Charles University in Prague
Faculty of Mathematics and Physics
Department of Mathematical Analysis

Barbora Batíková

COMPLETION OF SEMI-UNIFORM SPACES AND QUASI-TOPOLOGICAL GROUPS

(Doctoral Thesis)

Supervisor: Prof. RNDr. Miroslav Hušek, DrSc.
Branch: M 2 Geometry, Topology, Global Analysis and General Structures

Acknowledgement. I would like to express my gratitude to my supervisor, Prof. RNDr. Miroslav Hušek, DrSc., for his helpful sugestions and remarks considering the subject of the thesis. I want to thank him for his continuous support, encouragement and patience in all the time of writing of the work.

Prohlášení. Prohlašuji, že jsem tuto práci vypracovala samostatně a že jsem použila pouze citované zdroje.
V Praze dne Barbora Batíková

Contents

1 INTRODUCTION **4**
1.1 Semi-uniform and closure spaces . . . 4
1.2 Quasi-uniform spaces . . . 4
1.3 Paratopological groups . . . 6
1.4 T-semi-uniform spaces and quasi-topological groups . . . 6

2 T-SEMI-UNIFORM SPACES **8**
2.1 Basic properties . . . 8
2.2 Categories of semi-uniform spaces . . . 10
2.3 Initial and final t-semi-uniformities . . . 11
2.4 Special t-semi-uniformities on symmetric topological spaces . . . 15
2.4.1 Pointwise open and pointwise closed t-semi-uniformities . . . 15
2.4.2 Fine t-semi-uniformity . . . 16
2.4.3 Coarse t-semi-uniformity . . . 17
2.4.4 Cover t-semi-uniformity . . . 19
2.4.5 Categorical summary . . . 21

3 COMPLETENESS OF T-SEMI-UNIFORM SPACES **21**
3.1 Basic properties . . . 22
3.2 Convergence and Cauchy-like property . . . 23
3.3 Cauchy-like filters and clusters . . . 24
3.4 $\mathcal{L}$-filters . . . 26
3.5 Completeness . . . 27
3.6 Completeness of special t-semi-uniformities . . . 27
3.7 Absolute closedness and (P5) . . . 29
3.8 Categorical approach . . . 31
3.9 Emreflective subcategories . . . 32

4 Completion of t-semi-uniform spaces **33**
4.1 Minimal completion of t-semi-uniform spaces . . . 35
4.2 Maximal completion of t-semi-uniform spaces, weak reflection . . . 35

5 COMPLETION OF QUASI-TOPOLOGICAL GROUPS **36**
5.1 T-semi-uniformities of quasi-topological groups . . . 36
5.2 Initial t-semi-uniformities on groups . . . 38
5.3 Categories of Hausdorff complete groups . . . 39
5.4 Completion of quasi-topological groups . . . 39

1 INTRODUCTION

1.1 Semi-uniform and closure spaces

The most of basic definitions considering semi-uniform and closure spaces used in my thesis come from the monograph [3] by Eduard Čech, revised by Zdeněk Frolík and Miroslav Katětov.

Semi-uniformities are one of possible ways of gereralizing uniformities. They, contrary to uniformities, need not satisfy the "triangular inequality", and the induced closures need not be topological.

In [3] semi-uniform spaces are defined, and then described in terms of uniformly continuous semi-pseudometrics.

The relationship between semi-uniformities and the induced closures is examined. Many interesting properties of semi-uniformities are set, e.g. that a semi-uniform space is semi-pseudometrizable if and only if it has a countable base, or that a closure space is semi-uniformizable if and only if it is symmetric.

The concept of uniformly continuous mappings for semi-uniform spaces, and continuous homomorphisms for topological groups is given there.

Initial and final structures are defined and explored, here they are called projectively and inductively generated structures, namely subspaces, products, sums, quotients, suprema and infima of semi-uniformities or closures.

The category of topological spaces is shown to be bireflective subcategory of closure spaces, the bireflection is called topological or t-modification, the category of uniform spaces is shown to be bireflective subcategory of semi-uniform spaces, the bireflection is called uniform modification.

In Appendix of [3] completeness and compactness of uniform spaces and relationship between them are explored. Completeness is defined in terms of extension of uniformly continous mappings on dense subspaces. The equivalent conditions with Cauchy filters and ultrafilters are given. Cauchyness is proved to be invariant under uniformly continuous images. The category of complete uniform spaces is closed under the formation of products and closed subspaces.

Finally, every uniform space has a completion (complete uniform space, in which it is embedded).

Several results from uniform spaces are applied to topological groups, whose topology is induced by several uniformities. Every topological group is shown to have a group completion (complete in its two-sided uniformity group, in which it is embedded).

1.2 Quasi-uniform spaces

Another concept of generalized uniformities are quasi-uniformities, see e.g. [16]. They, contrary to uniformities, need not satisfy the symmetry condition.

Many basic facts about the category of quasi-uniform spaces and quasi-uniformly continuous maps are contained in the monograph [9] of P. Fletcher and W. F. Lindgren.

In the last three decades many results in the area of quasi-uniformities and quasi-pseudometrics were presented, many of them concerning extensions and completions of quasi-uniform spaces. Most of them can be found in the survey paper [16] by H. P. A. Künzi, see .

There are several types of completions of quasi-uniform spaces:

For the concept of standard completion, so called bicompletion, see e.g. [9]:

If $\mathcal{U}$ is a quasi-uniformity on X then so is the filter $\mathcal{U}^{-1} = \{U^{-1}, U \in \mathcal{U}\}$ (the conjugate quasi-uniformity of $\mathcal{U}$), and $\mathcal{U} \cup \mathcal{U}^{-1}$ is a subbase of a uniformity, which is the coarsest uniformity finer that $\mathcal{U}$, and is used to be denoted by $\mathcal{U}^s$. If this uniformity is complete the quasi-uniform space X is called bicomplete. Each T_0-space has (up to isomorphism) a T_0-bicompletion, in which it is embedded. Futhermore quasi-uniform mappings to other bicomplete spaces can be (uniquely) extended onto the bicompletions.

Some of the definitions, that we use in the thesis, are already known under other names:

Filters that we call semi-Cauchy are called PS (that is, Pervin-Sieber)-Cauchy, see e.g. [9], or $\mathcal{U}$-Cauchy, see e.g. [8], and a quasi-uniform space is called PS-complete (or complete) provided that each PS-Cauchy filter has a cluster point. For locally symmetric quasi-uniform spaces this is equvalent to the condition that each PS-Cauchy filter converges. This property is called convergence completeness in [9].

Filters that we call Cauchy are called left K-Cauchy, see e.g. [16]. A quasi-uniformity is called left K-complete provided that each left K-Cauchy filter converges.

Relationships between various completions were examined, we recall some of them, which can be found e.g. in [16]:

A quasi-uniform space is compact if and only if it is precompact and left K-complete.

A regular left K-complete quasi-pseudometric space is a Baire space.

PS-completeness implies left K-completeness.

It seems unlikely that a simple theory of K-comletion exists, since in general convergent filters are not left K-Cauchy. The same holds in semi-uniformities. Every convergent filter is semi-Cauchy, but it need not be Cauchy.

M. B. Smyth, see [24], endowed a quasi-uniform space $(X,\mathcal{U})$ with a topology τ that is not necessarily the topology induced by $\mathcal{U}$, but is linked to $\mathcal{U}$ by some additional axioms. He defined Cauchy (so called S-Cauchy) filters on such (so called topological quasi-uniform) spaces and called them complete (S-complete) if every round S-Cauchy filter is a τ-neihgborhood filter of a (unique) point. The idea of S(Smyth)-completion of topological quasi-uniforms T_0-spaces yields a satisfactory theory from the categorical point of view.

D. Doitchinov investigated a D(Doitchinov)-completion: A filter $\mathcal{G}$ is called D-Cauchy on a quasi-uniform if there exists a co-filter $\mathcal{F}$ on X (that is, if for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ that $F \times G \subset U$). Then $(\mathcal{F}, \mathcal{G})$ is called a Cauchy filter pair.

A quasi-uniformity is called D-complete provided each D-Cauchy filter converges.

Dotchinov showed, see [5], that each quiet quasi-uniform T_0-space has a standard D-completion. Each quasi-uniformly continuous map into an arbitrary quiet D-complete quasi-uniform T_0-space has a unique quasi-uniformly continuous extension.

Quietness is a rather restrictive property, thus other classes of quasi-uniform spaces that have canonical D-complete extensions were considered.

A filter $\mathcal{F}$ on a quasi-uniform space $(X.\mathcal{U})$ is called stable if $\bigcap\limits_{F\in\mathcal{F}} U(F) \in \mathcal{F}$ whenever $U \in \mathcal{U}$.

A quasi-uniform space is called stable provided each D-Cauchy filter is stable, see [6]. Doitchinov constructed a canonical D-complete extension for an arbitrary stable quasi-uniform T_0-space.

Stability is a strong property for quasi-pseudometrizable quasi-uniform spaces, see [11]. There the following can be found among other things:

Each completely regular pseudocompact stable quasi-pseudometric space is compact, and for a stable quasi-pseudometric space the following conditions are equivalent: separable, ccc, weakly Lindelöf and pseudo-$\aleph_1$*-compact.*

In the literature several modifications of completeness properties were considered where (co-)stable filters were replaced by weakly Cauchy filters, which means something else than our weak Cauchy filters (a filter $\mathcal{F}$ is weakly Cauchy on a quasi-uniform space $(X,\mathcal{U})$ provided $\bigcap\limits_{F\in\mathcal{F}} U^{-1}(F) \neq \emptyset$, for each $U \in \mathcal{U}$).

Authors also created further completeness properties by using supconvergence instead of ordinary convergence, see e.g. [20] or [23]. Other autors worked under weakened forms of quietness, see e.g. [4] or [22].

1.3 Paratopological groups

The concept of topological groups is connected with uniformities as well as paratopological groups are connected with quasi-uniformities. Most about the research on paratopological groups can be found in the survey paper [15] of H. P. A. Künzi.

A paratopological group is a group equipped with a topology such that its group operation (multiplication) is continuous.

Clearly, the topology of a paratopological group is induced by several (e.g. two-sided or right) quasi-uniformities, and its topology has a conjugate topology that is defined by algebraic inversion. The associated supremum topology gives rise to a topological group.

Generalizing many results from the theory of topological groups, J. Marín and S. Romaguera conducted a systematic study of quasi-uniformities on paratopological groups. In [19] they proved for instance the following theorem:

The ground set of the bicompletion of the two-sided quasi-uniformity of a paratopological (T_0-)group carries the structure of a paratopological group. Moreover the quasi-uniformity of that bicompletion yields the two-sided quasi-uniformity of the constructed paratopological group.

Other concepts of completion, e.g. D-completion of the two-sided quasi-uniformity of paratopological groups, were examined, too, see [17]:

The two-sided quasi-uniformity of a regular paratopological group is quiet. The D-completion of the two-sided quasi-uniformity of an abelian regular paratopological group can be considered an abelian paratopological group.

The research about quasi-uniformities on homeomorphism groups were conducted, see e.g. [14], where H. P. A. Künzi proved the following theorem:

If $(X,\mathcal{U})$ is a quasi-uniform space and $\mathcal{Q}(X)$ is the group of all quasi-uniform self-isomorphisms of $(X,\mathcal{U})$ equipped with the quasi-uniformity $\mathcal{Q}_{\mathcal{U}}$ of quasi-uniform convergence, then $\mathcal{Q}_{\mathcal{U}}$ is the right quasi-uniformity of the paratopological group $(\mathcal{Q}(X),\circ,\tau(\mathcal{Q}_{\mathcal{U}}))$; its two-sided quasi-uniformity is bicomplete if $\mathcal{U}$ is bicomplete.

1.4 T-semi-uniform spaces and quasi-topological groups

If a semi-uniformity is a uniformity then the induced closure is topological, but the condition is not necessary. This fact gives rise to the concept of t-semi-uniform spaces, which are precisely those semi-uniform spaces whose closures are topological, and are dealt with in the thesis.

We study completion of t-semi-uniform spaces. Completion is defined in terms of Cauchy filters. A t-semi-uniform space is called complete provided every Cauchy filter converges. We examine several types of Cauchy-like properties, which are equivalent in uniform spaces. All the Cauchy-like properties, contrary to the situation in quasi-uniformities, give a satisfactory theory from the categorical point of view:

The category of Hausdorff complete semi-uniform spaces is shown to be epireflective subcategory of Hausdorff semi-uniform spaces. But there is no reflective subcategory of Hausdorff semi-uniform spaces, in which all reflection arrows are embeddings.

On the other hand if we omit the T_1-property, the situation is not the same for all the Cauchy-like properties:

The category of all complete t-semi-uniform spaces is an almost reflective subcategory of all semi-uniform spaces, for some of used Cauchy-like properties. All the almost reflection maps can be chosen to be embeddings.

The same is true for T_1-spaces.

For the other Cauchy-like properties we show examples of spaces that do not have completion, in which they are embedded.

In the last sections of the thesis we study quasi-topological groups, which are another concept of generalized topological groups. Their topologies are induced by semi-uniformities.

We use the theory of Hausdorff complete semi-uniform spaces and show that Hausdorff complete in their two-sided semi-uniformity quasi-topological groups form an epireflective subcategory of Hausdorff quasi-topological groups. We show that reflection arrows need not be embeddings.

For several types of Cauchy-like properties we show examples of quasi-topological groups that cannot be embedded into a group that is complete in its two-sided semi-uniformity.

Unfortunately, the construction of an almost reflection in complete spaces (for the other Cauchy-like properties) for non-Hausdorff semi-uniformities cannot be used in quasi-topological groups, and we still do not know whether there is another possibility to construct group completions with embeddings for non-Hausdorff quasi-topological groups.

2 T-SEMI-UNIFORM SPACES

2.1 Basic properties

The definitions of semi-uniform and uniform spaces can be found in [3]. It was shown there that a semi-uniformity on a set induces a closure and a uniformity induces a topology. But it is not necessary to suppose that the explored semi-uniformity is a uniformity in order to induce a topology. We will describe the semi-uniformities inducing topologies and call them t-semi-uniformities. The t-semi-uniformities will be the main subject of our investigation in this work. The t-semi-uniformities together with uniformly continuous maps form a category `TSUnif` of t-semi-uniform spaces. It is a concrete category over the category `Top` of topological spaces and it is a bireflective subcategory of the category `SUnif` of all semi-uniform spaces (and contains `Unif` as a bireflective subcategory).

First let us recall the definitions of a semi-uniformity and a uniformity and add the definition of a t-semi-uniformity.

Let $\mathcal{U}$ be a set of some relations on a set X. Denote

$$U[x] = \{y; (x, y) \in U\} \text{ for } x \in X, U \in \mathcal{U},$$

$$U[A] = \bigcup_{x\in A} U[x] \text{ for } A \subset X, U \in \mathcal{U},$$

$$\mathcal{U}_x = \{U[x]; U \in \mathcal{U}\} \text{ for } x \in X,$$

$$U^{-1} = \{(x, y); (y, x) \in U\},$$

$$U^2 = \{(x, y); (x, z), (z, y) \in U \text{ for some } z \in X\},$$

$$\mathcal{P}(X) = \{A; A \subset X\} \ .$$

Definition 2.1 A nonvoid set $\mathcal{U} \subset \mathcal{P}(X \times X)$ is called *a semi-uniformity* on a set X if

(i) $(\forall U \in \mathcal{U})(\Delta_X \subset U)$, where $\Delta_X = \{(x, x); x \in X\}$,

(ii) $(\forall U, V \in \mathcal{U})(U \cap V \in \mathcal{U})$,

(iii) $(\forall U \in \mathcal{U})(\forall V \in \mathcal{P}(X \times X))(U \subset V \Rightarrow V \in \mathcal{U})$,

(iv) $(\forall U \in \mathcal{U})(U^{-1} \in \mathcal{U})$.

The couple $(X,\mathcal{U})$ is called *a semi-uniform space.*

If moreover:

(v) $(\forall U \in \mathcal{U})(\forall x \in X)(\exists V \in \mathcal{U})(\forall y \in V[x])(U[x] \in \mathcal{U}_y)$

then $\mathcal{U}$ is called *t-semi-uniformity* on the set X and the couple $(X,\mathcal{U})$ is called *a t-semi-uniform space*, where the letter *t* stands for *topological.*

If in addition to (i)-(iv)

(vi) $(\forall U \in \mathcal{U})(\exists V \in \mathcal{U})(V^2 \subset U)$

then $\mathcal{U}$ is called a *uniformity* on the set X and the couple $(X,\mathcal{U})$ is called a *a uniform space.*

We shall now prove some basic properties of t-semi-uniformities and recall properties of semi-uniformities from [3]. Examples of various t-semi-uniformities are given in Subsection 2.4.

The next assertion is trivial.

Proposition 2.2 *Every uniformity on X is a t-semi-uniformity on X.*

`Proof:` We will show that: (vi) $\Rightarrow$ (v). Let $U \in \mathcal{U}, x \in X$. If (vi) holds then there exists $V \in \mathcal{U}$ such that $V^2 \subset U$. For $y \in V[x], z \in V[y]$ one has $z \in U[x]$. $V[y]$ is a subset of $U[x]$ thus $U[x] \in \mathcal{U}_y$.

◇

Definition 2.3 Let $(X,\mathcal{U})$ be a semi-uniform space. A set $\mathcal{B} \subset \mathcal{U}$ is called *a base* of the semi-uniformity $\mathcal{U}$ if for each set $U \in \mathcal{U}$ there exists a set $B \in \mathcal{B}$ such that $B \subset U$.

A set $\mathcal{S} \subset \mathcal{U}$ is called *a subbase* of the semi-uniformity $\mathcal{U}$ if for each set $U \in \mathcal{U}$ there exists a finite subset $\mathcal{T} \subset \mathcal{S}$ such that $\bigcap \mathcal{T} \subset U$.

Clearly, the system of symmetric elements of a semi-uniformity $\mathcal{U}$ is its base. The next two results can be found in [3].

Proposition 2.4 *Let X be a set and $\mathcal{B} \subset \mathcal{P}(X \times X), \mathcal{B} \neq \emptyset$. Then $\mathcal{B}$ is a base for a semi-uniformity on X if and only if:*

(a) $(\forall B \in \mathcal{B})(\Delta_X \subset B)$,

(b) $(\forall B, C \in \mathcal{B})(\exists D \in \mathcal{B})(D \subset B \cap C)$,

(c) $(\forall B \in \mathcal{B})(\exists C \in \mathcal{B})(C \subset B^{-1})$.

Corollary 2.5 *Every system of reflexive and symmetric relations on a set X (i.e., of symmetric subsets of $X \times X$ containing the diagonal of X) is a subbase for a semi-uniformity.*

Proposition 2.6 *Let X be a set and $\mathcal{B} \subset \mathcal{P}(X \times X), \mathcal{B} \neq \emptyset$. Then $\mathcal{B}$ is a base for a uniformity on X if and only if*

(a) $(\forall B \in \mathcal{B})(\Delta_X \subset B)$,

(b) $(\forall B, C \in \mathcal{B})(\exists D \in \mathcal{B})(D \subset B \cap C)$,

(c) $(\forall B \in \mathcal{B})(\exists C \in \mathcal{B})(C \subset B^{-1})$,

(d) $(\forall B \in \mathcal{B})(\exists C \in \mathcal{B})(C^2 \subset B)$.

For t-semi-uniformities, the assertion about bases is similar:

Proposition 2.7 *Let X be a set and $\mathcal{B} \subset \mathcal{P}(X \times X), \mathcal{B} \neq \emptyset$. Then $\mathcal{B}$ is a base for a t-semi-uniformity on X if and only if*

(a) $(\forall B \in \mathcal{B})(\Delta_X \subset B)$,

(b) $(\forall B, C \in \mathcal{B})(\exists D \in \mathcal{B})(D \subset B \cap C)$,

(c) $(\forall B \in \mathcal{B})(\exists C \in \mathcal{B})(C \subset B^{-1})$,

(d) $(\forall B \in \mathcal{B})(\forall x \in X)(\exists C \in \mathcal{B})(\forall y \in C[x])(B[x] \in \mathcal{U}_y)$.

As shown in [3], every semi-uniformity $\mathcal{U}$ on a set X induces a closure operator u on X:

$$u(A) = \overline{A}^{\mathcal{U}} = \bigcap \mathcal{U}[A], \quad \text{for } A \subset X\,.$$

Equivalently, the neighborhood system at $x \in X$ coincides with $\mathcal{U}_x$. Clearly, one can use a base $\mathcal{B}$ instead of $\mathcal{U}$ in the description of u (for instance, all symmetric members of $\mathcal{U}$).

Theorem 2.8 *A semi-uniform space $(X,\mathcal{U})$ is t-semi-uniform if and only if the closure in the semi-uniformity $\mathcal{U}$ is topological, i.e. it satisfies the condition*

$$(*) \qquad (\forall A \subset X)(\overline{(\overline{A}^{\mathcal{U}})}^{\mathcal{U}} = \overline{A}^{\mathcal{U}})\,.$$

`Proof:` Suppose first that a semi-uniform space $(X,\mathcal{U})$ is topological. Let $x \in \overline{(\overline{A}^{\mathcal{U}})}^{\mathcal{U}}$. We will show that $x \in \overline{A}^{\mathcal{U}}$, i.e., there exists a point $z \in U[x] \cap A$, for each symmetric $U \in \mathcal{U}$.

Choose an arbitrary symmetric set $U \in \mathcal{U}$.

Since $\mathcal{U}$ satisfies (v) in the Definition 2.1 of the t-semi-uniformity there exists $V \in \mathcal{U}$ such that $(\forall y \in V[x])(\exists W_y \in \mathcal{U})(W_y[y] \subset U[x])$. We can assume all W_y to be symmetric.

As $x \in \overline{(\overline{A}^{\mathcal{U}})}^{\mathcal{U}}$ there must exist $y_0 \in V[x] \cap \overline{A}^{\mathcal{U}}$, which means that for each symmetric $W' \in \mathcal{U}$ there exists a point $z \in W'[y_0] \cap A$.

Take $W' = W_{y_0}$. Then there exists a point $z \in W_{y_0}[y_0] \cap A \subset U[x] \cap A$.

The converse implication:

For the contrary suppose that a semi-uniform space $(X, \mathcal{U})$ satisfies $(*)$ and is not topological. If $(X, \mathcal{U})$ is not topological then there is such a set $U \in \mathcal{U}$ and $x \in X$ that for each symmetric $V \in \mathcal{U}, V \subset U$ there exists $y_V \in V[x]$ (equivalently $x \in V[y_V]$) such that $U[x] \notin \mathcal{U}_{y_V}$, which means that for each symmetric $W \in \mathcal{U}$ there is $z \in W[y_V] - U[x]$. Thus $y_V \in W[z]$ (because W is symmetric) for some $z \in X \setminus U[x]$, for every $W \in \mathcal{U}$. That means that $y_V \in \bigcap_{W \in \mathcal{U}} W[X - U[x]]$.

Recall that $x \in V[y_V]$, for all V, because V are symmetric.

Thus $x \in \bigcap_{V \in \mathcal{U}} V[\bigcap_{W \in \mathcal{U}} W[X - U[x]] = \overline{\overline{X - U[x]}^{\mathcal{U}}}^{\mathcal{U}}$. The equality $(*)$ implies that $x \in \overline{X - U[x]}^{\mathcal{U}}$ which means that there is $y' \in U[x] \cap (X - U[x]) = \emptyset$. ◇

The topology induced by a t-semi-uniformity $\mathcal{U}$ is denoted by $\tau_{\mathcal{U}}$.

The previous proof could be given by means of a characterization of topological spaces among the closure ones by means of neighborhoods which is, in fact, described in the property (v).

Recall from [3] that a closure space X is semi-uniformizable (i.e., there is a semi-uniformity on the set X inducing the closure of X) iff it is *symmetric*, i.e. $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$, for every couple of points $x, y \in X$. We shall repeat the assertion for t-semi-uniformities.

Proposition 2.9 *A topological space is semi-uniformizable iff it is symmetric.*

Uniformly continuous mappings Let us recall the definition of a uniformly continuous mapping from [3]:

Definition 2.10 Let $(X, \mathcal{U}), (Y, \mathcal{V})$ be semi-uniform spaces. A mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called *uniformly continuous* if for each set $V \in \mathcal{V}$ there exists a set $U \in \mathcal{U}$ such that $(x, y) \in U \Rightarrow (f(x), f(y)) \in V$, for any couple of points $x, y \in X$.

By [3] any composite of two uniformly continuous mappings is uniformly continuous. Another result in [3] asserts that every uniformly continuous mapping is continuous between the corresponding induced closures. For the sake of completeness we shall repeat it for our t-semi-uniform spaces.

Proposition 2.11 *If a mapping f of a t-semi-uniform space $(X, \mathcal{U})$ into a t-semi-uniform space $(Y, \mathcal{V})$ is uniformly continuous then f is continuous with respect to the topologies $\tau_{\mathcal{U}}, \tau_{\mathcal{V}}$.*

Definition 2.12 Semi-uniform spaces $(X, \mathcal{U}), (Y, \mathcal{V})$ are called *uniformly equivalent (isomorphic)* if there exists a bijective mapping $e : (X, \mathcal{U}) \to (Y, \mathcal{V})$ such that both e and its inverse e^{-1} are uniformly continuous. The mapping e is then called a *uniform equivalence (isomorphism).*

A uniformly continuous mapping $e : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called a *uniform embedding* if the restriction $e : X \to e(X)$ is a uniform equivalence.

2.2 Categories of semi-uniform spaces

Let us denote by `Top` the category whose objects are topological spaces and morphisms are continuous mappings, `Clos` the category whose objects are closure spaces and morphisms are continuous mappings, `Unif` the category whose objects are uniform spaces and morphisms are uniformly continuous mappings, `TSUnif` the category whose objects are t-semi-uniform spaces and morphisms are uniformly continuous mappings, `SUnif` the category whose objects are semi-uniform spaces and morphisms are uniformly continuous mappings.

Theorem 2.13 *We have the following diagram, where all the horizontal arrows denote full bireflective embedding functors and vertical arrows denote forgetful functors assigning $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ to $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$.*

$$\begin{array}{ccccc} \mathtt{Unif} & \longrightarrow & \mathtt{TSUnif} & \longrightarrow & \mathtt{SUnif} \\ \downarrow & & \downarrow & & \downarrow \\ \mathtt{Top} & = & \mathtt{Top} & \longrightarrow & \mathtt{Clos} \end{array}$$

`Proof:` The facts about vertical arrows follow from Proposition 2.11 and the comment before it. The definition of our categories gives directly that all the horizontal arrows are full embeddings. It remains to prove the bireflectivity, which is done in [3] for `Unif` $\to$ `SUnif` and `Top` $\to$ `Clos`. We must prove that `TSUnif` is bireflective in `SUnif`.

Take a semi-uniform space $(X, \mathcal{U})$ and define a sequence $t_\alpha \mathcal{U}$ for ordinals $\alpha < (2^{2^{|X|}})^+$ in the following way:

Let $t_1 \mathcal{U}$ be a semi-uniformity with the base
$\{U = U^{-1} \in \mathcal{U}; (\forall x \in X)(\exists V = V^{-1} \in \mathcal{U})(\forall y \in V[x] : U[x] \in \mathcal{U}_y)\}$,
$t_{\alpha+1}\mathcal{U} = t_1(t_\alpha \mathcal{U})$,
$t_\alpha \mathcal{U} = \bigcap\limits_{\beta<\alpha} t_\beta \mathcal{U}$ for a limit α.
There exists such an κ that $t_{\kappa+1}\mathcal{U} = t_\kappa \mathcal{U}$ because there are at most $2^{2^{|X|}}$ semi-uniformities on the set X.

If $\mathcal{U}$ is a semi-uniformity then so is $t_1\mathcal{U}$. The conditions $(i), (iii), (iv)$ in Definition 2.1 of the semi-uniformity $t_1\mathcal{U}$ are clear. We must prove (ii). Let $U, V \in t_1\mathcal{U}, x \in X$. Then there are $U_1, V_1 \in \mathcal{U}$ such that $U[x] \in \mathcal{U}_y$ for every $y \in U_1[x]$, $V[x] \in \mathcal{U}_y$ for every $y \in V_1[x]$. Thus $U[x] \cap V[x] \in \mathcal{U}_y$ for every $y \in U_1[x] \cap V_1[x] = (U1 \cap U2)[x]$ and $U \cap V \in t_1\mathcal{U}$.

Clearly $t_\alpha \mathcal{U}$ is a semi-uniformity for every limit α and $t\mathcal{U} = t_\kappa \mathcal{U}$ is a t-semi-uniformity.
It is a maximal t-semi-uniformity contained in $\mathcal{U}$. Every uniformly continuous map f from $(X, \mathcal{U})$ into a t-semi-uniform space $(Y, \mathcal{V})$ is uniformly continuous on $(X, t\mathcal{U})$.

Thus $(X, t\mathcal{U})$ is a bireflection of $(X, \mathcal{U})$ in t-semi-uniform spaces. ◇

Instead of bireflection in `TSUnif` of a semi-uniformity $\mathcal{U}$ we often speak about *t-modification* of $\mathcal{U}$.

Corollary 2.14 *The category* `TSUnif` *is complete and hereditary in* `SUnif`*.*

Hereditariness means closedness under taking subspaces (see the next subsection).

2.3 Initial and final t-semi-uniformities

As we said in the previous part the category `TSUnif` is a bireflective subcategory of the category `SUnif`. Thus it is initially closed in `SUnif` and also finally complete (the finally generated structure in `TSUnif` is the bireflection in `TSUnif` of the finally generated structure in `SUnif`). Example 2.27 shows that `TSUnif` is not closed under the formation of quotients in `SUnif`. For readers' convenience we describe initial and final t-semi-uniformities.

Initial t-semi-uniformities The family of all t-semi-uniformities on a set X is ordered by the inclusion $\subset$. Instead of being smaller or bigger for semi-uniformities or t-semi-uniformities, one often says *coarser* and *finer*.

Proposition 2.15 *Let $(X_i, \mathcal{U}_i)$ be a t-semi-uniform space for each $i \in I$. Let X be a set and $f_i : X \to X_i$ be a map for each $i \in I$. Then there exists a coarsest t-semi-uniformity $\mathcal{U}$ such that the map $f_i : (X, \mathcal{U}) \to (X_i, \mathcal{U}_i)$ is uniformly continuous for each $i \in I$; $\mathcal{U}$ has for its subbase $\mathcal{S} = \{(f_i \times f_i)^{-1}(U_i); i \in I, U_i \in \mathcal{U}_i\}$.*

`Proof:` By Corollary 2.5 the system $\mathcal{U} = \{(f_i \times f_i)^{-1}(U_i); i \in I, U_i \in \mathcal{U}_i\}$ is a subbase for a semi-uniformity $\mathcal{U}$. We shall show that $\mathcal{U}$ is a t-semi-uniformity. Let $U \in \mathcal{U}, x \in X$. Then there is a finite subset $F \subset I$ such that $\bigcap\limits_{i \in F} (f_i \times f_i)^{-1}(U_i) \subset U$. As each $(X_i, \mathcal{U}_i)$ is a t-semi-uniform space there exists

$V_i \in \mathcal{U}_i$ such that $U_i[f_i(x)] \in \mathcal{U}_{iy_i}$ for each $y_i \in V_i[f_i(x)]$. But now $\bigcap_{i\in F}(f_i \times f_i)^{-1}(U_i)[x] \in \mathcal{U}_y$ and $U[x] \in \mathcal{U}_y$ for each $y \in V[x]$ where $V = \bigcap_{i\in F}(f_i \times f_i)^{-1}(V_i)$. Thus the condition (v) in Definition 2.1 is satisfied.

If $\mathcal{V}$ is another t-semi-uniformity on X such that all $f_i : (X, \mathcal{V}) \to (X_i, \mathcal{U}_i)$ are uniformly continuous then each set $(f_i \times f_i)^{-1}(U_i)$ must be in $\mathcal{V}$. Thus $\mathcal{U} \subset \mathcal{V}$. ◇

Definition 2.16 The t-semi-uniformity $\mathcal{U}$ from the previous proposition is called the *initial t-semi-uniformity on X with respect to the maps $f_i : X \to (X_i, \mathcal{U}_i), i \in I$.*

Since t-semi-uniformities are initially closed in semi-uniformities, the next result follows directly from the corresponding result for semi-uniformities and closures.

Proposition 2.17 *Let $(X_i, \mathcal{U}_i)$ be a t-semi-uniform space for each $i \in I$. Let $\mathcal{U}$ be the initial t-semi-uniformity on a set X with respect to the maps $f_i : X \to (X_i, \mathcal{U}_i), i \in I$. Then the topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$ coincides with the initial topology on X with respect to the maps $f_i : X \to (X_i, \tau_{\mathcal{U}_i})$, where $\tau_{\mathcal{U}_i}$ is the topology on X_i induced by $\mathcal{U}_i$ for each $i \in I$.*

The following result is a special case of general characterizations of initial structures formulated for t-semi-uniformities.

Proposition 2.18 (Universal property) *Let $(X_i, \mathcal{U}_i), i \in I$, be a family of t-semi-uniform spaces, $\mathcal{U}$ a t-semi-uniformity on a set X, $f_i : X \to X_i$ a map for each $i \in I$. Then the following statements are equivalent:*

(a) *$\mathcal{U}$ is an initial t-semi-uniformity on X with respect to the maps $f_i : X \to (X_i, \mathcal{U}_i), i \in I$.*

(b) *For each t-semi-uniform space $(Y, \mathcal{V})$ a map $g : (Y, \mathcal{V}) \to (X, \mathcal{U})$ is uniformly continuous iff all the maps $f_i g : (Y, \mathcal{V}) \to (X_i, \mathcal{U}_i), i \in I$, are uniformly continuous.*

Definition 2.19 Let $(X, \mathcal{U})$ be a t-semi-uniform space, $S \subset X$. The initial t-semi-uniformity $\mathcal{U}/S$ on S with respect to the inclusion map $: S \to (X, \mathcal{U}) : x \mapsto x$ is called the *subspace t-semi-uniformity* on S induced by $\mathcal{U}$ and $(S, \mathcal{U}/S)$ a *subspace* of $(X, \mathcal{U})$.

Products of structures are defined categorically. The next result describes products by means of initial structures and follows again from the corresponding result for semi-uniformities.

Proposition 2.20 *Let $(X_i, \mathcal{U}_i), i \in I$, be a family of t-semi-uniform spaces. The initial t-semi-uniformity $\mathcal{U}$ on the cartesian product $X = \prod_{i\in I} X_i$ with respect to the canonical projections $\pi_k : X \to (X_k, \mathcal{U}_k) : (x_i)_{i\in I} \mapsto x_k, k \in I$, is the product t-semi-uniformity on X. Thus $\mathcal{U}$ has for its subbase*

$$\{(\pi_k \times \pi_k)^{-1}(U_k); k \in I, U_k \in \mathcal{U}_k\}.$$

It is known that the set of all semi-uniformities on a set X is a complete lattice when ordered by inclusion. Since t-semi-uniformities are bireflective in `SUnif`, the set of all t-semi-uniformities on a set X is also a complete lattice when ordered by inclusion. Moreover, the suprema of t-semi-uniformities in `TSUnif` are the same as in `SUnif`, because of initial closedness. Again for completeness we describe the suprema here:

Proposition 2.21 *Let X be a set and $(\mathcal{U}_i), i \in I$, be a family of t-semi-uniformities on X. The supremum of the t-semi-uniformities $\mathcal{U}_i, i \in I$, is the initial t-semi-uniformity on X with respect to the identity maps $id_X : X \to (X, \mathcal{U}_i) : x \mapsto x, i \in I$ and has for its subbase the collection $\bigcup_{i\in I} \mathcal{U}_i$.*

Corollary 2.22 *Every symmetric topological space has the finest semi-uniformity inducing it.*

We shall investigate the above finest semi-uniformities in Subsection 2.4.

At he end of this part we want to mention that every bireflective subcategory of `TSUnif` is closed under taking initial structures by maps having codomain in the subcategory. Thus it is productive and hereditary and contains the indiscrete structures (in fact, these properties characterize bireflectivity).

Final t-semi-uniformities Final structures are dual to the initial ones and so we shall summarize needed results and not repeat general details from the previous part. As for the description of final structures in `TSUnif`, they are t-modifications of the final structures constructed in `SUnif`.

Proposition 2.23 *Let* $(X_i,\mathcal{U}_i)$ *be a t-semi-uniform space for each* $i \in I$*. Let* X *be a set and* $f_i : X_i \to X$ *be a map for each* $i \in I$*. Then there exists a finest t-semi-uniformity* $\mathcal{U}$ *such that the map* $f_i : (X_i,\mathcal{U}_i) \to (X,\mathcal{U})$ *is uniformly continuous for each* $i \in I$ *and it coincides with the t-modification of the semi-uniformity* $\{U \subset X \times X; (f_i \times f_i)^{-1}(U) \in \mathcal{U}_i \text{ for each } i\}$.

`Proof:` Let $\mathcal{M}$ be the set of all t-semi-uniformities $\mathcal{U}$ on X such that the map $f_i : (X_i,\mathcal{U}_i) \to (X,\mathcal{U})$ is uniformly continuous for each $i \in I$. Let $\mathcal{U} = \sup_{\mathcal{V}\in M} \mathcal{V}$. As $\mathcal{U}$ satisfies the universal property it is also in $\mathcal{M}$ and it is the finest t-semi-uniformity with this property. ◇

Definition 2.24 The t-semi-uniformity $\mathcal{U}$ from the previous proposition is called the *final t-semi-uniformity on* X *with respect to the maps* $f_i : (X_i,\mathcal{U}_i) \to X, i \in I$.

We will see in Example 2.28 that the corresponding result to Proposition 2.17 is not true; $\tau_{\mathcal{U}}$ is coarser than the final topology τ on the space X with respect to maps $f_i : (X_i, \tau_{\mathcal{U}_i}) \to X, i \in I$ and it may be strictly coarser.

The corresponding dual to Proposition 2.18:

Proposition 2.25 (Universal property) *Let* $(X_i,\mathcal{U}_i), i \in I$*, be a family of t-semi-uniform spaces,* $\mathcal{U}$ *a t-semi-uniformity on a set* X*,* $f_i : X_i \to X$ *a map for each* $i \in I$*. Then the following statements are equivalent:*

(a) $\mathcal{U}$ *is a final t-semi-uniformity on* X *with respect to the maps* $f_i : (X_i,\mathcal{U}_i) \to X, i \in I$.

(b) *For each t-semi-uniform space* $(Y,\mathcal{V})$ *a map* $g : (X,\mathcal{U}) \to (Y,\mathcal{V})$ *is uniformly continuous iff all the maps* $gf_i : (X_i,\mathcal{U}_i) \to (Y,\mathcal{V}), i \in I$*, are uniformly continuous.*

Definition 2.26 Let $(X,\mathcal{U})$ be a t-semi-uniform space and $q : X \to Y$ a map onto a set Y. The final t-semi-uniformity on Y with respect to $q : (X,\mathcal{U}) \to Y$ is called the *quotient* t-semi-uniformity on Y along q.

Example 2.27 (`TSUnif` is not quotient stable in `SUnif`.) Take the real line $\mathbb{R}$ with its standard uniformity which has a base consisting of the sets $U_\varepsilon = \{(x,y) \in \mathbb{R}^2; |x-y| < \varepsilon\}, \varepsilon > 0$, and a quotient mapping $q : \mathbb{R} \to \mathbb{P} \cup \{0\}$ such that $q(x) = 0$ for x rational, $q(x) = x$ for x irrational ($\mathbb{P}$). Let $U'_\varepsilon = (q \times q)(U_\varepsilon)$ for every $\varepsilon > 0$. Then $U'_\varepsilon(0) = \mathbb{P} \cup \{0\}$ for every $\varepsilon > 0$ and $0 \in U'_\varepsilon(x)$ for every x irrational. But none of $U'_\varepsilon(x) \neq \mathbb{P} \cup \{0\}, x \in \mathbb{P}$, is a neighborhood of 0 and so, the quotient in `SUnif` of $\mathbb{R}$ along q is not a t-semi-uniformity.

Example 2.28 (Quotients do not preserve topology in `TSUnif`.) Take a discrete space D and its coarse t-semi-uniformity $\mathcal{U}$ (See the subsection 1.5.3.), which has a subbase formed by sets $S = \{(x,x)\} \cup (D \setminus \{x\}) \times (D \setminus \{x\}), x \in X$. Let q be a mapping D onto a set X that is not finite-to-one. Let $\{y\} = q(A)$ be a singleton for some infinite set $A \subset D$ such that $D \setminus A$ is infinite. The quotient topology on X is clearly discrete. If this topology is induced by a t-semi-uniformity $\mathcal{V}$ on X then $V = \{(y,y)\} \cup (X \setminus \{y\}) \times (X \setminus \{y\}) \in \mathcal{V}$. But $(q \times q)^{-1}(V) = A \times A \cup (D \setminus A) \times (D \setminus A)$ because $A = q^{-1}(y)$. As $D \setminus A$ is infinite V is not in $\mathcal{U}$.

We shall now describe disjoint sums (coproducts) in `STUnif`.

Proposition 2.29 *Let* $(X_i,\mathcal{U}_i), i \in I$*, be a family of t-semi-uniform spaces. The final t-semi-uniformity* $\mathcal{U}$ *on the disjoint sum* $X = \sum_{i\in I} X_i$ *with respect to the canonical injections* $inj_i : (X_i,\mathcal{U}_i) \to X, i \in I$ *is the coproduct t-semi-uniformity on* X*. Thus* $\mathcal{U}$ *equals to* $\{U \subset X \times X; U \cap (X_i \times X_i) \in \mathcal{U}_i\}$.

It follows directly from the last Proposition that forming disjoint sums of t-semi-uniform spaces is the same in both `SUnif` and `TSUnif`. Also it commutes with topology, i.e., the topology of a disjoint sum of t-semi-uniformities is the disjoint sum of the topologies induced by the corresponding t-semi-uniformities.

Proposition 2.30 *Let X be a set and $\{\mathcal{U}_i\}, i \in I$, a family of t-semi-uniformities on X. The final t-semi-uniformity $\inf\limits_{i\in I} \mathcal{U}_i$ on X with respect to the identity maps $id_X : (X,\mathcal{U}_i) \to X : x \mapsto x, i \in I$, is the infimum of the t-semi-uniformities $\mathcal{U}_i, i \in I$. It is the t-modification of $\bigcap\limits_{i\in I} \mathcal{U}_i$.*

Example 2.31 (`TSUnif` is not infimum stable in `SUnif`.)
Let τ, σ be such symmetric non-indiscrete topologies on an infinite set X that he infimum topology $\tau \wedge \sigma = \tau \cap \sigma$ is the indiscrete topology on X, take for instance the coarse T_1 topology and the topology consisting precisely of the whole space, empty set and an infinite set A and its infinite complement. Take any semi-uniformities $\mathcal{U}_\tau$, resp. $\mathcal{U}_\sigma$, on τ, resp. σ. Neighbourhoods of points $x \in X$ in the infimum semi-uniformity $\mathcal{U}_\tau \wedge \mathcal{U}_\sigma$ are of the form $G \cup H$ where G, H resp., are neighbourhoods of x in τ, σ, resp., for instance a set containing the set A and having an infinite complement is a neighbourhood of a point $a \in A$. But neighbourhoods in $\tau \cap \sigma$ are just X and $\emptyset$.

As one can see from Example 2.31, the category `SUnif` is not infimum stable. But infimum in `TSUnif` preserves topologies:

Proposition 2.32 *The topology induced by the infimum t-semi-uniformity coincides with the infimum topology.*

`Proof:`
Let X be a set and $\{\mathcal{U}_i\}, i \in I$, a family of t-semi-uniformities on X. Then $\mathcal{U} = \inf\limits_{i\in I} \mathcal{U}_i$ is the t-modification of $\bigcap\limits_{i\in I} \mathcal{U}_i$, which is also the supremum of all t-semi-uniformities contained in $\bigcap\limits_{i\in I} \mathcal{U}_i$.

Let $\tau = \bigcap\limits_{i\in I} \tau_i$ be the infimum topology on X, where τ_i are topologies induced by $\mathcal{U}_i$, $i \in I$, $\tau_{\mathcal{U}}$ the topology induced by $\mathcal{U}$.

Denote by $\mathcal{V} \subset \bigcap\limits_{i\in I} \mathcal{U}_i$ a t-semi-uniformity inducing τ. Such a semi-uniformity V exists. In fact, there is some semi-uniformity $\mathcal{W}$ inducing τ because (X, τ) is symmetric. Take $\mathcal{V} = \mathcal{W} \cap (\bigcap\limits_{i\in I} \mathcal{U}_i)$, which is the infimum of $\mathcal{W}$ and $\bigcap\limits_{i\in I} \mathcal{U}_i$ in `SUnif`. It induces τ because infima in `SUnif` preserve closures. It holds $\mathcal{V} \subset \mathcal{U}$, and thus $\tau \subset \tau_{\mathcal{U}}$.

But from the definition of the infimum topology τ it follows that $\tau_{\mathcal{U}} \subset \tau$. Thus $\tau_{\mathcal{U}} = \tau$. $\diamond$

Corollary 2.33 *Let $\mathcal{U}_i$ be a semi-uniformity on a symmetric topological space X, for every $i \in I$. Then $\inf\limits_{i\in I} \mathcal{U}_i$ induces the original topology on X and equals to $\bigcap\limits_{i\in I} \mathcal{U}_i = \{\bigcup\limits_{U_i \in \mathcal{U}_i} U_i\}$.*

Corollary 2.34 *Every symmetric topological space has the coarsest semi-uniformity inducing it.*

Recall that the corresponding assertion does not hold for uniform spaces on completely regular spaces. We shall deal with the above coarsest semi-uniformity on X in Subsection 2.4.

Theorem 2.35 *Let X be a symmetric topological space. Then the set* `TSUnif`(X) *of all semi-uniformities on X is a complete lattice.*

`Proof:` See 2.21 and 2.33. $\diamond$

Unlike the initial case, final generation does not commute with inducing topologies. It is easy to see that the topology of a disjoint sum of t-semi-uniformities is a disjoint sum (in `Top`) of the induced topologies. But quotients do not commute with taking the topology. Another characterization of bireflective subcategories of `TSUnif` is that they are closed under suprema and under forming initial structures generated by one map.

2.4 Special t-semi-uniformities on symmetric topological spaces

In this part, X is a symmetric topological space. First we shall define several new classes of t-semi-uniformities, namely t-semi-uniformities having open–like or closed–like bases (all of them contain the class of uniformities. Let $\mathtt{TSUnif}(X)$ be the set of all semi-uniformities on the space X. We showed in Theorem 2.35 that $(\mathtt{TSUnif}(X), \subset)$ is a complete lattice. We shall now describe the minimal and maximal elements of $(\mathtt{TSUnif}(X), \subset)$ (they will be called the fine and the coarse semi-uniformities on X). A cover semi-uniformity on a symmetric topological space will also be defined. The fine semi-uniformity is shown to be a discrete object in the concrete category `TSUnif` over the category `Top`.

2.4.1 Pointwise open and pointwise closed t-semi-uniformities

Uniform spaces have bases consisting of open sets or of closed sets and those properties were shown to be very useful. Our t-semi-uniform spaces need not have such bases (e.g., some coarse spaces defined later in this part) and so it may be convenient to investigate subclasses of t-semi-uniform spaces having at least a weak form of those properties.

Definition 2.36 We say that a set $U \subset X \times X$ is *pointwise open* (or *pointwise closed*) if $U[x]$ is open (or closed, resp.) in X for each point $x \in X$. We say that a t-semi-uniform space $(X, \mathcal{U})$ is *pointwise open* (or *pointwise closed* or *open*) if there exists a base of $\mathcal{U}$ consisting of pointwise open sets (or pointwise closed sets, or open subsets of $X \times X$ resp.).

Clearly, it suffices to assume that a subbase has the corresponding property instead of a base. It is also clear that every discrete or indiscrete semi-uniform space is open (thus pointwise open) and pointwise closed.

We do not know an ansver to the natural question:

Question 1 *Has every pointwise open space a basis consisting of pointwise open and symmetric sets?*

Obviously, every uniformity is open and pointwise closed.

We shall see in Example 2.44 that not every t-semi-uniform space is pointwise open, and in Example 2.53 that not every regular t-semi-uniform space is pointwise closed.

Proposition 2.37 *The full subcategory of pointwise open t-semi-uniform spaces is bireflective in* `TSUnif`*.*

`Proof:` Suprema of pointwise open t-semi-uniformities are pointwise open. If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is initial and $\mathcal{V}$ is pointwise open, then $\mathcal{U}$ has the same property because $((f \times f)^{-1}(V))(x) = f^{-1}(V(f(x)))$ and f is continuous. $\diamond$

The same assertion with almost the same proof is true for pointwise closed t-semi-uniformities and for open semi-uniformities.

Proposition 2.38 *The full subcategory of pointwise closed t-semi-uniform spaces is bireflective in* `TSUnif`*.*

Corollary 2.39 *The full subcategory of t-semi-uniform spaces that are both pointwise open and closed is bireflective in* `TSUnif`*.*

Proposition 2.40 *The full subcategory of semi-uniform spaces having an open base is bireflective in* `TSUnif`*.*

Proposition 2.41 *The classes of pointwise open or of pointwise closed or of open semi-uniformities are productive and hereditary in* `TSUnif`*. They are closed under disjoint unions and not under quotients.*

`Proof:` The first assertion follows from the bireflectivity of our classes. The preservation under disjoint unions is trivial. The last assertion follows from the fact that every semi-uniformity is a quotient of a discrete semi-uniformity (see [3], 37.A.8). $\diamond$

2.4.2 Fine t-semi-uniformity

Proposition 2.42 *Let X be a symmetric topological space. Then $\mathcal{U}_f = \{U \in \mathcal{P}(X \times X); U[x], U^{-1}[x]$ are neighborhoods of x in X for each $x \in X\}$ is the largest t-semi-uniformity on X with respect to the order $\subseteq$.*

`Proof:` We will show that

1. $\mathcal{U}_f$ is a semi-uniformity on the set X,
2. $\mathcal{U}_f$ induces the original topology on X,
3. $\mathcal{U}_f$ is the largest semi-uniformity that induces the original topology on X.

1. The conditions from Definition 2.1 of semi-uniformities follow almost immediately from the properties of neighborhoods of points (a point x is in every its neighborhood, intersection of two neighborhoods of x is a neighborhood of x). The item (iv) follows from the fact $(U^{-1})^{-1}[x] = U[x]$.
2. Let τ be the topology of X and $U \in \mathcal{U}_f, a \in X$. It is clear that $U[a]$ is a neighborhood of a in τ. Conversely, let an open set G_a be a neighborhood of a point $a \in X$ in τ. Since X is a symmetric topological space, for every $y \notin G_a$ there exists a neighborhood G_y such that $a \notin G_y$. Let us take for every $y \notin G_a$ such a neighborhood G_y and for every $z \in G_a$ put $G_z = G_a$. Take $U = (\bigcup_{x\in X} \{x\} \times G_x) \cup (\bigcup_{x\in X} G_x \times \{x\}) \in \mathcal{U}_f$. It is now easy to see that $U[a] = \{y \in X; y \in G_a \text{ or } a \in G_y\} = G_a$.
3. If $\mathcal{V}$ is a semi-uniformity on X then $V[x], V^{-1}[x]$ are neighborhoods of x for each $x \in X$ and $V \in \mathcal{V}$. Then $V \in \mathcal{U}_f$, and $\mathcal{U}_f$ is the largest member of `TSUnif`(X).

◇

Definition 2.43 The t-semi-uniformity described in the previous proposition will be called the *fine t-semi-uniformity* on X.

Example 2.44 (Fine t-semi-uniformity that has no pointwise open base.) Let X be the space of all real numbers with the standard interval topology. Let $V = \{(x, y) \in X \times X; x = 0 \vee y = 0 \vee x < 2y < 4x\} \in \mathcal{U}_f$. Now V is a symmetric set, $V[x] = (\frac{1}{2}x; 2x) \cup \{0\}$ for each $x > 0$, $V[x] = (2x; \frac{1}{2}x) \cup \{0\}$ for each $x < 0$ and $V[0] = X$. Thus $V[x]$ is a neighborhood of x for each $x \in X$, and V is in the fine t-semi-uniformity on X. There is no pointwise open set contained in V. Let there be such a set $U \in \mathcal{U}_f$. Then there is a symmetric $W \in \mathcal{U}_f$ such that $W \subset U$. If $0 \neq z \in W[0]$ then $0 \in U[z]$. But there is no neighborhood of 0 contained in $U[z]$ and $U[z]$ is not open. Thus U is not pointwise open, and $\mathcal{U}_f$ has no pointwise open base.

Theorem 2.45 *Let X be a symmetric topological space, $\mathcal{U}_f$ its fine t-semi-uniformity. Then any continuous mapping of X to a t-semi-uniform space $(Y, \mathcal{V})$ is uniformly continuous with respect to $\mathcal{U}_f$ and $\mathcal{V}$.*

`Proof:` Let $V \in \mathcal{V}$ such that V is symmetric. Then $U = (f \times f)^{-1}(V)$ is symmetric and $U[x] = f^{-1}(V[f(x)])$. Since f is continuous and $V[f(x)]$ is a neighborhood of $f(x)$, the sets $U[x] = U^{-1}[x]$ are neighborhoods of x. Thus $U \in \mathcal{U}_f$. ◇

The previous theorem says that any t-semi-uniform space $(X, \mathcal{U}_f)$ is a discrete object in the concrete category `TSUnif` over the category `Top`. It also implies the next assertion:

Corollary 2.46 *The full subcategory of fine t-semi-uniform spaces is coreflective in* `TSUnif` *and, thus, closed under disjoint unions and quotients.*

The full subcategory of fine t-semi-uniform spaces is not productive (see Example 2.47); unlike the situation in `Unif`, it is hereditary (see Corollary 3.32).

Example 2.47 (Product of fine t-semi-uniformities that is not a fine t-semi-uniformity.)
Take the positive real line $\mathbb{R}^+$ with the Euklidean metric d, let $X_1, X_2 = \mathbb{R}^+$, and let $\mathcal{U}_1(\mathcal{U}_2)$ be the fine semi-uniformity on $X_1(X_2)$. Let Y be the cartesian product $Y = X_1 \times X_2$ with its fine semi-uniformity $\mathcal{U}$.

Take an arbitrary point x_0 in X_1.

Denote by $U \subset Y \times Y$ such a set that $U[(x_0, y)] = \{(u, v) \in Y; D((x_0, y), (u, v)) < \frac{1}{y}\}$, where D is the Euclidean metric on Y, $U[(x, y)] = Y$ for $x \neq x_0$, $y \in X_2$.

We will show that U is in the fine semi-uniformity $\mathcal{U}$ on Y but not in the product semi-uniformity $\mathcal{U}_1 \times \mathcal{U}_2$.

The set U is not symmetric. To prove that $U \in \mathcal{U}$ it suffices to show that $U[(x, y)]$ and $U^{-1}[(x, y)]$ are neighbourhoods of (x, y), for every $(x, y) \in Y$:

It follows from the definition of U that $U[(x, y)]$ is a neighbourhood of (x, y).

For $(u, v) \in Y$ it holds: $U^{-1}[(u, v)] = \{(x, y) \in Y; (x \neq x_0) \vee (D((x, y), (u, v)) < \frac{1}{y})\}$.

If $u \neq x_0$ then there is a positive number r that $x \neq x_0$ whenever $d(x, u) < r$, which implies that $(u, v) \in U[(x, y)] = Y$ whenever $d(u, x) < r$. Thus the neighborhood $\{(x, y) \in Y; d(x, u) < r\}$ of (u, v) is a subset of $U^{-1}[(u, v)]$.

If $u = x_0$ then the neighborhood $G = \{(x, y) \in Y; d(y, v) < \frac{1}{v+1}\}$ of (u, v) is a subset of $U^{-1}[(u, v)]$. In fact, if $d(y, v) = |v - y| < \frac{1}{v+1}$ then $y < v + \frac{1}{v+1} < v + 1$ and then $\frac{1}{v+1} < \frac{1}{y}$.
Now if $d(y, v) < \frac{1}{v+1}$ then $d(y, v) < \frac{1}{y}$. Thus if $(x, y) \in G$ then either $x \neq x_0$ and $(u, v) \in U[(x, y)] = Y$, or $x = x_0$ and $D((x, y), (u, v)) = d(u, v) < \frac{1}{y}$ and $(u, v) \in U[(x, y)]$.

Finally we show that U is not in the product semi-uniformity $\mathcal{U}_1 \times \mathcal{U}_2$. For the contrary let there be such sets $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ that

$$(\pi_1 \times \pi_1)^{-1}(U_1) \cap (\pi_2 \times \pi_2)^{-1}(U_2) \subset U.$$

As $U_1 = \bigcup_{x \in X_1} \{x\} \times U_1[x]$, $\quad U_2 = \bigcup_{x \in X_2} \{x\} \times U_2[x]$ $\quad$ it is $\quad U \supset \bigcup_{x \in X_1, y \in X_2} \{x\} \times \{y\} \times U_1[x] \times U_2[y]$.

Thus $U_1[x_0]$ is such a neighbourhood of x_0 that $\{x_0\} \times \{y\} \times U_1[x_0] \times U_2[y] \subset U$ for every $y \in X_2$, i.e. $U_1[x_0] \times U_2[y] \subset U[(x_0, y)]$. But for any $x \in U_1[x_0], x \neq x_0$, we find $y \in X_2$ that $D((x_0, y), (x, y)) = d(x_0, x) \geq \frac{1}{y}$. Thus $(x, y) \notin U[(x_0, y)]$, which is the contrary.

2.4.3 Coarse t-semi-uniformity

Proposition 2.48 *Let X be a symmetric topological space. For $a \in X$ and its neighborhood G define $U_{a,G} = (G \times G) \cup (X \setminus \{a\}) \times (X \setminus \{a\})$. Then $\{U_{a,G}; a \in X, G$ a neighborhood of $a\}$ forms a subbasis for the smallest t-semi-uniformity on X.*

`Proof:` Let us denote by $\mathcal{S}$ the suggested subbase and $\mathcal{U}_c$ the filter generated by $\mathcal{S}$. By Corollary 2.5, $\mathcal{U}_c$ is a semi-uniformity. We shall now show that $\mathcal{U}_c$ induces the topology of X. Let $U = U_{a,G} \in \mathcal{S}, x \in X$. If $x = a$ then $U[x] = G$. If $x \neq a$ and $x \in G$ then $U[x] = X$. If $x \notin G$ then $U[x] = X \setminus \{a\}$. In the last case, assume that $U[x]$ is not a neighborhood of x. Then a belongs to every neighborhood of x. As X is symmetric, x belongs to every neighborhood of a, which contradicts to $x \notin G$. Conversely, let a set G be a neighborhood of a point $x \in X$ in the original topology τ. Then $G = U[x]$ for $U = U_{x,G}$.

If now $\mathcal{V}$ is a semi-uniformity on X and $U = U_{a,G} \in \mathcal{S}$ then there is a symmetric set $V \in \mathcal{V}$ such that $V[a] \subset G$, which implies $V \subset U$. Then $U \in \mathcal{V}$ and $\mathcal{U}_c$ is thus the smallest semi-uniformity on X. ◇

Definition 2.49 The t-semi-uniformity described in the previous proposition will be called the *coarse* semi-uniformity on X.

Recall that a completely regular space has a coarse uniformity iff it is locally compact.

Proposition 2.50 *Any coarse t-semi-uniformity on a has an open base.*

Proof: Let $U = U_{a,G} \in \mathcal{S}$ such that G is an open neighborhood of a. Then $U = (G \times G) \cup (X \setminus \{a\}) \times (X \setminus \{a\})$ is an open set in $X \times X$. ◇

Corollary 2.51 *The pointwise open and open bireflections preserve T_1-topologies.*

Therefore for every T_1-topology there exist the finest pointwise open and the finest open semi-uniformities among all semi-uniformities inducing the topology. This assertion is valid for every symmetric topology, as we shall see in 2.64.

Example 2.52 (Coarse t-semi-uniform space that is not pointwise open.)
Let in a symmetric topological space X exist a point x and a nbhd G of x such that $\overline{\{x\}}, X \setminus G$ are infinite. Then the set $U_{x,G}$ is not pointwise open. If the coarse t-semi-uniformity $\mathcal{U}_c$ on X is pointwise open then there is a pointwise open set $V \in \mathcal{U}_c$ such that $V \subset U_{x,G}$. But that means that $V[y] \subset X \setminus \overline{\{x\}}$ for each $y \in X \setminus G$. That means that $V[y]$ has an infinite complement for infinitely many points, which is a contradiction to the definition of the coarse t-semi-uniformity $\mathcal{U}_c$.

The situation is different for pointwise closed semi-uniformities. It is seen from the next description that coarse semi-uniformities are rarely pointwise closed and that the pointwise closed bireflections need not preserve topology.

Example 2.53 (Coarse t-semi-uniformity on $\mathbb{R}$ is not pointwise closed.)
Let us take the real line with its standard topology. Let $\mathcal{U}_c$ be its coarse t-semi-uniformity. Then there is no pointwise closed set $U \in \mathcal{U}_c$ contained in $]-1,1[\times]-1,1[\cup(\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$.

We showed that every continuous mapping is uniformly continuous with respect to the fine t-semi-uniformities. The corresponding result is not true for coarse t-semi-uniformities, as the next example shows.

Example 2.54 Let X, Y be symmetric topological spaces, $\mathcal{U}_c, \mathcal{V}_c$ their coarse semi-uniformities. Take such a mapping $f : (X, \mathcal{U}_c) \to (Y, \mathcal{V}_c)$ that there is a point $y \in Y$ with an infinite $f^{-1}(y)$. Let H be such a neighborhood of y that $X \setminus f^{-1}(H)$ is infinite. Let there be $U \in \mathcal{U}_c$ such that $(f(x), f(x')) \in U_{y,H}$ for each $(x, x') \in U$. Then $U[x] \subset f^{-1}(H)$ for each $x \in f^{-1}(y)$. Thus $X \setminus U[x]$ is infinite for infinitely many $x \in f^{-1}(y)$. That is a contradiction.
For instance, the map $\sin : \mathbb{R} \to \mathbb{R}$ satisfies the above assumptions.

One could see from the previous example, which continuous maps into coarse semi-uniformities may be uniformly continuous.

Proposition 2.55 *Let X, Y be symmetric topological spaces, $\mathcal{V}_c$ the coarse t-semi-uniformity on Y and $f : X \to Y$ be a continuous finite-to-one map. Then $f : (X, \mathcal{U}) \to (Y, \mathcal{V}_c)$ is uniformly continuous for every semi-uniformity $\mathcal{U}$ on X.*

Proof: Let X, Y be symmetric topological spaces, $\mathcal{U}_c$, resp. $\mathcal{V}_c$, the coarse semi-uniformity on X, resp. Y, $\mathcal{U}$ a semi-uniformity on X. Take such a mapping $f : X \to Y$ that all points $y \in Y$ have finite $f^{-1}(y)$. Take $V = V_{y,H} \in \mathcal{V}$. Then $U = (f \times f)^{-1}(V) = (f^{-1}(H) \times f^{-1}(H)) \cup \bigcap_{x \in f^{-1}(y)} ((X \setminus \{x\}) \times (X \setminus \{x\})) \in \mathcal{U}_c \subset \mathcal{U}$.
◇

Corollary 2.56 *Restrictions of coarse t-semi-uniformities to subsets are coarse.*

It is not difficult to show that coarse t-semi-uniformities are not productive (see Example 2.57) and not closed under infinite disjoint sums (see Example 2.58), but closed under quotients (see Proposition 2.59).

Example 2.57 (Coarse t-semi-uniformities are not productive.)
Take $X,\ Y = \mathbb{R}$, $\mathcal{C}_X, \mathcal{C}_Y$ their coarse semi-uniformities. Let G, H resp., be some open sets in X, Y resp., with infinite complements. Denote by $\mathcal{C}$ the coarse semi-uniformity on $X \times Y$, $\mathcal{C}_X \times \mathcal{C}_Y$ the product semi-uniformity of $\mathcal{C}_X$ and $\mathcal{C}_Y$ on $X \times Y$.

Take some points $a \in G,\ b \in H$. Denote $C_1 = G \times G\ \cup (X \setminus \{a\}) \times (X \setminus \{a\}) \in \mathcal{C}_X$, $C_2 = H \times H\ \cup (Y \setminus \{b\}) \times (Y \setminus \{b\}) \in \mathcal{C}_Y$.

Take $C = \{((x, y,)(u, v)) \in X \times Y \times X \times Y; (x, u) \in C_1 \wedge (y, v) \in C_2\} \in \mathcal{C}_X \times \mathcal{C}_Y$.

Let for the contrary $C \in \mathcal{C}$. Then it contains a set $D = G' \times H' \times G' \times H'\ \cup (X \times Y \setminus \{(c, d)\}) \times (X \times Y \setminus \{(c, d)\})$, where $G'(H')$ is a neighbourhood of $x(y)$ in $X(Y)$ and $(c, d) \in X \times Y$. We can assume that $G' \subset G, H' \subset H$.

If $(a, b) \neq (c, d)$ then $D[(a, b)] = X \times Y$ or $D[(a, b)] = X \times Y \setminus \{(c, d)\}$, which is not contained in $C[(a, b)] = G \times H$, and that is the contrary.

If $(a, b) = (c, d)$ then take $x = a,\ y \notin H,\ u \notin G,\ v = b$. Now the point $((x, y), (u, v)) \in D \setminus C$, and that is the contrary.

Example 2.58 (Coarse t-semi-uniformities are not infinitely coproductive.)
It is clear beacuse an infinite sum of sets of the form $G_i \times G_i \cup (X_i \setminus \{x_i\}) \times (X_i \setminus \{x_i\})$ is not in the coarse semi-uniformity on the topological sum of spaces $X_i,\ i \in I$.

Proposition 2.59 *Quotients of coarse t-semi-uniformities are coarse t-semi-uniformities.*

`Proof:` Take a symmetric topological space X and its coarse semi-uniformity $\mathcal{U}_c$. Take a set Y and a mapping q of X onto Y.

Take a symmetric set U from the quotient t-semi-uniformity w.r.t. q on Y. There are such sets $C_i = G_i \times G_i \cup (X \setminus \{x_i\}) \times (X \setminus \{x_i\})$, where G_i is a neighbourhood of $x_i \in X, i = 0, .., n$, and $x_i \neq x_j$ for $i \neq j$, that $C = \bigcap_{i=0}^{n} C_i \subset (q \times q)^{-1}(U)$. (For every V there are such C_i because the quotient t-semi-uniformity is contained in the quotient semi-uniformity.)

As X is symmetric we can take such neighborhoods G_i that $x_j \notin G_i$ for $i \neq j, i = 0, .., n$.

First we show that $D = \bigcap_{i=0}^{n} (\quad q(G_i) \times q(G_i) \quad \cup \quad (Y \setminus \{q(x_i)\}) \times (Y \setminus \{q(x_i)\})) \subset U$.
Take $(u, v) \in D$ and show that $(u, v) \in U$.
If $(u, v) \in q(G_i) \times q(G_i) = (q \times q)(G_i \times G_i)$ for some $i = 0, .., n$ then $(u, v) \in (q \times q)(C) \subset U$. In fact, $G_i \times G_i \subset C$ for every $i = 0, .., n$ because $x_j \notin G_i$ for $i \neq j, i, j = 0, .., n$.

If $(u, v) \notin q(G_i) \times q(G_i) = (q \times q)(G_i \times G_i)$ for every $i = 0, .., n$ then $(u, v) \in \bigcap_{i=0}^{n} (q \times q)((X \setminus \{x_i\}) \times (X \setminus \{x_i\})) \subset (q \times q)(C) \subset U$.

The set $q(G_i)$ need not be a neighborhood of $q(x_i)$ in the quotient topology on Y. But if we take $U[q(x_i)]$ instead of $q(G_i)$ in the definition of D we only add points $(q(x_i), y)$ and $(y, q(x_i))$, where $y \in U[q(x_i)]$, which are in U. Thus $U \supset \bigcap_{i=0}^{n} (\quad U[q(x_i)] \times U[q(x_i)] \quad \cup \quad (Y \setminus \{q(x_i)\}) \times (Y \setminus \{q(x_i)\}))$ and U is in the coarse semi-uniformity inducing the quotient topology on Y. $\diamond$

2.4.4 Cover t-semi-uniformity

Definition 2.60 A semi-uniformity on a symmetric topological space X is called a *cover semi-uniformity* if it has a base consisting of sets of the form $\bigcup_{H \in \mathcal{H}} H \times H$, where $\mathcal{H}$ is an open cover of X.

Clearly, one can take a subbase in the previous definition instead of a base. It is known that every uniformity is a cover semi-uniformity. Clearly, every cover semi-uniformity on a space X has an open base.

We should notice in the above definition that one must take such a system of open covers of X that the corresponding cover semi-uniformity induces the topology of X (i.e., every neighborhood of any point must contain its star with respect to a chosen cover).

The next result shows that there exists the finest cover semi-uniformity on every symmetric topological space.

Proposition 2.61 *Let X be a symmetric topological space. The system $\mathcal{U}_{cv}$ having a base $\{\bigcup_{H\in\mathcal{H}} H \times H$, $\mathcal{H}$ is an open cover of $X\}$, is a semi-uniformity on X that is the finest cover semi-uniformity on X.*

`Proof:` Let $\mathcal{B} = \{\bigcup\{H \times H; H \in \mathcal{H}\}; \mathcal{H}$ is an open cover of $X\}$. Then $\mathcal{B}$ is a base for a semi-uniformity $\mathcal{U}$ inducing the original topology τ on X. Indeed, if G is an open neighborhood of $x \in X$ then $G = B[x]$, where $B = (G \times G) \cup ((X \setminus \overline{\{x\}}) \times (X \setminus \overline{\{x\}}))$; conversely, $B[x]$ is an open neighborhood of x, for every $x \in X$ and $B \in \mathcal{B}$. Thus $\mathcal{U}$ induces the topology τ.

The assertion that $\mathcal{U}_{cv}$ is the finest among all cover semi-uniformities on X follows from the fact that we take all the open covers for its description. ◇

Let us remark that if X is paracompact then the cover t-semi-uniformity $\mathcal{U}_{cv}$ is the finest uniformity on X. If X is not paracompact, then $\mathcal{U}_{cv}$ is not a uniformity.

The next result is analogous to that for fine semi-uniformities.

Proposition 2.62 *Let $(X, \tau), (Y, \sigma)$ be symmetric topological spaces, let $\mathcal{U}_{cv}$ be the finest cover t-semi-uniformity of (X, τ) and $\mathcal{V}$ be a cover semi-uniformity on (Y, σ). Then every continuous mapping $f : X \to Y$ is uniformly continuous with respect to $\mathcal{U}_{cv}$ and $\mathcal{V}$.*

`Proof:` Let $V \in \mathcal{V}$. Then there is an open cover $\mathcal{H}$ of Y such that $\bigcup_{H\in\mathcal{H}} (H \times H) \subset V$. Then $(f \times f)^{-1}(V) \supset \bigcup_{H\in\mathcal{H}} (f^{-1}(H) \times f^{-1}(H)) \in \mathcal{U}_{cv}$ because f is continuous (and thus $\{f^{-1}(H); H \in \mathcal{H}\}$ is an open cover of X). ◇

As we could see in Example 2.52, the coarse semi-uniformity on X need not be pointwise open and, thus, a cover semi-uniformity. For T_1-spaces X, the coarse semi-uniformity on X is a cover semi-uniformity, because the basic sets $U_{a,G}$ can be expressed in the form $(G \times G) \cup (X \setminus \{a\}) \times (X \setminus \{a\})$, and both sets G and $X \setminus \{a\}$ form an open cover of X. Such an expression suggests a description of the coarse cover semi-uniformity on X, which will be also the coarsest open and the coarsest pointwise open t-semi-uniformity on the space X:

Proposition 2.63 *For a symmetric topological space X, the sets $(G \times G) \cup (X \setminus \overline{\{a\}}) \times (X \setminus \overline{\{a\}})$, where G is open in X and $a \in G$, form a subbase for the coarsest pointwise open t-semi-uniformity on X.*

`Proof:` Let $\mathcal{U}$ be a pointwise open semi-uniformity on the space X. Let $a \in X, G$ be an open nbhd of a. Denote $V = (G \times G) \cup (X \setminus \overline{\{a\}}) \times (X \setminus \overline{\{a\}})$. We will show that $V \in \mathcal{U}$. We know that there is $U = U^{-1} \in \mathcal{U}$ such that $U[a] \subset G$ because $\mathcal{U}$ induces the original topology on X. As $\mathcal{U}$ is pointwise open there is a pointwise open $U' \in \mathcal{U}$ such that $U' \subset U$. We will show that $U'[x] \subset V[x]$ for every $x \in X$.

(i) For $x = a$ it is clear.

(ii) Let $x \neq a, x \in G$. Then $V[x] = X \supset U'[x]$.

(iii) Let $x \notin G \supset \overline{U[a]}$. Then $a \notin U[x]$ because U is symmetric. Then $a \notin U'[x]$, which is open. And thus $U'[x] \cap \overline{\{a\}} = \emptyset$. Now $U'[x] \subset X \setminus \overline{\{a\}} = V[x]$.

◇

Proposition 2.64 *The cover semi-uniformity on X described in the previous proposition is also the coarsest t-semi-uniformity on X having an open base and the coarsest cover t-semi-uniformity on X.*

`Proof:` It follows from the fact, that every cover t-semi-uniformity is open and every open t-semi-uniformity is pointwise open. And the described t-semi-uniformity is a cover t-semi-uniformity on X. ◇

Proposition 2.65 *The cover t-semi-uniformities are productive, hereditary, coproductive and they are not closed under quotients.*

Proof:

The subbase of the product semi-uniformity of semi-uniformities $\mathcal{U}_i$, $i \in I$, consists of sets $U = (\pi_i \times \pi_i)^{-1}(U_i)$, $U_i \in \mathcal{U}_i$, $i \in I$. If $\mathcal{U}_i$ are cover semi-uniformities then $U = (\pi_i \times \pi_i)^{-1}(\bigcup_{H_i \in \mathcal{H}_i} H_i \times H_i) = \bigcup_{H_i \in \mathcal{H}_i} \pi_i^{-1}(H_i) \times \pi_i^{-1}(H_i)$, where $\mathcal{H}_i$ is an open cover of X_i, and thus $\{\pi_i^{-1}(H_i)\}$ is an open cover of the product of the spaces X_i, $i \in I$.

$\{H \cap X, H \in \mathcal{H}\}$ is an open cover of a subspace X of the space Y, whenever $\mathcal{H}$ is an open cover of Y. Thus the subspace semi-uniformity of the cover semi-uniformity is a cover semi-uniformity.

$\{\sum_{i \in I} H_i; H_i \in \mathcal{H}_i\}$ is an open cover of a the coproduct of the spaces X_i, $i \in I$, whenever $\mathcal{H}_i$ are open covers of X_i, $i \in I$. Thus the coproduct semi-uniformity of the cover semi-uniformities is a cover semi-uniformity.

The last assertion follows from the fact that every semi-uniformity is a quotient of a discrete semi-uniformity (see [3], 37.A.8), which is a cover semi-uniformity.

$\diamond$

2.4.5 Categorical summary

We shall denote by $\mathtt{TSUnif}_{po}$ (or $\mathtt{TSUnif}_o$, $\mathtt{TSUnif}_c$) the full subcategory of TSUnif consisting of pointwise open semi-uniformities (or uniformities having an open base, or cover semi-uniformities, resp.).

Theorem 2.66 *All the arrows in the next diagram are full bireflective embeddings:*

$$\mathtt{Unif} \longrightarrow \mathtt{TSUnif}_c \longrightarrow \mathtt{TSUnif}_o \longrightarrow \mathtt{TSUnif}_{po} \longrightarrow \mathtt{TSUnif}\,.$$

Proof: We show that $\mathtt{TSUnif_c}$ is a bireflective subcategory of TSUnif. It is sufficient to show that it is closed under suprema and under forming initial structures generated by one map. (See the remark below 2.35). The first one follows from the fact that finite intersections of members of open covers form an open cover. The other one is clear because $(f \times f)^{-1}(H \times H) = f^{-1}(H) \times f^{-1}(H)$, and $f^{-1}(H)$ is open for an open set H. $\diamond$

As one can see from Example 2.52, the above bireflections preserve T_1-topologies and need not preserve non-T_1-topologies (indeed, the coarse semi-uniformity need not be pointwise open and so, e.g., its pointwise open bireflection is strictly coarser in this case and cannot induce the same topology).

But the bireflections of the fine semi-uniformities preserve topologies since they are finer than the coarsest cover semi-uniformity. Consequently, for any of the above categories $\mathcal{K}$, every symmetric topological space X is induced by the finest semi-uniformity belonging to $\mathcal{K}$. Moreover, this finest semi-uniformity is a discrete object with respect to $\mathcal{K}$ over $\mathtt{Top}(X)$.

3 COMPLETENESS OF T-SEMI-UNIFORM SPACES

In this chapter we will follow the classical approach to study the completeness in t-semi-uniform spaces. It means first to define a class of filters that we want to play a role of Cauchy filters. Complete spaces will be those t-semi-uniform spaces in which every corresponding filter from our class converges. One of the properties we want to be fulfilled is that such Cauchy-like filters coincide with Cauchy filters on uniform spaces and that a uniform space is complete as a t-semi-uniform space iff it is complete in the classical sense.

Characterizations of Cauchy filters in uniform spaces may not longer be equivalent when we transfer them to t-semi-uniform spaces. In the next we will restrict our consideration to several such basic characterizations. Certainly our selection is not exhaustive.

We would like to maintain properties of completeness for uniform spaces also in t-semi-uniform setting, mainly the existence of completion. We will see later that it is not always possible.

We will compare this approach with another generalization of completeness, namely with absolute closedness. Recall that in uniform Hausdorff spaces, a space is complete iff it is absolutely closed, i.e. closed in every uniform superspace. Some relations between our completeness and absolute closedness will be proved, e.g., every (Cauchy) complete pointwise open t-semi-uniform space is absolutely closed in Hausdorff t-semi-uniform spaces, but not vice versa.

3.1 Basic properties

Recall that a filter $\mathfrak{f}$ in a uniform space $(X,\mathcal{U})$ is said to be Cauchy if for each $U \in \mathcal{U}$ there is an $F \in \mathfrak{f}$ such that $F \times F \subset U$. That is iff for each $U \in \mathcal{U}$ there is an $x \in X$ such that $U[x] \in \mathfrak{f}$. Clearly, the first defining property of Cauchy filters implies the second one also in t-semi-uniform spaces, but not conversely. If we take any property between those two, it will certainly coincide with Cauchy property on uniform spaces.

Definition 3.1 Let $(X,\mathcal{U})$ be a t-semi-uniform space.

1. A filter $\mathfrak{f}$ on a space X is called *a classic Cauchy filter on* $(X,\mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $F \in \mathfrak{f}$ such that $F \times F \subset U$.
2. A filter $\mathfrak{f}$ on a space X is called *a Cauchy filter on* $(X,\mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $F \in \mathfrak{f}$ such that for every $x \in F$ holds $U[x] \in \mathfrak{f}$.
3. A filter $\mathfrak{f}$ on a space X is called *a weak Cauchy filter on* $(X,\mathcal{U})$ if for every $U \in \mathcal{U}$ and every $F \in \mathfrak{f}$ there exists $x \in F$ such that $U[x] \in \mathfrak{f}$.
4. A filter $\mathfrak{f}$ on a space X is called *a semi-Cauchy filter on* $(X,\mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U[x] \in \mathfrak{f}$.

Clearly, it suffices to take U from a base of $\mathcal{U}$ and, in the first two cases, from a subbase of $\mathcal{U}$.

It may be convenient to realize that if a filter $\mathfrak{f}$ on X has one of our Cauchy-like properies except the third one then every larger filter on X has the same property. Every filter containing a singleton has each of our Cauchy-like properties.

Some of the previous Cauchy-like filters are known under other names in other situations. E.g., for quasi-uniform spaces Cauchy filters in our sense are sometimes called K-Cauchy (in fact, left and right K-Cauchy filters are defined there), our semi-Cauchy fiters are sometimes called PS-Cauchy or $\mathcal{U}$-Cauchy filters in quasi-uniform space $(X,\mathcal{U})$ - see [9].

Proposition 3.2 *Let X be a t-semi-uniform space, $\mathfrak{f}$ a filter in X.*

1. *If $\mathfrak{f}$ is convergent in X then it is semi-Cauchy on X.*
2. *If $\mathfrak{f}$ is classic Cauchy on X then it is Cauchy on X.*
3. *If $\mathfrak{f}$ is Cauchy on X then it is weak Cauchy on X.*
4. *If $\mathfrak{f}$ is weak Cauchy on X then it is semi-Cauchy on X.*

`Proof:` (1) If a filter $\mathfrak{f}$ converges to a point $x \in X$ then $U[x] \in \mathfrak{f}$ for each $U \in \mathcal{U}$. (2)-(4) are clear. ◇

Diagram. Let X be a t-semi-uniform space, let *cl.Cauchy*, *Cauchy*, *w.Cauchy*, *s-Cauchy*, *conv* be sets of all classic Cauchy, Cauchy, weak Cauchy, semi-Cauchy, converging filters on X, respectively. From the previous proposition we get the following diagram, in which all the arrows denote inclusions.

$$cl.Cauchy \longrightarrow Cauchy \longrightarrow w.Cauchy \longrightarrow s{-}Cauchy \longleftarrow conv$$

Example 3.5 shows a convergent and thus semi-Cauchy filter that is not weak Cauchy. Example 5.18 shows a weak Cauchy filter that is not Cauchy. Example 3.6 shows a Cauchy filter that is not classic Cauchy. Thus no of the arrows in the previous diagram can be reverted. As it is well-known, for uniform spaces all the arrows except the last one are identities.

Later we will need the preservation of Cauchy-like filters by uniformly continuous mappings.

Lemma 3.3 *Let $(X,\mathcal{U}), (Y,\mathcal{V})$ be semi-uniform spaces. Let f be a uniformly continuous map with respect to $\mathcal{U}$ and $\mathcal{V}$. Let $\mathfrak{f}$ be a classic Cauchy (resp. Cauchy, resp. weak Cauchy, resp. semi-Cauchy) filter on $(X,\mathcal{U})$. Then $f([\mathfrak{f}]) := \{f(F); F \in \mathfrak{f}\}$ is a classic Cauchy (resp. Cauchy, resp. weak Cauchy, resp. semi-Cauchy) filter base on $(Y,\mathcal{V})$.*

`Proof:` For a classic Cauchy filter: Let $\mathfrak{f}$ be a classic Cauchy filter on X, $V \in \mathcal{V}$. There is $U \in \mathcal{U}$ such that $f(U[x]) \subset V[f(x)]$ for each $x \in X$. And as $\mathfrak{f}$ is classic Cauchy there is a set $F \in \mathfrak{f}$ such that $F \times F \subset U$. Let $G = f(F)$. Now $G \times G \subset (f \times f)(U) \subset V$.

For a Cauchy filter: Let $\mathfrak{f}$ be a Cauchy filter on X, $V \in \mathcal{V}$. There is $U \in \mathcal{U}$ such that $f(U[x]) \subset V[f(x)]$ for each $x \in X$. And as $\mathfrak{f}$ is Cauchy there is a set $F \in \mathfrak{f}$ such that $U[x] \in \mathcal{U}$ for each $x \in F$. Let $G = f(F)$. Now $V[y] \supset f(U[x]) \in f([\mathfrak{f}])$ for each $y = f(x) \in G \in f([\mathfrak{f}])$.

For a weak Cauchy filter: Let $\mathfrak{f}$ be a weak Cauchy filter on X, $V \in \mathcal{V}, G \in f([\mathfrak{f}])$. There is $U \in \mathcal{U}$ such that $f(U[x]) \subset V[f(x)]$ for each $x \in X$ and $F \in \mathfrak{f}$ such that $f(F) = G$. And as $\mathfrak{f}$ is weak Cauchy there is $x \in F$ such that $U[x] \in \mathfrak{f}$. Now $f(x) \in G$ and $V[f(x)] \supset f(U[x]) \in f([\mathfrak{f}])$.

For a semi-Cauchy filter: Let $\mathfrak{f}$ be a semi-Cauchy filter on X, $V \in \mathcal{V}$. There is $U \in \mathcal{U}$ such that $f(U[x]) \subset V[f(x)]$ for each $x \in X$. And as $\mathfrak{f}$ is semi-Cauchy there is a point $x \in X$ such that $U[x] \in \mathfrak{f}$. Let $y = f(x)$. Now $V[y] \supset f(U[x]) \in f([\mathfrak{f}])$. ◇

3.2 Convergence and Cauchy-like property

In uniform spaces every convergent filter is Cauchy. We shall show now that in t-semi-uniform spaces this holds for semi-Cauchy property only, or for special t-semi-uniform spaces defined in the previous section.

Proposition 3.4 *Let $\mathfrak{f}$ be a convergent filter in a t-semi-uniform space $(X,\mathcal{U})$.*

1. *$\mathfrak{f}$ is semi-Cauchy.*
2. *If $\mathcal{U}$ is a pointwise open semi-uniformity then $\mathfrak{f}$ is Cauchy.*
3. *If $\mathcal{U}$ has an open base then $\mathfrak{f}$ is classic Cauchy.*

`Proof:`

1. coincides with Proposition 3.2.1.
2. Let $(X,\mathcal{U})$ be a pointwise open t-semi-uniform space, a filter $\mathfrak{f}$ converge to a point $x \in X$. Take a pointwise open $U \in \mathcal{U}$. There is a symmetric set $V \in \mathcal{U}$ such that $V \subset U$. If $y \in F = V[x] \in \mathfrak{f}$ then $x \in V[y] \subset U[y]$ and $U[y] \in \mathfrak{f}$ because it is a neighborhood of all its points. Thus $\mathfrak{f}$ is Cauchy.
3. If $\mathfrak{f}$ converges to a and $U \in \mathcal{U}$ is an open subset of $X \times X$, then there is an open neighborhood G of a with $G \times G \subset U$. Clearly, $G \in \mathfrak{f}$ and the proof is finished.

◇

We shall now show that the previous assertions cannot be improved (in the realm of our special properties).

Example 3.5 (Convergent filter that is not weak Cauchy.)

Let X be the space of all real numbers with its standard interval topology, let $\mathcal{U}_f$ be the fine t-semi-uniformity on X. Let $\mathfrak{f}$ be a filter with the base $\{U[0] \setminus \{0\}; U \in \mathcal{U}\}$. The filter $\mathfrak{f}$ is convergent but not weak Cauchy on $(X,\mathcal{U}_f)$. In fact, let $V = \{(x,y) \in X \times X; x = 0 \vee y = 0 \vee x < 2y < 4x\} \in \mathcal{U}_f$. Now $V[x] = (\frac{1}{2}x; 2x) \cup \{0\}$ for each $x > 0$, $V[x] = (2x; \frac{1}{2}x) \cup \{0\}$ for each $x < 0$ and $V[0] = X$. If $\mathfrak{f}$ is weakly Cauchy then for $U \in \mathcal{U}_f$ there is $x \in U[0] \setminus \{0\}$ such that $V[x] \in \mathfrak{f}$. But for each $x \neq 0$ there is no set $U' \in \mathcal{U}$ such that $(U'[0] \setminus \{0\}) \subset V[x]$. Thus $V[x] \notin \mathfrak{f}$ and $\mathfrak{f}$ is not weakly Cauchy.

Example 3.6 (Convergent in a pointwise open semi-uniform space and thus Cauchy filter that is not classic Cauchy.)

Let X be the space of all real numbers with its standard interval topology, let $\mathcal{U}_{\mathcal{R}}$ be its standard uniformity and $V = \{(x,y) \in X \times X; \frac{1}{2} < \frac{y}{x} < 2 \vee -\frac{1}{4} < \frac{y}{x} < \frac{1}{4} \vee -\frac{1}{4} < \frac{x}{y} < \frac{1}{4}\} \cup \{(0,0)\}$. Let $\mathcal{U}$ be the t-semi-uniformity with the subbase $\mathcal{U}_{\mathcal{R}} \cup \{V\}$. It is easy to show that it is a pointwise open semi-uniformity inducing the standard interval topology on the real line.

The filter $\mathcal{U}_0$ is not classic Cauchy. In fact, there is no set $F \in \mathcal{U}_0$ such that $F \times F \subset V$.

In the next example we construct a t-semi-uniform space that is not pointwise open but every its convergent filter is classic Cauchy.

Example 3.7 Let X be the set of all real numbers with the standard interval topology. Let $\mathcal{U}_c$ be its coarsest t-semi-uniformity. Let $\mathcal{S} = \mathcal{U}_c \cup \{U_0\}$, where $U_0 = \{(x,y) \in X \times X; |x-y| \leq 1\}$. $\mathcal{S}$ is a subbase for a semi-uniformity $\mathcal{U}$ on the set X.

$\mathcal{U}$ is a t-semi-uniformity on X because it induces the original topology on X. In fact, any neighborhood G of a point $x \in X$ is equal $U[x]$ for some $U \in \mathcal{U}_c$. On the other hand, $U[x]$ is a neighborhood of x for every $x \in X$ and $U \in \mathcal{U}_c$. And $U_0[x] = [x-1, x+1]$ is also a neighborhood of x for every $x \in X$.

Let $\mathfrak{f}$ be a filter on X that converges to a point $x \in X$. Is $\mathfrak{f}$ classic Cauchy? Let $U = U_{a,G} \in \mathcal{U}_c$. If $x = a$ then $G \in \mathfrak{f}$ and $G \times G \subset U$. If $x \neq a$ then $H = X \setminus \{a\}$ is a neighborhood of x such that $H \times H \subset U$. For U_0 take a set $G =]x-0.5, x+0.5[$ that is an open neighborhood of x, and thus $G \in \mathfrak{f}$. Clearly $G \times G \subset U_0$ and $\mathfrak{f}$ is classic Cauchy.

$(X,\mathcal{U})$ is not a pointwise open space. Indeed, U_0 contains no subset $V \in \mathcal{U}$ such that for each $x \in \mathbb{R}$ the set $U_0[x]$ is a neighborhood of $V[x]$, since there are always points x with $U_0[x] = [x-1, x+1]$.

Diagram. Let $\mathtt{TSUnif}_{cC}$ (or $\mathtt{TSUnif}_{C}$ or $\mathtt{TSUnif}_{wC}$ or $\mathtt{TSUnif}_{sC}$) be the full subcategory of $\mathtt{TSUnif}$ generated by those t-semi-uniform spaces in which every convergent filter is classic Cauchy (or Cauchy, or weak Cauchy, or semi-Cauchy, resp.).

From Proposition 3.4 we get the following diagram, in which all the arrows denote embedding functors. According the previous examples, no diagonal arrow going down left can exists, and the last example implies that no vertical arrow is onto (thus no arrow is onto).

$$\begin{array}{ccccc} \mathtt{TSUnif}_o & \longrightarrow & \mathtt{TSUnif}_{po} & \longrightarrow & \mathtt{TSUnif} \\ \downarrow & & \downarrow & & \| \\ \mathtt{TSUnif}_{cC} & \longrightarrow & \mathtt{TSUnif}_{C} & \longrightarrow & \mathtt{TSUnif}_{\mathtt{sC}} \end{array}$$

3.3 Cauchy-like filters and clusters

In uniform spaces, every Cauchy filter converges to its cluster point. That need not be true for our Cauchy-like filters on t-semi-uniform spaces.

First we will show that there are cover semi-uniform spaces containing Cauchy-like filters that do not converge to any of its clusters. To show that we need a lemma:

Lemma 3.8 *Let X be a symmetric topological space, $\mathcal{U}_c$ its coarse semi-uniformity.*

(i) *Classic Cauchy filters on $(X,\mathcal{U}_c)$ coincide with Cauchy filters on $(X,\mathcal{U}_c)$ (which are filters on X that either have an empty intersection or converge to every point of their intersection).*

(ii) *Weak Cauchy filters on $(X,\mathcal{U}_c)$ are filters that contain a singleton provided their intersection contains a point isolated in some member of the filter.*

(iii) *Semi-Cauchy filters on $(X,\mathcal{U}_c)$ are filters that contain a singleton provided their intersection contains an isolated point in X.*

`Proof:` Recall that $\mathcal{U}_c$ has for its subbase all the sets $U = U_{a,G} = (G \times G) \cup (X \setminus \{a\}) \times (X \setminus \{a\})$ where $a \in X$ and G is a neighborhood of a.

A filter containing a singleton has each of our Cauchy-like property.

(i) Denote by $\mathfrak{K}$ the filter on X consisting of cofinite subsets. Clearly, $\mathfrak{K}$ is classic Cauchy and, thus, every filter on X having empty intersection is classic Cauchy. If $x \in \bigcap \mathfrak{k}$ and $\mathfrak{k} \to x$ then every neighborhood V of x belongs to $\mathfrak{k}$. If we choose $U_{a,G}$ and $x \in G$ then $G \times G \subset U_{a,G}$; if $x \notin G$ then $F = X \setminus \{a\} \in \mathfrak{k}$ and again we have $F \times F \subset U_{a,G}$. So, such filters are classic Cauchy.

Suppose conversely, that $\mathfrak{f}$ is Cauchy and has a nonempty intersection. If $\mathfrak{f}$ is Cauchy then for every $U \in \mathcal{U}_c$ there is a set $F \in \mathfrak{f}$ that $U[x] \in \mathfrak{f}$ for all $x \in F$. Let $x \in \bigcap \mathfrak{f}$. Then $x \in F$ and $U[x] \in \mathfrak{f}$.

(ii) By (i), every filter on X having empty intersection is weak Cauchy. Let $\mathfrak{f}$ be a filter containing no singleton and having a nonempty intersection. Then no point of $\bigcap \mathfrak{f}$ is isolated in some $F \in \mathfrak{f}$. Take $U = U_{a,G}$.
If $a \in \bigcap \mathfrak{f}$ and $F \in \mathfrak{f}$ then there is a point $b \neq a$ that $b \in F \cap G$. Then $U[b] = X \in \mathfrak{f}$.
If $a \notin \bigcap \mathfrak{f}$ then there is a set $F \in \mathfrak{f}$ that $a \notin F$. Then either $U[x] = X \in \mathfrak{f}$ or $U[x] = X \setminus \{a\} \supset F \in \mathfrak{f}$, for every $x \in F$. Consequently, $\mathfrak{f}$ is weak Cauchy.

Suppose conversely, that $\mathfrak{f}$ is weak Cauchy and has a nonempty intersection that contains a point x isolated in some $F \in \mathfrak{f}$. Thus there is an open set G such that $G \cap F = \{x\}$. There must be some $y \in F$ such that $U_{x,G}[y] \in \mathfrak{f}$. Clearly, y must coincide with x and, thus, $G \in \mathfrak{f}$.

(iii) By (i), every filter on X having an empty intersection is semi-Cauchy. Let $a \in \bigcap \mathfrak{f}$. Let $U = U_{a,G}$. If a is an isolated point in X then, by the assumption, the filter contais a singleton and is semi-Cauchy. If a is not an isolated point in X then for every nbhd G of a there is $y \in G \smallsetminus \{a\}$, and $U[y] = X \in \mathfrak{f}$.

Suppose conversely, that $\mathfrak{f}$ is semi-Cauchy and has a nonempty intersection that contains a point x isolated in X. Let $G = \{a\}$, $U = U_{a,G}$. Now $U[a] = \{a\}$, $U[x] = X \smallsetminus \{a\}$ for $x \neq a$. Then $\{a\} \in \mathfrak{f}$ because $X \smallsetminus \{a\} \notin \mathfrak{f}$.

$\diamond$

Example 3.9 (Classic Cauchy filter in a cover semi-uniformity that does not converge to its clusters.)

Let X be an infinite Hausdorff space without isolated points. Let $\mathcal{U}_c$ be the coarsest t-semi-uniformity on X. Let $\mathfrak{f}$ be a filter of sets with finite complements in X. By the previous Lemma (i), $\mathfrak{f}$ is classic Cauchy. Clearly, every point in X is a cluster of $\mathfrak{f}$ and, thus, $\mathfrak{f}$ has no limit point according our assumption on X.

There are nice classes of t-semi-uniform spaces where every Cauchy filter converges to any of its clusters as we can see from the next two assertions.

Proposition 3.10 *Let $(X,\mathcal{U})$ be a pointwise closed t-semi-uniform space. Then each Cauchy filter on $(X,\mathcal{U})$ converges to its clusters.*

`Proof:` Let $\mathfrak{f}$ be a Cauchy filter on X, $x \in X$ be its cluster. Thus $x \in \overline{F}$ for each $F \in \mathfrak{f}$. Let us choose an arbitrary set $V \in \mathcal{U}$. Without loss of generality we can assume that V is symmetric. Then there exists a pointwise closed set $U \in \mathcal{U}$ such that $U \subset V$. As the filter $\mathfrak{f}$ is Cauchy we can find a set $F \in \mathfrak{f}$ such that $U[y] \in \mathfrak{f}$ for each $y \in F$. But $x \in U[y]$ for each $y \in F$ because U is pointwise closed and x is a cluster of $\mathfrak{f}$. As V is symmetric $y \in V[x]$ for each $y \in F$. Thus $F \subset V[x]$ and, consequently, $V[x] \in \mathfrak{f}$, which was to prove. $\diamond$

Proposition 3.11 *Let X be a regular space and $\mathcal{U}$ be a semi-uniformity on X containing the sets $(G \times G) \cup ((X \smallsetminus \overline{H}) \times (X \smallsetminus \overline{H}))$ for all open sets G, H with $\overline{H} \subset G$. Then every semi-Cauchy filter on $(X,\mathcal{U})$ converges to its clusters.*

`Proof:` Let $\mathfrak{f}$ be a semi-Cauchy filter on $(X,\mathcal{U})$, let $x \in X$ be its cluster. Let $G \subset X$ be an open neighborhood of x. If X is a regular space then there exist open neighborhoods K, H of x such that $K \subset \overline{K} \subset H \subset \overline{H} \subset G$. Then the sets $U = (G \times G) \cup ((X \smallsetminus \overline{H}) \times (X \smallsetminus \overline{H}))$, $V = (H \times H) \cup ((X \smallsetminus \overline{K}) \times (X \smallsetminus \overline{K}))$ are in $\mathcal{U}$. If $\mathfrak{f}$ is semi-Cauchy then there is $y \in X$ such that $(U \cap V)[y] \in \mathfrak{f}$. Thus $U[y] \in \mathfrak{f}$ and $V[y] \in \mathfrak{f}$. Now $U[y] = G$ for $y \in H$, $V[y] = X \smallsetminus \overline{K}$ for $y \in X \smallsetminus H$. As x is a cluster of $\mathfrak{f}$ and $K \cap (X \smallsetminus \overline{K}) = \emptyset$, the second possibility is excluded. Thus $G \in \mathfrak{f}$ and $\mathfrak{f}$ converges to x. $\diamond$

Every fine semi-uniformity or fine cover semi-uniformity on a regular topological space satisfies the condition of the previous Proposition.

The following example shows that Cauchyness in Proposition 3.10 cannot be weakened.

Example 3.12 (Weak Cauchy filter on poinwise closed t-semi-uniform space that does not converge to its clusters.)
Take a quasi-topological group from Example 5.22. There is a non-converging weak Cauchy filter with a non-void intersection described. Points from the intersection are clusters of the filter. If we take for X in this example a compact space, the group will be a subspace of the product of compact spaces, and thus regular. In quasi-topological groups regularity and pointwise closedness of the induced semi-uniformity (see the section 4.1) are equivalent.

The regularity in Proposition 3.11 also cannot be weakened:

Example 3.13 Take an infinite Abelian group X with the cofinite (thus not regular) topology. In Example 5.18 a weak Cauchy filter in every group semi-uniformity is described that has two points in its intersection (thus two clusters) and no limit point. There are no open sets H, G that $\overline{H} \subset G$ except for $H = G = X$. Thus every semi-uniformity on X satisfies the condition in the Proposition 3.11.

3.4 $\mathcal{L}$-filters

To avoid repeating various properties for Cauchy-like filters, we will now comprise several basic properties that are valid in uniform spaces and are needed to get completions with requested properties. We will refer to this section also in the part where we deal with categorical approach to completeness. Because our main task is to deal with completions (also from categorical point of view), we shall often restrict our consideration to Hausdorff spaces. The category of Hausdorff t-semi-uniform spaces from a category $\mathcal{K}$ will be denoted as $\mathcal{K}_2$ (e.g., $\mathtt{TSUnif}_2$ or $\mathtt{TSUnif}_{po_2}$).

Definition 3.14 Let $\mathcal{K}$ be a full subcategory of $\mathtt{TSUnif}_2$ and $\mathcal{L}$ assign to every space X from $\mathcal{K}$ a class of filters $\mathcal{L}(X)$ on X (such a map will be called a *filter operator*). The following is a definition of properties of $\mathcal{L}$ on $\mathcal{K}$ (we assume that $X, Y \in \mathcal{K}$): $\mathcal{L}$ is said to have

(P1) if $\mathcal{L}(X)$ contains all convergent filters in X;

(P2) if every $\mathfrak{f} \in \mathcal{L}(Y)$ is a base of a filter from $\mathcal{L}(X)$ whenever $Y \subset X, Y \in \mathcal{K}$;

(P3) if $\mathfrak{G} \in \mathcal{L}(X)$ whenever $\mathfrak{G}$ is a filter on X containing $\mathfrak{f} \in \mathcal{L}(X)$;

(P4) if $[\mathfrak{f}] \cap Y = \{F \cap Y; F \in \mathfrak{f}\} \in \mathcal{L}(Y)$ whenever $\mathfrak{f} \in \mathcal{L}(X)$, $Y \subset X, Y \in \mathcal{K}$, $[\mathfrak{f}] \cap Y$ is a filter on Y;

(P5) if the space X is a subspace of a t-semi-uniform space $Y \in \mathcal{K}$ such that $\mathfrak{f}$ converges in Y whenever $\mathfrak{f} \in \mathcal{L}(X)$ is a non-converging filter in X ;

(P6) if $f([\mathfrak{f}]) = \{f(F); F \in \mathfrak{f}\}$ is a base of a filter from $\mathcal{L}(Y)$ whenever $f : X \to Y, Y \in \mathcal{K}$ is uniformly continuous mapping, $\mathfrak{f} \in \mathcal{L}(X)$.

In case $\mathcal{K} = \mathtt{TSUnif}_2$ the category $\mathcal{K}$ need not be specified.

An investigation of (P5) is postponed to Proposition 3.38.

Proposition 3.15

(1) Classic Cauchy filters have properties (P2), (P3), (P4), (P6)(and (P1) on $\mathtt{TSUnif}_{o_2}$*).*

(2) Cauchy filters have properties (P2), (P3), (P4), (P6) (and (P1) on $\mathtt{TSUnif}_{po_2}$*).*

(3) Weak Cauchy filters have properties (P2), (P6) (and (P1) on $\mathtt{TSUnif}_{po_2}$*).*

(4) Semi-Cauchy filters have properties (P1), (P2), (P3), (P6).

`Proof:` The facts about (P1) follow from Proposition 3.4 and those about (P6) follow from Lemma 3.3.

(1)

(P2) If $F \times F \subset U \cap (A \times A)$ then $F \times F \subset U$.

(P3) is clear.

(P4) If $F \in \mathfrak{f}$ and $F \times F \subset U$ then $G = F \cap A \in [\mathfrak{f}] \cap A$ and $G \times G \subset U \cap (A \times A)$.

(2)

(P2) If $U[x] \cap A \in \mathfrak{f}$ for each $x \in F \in \mathfrak{f}$ then $U[x] \in \mathfrak{f}$ for each $x \in F \in \mathfrak{f}$.

(P3) is clear.

(P4) If $U[x] \in \mathfrak{f}$ for each $x \in F \in \mathfrak{f}$ then $U[x] \cap A \in [\mathfrak{f}] \cap A$ for each $x \in F \cap A \in [\mathfrak{f}] \cap A$.

(3)

(P2) If $U[x] \cap A \in \mathfrak{f}$ then $U[x] \in \mathfrak{f}$.

(4)

(P2) If $U[x] \cap A \in \mathfrak{f}$ then $U[x] \in \mathfrak{f}$.

(P3) is clear. ◇

It follows from Example 3.5 that the class of weak Cauchy filters (and thus of Cauchy or classic Cauchy filters) does not satisfy (P1).

The next example shows that weak Cauchy filters need not satisfy (P3), (P4) and that semi-Cauchy filters need not satisfy (P4).

Example 3.16 Let $(X, \mathcal{U}_\mathcal{R})$ be the uniform space of real numbers with its standard uniformity. Let $V = \{(x, y) \in X \times X; x = 0 \vee y = 0 \vee x < 2y < 4x\}$. The set V is symmetric. Let $\mathcal{U}$ be the t-semi-uniformity with the subbase $\mathcal{U}_\mathcal{R} \cup \{V\}$. $V[x] = (\frac{1}{2}x; 2x) \cup \{0\}$ for each $x > 0$, $V[x] = (2x; \frac{1}{2}x) \cup \{0\}$ for each $x < 0$ and $V[0] = X$. Thus $V[x]$ is a neighborhood of x for each $x \in X$. The filter $\mathfrak{f} = \mathcal{U}_0$ is a weak Cauchy filter because in every $U[0]$ there is $x = 0$ such that $U[x] \in \mathfrak{f}$.

The filter $\mathfrak{G}$ with the base $\{U[0] \setminus \{0\}\}_{U \in \mathcal{U}}$ si not weak Cauchy because for V there is no $x \in U[0] \setminus \{0\}$ such that $V[x] \in \mathfrak{G}$. Therefore, (P3) is not satisfied.

Let $A = X \smallsetminus \{0\}$. Now $[\mathfrak{f}] \cap A = \{U[0] \smallsetminus \{0\}\}_{U \in \mathcal{U}}$ is a filter on A but it is not semi-Cauchy on A. In fact, let there be a point $x \in X \smallsetminus \{0\}$ such that $V[x] \cap A \in [\mathfrak{f}] \cap A$. Since $V[x] \cap A = (\frac{1}{2}x; 2x)$ for each $x > 0$, $V[x] \cap A = (2x; \frac{1}{2}x)$ for each $x < 0$, one has $V[x] \notin [\mathfrak{f}] \cap A$ for each $x \neq 0$. Thus $[\mathfrak{f}] \cap A$ is not semi-Cauchy in A.

3.5 Completeness

This part is devoted to the completeness of t-semi-uniform spaces defined classically by means of Cauchy-like filters.

Definition 3.17 Let $\mathcal{L}(X)$ be a class of filters on a t-semi-uniform space $(X,\mathcal{U})$. The space $(X,\mathcal{U})$ is called *$\mathcal{L}$–complete* if every filter from $\mathcal{L}(X)$ converges. In particular (we omit the word "Cauchy"):

(1) A semi-uniform space $(X,\mathcal{U})$ is called *classically complete* if every classic Cauchy filter in $(X,\mathcal{U})$ converges.

(2) A semi-uniform space $(X,\mathcal{U})$ is called *complete* if every Cauchy filter in $(X,\mathcal{U})$ converges.

(3) A semi-uniform space $(X,\mathcal{U})$ is called *weakly complete* if every weak Cauchy filter in $(X,\mathcal{U})$ converges.

(4) A semi-uniform space $(X,\mathcal{U})$ is called *semi-complete* if every semi-Cauchy filter in $(X,\mathcal{U})$ converges.

There are several general results about $\mathcal{L}$-completeness:

Proposition 3.18 *Let $\mathcal{L}(X), \mathcal{K}(X)$ be classes of filters on a t-semi-uniform space $(X,\mathcal{U})$. Let $(X,\mathcal{U})$ be $\mathcal{L}$ complete. If $\mathcal{K}(X) \subset \mathcal{L}(X)$ then $(X,\mathcal{U})$ is $\mathcal{K}$ complete, too.*

Corollary 3.19

(1) If a t-semi-uniform space $(X,\mathcal{U})$ is semi-complete then it is weakly complete.

(2) If a t-semi-uniform space $(X,\mathcal{U})$ is weakly complete then it is complete.

(3) If a t-semi-uniform space $(X,\mathcal{U})$ is complete then it is classically complete.

Proposition 3.20 *If $\mathcal{L}$ has (P6), $(X,\mathcal{U})$ is $\mathcal{L}$-complete and $\mathcal{V}$ is finer than $\mathcal{U}$ and both induce the same topology, then $(X,\mathcal{V})$ is $\mathcal{L}$-complete.*

`Proof:` If $(X,\mathcal{U})$ is $\mathcal{L}$-complete and $\mathcal{V}$ is finer than $\mathcal{U}$ then the identity map $id : (X,\mathcal{V}) \to (\mathcal{X},\mathcal{U})$ is uniformly continuous. Thanks to (P6) every $\mathcal{L}-$filter on $(X,\mathcal{V})$ is an $\mathcal{L}-$filter on $(X,\mathcal{U})$ and thus it converges. $\diamond$

3.6 Completeness of special t-semi-uniformities

We examine completeness of fine and coarse t-semi-uniformities.
We shall use total boundedness and precompactness in the following meaning:

Definition 3.21

A t-semi-uniform space $(X,\mathcal{U})$ is called *totally bounded* if for every $U \in \mathcal{U}$ one can find a finite K with $U[K] = X$.

A t-semi-uniform space $(X,\mathcal{U})$ is called *precompact* if for every $U \in \mathcal{U}$ one can find a finite cover $\{A_1, ..., A_n\}$ of X such that $\bigcup_{i=1}^{n} A_i \times A_i \subset U$.

The next assertion is trivial but useful.

Lemma 3.22 *If every ultrafilter on X belongs to $\mathcal{L}(X,\mathcal{U})$ and $(X,\mathcal{U})$ is $\mathcal{L}$-complete, then $(X,\mathcal{U})$ is compact.*

Corollary 3.23

If $(X,\mathcal{U})$ is totally bounded and semi-complete, then $(X,\mathcal{U})$ is compact.

If $(X,\mathcal{U})$ is precompact and classically complete, then $(X,\mathcal{U})$ is compact.

First we look at the completeness of coarse t-semi-uniformities. Every coarse t-semi-uniform space is precompact, so it is compact whenever it is classically complete. The converse need not be true:

Theorem 3.24 *Let X be a T_2-topological space, $\mathcal{U}_c$ the coarse t-semi-uniformity on X. Then the following conditions are equivalent:*

1. *$(X,\mathcal{U}_c)$ is classically complete.*
2. *$(X,\mathcal{U}_c)$ is complete.*
3. *$(X,\mathcal{U}_c)$ is weakly complete.*
4. *$(X,\mathcal{U}_c)$ is semi-complete.*
5. *X is a compact space having at most one accumulation point.*

`Proof:` Clearly, $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. Assume now that $(X,\mathcal{U}_c)$ is classically complete. The compactness of X follows from the preceding note. By Proposition 3.8, the filter $\mathfrak{k}$ of cofinite subsets of X is a classic Cauchy filter and, so, it converges to a point a. It means that the neighborhood system of a coincides with complements of finite sets $K \not\ni a$. Thus, X is either finite or it has the unique accumulation point, namely the point a.

Suppose now that X is an infinite compact space having a unique accumulation point a. Let $\mathfrak{f}$ be a semi-Cauchy filter on $(X,\mathcal{U}_c)$. Then, by Proposition 3.8, $\mathfrak{f}$ either has an empty intersection (and then it converges to a) or an isolated point b belongs to $\mathfrak{f}$ (then $\mathfrak{f}$ converges to b) or the intersection of $\mathfrak{f}$ is the point a (then $\mathfrak{f}$ converges to a). $\diamond$

For T_1-space the above characterization is not true:

Proposition 3.25 *The coarse semi-uniformity of an infinite coarse T_1-space X is complete (thus classically complete) and not weakly complete (thus not semi-complete).*

`Proof:` By Proposition 3.8, the only non-convergent Cauchy filters could be those with empty intersection. Such filters converge to any point on coarse T_1-spaces.

By the same Proposition, every filter $\{F \subset X; A \subset F\}$, where $A, X \setminus A$ are infinite, is weak Cauchy and non-convergent. $\diamond$

Now we look at the completeness of fine t-semi-uniformities.

Since every paracompact space (in fact, every Dieudonné complete space) has a complete fine uniformity, we have the following result following from Proposition 3.20:

Proposition 3.26 *The fine t-semi-uniformity of a paracompact topological space is semi-complete.*

The next two examples shows the expected situation that there is a non-paracompact space the fine t-semi-uniformity of which is complete in some sense.

Example 3.27 ($\omega + 1$ has semi-complete (thus weakly complete, complete, classically complete) fine semi-uniformity.)
Take the space $X = \omega + 1$ with its standard interval topology, $\mathcal{U}$ the fine semi-uniformity on X.
Let $\mathfrak{f}$ be a semi-Cauchy filter on $(X,\mathcal{U})$. Let $\mathfrak{f}$ has no limit point in X.
For every $x \in X$ there is a neighbourhood M_x of x that does not belong to the filter $\mathfrak{f}$. For $n \in \omega$ we can take $M_n = \{n\}$. Denote $U = \bigcup_{x \in X} M_x$. Then $V = U \cup U^{-1}$ is in the fine semi-uniformity $\mathcal{U}$ on X.
If $\mathfrak{f}$ is semi-Cauchy then there must be a point $x \in X$ such that $V[x] \in \mathfrak{f}$. For ω it is $V[\omega] = U[\omega] \notin \mathfrak{f}$, because there is neither $n \in \mathbb{N}$ that $\omega \in U[n] = \{n\}$. Thus there is $n_0 \in \omega$ that $V[n_0] = \{n_0\} \cup \{\omega\} \in \mathfrak{f}$.
Take $M'_\omega = M_\omega \setminus \{n_0\} \notin \mathfrak{f}$, $M'_n = M_n$ for $n \in \omega$, $U' = \bigcup_{x \in X} M'_x$. Then $V' = U' \cup (U')^{-1}$ is in the fine semi-uniformity on X. And there must be a point $x' \in X$ such that $V'[x'] \in \mathfrak{f}$. Similarly as for V we have $n'_0 \in \omega$ that $V'[n'_0] = \{n'_0\} \cup \{\omega\} \in \mathfrak{f}$. Clearly $n'_0 \neq n_0$. As $\mathfrak{f}$ is a filter the set $V[n] \cap V'[n'] = \{\omega\}$ has to be in $\mathfrak{f}$, which is a contrary.

We show one more example of a non-paracompact space that is complete in some sense:

Example 3.28 (Mrówka space has weak complete (thus complete and classically complete) fine semi-uniformity.)
Take the Mrówka space described in [7],3.6.I(a):
Let $\{\mathbb{N}_s\}_{s\in S}$ be an infinite family of infinite subsets of $\mathbb{N}$ ($\mathbb{N}$ is the set of positive integers) where S is an uncountable set such that $S \cap \mathbb{N} = \emptyset$ and $\mathbb{N}_s \cap \mathbb{N}_t$ is finite for every couple s, t of distinct elements of S. The Mrówka space is the space $X = \mathbb{N} \cup S$ with the topology generated by the neighbourhood bases $\{\mathcal{B}(x)\}_{x\in X}$, where $\mathcal{B}(n) = \{\{n\}\}$ for $n \in \mathbb{N}$ and $\mathcal{B}(s) = \{\{s\} \cup (\mathbb{N}_s \setminus \{1, 2, .., i\})\}_{i=1}^{\infty}$ for $s \in S$. X is non-normal, thus non-paracompact.

Let $\mathfrak{f}$ be a weak Cauchy filter on $(X,\mathcal{U})$, where $\mathcal{U}$ is the fine semi-uniformity on X. Let $\mathfrak{f}$ has no limit point in X. First we show that $S \in \mathfrak{f}$.
For every $x \in X$ there is a neighbourhood M_x of x that does not belong to the filter $\mathfrak{f}$. For $n \in \mathbb{N}$ we can take $M_n = \{n\}$. Denote $U = \bigcup_{x\in X} M_x$. Then $V = U \cup U^{-1}$ is in the fine semi-uniformity on X. If $\mathfrak{f}$ is weak Cauchy then it is semi-Cauchy and there must be a point $x \in X$ such that $V[x] \in \mathfrak{f}$. For $s \in S$ it is $V[s] = U[s] \notin \mathfrak{f}$, because there is neither $n \in \mathbb{N}$ that $s \in U[n] = \{n\}$ nor $s' \in S, s' \neq s$ that $s \in U[s'] \subset \{s'\} \cup \mathbb{N}$. Thus there is $n_0 \in \mathbb{N}$ that $V[n_0] = \{n_0\} \cup \{s; n_0 \in M_s\} \in \mathfrak{f}$. Take $M'_s = M_s \setminus \{1, .., n_0\} \notin \mathfrak{f}$ for $s \in S$, $M'_n = M_n$ for $n \in \mathbb{N}$, $U' = \bigcup_{x\in X} M'_x$. Then $V' = U' \cup (U')^{-1}$ is in the fine semi-uniformity on X. And there must be a point $x' \in X$ such that $V'[x'] \in \mathfrak{f}$. Similarly as for V we have $n' \in \mathbb{N}$ that $V'[n'] = \{n'\} \cup \{s; n' \in M'_s\} \in \mathfrak{f}$. Clearly $n' > n_0$. As $\mathfrak{f}$ is a filter the set $V[n_0] \cap V'[n'] \in \mathfrak{f}$. Then $S \in \mathfrak{f}$ because S contains $V[n_0] \cap V'[n']$.

From the fact that $\mathfrak{f}$ is weak Cauchy it follows that there must exist $s \in S$ such that $V[s] = U[s] \in \mathfrak{f}$, which is a contrary.

3.7 Absolute closedness and (P5)

We shall define absolutely closed structures. All the definitions and results in the next part can be achieved in a more generally situations, namely for categories having a closure operator and embedding morphisms satisfying some properties. Nevertheless, in this part all the categories will be full subcategories of `TSUnif` or of `Top` with the usual closures of their (induced) topologies and embeddings.

Definition 3.29 Let $\mathcal{B}$ be a category. An object $A \in \mathcal{B}$ is called *$\mathcal{B}$-closed* if A is closed in any object $B \in \mathcal{B}$ in which it is embedded.
For $\mathcal{B} = Top_2$, $\mathcal{B}$-closed objects are called *H-closed* spaces.

Proposition 3.30 *If $\mathcal{M}$ is a subcategory of $\mathcal{B}$ then each $\mathcal{B}$-closed object of $\mathcal{M}$ is also $\mathcal{M}$-closed.*

In our situation, $\mathcal{B}$-closed objects have a reasonable sense for structures bearing Hausdorff topology only. For instance, if $\mathcal{B} =$ `TSUnif` then $\mathcal{B}$-closed objects are empty spaces only.
It should be recalled that the next lemma does not hold for uniform spaces (e.g., for some fine uniformities).

Lemma 3.31 *Let $(X,\mathcal{U})$ be a t-semi-uniform space and the corresponding topological space X be a topological subspace of a symmetric topological space Y. Then there exists such a t-semi-uniformity $\mathcal{V}$ on Y that $\mathcal{U}$ is the subspace semi-uniformity on X induced by $\mathcal{V}$.*

`Proof:` There exists the coarsest semi-uniformity $\mathcal{W}_2$ on the underlying set $|Y|$ of the topological space Y such that $\mathcal{U}$ is the subspace semi-uniformity on X with respect to $\mathcal{W}_2$. Indeed, $\mathcal{W}_2 = \{U \cup (Y \times (Y - X)) \cup ((Y - X) \times Y); U \in \mathcal{U}\}$. Take $\mathcal{V} = \sup\{\mathcal{W}_1, \mathcal{W}_2\}$, where $\mathcal{W}_1$ is the coarse t-semi-uniformity on Y. Clearly, $\mathcal{U}$ is finer than the restriction of $\mathcal{V}$ to X since the restriction of $\mathcal{W}_1$ to X is the coarse semi-uniformity on X (see Corollary 2.56), and it is also coarser since the restriction of $\mathcal{W}_2$ to X coincides with $\mathcal{U}$.
It remains to show that $\mathcal{V}$ induces the original topology on Y. But $\mathcal{W}_2$ induces a coarser closure on Y and $\mathcal{W}_1$ induces the topology of Y. Since suprema in `SUnif` preserve closures, the semi-uniformity $\mathcal{V}$ induces the topology of Y. ◇

Corollary 3.32 *Restrictions of fine t-semi-uniformities to subsets are fine.*

`Proof:` Let $\mathcal{U}$ be the subspace semi-uniformity on the set $X \subset Y$ with respect to the fine semi-uniformity $\mathcal{V}_f$ on Y. Let $\mathcal{V}$ be the superspace semi-uniformity on Y with respect to $\mathcal{U}$ defined in the proof of Lemma 3.31. And let $\mathcal{V}'$ be the superspace semi-uniformity on Y with respect to $\mathcal{U}_f$ defined in the proof of Lemma 3.31, where $\mathcal{U}_f$ is the fine semi-uniformity on X.

As $\mathcal{U} \subset \mathcal{U}_f$ it is $\mathcal{V} \subset \mathcal{V}' \subset \mathcal{V}_f$ (see the proof of Lemma 3.31). And the same holds for subspace semi-uniformities:

$\mathcal{U} \subset \mathcal{U}_f \subset \mathcal{U}$.

◇

Corollary 3.33 *A t-semi-uniform space is* $\mathtt{TSUnif}_2$*-closed if and only if its topology is H-closed.*

`Proof:` Let $(X,\mathcal{U})$ be a $\mathtt{TSUnif}_2$-closed space, Y a Hausdorff topological superspace of X. Then by the previous lemma there is a t-semi-uniformity $\mathcal{V}$ inducing the topology of Y such that $\mathcal{U}$ is the subspace semi-uniformity on X with respect to $\mathcal{V}$. As $(X,\mathcal{U})$ is $\mathtt{TSUnif}_2$-closed then X is closed in Y. The other implication is clear. ◇

For corresponding definitions used in the next results see Subsection 3.4.

Proposition 3.34 *If a filter operator property $\mathcal{L}$ has properties (P1) and (P4) in $\mathcal{B}$ then every Hausdorff $\mathcal{L}$-complete space $X \in \mathcal{B}$ is $\mathcal{B}$-closed.*

`Proof:` Let $X \in \mathcal{B}$ be a subspace of a Hausdorff t-semi-uniform space $(Y,\mathcal{U}) \in \mathcal{B}$. Let $y \in \overline{X}$. Then by (P1) the filter $\mathcal{U}_y$ is Cauchy on $(Y,\mathcal{U})$. Since X is dense in $\overline{X}$ and the (P4) holds then the filter $[\mathcal{U}_y] \cap X = \{U[y] \cap X; U \in \mathcal{U}\}$ is a filter and $[\mathcal{U}_y] \cap X \in \mathcal{L}(X)$. Then $[\mathcal{U}_y] \cap X$ converges to some $x \in X$ in X and, thus, in Y. Since Y is Hausdorff we have $x = y$. Consequently, $X = \overline{X}$ and X is $\mathcal{B}$-closed. ◇

After Proposition 3.15 we showed that none of the properties defined in Subsection 3.4 satisfies both (P1) and (P4) in $\mathtt{TSUnif}_2$. Classic Cauchy filters or Cauchy filters satisfy (P1) and (P4) in $\mathtt{TSUnif}_{o_2}$ or in $\mathtt{TSUnif}_{po_2}$, respectively.

Corollary 3.35

Every Hausdorff pointwise open semi-uniform space X that is complete is H-closed.

Every Hausdorff open semi-uniform space X that is classically complete is H-closed.

The next result shows a converse situation to Proposition 3.34.

Proposition 3.36 *If a property $\mathcal{L}$ has the property (P5) in $\mathcal{B}$ then every $\mathcal{B}$-closed space is $\mathcal{L}$-complete.*

`Proof:` Let X be a $\mathcal{B}$-closed space. Let $\mathfrak{f} \in \mathcal{L}(X)$. Let $(Y,\mathcal{U})$ be such a t-semi-uniform space that X is a subspace of $(Y,\mathcal{U})$ (with its subspace semi-uniformity w.r.t. $\mathcal{U}$) and that $\mathfrak{f}$ converges in Y. By (P5) such a space $(Y,\mathcal{U})$ exists. If $\mathfrak{f}$ converges to $y \in Y$ then $y \in \overline{X}$. Since X is absolutely closed $\overline{X} = X$ and X is $\mathcal{L}$-complete. ◇

Corollary 3.37 *If a property $\mathcal{L}$ has the property (P1), (P2) and (P5) in $\mathcal{B}$ then a Hausdorff t-semi-uniform space is $\mathcal{B}$-closed space iff it is $\mathcal{L}$-complete.*

Unfortunately, none of our special completeness is of such a type:

Proposition 3.38 *Neither classic Cauchy or Cauchy or weakly Cauchy or semi-Cauchy filters satisfy the property (P5) in* $\mathtt{TSUnif}_2$ *(or in* $\mathtt{TSUnif}_{po_2}$, $\mathtt{TSUnif}_{o_2}$*).*

`Proof:` It follows from Theorem 3.24 that there are compact Hausdorff spaces (e.g., the interval $[0,1]$), thus H-closed spaces, having its coarse semi-uniformity (thus an open uniformity) non-complete. ◇

In fact, it follows from the preceding proof that even in much more restrictive classes of semi-uniform spaces the property (P5) is not satisfied for our Cauchy–like filters.

3.8 Categorical approach

The second assertion of the next Proposition is, in fact, Proposition 3.34. We repeat it because of completeness.

Proposition 3.39

(i) Let X be a closed subspace of an $\mathcal{L}$-complete space Y. If $\mathcal{L}$ satisfies (P2) then X is $\mathcal{L}$ complete.

(ii) Let X be an $\mathcal{L}$-complete subspace of a Hausdorff semi-uniform space Y. If $\mathcal{L}$ satisfies (P1) and (P4) then X is closed in Y.

`Proof:`

(i) Let Y be an $\mathcal{L}$-complete t-semi-uniform space and X be a closed subset of Y. Let $\mathfrak{f}$ be an $\mathcal{L}$ filter on X. Then by (P2) $\mathfrak{f}$ is an $\mathcal{L}$-filter base on Y. Since Y is $\mathcal{L}$ complete $\mathfrak{f}$ converges to some point $x \in Y$. If X is closed then $x \in X$ and $U[x] \cap X \in \mathfrak{f}$ for each $U \in \mathcal{U}$. Thus $\mathfrak{f}$ is convergent in X and X is $\mathcal{L}$-complete.

(ii) See Proposition 3.34.

$\diamond$

Proposition 3.40 *Let $\mathcal{L}$ satisfy (P6) on t-semi-uniform spaces $X_i, i \in I$. If all of the spaces X_i are $\mathcal{L}$-complete then their product $\prod_{i\in I} X_i$ is also $\mathcal{L}$-complete.*

`Proof:` Denote $X = \prod_{i\in I} X_i$. Let $\mathfrak{f}$ be an $\mathcal{L}$-filter on X. Then $\pi_i(\mathfrak{f})$ is an $\mathcal{L}$-filter on X_i for each $i \in I$ because π_i are uniformly continuous. As X_i is $\mathcal{L}$-complete for each $i \in I$ then $\pi_i(\mathfrak{f})$ converges to a point $x_i \in X_i$. We will prove that $\mathfrak{f}$ converges to the point $(x_i)_{i\in I}$.

Take semi-uniform neighborhoods U_i of Δ_{X_i} for a finitely many indices i. Then $U_i[x_i] \in \pi_i(\mathfrak{f})$ for each $U_i \in \mathcal{U}_i$ and $i \in I$, which implies $\pi_i^{-1}(U_i[x_i]) \in \mathfrak{f}$. As $\mathfrak{f}$ is a filter, the finite intersection of the last sets belong to $\mathfrak{f}$. These finite intersections form a base of neighborhoods of the point $(x_i)_{i\in I}$. Consequently, the point $(x_i)_{i\in I}$ is a limit of $\mathfrak{f}$. $\diamond$

Definition 3.41 The category of all Hausdorff $\mathcal{L}$-complete t-semi-uniform spaces with uniformly continuous mappings will be denoted by $\mathtt{Compl}_{\mathcal{L}}$.

Theorem 3.42 *If the property $\mathcal{L}$ satisfy (P2) and (P6) then the category $\mathtt{Compl}_{\mathcal{L}}$ is an epireflective subcategory of the category $\mathtt{TSUnif}_2$.*

`Proof:` The proof follows from a general categorical result that $\mathtt{Compl}_{\mathcal{L}}$ is closed under the formation of products and closed subspaces in $\mathtt{TSUnif}_2$. For readers' convenience we shall give a proof here.

Let X be an arbitrary Hausdorff t-semi-uniform space. First let us remind that the cardinality of the closure of a space X in any Hausdorff superspace is less or equal to $2^{2^{|X|}}$ where $|X|$ denotes the cardinality of X. Let $\mathcal{M}$ be the set of all uniformly continuous maps X to an $\mathcal{L}$-complete t-semi-uniform space Y where $|Y| \le 2^{2^{|X|}}$ and $\mathcal{O}$ will be a set of all such spaces Y each of them as many times as many uniformly continuous maps X to Y exist. Let a map r' be the product $\prod_{f\in\mathcal{M}} f : X \to \prod_{Y\in\mathcal{O}} Y$. The reflection arrow for X will be the restriction $r' : X \to \overline{r'(X)}$.
From the previous lemmas we know that $\prod_{Y\in\mathcal{O}} Y$ is an $\mathcal{L}$ complete t-semi-uniform space and so is its closed subspace $\overline{r'(X)}$.
Now let Y be any space from $\mathcal{C}_2$ and $f : X \to Y$ a uniformly continuous map. We know that $\overline{|f(X)|} \le 2^{2^{|X|}}$. Thus $\overline{f(X)}$ has its representative in $\mathcal{O}$. Then there is a unique map (projection) $\pi : \overline{r'(X)} \to Y$ such that $f = \pi r$. $\diamond$

Corollary 3.43 *Either of the classes of classically complete or complete or weakly complete or semi-complete spaces, resp., is epireflective in* $\mathtt{TSUnif}_2$.

`Proof:`According to Proposition 3.15, every mentioned filter operator has properties (P2) and (P6). $\diamond$

3.9 Emreflective subcategories

The previous Subsection brought the result about epireflectivity of our completeness. Nevertheless, completeness in uniform spaces has one more property, namely the reflective maps are embeddings. We shall now look at that possibility.

In the next, let $\mathcal{B}$ be a subcategory of $\mathtt{Top}_2$ or of $\mathtt{TSUnif}_2$ (as usually our subcategories are full and isomorphism closed).

Recall that monomorphisms in $\mathtt{Top}_2$ are one-to-one continuous maps, monomorphisms in $\mathtt{TSUnif}_2$ are one-to-one uniformly continuous maps, epimorphisms in $\mathtt{Top}_2$ are continuous maps with dense images.

Proposition 3.44 *Epimorphisms in $\mathtt{TSUnif}_2$ are uniformly continuous maps with dense images.*

`Proof:` If a map has a dense image then it is clearly an epimorphism in $\mathtt{TSUnif}_2$ (because it is an epimorphism in $\mathtt{Top}_2$, see, e.g., [1]).

If a map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ has not a dense image we define two different uniformly continuous maps $g, h : (Y, \mathcal{V}) \to (Z, \mathcal{W})$ such that $gf = hf$.

Let Z be the topological quotient of the disjoint sum $Y_1 + Y_2$, where $Y_1 = Y_2 = Y$ (simply $Y + Y$), obtained by identifying the same points of $\overline{f(X)}$, denote q the corresponding quotient map.

It is clear (and known) that the topology of Z is Hausdorff.

Denote by $\mathcal{W}$ the image of $\mathcal{V}+\mathcal{V}$ under $q\times q$, where $\mathcal{V}+\mathcal{V}$ is the sum semi-uniformity of $(Y,\mathcal{V})+(Y,\mathcal{V})$. Then it is easily checked that the closure induced by $\mathcal{W}$ is equal to the closure induced by the topology of Z (cf. also Theorem 37D.11 in [3]).

Define $g = q\ j_1$, $h = q\ j_2$, where $j_i : Y \to Y_1 + Y_2$ is the i-th injection, $i = 1, 2$.

◇

Definition 3.45 A reflective subcategory $\mathcal{R}$ of a category $\mathcal{B}$ is called *emreflective subcategory of $\mathcal{B}$* if all reflection arrows are embeddings.

Lemma 3.46 *Let $\mathcal{R}$ be an epireflective subcategory of the category $\mathcal{B}$. Let X be a $\mathcal{B}$-closed object such that its reflection arrow is an embedding. Then $X \in \mathcal{R}$.*

`Proof:` Let X be such an object and $r : X \to rX$ be a reflection arrow of X. As r is an epimorphism we can take $\overline{r(X)} = rX \in \mathcal{R}$.

As X is $\mathcal{B}$-closed it is $r(X) = \overline{r(X)}$.

As r is an embedding we can identify $X = r(X) = rX$ and, therefore, $X \in \mathcal{R}$. ◇

Lemma 3.47 *Every emreflective subcategory $\mathcal{R}$ of $\mathcal{B}$ is epireflective.*

`Proof:` Since every embedding is a monomorphism, $\mathcal{R}$ is monoreflective. And every monoreflective subcategory of $\mathcal{B}$ is bireflective. ◇

The previous lemmas entail the following result.

Theorem 3.48 *Let $\mathcal{R}$ be an emreflective subcategory of $\mathcal{B}$. Then every $\mathcal{B}$-closed object belongs to $\mathcal{R}$.*

Corollary 3.49 *If each object of a category $\mathcal{B}$ embedds as a closed subspace into a product of $\mathcal{B}$-closed objects, then the only emreflective subcategory of $\mathcal{B}$ is $\mathcal{B}$ itself.*

Example 3.50 Let $\mathcal{B}$ be the category of completely regular Hausdorff spaces. Since there are nontrivial emreflective subcategories (e.g., the subcategory of compact spaces), not every completely regular Hausdorff space embeds as a closed subspace into a product of $\mathcal{B}$-closed spaces. Of course, in our case $\mathcal{B}$-closed spaces coincide with compact spaces and they form the least emreflective subcategory of $\mathcal{B}$.

Example 3.51 Let $\mathcal{B}$ be the category of Hasdorff topological spaces. It is not difficult to show that every Hausdorff space embedds as a closed subspace into an H-closed space. Indeed, for any Hausdorff space X take its Alexandroff double Y ($Y = X \times \{0,1\}$ with X embedded onto $X \times \{0\}$, the subspace $X \times \{1\}$ open discrete, and if U is a neighbourhood of x in X, then $U \times \{0,1\} - \{(x,1)\}$ is a typical neighbourhood of $(x, 0)$. Then X is closed in the Katětov hull τY of Y.

The previous example gives the following (known) result.

Proposition 3.52 *There is no emreflective subcategory of the category* $\mathtt{Top}_2$ *different from* $\mathtt{Top}_2$*.*

Example 3.53 Let $\mathcal{B} = \mathtt{TSUnif}_2$. Let $X \in \mathtt{TSUnif}_2$. By Corollary 3.33 the situation is the same as in the category $\mathtt{Top}_2$. If we take the space τY from the previous example then by Lemma 3.31 there is a t-semi-uniformity on τY such that X is a t-semi-uniform subspace of it. Since τY is H-closed it is also $\mathtt{TSUnif}_2$-closed.

The previous example gives the following result.

Proposition 3.54 *There is no emreflective subcategory of the category* $\mathtt{TSUnif}_2$ *different from* $\mathtt{TSUnif}_2$*.*

Example 3.55 Let $\mathcal{B} = \mathtt{Unif}_2$. We know that there is an emreflective subcategory of $\mathcal{B}$, namely the subcategory of complete separated uniform spaces. Then by Corollary 3.49 there exists a separated uniform space that cannot be embedded as a closed subspace into a $Unif_2$-closed uniform space. This assertion is clear because a separated uniform space is $\mathtt{Unif}_2$-closed iff it is complete iff it can be embedded into some $\mathtt{Unif}_2$-closed space as a closed subspace.

Example 3.56 Let $\mathcal{B} = \mathtt{TopGrp}_2$ be the category of all Hausdorff topological groups with continuous homomorphisms. We know that there is an emreflective subcategory $\mathcal{R}$ of $\mathcal{B}$ where reflective arrows have dense images, namely the subcategory of Hausdorff topological groups complete in their two-sided uniformities. Each reflection arrow in $\mathcal{R}$ is a homomorphic embedding with a dense range, and, thus, despite the fact that epimorphisms in $\mathtt{TopGrp}_2$ need not have dense ranges, see [25], the emreflective subcategory $\mathcal{R}$ is closed under the closed subojects.

We can substitude "reflective with reflective arrows with dense images" for "epireflective" in Lemma 3.46. In the proof $\overline{r(X)} = rX$ because r has a dense image.

We can substitude "emreflective with reflective arrows with dense images" for "emreflective". in Theorem 3.48 and Corollary 3.49.

Proposition 3.57 *There exists a Hausdorff topological group that cannot be embedded as a closed subspace into a (product of)* $\mathtt{TopGrp}_2$*-closed Hausdorff topological group.*

The following corollary follows also from Proposition 3.38.

Corollary 3.58 *If* $\mathcal{L}$ *is any of our four special filter operators, the category* $\mathtt{Compl}_{\mathcal{L}}$ *of all* $\mathcal{L}$*-complete Hausdorff t-semi-uniform spaces with uniformly continuous mapping is epireflective but not emreflective subcategory of the category* $\mathtt{TSUnif}_2$ *of all Hausdorff t-semi-uniform spaces with uniformly continuous mappings.*

4 Completion of t-semi-uniform spaces

We have seen that there is no possibility to construct Hausdorff completions of Hausdorff t-semi-uniform spaces that would all be embeddings. Even if we change to other possibilities of Cauchy filters or to other definitions of completeness (e.g., to weaken convergence of Cauchy filters), we cannot get the requested result.

There is still chance to construct non-Hausdorff completions that would be embeddings. In the former case, categorical methods do not help too much and so, we now return to classical procedures, i.e., adding to X non-converging Cauchy-like filters on X.

We can see in Example 5.16 a T_1-semi-uniform space (a quasi-topological qroup) that has not a weak, and thus not a semi-completion. On the other hand we construct classic completion and completion for all t-semi-uniform spaces.

Definition 4.1 *Let* $\mathcal{L}$ *be a filter operator on* $\mathtt{TSUnif}$ *and* X *be a t-semi-uniform space. Every* $\mathcal{L}$*-complete space* Y *containing* X *as a dense t-semi-uniform subspace is called an* $\mathcal{L}$*-completion of* X*.* In particular (we omit the word "Cauchy"): Every (classic, weak, semi-) complete space Y containing X as a dense t-semi-uniform subspace is called *a (classic, weak, semi-) completion of* X*.*

First we will try to create a semi-uniformity on the set of all Cauchy filters in the same way as for uniformities (See [21].). We do not examine completeness of the semi-uniform space $(\hat{X}, \hat{\mathcal{U}})$ because it need not be topological as we see in Examples 4.7 and 4.8.

Definition 4.2 Let $(X,\mathcal{U})$ be a t-semi-uniform space. Let us define sets $\hat{X},\hat{\mathcal{U}}$ in the following way:
$\hat{X} = \{\mathfrak{f}; \mathfrak{f}$ is a Cauchy filter on$(X,\mathcal{U})\}$;
$\hat{\mathcal{U}}$ has a base consisting of sets $\hat{U}$ that are defined in the following way:
$(\mathfrak{f},\mathfrak{g}) \in \hat{U} \;\Leftrightarrow ((\exists F \in \mathfrak{f}\cap\mathfrak{g})(\forall z \in F)(U[z] \in \mathfrak{f}\cap\mathfrak{g}))$, where $U \in \mathcal{U}$.

Theorem 4.3 *Let $(X,\mathcal{U})$ be a t-semi-uniform space. The couple $(\hat{X},\hat{\mathcal{U}})$ is a semi-uniform space.*

`Proof:` We will show all the four conditions in the definition of the semi-uniformity $\hat{\mathcal{U}}$ are satisfied.

(i) It is clear that $(\mathfrak{f},\mathfrak{f}) \in \hat{U}$ for each Cauchy filter $\mathfrak{f} \in \hat{X}$ and $U \in \mathcal{U}$.

(ii) We will show that $\hat{U}\cap\hat{V} = (\widehat{U\cap V})$:
$(\mathfrak{f},\mathfrak{g}) \in \hat{U}\cap\hat{V}$ iff there are sets $F,G \in \mathfrak{f}\cap\mathfrak{g}$ such that $U[x],V[y] \in \mathfrak{f}\cap\mathfrak{g}$ for each $x \in F, y \in G$. Then $(U\cap V)[x] = U[x]\cap V[x] \in \mathfrak{f}\cap\mathfrak{g}$ for each $x \in F\cap G \in \mathfrak{f}\cap\mathfrak{g}$.

(iii) It is clear.

(iv) Each $\hat{U}$ is symmetric.

$\diamond$

Proposition 4.4 *The mapping $1_X : X \to \hat{X} : x \mapsto \{F; x \in F\}$ is a uniform embedding and $1_X(X)$ is a dense subspace of $\hat{X}$.*

`Proof:`For the first part it is sufficient to prove that
$(x,y) \in U$ iff $(1_X(x),1_X(y)) \in \hat{U}$ for each couple $x,y \in X$ and each symmetric $U \in \mathcal{U}$.
Let $(x,y) \in U$. Then we can define the required set $F = \{x,y\}$ and now $U[z] \in 1_X(x)\cap 1_X(y)$ for each $z \in F$.
Let $(1_X(x),1_X(y)) \in \hat{U}$ which means that there is a set $F \in 1_X(x)\cap 1_X(y)$ such that $U[z] \in 1_X(x)\cap 1_X(y)$, equivalently $(x,z),(y,z) \in U$, for each $z \in F$. Then $(x,y) \in U$ because $F \in 1_X(y)$, equivalently $y \in F$.

Let us prove the other part. Let $\mathfrak{f} \in \hat{X}$ and $U \in \mathcal{U}$. Let us assume that U is symmetric. Is there $x \in X$ such that $1_X(x) \in \hat{U}[\mathfrak{f}]$?
As $\mathfrak{f}$ is a Cauchy filter on X there exists a set $F' \in \mathfrak{f}$ such that $U[x] \in \mathfrak{f}$ for each $x \in F'$. Let us choose such a point x and define a set $F = F'\cap U[x]$. Then $z \in U[x]$ and $U[z] \in \mathfrak{f}$ for each $z \in F$. But $z \in U[x]$ if and only if $x \in U[z]$ if and only if $U[z] \in 1_X(x)$. Thus $U[z] \in \mathfrak{f}\cap 1_X(x)$ for each $z \in F$ and $1_X(X)$ is dense in $\hat{X}$. $\diamond$

Since the mapping 1_X from the previous proposition is a uniform embedding we will identify $1_X(x)$ and x for every $x \in X$, and $1_X(X)$ and X.

Proposition 4.5 *Each Cauchy filter $\mathfrak{f}$ on X converges to the point $\mathfrak{f} \in \hat{X}$.*

`Proof:` Let $\mathfrak{f}$ be a Cauchy filter on X. Let us take an arbitrary symmetric $U \in \mathcal{U}$. Then there is a set $G \in \mathfrak{f}$ such that $U[z] \in \mathfrak{f}$ for each $z \in G$. Now for any $z \in G$, we have $H = G\cap U[z] \in \mathfrak{f}\cap 1_X(z)$ and $U[v] \in \mathfrak{f}\cap 1_X(z)$ for each $v \in H$, and then $(1_X(z),\mathfrak{f}) \in \hat{U}$. Thus $\hat{U}[\mathfrak{f}] \supset G \in \mathfrak{f}$ for each $U \in \mathcal{U}$ and $\mathfrak{f}$ converges to $\mathfrak{f}$. $\diamond$

Proposition 4.6 *Let $(X,\mathcal{U})$ be a t-semi-uniform space. Then $R = \bigcap_{U\in\mathcal{U}} U$ is an equivalence relation on X ($(x,y) \in R \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}$).*

`Proof:` R is reflexive because $(x,x) \in U$ for each $x \in X$ and $U \in \mathcal{U}$.
R is symmetric because of (iv) in the definition of the semi-uniformity $\mathcal{U}$.
Let us show that R is transitive.
Let $(x,y),(y,z) \in U$ for each $U \in \mathcal{U}$, which means that $y \in \overline{\{x\}}, z \in \overline{\{y\}}$, and then $z \in \overline{\{x\}}$. $\diamond$

As we see in the next example there are spaces where the relation R defined in the previous Proposition is not an equivalence, thus the semi-uniform space $(\hat{X},\hat{\mathcal{U}})$ defined in Definition 4.2 is not topological.

Let us remark that $x/R = \overline{\{x\}}$ for every $x \in X$.

Example 4.7 (R need not be an equivalence on $(\hat{X},\hat{\mathcal{U}})$, thus $(\hat{X},\hat{\mathcal{U}})$ is not a t-semi-uniform space.)
First let us observe that $(\mathfrak{f},\mathfrak{g}) \in R$ iff $\mathfrak{f} \cap \mathfrak{g}$ is a Cauchy filter on X. We will construct a t-semi-uniform space $(X,\mathcal{U})$ and three filters $\mathfrak{f},\mathfrak{g},\mathfrak{h}$ such that $(\mathfrak{f},\mathfrak{g}),(\mathfrak{g},\mathfrak{h}) \in R, (\mathfrak{f},\mathfrak{h}) \notin R$. That means that $\mathfrak{f}\cap\mathfrak{g}$ and $\mathfrak{h}\cap\mathfrak{g}$ are Cauchy, but $\mathfrak{f}\cap\mathfrak{h}$ is not.

Take $X = (\mathbb{R} \setminus \{0\})^2$. Denote $I = \{(x,y) \in X; x > 0 \wedge y > 0\}, II = \{(x,y) \in X; x < 0 \wedge y > 0\}, III = \{(x,y) \in X; x < 0 \wedge y < 0\}, IV = \{(x,y) \in X; x > 0 \wedge y < 0\}$. Take $\mathcal{U}$ a t-semi-uniformity on X with the subbase consisting of all sets of standard uniformity $\mathcal{U}_s/X$ on X ($\mathcal{U}_s$ is the standard uniformity on $\mathbb{R}^2$) and the set $U_0 = (X \times X) \setminus (I \times III) \setminus (III \times I)$. That is a pointwise open and poitwise closed semi-uniformity inducing the standard topology on X. Take the filters
$\mathfrak{g} = \{U[0] \cap X \cap II; U \in \mathcal{U}_s\}$,
$\mathfrak{f} = \{U[0] \cap X \cap (I \cup II \cup IV); U \in \mathcal{U}_s\}$,
$\mathfrak{h} = \{U[0] \cap X \cap (II \cup III \cup IV); U \in \mathcal{U}_s\}$.
Now $\mathfrak{f} \cap \mathfrak{g} = \mathfrak{f}$ and $\mathfrak{h} \cap \mathfrak{g} = \mathfrak{h}$ are Cauchy filters on $(X,\mathcal{U})$. In fact, take $U \in \mathcal{U}s$. Then $U[x] \cap X \in \mathfrak{f}$ for every $x \in U[0] \cap X \cap (I \cup II \cup IV) \in \mathfrak{f}$. And $U_0[x] \supset I \cup II \cup IV \in \mathfrak{f}$ for every $x \in I \cup II \cup IV \in \mathfrak{f}$. For $\mathfrak{g}$ the proof is simillar.

But the filter $\mathfrak{f} \cap \mathfrak{h} = \{U[0] \cap X\}$ is not Cauchy on $(X,\mathcal{U})$ because no $U_0[x] \in \{U[0] \cap X\}$.

We will see in the next that we can construct a one-point T_1-completion for every T_1-space . However the construction of $(\hat{X},\hat{\mathcal{U}})$ from above does not give a t-semi-uniform space for coarse t-semi-uniformities.

Example 4.8 (R need not be an equivalence on a coarse t-semi-uniformity.)
As in the previous Example we find three filters $\mathfrak{f},\mathfrak{g},\mathfrak{h}$ such that $(\mathfrak{f},\mathfrak{g}),(\mathfrak{g},\mathfrak{h}) \in R, (\mathfrak{f},\mathfrak{h}) \notin R$. That means that $\mathfrak{f}\cap\mathfrak{g}$ and $\mathfrak{h}\cap\mathfrak{g}$ are Cauchy, but $\mathfrak{f}\cap\mathfrak{h}$ is not.

Take a coarse t-semi-uniformity $\mathcal{U}$ on a space X such that there is a point $x \in X$ with a non-cofinite nbhd H.

Remark that $(\mathfrak{f},\mathfrak{g}) \in \hat{U}_{a,G}$ iff $G \in \mathfrak{f}\cap\mathfrak{g}$ or $X \setminus \{a\} \in \mathfrak{f}\cap\mathfrak{g}$.

Take a filter $\mathfrak{f} = \mathcal{U}_x$, a cofinite filter $\mathfrak{h}$, a filter $\mathfrak{g} = \mathfrak{f} \cup \mathfrak{h}$.

Now $\mathfrak{f}\cap\mathfrak{g} = \mathfrak{f}$ and $\mathfrak{h}\cap\mathfrak{g} = \mathfrak{h}$ are Cauchy filters. But the filter $\mathfrak{f}\cap\mathfrak{h}$ is not Cauchy. In fact, $H \notin \mathfrak{h}$ and $X \setminus \{x\} \notin \mathfrak{f}$. Thus $(\mathfrak{f},\mathfrak{g}) \notin \hat{U}_{x,H}$ and $\mathfrak{f}\cap\mathfrak{h}$ is not Cauchy on $(X,\mathcal{U})$.

4.1 Minimal completion of t-semi-uniform spaces

If one omits the T_2-property, it is possible to construct one-point completions in a similar way to which one constructs one-point non-Hausdorff compactifications. Since we work in symmetric topological spaces, not every compactification can serve as a model for a completion. Moreover, a compact space need not be complete but the symmetric one-point compactification is complete:

Theorem 4.9 *Let X be a non-complete t-semi-uniform space. Take $\hat{X} = X \cup \{\infty\}$, $\hat{\mathcal{U}}$ be a filter on $\hat{X} \times \hat{X}$ with the base consisting of all sets $\hat{U} = U \cup (\{\infty\} \times (\hat{X} \setminus \overline{A})) \cup ((\hat{X} \setminus \overline{A}) \times \{\infty\})$ for $U \in \mathcal{U}$, and finite subsets A of X. Then $(\hat{X},\hat{\mathcal{U}})$ is a completion of $(X,\mathcal{U})$ that is T_1 provided X is T_1.*

`Proof:` Clearly, $\hat{\mathcal{U}}$ is a semi-uniformity on the set $\hat{X}$ because it has a symmetric base.

It is topological and contains X as a dense subspace. Indeed, the induced topology on $\hat{X}$ has the base consisting of open sets in X and of complements in $\hat{X}$ of closures of finite subsets of X.

Since every Cauchy filter converges to each point of its intersection, take a Cauchy filter $\mathfrak{f}$ on $\hat{X}$ with $\bigcap \mathfrak{f} = \emptyset$. Then $\mathfrak{f}$ contains the filter $\mathfrak{c}$ consisting of complements of finite sets in X. Clearly, $\mathfrak{c}$ converges to ∞ and so does $\mathfrak{f}$. $\diamond$

4.2 Maximal completion of t-semi-uniform spaces, weak reflection

The completion from the previous subsection is the coarsest (or minimal) completion of X. Of course, one-point completions cannot have (up to some exceptions) the property of extending uniformly continuous maps. To obtain completions having that property one must add much more points than just one.

Theorem 4.10 *Let $(X,\mathcal{U})$ be a non-complete t-semi-uniform space. Denote $cX = X \cup \{\mathfrak{f}; \mathfrak{f}$ is a non-convergent Cauchy filter on $(X,\mathcal{U})\}$ and $c\mathcal{U}$ be a filter on $cX \times cX$ having the base $\mathcal{B}$ consisting of symmetric sets V such that $V \cap (X \times X) \in \mathcal{U}$ and $V[\mathfrak{f}] = \{\mathfrak{f}\} \cup F$, where F is an open set from the filter $\mathfrak{f}$, for each $\mathfrak{f} \in cX \setminus X$.*
Then $(cX, c\mathcal{U})$ is a completion of $(X,\mathcal{U})$ that is T_1 provided X is T_1.

`Proof:` It is clear that $(cX, c\mathcal{U})$ is a semi-uniformity for $\mathcal{B}$ consists of symmetric sets and it is closed under finite intersections.

As $\mathfrak{f} \in cX \setminus X$ does not converge in X, it contains the cofinite filter on X. Then $G = V[x]$ for $V = (G \times G) \cup ((cX \setminus \{x\}) \times (cX \setminus \{x\})) \in c\mathcal{U}$ for every neighborhood G of x in $(X, \mathcal{U})$. Thus for $x \in X$ the neighborhood filter in $\mathcal{U}$ coincides with the base of the neighborhood filter of x in $c\mathcal{U}$. For a filter $\mathfrak{f} \in cX \setminus X$ and $V \in \mathcal{B}$ clearly $V[\mathfrak{f}]$ is a neighborhood of each its point. Thus $c\mathcal{U}$ is a t-semi-uniformity on cX.
Clearly X is dense in cX and cX is T_1 provided X is T_1.

It remains to prove that $(cX, c\mathcal{U})$ is complete. Let $\mathfrak{F}$ be a Cauchy filter on $(cX, c\mathcal{U})$. If $X \in \mathfrak{F}$ then $\mathfrak{f} = [\mathfrak{F}] \cap X = \{F \cap X; F \in \mathfrak{F}\}$ is Cauchy on X and it is a base of $\mathfrak{F}$. Either $\mathfrak{f}$ converges in X to a point x (then $\mathfrak{F}$ converges to x as well) or $\mathfrak{f} \in cX \setminus X$ and, clearly, $\mathfrak{F}$ converges to $\mathfrak{f}$.

If $X \notin \mathfrak{F}$ then for every $V \in \mathcal{B}$ there exists $F_V \in \mathfrak{F}$ such that $V[x] \in \mathfrak{F}$ for every $x \in F_V$. Since $X \notin \mathfrak{F}$, the set F_V meets $cX \setminus X$; for every $\mathfrak{f}$ in this intersection we have $V[\mathfrak{f}] = \{\mathfrak{f}\} \cup G$ for some open G in X. It follows that the intersection $F_V \cap (cX \setminus X)$ consists of a unique point $\mathfrak{f}$ that is the same for every V. Consequently, $\mathfrak{F}$ converges to $\mathfrak{f}$. ◇

Theorem 4.11 *Let $(cX, c\mathcal{U})$ be the completion of a t-semi-uniform space $(X, \mathcal{U})$ defined in the previous theorem. Let f be a uniformly continuous mapping of the space $(X, \mathcal{U})$ to a complete t-semi-uniform space $(Y, \mathcal{V})$. Then there exists a uniformly continuous extension $\hat{f}$ of f onto the completion $(cX, c\mathcal{U})$.*

`Proof:` Let f be a uniformly continuous map of $(X, \mathcal{U})$ to a complete t-semi-uniform space $(Y, \mathcal{V})$. Take $\hat{f}(x) = f(x)$ for $x \in X$ and $\hat{f}([\mathfrak{f}]) \in \lim f[\mathfrak{f}]$ where $=f[\mathfrak{f}] = \{f(F); F \in \mathfrak{f}\}$ for $\mathfrak{f} \in cX \setminus X$. Let us remark that $f[\mathfrak{f}]$ is a base of a Cauchy filter according to Proposition 3.15 and thus a convergent filter base on $(Y, \mathcal{V})$. Take an arbitrary symmetric set $V \in \mathcal{V}$. Then $(\hat{f} \times \hat{f})^{-1}(V) \supset U = (f \times f)^{-1}(V) \cup (\bigcup_{\mathfrak{f} \in cX \setminus X} \{\mathfrak{f}\} \times f^{-1}(V[\hat{f}([\mathfrak{f}])])) \cup (\bigcup_{\mathfrak{f} \in cX \setminus X} f^{-1}(V[\hat{f}([\mathfrak{f}])] \times \{\mathfrak{f}\}) \in c\mathcal{U}$. In fact, $V[\hat{f}([\mathfrak{f}])] \in f[\mathfrak{f}]$ because $\hat{f}([\mathfrak{f}]) \in \lim f[\mathfrak{f}]$. As $(Y, \mathcal{V})$ is a t-semi-uniform space there is an open neighborhood $H \in f[\mathfrak{f}]$ of $\hat{f}([\mathfrak{f}])$ such that $H \subset V[\hat{f}([\mathfrak{f}])]$. Now $f^{-1}(V[\hat{f}([\mathfrak{f}])]) \supset \{\mathfrak{f}\} \cup f^{-1}(H)$ and it is an open neighborhood of $\mathfrak{f}$. ◇

The next definition comes from [10].

Definition 4.12 *A subcategory $\mathcal{R}$ of a category $\mathcal{K}$ is said to be almost reflective if it is closed under retracts, and for every object $X \in \mathcal{K}$ there exists a morphism r (almost reflection) from X into $rX \in \mathcal{R}$ such that every morphism f from X into $Y \in \mathcal{R}$ can be decomposed as gr for some morphism g.*

Theorem 4.13 *The category of all complete t-semi-uniform spaces is an almost reflective subcategory of* `TSUnif`*. All the almost reflection maps can be chosen to be embeddings.*
The same is true for T_1-spaces.

`Proof:` Because of Theorem 4.11 it remains to prove that the category of all complete t-semi-uniform spaces is closed under the formation of retracts. Let $r : X \to X$ be a retraction of X. Let $\mathfrak{f}$ be a Cauchy filter on $r(X)$. As Cauchy filters have the property (P1) (see Proposition 3.15.) then $\mathfrak{f}$ is a base of a Cauchy filter $\mathfrak{F}$ on X. Then $\mathfrak{F}$ converges to a point $y \in X$.

As r is a retraction then $\mathfrak{f} = r([\mathfrak{F}]) = \{r(F); F \in \mathfrak{F}\}$. As r is continuous $\mathfrak{f}$ converges to $r(y) \in r(X)$ and $r(X)$ is complete. ◇

Since every classic Cauchy filter is Cauchy, it is seen from the procedures, that the previous results of this section are valid for classic completeness, too. The classic completion cX is then formed from classic Cauchy filters only.

5 COMPLETION OF QUASI-TOPOLOGICAL GROUPS

5.1 T-semi-uniformities of quasi-topological groups

On topological groups there are several natural uniformities inducing the original group topology, namely right, left, upper (two-sided) and lower (Roelke) uniformities. The situation is similar if we demand only separate continuity of the multiplication mapping and instead of uniformities consider semi-uniformities. First recall some basic definitions of topological structures on groups.

Definition 5.1 Let X be a group with a topology τ.

We say that X is *a right topological group* if the mapping $m : X \times X \to X : (x, y) \mapsto xy$ is continuous for the left variable, which means that for each $y \in X$ the mapping (*the right translation*) $r_y : x \mapsto xy$ is continuous.

We say that X is *a left topological group* if the mapping m is continuous for the right variable, which means that for each $x \in X$ the mapping (*the left translation*) $l_x : y \mapsto xy$ is continuous.

We say that X is *a semi-topological group* if the mapping m is separately continuous, which means that it is continuous for the right and left variable.

We say that X is *a quasi-topological group* if it is a semi-topological group and the inversion mapping $i : X \to X : x \mapsto x^{-1}$ is continuous.

For $A, B \subset X$ let us denote $AB = \{xy; x \in A \ \wedge y \in B\}$.
For $A \subset X, x \in X$ let us denote $xA = \{x\}A$, $Ax = A\{x\}$.
For $A \subset X$ let us denote $A^{-1} = \{x^{-1}; x \in A\}$.

Proposition 5.2 *A group X with a topology is a right, resp. left topological group iff all the right, resp. left translations are homeomorphisms.*

`Proof:` Every r_y is injective and $r_{y^{-1}} = r_y^{-1}$. ◇

Proposition 5.3 *Any right topological group with a continuous inversion map is a quasi-topological group.*

`Proof:` Let X be a right topological group with a continuous inversion map. That means that the mappings $r_y : x \mapsto xy$ and $i : x \mapsto x^{-1}$ are continuous for every $y \in X$. But $l_y : x \mapsto yx = (x^{-1}y{-}1)^{-1}$, thus $l_y = i \circ r_{y^{-1}} \circ i$ is also continuous for every $y \in X$. ◇

Corollary 5.4 *A group X with a topology τ, in which $\mathcal{U}_e$ is a neighbourhood filter of the unit e, is a quasi-topological group if and only if the following conditions are satisfied:*

(1) $U^{-1} \in \mathcal{U}_e$ for each $U \in \mathcal{U}_e$,

(2) $\{Uy; U \in \mathcal{U}_e\}$ is a neighbourhood filter of y for each $y \in X$,

(3) $\{yU; U \in \mathcal{U}_e\}$ is a neighbourhood filter of y for each $y \in X$.

Lemma 5.5 *Let X be a quasi-topological group with a topology τ, in which $\mathcal{U}_e$ is a neighbourhood filter of the unit e. Then the system of sets $\mathcal{R} = \{R_U; U \in \mathcal{U}_e\}$, resp. $\mathcal{L} = \{L_U; U \in \mathcal{U}_e\}$, where $R_U = \{(x, y) \in X \times X; yx^{-1} \in U\}$, $L_U = \{(x, y) \in X \times X; x^{-1}y \in U\}$, is a semi-uniformity on X.*

`Proof:` Let $\mathcal{R}$ be such a system. We will prove the four conditions in the definition of a semi-uniformity.

(i) Clear because $xx^{-1} = e \in U, \forall U \in \mathcal{U}_e$.

(ii) Clear because for $U_1, U_2 \in \mathcal{U}_e$ is $U_1 \cap U_2 \in \mathcal{U}_e$.

(iii) is clear.

(iv) If the mapping i is continuous, then $U^{-1} \in \mathcal{U}_e$ whenever $U \in \mathcal{U}_e$.
It holds: $R_U^{-1} = \{(x, y) \in X \times X; yx^{-1} \in U\} = \{(x, y) \in X \times X; xy^{-1} \in U^{-1}\} = R_{U^{-1}}$.

For $\mathcal{L}$ the proof is similar.

◇

Definition 5.6 The semi-uniformity $\mathcal{R}$, resp. $\mathcal{L}$ from Lemma 5.5 will be called the *right,* resp. *left semi-uniformity* on the group X.

Supremum $\mathcal{L} \vee \mathcal{R}$ is also a semi-uniformity on X, see Proposition 2.21, and it is called the *two-sided* (or *upper) t-semi-uniformity* on X.

Infimum $\mathcal{L} \wedge \mathcal{R}$ is also a semi-uniformity on X, see Propositions 2.30 and 2.33, and it is called the *Roelke* (or *lower) t-semi-uniformity* on X.

Clearly, semi-uniformities $\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}$ on any quasi-topological group are t-semi-uniformities (they induce the original topology on the group) and they are pointwise open.
Regularity and pointwise closedness of semi-uniformities $\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}$ are equivalent ($\overline{U}x = \overline{Ux}, \overline{x}U = \overline{xU}$, for every x from the group and every U from the neighbourhood filter of the neutral element.

In the next we examine only the two-sided semi-uniformities, which are used in the definition of complete quasi-topological groups.

Proposition 5.7 *Let X, Y be quasi-topological groups, $f : X \to Y$ a continuous homomorphism. Then f is uniformly continuous with respect to the right, resp. to the left, resp. to the two-sided, resp. to the lower semi-uniformities on X and Y.*

`Proof:` Let e_X, resp. e_Y be the neutral element in X, resp. Y. Let $\mathcal{U}_{e_X}$, resp. $\mathcal{V}_{e_Y}$ be the neighbourhood filter of e_X in X, resp. e_Y in Y. Let $V \in \mathcal{V}_{e_Y}$. If f is continuous then there exists a set $U \in \mathcal{U}_{e_X}$ such that $f(U) \subset V$.
Now $(x, y) \in R_U \Leftrightarrow yx^{-1} \in U \Rightarrow f(yx^{-1}) = f(y)f(x^{-1}) = f(y)(f(x))^{-1} \in V \Leftrightarrow (f(x), f(y)) \in R_V$. The mapping f is uniformly continuous with respect to the right t-semi-uniformities.
The proof for the other semi-uniformities are similar. ◇

5.2 Initial t-semi-uniformities on groups

Proposition 5.8 *Let (X_i, τ_i) be a quasi-topological group for each $i \in I$. Let X be a group and $f_i : X \to X_i$ be a homomorphism, for each $i \in I$, τ the initial topology on X w.r.t. the maps $f_i : X \to (X_i, \tau_i)$. Then (X, τ) is a quasi-topological group. Its right t-semi-uniformity coinsides with the initial semi-uniformity w.r.t the maps $f_i : X \to (X_i, \mathcal{R}_i)$, where $\mathcal{R}_i$ is the right t-semi-uniformity on $(X_i, \tau_i), i \in I$. The same holds for the left and two-sided t-semi-uniformities.*

`Proof:` From the universal property of the initial topology τ it follows that the right translation r_y is continuous iff the composition $f_i \circ r_y = r_{f_i(y)} \circ f_i$ is continuous for all f_i. Similarly for the inversion map i.

The fact about the right semi-uniformity holds because for every basic neighbourhood $U = \bigcap\limits_{i=1}^{n} f_i^{-1}(U_i)$ of the neutral element e is $y \in Ux$ iff $f_i(y) \in U_i f_i(x)$, for all $i = 1, .., n$. ◇

Proposition 5.9

(a) The subgroup of a quasi-topological group with the subspace topology is a quasi-topological group.

(b) Let (X, τ_i) be quasi-topological groups, $i \in I$. The group (X, τ) with the supremum topology $\tau = \sup\limits_{i \in I} \tau_i$ is a quasi-topological group.

(c) Let (X, τ_i) be quasi-topological groups, $i \in I$. The topological product $(\prod\limits_{i \in I} X_i, \prod\limits_{i \in I} \tau_i)$ is a quasi-topological group.

Proposition 5.10 *A closure of a subgroup X of a quasi-topological group Y is a quasi-topological subgroup of Y.*

`Proof:` Let X is a subgroup of a quasi-topological group Y.

Let $x, y \in \overline{X}$. Then there are nets $\{x_\alpha\}$, resp. $\{y_\beta\}$ in X converging to x, resp. to y. Then $xy_\beta \in \lim x_\alpha y_\beta$, and thus each xy_β is in $\overline{X}$. Thus $xy \in \lim xy_\beta \subset \overline{X}$.

Let $x \in \overline{X}$. Then there is a net $\{x_\alpha\}$ in X converging to x. Then $x^{-1} \in \lim x_\alpha^{-1} \subset \overline{X}$.

◇

5.3 Categories of Hausdorff complete groups

Recall that every (Hausdorff) topological group is embedded in a (Hausdorff) topological group that is complete in its two-sided uniformity. The Hausdorff complete (in the two-sided uniformity) groups form an epireflective subcategory of the category of all Hausdorff topological groups with continuous homomorphisms, the reflection arrows are embeddings. We will examine the situation in quasi-topological groups.

Definition 5.11 A quasi-topological group is called *(clacsic, semi, weak) complete* if it is (classic, semi, weak) complete in its two-sided semi-uniformity.

Notation. Denote by $(\mathtt{cl}, \mathtt{s}, \mathtt{w})\mathtt{Compl}_{\mathtt{Grp}}$ the category of all (classic, semi, weak) complete Hausdorff quasi-topological groups with continuous homomorphisms, $\mathtt{QGrp}$ the category of all Hausdorff quasi-topological groups with continuous homomorphisms.

Theorem 5.12 *The categories* $\mathtt{clCompl}_{\mathtt{Grp}}, \mathtt{sCompl}_{\mathtt{Grp}}, \mathtt{wCompl}_{\mathtt{Grp}}, \mathtt{Compl}_{\mathtt{Grp}}$ *are epireflective subcategories of* $\mathtt{QGrp}$.

`Proof:` It follows from the fact that complete spaces are closed under products and closed subspaces in quasi-topological groups and closures of subgroups are subgroups (see 5.10). ◇

5.4 Completion of quasi-topological groups

We showed that Hausdorff complete (for all of our definitions) quasi-topological groups form an epireflective subcategory of Hausdorff quasi-topological groups. In the category of Hausdorff topological groups the reflection arrows (to complete reflections) are embeddings. Example 5.14 shows that in all the categories $(\mathtt{cl}, \mathtt{s}, \mathtt{w})\mathtt{Compl}_{\mathtt{Grp}}$ the situation is different.

Definition 5.13 Let X be a quasi-topological group. A (classic, semi, weak) complete quasi-topological group that contains X as a dense subgroup is *a group (classic, weak, semi-) completion* of the group X.

We show that not every Hausdorff quasi-topological group has a Hausdorff group classic completion, which means that the reflection arrows in Hausdorff (classic, weak, semi-) complete quasi-topological groups need not be embeddings:

Example 5.14 Take the Abelian group $\mathbb{R}\times\mathbb{R}$. Denote $e = (0;0)$ and $a_n = (\frac{1}{n}, \sin\frac{1}{n}) \in \mathbb{R}\times\mathbb{R}, n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Denote $B_\varepsilon = \{x \in \mathbb{R}\times\mathbb{R}; d(x,e) < \varepsilon\}$, where d is the Euclidean metric on $\mathbb{R}\times\mathbb{R}$.
For all points $x \in \mathbb{R}\times\mathbb{R}$ define sets $\mathcal{B}_x$:

for $x = e \in \mathbb{R}\times\mathbb{R}$ let $\mathcal{B}_x = \{B_\varepsilon \setminus \{a_n, -a_n\}_{n\in\mathbb{N}}\}_{\varepsilon>0}$,

for $x \in \mathbb{R}\times\mathbb{R}$ let $\mathcal{B}_x = \{G_e + x\}_{G_e\in\mathcal{B}_e}$.

First we show that the system $\{\mathcal{B}_x\}_{x\in X}$ generates a topology on $\mathbb{R}\times\mathbb{R}$. It suffices to show that if $y \in G_x \in \mathcal{B}_x$ then there is a $G_y \in \mathcal{B}_y$ that $G_y \subset G_x$, see Proposition 1.2.3 in [7]. In fact, if $y \in G_x = G \setminus \{a_n + x, -a_n + x\}_{n\in\mathbb{N}}, y \neq x$, where G is a standard open neighborhood of x in $\mathbb{R}\times\mathbb{R}$, then there is a standard open neighborhood H of y that $H \subset G$. As $\{a_n + x\}_{n\in\mathbb{N}}$ and $\{-a_n + x\}_{n\in\mathbb{N}}$ converge to x in the standard topology, which is Hausdorff, we can assume that H contains only finitely many $a_n + x, -a_n + x$, and thus we can take H without points $a_n + x, -a_n + x$. Now $H \subset G_x$.

Take $X = \mathbb{R}\times\mathbb{R}$ with the the topology τ generated by the neighborhood system $\{\mathcal{B}_x\}_{x\in X}$. Clearly, it is a Hausdorff quasi-topological group, see Corollary 5.4.

Take the filter $\mathfrak{f}$ generated by the set $\{\{a_n; n \geq n_0\}, n_0 \in \mathbb{N}\}$.

We show that $\mathfrak{f}$ is classic Cauchy on X (with the right (denoted by $\mathcal{R}$), left or two-sided semi-uniformities, which are the same on Abelian groups).

Take $U \in \mathcal{B}_e$ and $R_U = \{(x,y) \in X\times X; x - y \in U\} \in \mathcal{R}$. We find a set $F \in \mathfrak{f}$ that $F \times F \subset R_U$, which means that we find a number $n_0 \in \mathbb{N}$ that $(a_n, a_m) \in R_U$ for all $n, m \geq n_0$. As $U = G \setminus \{a_n, -a_n\}_{n\in\mathbb{N}}$ for some standard neighborhood G of e and the sequence $\{a_n, n \in \mathbb{N}\}$ converges to e, and thus it is Cauchy, in the standard topology, there is an $n_0 \in \mathbb{N}$ that $a_n - a_m \in G$ for all $n, m \geq n_0$.

There is no $p \in \mathbb{N}$ that $a_n - a_m = a_p$. In fact, it would mean that there is a $p \in \mathbb{N}$ that $\frac{1}{n} - \frac{1}{m} = \frac{1}{p}$ and $\sin\frac{1}{n} - \sin\frac{1}{m} = \sin\frac{1}{p}$, and thus $\sin\frac{1}{n} - \sin\frac{1}{m} = \sin\left(\frac{1}{n} - \frac{1}{m}\right) = \sin\frac{1}{n}\cos\frac{1}{m} - \sin\frac{1}{m}\cos\frac{1}{n}$, which is impossible because the function $h(x) = \frac{\sin x}{1-\cos x}$ is injective in the interval $[0; \frac{\pi}{2}]$. Thus $a_n - a_m \in U$ for all $n, m \geq n_0$, and $\mathfrak{f}$ is classic Cauchy, and, of course, it has no limit in X.

If there is a Hausdorff classic completion Y of the space X the filter $\mathfrak{f}$ is classic Cauchy on Y, see Proposition 3.15, and it converges to a point $y \in Y$. We show that e and y cannot be separated. Let there be disjoint open sets $G, H \subset Y$ such that $e \in G, y \in H$. First remind that every neighborhood G of e intersects every neighborhood of every point $a_n, n \geq n_0$ for some $n_0 \in \mathbb{N}$. As $\mathfrak{f}$ converges to y, and thus the sequence $\{a_n, n \in \mathbb{N}\}$ also converges to y, we can take such n_0 that $a_n \in H$ for all $n \geq n_0$, and thus H is a neighborhood of all $a_n, n \geq n_0$. That is a contrary.

The quasi-topological group X has no (classic, weak, semi-) completion containing X as a subgroup.

In the previos section we constructed a completion (weak reflection) for all non-complete semi-uniform spaces. But there constructed completion cannot be a quasi-topological group:

Lemma 5.15 *Let X be a quasi-topological group that is not complete. The weak reflection constructed in Theorem 4.10 is not a quasi-topological group, for any semi-uniformity on X.*

`Proof:` Let e be the neutral element e of X. The set X is an open neighborhood of e in the completion cX defined in Theorem 4.10. Then $\mathfrak{f}X$ must be an open neighborhood of a non-convergent Cauchy filter $\mathfrak{f} \in cX$, and $\mathfrak{f}X \subset cX \setminus X$ because X should be a sugroup of cX. That is a contrary to the fact that X is dense in cX. ◇

So, there is a question, whether another weak reflection in groups exists.

We show an easy example of T_1 quasi-topological group that has not a weak (and thus not a semi-) completion (in t-semi-uniform spaces), and consequently it has not a group weak (and thus not a semi-) completion:

Example 5.16 (T_1 quasi-topological group that has not a weak (and thus not a semi-) completion.)

Let X be an Abelian quasi-topological group of integers with the cofinite topology, which is T_1.

The filter $\mathfrak{f} = \{F \subset X; |X \setminus F| < \omega \wedge \{a, b\} \subset F\}$ is a weak Cauchy filter on the right (equivalently on the left, two-sided or lower) t-semi-uniformity on X, for every couple $a, b \in X, a \neq b$.
In fact, let U be an arbitrary neighbourhood of the neutral element 0, $F \in \mathfrak{f}$.
The sets $a - U, b - U, F$ have finite complements, and, thus, a non-empty intersection. Take an element z from this intersection. Now $z \in F$ and $a, b \in z + U$. Then $U[z] = z + U \in \mathfrak{f}$, and $\mathfrak{f}$ is weak Cauchy.

If the space X has a weak completion Y, then the filter $\mathfrak{f}$ is a base of a weak Cauchy filter $\mathfrak{F}$ on Y, see Proposition 3.15, (P2). As Y is weakly complete, the filter $\mathfrak{F}$ converges to a point $y \in Y$. This means that every neighborhood of y belongs to the filter $\mathfrak{F}$, and thus it contains the points a, b. Now $y \in \overline{\{a\}}^Y$, which is equivalent to $a \in \overline{\{y\}}^Y$, because the space Y is symmetric. From $a \in \overline{\{y\}}^Y$ and $y \in \overline{\{b\}}^Y$ it follows that $a \in \overline{\{b\}}^Y$. This is a contradiction to the topology of X.

Proposition 5.17 *If a pointwise open t-semi-uniform space X has a pointwise open semi-completion, resp. a pointwise open weak completion, then every semi-Cauchy, resp. weak Cauchy, filter on X must be Cauchy on X.*

`Proof:` Let Y be a pointwise open semi-completion of X, $\mathfrak{f}$ be a semi-Cauchy filter on X. Then $\mathfrak{f}$ generates a filter $\mathfrak{f}'$ that is semi-Cauchy on Y (see 3.15) and thus convergent on Y. Then $\mathfrak{f}'$ is Cauchy on Y (see 3.4) and thus Cauchy on X (see 3.15).

The proof for the weak completion is the same. ◇

Recall that each of t-semi-uniformities $\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}$ on a quasi-topological group is pointwise open.

Example 5.18 (Weak (and semi-) Cauchy filter which is not Cauchy.)
Let X be an Abelian quasi-topological group of integers with the cofinite topology. The filter $\mathfrak{f} = \{F \subset X; |X \setminus F| < \omega \wedge \{a, b\} \subset F\}$ is a weak Cauchy but not a Cauchy filter in the right (equivalently in the left, two-sided or lower) t-semi-uniformity on X for every couple $a, b \in X, a \neq b$.

`Proof:` In Example 5.16 we showed that $\mathfrak{f}$ is weak Cauchy.
Let us prove that $\mathfrak{f}$ is not Cauchy.
For a contrary let $\mathfrak{f}$ be a Cauchy filter. That means that it converges to any point in $\bigcap \mathfrak{f} = \{a, b\}$. That is a contrary because $X \setminus \{b\}$ is a neighbourhood of a that is not in $\mathfrak{f}$. $\diamond$

The previous example also shows that there described T_1 quasi-topological group has not a group weak (and thus not semi-) completions, which we proved in Example 5.16. In fact, if it had a group weak (semi-) completion, weak (semi-) Cauchy filters would coincide with Cauchy ones, see Proposition 5.17 and the comment after it.

The next examples show that in Hausdorff quasi-topological groups there are also groups that do not have group (classic, weak and semi-) completions.

First recall examples of quasi-topological groups described in [2] and [12]:

Example 5.19 (Orbits as quasi-topological groups.)
Let $(G, +)$ be an Abelian group, Z a Hausdorff topological space. If $\phi : G \times Z \to Z$ is an action (i.e., $\phi(0, z) = z$, $\phi(a, \phi(b, z)) = \phi(a + b, z)$ for every $a, b \in G, z \in Z$, and $\phi(a, -) : Z \to Z$ is a continuous mapping for every $a \in G$), then every orbit $(O(z_0) = \{\phi(a, z_0); a \in G\} \subset Z)$ form a semi-topological group. (See[2] par. 3.)

If G is an Abelian group of order 2, the orbits will be quasi-topological groups.

A special orbit is described in [12], Theorem 4:

Example 5.20 (Korovin's orbit.)
Let $(G, +)$ be an Abelian group, X a Hausdorff topological space such that $|G^\omega| = |G| \geq |X| \cdot \omega$. In the previous example let $Z = X^G$, ϕ be the shift $\phi(a, f)(b) = f(a + b)$.

Denote by $\mathcal{A}$ the set of all finite subsets of G, $\mathcal{F}(A)$ the set of all mappings of A to X for $A \in \mathcal{A}$, $\mathcal{F} = \bigcup\{\mathcal{F}(A); A \in \mathcal{A}\}$. Denote by $A(f)$ such a set from $\mathcal{A}$ that $f \in \mathcal{F}(A(f))$, for $f \in \mathcal{F}$.
Denote $\tau = |G| = |\mathcal{A}| = |\mathcal{F}|$.
Take $\{f_\alpha; \alpha \in \tau\}$ a numeration of the set $\mathcal{F}$.

By transfinite induction we pick a subset $\{g_\alpha : \alpha \in \tau\}$ of G that the family $\{g_\alpha + A(f_\alpha) : \alpha \in \tau\}$ is disjoint:

Suppose that for an ordinal $\beta \in \tau$ we have a subset $\{g_\alpha : \alpha \in \beta\}$ of G such that the family $\{g_\alpha + A(f_\alpha) : \alpha \in \beta\}$ is disjoint.

Take an element $g_\beta \in G \setminus H$, where H is a minimal subgroup of G containing the set $\{A(f_\alpha) : \alpha \leq \beta\} \cup \{g_\alpha : \alpha < \beta\}$. $G \setminus H \neq \emptyset$ because $|H| \leq |\beta| \cdot \omega < \tau = |G|$.

Now the family $\{g_\alpha + A(f_\alpha) : \alpha \leq \beta\}$ is disjoint.

Then take a mapping $f \in X^G$ such that $f|_{g_\alpha + A(f_\alpha)} = f_\alpha \circ l_{g_\alpha}^{-1}|_{g_\alpha + A(f_\alpha)}$ for any $\alpha \in \tau$.

The subspace $Y = \{f_g = f \circ l_g; g \in G\}$ of the space X^G is a semi-topological group, its elements are functions $f_g, g \in G$, where $f_g(a) = f(g + a)$, for $a \in G$, with the group operation $f_g + f_h = f_{g+h}$. The neutral element is $f_0 = f$, invers elements are $-f_g = f_{-g}$.

The subbasic neighbourhoods of the neutral element $f = f_0$ are sets $U_c^f = \{f_g; f_g(c) = f(g + c) \in U\}$, where U are neighbourhoods of $f(c)$ in X. The subbasic neighbourhoods of an element f_g are sets $U_c^{f_g} = \{f_h; f_h(c) = f(h + c) \in U\} = \{f_{g+k}; f_{g+k}(c) = f(g + k + c) = f_k(g + c) \in U\} = f_g + U_{g+c}^f$, where U are neighbourhoods of $f_g(c) = f(g + c)$ in X.

If we take for G a group of order 2 then Y is a quasi-topological group. (Here $l_g^{-1} = l_g$.)

Lemma 5.21 *Take G, X, f from the previous example. For any finite sets $\{x_0, ..., x_m\} \subset X$ and $\{d_0, ..., d_m\} \subset G$, where $d_i \neq d_j$ for $i \neq j, i, j = 0, ..., m$, there is a $g_\alpha \in G$ (chosen by the transfinite induction in the previous example) such that $f_{g_\alpha}(d_i) = f(g_\alpha + d_i) = x_i, i = 0, 1, ..., m$. Moreover $g_\alpha + d_i \neq 0$ for all $i = 0, ..., m$.*

`Proof:` For the finite mapping h assigning x_i to d_i, $i = 0, ..., m$, there must exist an $\alpha \in \tau$ that $h = f_\alpha \in \mathcal{F}$ and $A(f_\alpha) = \{d_0, ..., d_m\}$. From $f|_{g_\alpha + A(f_\alpha)} = f_\alpha \circ l_{g_\alpha}^{-1}|_{g_\alpha + A(f_\alpha)}$ it follows that $f_{g_\alpha}(d_i) = f(g_\alpha + d_i) = x_i$. Clearly we can suppose that $\alpha \neq 0$. Thanks to the construction of the family $\{g_\alpha + A(f_\alpha)\}$ in the Korovin's orbit we have g_α that is not in the minimal subgroup generated by $A(f_\alpha)$. Thus $g_\alpha, d_1 + g_\alpha, ..., d_m + g_\alpha \neq 0$. $\diamond$

Example 5.22 (A Hausdorff quasi-topological group that has not a group weak (and semi-) completion.)

Take the Korovin's orbit Y described in the previous example. Let G be a group of order 2 so that Y would be quasi-topological.

Denote by $\mathfrak{f}$ the filter generated by the set consisting of all the sets $F_{c_0,\dots,c_n} = \{f_g; f_g(c_i) = f(c_i), i = 0,\dots,n\}$, where $c_0,\dots,c_n \neq 0$.

For $\mathfrak{f}$ is $F_{c_0,\dots,c_n} \cap F_{d_0,\dots,d_m} = F_{c_0,\dots,c_n,d_0,\dots,d_m} \in \mathfrak{f}$ and f is in $\bigcap \mathfrak{f}$. Thus $\mathfrak{f}$ is a filter on Y.

First we show that $\mathfrak{f}$ is weak Cauchy. Take an arbitrary set $F_{c_0,\dots,c_n} \in \mathfrak{f}$ and a basic neighbourhood $U^f_{d_0,\dots,d_m} = \{f_g; f_g(d_i) \in U_i, i = 0,\dots,m\}$ of $f_0 = f$, where U_i are neighbourhoods of $f(d_i)$ in X, $d_i \in G$, $i = 0,\dots,m$.

If $d_i \neq 0$ for all $i = 0,\dots,m$ then $U^f_{d_0,\dots,d_m} \supset F_{d_0,\dots,d_m} \in \mathfrak{f}$. We have $f_0 = f \in F_{c_0,\dots,c_n}$ and $f_0 + U^f_{d_0,\dots,d_m} = U^f_{d_0,\dots,d_m} \in \mathfrak{f}$.

Take now $d_0 = 0$. Thanks to the previous Lemma we find such an $\alpha \neq 0$ that $f_{g_\alpha}(c_i) = f(c_i), i = 0,\dots,n$ (which means that $f_{g_\alpha} \in F_{c_0,\dots,c_n}$) and $f_{g_\alpha}(d_i) = f(d_i), i = 0,\dots,m$ and $g_\alpha, d_1+g_\alpha,\dots,d_m+g_\alpha \neq 0$. The set $F_{g_\alpha,d_1+g_\alpha,\dots,d_m+g_\alpha} = \{f_g; f_g(d_i+g_\alpha) = f(d_i+g_\alpha) = f(d_i) = f_{g_\alpha}(d_i+g_\alpha), i = 0,\dots,m\} \in \mathfrak{f}$. The last set is included in a neighbourhood $U^{f_{g_\alpha}}_{d_0+g_\alpha,\dots,d_m+g_\alpha} = U^f_{d_0,\dots,d_m} + f_{g_\alpha}$ of f_{g_α}, which finaly must belong to the filter $\mathfrak{f}$.

Filter $\mathfrak{f}$ is weak Cauchy.

It cannot be Cauchy. In fact the function $f = f_0$ is in $\bigcap \mathfrak{f}$ and if $\mathfrak{f}$ were Cauchy it would converge to f. It does not because thanks to the previous Lemma for any point $x \in X \setminus V$, where V is a neighbourhood of $f(0)$ in X, and any $F_{c_0,\dots,c_n}$, where $c_0,\dots,c_n \neq 0$, we find an $\alpha < \tau$ such that $f(c_i+g_\alpha) = f(c_i), i = 0,\dots,n$, and $f(g_\alpha) = x$. Thus the neighbourhood $V^f = \{f_d; f_d(0) \in V\}$ of f is not in the filter $\mathfrak{f}$ because for every $F_{c_0,\dots,c_n}$ from the base of the filter $\mathfrak{f}$ we can find a function $f_{g_\alpha} \in F_{c_0,\dots,c_n} \setminus V^f$.

From the last example it follows that Korovin's orbits are not topological groups because in topological groups Cauchy, weak Cauchy and semi-Cauchy filters coincide.

There still remains an important question:

Question 2 *Do quasi-topological groups have group classic completions or completions?*

References

[1] J. Adámek, H. Herrlich, G. E. Strecker: *Abstract and Concrete Categories*, Wiley, New York 1990.

[2] A. V. Archangel'skii, M. Hušek: *Extensions of topological and semitopological groups nad the operation*, Comment. Math. Univ. Carolinae 42, 1 (2001), 173-186.

[3] E. Čech: *Topological Spaces*, Academia, Prague 1966.

[4] J. Deák *A survey of compatible extensions (presenting 77 unsolved problems)*, Colloq. Math. Soc. János Bolyai 55, Pécs (Hungary), 1989, 127-175.

[5] D. Doitchinov: *A concept of completeness of quasi-uniform spaces*, Topology Appl., 30, (1991), 205-217.

[6] D. Doitchinov: *E-completions of quasi-uniform spaces*, Proc. Symposium on Categorical Topology, University of Cape Town 1994, Banaschewski, B., C.R.A.Gilmour adn H.Herrlich, eds., Dept. Math., University of Cape Town, 89-102.

[7] R. Engelking: *General Topology*, PWN - Polish scientific publishers, Warszawa 1977.

[8] P. Fletcher, H. P. A. Künzi: *Extension properties induced by complete quasi-uniformities*, Pacific J. Math., 120 (1985), 357-384.

[9] P. Fletcher, W. F. Lindgren: *Quasi-uniform Spaces*, Marcel Dekker, New York 1982.

[10] H. Herrlich: *Almost reflective subcategories of Top*, Topology Appl.49 (1993), 251-264.

[11] H. J. K. Junnila, H.P.A.Künzi: *Stability in quasi-uniform spaces and the inverse problem*, Topology Appl., 49, (1993), 175-189.

[12] A. V. Korovin: *Continuous actions of pseudocompact groups and axioms of topological groups*, Comment. Math. Univ. Carolinae 33, 2 (1992), 335-343.

[13] H. P. A. Künzi: *Nonsymmetric topology*, Topology with Applications, Bolyai Soc. Math.Studies, Vol.4, Scekszárd, (1993), 303-338.

[14] H. P. A. Künzi: *Quasi-uniform isomorphism groups*, Recent Progress in Function Spaces, G. Di Maio and L. Holá, eds., Quaderni di matematica, Dip. Mat. Sec. Univ. Napoli, 3, (1998), 187-220.

[15] H. P. A. Künzi: *Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology.*, Handbook of the History of General Topology, Vol.3, Aull, C. E. and R. Lowen, eds., Kluwer, Dordrecht, (2001), 853-968.

[16] H. P. A. Künzi: *Quasi-uniform spaces in the year 2001*, in Recent Progress in gen.Top.II (ed. M.Hušek, J.van Mill), North Holland, Amsterdam (2002), 313–344.

[17] H. P. A. Künzi, S. Romaguera and O.V.Sipacheva: *The Doitchinov completion of a regular paratopological group*, Serdica Math. J., 24, (1998), 73-88.

[18] E. Lowen–Colebunders: *Function Classes of Cauchy Continuous Maps*, Marcel-Dekker, New York and Basel (1989).

[19] J. Marín and S. Romaguera: *A bitopological view of quasi-topological groups*, Indian J. Pure Appl. Math., 28, (1996), 393-405.

[20] M. J. Pérez-Peñalver: *Weakly Cauchy filters and quasi-uniform completeness*, Acta Math. Hungar., 82, (1999), 217-228.

[21] W. Roelke, S. Dierolf: *Uniform Structures on Topological Groups and their Quotients*, McGraw-Hill, New York 1981.

[22] S. Romaguera: *A new class of quasi-uniform spaces*, Math. Pannonica, 11(1), (2000), 17-28.

[23] S. Romaguera, M. P. Schellekens: *Cauchy filters and strong completeness of quasi-uniforms spaces*, Rostock. Math. Kolloq., 54, (2000), 69-79.

[24] M. B. Smyth: *Completeness of quasi-uniform and syntopological spaces*, J. London Math. Soc., 49, (1994), 385-400.

[25] V. V. Uspenskij: *The epimorphism problem for Hausdorff topological groups*, Topology Appl., 57, (1994), 287-294.