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Prime geodesic theorem for the Picard manifold

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Abstract: The goal of this thesis is to obtain a weighted first moment of the error term of the approximation of the counting function of prime geodesics on the Picard variety $\mathrm{SL}(2,\mathbb{Z}[i])\backslash\mathbb{H}^3$. The group $\mathrm{SL}(2,\mathbb{C})$ acts on the 3-dimensional hyperbolic space $\mathbb{H}^3$. We choose a discrete subgroup $\mathrm{SL}(2,\mathbb{Z}[i])$ called the Picard group and study its action on the hyperbolic space. A matrix that fixes two points on the boundary of $\mathbb{H}^3$ is called hyperbolic or loxodromic. These matrices have a similar asymptotic behaviour as primes in number theory. The counting function $\psi_g(X)$ counts the number of conjugacy classes of these matrices with norm less than X. This function asymptotically grows as X and the error term is the difference $\psi_g(X)-X$. The error can be explicitly written using the Selberg trace formula which relates geometrical information with the spectrum of the Laplace operator on the Picard manifold. This is used to calculate the weighted first moment of the error explicitly.

Keywords: prime geodesic theorem Picard manifold hyperbolic geometry

Contents

1. The prime geodesic theorem

Let Γ be the modular group of $\mathrm{SL}(2,\mathbb{Z})$. This group acts on the upper half-plane by the Möbius transforms. The elements of Γ that fix exactly 2 elements on the boundary $\mathbb{R}\cup\{\infty\}$ are called hyperbolic. Every such element M can be diagonalized to the form

$$SMS^{-1} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a > 1 \tag{1.1}$$

The number $|a|^2$ is called the norm $N(M)$. The norm is constant on conjugacy classes. A hyperbolic matrix is called *primitive* if it cannot be written as a power of some hyperbolic matrix. Every hyperbolic element is a power of some primitive hyperbolic element $M = M_0^n$. Let $\pi(X)$ be the number of primitive hyperbolic conjugacy classes with norm $\leq X$. This in fact describes geometric data about the quotient variety $\Gamma\backslash\mathbb{H}$. The norms of the hyperbolic conjugacy groups describe the hyperbolic lengths of closed geodesics in the quotient variety.
This function behaves similarly to the prime counting function from number theory. The analogue of the prime number theorem holds:

$$\pi(X) \sim \mathrm{li}(X) \tag{1.2}$$

where $\mathrm{li}(N)$ is the logarithmic integral $\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}$. As for the prime number theorem we first define the auxiliary function

$$\psi_\Gamma(X) = \sum_{\{M\}\text{ hyper. } \leq X} \Lambda(M). \tag{1.3}$$

where the sum is over the conjugacy classes of hyperbolic matrices with norm $\leq X$ and $\Lambda(M)$ is $\log(N(M_0))$ if $M = M_0^n$ for some primitive M_0 and 0 otherwise. This is the analogue of the von Mangoldt function in number theory. Then the equation (1.2) is equivalent to

$$\psi_\Gamma(X) \sim X$$

as is the case in analytic number theory.
We can use the spectral theory of the Laplace operator on hyperbolic spaces to approximate $\psi_\Gamma(X)$. We define the main term as

$$M_\Gamma(X) = \sum_{\frac{1}{2}<s_i\leq 1} \frac{X^{s_i}}{s_i}.$$

where $\lambda_i = s_i(1-s_i), 0 \leq \lambda_i < \frac{1}{4}$ are the eigenvalues of the Laplace operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$ acting on $L^2(\Gamma\backslash\mathbb{H}^3)$. This term is a good approximation of $\psi_\Gamma(X)$ and we can study the remainder $E_\Gamma(X) = \psi_\Gamma(X) - M_\Gamma(X)$.

In the two dimensional case the most studied is the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and there are multiple results for a pointwise bound for the remainder E_Γ. We can use a similar method as for the prime number case. The Selberg zeta function is defined as a product regarding the norms $N(T)$ and the numbers $a(T)$

of primitive hyperbolic conjugacy classes. The non-trivial zeros correspond to the discrete spectrum of the Laplace operator. One could think that we can use a version of the Riemann hypothesis to prove that the error term is $O_\epsilon(X^{1/2+\epsilon})$, where the implied constant can depend on ϵ. But this doesn't work, the Riemann hypothesis has been proven for the Selberg zeta function, but the density of zeros on the critical line is higher. So the best exponent we get from this method is $3/4+\epsilon$, but it is assumed that it actually is $1/2+\epsilon$, so more precise methods need to be used.
The first to break the 3/4 border was Iwaniec in (Iwa84) with the exponent $35/48+\epsilon$ and he noticed that an exponent of $2/3+\epsilon$ would follow from a version of the Lindeföf hypothesis for Dirichlet L-functions. His methods use the Selberg trace formula, the Kuznetsov trace formulas and some bounds of the symmetric-square L-functions to go between them. The bound on the L-functions was improved in (LS95) to obtain the exponent $7/10+\epsilon$. The best unconditional result known now is found in (SY13).

$$E_\Gamma(X) = O_\epsilon(X^{25/36+\epsilon}) \tag{1.4}$$

for some $\epsilon > 0$. The best conditional result is the exponent $5/8+\epsilon$ assuming a conjecture for Zagier L-series found in (BFR22). The best conjectured exponent is still $1/2+\epsilon$.
There are also results on the second moment of the remainder that support the conjecture. For example in (BBHM19) it is proved that the second moment satisfies

$$\frac{1}{A}\int_A^{2A} |E_\Gamma(X)|^2 dX \ll A^{7/6+\epsilon} \tag{1.5}$$

for $A \gg 1$ and every $\epsilon > 0$. This shows that the exponent $7/12+\epsilon$ holds on average.

We can also study groups acting on the 3-dimensional hyperbolic space $\mathbb{H}^3 = \{(x,y,r) \in \mathbb{R}^3 | r > 0\}$. We will be mostly interested in the subgroup $\Gamma = \mathrm{SL}(2,\mathbb{Z}[i])$ of the group $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathbb{H}^3$ which is called the Picard group. A general discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ is usually called a Kleinian group. The elements are divided into four categories: elliptic, parabolic, hyperbolic and loxodromic, depending on the fixed points. Hyperbolic and loxodromic can be written as a power of primitive elements and we can define $\pi(N)$ as the number of hyperbolic and loxodromic primitives with norm $\leq N$. Similarly we can define the ψ function. The main term will be defined as

$$M_\Gamma(X) = \sum_{0\leq s_j \leq 1} \frac{X^{1+s_j}}{1+s_j}. \tag{1.6}$$

where $\lambda_j = 1 - s - j^2, 0 \leq \lambda_j \leq 1$ are the eigenvalues of the Laplace operator on $\Gamma\backslash\mathbb{H}^3$. In the three dimensional case there have been similar results as in the two dimensional case. The first non-trivial bound was found by (Sar83) for a general cofinite Kleinian group (that is the quotient space has finite volume.)

$$E_\Gamma(X) = O_\epsilon(X^{5/3+\epsilon}) \tag{1.7}$$

for any $\epsilon > 0$. For the Picard group $\mathrm{SL}(2,\mathbb{Z}[i])$ this has been improved to an exponent $11/7+\epsilon$ by (Koy01) assuming the Lindelöf hypothesis for L-functions

attached to Maass forms on the Picard variety. This was further improved to $3/2+\epsilon$ in (BF22) and finally to $3/2-1/46+\epsilon$ in (BBCL22). An unconditional bound was improved to $13/8+\epsilon$ by (BCC$^+$18) then to $67/42+\epsilon$ in (BBCL22) and most recently to $376/237+\epsilon$ in (BF22).
The methods used for these results usually work by finding an estimate for a sum involving the discrete spectrum of the Laplacian like the spectral exponential sum

$$S(T,X)=\sum_{0<s_j\leq T} X^{is_j}. \tag{1.8}$$

A bound for the sum then gives a bound for the remainder E_Γ. A simple bound given by the Weyl law which gives a density of the eigenvalues gives the bound $5/3+\epsilon$. A better exponent can be given with some cancellation, for example the Kuznetsov formula which relates Fourier coefficients of cusp forms on the quotient variety and Kloosterman sums which give arithmetic information about the Picard group.

It is not known what the best value of the exponent should be, it might go as low as $1+\epsilon$, but there are reasons indicating it is higher, see Remark 3.1 of (BCC$^+$18).

One can also study the second moment of the remainder. In (CCL22) a bound of the integral of the second power of the remainder has the following form

$$\frac{1}{Y}\int_V^{V+Y}|E_\Gamma(X)|^2dX \ll V^{3+\epsilon}\left(\frac{V}{Y}\right)^{2/3} \tag{1.9}$$

for $V\geq Y\gg 1$ and $\epsilon>0$. This shows that the exponent $3/2$ holds on average unconditionally. This was similarly studied in (Kan20) which gives more information and additional tests.

In this thesis we will use a slightly modified counting function ψ_g which is better suited to be used with the Selberg trace formula. Our goal is to add more information about the remainder by computing the first moment of the difference of the counting function $\psi(X)$ and its asymptotic approximation given by the sum $M_\Gamma(X)$ weighted by the factor $\frac{1}{X}$. The function ψ_g is expected to grow as X which is the main term of M_Γ

$$\int_1^V\left(\psi_g(X)-X\right)\frac{dX}{X} \tag{1.10}$$

But we will need to alter the integral by adding a smoothing factor. We will change the variables $T=\log V$ and $\rho=\log X$

$$\int_0^T\left(\psi_g(e^\rho)-e^\rho\right)d\rho \tag{1.11}$$

We can choose a non-negative smooth function ω with support on the interval $(0,1)$ and with mass 1. We can then add a smoothing factor $\omega_T(x)=\omega(\frac{x}{T})$ and introduce a normalizing factor $\frac{1}{T^2}$. This brings us to the main theorem.

Theorem 1.0.1. *The weighted average of the difference of the counting function* $\psi_\delta(e^\rho)$ *and its approximation* e^ρ *for the modular group* $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$ *satisfies*

$$\lim_{T\to\infty} \frac{1}{T^2} \int_0^T \omega_T(\rho)\bigg(\psi_g(e^\rho) - e^\rho\bigg) d\rho = 2\kappa. \tag{1.12}$$

where κ *is a positive constant that depends only on* $\omega(x)$*.*

In Chapter 2 we will begin by defining the action of the group $\mathrm{SL}(2, \mathbb{C})$ on the three-dimensional half-space with a hyperbolic metric and describing the different kinds of matrices by their actions. Then we will restrict the action on the discrete subgroup $\mathrm{SL}(2, \mathbb{Z}[i])$ and look at the functions that are invariant under the action, in other way the functions on the quotient space $\mathrm{SL}(2, \mathbb{Z}[i])\backslash\mathbb{H}^3$. In particular we will be interested in functions that are also eigenfunctions of a differential operator on the quotient space, the Laplace operator.
Then we will describe the spectrum of the Laplace operator on the space of square integrable functions. Chapter 3 is the core of the thesis, the Selberg trace formula, which gives a connection between the spectral data and geometrical data of the Picard group $\mathrm{SL}(2, \mathbb{Z}[i])$. A suitable test function will be used in the trace formula to write the remainder E_Γ explicitly. Then we will calculate the weighted first moment of this explicit remainder to get our main result. Finally in Chapter 4 we will talk about extending this result to other possible discrete subgroups of $\mathrm{SL}(2, \mathcal{O})$ for some quadratic imaginary number field, usually called the Bianchi groups.

2. Preliminaries

Most of the theorems and proofs will be taken from (EGM98). The group $\Gamma = \mathrm{SL}(2, \mathbb{C})$ acts on the hyperbolic space $\mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 | r > 0\}$ as follows: represent the point (x, y, r) as a quaternion $Q = x + iy + jr$ and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

act as:

$$MQ = (aQ + b)(cQ + d)^{-1} \tag{2.2}$$

using the standard quaternion division. This is in analogy to the classical two-dimensional case where the action is the same just with the complex numbers instead of the quaternions. The action can more explicitely be described as

$$\begin{aligned} z' &= \frac{(az + b)(\overline{cz} + \overline{d}) + a\overline{c}r^2}{|cz + d|^2 + |c|^2 r^2} \\ r' &= \frac{r}{|cz + d|^2 + |c|^2 r^2} \end{aligned} \tag{2.3}$$

with $MQ = z' + jr'$. From this we can verify that it maps $\mathbb{H}^3$ to $\mathbb{H}^3$. As in the two-dimensional case we have that $M(NQ) = (MN)Q$ so it forms a well defined group action.
The space $\mathbb{H}^3$ has a hyperbolic metric $ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$ and a volume element $dv = \frac{dxdydr}{r^3}$. The geodesics of this metric are half-lines and half- circles orthogonal to the boundary $\mathbb{C} \cup \{\infty\}$. The hyperbolic distance between two points x, y will be denoted $d(x, y)$.
We also define the associated Laplace-Beltrami (or simply Laplace) operator

$$\Delta = r^2\{\partial_x^2 + \partial_y^2 + \partial_r^2\} - r\partial_r. \tag{2.4}$$

One can verify that the metric is invariant under the group action.
Let $M \in \mathrm{SL}(2, \mathbb{C})$. We say that M is

- parabolic if $|\mathrm{tr}M| = 2$ and $\mathrm{tr}M \in \mathbb{R}$
- elliptic if $|\mathrm{tr}M| < 2$ and $\mathrm{tr}M \in \mathbb{R}$
- hyperbolic if $|\mathrm{tr}M| > 2$ and $\mathrm{tr}M \in \mathbb{R}$
- loxodromic otherwise.

These definitions have geometric meaning in regards to the action on $\mathbb{H}^3$ and the boundary $\mathbb{P}^1\mathbb{C}$.

Theorem 2.0.1. *Let $M \neq I$ be an element of $\mathrm{SL}(2, \mathbb{C})$. Then the following holds:*

- *M is parabolic iff it has exactly one fixed point in $\mathbb{P}^1\mathbb{C}$.*
- *M is elliptic iff it has two fixed points in $\mathbb{P}^1\mathbb{C}$ and if the points on the geodesic line in $\mathbb{H}^3$ joining the two points are also left fixed. M is a rotation around this line.*

- *M is hyperbolic iff it has two fixed points in $\mathbb{P}^1\mathbb{C}$ and if any circle in $\mathbb{P}^1\mathbb{C}$ through these two points together with its interior is left invariant. The line in $\mathbb{H}^3$ joining these two fixed points is then left invariant, but M has no fixed point in $\mathbb{H}^3$.*

- *M is loxodromic in all other cases. M has then two fixed points in $\mathbb{P}^1\mathbb{C}$ and no fixed points in $\mathbb{H}^3$. The geodesic joining the 2 fixed points is the geodesic in $\mathbb{H}^3$ which is left invariant. M may leave the circles joining the two fixed points invariant, but then it interchanges exterior and interior.*

Proof. Proposition 1.4 Chapter 2 in (EGM98). □

Every hyperbolic or loxodromic matrix T can be conjugated by an element of $\mathrm{SL}(2,\mathbb{C})$ to be in the following form:

$$DTD^{-1} = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix} \quad a(T) \in \mathbb{C}, |a(T)| > 1. \tag{2.5}$$

We call the number $N(T) = |a(T)|^2$ the norm of T. The norm has a geometric interpretation: we can define the length of a matrix

$$\ell(T) = \inf_{z \in \mathbb{H}^3} d(z, Tz). \tag{2.6}$$

Note that for elliptic or parabolic matrices this is 0.
Then $\log N(T) = \ell(T)$ for all hyperbolic or loxodromic T. If P is primitive, the norm of P describes the length of the geodesic connecting the two fixed points of P in the Picard variety.
The stabilizer subgroup of Γ that fixes a point z on the boundary is denoted Γ_z. The subgroup consisting of parabolic elements and the identity is denoted Γ'_z. By conjugating with an element of $\mathrm{SL}(2,\mathbb{C})$ we can move z to ∞. There are 3 possible options for Γ'_z:

- $\Gamma'_\infty = I$ then Γ_∞ is either cyclic generated by an elliptic element or a product of a cyclic group and a free group generated by a hyperbolic or loxodromic element with the same fixed boundary points

- $\Gamma'_\infty = \{\begin{pmatrix} 1 & k\omega \\ 0 & 1 \end{pmatrix} | k \in \mathbb{Z}\}$

- Γ'_∞ is a lattice in $\{\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} | w \in \mathbb{C}\}$

In the third case we call z a cusp of Γ. We will denote $C(T)$ the centralizer of T and from the previous theorem we can see that $C(T) = T_0^n E^k$ for a primitive T_0 and some elliptic E.

2.1 Picard group

If Γ is a discrete group, acting fixed-point freely on $\mathbb{H}^3$ then the quotient space $\Gamma\backslash\mathbb{H}^3$ inherits from $\mathbb{H}^3$ the structure of an orientable Riemannian manifold. We call Γ cocompact if the quotient space is compact and cofinite if the quotient has

finite volume. The group Γ will be called the Picard group and the manifold $\Gamma\backslash\mathbb{H}^3$ will be called the Picard variety.
Let $\Gamma < \mathrm{SL}(2,\mathbb{C})$ be a Picard group. We want to find a connected subset $F \subset \mathbb{H}^3$ such that:

- F is closed
- F meets each Γ orbit at least once
- the interior meets each Γ orbit at most once
- the boundary of F has Lebesgue measure zero

Then F is called the fundamental domain of Γ. We will describe the fundamental domain of the Picard group $\mathrm{SL}(2,\mathbb{Z}[i])$ in the next chapter.

2.2 Automorphic functions

A big part of the study of analytic number theory are automorphic functions. We call a function $f : \mathbb{H}^3 \to \mathbb{C}$ automorphic if it is invariant under the group action $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$. This makes f a well defined function on the quotient space $\Gamma\backslash\mathbb{H}^3$. We will be mostly interested in the Hilbert space $L^2(\Gamma\backslash\mathbb{H}^3)$ of square integrable automorphic functions, that is the functions such that:

$$\int_F |f|^2 dv < \infty \tag{2.7}$$

where F is a fundamental region of Γ. This space has the standard scalar product

$$\langle f, g\rangle = \int_F f\bar{g}dv. \tag{2.8}$$

Recall that if $L : U \to L^2(M)$ is a linear operator on a dense subspace $U \subset H$ of a Hilbert space, its adjoint is defined as $L^* : U^* \to L^2(M)$ where U^* is the subset of H such that for every $f, g \in U$ there is $h \in U^*$ such that $\langle Lf, g\rangle = \langle f, h\rangle$ and L^* associates to every g such h. The operator L is called symmetric if $U \subset U^*$ and $L^*|_{U^*} = L$. It is called self-adjoint if $L = L^*$ and essentially self-adjoint if L^* is self-adjoint.
The Laplace operator commutes with the group action, so it acts on automorphic functions. Furthermore if we pick a nice enough set of automorphic functions the Laplace operator is essentially self-adjoint.

Theorem 2.2.1. *Let D be the set of $f \in L^2(\Gamma\backslash\mathbb{H}^3) \cap C^2$ automorphic functions such that $\Delta f \in L^2(\Gamma\backslash\mathbb{H}^3)$. Then this is a dense subset of $L^2(\Gamma\backslash\mathbb{H}^3)$ and $-\Delta$ is symmetric and nonnegative on D, that is $\langle -\Delta f, g\rangle = \langle f, -\Delta g\rangle, \langle -\Delta f, f\rangle \geq 0$ for all $f, g \in D$. Furthermore Δ is essentially self-adjoint.*

Proof. We will prove this for groups of finite covolume and with one class of cusps (the proof for multiple cusps is similar). Define

$$F(R) = \left\{ z + jr \in F \middle| r > R \right\} \tag{2.9}$$

and we also define $F_R = F\backslash F(R)$ and $Q(R) = \left\{ z + jr \in F \middle| r = R \right\}$.
We will show that

$$J(R) = \int_{F_R} (\Delta f)\overline{f} + r^2(\partial_x f \partial_x \overline{g} + \partial_y f \partial_y \overline{g} + \partial_r f \partial_r \overline{g}) \frac{dxdydr}{r^3} \tag{2.10}$$

goes to 0 as $R \to \infty$. It is enough to prove this for $f = g$ since we can use the Cauchy-Schwartz inequality.
The function $J(R)$ can be written as the integral of a closed form, so we can use the Stokes' theorem

$$J(R) = \int_{F_R} d\omega = \int_{\partial F_R} \omega = \int_{Q(R)} \omega \tag{2.11}$$

where

$$\omega = \partial_x f \cdot \overline{f} \frac{dydr}{r} + \partial_y f \cdot \overline{f} \frac{drdx}{r} + \partial_r f \cdot \overline{f} \frac{dxdy}{r}. \tag{2.12}$$

The last equality in 2.11 holds for R high enough because integrals of the other boundary surfaces cancel. We have the inequality

$$\begin{aligned}
&\left| \int_Y^R J(r) \frac{dr}{r} \right| = \left| \int_Y^R \int_{Q(r)} \omega \frac{dt}{t} \right| \\
&\left| \int_Y^R \int_{Q(t)} \partial_r f \cdot \overline{f} \frac{dxdy}{r} \right| \\
&\leq \left(\int_{P\times[Y,R]} |r\partial_r f|^2 dv \right)^{1/2} \cdot \left(\int_{P\times[Y,R]} |f|^2 dv \right)^{1/2} \\
&\leq \left(\int_{F_R} |r\partial_r f|^2 dv \right)^{1/2} \cdot ||f|| \\
&\leq \left(\int_{F_R} |r\partial_x f|^2 + |r\partial_x f|^2 + |r\partial_r f|^2 dv \right)^{1/2} \cdot ||f||
\end{aligned} \tag{2.13}$$

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It is thus enough to prove that

$$\left(\int_{F_R} |r\partial_x f|^2 + |r\partial_x f|^2 + |r\partial_r f|^2 dv \right)^{1/2} < \infty \tag{2.14}$$

Assume the contrary. We can define the function

$$\phi(R) = \mathrm{Re} \int_{R_0}^R J(r) \frac{dr}{r}. \tag{2.15}$$

From the definition of $J(R)$ we can see that ϕ goes to infinity since the functions $\Delta f, f$ are square integrable. We therefore have

$$R\phi'(R) = \mathrm{Re} J(R) \geq c \int_{F_R} |r\partial_x f|^2 + |r\partial_x f|^2 + |r\partial_r f|^2 dv > 0 \tag{2.16}$$

for some constants $c > 0, R_1$ and all $R > R_1$. Using the inequality 2.13 we can see that

$$0 < \phi(R) \leq C\sqrt{R\phi'(R)} \tag{2.17}$$

for some constant C and $R > R_1$. Thus

$$\frac{1}{R} \leq C^2 \frac{\phi'(R)}{\phi^2(R)}. \tag{2.18}$$

If we integrate this we get

$$\log R - \log R_1 \leq C^2 \left(\frac{1}{\phi(R_1)} - \frac{1}{\phi(R)} \right) \tag{2.19}$$

which implies that $\phi(R)$ can't go to as $R \to \infty$. This is a contradiction. □

The adjoint of $-\Delta$ is denoted $-\tilde{\Delta} : \tilde{D} \to L^2(\Gamma \backslash \mathbb{H}^3)$. This in particular means that all eigenvalues of $-\Delta$ are nonnegative real numbers.

2.3 Eisenstein series

The most important example of an automorphic function is the Eisenstein series. Let $\zeta = A^{-1}\infty$ be a cusp and $P = z(P) + r(P)j \in \mathbb{H}^3$. Then

$$E_A(P, s) = \sum_{M \in \Gamma'_\zeta \backslash \Gamma} r(AMP)^{1+s}. \tag{2.20}$$

where $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$. This is a well defined automorphic function since $r(P)$ is invariant under Γ'_∞ and converges to a holomorphic function when $\mathrm{Re} s > 1$. The definition is similar to the definition of the two-dimensional case with modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ to the non-holomorphic Eisenstein series

$$E(s, z) = \pi^{-s}\Gamma(s)\zeta(2s) \sum_{m \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(Mz)^s \tag{2.21}$$

Furthermore the Eisenstein series is an eigenfunction of the Laplace operator for all such s, more concretely

$$(-\Delta - (1 - s^2))E_\zeta(P, s) = 0. \tag{2.22}$$

2.4 Fourier expansion of automorphic functions

Similarly to classical two-dimensional automorphic forms, automorphic functions that satisfy certain growth conditions at the cusps can be expanded in a Fourier series. We will be mainly interested in the expansion of the Eisenstein series.

Theorem 2.4.1. *Suppose L is a lattice in $\mathbb{C}$. Let $f : \mathbb{H}^3 \to \mathbb{C}$ be a L-invariant C^2 function satisfying the differential equation $-\Delta f = \lambda f$. Choose $s \in \mathbb{C}$ with $\lambda = 1 - s^2$. Assume further that f has a polynomial growth at ∞, that is:*

$$f(z + rj) = O(r^k) \text{ as } r \to \infty \tag{2.23}$$

for some constant k uniformly with respect to $z \in \mathbb{C}$. Then for $s \neq 0$ f has a Fourier expansion

$$f(z + rj) = a_0 r^{1+s} + b_0 r^{1-s} + \sum_{0 \neq \mu \in L^\vee} a_\mu r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle} \tag{2.24}$$

where $L^\vee$ is the dual lattice with respect to the standard scalar product on $\mathbb{C}$ and K_s is the modified Bessel function.

Proof. The function $f(z+jr)$ is a real-analytic function in z periodic with respect to the lattice L. Therefore it has the Fourier expansion

$$f(z+jr) = \sum_{\mu \in L^\vee} g_\mu(r) e^{2\pi i \langle \mu, z\rangle} \tag{2.25}$$

The Laplace equation in this form is

$$\left(r^2\partial_r^2 - r\partial_r + \lambda - 4|\mu|^2 r^2\right) g_\mu(r) = 0 \tag{2.26}$$

If $\mu = 0, s \neq 0$ the functions $g_\mu(r) = r^{1+s}, r^{1-s}$. This gives the first two terms on the right-hand side.
If $\mu \neq 0$ we have to discuss the Bessel's equation. The functions

$$y = u^\alpha K_\nu(\beta x), \quad y = u^\alpha I_\nu(\beta u) \tag{2.27}$$

solve the equation

$$x^2\partial_x^2 y + (1-2\alpha)x\partial_x y + (\beta^2 u^2 + \alpha^2 - \nu^2)y = 0 \tag{2.28}$$

where K_ν and I_ν are the modified Bessel functions

$$\begin{aligned} I_\nu(x) &= \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu} \\ K_\nu(x) &= \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\alpha\pi)}. \end{aligned} \tag{2.29}$$

For our equation we need to chose $\alpha = 1, \beta = 2\pi i|\mu|, \nu = s$. Hence we arrive at the solution

$$g_\mu(r) = a_\mu r K_s(2\pi|\mu|r) + b_\mu r I_s(2\pi|\mu|r) \tag{2.30}$$

with some constants a_μ, b_μ. Since the Bessel function I_ν increases exponentially and the function K_ν decreases exponentially as $x \to \infty$, the growth condition on f forces $b_\mu = 0$. □

This defines the expansion for the cusp ∞. For a cusp $\zeta = A^{-1}\infty$ the expansion of $f(s)$ at ζ is the expansion of $f(A^{-1}s)$ at ∞.
When the coefficients a_0, b_0 are 0 for all cusps, we say that f is a cusp form. For example if f is square integrable and an eigenfunction with eigenvalue $\lambda \leq 1$ then σ is purely imaginary and f has to be a cusp form. Now we will describe the expansion of the Eisenstein series.

Theorem 2.4.2. *Let $\zeta = A^{-1}\infty$ and $\eta = B^{-1}\infty$ be 2 cusps of Γ. Let L be the lattice corresponding to $(B\Gamma B^{-1})'_\infty$. Then for $Res > \sigma$ the Eisenstein series has the expansion*

$$\begin{aligned} E_A(B^{-1}P, s) =& \delta_{\eta,\zeta}[\Gamma_\zeta : \Gamma'_\zeta]|d_0|^{-2-2s} r^{1+s} \\ &+ \frac{\pi}{|P|s}\left(\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in R} |c|^{-2-2s}\right) r^{1-s} \\ &+ \frac{2\pi^{1+s}}{|P|\Gamma(1+a)} \sum_{0 \neq \mu \in L^\vee} |\mu|^s \\ &\cdot \left(\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in R} \frac{e^{2\pi i \langle \mu, \frac{d}{c}\rangle}}{|c|^{2+2s}}\right) r K_s(2\pi|\mu|r) e^{2\pi i\langle\mu,z\rangle} \end{aligned} \tag{2.31}$$

where the following notation is used. R denotes a system of representatives $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ of the double cosets in

$$A\Gamma'_\zeta A^{-1} \backslash A\Gamma B^{-1} / B\Gamma'_\eta B^{-1} \tag{2.32}$$

with $c \neq 0$. P is a fundamental parallelogram fro L with area $|P|$. If ζ and η are equivalent, let G be such that $G\eta = \zeta$, then d_0 is defined by

$$AGB^{-1} = \begin{pmatrix} \cdot & \cdot \\ 0 & d_0 \end{pmatrix}. \tag{2.33}$$

Proof. Theorem 4.1 in Chapter 3.4 in (EGM98). □

2.5 Meromorphic extension of the Eisenstein series

Let Γ be a non-cocompact but cofinite discrete subgroup. The Eisenstein series $E_\zeta(P,s)$ defines a holomorphic function on some half-plane $(Re)s > \sigma$. We shall show in this section that the series can be meromorphically continued to all of $\mathbb{C}$. We shall also show a functional equation it satisfies.
Choose a representative set of Γ-classes of cusps $\zeta_1, \zeta_2, \cdots, \zeta_n$ and matrices $B_1, B_2, \cdots, B_n \in \mathrm{SL}(2,\mathbb{C})$ such that $\zeta_i = B_i^{-1}\infty$.
Normalize the Eisenstein series as

$$E_\nu(P,s) = \frac{1}{[\Gamma_{\zeta_\nu} : \Gamma'_{\zeta_\nu}]} E_{B_\nu}(P,s). \tag{2.34}$$

From the Fourier expansion we can see that

$$E_\nu(B_\nu^{-1}P,s) = r^{1+s} + \phi_{\nu\nu}(s) r^{1-s} + \cdots \tag{2.35}$$

and in the case $\nu \neq \mu$

$$E_\nu(B_\mu^{-1}P,s) = \phi_{\nu\mu}(s) r^{1-s} + \cdots \tag{2.36}$$

Now we describe the functional equation satisfied by the series.

Definition 1. *Let Γ be a cofinite group. We define*

$$\mathcal{E}(P,s) = \begin{pmatrix} E_1(P,s) \\ \cdots \\ E_n(P,s) \end{pmatrix} \tag{2.37}$$

$$\Phi(s) = (\phi_{\nu\mu}(s)) \tag{2.38}$$

This is called the scattering matrix for Γ. In our case we will usually only have one class of cusps, so the scattering matrix only has one element, which is a holomorphic function $\Phi(s)$.

Theorem 2.5.1. *The form of the scattering matrix (which has only one component) for the group $\mathrm{SL}(2,\mathbb{Z}[i])$ has the form*

$$\Phi(s) = \pi \frac{\zeta_{\mathbb{Q}[i]}(s)}{s\zeta_{\mathbb{Q}[i]}(1+s)}. \tag{2.39}$$

Proof. See Theorem 3.10 Chapter 8 of (EGM98). □

Theorem 2.5.2. *Let* Γ *be a cofinite non-cocompact gruop. Then both* $\Phi(s)$ *and* $\mathcal{E}(P,s)$ *have meromorphic continuations to all of* $\mathbb{C}$ *in the following sense: There is a holomorphic function* $g : \mathbb{C} \to \mathbb{C}$ *with* $g \neq 0$ *such that for every* ν *the product* $g(s)E_\nu(P,s)$ *can be continued to a function on* $\mathbb{H}^3 \times \mathbb{C}$ *which is real analytic in* P *and holomorphic in* s*. The continued functions satisfy*

$$\mathcal{E}(P,-s) = \Phi(s)\mathcal{E}(P,s), \quad \Phi(s)\Phi(-s) = I. \tag{2.40}$$

The continued components of $\mathcal{E}(P,s)$ *satisfy*

$$(-\Delta - (1-s^2))E_\nu(P,s) = 0 \tag{2.41}$$

if s *is not a pole of* $E_\nu(P,s)$.

Proof. Theorem 1.2 Chapter 6.1 in (EGM98). □

2.6 Selberg zeta function

This section is not essential for our use, but is included for completeness. We will be interested in the spectrum of the Laplace operator. The Selberg zeta function gives a connection between the spectrum and the hyperbolic and loxodromic elements of Γ.
Suppose that Γ is a cofinite discrete group with fundamental domain F. Then the set of eigenvalues of $-\Delta$ is discrete $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$.
If $\lambda_n < 1$ we denote $s_n = \sqrt{1-\lambda_n}$ and if $\lambda \geq 1$ we denote $s_n = i\sqrt{\lambda_n - 1}$.
If $T \in \Gamma$ is a hyperbolic or loxodromic element, let $C(T)$ be its centralizer in Γ and $E(T)$ be the subgroup of elements of finite order which is cyclic. Every hyperbolic or loxodromic element if a power of some T_0 of minimal norm. Denote $m(T) = |E(T)|$.

Definition 2. *For Res* > 1, *the Selberg zeta function for* Γ *is defined by*

$$Z_\Gamma(s) = \prod_{\{T_0\}\in R} \prod_{\substack{k,l\leq 0,\\ k=l \ mod\ m(T_0)}} (1 - a(T_0)^{-2k}\overline{a(T_0)}^{-2l} N(T_0)^{-s-1}) \tag{2.42}$$

where R *is a maximal system of conjugacy classes of primitive hyperbolic and loxodromic elements with different centralizers.*

Theorem 2.6.1. *Then the Selberg zeta function is an entire function of* s*. The zeros of* $Z(s)$ *are the numbers* $\pm s_n, n \leq 0$*. If* $\lambda_n \neq 1$*, both* $s_n, -s_n$ *are zeros whose multiplicities are equal to the multiplicity of the eigenvalue of* λ_n*. If* $\lambda_n = 1$ *then* s_n *is a zero with multiplicity twice of* λ_n*. Furthermore it satisfies the function equation*

$$Z(-s) = \exp(-\frac{v(F)}{3\pi}s^3 + 2Es)Z(s) \tag{2.43}$$

where $v(F)$ *is the area of the fundamental domain of* Γ *and* E *is some constant.*

2.7 The spectrum of the Laplace operator

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a discrete subgroup. We will now describe the spectrum of the adjoint of the Laplace operator $\tilde{\Delta}$ on $L^2(\Gamma\backslash\mathbb{H}^3)$. If Γ is cocompact, then the spectrum is discrete and the eigenvalues are real numbers $\lambda \geq 0$. If Γ is only cofinite the spectrum has a discrete and a continuous part. The orthogonal complement of the space of eigenfunction of Δ in $L^2(\Gamma\backslash\mathbb{H}^3)$ is called the continuous part. This space is generated by elements called eigenpackets.

Definition 3. *An eigenpacket of an operator L is a map $\nu : \mathbb{R} \to D_L$, $\lambda \mapsto \nu_\lambda$ with the following properties:*

1. $\nu_0 = 0$
2. *ν is continuous in the norm sense*
3. $L\nu_\lambda = \lambda\nu_\lambda - \int_0^\lambda \nu_\lambda d\mu$

The eigenpackets act as a basis for the continuous part of the Hilbert space. Together with the eigenfunctions they form a basis of $L^2(\Gamma\backslash\mathbb{H}^3)$.

Theorem 2.7.1. *Let Γ be a cofinite group. Then there are eigenfunctions e_m and eigenpackets ν_n of the essentially self-adjoint operator $-\Delta$ which form an orthogonal basis of $L^2(\Gamma\backslash\mathbb{H}^3)$.*

In the case of $L^2(\Gamma\backslash\mathbb{H}^3)$ the eigenpackets can be described as the analytically continued Eisenstein series $E_\nu(\cdot, it)$. Therefore every automorphic function has the following expansion:

Theorem 2.7.2. *Let Γ be a cofinite group and e_m be an orthogonal basis of eigenvectors of Δ. Then every $f \in \tilde{D}$ with $f(\cdot)\overline{E_\mu(\cdot, it)}$ absolutely integrable has an L^2 convergent expansion*

$$f = \sum_m \langle f, e_m\rangle e_m + \frac{1}{4\pi} \sum_{\mu=1}^{n} \frac{[\Gamma_\mu : \Gamma'_\mu]}{\Lambda_\mu}$$
$$\cdot \int_{-\infty}^{\infty} \langle f, E_\mu(\cdot, it)\rangle E_\mu(\cdot, it) dt$$

Proof. See Chapter 6.3 Theorem 3.4 of (EGM98). □

The basis described here is used in the proof of the Selberg trace formula introduced in the next chapter.

3. Selberg trace formula

The Selberg trace formula gives an identity between spectral and geometric data on the Picard variety $\Gamma\backslash\mathbb{H}^3$. On one side we have a test function evaluated on the discrete and continuous spectrum, and on the other side we have a sum of terms regarding different classes of matrices in Γ. The term we want to approximate is the sum of the hyperbolic and loxodromic matrices and the most dominant term will be the sum of the values of the test function on the discrete spectrum. By choosing the test function correctly we can make the term corresponding to the eigenvalue $\lambda = 0$ the main term and make the contributions from the other term small enough to non be important in the final result.
Let $k : [1, \infty) \to \mathbb{C}$ be a function with suitable growth hypothesis. We define the function $K(P, Q) = k(\delta(P, Q))$. We obtain the integral operator

$$\tilde{K}f(P) = \int_{\mathbb{H}}^{3} K(P, Q)f(Q)dv(Q) \tag{3.1}$$

If f is an eigenfunction of the Laplace operator, so $-\Delta f = \lambda f$ then

$$\tilde{K}f(P) = \int_{\mathbb{H}}^{3} K(P, Q)f(Q)dv(Q) = h(P)f(P) \tag{3.2}$$

where h is the Selberg transform of k given by

$$h(1 - s^2) = \frac{\pi}{s} \int_1^\infty k\Big(\frac{1}{2}\Big(u + \frac{1}{u}\Big)\Big)(u^s - u^{-s})(u - \frac{1}{u})\frac{du}{u}. \tag{3.3}$$

See (EGM98) Chapter 3.5 equations 5.2 and 5.3. For Γ cofinite we define

$$K_\Gamma(P, Q) = \sum_{\gamma \in \Gamma} K(P, \gamma Q) \tag{3.4}$$

Now the operator $\tilde{K}_\Gamma$ can be defined as

$$\tilde{K}_\Gamma f(P) = \int_F K_\Gamma(P, Q)f(Q)dv(Q) \tag{3.5}$$

for Γ-invariant functions. If e_m is a basis for the space L^2_{disc} with eigenvalue λ_m, then

$$\langle K_\Gamma(P, \cdot), e_m\rangle = h(\lambda_m)\overline{e_m(P)}. \tag{3.6}$$

Using the spectral decomposition of the Hilbert space we can write the operator as

$$\begin{aligned} K_\Gamma(P, Q) &= \sum_{m \in D} h(\lambda_m)e_m(Q)\overline{e_m(P)} \\ &+ \frac{1}{4\pi} \sum_{\nu=1}^{n} \frac{[\Gamma : \Gamma'_\nu]}{\Lambda_\nu} \int_{-\infty}^{\infty} h(1 + t^2)E_\nu(Q, it)\overline{E_\nu(P, it)}dt. \end{aligned} \tag{3.7}$$

In the proof of the main theorem we will use the Selberg trace formula. We refer to (EGM98) chapter 6.5 as a reference. The trace formula is obtained by computing the trace of $\tilde{K}_\Gamma$ in two different ways. One way is by integrating the right-hand side of Equation 3.7 over F. The second way is by writing $K_\Gamma(P)$ as a sum of different elements

$$K_\Gamma = K_\Gamma^{\text{id}} + K_\Gamma^{\text{par}} + K_\Gamma^{\text{ce}} + K_\Gamma^{\text{nce}} + K_\Gamma^{\text{loc}} \tag{3.8}$$

where the summands are defined similarly to K_Γ with summation extended over the identity, all parabolic conjugacy classes, the classes of elements of $\gamma_\zeta \backslash \Gamma'_\zeta$, elliptic classes not stabilizing a cusp, and all hyperbolic and loxodromic classes respectively. After integrating all of the kernels is computed, we get the trace formula.

Theorem 3.0.1. *Let $\Gamma < PSL(2,\mathbb{C})$ be a cofinite group. Let h be a function holomorphic in a strip of width strictly greater than 2 around the real axis satisfying the growth condition $h(1+z^2) = O((1+|z|^2)^{-3-\epsilon})$ for $|z| \to \infty$ uniformly in the strip. Let g be the cosine transform*

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)cos(xt)dt. \tag{3.9}$$

For all cusps $\zeta_i, 1 \leq i \leq n$ there is a number $\ell_i \in \mathbb{N}$ and constants $c, \tilde{c}$, $d, d(i,j), \alpha(i,j)$ such that the following identity holds with all sums being absolutely convergent:

$$\begin{aligned}
\sum_m h(\lambda_m) &= \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{-\infty}^{\infty} h(1+t^2)t^2 dt + \\
&+ \sum_{\{R\}nce} \frac{\pi g(0)\log N(T_0)}{|E(R)|\sin^2(\frac{\pi k}{m(R)})} + \sum_{\{T\}lox} \frac{4\pi g(\log N(T))\log N(T_0)}{|E(R)||a(T)-a(T)^{-1}|} \\
&+ cg(0) + \tilde{c}h(1) - \frac{\mathrm{tr}\Phi(0)h(1)}{4} + \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\phi'}{\phi}(it)dt - d\int_{-\infty}^{\infty} h(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{\ell_i} d(i,j)\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i,j)}dx
\end{aligned} \tag{3.10}$$

The first sum on the second line is over the conjugacy classes of elliptic elements not stabilizing a cusp. The second is over the classes of hyperbolic and loxodromic elements.

Note 1. *The term*

$$\sum_{\{T\}lox} \frac{4\pi g(\log N(T))\log N(T_0)}{|E(R)||a(T)-a(T)^{-1}|} \tag{3.11}$$

in the Selberg trace formula has the factor 4π in the numerator. This appears to be a mistake in (EGM98) as it would imply an incorrect growth of the ψ_g function. This was also noted in (BCC+ 18). Therefore we will ignore the 4π factor henceforth.

3.1 The case of the groups $\mathrm{SL}(2,\mathcal{O})$

In this section we will focus on the groups $\mathrm{SL}(2,\mathcal{O}) \subset \mathrm{SL}(2,\mathbb{C})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic number field K, especially $\mathrm{SL}(2,\mathbb{Z}[i])$. We can see that the groups are discrete, but they are not cocompact since the stabilizer of ∞ contains the abelian group

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathcal{O} \right\}. \tag{3.12}$$

In fact the cusps are exactly the points $\mathbb{P}^1 K$. The equivalence classes of cusps correspond to classes of fractional ideals in K.

Theorem 3.1.1. *The map $f : \mathbb{P}^1 K \to Cl_K$ to the group of fractional ideals modulo principal ideals of K defined by*

$$f([a, b]) = (a, b) = a\mathcal{O} + b\mathcal{O} \tag{3.13}$$

gives a bijection $\mathbb{P}^1 K/\mathrm{SL}(2, \mathbb{Z}[i]) \to Cl(K)$ of the classes of cusps to the ideal class group.

Proof. We will prove the theorem only in the case when K has class number 1. Since ever ideal in a number field is generated by 2 elements the map is surjective. Every ideal is principal so we should be able to find a matrix which transforms any $[a, b]$ to the cusp at infinity $[1, 0]$.We can choose a, b to be in $\mathcal{O}$ and coprime. We have $(a, b) = (1)$ so $xa + yb = 1$ for some $x, y \in \mathcal{O}$. Therefore the matrix

$$M = \begin{pmatrix} a & -y \\ b & x \end{pmatrix} \tag{3.14}$$

has determinant 1 and maps $[1, 0]$ to $[a, b]$. Its inverse is the wanted matrix. □

We will now describe the fundamental domain of the group $\mathrm{SL}(2, \mathbb{Z}[i])$.

Theorem 3.1.2. *The fundamental domain F of* $\mathrm{SL}(2, \mathbb{Z}[i])$ *is described as*

$$\begin{aligned} B &= \{z + rj \in \mathbb{H}^3 | \quad |z|^2 + r^2 \geq 1\} \\ P &= \{z \in \mathbb{C} | \quad 0 \leq |\mathrm{Re} z| \leq 1/2, \quad 0 \leq \mathrm{Im} z \leq 1/2\} \\ F &= \{z + rj \in B | \quad z \in P\} \end{aligned} \tag{3.15}$$

This also shows that the group is cofinite.

Proof. The region P is the fundamental domain of the action of $\mathrm{SL}(2, \mathbb{Z}[i])_\infty$ which is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$. Define the region

$$C = \{z + jr \in \mathbb{H}^3 | \quad |cz + d|^2 + |cr|^2 \geq 1 \forall c, d \in \mathcal{O}\} \tag{3.16}$$

We can see that $P = z + jr \in C$ iff for all $\sigma \in \mathrm{SL}(2, \mathbb{Z}[i])$ and $\sigma P = z' + jr'$ we have $r' \leq r$. Also for every $P \in \mathbb{H}^3$ there is an matrix $\sigma \in \mathrm{SL}(2, \mathbb{Z}[i])$ such that $\sigma P \in C$ since there is only a finite number of numbers c, d such that $|cz + d|^2 + |cr|^2 < k$ for any constant k so we can choose a matrix with minimal values.
For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, if $c = 0$ we have $MC = C$ and otherwise $MC \cap C = S(c, d)$ where $S(c, d)$ is the halfsphere centered at $\frac{-d}{c}$ with radius $\frac{1}{|c|}$.
The region C can be also described as

$$C = \{z + jr \in \mathbb{H}^3 | \quad |z + d|^2 + |r|^2 \geq 1 \forall d \in \mathcal{O}\} \tag{3.17}$$

This is because C is $\mathbb{H}^3$ minus the union of half-spheres of radius $\frac{1}{|c|}$ with center at $\frac{-d}{c}$. It is enough to do the intersection of just the spheres of radius 1 since the lowest r can be on this region is $\frac{1}{\sqrt{2}}$ which is more than $\frac{1}{|c|}$ for $|c| > 1$. The

intersection of C and P is therefore the fundamental region F.
We infer that the intersection $F = P \cap C$ satisfies $F^o \cap \sigma F^o = \emptyset$ for all nontrivial $\sigma \in \mathrm{SL}(2, \mathbb{Z}[i])$ since the intersection $F \cap \sigma F$ is a subset of $S(c, d)$ for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ if $c \neq 0$ or a vertical plane otherwise. The sets C and F are intersections of closed sets so they are closed. From the properties of the set C we have

$$\mathbb{H}^3 = \bigcup_{\sigma \in \mathrm{SL}(2,\mathbb{Z}[i])} \sigma F \tag{3.18}$$

The boundary of F is included in an intersection of vertical planes and sets of the type $S(c, d)$ and therefore has Lebesgue measure 0.

□

3.2 The Selberg trace formula for $\mathrm{SL}(2, \mathbb{Z}[i])$

We will first calculate the exact form of the scattering function in the case of $\mathrm{SL}(2, \mathbb{Z}[i])$.

Theorem 3.2.1. *The scattering function $\Phi(s)$ approaches the value -1 as $s \to 0$ for the group $\mathrm{SL}(2, \mathbb{Z}[i])$.*

Proof. For the group $\mathrm{SL}(2, \mathbb{Z}[i])$ the function Φ has the form

$$\Phi(s) = \pi \frac{\zeta_K(s)}{s\zeta_K(s+1)}. \tag{3.19}$$

We can compute limit $s \to 0$ of the denominator using the Class number formula. The Dedekind zeta function of the number field has a simple pole at 1 with residue

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}h}{w\sqrt{|\Delta|}} \tag{3.20}$$

where r_1, r_2 are the number of real and complex embeddings, Reg is the regulator of the number field, h is the class number, w is the number of roots of unity in K and Δ is the discriminant. In our case $\mathrm{Res}_{s=0}\zeta_{\mathbb{Q}[i]}(s) = \frac{\pi}{4}$. The numerator can then be computed via the functional equation of the zeta function. In the case of $K = \mathbb{Q}[i]$ the functional equation has the form

$$\pi^{-s}\Gamma(s)\zeta_K(s) = \pi^{s-1}\Gamma(1-s)\zeta_K(1-s) \tag{3.21}$$

from this we get that $\zeta_{\mathbb{Q}[i]}(0) = -\frac{1}{4}$. Therefore

$$\Phi(0) = \pi \frac{-1/4}{\pi/4} = -1. \tag{3.22}$$

□

We can write the exact form the Selberg trace formula will have for $\mathrm{SL}(2, \mathbb{Z}[i])$.

Theorem 3.2.2. *The formula satisfied in Theorem 3.0.1 has the form:*

$$\begin{aligned}
\sum_m h(\lambda_m) &= \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{-\infty}^{\infty} h(1+t^2)t^2 dt + \\
&+ \sum_{\{R\}nce} \frac{\pi g(0)\log N(T_0)}{|E(R)|\sin^2(\frac{\pi k}{m(R)})} + \sum_{\{T\}lox} \frac{4\pi g(\log N(T))\log N(T_0)}{|E(R)||a(T)-a(T)^{-1}|} \\
&+ cg(0) + \frac{h(1)}{2} \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\phi'}{\phi}(it)dt - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \int_0^{\infty} g(x)\frac{\sinh x}{\cosh x+1}dx
\end{aligned} \tag{3.23}$$

Note that we won't need the exact forms of the constants c and vol(Γ).

Proof. The number field $\mathbb{Q}(i)$ has class number one, so there is only 1 class of cusps ($n = 1$). The constants $\tilde{c}$ and d are obtained from Proposition 5.3 Chapter 6 in (EGM98). From the same proposition we can see that the constants $\alpha(i, j)$ and $d(i, j)$ depend on the conjugacy classes of elliptic elements that fix the cusp at infinity. There are 4 such classes generated by:

$$\begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} -i & -i+1 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{3.24}$$

For proof see (YC00). From this we can calculate the constants $\alpha(1, j)$ and $d(1, j)$. We have $d(1, j) = \frac{1}{(1-i^2)} = \frac{1}{4}$ and $\alpha(1, j) = 1 - i^2 = 2$ for all $j = 1, \cdots, 4$. The constant $\mathrm{tr}\Phi(0)$ was calculated in the previous theorem.

□

3.3 Prime counting function

The prime counting function $\pi_\Gamma(X)$ is defined as the number of primitive hyperbolic and loxodromic conjugacy classes of Γ with norm $\leq X$. We can expect the function to have similar behaviour as the counting function for primes $\pi_\Gamma(X) \sim \mathrm{li}(X)$. We also define an auxiliary function

$$\psi_\Gamma(X) = \sum_{\{P\}lox, N(P)\leq X} \log(NP_0) \tag{3.25}$$

where $P = P_0^n$ for some primitive P_0. This function will be the main focus of our study. It can be expected to grow as $\psi_\Gamma(X) \sim \frac{X^2}{2}$. Its behaviour can be estimated by using the Selberg trace formula.
One of the terms in the Selberg trace formula is

$$\sum_{\{T\}lox} \frac{g(\log N(T))\log N(T_0)}{|E(T)||a(T)-a(T)^{-1}|^2}. \tag{3.26}$$

By choosing a suitable g, we can get a function similar to ψ_Γ.
Let X be a constant and $\rho = \log X$. We will choose

$$g(t) = g_\rho(t) = \mathbb{I}_{(0,\rho)}(|t|) \tag{3.27}$$

where $\mathbb{I}$ is the interval indicator function. We will define the counting function for g:

$$\psi_g(X) = \sum_{\{T\} lox, N(T) \le X} \frac{g(\log N(T)) \log N(T_0)}{|E(T)||a(T) - a(T)^{-1}|^2} \tag{3.28}$$

Plugging in our choice of g we get:

$$\psi_{g_\rho}(X) = \psi_g(X) = \sum_{\{T\} lox, N(T) \le X} \frac{\log N(T_0)}{|E(T)||a(T) - a(T)^{-1}|^2} \tag{3.29}$$

This looks similar to our definition of ψ_Γ. We will use this ψ_g as our counting function instead of ψ_Γ since it is more convenient to use with the Selberg trace formula.

Note that the term $E(T)$ is different from 1 in only finitely many cases. This is because there are only finitely many elliptic conjugacy classes fixing the cusp and an elliptic element is in the centralizer of T iff they have the same fixed points. It is thus not important for our discussion and can be omitted.

This counting function $\psi_g(X)$ is expected to grow as X. We would like to estimate the weighted difference between it and the main term X. A simple way to do this would be to consider this integral

$$\frac{1}{f(V)} \int_1^V (\psi_g(X) - X) \frac{dX}{X} \tag{3.30}$$

for some normalizing function $f(X)$. But we will need to modify this slightly to make it work in our case. Writing $X = e^\rho$ and $V = e^T$ this transforms to

$$\frac{1}{f(e^T)} \int_0^T (\psi_g(e^\rho) - e^\rho) d\rho. \tag{3.31}$$

Now we can introduce a smoothing factor that will help us with convergence in the next steps. Let ω be a nonnegative smooth function with compact support on $(0, 1)$ and mass 1. Then let

$$\omega_T(x) = \omega(\frac{x}{T}). \tag{3.32}$$

We will consider the modified integral

$$\frac{1}{f(e^T)} \int_0^T \omega_T(\rho_)(\psi_g(e^\rho) - e^\rho) d\rho. \tag{3.33}$$

This shouldn't be much different from the original and is similar to the modification done in (Phi95) in the two dimensional case. In this version we will use with the normalizing factor $\frac{1}{T^2}$.

The Fourier transform of $g(t)$ is the function $h(1 + t^2)$.

$$h(1 + t^2) = \int_{-\infty}^{\infty} g(s) e^{ist} ds = 2 \int_0^\rho \cos(st) ds = \frac{2\sin(t\rho)}{t} = \frac{X^{it}}{it} + \frac{X^{-it}}{it} \tag{3.34}$$

The function $g(s)$ by itself is not enough since $h(1+t^2)$ doesn't satisfy the growth condition of the Selberg trace formula. Therefore we must modify $g(s)$ to make $h(1+t^2)$ decrease quicker and find some inequality for the modified g and the old g to be able to get an estimate for ψ_g. We will smooth out g by convoluting it with some function with compact support.
Let $q(x)$ be an even, smooth, non-negative function with support in the interval $[-1,1]$ and with integral 1 over the interval. For $\delta > 0$ we define

$$q_\delta(x) = \frac{1}{\delta} q(\frac{x}{\delta}) \tag{3.35}$$

If we take the convolution of g with q_δ, the function $h(1+t^2)$ is multiplied by the Fourier transform of q_δ. We shall choose

$$\delta = \frac{1}{e^T} \tag{3.36}$$

where T is the averaging factor for E_Γ. This value will make everything work for us in the future.
We will find out that the sum $\sum_m h(\lambda_m)$ depends only on the eigenvalues $\lambda_m \leq 1$. We need an estimate for the Fourier transform of q_δ to be used in the future.

Lemma 3.3.1. *For $0 < \delta < \frac{1}{4}$ and any integer k the Fourier transform of the function q_δ for all t with $|\mathrm{Im}(t)| \leq M$ for some M satisfies:*

$$|\hat{q}_\delta(t)| < \frac{c}{1 + |\delta t|^k}$$

for some constant c depending on k.

Proof. We use integration by parts. If $t \neq 0$

$$|\hat{q}_\delta(t)| = \left| \int_{\mathbb{R}} q(x) e^{-itx\delta} dx \right| = \left| \frac{1}{(it\delta)^k} \int_{\mathbb{R}} e^{-itx\delta} \partial_x^k q(x) dx \right| < \frac{c}{|t\delta|^k}. \tag{3.38}$$

Also $|\hat{q}_\delta(t)| < c'$, so we combine it to the wanted expression. □

We will define two functions g_+, g_- that we will use to estimate g.

$$g_\pm(x) = g_{\rho\pm\delta} * q_\delta(x) = \int_{-\infty}^{\infty} g_{\rho\pm\delta}(x-y) q_\delta(y) dy. \tag{3.39}$$

The related functions $h_\pm(1+t^2)$ are defined as the Fourier transforms of $g_\pm$. There is a relation between $g_\pm$ and g for $x \geq 0$

$$g_\pm(x) = \int_{\mathbb{R}} g_{\rho\pm\delta}(x-y) q_\delta(y) dy = \mathbb{I}_{[0,\rho\pm\delta+\delta]}(x) \int_Q q_\delta(y) dy \tag{3.40}$$

where $Q = [\max(-\delta, x - (\rho \pm \delta)), \delta]$. We can see that for $g_+(x)$ with $|x| \leq \rho$ we have $g_+ = g_\rho$ and $g_+ \leq g_\rho$ otherwise.

For $x > \delta$ we have

$$g_\pm(x) = \mathbb{I}_{[0,\rho\pm\delta+\delta]}(x) \int_Q q_\delta(y) dy \leq \mathbb{I}_{[0,\rho\pm\delta+\delta]}(x) \tag{3.41}$$

Therefore we get

$$g_- \le g_\rho. \tag{3.42}$$

We have the following estimates:

$$g_-(x) \le g_\rho(x) \le g_+(x). \tag{3.43}$$

We can use this to obtain an estimate for g_δ using estimates for $g_\pm$.

$$\psi_-(X) \le \psi_g(X) \le \psi_+(X) \tag{3.44}$$

where $\psi_\pm$ is the counting function ψ_g with $g_\pm$ instead of g. We will use these test functions and the corresponding $h_\pm$ in the Selberg trace formula.

Plugging this in $h_\pm$ the sum on the left-hand side of (3.0.1) will get us the main term.

$$\sum_{\lambda_i} h_\pm(\lambda_i) \tag{3.45}$$

For our Picard group there are no small eigenvalues with $\lambda \le 1$ with the exception of 0.

Theorem 3.3.2. *For the Picard group* $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$ *there are no eigenvalues* λ *in the interval* $(0, 1]$.

Proof. The modular group has only one cusp. Let f be a square integrable automorphic function with eigenvalue λ and $\int_F |f(z)|^2 dv = 1$. The automorphic function has the expansion

$$f(z) = \sum_{m,n\in\mathbb{Z}} a_{m,n}(r) e^{2\pi i(mx+ny)}. \tag{3.46}$$

The Stokes' theorem similarly like in Theorem 2.2.1 has the form

$$\begin{aligned}&\int_F \frac{1}{r}\left(\partial_x^2 f + \partial_y^2 f + \partial_r^2 f\right)\overline{f} - \frac{1}{r^2}\partial_r f\overline{f} + \frac{1}{r}\left(\partial_x f\partial_x\overline{f} + \partial_y f\partial_y\overline{f} + \partial_r f\partial_r\overline{f}\right)dxdydz\\ &= \int_F \partial_x(\frac{1}{r}\partial_x f \cdot \overline{f}) + \partial_y(\frac{1}{r}\partial_y f \cdot \overline{f}) + \partial_r(\frac{1}{r}\partial_r f \cdot \overline{f})dxdydz = 0\end{aligned} \tag{3.47}$$

Therefore

$$\lambda = \int_F -\Delta f \cdot \overline{f} dv = \int_F r^2\left(|\partial_x f|^2 + |\partial_y f|^2 + |\partial_r f|^2\right)dv. \tag{3.48}$$

We define the domain

$$C = \{x + iy + jr| - \frac{1}{2} < x < \frac{1}{2}, 0 < y < \frac{1}{2}, r > \frac{1}{\sqrt{2}}\}. \tag{3.49}$$

Let A, B be the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} i & -1 \\ 0 & -1 \end{pmatrix}. \tag{3.50}$$

Then we have the inclusions $F \subset C \subset F \cup ABF \cup BF \cup BVF \cup A^{-1}BF$.
Therefore

$$
\begin{aligned}
5\lambda &> \int_C r^2 \Big(|\partial_x f|^2 + |\partial_y f|^2 + |\partial_r f|^2 \Big) dv \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \Big(|\partial_x f|^2 + |\partial_y f|^2 + |\partial_r f|^2 \Big) \frac{drdydx}{r} \\
&> \sum_{m,n\in\mathbb{Z}} 2\pi^2(m^2+n^2) \int_{\frac{1}{\sqrt{2}}}^{\infty} |a_{m,n}(r)|^2 \frac{dr}{r} \qquad (3.51) \\
&\pi^2 \sum_{m,n\in\mathbb{Z}} \int_{\frac{1}{\sqrt{2}}}^{\infty} |a_{m,n}(r)|^2 \frac{dr}{r^3} \\
&\geq 2\pi^2 \int_C |f|^2 dv \\
&> 2\pi^2
\end{aligned}
$$

Therefore $\lambda > \frac{2}{5}\pi^2$ and in particular $\lambda > 1$. □

Therefore the most dominant term of the sum will be for $\lambda_j = 0$

$$h_{\pm}(0) = \hat{q}_\delta(i)\frac{2i\sinh(\rho_\pm)}{i} = \hat{q}_\delta(i)(e^{\rho_\pm} + e^{-\rho_\pm}) \qquad (3.52)$$

with $\rho_\pm = \rho \pm \delta$.
The value of $\hat{q_\delta}(i)$ can be calculated

$$\hat{q_\delta}(i) = \int_{\mathbb{R}} q_\delta(t)e^t dt = \int_{\mathbb{R}} q(t)e^{\delta t}dt = \int_{\mathbb{R}} q(t)\Big(1 + O(\delta)\Big)dt = 1 + O(\delta) \qquad (3.53)$$

as $\delta \to 0$.
Therefore we see that the main term is indeed $X = e^\rho$.
The other terms for real t will be of the form

$$\hat{q}_\delta(s_j)\frac{2\sin(s_j\rho)}{s_j} \qquad (3.54)$$

with $\lambda_j = 1 + s_j^2$.
Therefore we have

$$\sum_{\lambda_j} h_\pm(\lambda_j) = e^\rho + O(1) \qquad (3.55)$$

in the limit as $\delta \to 0$. We would like to estimate

$$\frac{1}{T^2}\int_0^T \omega_T(\rho)(e^\rho - \sum_{\lambda_j} h_\pm(\lambda_j))d\rho. \qquad (3.56)$$

Exchanging the sum with the integral we can look at individual terms

$$\frac{1}{T^2}\sum_{\lambda_j>0} \hat{q}_\delta(s_j)\int_0^T \omega_T(\rho)\frac{2\sin(\rho_\pm s_j)}{s_j}d\rho \qquad (3.57)$$

with $\rho_\pm = \rho \pm \delta$.
The reason we introduced the smoothed integral by adding ω_T is to use integration

by parts multiple times to make the sum converge. Since ω_T has a compact support we have

$$\int_0^T \omega_T(\rho)\frac{2\sin(\rho_\pm s_j)}{s_j}d\rho = \pm\int_0^T \omega_T^{(n)}(\rho)\frac{2\cos^{(n)}(\rho_\pm s_j)}{s_j^{1+n}}d\rho \tag{3.58}$$

where $\cos^{(n)}$ is sine for even n and cosine for odd n and $\omega_T^{(n)}$ is the n-th derivate of ω_T. This can be bounded

$$\left|\int_0^T \omega_T^{(n)}(\rho)\frac{2\cos^{(n)}(\rho_\pm s_j)}{s_j^{1+n}}d\rho\right| \leq \frac{2}{s_j^{1+n}T^{n-1}}\left|\int_0^1 \partial^n\omega(\rho)d\rho\right|. \tag{3.59}$$

Now we need to use the Weyl bound on the density of the spectrum.

Theorem 3.3.3. *Let $A(\Gamma, T)$ be the number of eigenvalues λ in the spectrum of the Laplace operator on $\Gamma\backslash\mathbb{H}^3$ with $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$ such that $\lambda \leq T$. Then $A(\Gamma, T)$ behaves asymptotically as*

$$A(\Gamma, T) \sim \frac{\mathrm{vol}(\Gamma)}{6\pi^2}T^{\frac{3}{2}}. \tag{3.60}$$

Proof. Theorem 9.1 Chapter 8 of (EGM98). □

This shows that the number of eigenvalues $\lambda_j = 1 + s_j$ with $s_j \leq T$ grows as T^3. We can use this to prove that the sum converges.

$$\frac{1}{T^2}\sum_{\lambda_j>0}\hat{q}_\delta(s_j)\int_0^T \omega_T(\rho)\frac{2\sin(\rho_\pm s_j)}{s_j}d\rho \tag{3.61}$$

can be bounded as

$$\frac{1}{T^2}\sum_{\lambda_j>0}\frac{C}{s_j^{1+n}T^{n-1}} \tag{3.62}$$

with some constant C. This converges for $n > 2$ and goes to 0 as $T \to \infty$. Therefore we can ignore all but the first term in the sum.

Our main problem will be to estimate the quantity

$$\frac{1}{T^2}\int_0^T \omega_T(\rho)\Big(\psi_g(e^\rho) - e^\rho\Big)d\rho \tag{3.63}$$

in the limit as $T \to \infty$. Now we can use the Selberg trace formula to get the estimate.
The first term in the Selberg trace formula is

$$IE = \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{-\infty}^{\infty} h_\pm(1+t^2)t^2dt \tag{3.64}$$

To make this term easier to calculate, we will calculate its moment directly

$$\frac{1}{T^2}\frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_0^T \omega_T(\rho)\int_{-\infty}^{\infty} h_\pm(1+t^2)t^2dtd\rho \tag{3.65}$$

This integral converges absolutely because of the $\hat{q}_\delta$ and thus we can change the order of integration. The integral over t is even so we can integrate from 0. Then we use integration by parts twice.

$$\begin{aligned}&\frac{\mathrm{vol}(\Gamma)}{T^2 2\pi^2}\int_0^\infty\int_0^T \omega_T(\rho)\hat{q}_\delta(t)\sin(\rho_\pm t)td\rho dt=\\=&\frac{\mathrm{vol}(\Gamma)}{T^2 2\pi^2}\int_0^\infty\int_0^T \omega_T^{(1)}(\rho)\hat{q}_\delta(t)\cos(\rho_\pm t)d\rho dt\\=&\frac{-\mathrm{vol}(\Gamma)}{T^2 2\pi^2}\int_0^\infty\int_0^T \omega_T^{(2)}(\rho)\hat{q}_\delta(t)\frac{\sin(\rho_\pm t)}{t}d\rho dt\end{aligned} \tag{3.66}$$

We switch the order of integration back. We can divide the integral over the real line to interval $[0,1/\rho]$ and the rest. For the rest we bound $\hat{q}_\delta$ by $\frac{1}{1+|\delta t|^k}$ and the rest by a constant. We get an integral of the form

$$\int_{1/\rho}^\infty \frac{dt}{t(1+|\delta t|^k)} = O(\log(\frac{1}{\rho\delta})) = T-\log(\rho) \tag{3.67}$$

since $\delta = e^{-T}$.
For the first interval part we bound $\hat{q}_\delta$ by a constant and $\sin(x)$ by $|x|$. We have

$$\int_0^{1/\rho}\frac{\sin(\rho_\pm t)}{t}dt \le \int_0^{1/\rho}|\rho_\pm|dt = \frac{|\rho\pm\delta|}{\rho}. \tag{3.68}$$

Taking the integral over ρ.

$$\begin{aligned}&\frac{1}{T^2}\int_0^T \omega_T^{(2)}(\rho)\left(T-\log(\rho)+\frac{1\pm\delta}{\rho}\right)d\rho\\=&\frac{1}{T^4}\int_0^T \omega^{(2)}(\rho/T)\left(T-\log(\rho)+\frac{|\rho\pm\delta|}{\rho}\right)d\rho\\=&O\left(\frac{1}{T^3}\int_0^T\omega^{(2)}(\rho/T)d\rho\right)=O\left(\frac{1}{T^2}\int_0^1\omega^{(2)}(\rho)d\rho\right)\end{aligned} \tag{3.69}$$

which goes to 0.

Now we will calculate the first moment of the term

$$\frac{1}{2\pi}\int_{-\infty}^\infty h_\pm(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt. \tag{3.70}$$

The function $\frac{\Gamma'}{\Gamma}(1+it)$, also known as the digamma function $\psi^{(0)}(1+it)$ grows as $|\psi^{(0)}(1+it)| = O(\log t)$ as t goes to ∞. As the integral converges absolutely we can exchange the order of integration to get:

$$\frac{1}{T2\pi}\int_{-\infty}^\infty\int_0^T \omega_T(\rho)\hat{q}_\delta(t)\frac{2\sin(t(\rho\pm\delta))}{t}\frac{\Gamma'}{\Gamma}(1+it)d\rho dt \tag{3.71}$$

We can integrate with respect to ρ by parts and we get this bound

$$\begin{aligned}&\frac{1}{T^2 2\pi}\int_{-\infty}^\infty\int_0^T\left|\omega_T^{(1)}(\rho)\hat{q}_\delta(t)\frac{2\cos(t(\rho\pm\delta))}{t^2}\frac{\Gamma'}{\Gamma}(1+it)\right|d\rho dt\\\le&\frac{1}{2\pi T^2}\int_{-\infty}^\infty\left|\hat{q}_\delta(t)\frac{2}{t^2}\frac{\Gamma'}{\Gamma}(1+it)C\right|dt\end{aligned} \tag{3.72}$$

with $C = ||\omega^{(1)}||$. This integral converges as $\int_\infty^\infty \frac{\log t dt}{t^2}$ converges. Therefore this term doesn't contribute.

For the term

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\phi'}{\phi}(it)dt \tag{3.73}$$

we can use a similar method. The function ϕ has the form

$$\phi(s) = \pi\frac{\zeta_{\mathbb{Q}[i]}(s)}{s\zeta_{\mathbb{Q}[i]}(s+1)} \tag{3.74}$$

so its logarithmic derivative can be written as

$$\frac{\phi'(s)}{\phi(s)} = \frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} - \frac{1}{s} - \frac{\zeta'_{\mathbb{Q}[i]}(1+s)}{\zeta_{\mathbb{Q}[i]}(1+s)} \tag{3.75}$$

Note that the term $\frac{1}{s}$ erases the pole of the term $\frac{\zeta'_{\mathbb{Q}[i]}(1+s)}{\zeta_{\mathbb{Q}[i]}(1+s)}$ since $\zeta_{\mathbb{Q}[i]}(s)$ has a simple zero at $s = 1$.
A asymptotic bound on the logarithmic derivative of the Dedekind zeta function is given by the following theorem:

Theorem 3.3.4. *Let L be a number field and $\zeta_L(s)$ be its zeta function. Then there are positive constants k, k' such that*

$$\frac{\zeta'_L}{\zeta_L}(it) = O((\log t)^k) \tag{3.76}$$

and

$$\frac{\zeta'_L}{\zeta_L}(1+it) = O((\log t)^{k'}) \tag{3.77}$$

as $t \to \infty$.

Proof. See Lemma 9.2 Chapter 8 of (EGM98). □

We exchange the order or integration and integrate by parts. After that the integrand goes as $\frac{(\log t)^k}{t^2}$ as $t \to \infty$. Since the integral of $\frac{(\log t)^k}{t^2}$ converges. Therefore this term doesn't contribute to the final sum.
The term

$$CS_\pm = \int_0^\infty g_\pm(x)\frac{\sinh x}{\cosh x + 1}dx \tag{3.78}$$

for g the integral can be estimated as

$$\begin{aligned} CS = \int_0^\rho \frac{\sinh x}{\cosh x + 1}dx = \log(\cosh x + 1)|_0^\rho = \\ = \log(e^\rho + e^{-\rho} + 1) = \rho + O(1/\rho). \end{aligned} \tag{3.79}$$

If we integrate these terms over ρ we get:

$$\frac{1}{T^2}\int_0^T \omega_T(\rho)\Big(\rho + O(1/\rho)\Big)d\rho \tag{3.80}$$

We can ignore the $O(1/\rho)$ term because it contributes 0 in the limit. We can use integration by parts by choosing a primitive function $\Omega(x)$ of $\omega(x)$ which has the value 0 at $x = 1$. The boundary term then vanishes and we get

$$\frac{1}{T^2}\int_0^T \omega_T(\rho)\rho d\rho = \frac{-1}{T}\int_0^T \Omega(\frac{\rho}{T})d\rho = \kappa \tag{3.81}$$

with $\kappa = -\int_0^1 \Omega(\rho)d\rho$ which is positive. If we chose ω as the indicator function of the interval $(0,1)$ we would get $= \frac{1}{2}$ and by choosing ω close to the indicator function we can get κ close to $\frac{1}{2}$.

To get CS_+ we use the bound $g \leq g_+$ to get $CS \leq CS_+$. By choosing $\rho' = \rho + 2\delta$ we have that $g'_- = g_+ \leq g_{\rho+2\delta}$ so $CS_+ \leq CS + O(\delta)$ which gives that CS_+ also gives κ and similarly CS_-.

The terms

$$\sum_{\{R\}nce} \frac{\pi g_\pm(0)\log N(T_0)}{|E(R)|\sin^2(\frac{\pi k}{m(R)})} \qquad c g_\pm(0) \tag{3.82}$$

are constants independent of X and they don't contribute since

$$\frac{1}{T^2}\int_0^T \omega_T(\rho) k d\rho \to 0 \tag{3.83}$$

as $T \to \infty$ for a constant k.
Finally the term $\frac{h_\pm(1)}{2} = \rho_\pm$ contributes κ since

$$\frac{1}{T^2}\int_0^T \omega_T(\rho)\rho_\pm d\rho \to \kappa \tag{3.84}$$

as $T \to \infty$ (as we have $\delta = \frac{1}{e^T}$).

3.4 Main result

We can put all the computations done in the previous sections together to obtain the main theorem of this thesis.

Theorem 3.4.1. *The weighted average of the difference of the counting function $\psi_g(e^\rho)$ and its approximation e^ρ for the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$ satisfies*

$$\lim_{T\to\infty} \frac{1}{T^2}\int_0^T \omega_T(\rho)\bigg(\psi_g(e^\rho) - e^\rho\bigg)d\rho = 2\kappa. \tag{3.85}$$

Proof. We plug the functions $g_\pm$ into the Selberg trace formula. We note that $h_\pm$ is holomorphic on the entire complex plane and satisfies the growth condition of the trace formula because $\hat{q}_\delta$ satisfies Theorem 3.3.1.
As we have discussed in the equation (3.61) the difference between the main term e^ρ and the left-hand side of the Selberg trace formula gets annihilated by the integral. We can thus use e^ρ insted of the left-hand side.
We can write $\psi_\pm(e^\rho) - e^\rho$ as a sum of terms in the trace formula. We saw that the only terms that contribute are the terms CS and $h(1)/2$ which give the contribution 2κ.
To get the formula for ψ_g we use the bounds in equation (3.44). This gives us the same result for the chose counting function ψ_g. $\square$

4. Extension to other number fields

The machinery of the previous chapter can be used for other number fields. Let us look at the case of K a quadratic imaginary number field with class number 1. The group is $\mathrm{SL}(2,\mathcal{O})$ which is usually called the Bianchi group. We still have only one cusp in the quotient space $\mathbb{H}^3/\mathrm{SL}(2,\mathcal{O})$. The Selberg trace formula has the form:

Theorem 4.0.1. *The formula satisfied in Theorem 3.0.1 has the form:*

$$\begin{aligned}
\sum_m h(\lambda_m) &= \frac{\mathrm{vol}(\Gamma)}{4\pi^2}\int_{-\infty}^{\infty} h(1+t^2)t^2dt+ \\
&+ \sum_{\{R\}nce} \frac{\pi g(0)\log N(T_0)}{|E(R)|\sin^2(\frac{\pi k}{m(R)})} + \sum_{\{T\}lox} \frac{g(\log N(T))\log N(T_0)}{|E(R)||a(T)-a(T)^{-1}|} \\
&+ cg(0) + \frac{h(1)}{2} \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\phi'}{\phi}(it)dt - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \sum_i \frac{1}{4}\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i)}dx
\end{aligned}$$

where the last sum is over the conjugacy classess of elliptic elements that fix the cusp and $\alpha(i)$ depends on the order of the elliptic class.

The proof is similar as for the number field $\mathrm{SL}(2,\mathbb{Z}[i])$. The only term that is different is the term CS. The integral

$$\int_0^{\rho_\pm} \frac{\sinh x}{\cosh x - 1 + \alpha(i)}dx \tag{4.1}$$

will similarly give $\rho_\pm + O(1/\rho_\pm)$.
There can also be a different number of conjugacy classes of matrices fixing the cusp. If we call the number d the coefficient in front of the CS term will be $\frac{d}{4}$.

We will also need to assume that there are no small eigenvalues for the Bianchi group as we otherwise would have more main terms in the approximation.

If we put all of these changes together we will get the following version of the main theorem.

Theorem 4.0.2. *Assuming there are no eigenvalues $\lambda \le 1$ for the Bianchi group $\mathrm{SL}(2,\mathcal{O})$ the weighted average of the difference of the counting function $\psi_\delta(e^\rho)$ and its approximation e^ρ for the Bianchi group $\Gamma = \mathrm{SL}(2,\mathcal{O})$ for the number field K with class number 1 satisfies*

$$\lim_{T\to\infty}\frac{1}{T^2}\int_0^T \omega_T(\rho)\Big(\psi_g(e^\rho) - e^\rho\Big)d\rho = \kappa + \frac{d\kappa}{4}. \tag{4.2}$$

where d is the number of conjugacy classes of elliptic elements fixing the cusp.

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