Report on master thesis by Alexander Beneš:

Prime geodesic theorem for the Picard manifold,

reviewer: Giacomo Cherubini, thesis supervisor

Summary: This work explores a generalisation of the prime number theorem in the three-dimensional hyperbolic space.

The prime number theorem provides an asymptotic formula for the number of primes up to a given quantity. The "prime geodesic theorem" is a geometric analogue in the hyperbolic space, where –loosely speaking– closed curves (geodesics) play the role of primes. The "Picard manifold" is the quotient of the three-dimensional hyperbolic space $\mathbb{H}^3$ by the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$. It is the natural generalisation to three dimensions of the modular surface, which has been extensively studied in the literature.

In addition to an asymptotic formula, one would like to control the size of the remainder in the prime geodesic theorem. Partial results towards this are known, as the author explains in the introduction.

The main goal of the thesis is to compute (Theorem 1.0.1 and 3.4.1) the first moment of the remainder, thus showing that its fluctuations do not deviate too much from its average value.

The result is new and extends a theorem by Phillips (see the reference [Phi95]), who worked on the two-dimensional hyperbolic plane. It also complements several recent results in the area (see the references from [BBCL22] until [CCL22]). Because of this it should be possible, in my opinion, to publish the result in a peer-reviewed journal specialised in number theory, which is a strong positive feature of the thesis.

Evaluation: The thesis is well organized in the presentation of the material. It is overall well written, possibly a little dry at places, and it is mathematically correct. In Chapter 2 the results are cited from other sources giving appropriate references, while most of the results in Chapter 3 are proved in full details. Thanks to this, the strategy of proof of the main theorem can be followed without too much effort. In the more general situation described in Chapter 4, some familiarity with algebraic number theory and with the reference book [EGM98] is probably needed.

The material in the thesis is rather advanced and broad as it spans from the geometry of hyperbolic spaces to automorphic forms and to the spectral analysis of the hyperbolic Laplacian. The thesis attests that the student has mastered all these concepts as well as the other tools and techniques employed in the proofs.

I record below a few minor typos:

- there is a small inconsistency in terminology as both expressions "quadratic imaginary number field" and "imaginary quadratic number field" occurr in the thesis. The most common wording is in fact simply "imaginary quadratic fields".
- there is some unnecessary additional spacing near the left parenthesis in the typesetting of matrices, possibly due to an extra `&` before the first column.
- on p.21 "we will smooth out g by convoluting it" should be "by convolving it".
- it would be better to write the full bibliographic entry for the references, including volume, issue and page numbers.

Recommendation for the thesis: I recommend the thesis to be defended as Master's thesis and awarded the grade 1.

Prague, 15.08.2022

Giacomo Cherubini