Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Zuzana Vydrová

Lorentz Violation and Supersymmetry

Ústav teoretické fyziky

Vedoucí diplomové práce: Alfredo Iorio, Ph.D.

Studijní program: teoretická fyzika

2008

Charles University in Prague
Faculty of Mathematics and Physics

DIPLOMA THESIS

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2008

I would like to thank to my supervisor Alfredo Iorio for consulting, explaining and also for his great patience with me.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 30. 7. 08 Zuzana Vydrová

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Název práce: Lorentz Violation and Supersymmetry
Autor: Zuzana Vydrová
Katedra (ústav): Ústav teoretické fyziky
Vedoucí diplomové práce: Alfredo Iorio, Ph.D.
e-mail vedoucího: iorio@ipnp.troja.mff.cuni.cz

Abstrakt: Tato práce se zabývá zkoumáním narušení Lorentzovy symetrie modifikací standardního modelu (minimal standard model extension) z pohledu teorému Noetherové. Ukazuje se, že pro takovéto teorie nejsou pro všechna pole splněny Euler-Lagrangeovy rovnice a teorém Noetherové je třeba modifikovat. V takovém připadě přestává platit ekvivalence symetrie a zákona zachování.

Klíčová slova: standard model extension, nekomutativní teorie, narušení Lorentzovy symetrie, teorém Noetherové

Title: Lorentz Violation and Supersymmetry
Author: Zuzana Vydrová
Department: Institute of Theoretical Physics
Supervisor: Alfredo Iorio
Supervisor's e-mail address: iorio@ipnp.troja.mff.cuni.cz

Abstract: This work studies Lorentz Violation in modifications of Standard Model (minimal Standard Model Extension) using Noether theorem. For such theories there often is no way to fulfill Euler-Lagrange expressions for all fields involved: the Noether theorem needs to be modified. In such case equivalence of symmetry and conservation vanishes.

Keywords: Standard Model Extension, noncommutative theories, Lorentz Violation, Noether theorem

1 Motivation

Lorentz symmetry – group SO(3,1) – is a fundamental symmetry of relativistic physics.

Standard Model (SM) is a Lorentz invariant theory that describes known elementary particles and three of four basic interactions – all of them except gravity. SM is phenomenologically successful, but still leaves some questions to answer. It is assumed to be a limit of a more fundamental theory, which would involve all four interactions.

One of the ways of finding this theory is looking for slight effects, which are not described by SM. Such an effect could be *Lorentz violation* – SM is Lorentz invariant and any experimentally detected Lorentz violation would be a sign of a new physics, which would have to be governed by new theory. Such effects would, of course, be highly suppressed.

In higher dimensional theories – e. g. string theories – there occur a spontaneous Lorentz symmetry breaking (not always in the four dimensions of our spacetime). This breaking is spontaneous, that is why fundamental theory is still Lorentz invariant.

There are two basic sets of theories at which Lorentz violation is studied: NCFTs and SME:

Noncommutative field theory (NCFT) arises from assumption that coordinates does not commute, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, with θ tensor antisymmetric. The reason is that noncommutative algebra occurs in string theories. There are indications, that the theory of quantum gravity will not be local.

NCFT can be built from commutative theory by replacing fields with their noncommutative analogues and their products with Moyal product $f \star g = \exp(\frac{1}{2} i\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}) f(x)g(y)|_{x=y}$. See e.g. [3].

Standard model extension (SME) was developed by Don Colladay, Alan Kostelecký and Robertus Pointing. It is an effective theory, which contains both SM and general relativity with any higher-order couplings between them. SME comprises an endless number of Lagrangians widening the standard model; SME contains every possible Lorentz violating interaction terms involving particle fields of SM and gravitational fields in the generalised theory of gravity, that can be written as observer scalar – it is compatible with SM and with string theory.

Minimal SME is a subset of SM built in flat spacetime, which preserves gauge invariance and renormalisability. For more information on SME see e.g. [4].

2 Introduction

2.1 Transformations

First, we need to distinguish two different ways how to transform. *Particle transformations* are transformations changing parameters of a system, while *observer transformations* are just changes of our coordinate frame. Any physically relevant theory should confessedly be observer invariant.

In physics we can distinguish following types of transformations

– *discrete symmetries* such as CPT

– *continuous symmetries* described using a continuous parameter

  – *internal symmetries* only involve field indices (gauge symmetries)
  – *spatiotemporal symmetries* act on coordinates x

Spatiotemporal or space-time symmetries involve SUSY, and the *full conformal group* of symmetries, with associated generators:

– $f^\mu(x) = a^\mu$ translations, $P_\mu = \partial_\mu$

– $f^\mu(x) = \omega^\mu{}_\nu x^\nu$ Lorentz transformations, $D = x^\mu \partial_\mu$

– $f^\mu(x) = a x^\mu$ dilatations, $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$

– $f^\mu(x) = a^\mu x^2 - 2a^\alpha x_\alpha x^\mu$ special conformal transformations, $K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$

We are not going to use common transformation $f'(x') - f(x)$ but $f'(x) - f(x)$ – change in expression evaluated in the same coordinate. It's more convenient, as this type of transformation commute with derivative. This 'same point' transformation turns out to be the Lie derivative.

2.2 Lie derivative

When transforming the system, change in coordinate is $\delta x^\mu = x'^\mu - x^\mu = -f^\mu$. For Poincaré group of transformations $f^\mu = a^\mu$ for translations and $f^\mu = \omega^\mu{}_\nu x^\nu$ for Lorentz transformations.

Let's start with the simplest case, the transformation of a scalar field ϕ. First notice that

$$\phi'(x') = \phi(x) \tag{2.1}$$

The 'same point' transformation of scalar field obviously is (the first order calculation is sufficient, as all the changes are infinitesimal)

$$\delta_f \phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x) = \phi'(x') - \phi(x) - \frac{\partial \phi}{\partial x^\mu}\delta x^\mu = f^\mu \partial_\mu \phi \tag{2.2}$$

The partial derivative with respect to the coordinate $\frac{\partial}{\partial x^\mu}$ will be further denoted just by ∂_μ.

Vector field v^μ transforms as

$$v'^\mu(x') = v^\alpha(x)\frac{\partial x'^\mu}{\partial x^\alpha} \tag{2.3}$$

Change in the form of the field in the same coordinate is

$$\begin{aligned}\delta_f v^\mu \equiv v'^\mu(x) - v^\mu(x) &= v'^\mu(x') - v^\mu(x) - \delta x^\alpha \partial_\alpha v^\mu = \\ &= v^\alpha(x)\left(\frac{\partial x'^\mu}{\partial x^\alpha} - \delta^\mu_\alpha\right) + f^\alpha \partial_\alpha v^\mu = \\ &= v^\alpha \partial_\alpha (x'^\mu - x^\alpha \delta^\mu_\alpha) + f^\alpha \partial_\alpha v^\mu = f^\alpha \partial_\alpha v^\mu - v^\alpha \partial_\alpha f^\mu \end{aligned} \tag{2.4}$$

The case of a co-vector field v_μ is similar, result is

$$\delta_f v_\mu = f^\alpha \partial_\alpha v_\mu + v_\alpha \partial_\mu f^\alpha \tag{2.5}$$

Opposite sign comes from switching primed and non-primed coordinate in the transformation rule.

A little bit more tricky is the case of two rank tensor, whose transformation follows

$$t'_{\mu\nu}(x') = t_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \tag{2.6}$$

'Same point' transformation

$$\delta_f t_{\mu\nu} \equiv t'_{\mu\nu}(x) - t_{\mu\nu}(x) = t_{\alpha\beta}(x)\left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} - \delta^\alpha_\mu \delta^\beta_\nu\right) + f^\alpha \partial_\alpha t_{\mu\nu} \tag{2.7}$$

The bracket on the right hand side has the structure of

$$(\delta^\alpha_\mu + \mathcal{O}(\delta))(\delta^\beta_\nu + \mathcal{O}(\delta)) - \delta^\alpha_\mu \delta^\beta_\nu \tag{2.8}$$

so the zero order vanishes and there remain only two first order contributions. We get

$$\begin{aligned}\delta_f t_{\mu\nu} &= t_{\alpha\beta}(\delta^\alpha_\mu(\partial'_\nu x^\beta - \delta^\beta_\nu))(\delta^\beta_\nu(\partial'_\mu x^\alpha - \delta^\alpha_\mu)) + f^\alpha \partial_\alpha t_{\mu\nu} = \\ &= t_{\alpha\nu}\partial_\mu f^\alpha + t_{\mu\alpha}\partial_\nu f^\alpha + f^\alpha \partial_\alpha t_{\mu\nu}\end{aligned} \tag{2.9}$$

From what was done so far the pattern can be clearly seen: for higher rank tensors, either covariant or contravariant, the calculations look the same way.

This results correspond to the standard definition of the Lie derivative:

$$\delta_f T = \lim_{\lambda \leftarrow 0} \frac{\Phi^*_\lambda T - T}{\lambda} \tag{2.10}$$

where Φ^*_λ is the flow (group of diffeomorphisms) of vector f, which acts as a pullback on T: the Lie derivative thus compares value of the tensor T in two points, related by diffeomorphism.

In coordinates that gives:

$$\begin{aligned}\delta_f A^{\alpha\beta\dots}_{\mu\nu\dots} = f^\sigma \partial_\sigma A^{\alpha\beta\dots}_{\mu\nu\dots} - A^{\sigma\beta\dots}_{\mu\nu\dots}\partial_\sigma f^\alpha - A^{\alpha\sigma\dots}_{\mu\nu\dots}\partial_\sigma f^\beta - \dots \\ + A^{\alpha\beta\dots}_{\sigma\nu\dots}\partial_\mu f^\sigma + A^{\alpha\beta\dots}_{\mu\sigma\dots}\partial_\nu f^\sigma + \dots\end{aligned} \tag{2.11}$$

For Dirac spinor the transformation under Lorentz group is

$$\psi'(x') = \exp\left(\frac{1}{2}i\omega^{\alpha\beta}\Sigma_{\alpha\beta}\right)\psi(x) \cong \left(1 + \frac{1}{2}i\omega^{\alpha\beta}\Sigma_{\alpha\beta}\right)\psi(x) \tag{2.12}$$

where the matrix $\Sigma_{\alpha\beta} \equiv \frac{1}{4}i[\gamma_\alpha\gamma_\beta]$. Lie derivative then is

$$\begin{aligned}\delta_f\psi &\equiv \psi'(x) - \psi(x) = \\ &= \psi'(x') - \psi(x) + f^\mu\partial_\mu\psi = \psi(x) + \frac{1}{2}i\omega^{\alpha\beta}\Sigma_{\alpha\beta}\psi(x) - \psi(x) + f^\mu\partial_\mu\psi = \\ &= f^\alpha\partial_\alpha\psi(x) - \frac{1}{8}\omega^{\alpha\beta}[\gamma_\alpha\gamma_\beta]\psi(x) = f^\alpha\partial_\alpha\psi(x) - \frac{1}{8}\partial_\alpha f_\beta[\gamma^\alpha\gamma^\beta]\psi(x)\end{aligned} \tag{2.13}$$

Notice that with spinor indices properly written (capital Latine letters used to distinguish from tensor indices denoted by Greek aplhabet) this reads

$$\delta_f \psi^A = f^\alpha \partial_\alpha \psi^A - \frac{1}{8}\partial_\alpha f_\beta [\gamma^\alpha \gamma^\beta]^A_B \psi^B \tag{2.14}$$

For a complex conjugated spinor $\overline{\psi^A} \equiv \bar{\psi}^{\dot{A}} \equiv (\psi^{\dot{A}})^\dagger \gamma_0$ the first order transformation rule changes to

$$\bar{\psi}'(x') = \bar{\psi}(x)\left(1 + \frac{1}{2} i\omega^{\alpha\beta}\gamma_0 \Sigma_{\alpha\beta}\gamma_0\right) \tag{2.15}$$

so the Lie derivative reads

$$\delta_f \bar{\psi}^{\dot{A}} = \delta_f \bar{\psi}_A = f^\alpha \partial_\alpha \bar{\psi}_A + \frac{1}{8}\partial_\alpha f_\beta (\gamma_0 [\gamma^\alpha \gamma^\beta]\gamma_0)^B_A \psi_B \tag{2.16}$$

For a Weyl spinor everything works exactly the same way as for Dirac spinor, of course Dirac γ matrices are to be replaced by Pauli σ matrices.

The Lie derivative has common attributes of a derivative: linearity and Leibniz rule:

$$\delta_f(\varphi_i + c\psi_j) = \delta_f \varphi_i + c\delta_f \psi_j$$
$$\delta_f(\varphi_i \psi_j) = (\delta_f \varphi_i)\psi_j + \varphi_i \delta_f \psi_j$$

i, j being any set of indices; moreover, it has some interesting characteristics: it closes a group:

$$[\delta_f, \delta_g]\varphi_i = \delta_{[f,g]}\varphi_i \tag{2.17}$$

and, as was written above, it does commute with partial derivative ∂_μ:

$$\delta_f \partial_\mu \varphi_i = \partial_\mu \delta_f \varphi_i \tag{2.18}$$

2.3 The first Noether theorem

Noether variational problem is finding conditions, under which a given variation leaves the action invariant, $\delta\mathcal{A} = 0$.

Let us concern flat four dimensional space. Action is then given as:

$$\mathcal{A} = \int \mathrm{d}^4 x \ \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \tag{2.19}$$

Noether theorem says, that in if the action is invariant under a particular transformation

$$\mathcal{A}_\Omega \longrightarrow \mathcal{A}'_{\Omega'} = \int_{\Omega'} \mathrm{d}^4x' \; \mathcal{L}(\Phi'_i, \partial'_\mu \Phi'_i) \equiv \mathcal{A}_\Omega \tag{2.20}$$

then transformation is a symmetry; for any parameter of the symmetry there exist a conserved quantity, *Noether current*.

Transformations follow:

$$\begin{aligned} \delta x^\mu &= x'^\mu - x^\mu \\ \delta \phi_\mathrm{i} &= \phi_\mathrm{i}'(x') - \phi_\mathrm{i}(x) \\ \delta(\partial_\mu \phi_\mathrm{i}) &= \partial'_\mu \phi_\mathrm{i}'(x') - \partial_\mu \phi_\mathrm{i}(x) \end{aligned} \tag{2.21}$$

It is seen easily that the commutator $[\delta, \partial_\mu] \neq 0$.

Let us consider transformations in the same space-time point x (Lie derivative). Then:

$$\begin{aligned} \delta_f \phi_\mathrm{i} &= \phi_\mathrm{i}'(x) - \phi_\mathrm{i}(x) \\ \delta_f(\partial_\mu \phi_\mathrm{i}) &= \partial_\mu \phi_\mathrm{i}'(x) - \partial_\mu \phi_\mathrm{i}(x) \end{aligned} \tag{2.22}$$

In this case there is commutativity: $[\delta_f, \partial_\mu] = 0$.

Relationship of both transformations is:

$$\begin{aligned} \delta \phi_\mathrm{i} &= \delta_f \phi_\mathrm{i} + \delta x^\mu \partial_\mu \phi_\mathrm{i} \\ \delta(\partial_\mu \phi_\mathrm{i}) &= \delta_f \partial_\mu \phi_\mathrm{i} + \delta x^\nu \partial_\nu \partial_\mu \phi_\mathrm{i} \end{aligned} \tag{2.23}$$

Using Lie derivative δ_f the condition for transformation being a symmetry reads

$$\begin{aligned} 0 = \mathcal{A}_\Omega - \mathcal{A}'_{\Omega'} &= \int_{\Omega'} \mathrm{d}^4x' \; \mathcal{L}(\phi'_\mathrm{i}, \partial'_\mu \phi'_\mathrm{i}) - \int_\Omega \mathrm{d}^4x \; \mathcal{L}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}) = \\ &= \int_\Omega \mathrm{d}^4x \; \Big[\Big|\frac{x'}{x}\Big| \mathcal{L}(\phi_\mathrm{i}', \partial'_\mu \phi_\mathrm{i}') - \mathcal{L} \Big] \\ &= \int_\Omega \mathrm{d}^4x \; \Big[\Big|\frac{x'}{x}\Big| \mathcal{L}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}) + \frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}} \delta \phi_\mathrm{i} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_\mathrm{i})} \delta(\partial_\mu \phi_\mathrm{i}) - \mathcal{L} \Big], \end{aligned} \tag{2.24}$$

where $\left|\frac{x'}{x}\right| = \det \frac{x'}{x} = 1 + \partial_\mu \delta x^\mu + \mathcal{O}(\delta^2)$ is Jacobian of the transformation.

For infinitesimal transformation we consider only the first order in δ, getting equation:

$$
\begin{aligned}
0 = \int_\Omega \mathrm{d}^4 \Big[(1+\partial_\mu \delta x^\mu)\Big(\mathcal{L} + \frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}}(\delta_f \phi_\mathrm{i} + \delta x^\nu \partial_\nu \phi_\mathrm{i}) + \\
\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_\mathrm{i})}(\delta_f \partial_\mu \phi_\mathrm{i} + \delta x^\nu \partial_\nu \partial_\mu \phi_\mathrm{i})\Big) - \mathcal{L}\Big] = \\
= \int_\Omega \mathrm{d}^4 x \Big[\mathcal{L}\partial_\mu \delta x^\mu + \frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}}\delta_f \phi_\mathrm{i} + \Pi^\mu_\mathrm{i} \delta_f \partial_\mu \phi_\mathrm{i} + \Big(\frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}}\partial_\mu \phi_\mathrm{i} + \Pi^\nu_\mathrm{i} \partial_\mu \partial_\nu \phi_\mathrm{i}\Big)\delta x^\mu\Big] = \\
= \int_\Omega \mathrm{d}^4 x \Big[\partial_\mu(\Pi^\mu_\mathrm{i} \delta_f \phi_\mathrm{i} + \mathcal{L}\delta x^\mu) + \Big(\frac{\partial \mathcal{L}}{\partial \phi_\mathrm{i}} - \partial_\mu \Pi^{\mu i}\Big)\delta_f \phi_\mathrm{i}\Big] = \\
= \int_\Omega \mathrm{d}^4 x \Big[\partial_\mu J^\mu - \Psi[\phi_\mathrm{i}]\delta_f \phi_\mathrm{i}\Big] \qquad (2.25)
\end{aligned}
$$

where $\Pi^{\mu i} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}$ and Lagrange expression $\Psi[\phi_\mathrm{i}] \equiv \frac{\partial \mathcal{L}}{\partial \phi_\mathrm{i}} - \partial_\mu \Pi^{\mu i}$.

Notice that there is no need to use equations of motion to derive this result.

We have just derived the *first Noether theorem*:

$$\delta \mathcal{A} = \int_\Omega (\mathrm{d}^4 x \Big[\partial_\mu J^\mu - \Psi[\phi_\mathrm{i}]\delta_f \phi_\mathrm{i}\Big]) \qquad (2.26)$$

What does this mean? If given transformation is a symmetry – action does not change under it – and on shell, Euler-Lagrange equations are fulfilled, then $\partial_\mu J^\mu = 0$, there exists a conserved quantity. This works in both directions: whenever J^μ is conserved, corresponding transformation is a symmetry.

The *Noether current* conserved on shell ($\partial_\mu J^\mu = 0$) has the form:

$$
\begin{aligned}
J^\mu &= \Pi^\mu_i \delta_f \Phi_i + \mathcal{L}\delta x^\mu \qquad (2.27)\\
J_\mu &= \Pi^i_\mu \delta \Phi_i - [\Pi^i_\mu \partial_\nu \Phi_i - \eta_{\mu\nu}\mathcal{L}]\delta x^\nu \qquad (2.28)
\end{aligned}
$$

We can define Noether charge as

$$Q = \int \mathrm{d}^3 x \; J^0 \qquad (2.29)$$

Charges close an algebra, no matter whether there is or is not a conservation:

$$
\begin{aligned}
\{Q, Q'\} &= Q'' \\
\{G(\Phi, \Pi_\Phi), Q\} &= \frac{\delta G}{\delta \Phi}\delta\Phi\{\Phi, Q\} = \delta\Phi \qquad (2.30)
\end{aligned}
$$

The number of currents is the order of the symmetry group: e.g. for Poincaré group (ord $G = 10$) there are

- 4 for translations
- 3 for rotations
- 3 for boosts

Corresponding Noether currents are

- $\delta x^\mu = e^\mu \Longleftrightarrow T^{\mu\nu}$ for infinitesimal translations (4)
- $\delta x^\mu = \omega^\mu{}_\nu; \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \Longleftrightarrow M^{\mu\nu\lambda} \equiv T^{\mu\nu}x^\lambda - T\mu\lambda x^\nu$ for Lorentz transformations (6)

Conserving $M^{\mu\nu\lambda}$ means:

$$\begin{aligned} 0 &= \partial_\mu M^{\mu\nu\lambda} = (\partial_\mu T^{\mu\nu})x^\lambda + T^{\mu\nu}\delta^\lambda_\mu - (\partial_\mu T^{\mu\lambda})x^\nu + T^{\mu\lambda}\delta^\nu_\mu \\ 0 &= T^{\lambda\nu} - T^{\nu\lambda} \end{aligned} \tag{2.31}$$

that means $T^{\mu\nu}$ symmetric. Energy-momentum tensor $T^{\mu\nu}$ is in general not symmetric.

Generalisation of the Noether theorem for a more general Lagrangian involving higher derivatives of fields is straightforward; for $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi)$ we get:

$$\begin{aligned} 0 = \mathcal{A}_\Omega - \mathcal{A}'_{\Omega'} &= \int_\Omega \mathrm{d}^4x \, \Big[\Big|\frac{x'}{x}\Big|\mathcal{L}(\phi_\mathrm{i}', \partial'_\mu\phi_\mathrm{i}', \partial'_\mu\partial'_\nu\phi_\mathrm{i}') - \mathcal{L}\Big] = \\ &= \int_\Omega \mathrm{d}^4x \, \Big[(1 + \partial_\mu\delta x^\mu)\Big(\mathcal{L}(\phi_\mathrm{i}, \partial_\mu\phi_\mathrm{i}, \partial_\mu\partial_\nu\phi_\mathrm{i}) + \\ &\qquad + \frac{\delta\mathcal{L}}{\delta\phi_\mathrm{i}}\delta\phi_\mathrm{i} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_\mathrm{i})}\delta(\partial_\mu\phi_\mathrm{i}) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\phi_\mathrm{i})}\delta(\partial_\mu\partial_\nu\phi_\mathrm{i})\Big) - \mathcal{L}\Big] \\ &= \int_\Omega \mathrm{d}^4x \, \Big[\mathcal{L}\partial_\mu\delta x^\mu + \frac{\delta\mathcal{L}}{\delta\phi_\mathrm{i}}(\delta_f\phi_\mathrm{i} + \delta x^\alpha\partial_\alpha\phi_\mathrm{i}) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_\mathrm{i})}(\delta_f\partial_\mu\phi_\mathrm{i} + \delta x^\alpha\partial_\alpha\partial_\mu\phi_\mathrm{i}) + \\ &\qquad + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\phi_\mathrm{i})}(\delta_f\partial_\mu\partial_\nu\phi_\mathrm{i} + \delta x^\alpha\partial_\alpha\partial_\mu\partial_\nu\phi_\mathrm{i})\Big] = \\ &= \int_\Omega \mathrm{d}^4x \, \Big[\partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_\mathrm{i}}\delta_f\phi_\mathrm{i} - 2\partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_\mathrm{i}}\delta_f\phi_\mathrm{i} + \partial_\nu\Big(\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_\mathrm{i}}\delta_f\phi_\mathrm{i} + \mathcal{L}\delta x^\mu\Big)\Big) + \\ &\qquad + \Big(\frac{\delta\mathcal{L}}{\delta\phi_\mathrm{i}} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_\mathrm{i}} + \partial_\mu\partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_\mathrm{i}}\Big)\delta_f\phi_\mathrm{i}\Big] \end{aligned} \tag{2.32}$$

2.4 The Second Noether theorem

While the first Noether theorem applies to *global* symmetries associated with finite dimensional Lie groups, the second Noether theorem to *local* symmetries associated with infinite dimensional Lie group.

While first Noether theorem applies to an action invariant under a continuous group of transformations depending smoothly on constant parameters, the second to an action invariant under an infinite dimensional group of transformations depending smoothly on ρ arbitrary functions $p_\alpha(x^\mu)$, $\alpha = 1, 2 \ldots \rho$.

For infinitesimal transformation we have:

$$\delta\psi_i = \sum_\alpha [a_{\alpha i}(\psi_i, \partial_\mu \psi_i, x^\mu)\Delta p_\alpha(x^\mu) + b^\mu_{\alpha i}(\psi_i, \partial_\mu \psi_i, x^\mu)\partial_\mu(\Delta p_\alpha(x^\mu))]$$

Now, functions p_α are arbitrary, so we can choose such that they and their derivatives vanish on the boundary. Thus the interior condition must vanish independently of the boundary condition: this way we get *second Noether theorem* in a form:

$$\sum_i \Psi[\psi_i]a_{\alpha i} = \sum_i \partial_\mu(\Psi[\psi_i]b^\mu_{\alpha i}) \tag{2.33}$$

On Noether theorems see [1] or Noethers original paper [2].

3 Theorem

3.1 Theorem to prove

When considering an example of noncommutative electrodynamics, Lagrangian of the structure

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\nu}F_{\mu\beta}F^{\mu\nu} + \frac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\theta^2) \tag{3.1}$$

using Noether approach – theorem in the form

$$\partial_\mu J^\mu \equiv \partial_\mu(\Pi^{\mu\nu}_{A_\mu}\delta_f A_\mu - \mathcal{L}f^\mu) \equiv \Psi[A_\mu]\delta_f A_\mu = 0 \Leftrightarrow \text{symmetry with } f^\mu \tag{3.2}$$

suddenly there occurred an interesting identity (on shell for A_μ)

$$\partial_\mu J^\mu = \Psi[\theta_\mu]\delta_f \theta_\mu \tag{3.3}$$

Note: f^μ here means Poincaré transformations only. (See [5] and comment and reply to comment [6], [7].)

This equation is valid not only in first order in θ but holds for any order, which suggests to assume it is a general result. In more general form, for Lagrangian of the shape $\mathcal{L} = \mathcal{L}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}; \chi_\mathrm{j})$ here – ϕ_i denotes dynamical and χ_j non-dynamical fields:

$$\partial_\mu J^\mu = \sum_{\chi_\mathrm{j}} \Psi[\chi_\mathrm{j}]\delta_f \chi_\mathrm{j} \tag{3.4}$$

holding even when there is no way to introduce Lagrange equations of motion for non-dynamical fields so that the theory would still make sense.

Identity (3.4) introduces a relationship between dynamical and non-dynamical fields, eventually allowing us to express non-dynamical fields in terms of dynamical ones the same way as it is possible in SUSY.

It seems that the identity holds for any scalar Lagrangian. When there is a possibility for Euler-Lagrange expression of non-dynamical fields to hold, it is in perfect agreement with Noether theorem. However, if Euler-Lagrange expression of non-dynamical fields doesn't make sense (a case which E. Noether did not studied), it gives as an interesting result: there is *no more any equivalence of symmetry and conservation*! Even if we have invariant Lagrangian, $\delta\mathcal{L} = 0$, we still have nonzero derivative of the Noether charge, $\delta_\mu J^\mu = \psi[\chi]\delta_f \chi \neq 0$.

When testing on more Lagrangians of similar form, that means including non-dynamical fields, it seems that the result is really general. Next section is denoted to proof of (3.4) in some specific situations.

3.2 General remarks

For simplicity the following calculations are done only for Lagrangian containing just one dynamical field ϕ_i and one non-dynamical field χ_j. Generalisation for more fields is straightforward, in the calculations just a sum over the field should be added. We will so consider Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}, \chi_\mathrm{j}) \tag{3.5}$$

where $\mathcal{L}$ is any scalar function. Indices i and j denotes any nature of corresponding fields.

Rewriting Noether theorem (3.2) we get

$$\partial_\mu J^\mu = \partial_\mu \Pi^\mu_{\phi_\mathrm{i}} \delta_f \phi_\mathrm{i} + \Pi^\mu_{\phi_\mathrm{i}} \partial_\mu \delta_f \phi_\mathrm{i} - \left(\frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}} \partial_\mu \phi_\mathrm{i} + \frac{\delta \mathcal{L}}{\delta \partial_\alpha \phi_\mathrm{i}} \partial_\mu \partial_\alpha \phi_\mathrm{i} + \frac{\delta \mathcal{L}}{\delta \chi_\mathrm{j}} \partial_\mu \chi_\mathrm{j} \right) f^\mu \quad (3.6)$$

where we used properties of Poincaré transformations, especially that $\partial_\mu f^\mu$ vanishes both for translations and for Lorentz transformations.

Eventually (3.4) gives us (on shell – equations of motion for dynamical fields used)

$$\begin{aligned} 0 &\stackrel{?}{=} \partial_\mu J^\mu - \Psi[\chi_\mathrm{j}] \delta_f \chi_\mathrm{j} = \\ &= \frac{\delta \mathcal{L}}{\delta \phi_\mathrm{i}} (\delta_f \phi_\mathrm{i} - f^\mu \partial_\mu \phi_\mathrm{i}) + \frac{\delta \mathcal{L}}{\delta \partial_\alpha \phi_\mathrm{i}} (\delta_f \partial_\alpha \phi_\mathrm{i} - f^\mu \partial_\mu \partial_\alpha \phi_\mathrm{i}) + \frac{\delta \mathcal{L}}{\delta \chi_\mathrm{j}} (\delta_f \chi_\mathrm{j} - f^\mu \partial_\mu \chi_\mathrm{j}) \end{aligned} \quad (3.7)$$

In this the commutativity of Lie and coordinate derivative was used. Just this is enough to prove (3.4) for translations ($\partial_\mu f^\alpha = 0$). Lie derivative has the structure of

$$\delta_f \Phi_i = f^\mu \partial_\mu \Phi_i + \sum \partial_\alpha f^\mu \Phi_{j,\mu} \quad (3.8)$$

so obviously the first terms of the Lie derivatives are to vanish with the second terms in brackets in (3.7). No f^α alone but only terms with $\partial_\mu f^\alpha$ survives in (3.7), so the right-hand side is zero for translations.

Note that if the Lagrangian has the structure of sum, the (3.4) should be fulfilled for any term in the sum; specially, for Lagrangian being a sum of an invariant part $\mathcal{L}_{\text{inv}} = \mathcal{L}_{\text{inv}}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i})$ and a part containing non-dynamical field(s) $\bar{\mathcal{L}} = \bar{\mathcal{L}}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}, \chi_\mathrm{j})$, previous holds for $\bar{\mathcal{L}}$ alone, as for the invariant part both sides of (3.4) vanish separately.

3.3 Special form of Lagrangian used

Just as a motivation lets look at the very special (and not very physically relevant) Lagrangian of a structure:

$$\mathcal{L} = \mathcal{L}_{\text{inv}}(\phi_\mathrm{i}, \partial_\mu \phi_\mathrm{i}) + \bar{\mathcal{L}}(\chi_\mathrm{j}) \quad (3.9)$$

Apparently, this is the most simple case, as the dynamical and non-dynamical fields are completely decomposed. Equation (3.7) becomes much simpler, getting the form

$$0 \stackrel{?}{=} \frac{\delta \mathcal{L}}{\delta \chi_\mathrm{j}} (\delta_f \chi_\mathrm{j} - f^\mu \partial_\mu \chi_\mathrm{j}) \quad (3.10)$$

This is evidently fulfilled for $\chi_{\mathrm{j}} = \chi$ (scalar filed). For other possible nature of non-dynamical fields proof follows, but stop for a moment at this simplest case and notice that (3.4) doesn't work for a general transformation with $f^\mu \neq 0$, as for such a transformation terms coming from $\partial_\mu f^\mu$ in the derivative of the Noether current J^μ would occur.

For $\chi_{\mathrm{j}} = \chi_\mu$ (vector field) Lagrangian for being a scalar function needs to be of structure $\mathcal{L} = \mathcal{L}(\chi_\mu \chi^\mu) = \mathcal{L}(\chi^2)$). In such case equation (3.7) becomes

$$0 \stackrel{?}{=} \partial_\mu J^\mu - \Psi[\chi_\mu]\delta_f \chi_\mu = \frac{\delta \mathcal{L}}{\delta \chi_\mu} \partial_\mu f^\alpha \chi_\alpha; \qquad \frac{\delta \mathcal{L}}{\delta \chi_\mu} \sim \chi_\mu \tag{3.11}$$

As $\partial_\mu f^\alpha$ is antisymmetric in μ, α and the rest is symmetric, the right-hand side also vanishes.

For $\chi_{\mathrm{j}} = \chi_{\mu\nu}$ (tensor field) the only possible scalar Lagrangian is $\mathcal{L} = \mathcal{L}({\chi_\alpha}^\beta {\chi_\beta}^\gamma \cdots {\chi_\mu}^\alpha) = \mathcal{L}(\mathrm{Tr}(\chi^n))$, n here meaning just a power. Trying to prove (3.4) we get

$$0 \stackrel{?}{=} \partial_\mu J^\mu - \Psi[{\chi_\mu}^\nu]\delta_f {\chi_\mu}^\nu = {\omega_\alpha}^\mu {\chi_\mu}^\beta {\chi_\beta}^\gamma \cdots {\chi_\nu}^\alpha - {\omega_\mu}^\beta {\chi_\beta}^\gamma \cdots {\chi_\nu}^\alpha {\chi_\alpha}^\mu \tag{3.12}$$

which is evidently fulfilled - both terms on the right-hand side are equal just by renaming indices.[1]

For more complicated nature of non-dynamical field situation is similar, but the calculations are getting difficult. From former analysis can be seen how the procedure acts, though: Lagrangian is scalar function, so all the indices are coupled. Right-hand side of our theorem (3.4) becomes just a sum of terms, which all have the structure of Lagrangian itself with antisymmetric ${\omega_\mu}^\nu$ plunged between some two former coupled indices. Thanks to the very specific structure of Lagrangian, the part multiplying ${\omega_\mu}^\nu$ is symmetric.

3.4 Special types of fields in general Lagrangian

In following we are about to examine much more general Lagrangian

$$\mathcal{L} = \mathcal{L}((\partial_\mu \phi_{\mathrm{i}})^k (\phi_{\mathrm{i}})^l (\chi_{\mathrm{j}})^m) \tag{3.13}$$

Indices i, j denote any nature of corresponding field, k, l and m mean powers.

[1] Maybe not so clearly can be seen that both terms are zero separately, too.

For both ϕ and χ scalar fields power k has to be even, and equation (3.7) becomes just

$$0 \stackrel{?}{=} \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu f^\alpha \partial_\alpha \phi; \qquad \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \sim \partial_\mu \phi \tag{3.14}$$

Right-hand side vanishes from symmetry reasons again.

For ϕ scalar and χ_μ vector field we can have powers k, l either both even or both odd (fields coupled together). For both even the proof is evident, for both odd we need to fulfill

$$0 \stackrel{?}{=} \left(\frac{\delta \mathcal{L}}{\delta \partial_\alpha \phi} \partial_\mu \phi + \frac{\delta \mathcal{L}}{\delta \chi_\alpha} \chi_\mu \right) \omega^\alpha{}_\mu; \qquad \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \chi_\mu = \frac{\delta \mathcal{L}}{\delta \chi_\mu} \partial_\mu \phi \tag{3.15}$$

which is again nothing more than just a condition for a scalar Lagrangian.

A little more complicated is a case of both ϕ_μ and χ_μ vector fields: Lagrangians with k even or of a structure $\partial_\mu \phi^\mu$ are similar to foregone, structure $\partial_\alpha \phi^\beta \phi^\alpha \chi_\beta$ or $\partial_\alpha \phi^\beta \phi_\beta \chi^\alpha$ needs a little work:

$$0 \stackrel{?}{=} C(\phi^\alpha \chi_\beta (\omega^\mu{}_\beta \partial_\mu \phi^\beta \ - \omega^\beta{}_\mu \partial_\alpha \phi^\mu) - \partial_\alpha \phi^\beta \omega^\alpha{}_\mu \chi_\beta \phi^\mu + \partial_\alpha \phi^\beta \omega^\mu{}_\beta \chi_\mu \phi^\alpha) = 0 \tag{3.16}$$

where C is function (same for all terms) introduced by differentiating Lagrangian.

4 Minimal Standard Model Extension

4.1 Minimal SME Lagrangians

SM Lagrangians are Lorentz invariant.

Relevant sectors are:

$$\mathcal{L}_{\text{lepton}} = \frac{i}{2} \bar{L}_A \gamma^\mu \overleftrightarrow{D^\mu} L_A + \frac{i}{2} \bar{R}_A \gamma^\mu \overleftrightarrow{D^\mu} R_A \tag{4.1}$$

$$\mathcal{L}_{\text{Yukawa}} = -(G_L)_{AB} \bar{L}_A \phi R_B \tag{4.2}$$

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger (D_\nu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{3!} (\phi^\dagger \phi)^2 \tag{4.3}$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2} \text{Tr}(W^{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B^{\mu\nu} B^{\mu\nu} \tag{4.4}$$

where

$$L_A = \begin{pmatrix} \nu_A \\ l_A \end{pmatrix}_L \qquad R_A = (l_A)_R.$$

are left and right lepton multiplets, A denoting lepton generation: $l_A = (e, \mu, \tau)$ and $\nu_A = (\nu_e, \nu_\mu, \nu_\tau)$.

ϕ denotes Higgs field, B_μ and W_μ are U(1) and SU(2) gauge fields.

Lagrangian for quarks, with left and right quark fields

$$Q_A = \left(\begin{array}{c} u_A \\ d_A \end{array} \right)_L \qquad \begin{array}{c} U_A = (u_A)_R \\ D_A = (d_A)_R \end{array}$$

$u_A = (u, c, t)$, $d_A = (d, s, b)$, is equivalent to lepton sector:

$$\mathcal{L}_{\text{quark}} = \frac{i}{2}\bar{Q_A}\gamma^\mu \overleftrightarrow{D^\mu} Q_A + \frac{i}{2}\bar{U_A}\gamma^\mu \overleftrightarrow{D^\mu} U_A + \frac{i}{2}\bar{D_A}\gamma^\mu \overleftrightarrow{D^\mu} D_A \tag{4.5}$$

terms involving quark fields or SU(3) qauge field G_μ occure in Yukawa and gauge sectors:

$$\mathcal{L}_{\text{Yukawa}} = -(G_U)_{AB}\bar{Q_A}\phi^c U_B - (G_D)_{AB}\bar{Q_A}\phi D_B \tag{4.6}$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\text{Tr}(G^{\mu\nu}G^{\mu\nu}) \tag{4.7}$$

Minimal SME Lagrangians violate Lorentz symmetry.
CPT even minimal SME Lagrangians are:

Lepton sector:

$$\mathcal{L}_{\text{lepton}}^{\text{CPT}-\text{even}} = \frac{i}{2}(c_L)_{AB\mu\nu}\bar{L_A}\gamma^\mu \overleftrightarrow{D^\nu} L_B + \frac{i}{2}(c_R)_{AB\mu\nu}\bar{R_A}\gamma^\mu \overleftrightarrow{D^\nu} R_B \tag{4.8}$$

Yukawa sector:

$$\mathcal{L}_{\text{Yukawa}}^{\text{CPT}-\text{even}} = -\frac{1}{2}(H_L)_{AB\mu\nu}\bar{L_A}\phi\sigma^{\mu\nu}R_B \tag{4.9}$$

Higgs sector:

$$\mathcal{L}_{\text{Higgs}}^{\text{CPT}-\text{even}} = \frac{1}{2}(k_{\phi\phi})^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi) - \frac{1}{2}(k_{\phi B})^{\mu\nu}\phi^\dagger\phi B_{\mu\nu} - \frac{1}{2}(k_{\phi W})^{\mu\nu}\phi^\dagger W_{\mu\nu}\phi \tag{4.10}$$

Gauge sector:

$$\mathcal{L}_{\text{gauge}}^{\text{CPT-even}} = -\frac{1}{2}(k_W)_{\kappa\lambda\mu\nu}\text{Tr`}(W^{\kappa\lambda}W^{\mu\nu}) - \frac{1}{4}(k_B)_{\kappa\lambda\mu\nu}B^{\kappa\lambda}B^{\mu\nu} \tag{4.11}$$

CPT odd Lagrangians are:

For lepton sector:

$$\mathcal{L}_{\text{lepton}}^{\text{CPT-odd}} = -(a_L)_{AB\mu}\bar{L_A}\gamma^\mu L_B - (a_R)_{AB\mu}\bar{R_A}\gamma^\mu R_B \tag{4.12}$$

None for Yukawa sector.

Higgs sector:

$$\mathcal{L}_{\text{Higgs}}^{\text{CPT-odd}} = i(k_\phi)^\mu \phi^\dagger D_\mu \phi \tag{4.13}$$

Gauge sector:

$$\begin{aligned}\mathcal{L}_{\text{gauge}}^{\text{CPT-odd}} = (k_2)_\kappa \epsilon^{\kappa\lambda\mu\nu}\text{Tr}(W_\lambda W_{\mu\nu} + \frac{2}{3}igW_\lambda W_\nu W\mu)+ \\ + (k_1)_\kappa \epsilon^{\kappa\lambda\mu\nu}B_\lambda B_{\mu\nu} + k_0^\kappa B_\kappa\end{aligned} \tag{4.14}$$

Quark sector is an analogy to lepton sector:

$$\begin{aligned}\mathcal{L}_{\text{quark}}^{\text{CPT-even}} &= \frac{i}{2}(c_Q)_{AB\mu\nu}\bar{Q_A}\gamma^\mu \overleftrightarrow{D^\nu} Q_B + \\ &+ \frac{i}{2}(c_U)_{AB\mu\nu}\bar{U_A}\gamma^\mu \overleftrightarrow{D^\nu} U_B + \frac{i}{2}(c_D)_{AB\mu\nu}\bar{D_A}\gamma^\mu \overleftrightarrow{D^\nu} D_B\end{aligned} \tag{4.15}$$

$$\mathcal{L}_{\text{Yukawa}}^{\text{CPT-even}} = -\frac{1}{2}(H_U)_{AB\mu\nu}\bar{Q_A}\phi^c\sigma^{\mu\nu}U_B - \frac{1}{2}(H_D)_{AB\mu\nu}\bar{Q_A}\phi\sigma^{\mu\nu}D_B \tag{4.16}$$

$$\mathcal{L}_{\text{gauge}}^{\text{CPT-even}} = -\frac{1}{2}(k_G)_{\kappa\lambda\mu\nu}\text{Tr}(G^{\kappa\lambda}G^{\mu\nu}) \tag{4.17}$$

$$\mathcal{L}_{\text{quark}}^{\text{CPT-odd}} = -(a_Q)_{AB\mu}\bar{Q_A}\gamma^\mu Q_B - (a_U)_{AB\mu}\bar{U_A}\gamma^\mu U_B - (a_D)_{AB\mu}\bar{D_A}\gamma^\mu D_B \tag{4.18}$$

$$\mathcal{L}_{\text{gauge}}^{\text{CPT-odd}} = (k_3)_\kappa \epsilon^{\kappa\lambda\mu\nu}\text{Tr}(G_\lambda G_{\mu\nu} + \frac{2}{3}ig_3 G_\lambda G_\mu G_\nu) \tag{4.19}$$

For all these Lagrangians (3.4) holds, proof follows:

4.2 Proof of the theorem (3.4) for minimal SME

Let's prove (3.4) for minimal SME Lagrangians. We're going to be quite general about transformation group used, our f^μ could be any transformation of the full conformal group. We're going to use only the Leibniz rule for Lie derivative.

1) CPT-even cases:
(i) lepton sector
Both parts of the Lagrangians have the form

$$\mathcal{L} = \frac{i}{2} c_{\mu\nu} \bar{\psi}_A \gamma^\mu \overleftrightarrow{D^\nu} \psi_B = \frac{i}{2} c_{\mu\nu} (\bar{\psi}_A (\overleftarrow{\partial^\nu} + igA^\nu) \gamma^\mu \psi_B - \bar{\psi}_A \gamma^\nu (\partial^\nu + igA^\nu) \gamma^\mu \psi_B) \quad (4.20)$$

Of course spinors ψ have different structure for left and right part.

Corresponding Π^μ_φ, $\varphi = \psi, \bar{\psi}, A^\mu$, and Euler-Lagrange equations are:

$$\Pi^\mu_\psi = -\frac{i}{2} c_{\nu\mu} \bar{\psi}_A \gamma^\nu \Longrightarrow -\frac{i}{2} c_{\nu\mu} \partial^\mu \bar{\psi}_A \gamma^\nu = \frac{i}{2} c_{\nu\mu} D^{\mu\dagger} \bar{\psi}_A \gamma^\nu + \bar{\psi}_A \gamma^\nu igA^\mu$$
$$\Pi^\mu_{\bar{\psi}} = \frac{i}{2} c_{\nu\mu} \gamma^\nu \psi_B \Longrightarrow \frac{i}{2} c_{\nu\mu} \gamma^\nu \partial^\mu \psi_B = \frac{i}{2} c_{\nu\mu} igA^\mu \gamma^\nu - \frac{i}{2} c_{\nu\mu} \gamma^\nu D^\mu \psi_B$$
$$\Longrightarrow c_{\nu\mu} D^{\mu\dagger} \bar{\psi}_A \gamma^\nu = 0 \text{ and } c_{\nu\mu} \gamma^\nu D^\mu \psi_B = 0$$
$$\Pi^{\mu\nu}_{A^\mu} = 0 \Longrightarrow 0 = c_{\nu\mu} \bar{\psi}_A \gamma^\nu \psi_B$$

Derivative of the Noether current reads:

$$\begin{aligned}
\partial_\mu J^\mu &= \partial_\mu(\Pi^\mu_\varphi \delta_f \varphi - f^\mu \mathcal{L}) = \partial_\mu \Pi^\mu_\varphi \delta_f \varphi + \Pi^\mu_\varphi \partial_\mu \delta_f \varphi - \delta_f \mathcal{L} = \\
&= \frac{i}{2} c^{\nu\mu} (\bar{\psi_A} \gamma_\nu i g A_\mu \delta_f \psi_B + \delta_f \bar{\psi_A} i g A_\mu \gamma_\nu \psi_B) + \\
&\quad + \frac{i}{2} c^{\nu\mu} (-\bar{\psi_A} \gamma_\nu \delta_f \partial_\mu \psi_B + \delta_f \partial_\mu \bar{\psi_A} \gamma_\nu \psi_B) \\
&\quad - \frac{i}{2} \delta_f c^{\nu\mu} (\bar{\psi_A} \gamma_\nu \overleftrightarrow{D_\mu} \psi_B) \\
&\quad - \frac{i}{2} c^{\nu\mu} (\delta_f (D^\dagger_\mu \bar{\psi_A}) \gamma_\nu \psi_B + (D^\dagger_\mu \bar{\psi_A}) \delta_f \gamma_\nu \psi_B + (D^\dagger_\mu \bar{\psi_A}) \gamma_\nu \delta_f \psi_B) \\
&\quad + \frac{i}{2} c^{\nu\mu} (\delta_f \bar{\psi_A} \gamma_\nu (D_\mu \psi_B) + \bar{\psi_A} \delta_f \gamma_\nu (D_\mu \psi_B) + \bar{\psi_A} \gamma_\nu \delta_f (D_\mu \psi_B)) = \\
&= \Psi[c_{\mu\nu}] \delta_f c_{\mu\nu} + \Psi[\gamma_\mu] \delta_f \gamma_\mu + \\
&\quad + \frac{i}{2} c^{\nu\mu} (\bar{\psi_A} \gamma_\nu i g A_\mu \delta_f \psi_B + \delta_f \bar{\psi_A} i g A_\mu \gamma_\nu \psi_B - \bar{\psi_A} \gamma_\nu \delta_f \partial_\mu \psi_A + \delta_f \partial_\mu \bar{\psi_A} \gamma_\nu \psi_B) \\
&\quad - \frac{i}{2} c^{\nu\mu} (\delta_f \partial_\mu \bar{\psi_A} \gamma_\nu \psi_B + \delta_f \bar{\psi_A} i g A_\mu \gamma_\nu \psi_B + \bar{\psi_A} \delta_f (i g A_\mu) \gamma_\nu \psi_B \\
&\qquad\qquad - \bar{\psi_A} \gamma_\nu \delta_f \partial_\mu \psi_B - \bar{\psi_A} \gamma_\nu \delta_f (i g A_\mu) \psi_B - \bar{\psi_A} \gamma_\nu i g A_\mu \delta_f \psi_B) \\
&= \Psi[c_{\mu\nu}] \delta_f c_{\mu\nu} + \Psi[\gamma_\mu] \delta_f \gamma_\mu - \frac{i}{2} c^{\nu\mu} (\bar{\psi_A} \delta_f (i g A_\mu) \gamma_\nu \psi_B + \bar{\psi_A} \gamma_\nu \delta_f (i g A_\mu) \psi_B) \\
&= \Psi[c_{\mu\nu}] \delta_f c_{\mu\nu} + \Psi[\gamma_\mu] \delta_f \gamma_\mu + \Psi[A_\mu] \delta_f A_\mu
\end{aligned}$$

where of course $\Psi[A_\mu]\delta_f A_\mu = 0$ (field A_μ follows Euler-Lagrange expression), and also $\Psi[\gamma_\mu]\delta_f \gamma_\mu = 0$.

(ii) Yukawa sector: There are no derivatives in Yukawa Lagrangian, derivative of the Noether current is simple:

$$\partial_\mu J^\mu = -\partial_\mu (f^\mu \mathcal{L}) = -\delta_f \mathcal{L} = \frac{1}{2} \delta_f (H_{\mu\nu} \bar{L_A} \phi \sigma^{\mu\nu} R_B) = 0$$

Last from Euler Lagrange equation. Apparently here $\Psi[H_{\mu\nu}] = 0$, too. $H_{\mu\nu}$ follows it's Euler-Lagrange equation.

(iii) Higgs sector: First, Lagrangian $\mathcal{L} = \frac{1}{2} k^{\mu\nu} (D_\mu \phi)^\dagger (D_\nu \phi)$ Euler-Lagrange equations:

$$\begin{aligned}
\Pi^\nu_\phi &= \frac{1}{2} k^{\mu\nu} (D_\mu \phi)^\dagger \Longrightarrow \frac{1}{2} k^{\mu\nu} \partial_\nu (D_\mu \phi)^\dagger = -\frac{1}{2} k^{\mu\nu} i g A_\nu (D_\mu \phi)^\dagger \Longrightarrow k^{\mu\nu} (D_\nu D_\mu \phi)^\dagger = 0 \\
\Pi^\mu_{\phi^\dagger} &= \frac{1}{2} k^{\mu\nu} (D_\nu \phi) \Longrightarrow \frac{1}{2} k^{\mu\nu} \partial_\mu (D_\nu \phi) = \frac{1}{2} k^{\mu\nu} i g A_\mu (D_\nu \phi) \Longrightarrow k^{\mu\nu} (D_\mu D_\nu \phi) = 0
\end{aligned}$$

Derivative of Noether current:

$$
\begin{aligned}
\partial_\mu J^\mu = \frac{1}{2}k^{\mu\nu}(-igA_\nu(D_\mu\phi)^\dagger\delta_f\phi + (D_\mu\phi)^\dagger\delta_f\partial_\nu\phi + \\
+ igA_\mu\delta_f\phi^\dagger(D_\nu\phi) + \delta_f\partial_\mu\phi^\dagger(D_\nu\phi)) - \frac{1}{2}\delta_f(k^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi)) = \\
= \frac{1}{2}k^{\mu\nu}(-igA_\nu(D_\mu\phi)^\dagger\delta_f\phi + (D_\mu\phi)^\dagger\delta_f\partial_\nu\phi) + \\
+ \frac{1}{2}\delta_f(k^{\mu\nu}igA_\mu\phi^\dagger D_\nu\phi + k^{\mu\nu}\partial_\mu\phi^\dagger D_\nu\phi) - \\
- \frac{1}{2}\delta_f k^{\mu\nu}(igA_\mu\phi^\dagger D_\nu\phi + \partial_\mu\phi^\dagger D_\nu\phi) - \\
- \frac{1}{2}k^{\mu\nu}(\delta_f(igA_\mu)\phi^\dagger D_\nu\phi + igA_\mu\phi^\dagger\delta_f(D_\nu\phi) + \partial_\mu\phi^\dagger\delta_f(D_\nu\phi)) - \\
- \frac{1}{2}\delta_f(k^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi)) = \\
= \Psi[k^{\mu\nu}]\delta_f k^{\mu\nu} + k^{\mu\nu}((D_\mu\phi)^\dagger\delta_f(D_\nu\phi) + (D_\mu\phi)^\dagger\delta_f(igA_\nu)\phi - \\
- \delta_f(igA_\mu)\phi^\dagger D_\nu\phi - (D_\mu\phi)^\dagger\delta_f(D_\nu\phi)) = \\
= \Psi[k^{\mu\nu}]\delta_f k^{\mu\nu} + \Psi[A^\mu]\delta_f A^\mu
\end{aligned}
$$

Second part of Higgs sector Lagrangian:

$$
\mathcal{L} = -\frac{1}{2}k^{\mu\nu}\phi^\dagger\phi B_\mu\nu = -\frac{1}{2}k^{\mu\nu}\phi^\dagger\phi(\partial_\mu B_\nu - \partial_\nu B_\mu) \tag{4.21}
$$

Euler-Lagrange equations:

$$
\Pi^{\alpha\beta}_{B_\beta} = -\frac{1}{2}k^{\mu\nu}\phi^\dagger\phi(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) = -\frac{1}{2}\phi^\dagger\phi(k^{\alpha\beta} - k^{\beta\alpha})
$$

$$
\partial_\mu\Pi^{\mu\nu}_{B_\mu} = 0 \Longrightarrow \partial_\mu(\phi^\dagger\phi) = \delta_f(\phi^\dagger\phi) = 0
$$

So for $\partial_\mu J^\mu$:

$$
\begin{aligned}
\partial_\mu J^\mu &= \Pi^{\alpha\beta}_{B_\beta}\delta_f\partial_\mu B_\nu + \frac{1}{2}\delta_f(k^{\mu\nu}\phi^\dagger\phi B_{\mu\nu}) = \\
&= -\frac{1}{2}\phi^\dagger\phi k^{\mu\nu}\delta_f B_{\mu\nu} + \frac{1}{2}\delta_f k^{\mu\nu}\phi^\dagger\phi B_{\mu\nu} + \frac{1}{2}k^{\mu\nu}\phi^\dagger\phi\delta_f B_{\mu\nu} = \\
&= \Psi[k^{\mu\nu}]\delta_f k^{\mu\nu}
\end{aligned}
$$

Finally third part of Higgs sector:

$$
\mathcal{L} = -\frac{1}{2}k^{\mu\nu}\phi^\dagger W_\mu\nu\phi = -\frac{1}{2}k^{\mu\nu}\phi^\dagger(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon^{abc}W^b_\mu W^c_\nu)\phi \tag{4.22}
$$

Now we have

$$
\begin{aligned}
\Pi^{\alpha\beta}_{W^n_\beta} &= -\frac{1}{2}\phi^\dagger\phi(k^{\alpha\beta} - k^{\beta\alpha}) \\
\partial_\beta\Pi^{\alpha\beta}_{W^n_\beta} &= \frac{1}{2}\epsilon^{abn}\phi^\dagger A^a_\mu\phi(k^{\alpha\beta} - k^{\beta\alpha}) \\
\Pi^\mu_{\phi,\ \phi^\dagger} &= 0 \Longrightarrow k^{\mu\nu}W_{\mu\nu}\phi = k^{\mu\nu}\phi^\dagger W_{\mu\nu} = 0
\end{aligned}
$$

That means for $\partial_\mu J^\mu$:

$$
\begin{aligned}
\partial_\mu J^\mu &= \frac{1}{2}\epsilon^{abn}(k^{\mu\nu} - k^{\nu\mu})\phi^\dagger A^a_\mu\delta_f A^n_\nu\phi - \frac{1}{2}(k^{\mu\nu} - k^{\nu\mu})\phi^\dagger\delta_f\partial_\mu A^b_\nu\phi + \frac{1}{2}\delta_f(k^{\mu\nu}\phi^\dagger W_\mu\nu\phi) = \\
&= \frac{1}{4}(k^{\mu\nu} - k^{\nu\mu})\phi^\dagger(\epsilon^{abn}\delta_f(A^a_\mu A^n_\nu + \partial_\mu A^b_\nu - \partial_\nu A^b_\mu)\phi + \frac{1}{2}\delta_f(k^{\mu\nu}\phi^\dagger W_\mu\nu\phi) = \\
&= -\frac{1}{4}(k^{\mu\nu} - k^{\nu\mu})\phi^\dagger\delta_f W_{\mu\nu}\phi + \frac{1}{2}\delta_f(k^{\mu\nu}\phi^\dagger W_\mu\nu\phi) \\
&= -\frac{1}{2}k^{\mu\nu}\phi^\dagger\delta_f W_{\mu\nu}\phi + \frac{1}{2}\delta_f(k^{\mu\nu})\phi^\dagger W_\mu\nu\phi + \\
&\quad + \frac{1}{2}k^{\mu\nu}(\delta_f\phi^\dagger W_{\mu\nu}\phi + \phi^\dagger\delta_f W_{\mu\nu}\phi + \phi^\dagger W_{\mu\nu}\delta_f\phi) = \\
&= \Psi[k^{\mu\nu}]\delta_f k^{\mu\nu}
\end{aligned}
\tag{4.23}
$$

(iv) Gauge sector is little bit more complicated:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2}k^{\alpha\beta\gamma\delta}\mathrm{Tr}(W_{\alpha\beta}W_{\gamma\delta}) = \\
&= -\frac{1}{2}k^{\alpha\beta\gamma\delta}(\partial_\alpha W^a_\beta - \partial_\beta W^a_\alpha + \epsilon^{abc}W^b_\alpha W^c_\beta)(\partial_\gamma W^a_\delta - \partial_\delta W^a_\gamma + \epsilon^{ade}W^d_\gamma W^e_\delta)
\end{aligned}
$$

In following there are used first two conditions of Riemann tensor symmetry for $k^{\alpha\beta\gamma\delta}$. Demanding this symmetry is reasonable: Lagrangian is obviously antisymmetric in α, β and in γ, δ and symmetric when switching these pairs.

$$
\begin{aligned}
\Pi^{\mu\nu}_{W^n_\nu} &= -\frac{1}{2}k^{\alpha\beta\gamma\delta}((\delta^\mu_\alpha\delta^\nu_\beta\delta^{an} - \delta^\nu_\alpha\delta^\mu_\beta\delta^{an})W^a_{\gamma\delta} + W^n_{\alpha\beta}(\delta^\mu_\gamma\delta^\nu_\delta\delta^{an} - \delta^\nu_\gamma\delta^\mu_\delta\delta^{an})) = -2k^{\mu\nu\alpha\beta}W^n_{\alpha\beta} \\
\partial_\mu\Pi^{\mu\nu}_{W^n_\nu} &= -\frac{1}{2}k^{\alpha\beta\gamma\delta}(\epsilon^{abc}(\delta^\nu_\alpha\delta^{bn}A^c_\beta + \delta^\nu_\beta\delta^{cn}A^b_\alpha)W^a_{\gamma\delta} + W^a_{\alpha\beta}(\epsilon^{ade}(\delta^\nu_\gamma\delta^{dn}A^e_\delta + \delta^\nu_\delta\delta^{en}A^d_\gamma))) = \\
&\qquad 2k^{\mu\nu\alpha\beta}\epsilon^{nab}A^a_\mu W^b_{\alpha\beta}
\end{aligned}
$$

And finally

$$
\begin{aligned}
\partial_\mu J^\mu &= 2k^{\mu\nu\alpha\beta}\epsilon^{nab}W^b_{\alpha\beta}A^a_\mu\delta_f A^n_\nu - 2k^{\mu\nu\alpha\beta}W^n_{\alpha\beta}\delta_f(\partial_\mu A^n_\nu) + \frac{1}{2}\delta_f(k^{\mu\nu\alpha\beta}W^a_{\mu\nu}W^a_{\alpha\beta}) = \\
&= k^{\mu\nu\alpha\beta}(\epsilon^{abn}W^b_{\alpha\beta}\delta_f(A^a_\mu A^n_\nu) - W^n_{\alpha\beta}\delta_f(\partial_\mu A^n_\nu - \partial_\nu A^n_\mu)) + \frac{1}{2}\delta_f(k^{\mu\nu\alpha\beta}W^a_{\mu\nu}W^a_{\alpha\beta}) = \\
&= k^{\mu\nu\alpha\beta}W^n_{\mu\nu}\delta_f W^n_{\alpha\beta} + \frac{1}{2}\delta_f(k^{\mu\nu\alpha\beta}W^a_{\mu\nu}W^a_{\alpha\beta}) = \\
&= -\frac{1}{2}k^{\mu\nu\alpha\beta}\delta_f(W^n_{\mu\nu}W^n_{\alpha\beta}) + \frac{1}{2}\delta_f k^{\mu\nu\alpha\beta}W^a_{\mu\nu}W^a_{\alpha\beta} + \frac{1}{2}k^{\mu\nu\alpha\beta}\delta_f(W^a_{\mu\nu}W^a_{\alpha\beta}) \\
&= \Psi[k^{\mu\nu\alpha\beta}]\delta_f k^{\mu\nu\alpha\beta}
\end{aligned} \tag{4.24}
$$

2) CPT-odd cases:
(i) lepton sector

$$
\begin{aligned}
\Pi^\mu_\psi = 0 &\Longrightarrow 0 = a_\mu\bar{\psi_A}\gamma_\mu \\
\Pi^\mu_{\bar{\psi}} = 0 &\Longrightarrow 0 = a_\mu\gamma_\mu\psi_B
\end{aligned}
$$

Derivative of the Noether current:

$$
\begin{aligned}
\partial_\mu J^\mu &= -a_\mu\delta_f\bar{\psi_A}\gamma^\mu\psi_B - a_\mu\bar{\psi_A}\gamma^\mu\delta_f\psi_B - \delta_f\mathcal{L} \\
&= \delta_f\mathcal{L} + \delta_f(a_\mu)\bar{\psi_A}\gamma^\mu\psi_B + a_\mu\bar{\psi_A}\delta_f(\gamma^\mu)\psi_B - \delta_f\mathcal{L} = \\
&= \Psi[a_\mu]\delta_f a_\mu + \Psi[\gamma_\mu]\delta_f\gamma_\mu
\end{aligned}
$$

(ii) Higgs sector

$$
\Pi^\mu_\phi = ik^\mu\phi^\dagger; \quad \Pi^\mu_{\phi^\dagger} = 0 \Longrightarrow \partial_\mu\Pi^\mu_\phi = ik^\mu\phi^\dagger(-igA_\mu)
$$

Derivative of the Noether current:

$$
\begin{aligned}
\partial_\mu J^\mu &= ik^\mu\phi^\dagger(-igA_\mu)\delta_f\phi + ik^\mu\phi^\dagger\delta_f\partial_\mu\phi - \delta_f\mathcal{L} \\
&= ik^\mu\phi^\dagger\delta_f D_\mu\phi + \delta_f(igA_\mu)ik^\mu\phi^\dagger\phi - \delta_f\mathcal{L} = \\
&= \Psi[A_\mu]\delta_f A_\mu + \delta_f\mathcal{L} - \delta_f k^\mu\phi^\dagger D_\mu\phi - \delta_f\phi^\dagger(ik^\mu_\mu\phi) - \delta_f\mathcal{L} \\
&= \Psi[A_\mu]\delta_f A_\mu + \Psi[k_\mu]\delta_f k_\mu
\end{aligned} \tag{4.25}
$$

(iii) gauge sector, part one

$$\mathcal{L} = k_\kappa \epsilon^{\kappa\lambda\mu\nu} B_\lambda B_{\mu\nu} = k_\kappa \epsilon^{\kappa\lambda\mu\nu} B_\lambda (\partial_\mu B_\nu - \partial_\nu B_\mu) \tag{4.26}$$

Euler-Lagrange equation:

$$\Pi^{\mu\nu} = 2k_\kappa \epsilon^{\kappa\lambda\mu\nu} B_\lambda \delta_\alpha \Pi^{\alpha\beta} = k_\alpha \epsilon^{\alpha\beta\mu\nu} B_{\mu\nu}$$

Derivative of the Noether current:

$$\begin{aligned}\partial_\mu J^\mu &= k_\kappa \epsilon^{\kappa\lambda\mu\nu} B_{\mu\nu} \delta_f B_\lambda + 2k_\kappa \epsilon^{\kappa\lambda\mu\nu} B_\lambda \delta_f \partial_\mu B_\nu - \delta_f \mathcal{L} \\ &= k_\kappa \epsilon^{\kappa\lambda\mu\nu} (B_{\mu\nu} \delta_f B_\lambda + B_\lambda \delta_f B_{\mu\nu}) - \delta_f \mathcal{L} = \\ &= \Psi[k_\mu] \delta_f k_\mu\end{aligned}$$

Gauge sector, part two

$$\mathcal{L} = k_\kappa B^\kappa \tag{4.27}$$

Euler-Lagrange equation:

$$\Pi^\mu = 0 \Longrightarrow \delta_\mu \Pi^\mu = k_\mu = 0$$

Derivative of the Noether current:

$$\partial_\mu J^\mu = -\delta_f \mathcal{L} = -\delta_f k_\alpha B^\alpha - k_\alpha \delta_f B^\alpha = \Psi[k_\mu] \delta_f k_\mu$$

Gauge sector, part three

$$\mathcal{L} = k_\kappa \epsilon^{\kappa\lambda\mu\nu} W_\lambda W_{\mu\nu} = k_\kappa \epsilon^{\kappa\lambda\mu\nu} W^a_\lambda (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon^{abc} W^b_\mu W^c_\nu) \tag{4.28}$$

Euler-Lagrange equation:

$$\Pi^{\mu\nu} = 2k_\kappa \epsilon^{\kappa\lambda\mu\nu} W^a_\lambda \delta_\alpha \Pi^{\alpha\beta} = k_\alpha \epsilon^{\alpha\beta\mu\nu} W^a_{\mu\nu} - 2k_\alpha \epsilon^{\alpha\beta\mu\nu} \epsilon^{abc} W^b_\mu W^c_\nu$$

Derivative of the Noether current:

$$\begin{aligned}\partial_\mu J^\mu &= k_\kappa \epsilon^{\kappa\lambda\mu\nu} (W^a_{\mu\nu} - 2\epsilon^{abc} W^b_\mu W^c_\nu) \delta_f W^a_\lambda + 2k_\kappa \epsilon^{\kappa\lambda\mu\nu} W^a_\lambda \delta_f \partial_\mu W^a_\nu - \delta_f \mathcal{L} \\ &= k_\kappa \epsilon^{\kappa\lambda\mu\nu} (W^a_{\mu\nu} - 2\epsilon^{abc} W^b_\mu W^c_\nu) \delta_f W^a_\lambda + k_\kappa \epsilon^{\kappa\lambda\mu\nu} W^a_\lambda \delta_f W^a_{\mu\nu} - \\ &\quad - k_\kappa \epsilon^{\kappa\lambda\mu\nu} W^a_\lambda \delta_f (\epsilon^{abc} W^b_\mu W^c_\nu) - \delta_f \mathcal{L} = \\ &= k_\kappa \epsilon^{\kappa\lambda\mu\nu} \delta_f (W^a_\lambda W^a_{\mu\nu}) - \delta_f \mathcal{L} \\ &= \Psi[k_\mu] \delta_f k_\mu\end{aligned}$$

Gauge sector, part four

$$\mathcal{L} = \frac{2}{3} i g k_\kappa \epsilon^{\kappa\lambda\mu\nu} \mathrm{Tr}(W_\lambda W_\mu W_\nu) \tag{4.29}$$

Derivative of the Noether current:

$$\begin{aligned}\partial_\mu J^\mu &= \partial_\mu \Pi^{\mu\nu}\delta_f W_\nu - \delta_f \mathcal{L} = \\ &= \Psi[k_\alpha]\delta_f k_\alpha - \frac{2}{3} igk_\kappa \epsilon^{\kappa\lambda\mu\nu}\delta_f(\mathrm{Tr}(W_\lambda W_\mu W_\nu)) + 2igk_\kappa\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}(\delta_f(W_\lambda)W_\mu W_\nu) = \\ &= \Psi[k_\alpha]\delta_f k_\alpha\end{aligned}$$

Theorem (3.4) thus holds for any minimal SME Lagrangian.

5 Relic symmetries of minimal SME Lagrangians

5.1 Coefficients in minimal SME and their properties

SM Lagrangians are of course Lorentz invariant.

SME Lagrangians – written above – violate Lorentz symmetry. Could any subgroup be restored?

CPT-even Lagrangians need to examine:

(i) rank two tensors $((c_L)_{AB})_{\mu\nu}$, $((c_R)_{AB})_{\mu\nu}$ in lepton sector, $(H_{AB})_{\mu\nu}$ in Yukawa sector and $(k_{\phi\phi})_{\mu\nu}$, $(k_{\phi B})_{\mu\nu}$ and $(k_{\phi W})_{\mu\nu}$ in Higgs sector. Quark sector is an exact analogy to lepton sector. For quarks (tensors $((c_Q)_{AB})_{\mu\nu}$, $((c_U)_{AB})_{\mu\nu}$, $((c_D)_{AB})_{\mu\nu}$ and $((H_U)_{AB})_{\mu\nu}$, $((H_D)_{AB})_{\mu\nu}$) the same result would hold.

(ii) rank four tensors $k_B^{\alpha\beta\gamma\delta}$ and $k_W^{\alpha\beta\gamma\delta}$ in gauge sector (for quarks additional one $k_G^{\alpha\beta\gamma\delta}$).

These tensors all have zero derivative (the reason is shown below); moreover, $c^{\mu\nu}_{R,L,Q,U,D}$ are traceless, $H^{\mu\nu}_{L,U,D}$ are antisymmeric, and $k^{\mu\nu}_{\phi\phi}$ has symmetric real and antisymmetric imaginary part.

$k^{\kappa\lambda\mu\nu}_{B,W,G}$ have symmetries of the Riemann tensor.

CPT-odd Lagrangians involve rank one tensors $((a_L)_{AB})_\mu$, $((a_R)_{AB})_\mu$ in lepton sector, $(k_\phi)_\mu$ in Higgs sector and $(k_0)_\mu$, $(k_1)_\mu$, $(k_2)_\mu$ in gauge sectors. For quarks there are also tensors $((a_Q)_{AB})_\mu$, $((a_U)_{AB})_\mu$, $((a_D)_{AB})_\mu$ and finally $(k_3)_\mu$.

Coefficients $k_{0,1,2,3}$ are real and they are assumed to vanish, as the terms could generate energy instabilities.

5.2 Lepton, Yukawa and Higgs sector: relic Lorentz symmetries of a rank two tensor

As was shown before, for all the cases of interest, lepton, Yukawa and Higgs sectors, $\partial_\mu J^\mu = \Psi[c^{\mu\nu}]\delta_f c^{\mu\nu}$.

When $\partial_\mu J^\mu = 0$? Vanishing Lagrange expression doesn't make sense, so only way how this could be zero is vanishing Lie derivative of the tensor $c^{\mu\nu}$. We know, that tensor $c^{\mu\nu}$ fulfills:

$$\partial_\alpha c^{\mu\nu} = 0 \tag{5.1}$$

so the condition for relic symmetries reads:

$$c^{\alpha\nu}\partial_\alpha f^\mu + c^{\mu\alpha}\partial_\alpha f^\nu = 0 \tag{5.2}$$

It's obvious that because of condition (5.1) there is invariance under the group of translations. Let's look at Lorentz group. We get

$$c^{\alpha\nu}\omega^\mu{}_\alpha + c^{\mu\alpha}\omega^\nu{}_\alpha = 0 \tag{5.3}$$

with antisymmetric ω. Let's decompose $c^{\mu\nu}$ into its symmetric and antisymmetric part: $c^{\mu\nu} = c_S^{\mu\nu} + c_A^{\mu\nu}$.

It's useful to express $\omega^\mu{}_\nu$ and $c^{\mu\nu}$ using vectors:

$$\begin{aligned}
\vec{\omega} &= (\omega^0{}_1, \omega^0{}_2, \omega^0{}_3) = -(\omega_1{}^0, \omega_2{}^0, \omega_3{}^0) \\
\tilde{\vec{\omega}} &= (\omega^2{}_3, \omega^3{}_1, \omega^1{}_2) = -(\omega_3{}^2, \omega_1{}^3, \omega_2{}^1) \\
\vec{c_S} &= \frac{1}{2}(c^{01} + c^{10}, c^{02} + c^{20}, c^{03} + c^{30}) \\
\vec{c_A} &= \frac{1}{2}(c^{01} - c^{10}, c^{02} - c^{20}, c^{03} - c^{30}) \\
\tilde{\vec{c_S}} &= \frac{1}{2}(c^{23} + c^{32}, c^{31} + c^{13}, c^{12} + c^{21}) \\
\tilde{\vec{c_A}} &= \frac{1}{2}(c^{23} - c^{32}, c^{31} - c^{13}, c^{12} - c^{21})
\end{aligned} \tag{5.4}$$

We are going to consider nonzero trace of $c^{\mu\nu}$, too.

Case i) $\mu = \nu = 0$
We get

$$\begin{aligned} &c^{i0}\omega^0{}_i + c^{0i}\omega^0{}_i = 0 \\ &\vec{c_S} \cdot \vec{\omega} = 0 \end{aligned} \tag{5.5}$$

Case ii) $\mu = \nu = i$ (not to be summed over i)

$$\begin{aligned} &c^{0i}\omega^i{}_0 + c^{i0}\omega^i{}_0 + c^{ji}\omega^i{}_j + c^{ij}\omega^i{}_j = 0 \\ &c_S^{\mu i}\omega^i{}_\mu = 0 \end{aligned} \tag{5.6}$$

Case iii) $\mu = 0,\ \nu = i$

$$c^{ji}\omega^0{}_j + c^{00}\omega^i{}_0 + c^{0j}\omega^i{}_j = 0$$

first term involves the case $i = j$, that means the trace of $c^{\mu\nu}$. That will be expressed later.

Case iv) $\mu = i,\ \nu = 0$

$$c^{00}\omega^i{}_0 + c^{j0}\omega^i{}_j + c^{ij}\omega^0{}_j = 0$$

Cases iii and iv together give:

$$\begin{aligned} c^{00}\omega^i{}_0 =& c^{ji}\omega^0{}_j + c^{0j}\omega^i{}_j = c^{j0}\omega^i{}_j + c^{ij}\omega^0{}_j = 0 \\ &\tilde{\vec{c_A}} \times \vec{\omega} = \tilde{\vec{\omega}} \times \vec{c_A} \end{aligned} \tag{5.7}$$

Eventually for trace terms that means

$$\begin{aligned} 2(c^i_i - c^0_0)\omega^{0i} &= (\vec{\omega} \times \tilde{\vec{c_A}})^i + (\tilde{\vec{\omega}} \times \vec{c_A})^i + |\epsilon^{ijk}|(\vec{\omega})^j(\tilde{\vec{c_S}})^k + |\epsilon^{ijk}|(\tilde{\vec{\omega}})^j(\vec{c_S})^k \\ 2(c^i_i - c^0_0)\omega^{0i} &= |\epsilon^{ijk}|(\vec{\omega})^j(\tilde{\vec{c_S}})^k + |\epsilon^{ijk}|(\tilde{\vec{\omega}})^j(\vec{c_S})^k \end{aligned} \tag{5.8}$$

Metric $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is used for raising and lowering indices.

Case v) $\mu = i,\ \nu = j, \quad i \neq j$

$$\begin{aligned} &c^{i0}\omega^j{}_0 + c^{0j}\omega^i{}_0 + c^{ik}\omega^j{}_k + c^{kj}\omega^i{}_k = 0 \\ &(c^i_i - c^j_j)\omega^{ij} = c^{i0}\omega^j{}_0 + c^{0j}\omega^i{}_0 + c^{ik}\omega^j{}_k + c^{kj}\omega^i{}_k, \quad k \neq i,\ k \neq j \end{aligned} \tag{5.9}$$

Symmetric and antisymmetric (in i, j) part of the last expression gives:

$$\vec{c_A} \times \vec{\omega} = \tilde{\vec{c_A}} \times \tilde{\vec{\omega}} \tag{5.10}$$

$$|\epsilon^{ijk}|(c_i^i - c_j^j)\omega^{ij} = |\epsilon^{ijk}|(\vec{c_S})^i(\vec{\omega})^j + \epsilon^{ijk}(\tilde{\vec{c_S}})^i(\tilde{\vec{\omega}})^j \tag{5.11}$$

Equations (5.5)-(5.11) are conditions for $c^{\mu\nu}$ such that Lagrangian would have a relic Lorentz symmetry.

Could any subgroup of Lorentz group be restored? Let's concentrate on the antisymmetric $c^{\mu\nu}$ for simplicity. Our conditions now are:

$$\begin{aligned} \tilde{\vec{c_A}} \times \vec{\omega} &= \tilde{\vec{\omega}} \times \vec{c_A} \\ \vec{\omega} \times \vec{c_A} &= \tilde{\vec{\omega}} \times \tilde{\vec{c_A}} \end{aligned} \tag{5.12}$$

These are six conditions for components of four three dimensional vectors; one vector is fixed by the choice of the base and scale.

Which subgroup of Lorentz group could be found?

Of course simple subgroups involving just one invariant boost or one rotation are allowed; what about bigger subgroups?

Let's look at SO(2,1), that means two remaining boosts $\vec{\omega}$ and one rotation $\tilde{\vec{\omega}}$ perpendicular to both of them. But our condition means all four vector lay in the same plane; it is not possible to find two boosts and one rotation forming SO(2,1): rotation could never be perpendicular to two different boosts.

For symmetric $c^{\mu\nu}$ (still zero trace terms c^μ_μ). Conditions become:

$$\begin{aligned} |\epsilon^{ijk}|(\tilde{\vec{c_S}})^j(\vec{\omega})^k &= -|\epsilon^{ijk}|(\tilde{\vec{\omega}})^j(\vec{c_S})^k \\ |\epsilon^{ijk}|(\vec{\omega})^j(\vec{c_S})^k &= \epsilon^{ijk}(\tilde{\vec{\omega}})^j(\tilde{\vec{c_S}})^k \\ c_S^{\mu i}\omega^i{}_\mu &= 0 \end{aligned} \tag{5.13}$$

These are ten conditions for components of four three dimensional vectors; adding the freedom in the choice of the base the system looks overdefined; but one of the last four equations is dependent on the remaining three. There again exists a solution forming a group consisting of just one boost or just one rotation, and no bigger group.

5.3 Gauge sector: relic symmetries of the fourth-rank tensor $k^{\alpha\beta\gamma\delta}$

We know that $k^{\alpha\beta\gamma\delta}$ has the Riemann tensor symmetries:

$$\begin{aligned} k^{\alpha\beta\gamma\delta} = -k^{\beta\alpha\gamma\delta} &= \alpha\beta\delta\gamma && \text{antisymmetry in first and second pairs of indices} \\ k^{\alpha\beta\gamma\delta} &= k^{\gamma\delta\alpha\beta} && \text{symmetry in pairs of indices} \\ k^{\alpha\beta\gamma\delta} + k^{\alpha\gamma\delta\beta} + k^{\alpha\delta\beta\gamma} &= 0 && \text{zero cycle permutation in last three indices} \end{aligned} \tag{5.14}$$

Relic symmetries require:

$$\begin{aligned} \delta_f k^{\alpha\beta\gamma\delta} = 0 \ &= \omega^{\alpha}_{\ \mu} k^{\mu\beta\gamma\delta} + \omega^{\beta}_{\ \mu} k^{\alpha\mu\gamma\delta} + \omega^{\gamma}_{\ \mu} k^{\alpha\beta\mu\delta} + \omega^{\delta}_{\ \mu} k^{\alpha\beta\gamma\mu} = \\ &= \omega^{\alpha}_{\ \mu} k^{\mu\beta\gamma\delta} - \omega^{\beta}_{\ \mu} k^{\mu\alpha\gamma\delta} + \omega^{\gamma}_{\ \mu} k^{\mu\delta\alpha\beta} - \omega^{\delta}_{\ \mu} k^{\mu\gamma\alpha\beta} \end{aligned} \tag{5.15}$$

Second line got using symmetries of the Riemann tensor.

We can find following cases:
(i) $\alpha = \beta = \gamma = \delta$, so $k^{\alpha\beta\gamma\delta} = k^{\alpha\alpha\alpha\alpha}$
these are of course zero from the antisymmetry of $k^{\alpha\beta\gamma\delta}$.

(ii) $\alpha = \beta = \gamma \neq \delta$, so $k^{\alpha\beta\gamma\delta} = k^{\alpha\alpha\alpha\beta}$ and similar (any permutation of indices) are also zero for the same reason, so is any possibility with $\alpha = \beta$ or $\gamma = \delta$.

(iii) $\alpha = \beta \neq \gamma = \delta$, and permutations:
(iii a) $k^{\alpha\beta\gamma\delta} = k^{\alpha\alpha\beta\beta}$ is again zero
(iii b) $k^{\alpha\beta\gamma\delta} = k^{\alpha\beta\alpha\beta}$ and any other permutation: equation (5.15) gives:

$$\omega^{\alpha}_{\ \mu} k^{\mu\beta\alpha\beta} - \omega^{\beta}_{\ \mu} k^{\mu\alpha\alpha\beta} = 0 \tag{5.16}$$

(iv) $\alpha = \beta$, $k^{\alpha\beta\alpha\gamma}$ and permutations, which are not trivially zero
We get (again using Riemannian symmetry properties):

$$\omega^{\alpha}_{\ \mu} k^{\mu\beta\gamma\alpha} + \omega^{\alpha}_{\ \mu} k^{\mu\gamma\beta\alpha} = \omega^{\beta}_{\ \mu} k^{\mu\alpha\gamma\alpha} + \omega^{\gamma}_{\ \mu} k^{\mu\alpha\beta\alpha} \tag{5.17}$$

(v) finally general α, β, γ, δ: we can assume they all differ (other cases are above)

$$0 = \omega^{\alpha}{}_{\mu} k^{\mu\beta\gamma\delta} - \omega^{\beta}{}_{\mu} k^{\mu\alpha\gamma\delta} + \omega^{\gamma}{}_{\mu} k^{\mu\delta\alpha\beta} - \omega^{\delta}{}_{\mu} k^{\mu\gamma\alpha\beta} \tag{5.18}$$

Now, what does that mean?

Cases (i) - (iii a) are trivial, they produce no new information.

Case (iii b) gives six conditions on the components of $k^{\alpha\beta\gamma\delta}$ tensor; using denomination defined before:

$$\begin{aligned}
(\vec{\omega})_1 k^{1330} + (\vec{\omega})_2 k^{2330} &= (\tilde{\vec{\omega}})_1 k^{2003} - (\tilde{\vec{\omega}})_2 k^{1003} \\
(\vec{\omega})_3 k^{3220} + (\vec{\omega})_1 k^{1220} &= (\tilde{\vec{\omega}})_3 k^{1002} - (\tilde{\vec{\omega}})_1 k^{3002} \\
(\vec{\omega})_2 k^{2110} + (\vec{\omega})_3 k^{3110} &= (\tilde{\vec{\omega}})_2 k^{3001} - (\tilde{\vec{\omega}})_3 k^{2001}
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
(\vec{\omega})_1 k^{1220} + (\vec{\omega})_2 k^{2110} &= (\tilde{\vec{\omega}})_1 k^{2113} - (\tilde{\vec{\omega}})_2 k^{1223} \\
(\vec{\omega})_3 k^{3110} + (\vec{\omega})_1 k^{1330} &= (\tilde{\vec{\omega}})_3 k^{1332} - (\tilde{\vec{\omega}})_1 k^{3112} \\
(\vec{\omega})_2 k^{2330} + (\vec{\omega})_3 k^{3220} &= (\tilde{\vec{\omega}})_2 k^{3221} - (\tilde{\vec{\omega}})_3 k^{2331}
\end{aligned} \tag{5.20}$$

which is possible to write in a similar form as for 'Dirac' field:

$$\begin{aligned}
(\vec{\omega})_i k^{ikk0} &= \epsilon_{ijk} (\tilde{\vec{\omega}})^i k^{j00k} \\
|\epsilon_{ijk}| (\vec{\omega})^i k^{ijj0} &= \epsilon_{ijk} (\tilde{\vec{\omega}})^i k^{jiik}
\end{aligned} \tag{5.21}$$

in both these equations there is no sum over k.

Case (iv) – it's symmetric in β, γ – gives system of equations:

$$
\begin{aligned}
2(\vec{\omega})_1 k^{1(23)0} &= (\tilde{\vec{\omega}})_1(k^{2002} - k^{3003}) + (\vec{\omega})_2 k^{3220} + (\vec{\omega})_3 k^{2330} + (\tilde{\vec{\omega}})_3 k^{3001} - (\tilde{\vec{\omega}})_2 k^{2001} \\
2(\vec{\omega})_2 k^{2(13)0} &= (\tilde{\vec{\omega}})_2(k^{3003} - k^{1001}) + (\vec{\omega})_3 k^{1330} + (\vec{\omega})_1 k^{3110} + (\tilde{\vec{\omega}})_1 k^{1002} - (\tilde{\vec{\omega}})_3 k^{3002} \\
2(\vec{\omega})_3 k^{3(12)0} &= (\tilde{\vec{\omega}})_3(k^{1001} - k^{2002}) + (\vec{\omega})_1 k^{2110} + (\vec{\omega})_2 k^{1220} + (\tilde{\vec{\omega}})_2 k^{2003} - (\tilde{\vec{\omega}})_1 k^{1003} \\
2(\vec{\omega})_1 k^{1(23)0} &= (\tilde{\vec{\omega}})_1(k^{2112} - k^{3113}) - (\vec{\omega})_2 k^{3110} - (\vec{\omega})_3 k^{2110} + (\tilde{\vec{\omega}})_3 k^{3221} - (\tilde{\vec{\omega}})_2 k^{2331} \\
2(\vec{\omega})_2 k^{2(13)0} &= (\tilde{\vec{\omega}})_2(k^{3223} - k^{1221}) - (\vec{\omega})_3 k^{1220} - (\vec{\omega})_1 k^{3220} + (\tilde{\vec{\omega}})_1 k^{1332} - (\tilde{\vec{\omega}})_3 k^{3112} \\
2(\vec{\omega})_3 k^{3(12)0} &= (\tilde{\vec{\omega}})_3(k^{1331} - k^{2332}) - (\vec{\omega})_1 k^{2330} - (\vec{\omega})_2 k^{1330} + (\tilde{\vec{\omega}})_2 k^{2113} - (\tilde{\vec{\omega}})_1 k^{1223} \\
2(\tilde{\vec{\omega}})_1 k^{1(23)0} &= (\vec{\omega})_1(k^{0330} + k^{1331}) + (\vec{\omega})_2 k^{2331} - (\vec{\omega})_3 k^{3001} + (\tilde{\vec{\omega}})_3 k^{2330} - (\tilde{\vec{\omega}})_2 k^{3110} \\
2(\tilde{\vec{\omega}})_1 k^{1(23)0} &= -(\vec{\omega})_1(k^{0220} + k^{1221}) - (\vec{\omega})_3 k^{3221} + (\vec{\omega})_2 k^{2001} + (\tilde{\vec{\omega}})_2 k^{3220} - (\tilde{\vec{\omega}})_3 k^{2110} \\
2(\tilde{\vec{\omega}})_2 k^{2(13)0} &= (\vec{\omega})_2(k^{0110} + k^{2112}) + (\vec{\omega})_3 k^{3112} - (\vec{\omega})_1 k^{1002} + (\tilde{\vec{\omega}})_1 k^{3110} - (\tilde{\vec{\omega}})_3 k^{1220} \\
2(\tilde{\vec{\omega}})_2 k^{2(13)0} &= -(\vec{\omega})_2(k^{0330} + k^{2332}) - (\vec{\omega})_1 k^{1332} + (\vec{\omega})_3 k^{3002} + (\tilde{\vec{\omega}})_3 k^{1330} - (\tilde{\vec{\omega}})_1 k^{3220} \\
2(\tilde{\vec{\omega}})_3 k^{3(12)0} &= (\vec{\omega})_1(k^{0220} + k^{3223}) + (\vec{\omega})_1 k^{1223} - (\vec{\omega})_2 k^{2003} + (\tilde{\vec{\omega}})_2 k^{1220} - (\tilde{\vec{\omega}})_1 k^{2330} \\
2(\tilde{\vec{\omega}})_3 k^{3(12)0} &= -(\vec{\omega})_1(k^{0110} + k^{3113}) - (\vec{\omega})_2 k^{2113} + (\vec{\omega})_1 k^{1003} + (\tilde{\vec{\omega}})_1 k^{2110} - (\tilde{\vec{\omega}})_2 k^{1330}
\end{aligned}
\tag{5.22}
$$

Indices in round bracket () are symmetrised: $T^{(\mu\nu)} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$.

Case (v): there are only two independent possibilities of permutations of the indices, other depend through Riemannian symmetries.

We can have e.g. equations:

$$
\begin{aligned}
0 &= \omega^0{}_\mu k^{\mu 123} - \omega^1{}_\mu k^{\mu 023} + \omega^2{}_\mu k^{\mu 301} - \omega^3{}_\mu k^{\mu 201} \\
0 &= \omega^0{}_\mu k^{\mu 213} + \omega^1{}_\mu k^{\mu 302} - \omega^2{}_\mu k^{\mu 013} - \omega^3{}_\mu k^{\mu 102} \\
0 &= \omega^0{}_\mu k^{\mu 213} + \omega^1{}_\mu k^{\mu 203} - \omega^2{}_\mu k^{\mu 103} - \omega^3{}_\mu k^{\mu 203}
\end{aligned}
\tag{5.23}
$$

one of them still dependant.

For any possible μ we eventually get:

$$
\begin{aligned}
(\vec{\omega})_2(k^{1223} + k^{1003}) - (\vec{\omega})_3(k^{1332} + k^{1002}) &= (\tilde{\vec{\omega}})_2(k^{2110} - k^{2330}) + (\tilde{\vec{\omega}})_3(k^{3110} - k^{3220}) \\
(\vec{\omega})_3(k^{2331} + k^{2001}) - (\vec{\omega})_1(k^{2113} + k^{2003}) &= (\tilde{\vec{\omega}})_3(k^{3110} - k^{3220}) + (\tilde{\vec{\omega}})_1(k^{1220} - k^{1330}) \\
(\vec{\omega})_1(k^{3112} + k^{3002}) - (\vec{\omega})_2(k^{3221} + k^{3001}) &= (\tilde{\vec{\omega}})_1(k^{1220} - k^{1330}) + (\tilde{\vec{\omega}})_2(k^{2110} - k^{2330})
\end{aligned}
\tag{5.24}
$$

Sum of these three equations products a zero left hand side:

$$0 = (\tilde{\vec{\omega}})_1(k^{1220} - k^{1330}) + (\tilde{\vec{\omega}})_2(k^{2110} - k^{2330}) + (\tilde{\vec{\omega}})_3(k^{3110} - k^{3220}) \tag{5.25}$$

Any relic Lorentz symmetry of the tensor $k^{\alpha\beta\gamma\delta}$ must fulfill the conditions (5.19) – (5.25).

All the conditions are for $k^{\alpha\beta\gamma\delta}$ symmetric in second and third index, $k^{\alpha(\beta\gamma)\delta}$, that means antisymmetric part $k^{\alpha[\beta\gamma]\delta}$ could be any without violating Lorentz invariance.

However, conditions on presence of any (nontrivial) subgroup of Lorentz group are strict. For invariance under just one boost, for concreteness here boost ${\omega^0}_1$

$${\omega^0}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.26}$$

the conditions on $k^{\alpha\beta\gamma\delta}$ are:

$$\begin{aligned} k^{1(23)0} &= 0 \\ k^{1220} &= k^{1330} = 0 \\ k^{2110} &= k^{3110} = 0 \\ k^{3220} &= k^{2330} = 0 \\ k^{1002} &= k^{1332} = 0 \\ k^{1331} &= -k^{0330} = k^{1001} - k^{1003} \\ k^{1221} &= -k^{0220} = k^{3223} + k^{1223} \end{aligned} \tag{5.27}$$

that means large part of tensor must simply vanish.

Situation is similar for invariance under just one rotation ${\omega^2}_3$:

$${\omega^2}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.28}$$

conditions are:

$$\begin{aligned}
&k^{1(23)0} = 0 \\
&k^{1332} = k^{1223} = 0 \\
&k^{3110} = k^{3220} = 0 \\
&k^{2330} = k^{2110} = 0 \\
&k^{1002} = k^{1003} = 0 \\
&k^{2002} = k^{3003} \\
&k^{2112} = k^{3113}
\end{aligned} \tag{5.29}$$

again really strict conditions.

5.4 CPT-odd terms: rank one tensors

For CPT-odd terms rank one tensors, or vectors, occur; for four-vector $a_\mu = (a_0, \vec{a})$ the condition on relic Lorentz symmetry reads:

$$0 = a_\alpha \omega^\alpha{}_\mu \tag{5.30}$$

For $\mu = 0$ that imlies:

$$\begin{aligned}
0 &= a_i \omega^i{}_0 \\
0 &= \vec{a} \cdot \vec{\omega}
\end{aligned}$$

and for $\mu = i$

$$\begin{aligned}
0 &= a_0 \omega^0{}_i + a_j \omega^j{}_i \\
a_0 \vec{\omega}_i &= (\tilde{\vec{\omega}} \times \vec{a})_i
\end{aligned}$$

which would allow one boost or one rotation.

5.5 Conclusion: relic Lorentz symmetry of the minimal SME

From what was done above is quite clear, that for sum of all the minimal SME with no subgroup of Lorentz group is preserved, unless coefficients in SME modifications of SM are really special: one boost or one rotation could remain, if *all* the two-rank tensors *and* the four-rank tensor $k^{\alpha\beta\gamma\delta}$ allowed *the same boost or the same rotation*, but not both at same time.

6 Conclusion

What in next to be done is proving the theorem in general, (or finding a conditions on Lagrangians, under which it is valid).

For Minimal SME Lagrangians, for which it was proved, it leaded to a straihgt-forward way, how to decide about relic symmetries:
we found, that in general case there are no relic of the Lorentz symmetry left; to leave at least part of SO(3,1) there are really strict conditions to be applied on coefficients in Lagrangians of Minimal SME.

Next step could be moving beyond Minimal SME or study NCFT theories using this method, and studing SUSY in these models.

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