_{\kappa+1} = i^M(\kappa+1)$.

Since $\kappa = i_\kappa = i^M_\kappa$ is a Mahlo cardinal in both V and M, $\mathbb{P}$ factors to $\mathbb{P}_\kappa * \mathbb{P} \setminus \mathbb{P}_\kappa$, and as in Theorem 5.7, we have $\mathbb{P}_\kappa = j(\mathbb{P})_\kappa$ and so the generic filter G_κ be be used as an M-generic filter for $j(\mathbb{P})_\kappa$. By the definition of F, the next Mahlo cardinal $i(\kappa+1)$ above κ is strictly greater than $F(\kappa)$. It follows that in M, the next Mahlo cardinal $i^M(\kappa+1)$ after κ must be $\geq F(\kappa)$ (no new Mahlo cardinals can appear in the interval $(\kappa, F(\kappa))$ in M). Note that by the argument Observation 5.2 (the proof of Observation 5.2 works the same if the word "inaccessible" is replaced by "Mahlo") we cannot exclude the possibility that $F(\kappa)$ is a Mahlo cardinal in M.

As in the proof of Theorem 5.7, see the argument after Remark 5.8 before (5.3), work in $V[G_\kappa]$ and let $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ denote the forcing completely embeddable into $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda$ with the property that the length of the products in $\mathcal{Q}^+_\lambda$ and $\mathcal{Q}^M_\lambda$ of $M[G_\kappa]$ for λ in the interval $[\kappa, F(\kappa)]$ agrees.[12] This is possible since for all $\lambda \in [\kappa, F(\kappa)]$ it holds that $F(\lambda)$ has size at least $F(\kappa)$ in

[12]Since κ is Mahlo in M and V, the forcing $\mathcal{Q}^+_\lambda$ and $\mathcal{Q}^M_\lambda$ is actually empty at $\lambda = \kappa, \kappa^+$.

V and $j(F)(\lambda) < j(\kappa) < F(\kappa)^+$ has size at most $F(\kappa)$ in V. We obtain that $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ of $M[G_\kappa]$ is a subset of $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ of $V[G_\kappa]$. Let us denote as $g_{[\kappa,F(\kappa)]}$ the generic for $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda$ (note that by the complete embeddability of $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ into $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda$ this generic filter ensures the existence of a generic filter for $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$). We may conclude by arguments as in Lemma 5.9 and 5.10 (concentrating on the case when the cofinality of $F(\kappa)$ is $> \kappa^+$) that there is in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$, and hence in $V[G]$, an $M[G_\kappa]$-generic filter for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$, which we denote as $g^M_{[\kappa,i^M(\kappa+1))}$.

Since $\mathbb{P}$ and $j(\mathbb{P})$ are empty at κ, $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}]$ is clearly closed under κ-sequences in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$. Thus we may finish lifting j by the application of Lemma 5.15 to $j(\mathbb{P})_{j(\kappa)}$ (Note that if $F(\kappa) = i^M(\kappa+1)$ is a Mahlo cardinal in M then the forcing $j(\mathbb{P})$ is empty at $F(\kappa)$ and so the forcing $j(\mathbb{P}_\kappa) \setminus j(\mathbb{P})_{i^M(\kappa+1)}$ is still $F(\kappa)^+$-distributive in $M[G_\kappa * g^M_{[\kappa,i^M(\kappa+1))}$.) Now we my lift to the whole forcing $j(\mathbb{P})$ by applying Lemma 2.19 to $\mathbb{P} \setminus \mathbb{P}_\kappa$ which κ^+-distributive over $V[G_\kappa]$. We write $j^* : V[G] \to M[j^*(G)]$ for the lifted embedding.

It is clear that this j^* witnesses the first two conditions in Observation 6.16 from the following list (identifying V with our $V[G]$ and M with $M[j^*(G)]$, j^* with j, and $F(\kappa)$ with λ):

1. $M[j^*(G)]$ is closed under κ-sequences in $V[G]$
2. $j^*(\kappa) > F(\kappa)$
3. $([F(\kappa)]^{\kappa^+})^{V[G]} \subseteq M[j^*(G)]$
4. $|[F(\kappa)]^{\kappa^+}| = F(\kappa)$ in $M[j^*(G)]$ (and hence also in $V[G]$)
5. For some $f_{F(\kappa)} : \kappa \to \kappa$, $j^*(f_{F(\kappa)})(\kappa) = F(\kappa)$

Condition (5) can be assumed w.l.o.g. for the original j, so it holds for our j^* as well.

It remains to verify conditions (3) and (4).

To verify (3), let X be a subset of $F(\kappa)$ of size at most κ^+ in $V[G]$. By the definition of F, the cofinality of $F(\kappa)$ is at least κ^{++} and so there is some $\alpha < F(\kappa)$ such that $X \subseteq \alpha$. Since $\mathbb{P}$ is trivial at κ and κ^+, X must be added by $\mathbb{P}_\kappa$. But now (3) follows since if $\dot{X}$ is a nice $\mathbb{P}_\kappa$-name for $X \subseteq \alpha$, then $\dot{X} \in H(F(\kappa))$, and so $\dot{X} \in M$; as also $\mathbb{P}_\kappa = j(\mathbb{P})_\kappa$ and G_κ is M-generic, we obtain that $\dot{X}^{G_\kappa} = X \in M[j^*(G)]$.

Now also (4) follows easily: we argued in the proof of (3) that all subsets of $F(\kappa)$ of size at most κ^+ have nice names which are included in $H(F(\kappa))$; this is also true in M and since M satisfies GCH, $H(F(\kappa))$ has size $F(\kappa)$ in M, and so the number of nice $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$-names is $F(\kappa)$. This implies (4) as desired. (Lemma 6.23) $\square$

We know turn to the forcing $\mathbb{R}$ which will cofinalize elements of Θ and simultaneously realise F on elements of Θ (and their successors).

Definition 6.24 *$\mathbb{R}$ is defined just as $\mathbb{R}^{\text{Easton}}$ in Definition 6.3 such that if $\alpha \notin \Theta$ then $\dot{R}_\alpha$ is a trivial forcing, and if $\alpha \in \Theta$, then*

1. *If $F(\alpha) = \alpha^+$ and $\mathbb{R}_\alpha$ forces that α is measurable then $\dot{R}_\alpha$ is a name for the simple Prikry forcing* $\mathrm{Prk}(\alpha)$*;*
2. *If $F(\alpha) > \alpha^+$ and $\mathbb{R}_\alpha$ forces that there exists a nice system for $(\alpha, F(\alpha))$ then $\dot{R}_\alpha$ is a name for the extender based Prikry forcing* $\mathrm{Prk}_E(\alpha, F(\alpha))$*.*

This iteration $\mathbb{R}$ satisfies the Prikry property by a general argument for Prikry type forcing notions (see [11]).

Lemma 6.25 *Let G be a generic over V for $\mathbb{P}$ as in Lemma 6.23. Then $\mathbb{R}$ applied over $V[G]$ has the following properties:*

1. *Preserves all cardinals.*
2. *Preserves the continuum function in $V[G]$ on cardinals realised by the forcing $\mathbb{P}$.*
3. *Changes the cofinality of all elements of Θ to ω and realises F on Θ (and on their successors).*

Proof. **Ad (1)**. As in Lemma 6.4(2) it holds that if κ is a cardinal, then $\mathbb{R}$ factors to $\mathbb{R}_{\kappa+1}$ and $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ so that $\mathbb{R}_{\kappa+1}$ preserves all cardinals greater or equal to κ^+ and $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ does not add new subsets of κ^+. If κ is as in 6.4 (2) and is regular, then $\mathbb{R}_\kappa$ is κ^+-cc by the Easton support. If κ is a singular cardinal, then $\mathbb{R}_\kappa$ has at most $(2^\kappa)^+$-cc. Since $V[G]$ satisfies SCH at all singular cardinals, this means that $\mathbb{R}_\kappa$ is κ^{++}-cc. A special, but standard, argument for the iteration of Prikry type forcing notions implies that κ^+ is preserved as well.[13]

Note that the preservation of cardinals would be more problematic if $\mathbb{R}$ was defined with the full support as in Definition 6.2. Since it is no longer true as in Lemma 6.4 that any two conditions extending the empty condition are compatible iff they are compatible on the finite set of coordinates with the non-direct extensions, the chain condition of $\mathbb{R}_\kappa$ for a regular κ would be $(2^\kappa)^+$-cc. Even if $2^\kappa = \kappa^+$ is true in $V[G]$, the preservation of κ^+ would still not be automatic due to the regularity of κ.

Ad (2). Obvious.

[13] In fact, it is known from the results in inner model theory that it is very hard to collapse successors of singular cardinals. Thus if we for instance assume that there is no inner model with a Woodin cardinal in our universe, κ^+ cannot be collapsed by a general inner model argument.

Ad (3). It remains to verify that if κ is in Θ then the iteration $\mathbb{R}_\kappa$ up to κ forces that κ retains sufficient "largeness" so that the relevant Prikry forcing can be defined.

Case 1: Assume first that $F(\kappa) > \kappa^+$. By the definition of F, κ is not a limit point of Θ. This implies that there is some $\beta < \kappa$ such that $\mathbb{R}_\kappa \setminus \mathbb{R}_\beta$ is trivial. A standard argument shows that in this case it is easy to lift the embedding j^* in $V[G]$ witnessing the largeness of κ so that all conditions identified in Observation 6.16 still hold in $V[G]^{\mathbb{R}_\beta} = V[G]^{\mathbb{R}_\kappa}$ and so $\mathrm{Prk}_E(\kappa, F(\kappa))$ can be correctly defined here.

Case 2: Assume that $F(\kappa) = \kappa^+$ and that κ is a limit point of Θ (if it is not a limit point, reason as in Case 1). We argue essentially as in the proof 2 of Theorem 6.6 but with the following simplification (let $j^* : V[G] \to M[j^*(G)]$ be an embedding witnessing for the measurability of κ): Since GCH holds at κ in $V[G]$, we can list all nice $\mathbb{R}_\kappa$-names for subsets of κ in κ^+-steps and handle them in a compatible fashion, using the fact that $j^*(\mathbb{R}) \setminus j^*(\mathbb{R})_{\kappa+1}$ is κ^+-closed in the direct extension relation. Note that since κ is a limit point of $j^*(\Theta)$ in $M[j^*(G)]$, $j^*(\mathbb{R})$ is the simple Prikry forcing at κ, so any two direct extensions at κ are compatible (or alternatively we could use a $(j^*)' : V[G] \to M^*$ so that κ is not measurable in M^*). (Lemma 6.25) $\square$

This ends the proof of Theorem 6.21 (Theorem 6.21) $\square$

We end this section with some notes on possible generalizations of the Theorem. As a general comment notice that unlike the F's allowed in Theorem 6.6, which required that GCH fails often below κ's to be cofinalized, Theorem 6.21 requires the opposite, i.e. that F should retain some substantial degree of GCH below such κ's.

Remark 6.26 We believe that with more work, Theorem 6.21 could be improved to include a larger class of Easton functions F. The following list includes some tips, comments and restrictions which are relevant to Theorem 6.21 and may possibly suggest some improvements. For the purposes of this Remark, let us say that F is *suitable* if it satisfies the conditions in Theorem 6.21.

1. It seems accidental and not at all necessary to demand that some Mahlo cardinals end up with GCH (or equivalently, that F is not realised on some Mahlo cardinals). However, there are some technical considerations which make it difficult to eliminate this restriction. (i) It is hard to include all Mahlo cardinals in the forcing $\mathbb{P}$ or $\mathbb{R}$ since the lifting arguments for $\mathbb{P}$ or $\mathbb{R}$ may fail (for instance the forcing $\mathbb{R}_\kappa$ would no longer be bounded below $\kappa \in \Theta$ with $F(\kappa) > \kappa^+$). (ii) If we attempt to force F on these Mahlo cardinals in $V^{\mathbb{P}*\mathbb{R}}$ we may collapse cardinals since this model may not satisfy SCH: Assume for instance that κ is a singular limit of ω-many Mahlo cardinals κ_i's cofinal in κ

and $2^\kappa = \kappa^{+4}$. Let us force F on κ_i's for $i < \omega$, for instance by the iteration of Cohen forcings. Then the chain condition of the forcing up to κ would be $(2^\kappa)^+ = \kappa^{+5}$ and it is not obvious why for instance κ^{+3} should not be collapsed to κ^{+2}.

2. Assume $j : V \to M$ is an embedding with a critical point κ. The special role which is played by Mahlo cardinals in the definition of suitable F's is of course brought about by the fact that Mahlo cardinals are perhaps the simplest way how to "talk" about κ in M; compare this with the assumption of F-goodness in Definition 5.3. Since κ is in fact a limit of Mahlo cardinals in M, Theorem 6.21 might be improved by replacing "Mahlo" by "Mahlo limit of Mahlo cardinals".

3. The restriction on F demanding that suitable F's are trivial at successors of Mahlo cardinals is necessitated by the requirement that the $\leq_E$ relation in the definition of a nice system (see Definition 6.15) is κ^{++}-directed. A straightforward way how to ensure κ^{++}-directedness is to control the way new κ^+-sequences in $F(\kappa)$ are added (see the proof of 6.23, Case 2). In [10], M. Magidor and M. Gitik claim (without proof) that the definition of the extender based Prikry forcing $\mathrm{Prk}_E(\kappa, \lambda)$ can done just from the assumption of κ^+-directedness of $\leq_E$. Hence it seems possible that the restriction on the behaviour of a suitable F on successors of Mahlo cardinals could be eliminated.

4. It seems more difficult to eliminate the assumption that if κ is a limit point of Θ then $F(\kappa) = \kappa^+$. It seems impossible to argue that κ retains sufficient largeness for the definition of $\mathrm{Prk}_E(\kappa, F(\kappa))$ in $\mathbb{R}_\kappa$ by simply lifting to $\mathbb{R}_\kappa$ the embedding witnessing the existence of a nice system in $V^{\mathbb{P}}$ (as we do when $\mathbb{R}_\kappa$ is bounded below κ). A strategy with a chance of success should rather follow the argument as in Ad (3), Case 2 of Lemma 6.25 but instead of a single measure, define a nice system of measures. However, this seems to be very difficult if only because $j^*(\mathbb{R})$ at κ could be the extender based Prikry forcing which does not have the property that any two direct extensions of the empty conditions are compatible.

We can formulate more compelling results if we focus on Easton functions F which just "toggle" GCH on or off. We say that an Easton function F is *toggle-like* if for all regular cardinals α, $F(\alpha) = \alpha^+$ or α^{++}.

Before we prove a theorem about toggle-like Easton functions we state the following observation which follows from arguments in [15]:

Observation 6.27 *Assume GCH and let Σ be an arbitrary subclass of κ^{++}-hypermeasurable cardinals. Then there is a cardinal preserving generic extension where SCH fails exactly at elements of Σ. In particular, the pattern* *of the failure of SCH can be the same as the pattern of any subclass of κ^{++}-hypermeasurable cardinals.*

Proof. Let $\mathbb{R}$ be an iteration with the Easton support of the forcings $\mathrm{Prk}_E(\kappa,\kappa^{++})$ for $\kappa \in \Sigma$. It suffices to show that if κ is in Σ then $\mathbb{R}_\kappa$ forces that there is an embedding j^* satisfying the conditions in Observation 6.16. The key observation is that we can choose in V an embedding $j : V \to M$ witnessing that κ is κ^{++}-hypermeasurable such that the Mitchell order of κ in M is $\leq \kappa^{++}$. In particular, κ is not κ^{++}-hypermeasurable in M (which is equal to the Mitchell order $\kappa^{++}+1$). It follows that $j(\mathbb{R}_\kappa)$ is trivial at κ. Using arguments from [15] we can then argue that this j can be used to show that κ remains κ^{++}-hypermeasurable in $V^{\mathbb{R}_\kappa}$, and thus satisfies the conditions in Observation 6.16.

For completeness of the argument we briefly review here the relevant parts of [15]. Assume $2^\kappa = \kappa^+$ and let $j : V \to M$ witness κ^{++}-hypermeasurability of κ, $\kappa^{++} < j(\kappa) < \kappa^{+3}$. Assume further that $2^{\kappa^+} = \kappa^{++}$ in M (this is for instance true if GCH holds in V). W.l.o.g. j is an extender emebedding. Let $\mathbb{R}$ be the forcing above. We show that κ is still κ^{++}-hypermeasurable in $V^{\mathbb{R}_\kappa}$. Let G_κ be $\mathbb{R}_\kappa$-generic. Since $j(\mathbb{R}_\kappa)$ is trivial at κ, κ^+, and κ^{++}, the tail forcing $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ is $(\kappa^{+3})^M$-$\leq^*$-closed in $M[G_\kappa]$ (i.e. the direct extension relation is $(\kappa^{+3})^M$-closed). Thus we can hit all $\leq^*$-dense open sets in $j(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa$ in $M[G_\kappa]$ in κ^+-many steps using the following construction: If f is a function $f : [\kappa]^{<\omega} \to H(\kappa^+)$ with its range containing just $\leq^*$-dense open sets of $\mathbb{R}_\kappa$, then $j(f) \restriction [\kappa^{++}]^{<\omega}$ determines in $M[G_\kappa]$ a family $\{D_\alpha \mid \alpha < \kappa^{++}\}$ of κ^{++}-many $\leq^*$-dense open sets in $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$. By closure of the $\leq^*$-ordering, the intersection $\bigcap_{\alpha<\kappa^{++}} D_\alpha$ is $\leq^*$-dense. Since there are only κ^+-many functions from $[\kappa]^{<\omega} \to H(\kappa^+)$, we can build in $V[G_\kappa]$ a sequence of conditions $\langle r_\alpha \mid \alpha < \kappa^+\rangle$ in $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ hitting all $\leq^*$-dense open sets in $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$.
Definition of an extender. First notice that all countable subsets of κ^{++} in $V[G_\kappa]$ are included in $M[G_\kappa]$ (since all $\mathbb{R}_\kappa$-nice names for such subsets are in M and the forcing $j(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa$ does not add new countable subsets of κ^{++} over $M[G_\kappa]$). Thus if a is a countable subset of κ^{++} in $V[G_\kappa]$, let $\dot{a}$ be a nice $j(\mathbb{R}_\kappa)$-name for it in M. For such a define in $V[G_\kappa]$ a measure E_a as follows: if $\dot{X}$ is a $\mathbb{R}_\kappa$-name for a subset of $[\kappa]^{|a|}$, then

$$X \in E_a \text{ iff } \exists p \in G_\kappa, \exists \alpha \text{ s.t. } p^\frown r_\alpha \Vdash \dot{a} \in j(\dot{X}) \tag{6.27}$$

By compatibility of all r_α's it is routine to verify that each E_a is a κ-complete measure and moreover if $a \in V$, then E_a extends the original measure E'_a obtained from j by setting $X \in E'_a$ iff $a \in j(X)$. The natural projections $\pi_{a,b}$ between $a \subseteq b \in [\kappa^{++}]^{\leq\omega}$ determine a directed system of embeddings

$$\langle (\mathrm{Ult}(V[G_\kappa], E_a)), \pi_{a,b} \mid a \subseteq b \in [\kappa^{++}]^{\leq\omega}\rangle \tag{6.28}$$

with a direct limit M^*. Because we have built the directed system using countable a's and all such a's have their names in $j(\mathbb{R}_\kappa)$, the direct limit M^* must be wellfounded (otherwise we could find an ill-founded epsilon chain in one of the measure ultrapowers). Let $j^* : V[G_\kappa] \to M^*$ be the corresponding embedding, and $M^* = \{j^*(f)(a) \mid f \in V[G_\kappa], f : [\kappa]^{\leq\omega} \to V[G_\kappa], a \in [\kappa^{++}]^{\leq\omega}\}$. The embeddings j and j^* are connected in the following way:

$$M^* \models \varphi(j^*(f)(a), \ldots) \text{ iff } \exists p \in G_\kappa, \alpha \in \kappa^+, p^\frown r_\alpha \Vdash \varphi(j(\dot{f})(\dot{a}), \ldots) \tag{6.29}$$

$H(\kappa^{++})^V$ is included in M^*. The key claim is the following: Let $g \in V[G_\kappa]$ from $[\kappa]^{\leq\omega} \to H(\kappa)^V$ be such that for each inaccessible $\alpha < \kappa$, the restriction of g to $[\alpha^{++}]^{\leq\omega}$ is a function with its range included in $H(\alpha^{++})^V$. Then for each $a \in [\kappa^{++}]^{\leq\omega}$ there is some h in V and $b \in [\kappa^{++}]^{\leq\omega}$ in V such that

$$j^*(g)(a) = j^*(h)(b) \tag{6.30}$$

To argue for (6.30), notice that if $\dot{g}$ is a name for g, then we can assume

$$1_{j(\mathbb{R}_\kappa)} \Vdash j(\dot{g})(\dot{a}) \in (H(\kappa^{++}))^M = (H(\kappa^{++}))^V \tag{6.31}$$

By the closure under κ^{++}-sequences of the direct ordering in $j(\mathbb{R}_\kappa) \backslash \mathbb{R}_\kappa$, combined with the fact that $H(\kappa^{++})$ has size κ^{++} in $M[G_\kappa]$ (by the assumption that $2^{\kappa^+} = \kappa^{++}$ in M), there must be some $t \in H(\kappa^{++})^V$, $p \in G_\kappa$, and $\alpha < \kappa^+$ such that

$$p^\frown r_\alpha \Vdash j(\dot{g})(\dot{a}) = \check{t} \tag{6.32}$$

As such t can be expressed as $j(h)(b)$ for some $h \in V$ and $b \in V$ we obtain

$$p^\frown r_\alpha \Vdash j(\dot{g})(\dot{a}) = j(h)(b) \tag{6.33}$$

which by (6.29) means that $j^*(g)(a) = j^*(h)(b)$. Using the representation in (6.30), we can argue by induction on $H(\kappa^{++})^V$ that $j^*(h)(a) = j(h)(a)$ for each $j(h)(a) \in H(\kappa^{++})^V$, $h, a \in V$. Hence $H(\kappa^{++})^V$ is in M^* as desired.

$H(\kappa^{++})^{V[G_\kappa]}$ is included in M^*. It suffices to argue that G_κ is in M^*. We will argue that there is some $g \in V[G_\kappa]$ such that $G_\kappa \subseteq j^*(g)(\kappa)$ (this implies the desired claim). Let g be defined by $g(\alpha) = G_\kappa \cap \mathbb{R}_\alpha$. Let $p \in G_\kappa$. Clearly there is some α such that

$$p^\frown r_\alpha \Vdash p \in j(\dot{g})(\kappa) \tag{6.34}$$

Since p is in $H(\kappa)$, we can write $p = j(h)(a) = j^*(h)(a)$ for some h, a in V, and so

$$p^\frown r_\alpha \Vdash j(h)(a) \in j(\dot{g})(\kappa) \tag{6.35}$$

which by (6.29) proves the desired claim $p \in j^*(g)(\kappa)$. (Observation 6.27) □

Notice that in Observation 6.27 GCH holds everywhere in a generic extension by $\mathbb{R}$ except at elements of Σ . We may ask if we can achive the same result if we allow an arbitrary toggle-like behaviour of the continuum function on regular cardinals. In Theorem 6.28 we show that is indeed possible, with one small restriction.

Let us set for a toggle-like F: Let Σ be a subclass of κ^{++}-hypermeasurable cardinals, $\Sigma_A = \{\kappa \in \Sigma \,|\, \kappa$ is κ^{++}-hypermeasurable and there is some $j : V \to M$ such that $j(F)(\kappa) \geq F(\kappa)\}$ and $\Sigma_B = \Sigma \setminus \Sigma_A$.

Theorem 6.28 *Assume GCH. Let F be toggle-like and such that*

1. *If $\kappa \in \Sigma$ then $F(\kappa) = \kappa^{++}$.*
2. *If $\alpha = \kappa^+$, where κ is a measurable cardinal, then $F(\alpha) = \alpha^+$.*

Then there is a cardinal-preserving forcing extension realising F where all elements of Σ have cofinality ω (and hence fail SCH).

The proof will be given in a sequence of lemmas. The forcing notion will be of a form $\mathbb{P}*\mathbb{R}*\mathbb{Q}$, where $\mathbb{P}$ will be cofinality-preserving and will realise F on regular cardinals excluding Σ_B, $\mathbb{R}$ will realise F on Σ_B and simultaneously cofinalise elements of Σ_B, and finally $\mathbb{Q}$ will cofinalise elements of Σ_A.

Define $\mathbb{P}$ as $\mathbb{P}^F$ in Definition 5.4 with the following modifications:

1. $\mathbb{P}$ is trivial (empty) at all elements of Σ_B.
2. $\mathbb{P}$ is trivial (empty) whenever $F(\alpha) = \alpha^+$ for a regular α except in the following case when you do force with $\mathrm{Add}(\alpha, \alpha^+)$:
   - $\alpha = \kappa^{++}$ for $\kappa \in \Sigma$.

Lemma 6.29 $\mathbb{P}$ *is cofinality preserving and realises F on all regular cardinals except Σ_B. Moreover all elements of Σ remain κ^{++}-hypermeasurable (as witnessed by embeddings lifted from V) and there is a nice (κ, κ^{++})-system for all elements of Σ_B so that $\mathrm{Prk}_E(\kappa, \kappa^{++})$ can be defined.*

Proof. Let G be a generic filter for $\mathbb{P}$. $\mathbb{P}$ is obviously cofinality-preserving and realises F on all regular cardinals except at Σ_B. We argue that elements of Σ remain κ^{++}-hypermeasurable separately for Σ_A and Σ_B.

Elements of Σ_A: Let $\kappa \in \Sigma_A$ be fixed. In particular let $j : V \to M$ with the critical point κ witness $\kappa \in \Sigma_A$ such that $\kappa^{++} = F(\kappa) \leq j(F)(\kappa) = \kappa^{++}$. We argue as in Theorem 5.7 that j can be lifted to $j^* : V[G] \to M[j^*(G)]$ to preserve measurability of κ (note that this works even in the case when $F(\kappa^{++}) = \kappa^{+3}$ and $j(F)(\kappa^{++}) = \kappa^{+4M}$ as we do force at κ^{++} in V in this case). Moreover, since κ is measurable in both V and M, the forcings $\mathbb{P}$ and

$j(\mathbb{P})$ are trivial at κ^+, and so $H(\kappa^{++})$ of $V[G]$ is included in $M[j^*(G)]$, i.e. κ is still κ^{++}-hypermeasurable in $V[G]$.

Elements of Σ_B: Let $\kappa \in \Sigma_B$ be fixed. Let $j : V \to M$ witness the fact that $\kappa \in \Sigma_B$, in particular $\kappa^+ = j(F)(\kappa) < F(\kappa) = \kappa^{++}$. By the definition of $\mathbb{P}$, $\mathbb{P}$ is empty at κ and κ^+ in V, and also $j(\mathbb{P})$ is empty at κ and κ^+ in M (κ is measurable in M). It follows we can lift as in the argument for Σ_A. Arguing as in Lemma 6.23, at the end of Case 2, we also conclude that the conditions (1) to (5) on page 63 hold in $V[G]$ and so in particular there is a nice (κ, κ^{++}) system in $V[G]$ which allows for the correct definition of $\mathrm{Prk}_E(\kappa, \kappa^{++})$. (Lemma 6.29) □

Let us define $\mathbb{R}$ in $V[G]$ as the Easton-supported iteration of the extender based Prikry forcing $\mathrm{Prk}_E(\kappa, \kappa^{++})$ for $\kappa \in \Sigma_B$. We know that this forcing satisfies the Prikry property and preserves cardinals (see Lemma 6.25).

Lemma 6.30 *$\mathbb{R}$ applied applied over $V[G]$ realises F on elements of Σ_B and simultaneously cofinalises them to the cofinality ω. In particular F is realised in $V[G]^{\mathbb{R}}$.*

Proof. It suffices to show that if κ is a limit point of Σ_B then $\mathbb{R}_\kappa$ forces that there is a nice system for (κ, κ^{++}) so that $\mathrm{Prk}_E(\kappa, \kappa^{++})$ can be defined. Let us fix a $j^* : V[G] \to M[j^*(G)]$ witnessing for κ^{++}-hypermeasurability of κ in $V[G]$. The existence of a nice system for (κ, κ^{++}) in $V[G]^{\mathbb{R}_\kappa}$ follows exactly as in [15], see a brief review here in Observation 6.27, since $2^\kappa = \kappa^+$ in $V[G]$, $2^{\kappa^+} = \kappa^{++}$ in $M[j^*(G)]$, and $j(\mathbb{R}_\kappa)$ is trivial at κ since $j(F)(\kappa) < \kappa^{++}$. (Lemma 6.30) □

Let H be a generic for $\mathbb{R}$. Let $\mathbb{Q}$ be the full support iteration of the simple Prikry forcings $\mathrm{Prk}(\kappa)$ defined in $V[G * H]$ with the domain Σ_A.

Lemma 6.31 *$\mathbb{Q}$ applied in $V[G * H]$ preserves cardinals and cofinalises all elements of Σ_A.*

Proof. Exactly as in Theorem 6.6, proof 1. (Lemma 6.31) □

This concludes the proof of the theorem. (Theorem 6.28) □

Recall Remark 6.26, item (3), where we suggested that perhaps the ordering $\leq_E$ needs to be only κ^+-directed in order to define $\mathrm{Prk}_E(\kappa, \kappa^{++})$. This leads to the following conjecture.

Conjecture 6.32 *Assume GCH. Let F be toggle-like and Σ a subclass of κ^{++}-hypermeasurable cardinals. Assume that*

- *If $\kappa \in \Sigma$ then $F(\kappa) = \kappa^{++}$.*

Then there is a cardinal-preserving forcing extension realising F where all elements of Σ have cofinality ω (and hence fail SCH).

References

[1] James Cummings. Iterated forcing and elementary embeddings. To be included in the Handbook of Set Theory.

[2] James Cummings. A model where GCH holds at successors but fails at limits. *Transactions of the American Mathematical Society*, 329(1), 1992.

[3] James Cummings. Strong ultrapowers and long core models. *The Journal of Symbolic Logic*, 58(1), 1993.

[4] William B. Easton. Powers of regular cardinals. *Annals of Mathematical Logic*, 1, 1970.

[5] Sy David Friedman. Topics in set theory: Prikry type forcings. Course notes, winter term 2007, University of Vienna. Available on author's webpage.

[6] Sy David Friedman. *Fine Structure and Class Forcing.* Walter de Gruyter, 2000.

[7] Sy David Friedman and Radek Honzik. Easton's theorem and large cardinals. *Annals of Pure and Applied Logic*, 154(3), 2008.

[8] Sy David Friedman and Pavel Ondrejovič. The internal consistency of the Easton's theorem. To appear in *Annals of Pure and Applied Logic.*

[9] Sy David Friedman and Katherine Thompson. Perfect trees and elementary embeddings. To appear in the *Journal of Symbolic Logic.*

[10] M. Gitik and M. Magidor. *The singular cardinal hypothesis revisited,* In *Set theory of the continuum.* Springer, 1992.

[11] Moti Gitik. Prikry type forcings. To be included in the Handbook of Set Theory.

[12] Moti Gitik. Changing cofinalities and the nonstationary ideal. *Isreal Journal of Mathematics*, 56(3), 1986.

[13] Moti Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Annals of Pure and Applied Logic*, 43(3), 1989.

[14] Moti Gitik. On measurable cardinals violating the continuum hypothesis. *Annals of Pure and Applied Logic*, 63, 1993.

[15] Moti Gitik and Saharon Shelah. On certain indestructibility of strong cardinals and a question of Hajnal. *Archive for Mathematical Logic*, 28, 1989.

[16] Joel David Hamkins. Fragile measurability. *The Journal of Symbolic Logic*, 59(1), 1994.

[17] Joel David Hamkins. Gap forcing. *Israel Journal of Mathematics*, 125, 2001.

[18] Tomáš Jech. *Set Theory.* Springer, 2003.

[19] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19, 1980.

[20] Akihiro Kanamori. *The Higher Infinite.* Springer, 2003.

[21] Kenneth Kunen. Saturated ideals. *The Journal of Symbolic Logic*, 43(1), 1978.

[22] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs.* North Holland, 1980.

[23] Menachem Magidor. How large is the first strongly compact cardinal? A study on identity crises. *Annals of Mathematical Logic*, 10, 1976.

[24] Telis K. Menas. Consistency results concerning supercompactness. *Transactions of the American Mathematical Society*, 223, 1976.

[25] K.L. Prikry. Changing measurable into accessible cardinals. *Dissertationes Math. Rozprawy Mat.*, 68, 1970.