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Local properties of modules

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Abstract: This thesis introduces module properties of projectivity and flatness relative to classes of finitely presented modules, these being generalization of projectivity and pure projectivity. Then it gives proof of ascent and descent of these properties through non-commutative ring homomorphisms with certain properties, most importantly reflection of pure epimorphisms. For relative case ascent and descent through flat ring homomorphisms, which reflect pure epimorphisms, is given. Finally, these results are applied in the setting of homomorphisms arising as central extensions of pure and faithfully flat central ring homomorphisms.

Keywords: relative-projectivity pure-projectivity purity faithfully-flat pure-descent

Contents

Introduction

By classical result due to Raynaud and Gruson [Raynaud and Gruson, 1971] and its later corrected proof by Perry [Perry, 2010], projectivity of modules descends through faithfully flat ring homomorphisms. That is if $R \to S$ is a faithfully flat homomorphism of commutative rings and M is an R-module such that $M \otimes_R S$ is a projective S-modules, then M is a projective R-module. This result has important corollaries in algebraic geometry, where it among other things assures locality of (possibly infinite dimensional) vector bundles. This classical result has since been generalized. Notably the original setting doesn't contain commutative ring homomorphisms, which split as module homomorphisms. These need not be flat, but projectivity does descend through them. This leads to considering pure ring homomorphisms, which encompass both split and faithfully flat homomorphisms. This setting is further bolstered by the fact, that pure ring homomorphism are precisely *effective descent morphisms*, that is morphisms providing a version of descent in more abstract categorical setting, e.g. in Mesablishvili [2000], where an original unpublished proof is attributed to Joyal and Tierney. Projectivity *does* in fact descend through pure homomorphisms of commutative rings, as shown in [Angermüller, 2015], in this thesis it is proven as Theorem 7.14.

Further possible generalization is in showing descent for different module properties, e.g. pure projectivity, or, as this thesis attempts, projectivity relative to a class of finitely presented modules, of which pure projectivity and projectivity are special cases. Pure descent for pure projectivity is currently being studied in an, as of writing of this thesis, unpublished work [Herbera et al.]. We prove it independently as Theorem 7.14. Relative properties have been studied e.g. in [Mehdi, 2013], without the context of descent. Here we prove faithfully flat descent for relative projectivity as Theorem 7.16. The question, whether the hypothesis of flatness can be weakened, remains open.

Another generalization that we explore is a shift to homomorphisms of non-commutative rings. Non-commutative version of the classical proof of Raynaud and Gruson is mentioned in [Osofsky, 1979] for right pure ring homomorphisms in a priori presence of flatness, though the cited source doesn't provide the proof. In this thesis we provide a rigorous proof of descent of projectivity for ring homomorphisms which are pure as both left and right module homomorphisms and already descend flatness (Theorem 7.13 (A)). In the case of descent for pure projectivity a stronger variant of ring homomorphism purity is needed – that of reflecting pure epimorphisms. For such ring homomorphisms, we prove descent of pure projectivity (Theorem 7.13 (B)). For such homomorphisms that are also flat we prove the descent of relative projectivity (Corollary 7.15).

In this text Chapters 1 and 2 recall important facts about modules in general and Mittag-Leffler modules respectively. Next two chapters are concerned with relative module properties, Chapter 3 establishes construction of *Auslander-Bridger transpose*, which is used in Chapter 4, where relative module properties are defined and where we prove important facts about them, including variants of Kaplansky's theorem, Lazard's theorem and Drinfeld characterization of relatively projective modules.

Chapter 5 defines the notions of ascent and descent for (relative) module properties and sketches the connection to local properties. Chapter 6 is concerned with different variants of purity for homomorphisms of non-commutative rings.

Chapter 7 then proves ascent and descent for various module properties over ring homomorphisms satisfying various conditions. In Chapter 8 the results are applied to a particular type of non-commutative ring homomorphisms, namely central extensions of pure and faithfully flat central ring homomorphisms.

1. Preliminaries

All rings in this text are associative unitary. Pure homomorphisms, projective and flat modules have their usual meaning, though proper definitions are given later in Chapter 4, any undefined notation has standard meaning, generally as in [Göbel and Trlifaj, 2006].

1.1 Selection of general theorems

Some proofs later in this thesis use or are direct applications of theorems and facts from module theory, which hold in a more general setting and whose detailed proof might obstruct the idea of presented proof. Such theorems and facts are recounted and proven here. For this section we fix a ring R, all modules are considered right R-modules.

First of these is a general countable version of Kaplansky's theorem, which will be applied on case of relative projective modules.

Lemma 1.1. *Let $M \oplus N = \bigoplus_{i \in I} Q_i$, $m \in M$ be an element. Then there is a countable subset $I' \subseteq I$ and countable generated submodules $M' \subseteq M$, $N' \subseteq N$ such that $M' \oplus N' = \bigoplus_{i \in I'} Q_i$ and $m \in M'$.*

Proof. We construct chains $(M_n)_{n<\omega}$ and $(N_n)_{n<\omega}$ of countably generated submodules of M and N respectively and a chain of subsets $(I_n)_{n<\omega}$ such that $m \in M_0$ and for each $n < \omega$ we have $M_n \cap N_n = 0$, $M_n \oplus N_n \subseteq \bigoplus_{i \in I_n} Q_i$ and $\bigoplus_{i \in I} Q_i \subseteq M_{n+1} \oplus N_{n+1}$. Then putting $M' = \bigcup_{n<\omega} M_n$, $N' = \bigcup_{n<\omega} N_n$ and $I' = \bigcup_{n<\omega} I_n$ will be as required.

Put $M_0 = mR$, $N_0 = 0$. For $n < \omega$ if M_n, N_n are defined, put $I_n = \{i \in I \mid \pi_i(M_n \oplus N_n) \neq 0\}$ where π_i are canonical projections. Set I_n is countable, as $M_n \oplus N_n$ is countably generated and each of its generators $x \in \bigoplus_{i \in I} Q_i$ is only non-zero at finitely many components. If the set I_n is defined, we put $M_{n+1} = \pi_M\left(\bigoplus_{i \in I_n} Q_i\right)$ and $N_{n+1} = \pi_N\left(\bigoplus_{i \in I_n} Q_i\right)$ where π_N, π_M are the respective canonical projections. Modules M_{n+1} and N_{n+1} are countably generated as epimorphic images of a countably generated module $\bigoplus_{i \in I_n} Q_i$ and clearly $\bigoplus_{i \in I_n} Q_i \subseteq M_{n+1} \oplus N_{n+1}$. □

Theorem 1.2 (Kaplansky)**.** *Let R be a ring, $\mathcal{T}$ be a class of at most countably generated R-modules. Then a module $M \in$ Add$(\mathcal{T})$ is isomorphic to a direct sum of countably generated modules in* Add$(\mathcal{T})$*.*

Proof. Take $M \in \operatorname{Add}(\mathcal{T})$, that is $M \oplus N = \bigoplus_{i \in I} Q_i$ for some module N and modules $Q_i \in \mathcal{T}$. Choose a well ordering on $M = \{m_\alpha \mid \alpha < \kappa\}$ by a cardinal κ. Then construct by transfinite induction non-decreasing chains of modules $(M_\alpha)_{\alpha<\kappa}$, $(N_\alpha)_{\alpha<\kappa}$ and a non-decreasing chain $(I_\alpha)_{\alpha<\kappa}$ of subsets of I such that for every $\alpha < \kappa$ we have $M_\alpha \oplus N_\alpha = \bigoplus_{i \in I_\alpha} Q_i$.

Take $I_\alpha = \emptyset$, $M_0 = N_0 = 0$. If M_α for an $\alpha < \kappa$ is already constructed, construct $M_{\alpha+1}$ as follows. If $m_\alpha \in M_\alpha$, take $M_{\alpha+1} = M_\alpha$. Otherwise consider element $m_\alpha + M_\alpha \in M/M_\alpha \oplus N/N_\alpha = \bigoplus_{i \in I \setminus I_\alpha} Q_i$. Find M'_α, N'_α and I'_α as M', N'

and I' in the Lemma. Put $M_{\alpha+1} = M'_\alpha + M_\alpha$, $N_{\alpha+1} = N'_\alpha + N_\alpha$ and $I_{\alpha+1} = I_\alpha \cup I'_\alpha$. Because $\bigoplus_{i\in I_{\alpha+1}} Q_i = \bigoplus_{i\in I_\alpha} Q_i \oplus \bigoplus_{i\in I'_\alpha} Q_i$ necessarily already $M_{\alpha+1} = M'_\alpha \oplus M_\alpha$ and $N_{\alpha+1} = N'_\alpha \oplus N_\alpha$. For a limit ordinal $\beta < \kappa$ put $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$, $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$, $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$.

Now for $\alpha < \kappa$ we have $m_\alpha \in M_{\alpha+1}$, $M_{\alpha+1} = M_\alpha \oplus M'_\alpha$ for a countably generated $M'_\alpha \in \mathrm{Add}(\mathcal{T})$ and for a limit $\beta < \kappa$ we have $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$. Then clearly $M = \bigcup_{\alpha<\kappa} M_\alpha = \bigoplus_{\alpha<\kappa} M'_\alpha$ and the theorem holds. □

Next two lemmas deal with useful properties of directed limits with respect to finitely presented modules.

Lemma 1.3. *Let $M = \varinjlim_{i\in I} M_i$ be a direct limit of a directed system $(M_i; f_{ji})_{i\in I}$. Let Q be a finitely presented module and $g : Q \to M$ a module homomorphism. Then there exists an index $i \in I$ and a map $g' : Q \to M_i$ such that $g = f_i g'$, where $f_i : M_i \to M$ is the canonical map into the direct limit.*

Proof. By properties of direct limits $M = \bigcup_{i\in I} f_i(M_i)$. As Q is finitely generated, there is already an index $j \in I$ such that $g(Q) \subseteq f_j(M_j)$. Consider a presentation $0 \to K \to R^m \xrightarrow{p} Q \to 0$. Then there is a map $g_j = R^n \to M_j$ such that $gp = f_j g_j$. By properties of direct limits, for each of finitely many generators of the kernel $k \in K$ there is an index $l \in I$ such that $f_{lj} g_j(k) = 0$. Taking i as maximum of these, we get $K \subseteq \mathrm{Ker}(f_{ij} g_j)$. By homomorphism theorem then there is a homomorphism $g' : Q \to M_i$ such that $g'p = f_{ij} g_j$ and $f_i g' p = f_i f_{ij} g_j = gp$ and as p is an epimorphism also $f_i g' = g$. □

Lemma 1.4. *Let $\mathcal{C}$ be a class of finitely presented right modules, let M be a right module. Then the following are equivalent.*

(i) *$M \cong \varinjlim_{i\in I} C_i$ for some system $(C_i, f_{ji})_{j,i\in I}$ of modules from $\mathcal{C}$*

(ii) *For an arbitrary finitely presented module Q and a map $f : Q \to M$, there is a module $C \in \mathcal{C}$ and maps $g : Q \to C$, $h : C \to M$ such that $f = hg$.*

Proof. (i) implies (ii). Let $M = \varinjlim_{i\in I} C_i$ and let Q be a finitely presented module and $f : Q \to M$ a homomorphism. Then previous lemma immediately provides the required module and maps. (ii) implies (i). Express M as a direct limit of a directed system $(M_i, f_{ji})_{i\in I}$ of finitely presented modules with canonical maps $f_i : M_i \to M$. By property (ii) there is for each $i \in I$ a module $C_i \in \mathcal{C}$ and homomorphisms $g_i : M_i \to C_i$ and $h_i : C_i \to M$ such that $h_i g_i = f_i$. As C_i are finitely presented, there is by Lemma 1.3 also an index $j \in I$ homomorphism $h_i : C_i \to M_{j_i}$ such that $h'_i = f_{j_i} h_i$. It is possible to pick $j > i$, then $f_{j_i i} f_{j_i} = f_i = h_i g_i = f_{j_i i} h'_i g_i$ and by properties of direct limit and the fact that maps between finitely presented modules have finite generated kernels, there is an index $k_i > j_i$ such that $f_{k_i j_i} f_{j_i i} = f_{k_i j_i} h'_i g_i$. Then we can construct a new directed set $I' = I \sqcup \{l_i\}_{i\in I}$ with ordering given by the original ordering on I, and $i < l_i < k_i$ for each $i \in I$. Note, that I' is still an upwards directed set, and both I and $\{l_i\}_{i\in I}$ are cofinal in I'. Now we construct the directed system $(M'_l, f'_{ml})_{l<m\in I'}$ where $M'_i = M_i$, $M'_{l_i} = C_i$, for $i \in I'$, $f'_{ji} = f_{ji}$ for $i < j \in I$, and $f'_{l_i i} = g_i$ and $f'_{k_i l_i} = f_{k_i j_i} h'_i$ for $i \in I$, all other maps are constructed by needed composition. This is in fact a directed system and its limit is M. As modules C_i form a cofinal subsystem in $(M'_l, f'_{ml})_{l<m\in I'}$, M is a limit of a system of modules from $\mathcal{C}$. □

2. Mittag-Leffler modules

Important part in characterization of pure projectivity, projectivity and related module properties is played by Mittag-Leffler modules. This chapter recalls their definition and basic characterization as well as some properties. The proofs are mostly omitted.

Definition 2.1. Let $(A_i,\ f_{ij})_{i,j\in I}$ be an inverse system of sets and maps. The system will be called *inverse Mittag-Leffler system* if for any $i \in I$ the system $(\operatorname{Im}(f_{ij}))_{j\geq i}$ stabilizes, that is there is an element $k \in I$ $k \geq i$ such that for any $j \in I$, $j \geq k$ we have $\operatorname{Im} f_{ik} = \operatorname{Im} f_{ij}$.

Lemma 2.2. *Let*

$$0 \to A_i \to B_i \to C_i \to 0,\ i \in I$$

be a countable inverse system of exact sequences of R-modules. Let further $(A_i)_{i\in I}$ be an inverse Mittag-Leffler system. Then the inverse limit sequence

$$0 \to \varprojlim_{i\in I} A_i \to \varprojlim_{i\in I} B_i \to \varprojlim_{i\in I} C_i \to 0$$

is exact.

Proof. [Perry, 2010, Lemma 6.5]. This proof is done in commutative setting, commutativity of R is however not used. □

Theorem 2.3. *Let R be a ring, $(M_i, f_{ij})_{i,j\in I}$ be a directed system of finitely presented right modules $M = \varinjlim M_i$ be their direct limit. Then the following are equivalent.*

(i) *For an arbitrary right module N the inverse system $(\operatorname{Hom}_R(M_i, N))_{i\in I}$ is inverse Mittag-Leffler.*

(ii) *For each $i \in I$ there is $i \leq j \in I$ such that for each $i \leq k \in I$ there is a map $g : M_k \to M_j$ such that $f_ji = g \circ f_ik$*

(iii) *For each $i \in I$ there is $i \leq j \in I$ such that for an arbitrary left module N we have $\ker(f_i \otimes_R \mathrm{id}_N) \subseteq \ker(f_{ji} \otimes_R \mathrm{id}_N)$.*

(iv) *For each $i \in I$ there is $i \leq j \in I$ such that in the pushout diagram*

$$\begin{tikzcd} M_i \arrow[r, "f_{ji}"] \arrow[d, "f_i"'] & M_j \arrow[d, "h"] \\ M \arrow[r, "g"'] & N \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

the map h is a pure monomorphism.

Proof. [Perry, 2010, \$6.6, in particular Lemma 6.10 and Proposition 6.11] The proof is again done for R commutative but doesn't use commutativity of R □

Definition 2.4. A directed system of modules satisfying the equivalent conditions of Theorem 2.3 is called a *Mittag-Leffler (directed) system.*

Theorem 2.5. *Let R be a ring, (M_i, f_{ij}) be a directed system of finitely presented right modules $M = \varinjlim M_i$ be their direct limit. Then the following are equivalent.*

(i) *The system (M_i, f_{ij}) is Mittag-Leffler.*

(ii) *For any map $f : F \to M$ where F is a finitely presented module there are a finitely presented module F' and a map $g : F \to F'$ such that for any left module N we have $\ker(f \otimes_R \mathrm{id}_N) = \ker(g \otimes_R \mathrm{id}_N)$.*

(iii) *For an arbitrary system $(Q_k)_{k\in K}$ of left modules the map*

$$M \otimes_R \prod_{k\in K} Q_k \to \prod_{k\in K} (M \otimes_R Q_k)$$

$$m \otimes_R (q_k)_{k\in K} \mapsto (m \otimes_R q_k)_{k\in K}$$

is monic.

Proof. [Göbel and Trlifaj, 2006, Theorem 3.14] □

Definition 2.6. A module satisfying the equivalent conditions of Theorem 2.5 is called a *Mittag-Leffler module.* Class of Mittag-Leffler modules (over a fixed ring) will be denoted $\mathcal{ML}$.

Here we compile more useful properties of Mittag-Leffler modules.

Proposition 2.7. *Finitely presented modules are Mittag-Leffler.*

Proof. In condition (ii) of Theorem 2.5 it is enough to take $g = f$. □

Proposition 2.8. *Class $\mathcal{ML}$ is closed under direct sums and direct summands.*

Proof. Using the condition (iii) of Theorem 2.5, note that putting a direct sum in place of M produces the following map

$$\left(\bigoplus_{i\in I} M_i\right) \otimes_R \left(\prod_{k\in K} Q_k\right) \cong \bigoplus_{i\in I}\left(M_i \otimes_R \prod_{K\in k} Q_k\right)$$

$$\xrightarrow{\bigoplus_{i\in I}\nu_i}$$

$$\bigoplus_{i\in I}\prod_{k\in K}(M_i \otimes_R Q_k) \subseteq \prod_{k\in K}\bigoplus_{i\in I}(M_i \otimes_R Q_k).$$

Here the map $\bigoplus_{i\in I}\nu_i$ is injective if and only if all of the maps ν_i are injective. So, M is Mittag-Leffler if and only if all the modules M_i are Mittag-Leffler. □

Proposition 2.9. *Let M be a countably generated Mittag-Leffler module. Then M is countably presented.*

Proof. [Stacks project authors, 2022, Lemma 10.92.1]. This is again a proof done in commutative setting, but not using the commutativity of R. □

3. Classes of finitely presented modules and Auslander-Bridger transpose

Before giving definitions of relative module properties, we review a few facts about classes of finitely presented modules and in particular the construction of *Auslander-Bridger transpose*, which will play an important role.

Definition 3.1. Let R be a ring, then denote

$$\mathfrak{A}(\mathrm{mod}{-}R)$$

the system of classes of finitely presented right R-modules, closed under finite direct sums, direct summands and isomorphic images, containing finitely presented projective modules.

Note, that as $\mathrm{mod}{-}R$ is essentially small, $\mathfrak{A}(\mathrm{mod}{-}R)$ is a set, and it is partially ordered by inclusion. As such it is bounded, with the smallest element $\mathcal{P}_0^{<\omega}$ the class of finitely presented projective modules, and the largest element the whole of $\mathrm{mod}{-}R$.

Similarly we define the partially ordered set

$$\mathfrak{A}(R{-}\mathrm{mod}) = \mathfrak{A}(\mathrm{mod}{-}R^{op}).$$

The construction of Auslander-Bridger transpose comes from [Auslander and Bridger, 1969]. Notably, the way in which it is defined in this thesis, it does not constitute a functor or even produce unique transpose modules. It does, as proven in [Auslander and Bridger, 1969, Theorem 2.6] form a functor of stable categories of small modules. Setting of stable categories of modules is beyond the scope of this thesis, this theorem however provides a kind of "uniqueness" (Lemma 3.5), which is important.

Definition 3.2. Let M be a finitely presented right R module with presentation

$$R^m \xrightarrow{p} R^n \to M \to 0$$

note that p can be represented by a matrix $(r_{ij})_{i=1,j=1}^{n,m}$, denote $p^\top$ the map of left modules given by the transpose matrix. Note, that $p^\top = \mathrm{Hom}_R(p, R)$.

A left R-module N will be called an Auslander-Bridger transpose (or just transpose) of M if N is isomorphic to $\mathrm{Coker}(p^\top)$ for some presentation of M.

For $\mathcal{C}$ a class of finitely presented right R-modules let $\mathcal{C}^\top$ denote the class of left R-modules that are transposes of modules in $\mathcal{C}$

Lemma 3.3. *Let P be a finitely presented right R-module, let Q be a finitely presented left R-module which is an Auslander-Bridger transpose of P. Then P is projective if and only if Q is projective.*

Proof. Let P be projective, let $p : R^m \to R^n$ be a homomorphism such that $\mathrm{Coker}(p) \cong P$ and $\mathrm{Coker}(p^\top) \cong Q$. As $p = (p^\top)^\top = \mathrm{Hom}_R(p^\top, R)$, and the contravariant functor $\mathrm{Hom}_R(-, R)$ is left exact, we have $\mathrm{Ker}(p) = \mathrm{Hom}_R(Q, R)$. As P is projective, so is $\mathrm{Ker}(R^n \to P) = \mathrm{Im}(p)$ and hence $\mathrm{Hom}_R(Q, R)$ is a finitely presented projective module. As $\mathrm{Hom}_R(-, R)$ commutes with finite direct sums, Q must be a finitely presented projective left R-module. Playing the same game with map $p^\top$ yields the other implication. □

Lemma 3.4. *Let M and N be two finitely presented right R-modules. Let finitely presented left R-modules M' and N' be Auslander-Bridger transposes of M and N respectively. Then $M' \oplus N'$ is an Auslander-Bridger transpose of $M \oplus N$. From this follows, that if $\mathcal{C}$ is closed under finite direct sums, then so is $\mathcal{C}^\top$.*

Proof. Let p_M, p_N be presentations of the modules M and N respectively such that $M' = \mathrm{Coker}(p_M^\top)$ and $N' = \mathrm{Coker}(p_N^\top)$. Then $p = (p_M \oplus p_N)$ is a presentation of $M \oplus N$ and $\mathrm{Coker}(p^\top) = M' \oplus N'$. □

Lemma 3.5. *Let M be a right R-module and let*

$$R^{n_i} \xrightarrow{p_i} R^{m_i} \to M \to 0, \ \ i = 1, 2$$

be two of its presentations. Then there are finitely presented left R-modules P_1, P_2, such that $\mathrm{Coker}(p_1^\top) \oplus P_1 \cong \mathrm{Coker}(p_2^\top) \oplus P_2$.

Proof. Omitted. Version working for this setting can be found in [Maşiek, 2000, Proposition 4]. Alternatively it follows from [Auslander and Bridger, 1969, Theorem 2.6], which states that Auslander-Bridger transpose is a functor of stable categories of modules. □

Corollary 3.6. *Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$ be a class of finitely presented right R-modules. Then $\mathcal{C}^\top \in \mathfrak{A}(R{-}\mathrm{mod})$.*

Proof. The class $\mathcal{C}^\top$ is certainly a class of finitely presented left R-modules. By Lemma 3.4 it is closed under finite direct sums. It certainly contains the left regular module of R, as it is a transpose of the right regular module of R via presentation $R \xrightarrow{0} R \to R \to 0$. It remains to show, that $\mathcal{C}^\top$ is closed on direct summands. Let A and B be finitely presented left R-modules with presentations p_A, p_B, such that their direct sum $A \oplus B$ is an Auslander-Bridger transpose of some module $C \in \mathcal{C}$. Denote $A' = \mathrm{Coker}(p_A^\top)$ and $B' = \mathrm{Coker}(p_B^\top)$. Homomorphism $p_A \oplus p_B$ is a presentation of $A \oplus B$ and $\mathrm{Coker}(p_A^\top) \oplus \mathrm{Coker}(p_B^\top) = A' \oplus B'$ is a transpose of $A \oplus B$. Therefore by Lemma 3.5 there are finitely presented projective modules P_1, P_2 such that $C \oplus P_1 \cong A' \oplus B' \oplus P_2$. As $\mathcal{C}$ is closed under direct sums, direct summands and contains all finitely presented projectives, $A', B' \in \mathcal{C}$ and hence also their transposes $A, B \in \mathcal{C}^\top$. □

Corollary 3.7. *Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$. Then $(\mathcal{C}^\top)^\top = \mathcal{C}$. So, the map*

$$\mathfrak{A}(\mathrm{mod}{-}R) \to \mathfrak{A}(R{-}\mathrm{mod}) : \mathcal{C} \mapsto \mathcal{C}^\top$$

is an isomorphism of partially ordered sets.

Proof. We show two inclusions. Clearly $\mathcal{C} \subseteq (\mathcal{C}^\top)^\top$, as any module M is a transpose of any of its transposes. To show that $(\mathcal{C}^\top)^\top \subseteq \mathcal{C}$, consider a module $M \in (\mathcal{C}^\top)^\top$. Then there must be $M' \in \mathcal{C}^\top$ such that M is its transpose and then there must be $M'' \in \mathcal{C}$ such that M' is its transpose. Then by Lemma 3.5 there are finitely presented projective modules P_1, P_2 such that $M \oplus P_1 \cong M'' \oplus P_2$. By properties of $\mathcal{C}$ then already $M \in \mathcal{C}$. The fact that map $\mathcal{C} \mapsto \mathcal{C}^\top$ is monotonous follows immediately from the definitions. □

4. Relative properties

The two module properties whose interactions with ring homomorphisms this text seeks to explore are projectivity and pure projectivity. They both however can be viewed as extreme cases of more general property of relative projectivity. This chapter introduces the notion of relative projectivity as well as notions of relative flatness and relative pure exact sequences. All these properties are tied together in fashion similar to the usual notions of projectivity, purity and flatness. Notably, analogues of Kaplansky's theorem, Lazard's theorem, and Drinfeld characterization hold.

Results similar to this chapter have been published in doctoral thesis of Akeel Ramadan Mehdi [Mehdi, 2013]. The proofs in this text however are done independently and using more elementary machinery.

The properties are made relative by taking in their definition some full subcategory of finitely presented modules. Specifically these are elements of the partially ordered set $\mathfrak{A}(\mathrm{mod}{-}R)$ defined in the previous chapter, so subcategories closed under finite direct and direct summands and containing already all finitely presented projective modules.

For the remainder of this chapter let us fix a not necessarily commutative ring R.

Definition 4.1. Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$ be a class of finitely presented right R-modules. An exact sequence of right R-modules

$$0 \to A \xrightarrow{\nu} B \xrightarrow{\pi} C \to 0$$

is called *$\mathcal{C}$-pure* if for all modules $M \in \mathcal{C}$ the sequence

$$0 \to \mathrm{Hom}_R(M, A) \xrightarrow{\nu_*} \mathrm{Hom}_R(M, B) \xrightarrow{\pi_*} \mathrm{Hom}_R(M, C) \to 0$$

is exact. Then ν is called a *$\mathcal{C}$-pure monomorphism* and π is called a *$\mathcal{C}$-pure epimorphism.* The class of $\mathcal{C}$-pure sequences will be denoted $*_{\mathcal{C}}$.

A right R-module P is called *$\mathcal{C}$-projective* if for any $\mathcal{C}$-pure sequence

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$$

is exact. The class of $\mathcal{C}$-projective modules will be denoted $\mathcal{P}_{\mathcal{C}}$.

A left R-module F is called *$\mathcal{C}$-flat* if for any $\mathcal{C}$-pure sequence of right R-modules

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$$

is exact. The class of $\mathcal{C}$-flat modules will be denoted $\mathcal{F}_{\mathcal{C}}$.

Similarly, taking $\mathcal{C} \in \mathfrak{A}(R{-}\mathrm{mod})$ we define the $\mathcal{C}$-pure sequences of left R-modules, $\mathcal{C}$-projective left R-modules and $\mathcal{C}$-flat right R-modules.

Remark. Note, that if we use an arbitrary subcategory of finitely presented modules $\mathcal{S}$ in place of $\mathcal{C}$ in the previous definition, then adding to it finitely presented projectives, finite direct sums or direct summands does not change the class $*_{\mathcal{S}}$ and consequently the classes $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}}$. Hence, we only need to consider classes $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. Later, we will explicitly use both the fact that they contain finitely presented projectives and that they are closed under finite direct sums and direct summands.

The usual notions of exactness, projectivity and flatness, and the notion of pure exactness and pure projectivity are clearly edge cases of the relative properties.

Taking $\mathcal{C} = \text{mod}-R$ produces precisely the usual notion of purity and pure projectivity. This is the largest class $\mathcal{C}$ can be. In this case, all modules are $\mathcal{C}$-flat.

Taking $\mathcal{C} = \mathcal{P}_0^{<\omega}$ produces the usual notion of projectivity and flatness. This is the smallest class $\mathcal{C}$ can be.

For the needs of the rest of this thesis, these serve as definitions of the usual notions.

Definition 4.2. A right R-module monomorphism (epimorphism) will be called *pure* if it is $(\text{mod}-R)$-pure. A $(\text{mod}-R)$-projective module shall be called *pure projective*, A $\mathcal{P}_0^{<\omega}$-flat module will be called *flat* and $\mathcal{P}_0^{<\omega}$-projective will be called *projective.*

Lemma 4.3. *Let $\mathcal{C} \subseteq \mathcal{C}'$ be two elements of $\mathfrak{A}(\text{mod}-R)$. Then $*_{\mathcal{C}'} \subseteq *_{\mathcal{C}}$, $\mathcal{P}_{\mathcal{C}} \subseteq \mathcal{P}_{\mathcal{C}'}$, and $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{F}_{\mathcal{C}'}$*

Proof. This is immediate from the definitions. □

Corollary 4.4. *Let $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. Then any pure exact sequence of right R-modules is $\mathcal{C}$-pure, any projective right R-module is $\mathcal{C}$-projective and any $\mathcal{C}$-projective right R-module is pure projective, and any flat left R-module is $\mathcal{C}$-flat.*

The classes of modules with relative properties have the expectable closure properties.

Proposition 4.5. *Let $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. Then $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}$ are closed under arbitrary direct sums and direct summands. Class $\mathcal{F}_{\mathcal{C}}$ is further closed under direct limits. $*_{\mathcal{C}}$ is closed under direct limits.*

Proof. Let $P = \bigoplus_{i\in I} P_i$ be a right R-module, let π be an epimorphism. Using natural isomorphism $\text{Hom}_R(\bigoplus_{i\in I} P_i, M) \cong \prod_{i\in I} \text{Hom}_R(P_i, M)$ we see that $Hom_R(P, \pi)$ is onto if and only if all of $Hom_R(P_i, \pi)$ are onto. Considering this for all $\mathcal{C}$ pure epimorphisms we get that P is $\mathcal{C}$-projective if and only if each of P_i are $\mathcal{C}$-projective. Hence, $\mathcal{P}_{\mathcal{C}}$ is closed under direct sums and direct summands.

The fact that $\mathcal{F}_{\mathcal{C}}$ is closed under direct sums, direct summands and direct limits follows similarly from the fact, that tensor product commutes with these constructions and taking direct limit is an exact functor.

The fact, that $*_{C}$ is closed under direct limits is again argued similarly, using the natural isomorphism $\varinjlim(\text{Hom}_R(M, A_i)) \cong \text{Hom}_R(M, \varinjlim(A_i))$ for M finitely presented. □

4.1 Basic properties of relative projectivity

This section shows, that relative projective modules share some convenient properties with the usual projective modules. The main result here is a version of Kaplansky's theorem. What will later come useful, is the fact, that the class of relatively projective modules admits a relatively pure precover for every module.

In the rest of this chapter we fix a subcategory $\mathcal{C} \in \mathfrak{A}(\text{mod}{-}R)$.

Proposition 4.6. *Let M be a right R-module. Then there exists a $\mathcal{C}$-pure exact sequence*

$$0 \to K \to C \to M \to 0$$

where C is a direct sum $\bigoplus_{i \in I} C_i$ of modules $C_i \in \mathcal{C}$.

Proof. Category mod$-R$ is essentially small, so we can take a set of representatives $\{C_i\}_{i \in I}$ of the class $\mathcal{C}$. Take now a disjoint union $J = \bigsqcup_{i \in I} \mathrm{Hom}_R(C_i, M)$ and take direct sum $C = \bigoplus_{f \in J} C_f$ where $C_f \cong C_i$ for $f \in \mathrm{Hom}_R(C_i, M)$. Construct map $p : C \to M$ given on the summands as $f : C_f \to M$ for $f \in J$. Map p is onto, because $R_R \in \mathcal{C}$. Thus we obtain an exact sequence

$$0 \to \mathrm{Ker}(p) \to C \xrightarrow{p} M \to 0.$$

This sequence is $\mathcal{C}$-pure, as any map $g : C_i \to M$ certainly factorizes as $f = p \circ \nu_f$ where $\nu_f : C_f \to M$ is the canonical inclusion. □

Remark. In the edge cases of projectivity and pure projectivity there are simpler constructions of the precover. In the case of usual projectivity all that is needed is a free cover, as in this case the $\mathcal{C}$-purity reduces to exactness. In the case of pure projectivity, we use the presentation of M as a direct limit of finitely presented modules and the fact that the natural map $\bigoplus_{i \in I} M_i \to \varinjlim_{i \in I} M_i$ is pure.

Theorem 4.7. *Let M be a right R-module. M is $\mathcal{C}$-projective if and only if it is a direct summand in a direct sum of modules from $\mathcal{C}$, that is $\mathcal{P}_{\mathcal{C}} = \mathrm{Add}(\mathcal{C})$. Specially for $Q \in \mathcal{P}_{\mathcal{C}}$ finitely generated already $Q \in \mathrm{add}\,\mathcal{C} = \mathcal{C}$.*

Proof. Applying the previous proposition to a $\mathcal{C}$-projectve module P yields a $\mathcal{C}$-pure exact sequence

$$0 \to K \to C \to P \to 0$$

which splits and so P is a direct summand in C. Furthermore, if P is taken finitely generated, then it is already a direct summand in a finite direct sum. □

Corollary 4.8. *Let κ be an infinite cardinal and let P be an at most κ-generated $\mathcal{C}$-projective module. Then P is at most κ-presented.*

Proof. In the previous proof we note that if P is at most κ-generated, then it is already a direct summand in a direct sum of at most κ finitely presented modules, and is as such at most κ-presented. □

Corollary 4.9. *Projective modules are precisely direct summands in free modules, pure projective modules are precisely direct summand in direct sums of finitely presented modules.*

This lets us prove relative version Kaplansky's theorem.

Theorem 4.10. *Let P be a $\mathcal{C}$-projective module. Then $P \cong \bigoplus_{i\in I} P_i$ where P_i are countably presented $\mathcal{C}$-projective modules.*

Proof. We directly apply Theorem 1.2, taking $\mathcal{C}$ for $\mathcal{T}$. Thus we get $P \cong \bigoplus_{i\in I} P_i$, with P_i countably generated. However, by previous corollary, each P_i is already countably presented. □

4.2 Properties of relative purity

In this section we explore some properties of relatively pure homomorphisms. Importantly, the Auslander-Bridger transpose enters the scene, allowing for a definition of relatively pure monomorphisms through tensor product.

Note, that as $(\mathcal{C}^\top)^\top = \mathcal{C}$, in all of this section the sidedness of modules and the roles of $\mathcal{C}$ and $\mathcal{C}^\top$ could be reversed with no effect on the proofs.

Most important facts will follow from the following proposition.

Theorem 4.11. *Let*

$$0 \to A \xrightarrow{\nu} B \xrightarrow{\pi} C \to 0$$

be an exact sequence of right modules. Let $r : R^n \to R^m$ be a map of free right R-modules with matrix $(r_{ij})_{i=1,j=1}^{n,m}$

let $M = \mathrm{Coker}(r)$ and $Q = \mathrm{Coker}(r^\top)$. (This makes the right module M and the left module Q Auslander-Bridger transposes of each other). Then the following are equivalent.

(i) *The sequence $0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$ is exact.*

(ii) *In any commutative square of shape*

$$\begin{array}{ccc} R^m & \xrightarrow{r} & R^n \\ \downarrow & & \downarrow \\ A & \xrightarrow{\nu} & B \end{array}$$

a map $h : R^n \to A$ exists, making the top triangle commute.

(iii) *Understanding ν as an inclusion $A \subseteq B$, if $a_i \in A, i = 1, \ldots, m$ and $b_i \in B, j = 1, \ldots, n$ are such that*

$$\sum_{i=1}^{n} b_i \cdot r_{ij} = a_j, j = 1, \ldots, m,$$

then there are also $a'_j \in A, j = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} a'_i \cdot r_{ij} = a_j, j = 1, \ldots, m$$

(iv) *The sequence $0 \to A \otimes_R Q \to B \otimes_R Q \to C \otimes_R Q \to 0$ is exact.*

Proof. Ideas of this proof are taken from [Perry, 2010, Theorem 3.11], adjustments are made to fit the relative setting.

Firstly, (i) implies (ii). Take a commutative square as in (ii), then there is also a map of cokernels denoted f'' in the commutative diagram

$$\begin{array}{ccccccccc} & & R^m & \xrightarrow{r} & R^n & \xrightarrow{q} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & A & \xrightarrow{\nu} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

By (i) the map $\pi_* : \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is surjective and hence there is map $h' : M \to B$ such that $\pi \circ h' = f''$.

Take now map $h'q - f$ and note, that $\pi(h'q - f) = \pi h'q - \pi f = f''q - f''q = 0$. Therefore, $\mathrm{Im}(h'q - f) \subseteq \mathrm{Ker}(\pi) = \mathrm{Im}(\nu)$ and as ν is an injective map, there is a map $h : R^n \to A$ such that $\nu h = h'q - f$. Finally, $\nu hr = (h'q - f)r = fr = \nu f'$ and with ν being a monomorphism we have $hr = f'$, so h is the right map making the top triangle commute.

Next let's see that (ii) implies (i). It is enough to show that the map $\pi_* : \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is surjective. Let there be a map $g : M \to C$. In the commutative diagram

$$\begin{array}{ccccccccc} & & R^m & \xrightarrow{r} & R^n & \xrightarrow{q} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle g''} & & \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \xrightarrow{\nu} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

the map g' comes from the projective property of R^n and the map g'' comes from the projective property of R^m and the fact that $g'r = gqr = 0$ and hence $\mathrm{Im}(g'r) \subseteq \mathrm{Ker}(\pi) = \mathrm{Im}(\nu)$. By (ii) there is a map $h' : R^n \to A$ such that $h'p = g''$

Now take map $g' - \nu h'$ and notice that $(g' - \nu h)r = g'r - \nu h'r = \nu g'' - \nu g'' = 0$, and hence there is map $h : M \to B$ such that $hq = \nu h - g$. Finally, $\pi hq = \pi(g' - \nu h') = \pi g' - 0 = gq$ and with q being an epimorphism we have $\pi h = g$. Thus, we found for an arbitrary map $g : M \to C$ a preimage with respect to π_*.

The condition (iii) is a reformulation of condition (ii) in language of equations. The map $r : R^m \to R^n$ specifies the coefficients of the system of equations, map $R^m \to A$ is the same as a choice of elements a_i, $i = 1, \ldots, m$, map $R^n \to B$ is the same as a choice of elements b_j, $j = 1, \ldots, n$, map $h : R^n \to A$ is the same as a choice of elements $'_j$, $j = 1, \ldots, n$. The first system of equalities is then precisely the statement of the square commuting, while the second system is the statement of the upper triangle commuting after introducing the map h.

Let's now show the equivalence of (iii) and (iv). Tensoring map $A \xrightarrow{\nu} B$ and sequence $R^n \xrightarrow{r^\top} R^m \to Q \to 0$ produces diagram

$$\begin{array}{ccccccc} A^n & \xrightarrow{r_A} & A^m & \xrightarrow{q_A} & A \otimes_R Q & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \nu^n} & & \Big\downarrow{\scriptstyle \nu^m} & & \Big\downarrow{\scriptstyle \nu'} & & \\ B^n & \xrightarrow{r_B} & B^m & \xrightarrow{q_B} & B \otimes_R Q & \longrightarrow & 0 \end{array}$$

with exact rows and where

$$r_A = \mathrm{id}_A \otimes_R r^\top : (a_i)_{i=1}^n \mapsto \left(\sum_{i=1}^n a_i \cdot r_{ij}\right)_{j=1}^m$$

$$r_B = \mathrm{id}_B \otimes_R r^\top : (b_i)_{i=1}^n \mapsto \left(\sum_{i=1}^n b_i \cdot r_{ij}\right)_{j=1}^m$$

and $\nu \otimes_R \mathrm{id}_{R^k} = \nu^k$ for $k = n,\ m$. The condition (iv) boils down to the map $\nu' = \nu \otimes_R \mathrm{id}_Q$ being injective.

Assume first the condition (iii). To prove the injectivity of map ν' let us choose an element $x \in A \otimes_R Q$ such that $\nu'(x) = 0$. By surjectivity of q_A there is an element $a = (a_j)_{j=1}^m \in A^m$ such that $q_A(a) = x$. Then $q_B \nu^m(a) = \nu' q_A(a) = 0$ and by exactness of the bottom row there is an element $b = (b_i)_{i=1}^n \in B^n$ such that (interpreting ν as an inclusion)

$$\left(\sum_{i=1}^n b_i \cdot r_{ij}\right)_{j=1}^m = r_B(b) = \nu^m(a) = (a_j)_{j=1}^m$$

This is precisely the first system of equations in (iii), and therefore there is an element $a' = (a'_i)_{i=1}^n \in A^n$ such that

$$r_A(a') = \left(\sum_{i=1}^n a'_i \cdot r_{ij}\right)_{j=1}^m = (a_j)_{j=i}^m = a$$

Then by exactness of the top row $x = q_A(a) = q_A r_A(a') = 0$. So, the map ν' is injective.

Suppose on the other hand that (iv) holds and so ν' is injective. Let $a_j,\ j = 1, \ldots, m$ and b_i, $i = 1, \ldots, n$ be such that

$$\sum_{i=1}^n b_i \cdot r_{ij} = a_j, j = 1, \ldots, m$$

that is for elements $a = (a_j)_{j=1}^m \in A^m$ and $b = (b_i)_{i=1}^n \in B^N = n$ we have $r_B(b) = \nu^m(a)$. Then by exactness of the bottom row $\nu' q_A(a) = q_B \nu^m(a) = 0$, by injectivity of ν' we have $q_A(a) = 0$ and by exactness of the top row there is an element $a' = (a'_i)_{i=1}^n \in A^n$ such that $r_A(a') = a$ and so

$$\sum_{i=1}^n a'_i \cdot r_{ij} = a_j, j = 1, \ldots, m$$

and condition (iii) holds. □

Proposition 4.12. *In the previous theorem the condition* (ii) *can be strengthened as follows*

(ii') *In any commutative square of shape*

$$\begin{tikzcd} N \arrow[r, "p"] \arrow[d] & N' \arrow[d] \\ A \arrow[r, "\nu"] & B \end{tikzcd}$$

where $\mathrm{Coker}(p) \cong M$ *a map* $h : N' \to A$ *exists, making the top triangle commute.*

Proof. Clearly, (ii') implies (ii). It suffices to show that (i) implies (ii'). To show that, it is enough to notice, that the proof that (i) implies (ii) doesn't use any properties of map R beyond it's cokernel. □

Remark. The conditions (ii) or (iii) in the previous theorem could be easily taken as a definition of a relatively pure submodule, more akin to the model theoretic definition. Informally they say that a module is relatively pure submodule of a larger module, whenever certain systems of linear equations solvable in the larger module also admits a solution in it. Difference from the normal notion of purity comes from restricting our attention to a certain class of systems of linear equations, namely those which define modules of class $\mathcal{C}$.

The equivalence of (i) a (iv) then lets us give a tensor product definition of relative purity.

Corollary 4.13. *An exact sequence*

$$0 \to A \to B \to C \to 0$$

of right R-modules is $\mathcal{C}$-pure if and only if for any $Q \in \mathcal{C}^\top$ the sequence

$$0 \to A \otimes_R Q \to B \otimes_R Q \to C \otimes_R Q \to 0$$

is exact.

Taking $\mathcal{C} = \text{mod}-R$, we get a characterization of the usual notion of purity.

Corollary 4.14. *Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then the following are equivalent*

(i) *For a finitely presented right module M the sequence*

$$0 \to \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C) \to 0$$

is exact.

(ii) *For a finitely presented left module Q the sequence*

$$0 \to A \otimes_R Q \to B \otimes_R Q \to C \otimes_R Q \to 0$$

is exact.

(iii) *For an arbitrary left module N the sequence*

$$0 \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

is exact.

Proof. The equivalence of (i) and (ii) is precisely the statement of Proposition (4.11), for the if direction taking a presentation of M as r, for the only if direction taking a presentation of Q as $r^\top$.

The implication from (iii) to (ii) is immediate. The reverse implication arises from the fact that N can be taken as a direct limit of a system of finitely presented modules and taking direct limits commutes with tensor product and is an exact functor. □

Remark. This is of course one of the ways to define a pure exact sequence. This result will be needed in the further discussion of relative flatness.

The final lemma of this section ties together $\mathcal{C}$-pure sequences of right R-modules and $\mathcal{C}^\top$-pure sequences of left R-modules.

Lemma 4.15. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then it is $\mathcal{C}$-pure if and only if the sequence $0 \to C^* \to B^* \to A^* \to 0$, where $(-)^* = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : (\mathrm{Mod}{-}R)^{\mathrm{op}} \to R{-}\mathrm{Mod}$ is the character dual functor, is $\mathcal{C}^\top$-pure.*

Proof. Recall that the functor $(-)^* = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : (\mathrm{Mod}{-}R)^{\mathrm{op}} \to R{-}\mathrm{Mod}$ is faithfully exact. For an arbitrary left R-module we then have that the sequence

$$0 \to Q \otimes_R A \to Q \otimes_R B \to Q \otimes_R C \to 0$$

is exact if and only if the sequence

$$0 \to (Q \otimes_R C)^* \to (Q \otimes_R B)^* \to (Q \otimes_R A)^* \to 0$$

is exact. This sequence is by tensor-hom adjunction isomorphic to sequence

$$0 \to \mathrm{Hom}_R(Q, C^*) \to \mathrm{Hom}_R(Q, B^*) \to \mathrm{Hom}_R(Q, A^*) \to 0.$$

By Corollary 4.13 a sequence $0 \to A \to B \to C \to 0$ is $\mathcal{C}$-pure if and only if

$$0 \to Q \otimes_R A \to Q \otimes_R B \to Q \otimes_R C \to 0$$

is exact for any $Q \in \mathcal{C}^\top$, which is if and only if

$$0 \to \mathrm{Hom}_R(Q, C^*) \to \mathrm{Hom}_R(Q, B^*) \to \mathrm{Hom}_R(Q, A^*) \to 0$$

is exact for any $Q \in \mathcal{C}^\top$, which is precisely the definition of

$$0 \to C^* \to B^* \to A^* \to 0$$

being $\mathcal{C}^\top$-pure. □

4.3 Properties of relative flatness

Finally, relatively flat modules share many properties with the usual notion of flatness. Importantly a version of Lazard's theorem holds.

Following several propositions show basic relationship between $\mathcal{C}$-projectivity and $\mathcal{C}^\top$-flatness, these both being properties of right modules. From now on, whenever we speak of "relative flatness" we mean $\mathcal{C}^\top$-flatness of right modules. Now we show that relatively projective modules are already relatively flat and for finitely presented modules this is an equivalence.

Proposition 4.16. *Finitely presented module M is $\mathcal{C}$-projective if and only if Q, a transpose of M is $\mathcal{C}$-flat.*

Proof. It is enough to in Proposition 4.11 start with a finitely presented module M, taking for $r : R^m \to R^n$ arbitrary presentation of its. Then $Q = \mathrm{Coker}(r^\top)$ is a transpose of M. Take for the exact sequence $0 \to A \to B \to C \to 0$ an arbitrary $\mathcal{C}$-pure sequence. Then the equivalence of (i) and (iv) amounts precisely to this proposition. □

Proposition 4.17. *All $\mathcal{C}$-projective modules are $\mathcal{C}^\top$-flat.*

Proof. Take a $\mathcal{C}^\top$-pure exact sequence of left modules

$$0 \to A \to B \to C \to 0.$$

By Proposition 4.15 the sequence

$$0 \to C^* \to B^* \to A^* \to 0$$

is $\mathcal{C}$-pure.

Take now an arbitrary $\mathcal{C}$-projective module M. The sequence

$$0 \to \mathrm{Hom}_R(M, C^*) \to \mathrm{Hom}_R(M, B^*) \to \mathrm{Hom}_R(M, A^*) \to 0$$

is exact. Tensor-hom adjunction yields that also

$$0 \to (M \otimes_R C)^* \to (M \otimes_R B)^* \to (M \otimes_R A)^* \to 0,$$

is an exact sequence. As $(-)^*$ is faithfully exact, also

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is an exact sequence. Hence, M is $\mathcal{C}^\top$-flat. □

Corollary 4.18. *Let Q be a finitely presented right R-module. Then Q is $\mathcal{C}$-projective if and only if it is $\mathcal{C}^\top$-flat.*

Proof. The only if direction comes directly from the previous proposition. Now if Q is $\mathcal{C}^\top$-flat then a transpose $Q^\top$ of Q is $\mathcal{C}^\top$-projective and hence $\mathcal{C}$-flat. Then however Q is $\mathcal{C}$-projective. □

Corollary 4.19. *Let Q be a finitely presented $\mathcal{C}^\top$-flat module. Then $Q \in \mathcal{C}$.*

Proof. This is immediate consequence of the previous lemma and Theorem 4.7. □

Now we can prove relative version of Lazard's theorem.

Theorem 4.20. *Let F be a right R-module, then the following are equivalent.*

(i) *F is $\mathcal{C}^\top$-flat.*

(ii) *Any $\mathcal{C}$-pure exact sequence*

$$0 \to A \to B \to F \to 0$$

is already pure.

(iii) *For any finitely presented right module M and a map $f : M \to F$ there are maps g, h and module $C \in \mathcal{C}$ forming commutative triangle*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & C \arrow[dl, "h"] \\ F & \end{tikzcd}$$

(iv) $F = \varinjlim_{i \in I} C_i$ *for some directed system of modules* $C_i \in \mathcal{C}$.

Proof. (i) implies (ii). Let

$$0 \to A \to B \to F \to 0$$

be such $\mathcal{C}$-pure exact sequence. Let Q be an arbitrary left R-module. The proof amounts to showing that the sequence

$$0 \to A \otimes_R Q \to B \otimes_R Q \to F \otimes_R Q \to 0$$

is exact, namely that the map $A \otimes_R Q \to B \otimes_R Q$ is a monomorphism. Proposition 4.6 for the class $\mathcal{C}^\top$ provides a $\mathcal{C}^\top$-pure sequence

$$0 \to K \to C \to Q \to 0$$

with C $\mathcal{C}^\top$-projective and therefore $\mathcal{C}$-flat. Tensoring sequences

$$0 \to A \to B \to F \to 0 \text{ and } 0 \to K \to C \to Q \to 0$$

yields commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & F \otimes_R K \arrow[u] \arrow[r, "h"] & F \otimes_R C \arrow[u] \arrow[r] & F \otimes_R Q \arrow[u] \arrow[r] & 0 \\
 & B \otimes_R K \arrow[u, "j"] \arrow[r, "g"] & B \otimes_R C \arrow[u, "e"] \arrow[r, "c"] & B \otimes_R Q \arrow[u] \arrow[r] & 0 \\
 & A \otimes_R K \arrow[u, "i"] \arrow[r, "f"] & A \otimes_R C \arrow[u, "d"] \arrow[r, "b"] & A \otimes_R Q \arrow[u, "a"] \arrow[r] & 0 \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

with all rows and columns exact. Map $A \otimes_R Q \to B \otimes_R Q$ can be shown to be a monomorphism by diagram chase as follows.

Take an element $x \in A \otimes_R Q$ such that $a(x) = 0$. By surjectivity of b there is an element $x' \in A \otimes_R C$ such that $b(x') = x$ and so $cd(x') = ab(x') = 0$. Then by exactness of the middle row there is an element $y \in B \otimes_R K$ such that $g(y) = d(x')$. Furthermore, $hj(y) = eg(y) = ed(x') = 0$. By injectivity of h then $j(y) = 0$ and by exactness of the left column there is an element $x'' \in A \otimes_R K$ such that $i(x'') = y$ and so $df(x'') = gi(x'') = g(y) = d(x')$. By injectivity of d then $f(x'') = x'$ and $x = b(x') = bf(x'') = 0$.

(ii) implies (iii). Let F, M and f be as in the formulation. Applying Proposition 4.6 to F produces a $\mathcal{C}$-pure exact sequence

$$0 \to K \to \bigoplus_{i \in I} C_i \xrightarrow{\pi} F \to 0$$

with $C_i \in \mathcal{C}$. By (ii) it is already pure. Then however by properties of pure exact sequences (corollary) there is a map $g : M \to C$ in

$$\begin{tikzcd} & & & M \arrow[dl, "g"'] \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r] & C \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

Since M is finitely generated, the image of map h is already contained in a finite direct sum $C = \bigoplus_{i \in I'} C_i$ which is itself a module in $\mathcal{C}$. Taking $h = \pi|_C$ completes the proof.

(iii) implies (iv) by Lemma 1.4.

Finally, (iv) implies (i) because $\mathcal{C} \subseteq \mathcal{F}_{\mathcal{C}^\top}$ and $\mathcal{F}_{\mathcal{C}^\top}$ is closed under taking direct limits. $\square$

Condition (ii) in previous theorem provides a characterization of $\mathcal{C}^\top$-flat modules similar to a well known characterization of flat modules summed up in the following corollary.

Corollary 4.21. *A right module F is flat if and only if any exact sequence*

$$0 \to A \to B \to F \to 0$$

ending in F is pure.

Proof. This arises from taking $\mathcal{C} = \mathcal{P}_0^{<\omega}$ in the previous Theorem. $\square$

This characterization lets us show more expected closure properties of the class of relatively flat modules.

Proposition 4.22. *The class $\mathcal{F}_{\mathcal{C}^\top}$ is closed under pure epimorphic images, pure submodules and pure extensions.*

Proof. To show that $\mathcal{F}_{\mathcal{C}^\top}$ is closed under pure epimorphic images let M be a $\mathcal{C}^\top$-flat module, let $M \to M''$ be a pure epimorphisms and let Q be a finitely presented module and $Q \to M''$ a map. In the diagram

$$\begin{tikzcd} C \arrow[d] & Q \arrow[l] \arrow[d] \arrow[dl] & \\ M \arrow[r, "*"] & M'' \arrow[r] & 0 \end{tikzcd}$$

map $Q \to M$ making the bottom triangle commute exists as $M \to M''$ is a pure epimorphism, module $C \in \mathcal{C}$ and maps $Q \to C$ and $C \to M$ such that the upper triangle commutes come from M being $\mathcal{C}^\top$-flat. Composition $C \to M \to M''$ witnesses M'' being $\mathcal{C}^\top$-flat.

For closure under pure submodules let M be a $\mathcal{C}^\top$-flat module, let $N' \to M$ be a pure monomorphism, and let Q be a finitely presented module and $Q \to M'$

a homomorphism. We seek to factorize $Q \to M'$ through a module $C \in \mathcal{C}$. In the diagram

$$\begin{tikzcd} & Q \arrow[r] \arrow[d] & C \arrow[d] \arrow[dl] \\ 0 \arrow[r] & M' \arrow[r, "*"] & M \end{tikzcd}$$

module C and maps $Q \to C$ and $C \to M$ come from M being $\mathcal{C}^\top$-flat. Homomorphism $Q \to C$ has finitely presented cokernel and $M' \to M$ is a pure monomorphism, by Proposition 4.12 map $C \to M'$ exists making the top triangle commute. Maps $Q \to C$ and $C \to M'$ witness M' being $\mathcal{C}^\top$-flat.

To show that $\mathcal{F}_{\mathcal{C}^\top}$ is closed under pure extensions, let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence such that M' and M'' are $\mathcal{C}^\top$-flat. Let $A \to B$ be an arbitrary $\mathcal{C}^\top$-pure monomorphism of left R modules. Tensoring produces diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \otimes_R A \arrow[r] \arrow[d, hook] & M \otimes_R A \arrow[r] \arrow[d] & M'' \otimes_R A \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M' \otimes_R B \arrow[r] & M \otimes_R B \arrow[r] & M'' \otimes_R B \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact, vertical maps on sides are monomorphisms by M' and M'' being $\mathcal{C}^\top$-flat, and the middle vertical map is a monomorphism by five lemma. Thus, M is a $\mathcal{C}^\top$-flat module. $\square$

Corollary 4.23. . *The class $\mathcal{F}_{\mathcal{C}^\top}$ can thus be characterized as*

(i) *Pure epimorphic images of $\mathcal{C}$-projective modules,*

(ii) *direct limits of $\mathcal{C}$-projective, or*

(iii) *direct limits of modules from $\mathcal{C}$.*

4.4 Drinfeld characterization

Armed with Lazard's theorem, we can finish the main conclusion of this chapter and characterize relative projective modules as relatively flat, Mittag-Leffler, Kaplansky decomposable modules. Proofs in this section are adaptations of [Perry, 2010, Lemma 7.2, Theorem 7.4].

For the whole picture, recall that in the finitely presented case (where the Mittag-Leffler condition is void) the relative flatness and relative projectivity merge (Corollary 4.18).

In the countable infinite case the Mittag-Leffler property starts playing an important role.

Proposition 4.24. *Let $\mathcal{C}$ be a class of finitely presented right R-modules containing R, closed under finite direct sums and direct summands. Let P be a countably generated module then P is $\mathcal{C}$-projective if and only if*

(a) *P is $\mathcal{C}^{\top}$-flat*

(b) *P is Mittag-Leffler*

Proof. For the only if direction, immediately a $\mathcal{C}$-projective module is $\mathcal{C}^{\top}$-flat by Proposition 4.17 and Mittag-Leffler as it is by Proposition 4.7 a direct summand in a direct sum of finitely presented modules, which are by Proposition 2.7 Mittag-Leffler and by Proposition 2.8 $\mathcal{ML}$ is closed under direct sums and direct summands.

Let now P be a countably generated $\mathcal{C}^{\top}$-flat Mittag-Leffler module. By Proposition 4.20 it is a direct limit of a system $(M_i)_{i\in I}$ of modules $M_i \in \mathcal{C}$, by Proposition 2.9 it is countably presented, and therefore the system can be taken countable. Finally, it is Mittag-Leffler. Let now $0 \to A \to B \to C \to 0$ be a $\mathcal{C}$-pure exact sequence. As $M_i \in \mathcal{C}$, each of the exact sequences

$$0 \to \mathrm{Hom}_R(M_i, A) \to \mathrm{Hom}_R(M_i, B) \to \mathrm{Hom}_R(M_i, C) \to 0$$

is exact, and together they comprise an exact sequence of inverse systems. As the system (M_i) is Mittag-Leffler, the system $(\mathrm{Hom}_R(M_i, A))_{i\in I}$ is an inverse Mittag-Leffler system and by Lemma 2.2 taking the inverse limit yields exact sequence

$$0 \to \varprojlim_{i\in I}(\mathrm{Hom}_R(M_i, A)) \to \varprojlim_{i\in I}(\mathrm{Hom}_R(M_i, B)) \to \varprojlim_{i\in I}(\mathrm{Hom}_R(M_i, C)) \to 0$$

isomorphic to exact sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0.$$

So, module M is $\mathcal{C}$-projective. □

In the general case we finally need the Kaplansky decomposition, as without the countable presentation Mittag-Leffler property no longer makes the inverse limit exact.

Proposition 4.25. *Let $\mathcal{C}$ be a class of finitely presented right R-modules containing R, closed under finite direct sums and direct summands. Let P be a right R-module. Then P is $\mathcal{C}$-projective if and only if*

(a) *P is $\mathcal{C}^{\top}$-flat*

(b) *P is Mittag-Leffler*

(c) *$P = \bigoplus_{i\in I} P_i$ where the modules P_i are countably generated.*

Proof. For the only if direction let P be $\mathcal{C}$-projective. Then it is $\mathcal{C}^{\top}$ flat by Proposition 4.17, and it is Mittag-Leffler as it is a direct summand in a direct sum of finitely presented and therefore Mittag-Leffler modules. By Theorem 4.10 it is a direct sum of countably generated, even countably presented modules.

Let now P satisfy conditions (a), (b), and (c). By condition (c) it decomposes into a direct sum $P = \bigoplus_{i\in I} P_i$, with P_i countably generated. Furthermore, each of the modules P_i is also $\mathcal{C}^{\top}$-flat and Mittag-Leffler. By Proposition 4.24 the modules P_i are therefore $\mathcal{C}$-projective and, as $\mathcal{P}_{\mathcal{C}}$ is closed under direct sums, so is P. □

Corollary 4.26. *A right module P is pure projective if and only if it is a direct sum of countably generated modules, and it is Mittag-Leffler.*

Corollary 4.27. *A right module P is $\mathcal{C}$-projective if and only if it is pure projective and $\mathcal{C}^\top$-flat.*

Remark. There is an alternative and more immediate proof of this corollary. Taking P as $\mathcal{C}^\top$-flat renders the sequence

$$0 \to K \to C \to P \to 0$$

of Proposition 4.6 pure, taking P pure projective makes it split and makes P a direct summand in a $\mathcal{C}$-projective module.

This betrays that the crux of the proof lays in countably generated Mittag-Leffler modules being pure projective.

4.5 Another relativization of flatness

The condition (ii) of Theorem 4.20 and Corollary 4.21 raise a question, whether we might weaken the condition of flatness in the corollary in another way, asking for an arbitrary sequence ending in our module to be relatively pure. This leads to characterization through the Tor_1^R functor and turns out to be the "other half" of usual flatness, that we dropped when weakening it into a relative version. This section is illustrative and results from it will not be needed in the rest of this text.

Lemma 4.28. *Let R be a ring, let $\mathcal{C}$ be a class of finitely presented right R-modules closed under finite direct sums and direct summands and containing finitely presented projective modules, let M be a right R-module. Then the following are equivalent.*

(i) *For a finitely presented right R-module $C \in \mathcal{C}$ and a map $f : C \to M$ there is a free module F of finite rank and the maps making the following diagram commute*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & F \arrow[dl] \\ M & \end{tikzcd}$$

(ii) *Any short exact sequence*

$$0 \to A \to B \to M \to 0$$

is $\mathcal{C}$-pure.

(iii) $\mathrm{Tor}_1^R(M, \mathcal{C}^\top) = 0$

Proof. For implication from (i) to (ii) take a short exact sequence $0 \to A \to B \to M \to 0$ and a map $f : C \to M$ for some $C \in \mathcal{C}$. In the following commutative diagram maps $C \to F$ come from (i) and map $F \to B$ from the projective property of F.

$$\begin{tikzcd} & & F \arrow[d] \arrow[dr] & C \arrow[l] \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

For the inverse implication, consider the exact sequence $0 \to K \to F' \to M \to 0$ where F' is a suitable free module, and map $C \to M$ for a $C \in \mathcal{C}$. By (ii) the exact sequence is $\mathcal{C}$-pure and there is a map $C \to F'$ making the triangle commute.

$$\begin{tikzcd} & & & C \arrow[dl] \arrow[d] & \\ 0 \arrow[r] & K \arrow[r] & F' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

As C is finitely presented, the map already factorizes through a finite rank free submodule $F \subseteq F'$.

For the implication from (ii) to (iii) consider again exact sequence $0 \to K \to F' \to M \to 0$ where F' is a suitable free module. Take an arbitrary left module $Q \in \mathcal{C}^\top$. Tensoring the exact sequence with Q produces a long exact sequence

$$0 = \mathrm{Tor}_1^R(F, Q) \to \mathrm{Tor}_1^R(M, Q) \to K \otimes_R Q \hookrightarrow F \otimes_R Q \to M \otimes_R Q \to 0$$

where $K \otimes_R Q \hookrightarrow F \otimes_R Q$ is injective because the original sequence was $\mathcal{C}$-pure. Sandwiched between an injective map and zero module, $\mathrm{Tor}_1^R(M, Q) = 0$.

For the inverse implication consider an arbitrary exact sequence ending in M. Tensoring it with a left module $Q \in \mathcal{C}^\top$ produces a sequence

$$0 = \mathrm{Tor}_1^R(M, Q) \to A \otimes_R Q \to B \otimes_R Q \to M \otimes_R Q \to 0$$

making the sequence $0 \to A \otimes_R Q \to B \otimes_R Q \to M \otimes_R Q \to 0$ exact and the sequence $0 \to A \to B \to M \to 0$ $\mathcal{C}$-pure. $\square$

Proposition 4.29. *The class $\{M \in \mathrm{Mod}{-}R \mid \mathrm{Tor}_1^R(M, \mathcal{C}^\top)\}$ is closed under direct sum and direct summands, direct limits, pure submodules, pure epimorphic images and arbitrary extensions.*

Proof. Omitted, comes directly from properties of the Tor_1^R functor. $\square$

Proposition 4.30. *Let F be a right module such that $\mathrm{Tor}_1^R(F, \mathcal{C}^\top) = 0$ and it is $\mathcal{C}^\top$-flat. Then F is flat.*

Proof. Let F be one such module. Let $0 \to A \to B \to M \to 0$ be an arbitrary exact sequence ending in F. According to the Lemma 4.28 it is already $\mathcal{C}$-pure. Then however by Theorem 4.20, condition (ii) it is already pure. So, by the Corollary 4.21 module F is flat.

Alternatively we could use the condition (ii) of previous lemma and the condition (iii) of Theorem 4.20 which together compose into an analogue of (iii) for usual flatness. $\square$

5. Ad-properties and local properties

The main proof in this thesis is concerned with descent of module properties through certain types of ring homomorphisms. This chapter recalls the definitions of ascent and descent for module properties and generalizes them for the case of relative properties.

Let us recall that given a ring homomorphism $\varphi : R \to S$, the ring S is automatically imbued with a (R, R)-bimodule structure and becomes an algebra.

Recall further that there is a standard construction of making R-modules into S-modules via the tensor product functors

$$- \otimes_R S : \mathrm{Mod}{-}R \to \mathrm{Mod}{-}S$$

and

$$S \otimes_R - : R{-}\mathrm{Mod} \to S{-}\mathrm{Mod}$$

5.1 Ascending and descending properties

Let us denote a general property of modules as $\mathfrak{P}$, then we denote the class of right modules over a ring R as $\mathfrak{P}(\mathrm{Mod}{-}R)$.

If $\mathfrak{P}$ is a property relative to a subcategory of modules, let us denote the class of right R-modules possessing the property $\mathfrak{P}$ with respect to a class $\mathcal{C} \subseteq \mathrm{Mod}{-}R$ as $\mathfrak{P}_{\mathcal{C}}(\mathrm{Mod}{-}R)$.

Definition 5.1. Let $\varphi : R \to S$ be a ring homomorphism. We say that the property $\mathfrak{P}$ ascends through the homomorphism φ as a right module property, or that φ ascends $\mathfrak{P}$ from the right, if for any right R-module M, whenever $M \in \mathfrak{P}(\mathrm{Mod}{-}R)$ then also $M \otimes_R S \in \mathfrak{P}(\mathrm{Mod}{-}S)$.

We say that the property $\mathfrak{P}$ descends through φ as a right module property, or that φ descends $\mathfrak{P}$ from the right, if for any R-module M whenever $M \otimes_R S \in \mathfrak{P}(\mathrm{Mod}{-}S)$ than also $M \in \mathfrak{P}(\mathrm{Mod}{-}R)$.

We define ascending and descending as a left module property or ascending and descending a property from the left similarly.

Certain provisions need to be made in order to define ascent and descent for relative properties, namely we need to interpret the subcategories of finitely presented modules in their definitions over different rings. To this end we introduce the following natural definition.

Definition 5.2. Let $R \to S$ be a ring homomorphism, take $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$. Then we denote $\mathcal{C}_S = \mathrm{add}(\{C \otimes_R S \mid C \in \mathcal{C}\}) \in \mathfrak{A}(\mathrm{mod}{-}S)$.

Similarly, for $\mathcal{C} \in \mathfrak{A}(R{-}\mathrm{mod})$ we put $\mathcal{C}_S = \mathrm{add}(\{S \otimes_R C \mid C \in \mathcal{C}\}) \in \mathfrak{A}(S{-}\mathrm{mod})$.

Proposition 5.3. *The map $\mathfrak{A}(\mathrm{mod}{-}R) \to \mathfrak{A}(\mathrm{mod}{-}S) : \mathcal{C} \mapsto \mathcal{C}_S$ is monotonous. Furthermore, $(\mathcal{P}_0^{<\omega}(R))_S = \mathcal{P}_0^{<\omega}(S)$, that is it takes the class of finitely presented projective modules over R to the class of finitely presented projective modules over*

S. Finally, it commutes with taking the Auslander-Bridger transpose, that is for any $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$ there is an identity $(\mathcal{C}_S)^\top = (\mathcal{C}^\top)_S \in \mathfrak{A}(S-\text{mod})$.

Proof. Monotonicity is immediate from the construction. Class $(\mathcal{P}_0^{<\omega}(R))_S$ contains $S = R \otimes_R S$ and as it is closed under finite direct sums and direct summands, it also contains all the finitely presented projective S-modules. Now, for any finitely presented module P, the module $P \otimes_R S$ is a finitely presented projective module, as P is a direct summand in some R^n and so $P \otimes_R S$ is a direct summand in S^n.

Let now $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. To show that $(\mathcal{C}_S)^\top = (\mathcal{C}^\top)_S$ we prove two inclusions. For the inclusion $(\mathcal{C}^\top)_S \subseteq (\mathcal{C}_S)^\top$ it is enough to check that each module of form $S \otimes_R C$ for some $C \in \mathcal{C}^\top$ in fact is contained in $(\mathcal{C}_S)^\top$. For such module C there is a transpose $C' \in \mathcal{C}$. It is easily confirmed, that then also $S \otimes_R C$ and $Q' \otimes_R S$ are each other's transposes, as for a presentation $p : R^n \to R^m$ we have $(\text{id}_S \otimes_R p^\top) = (p \otimes_R \text{id}_S)^\top$.

To prove the inclusion $(\mathcal{C}_S)^\top \subseteq (\mathcal{C}^\top)_S$, take a module $Q \in (\mathcal{C}_S)^\top$. Let $Q' \in \mathcal{C}_S$ be a transpose of Q. Then Q' is a direct summand of module $C \otimes_R S$ for some $C \in \mathcal{C}$. By construction similar to that in Lemma 3.4 we construct a module Q'', which is a transpose of $(C \otimes_R S)$ and Q is a direct summand in Q''. Next we take a module C', which is a transpose of C. Then, similarly to previous part, $S \otimes_R C'$ is a transpose of $C \otimes_R S$, and clearly $S \otimes_R C' \in (\mathcal{C}^\top)_S$. By Lemma 3.5 there are finitely presented projective left S-modules P_1 and P_2, such that $(S \otimes_R C') \oplus P_1 \cong Q'' \oplus P_2$. Class $(\mathcal{C}^\top)_S$ contains all finitely presented projective left S-modules and is close under finite direct sums and direct summands, so $Q \in (\mathcal{C}^\top)_S$. $\square$

Now we can define the ascent and descent of properties relative to a class $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$.

Definition 5.4. Let $\varphi : R \to S$ be a ring homomorphism and fix $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$.

We say that the relative property $\mathfrak{P}$ with respect to subcategory $\mathcal{C}$ ascends through the homomorphism φ as a right module property, or that φ ascends $\mathfrak{P}$ from the right, if for any right R-module M, whenever $M \in \mathfrak{P}_{\mathcal{C}}(\text{Mod}-R)$ then $M \otimes_R S \in \mathfrak{P}_{\mathcal{C}_S}(\text{Mod}-S)$.

We say that the relative property $\mathfrak{P}$ with respect to subcategory $\mathcal{C}$ descends through φ as a right module property, or that φ descends $\mathfrak{P}$ from the right, if for any R-module M whenever $M \otimes_R S \in \mathfrak{P}_{\mathcal{C}_S}(\text{Mod}-S)$ than also $M \in \mathfrak{P}_{\mathcal{C}}(\text{Mod}-R)$.

If for all $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$ the relative property $\mathfrak{P}$ with respect to $\mathcal{C}$ ascends or descends through φ, we say in short that the relative property $\mathfrak{P}$ respectively descends or ascends through φ.

Similarly we define ascending and descending as a left module property or ascending and descending a property from the left.

We will be using formulation like '$\mathcal{C}$-projectivity descends through the homomorphism $R \to S$ (as a right module property)' or 'The morphism $R \to S$ ascends relative projectivity from the left'.

Definition 5.5. Let $\mathcal{R}$ be a subcategory of rings and let $\mathfrak{P}$ be a module property. If $\mathfrak{P}$ ascends and descends through all morphisms in $\mathcal{R}$ as a right (left) module property, we say that $\mathfrak{P}$ is a right (left) ad-property in $\mathcal{R}$. If $\mathfrak{P}$ is both right and left ad-property in $\mathcal{R}$, we simply say it is an as-property in $\mathcal{R}$.

Remark. If $\mathcal{R}$ is taken as a subcategory of commutative rings and faithfully flat ring homomorphism, the previous definitions amount to the usual commutative notions of ad-properties.

The definitions of ascent and descent of relative properties leave a question open, whether different relative versions of flatness or projectivity over the domain even ascend into different versions of it over the target ring. What we wish to know is whether the map $\mathfrak{A}(\text{mod}-R) \to \mathfrak{A}(\text{mod}-S) : \mathcal{C} \mapsto \mathcal{C}_S$ is injective. Notably, this isn't generally true, even for pure ring homomorphisms.

Example. For an example without requiring purity, take flat homomorphism $\mathbb{Z} \to \mathbb{Z}_{(p)}$ for a prime p. Taking $\mathcal{C} = \text{add}(\{\mathbb{Z}_p, \mathbb{Z}\})$, we get $\mathcal{C}_{\mathbb{Z}_{(p)}} = \mathcal{P}_0^{<\omega}(\mathbb{Z}_{(p)})$, because $\mathbb{Z}_p \otimes_R \mathbb{Z}_{(p)} = 0$. While flatness will later play a role in our proofs, this example shows that some properties of purity are also needed. In fact, whenever $R \to S$ is such that there is some finitely presented module M such that $M \otimes_R S = 0$

Example with both purity and flatness is given in it own section, as it is a rather important non-example for several parts of this thesis.

Clearly, if for some distinct classes $\mathcal{C}, \mathcal{C}' \in \mathfrak{A}(\text{mod}-R)$ classes $\mathcal{C}_S$ and $\mathcal{C}'_S$ are the same, then descent of relative properties becomes nonsensical.

Later we show, that the map $\mathcal{C} \mapsto \mathcal{C}_S$ is injective when $R \to S$ is a faithfully flat homomorphism of commutative rings (Corollary 6.17) or a faithfully flat homomorphism which recognizes pure epimorphisms (Corollary 7.7). The proofs in the main body are in a sense "putting the cart before the horse", as injectivity of $\mathcal{C} \mapsto \mathcal{C}_S$ will in fact follow from descend for relative flatness.

5.2 An important non-example

This is an example of a ring homomorphism, which is both pure and flat and in fact even splits, but no version of flatness or projectivity descends through it.

Let k be a field and let A be a finite dimensional k-algebra. Consider the ring $\text{Hom}_k(A, A)$ and give it a structure of A-algebra via homomorphism $\varphi : A \to \text{Hom}_k(A, A), a \mapsto a \cdot (-)$. Now, $\text{Hom}_k(A, A) \cong A^n : f \mapsto (f(e_i))_{i=1}^n$, where $n = \dim_k A$ and $(e_i)_{i=1}^n$ is a k-basis of A is an isomorphism of left A-modules, meaning φ is both pure and flat at least from one side (for proof cf Lemmas 6.4 and 6.5). As $\text{Hom}_k(A, A)$ is really a matrix algebra over field k, it is in fact semisimple and all modules over it are projective. Picking A not-semisimple, we acquire an example of a faithfully flat ring homomorphism, which does not descend projectivity, flatness or any of its relative versions.

One such choice is $A = k[x]/(x^2)$, over which the cyclic module $k[x]/(x)$ is not projective. Here the homomorphism $A \to \text{Hom}_k(A, A)$ is even split as both left and right A-module homomorphism. The homomorphism is explicitly defined by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

viewing $\text{Hom}_k(A, A)$ as the algebra of 2×2 matrices over k. The section as a left A-module homomorphism ψ_L and the section as a right A-module homomorphism ψ_R are defined as

$$\psi_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx, \; \psi_R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = c + ax$$

This example hints, that not even both sided faithful flatness is enough for descent of projectivity. Importantly, this homomorphism isn't central.

5.3 Local properties

The title of this thesis talks about local properties of modules. This name arises from a geometric setting, where an ad property of modules makes for a sensible property of sheaves over schemes or of modules over topological rings. Ring homomorphisms arising from this setting are faithfully flat, motivating examination of faithfully flat descent.

Example. Let R be commutative ring. Let $(f_i)_{i=1}^n$ be a set of its elements which generate the regular module R. Let $R_f = R[f^{-1}]$ be a localization of R in the element f. Then the homomorphism $R \to \prod_{i=1}^n R_{f_i}$ is faithfully flat. This homomorphism is called *Zariski covering* and correspond to an open set covering of the spectrum of R. Properties of modules over rings R_f correspond to properties of sheaves over their spectra. Ascent of properties can be interpreted as open subsets of spectrum inheriting some properties of the whole space, descend through the homomorphism $R \to \prod_{i=1}^n R_{f_i}$ corresponds to a situation, when a property holding for each set of an open covering is also possessed by the whole space. This motivates the name *local*.

Example. The homomorphism $\mathbb{Z} \to \hat{\mathbb{Z}}_p$ of p-adic completion is a faithfully flat homomorphism, corresponding to completion of the ring with respect to some topology. Ascend here means that a property satisfied on a dense subspace extends to the entire space. Descent correspond to a property restricting.

6. Morphisms of interest

In the case of commutative rings, it is pure and faithfully flat homomorphisms that provide setting for descent of various properties. Different parts of the proof then use various facets of their purity. In the non-commutative setting the picture becomes more complicated, as the key property of reflecting pure epimorphisms no longer follows from mere purity of the ring homomorphism. In this chapter various sides of pure ring homomorphisms are recounted and relationships between them are explored. At the end we discuss, how the setting of commutative rings makes everything simple.

6.1 Purity of non-commutative ring homomorphisms

In the following we fix a homomorphism $\varphi : R \to S$ of not necessarily commutative rings.

Definition 6.1. We say that φ

(a) is *left (right) pure*, if it is a pure monomorphism of left (right) R-modules,

(b) *reflects monomorphisms from the right (left)*, if whenever $\nu : A \to B$ is a map of right (left) R-modules such that $\nu \otimes_R \mathrm{id}_S$ ($\mathrm{id}_S \otimes_R \nu$) is a monomorphism of S-modules, then also ν is a monomorphism,

(c) *reflects epimorphisms from the right (left)*, if whenever $\pi : A \to B$ is a map of right (left) R-modules such that $\pi \otimes_R \mathrm{id}_S$ ($\mathrm{id}_S \otimes_R \pi$) is an epimorphism of S-modules, then also ν is an epimorphism,

(d) is *right (left) object faithful*, if whenever $M \otimes_R S = 0$ ($S \otimes_R M = 0$) for a right (left) R-module M, then also $M = 0$,

(e) is *right (left) faithful*, if whenever $A \to B \to C$ is such sequence of right (left) R-modules, that $0 \to A \otimes_R S \to B \otimes_R S \to C \otimes_R S \to 0$ (respectively $0 \to S \otimes_R A \to S \otimes_R B \to S \otimes_R C \to 0$) is an exact sequence of S modules, then also the sequence $0 \to A \to B \to C \to 0$ is exact,

(f) *reflects pure monomorphisms from the right (left)*, if whenever $\nu : A \to B$ is a map of right (left) R-modules such that $\nu \otimes_R \mathrm{id}_S$ ($\mathrm{id}_S \otimes_R \nu$) is a pure monomorphism of S-modules, then also ν is a pure monomorphism,

(g) *reflects pure epimorphisms from the right (left)*, if whenever $\pi : A \to B$ is a map of right (left) R-modules such that $\pi \otimes_R \mathrm{id}_S$ ($\mathrm{id}_S \otimes_R \pi$) is a pure epimorphism of S-modules, then also ν is a pure epimorphism,

(h) *left (right) flat*, if S is flat as a left (right) R-module.

(i) *right (left) faithfully-flat*, if it is right faithful and left flat (respectively vice versa)

The following set of lemmas draws connections between these properties.

Lemma 6.2. *The homomorphism φ is left pure if and only if it reflects monomorphisms from the right.*

Proof. For the only if direction let φ be left pure and let $A \to B$ be right R-module homomorphism, such that $A \otimes_R S \to B \otimes_R S$ is a monomorphism of S-modules. Tensoring $A \to B$ with φ from the right produces the following commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ A \otimes_R S \arrow[r, hook] & B \otimes_R S \end{tikzcd} .$$

In it the bottom composition is injective, therefore the top map $A \to B$ must be a monomorphism.

For the if direction we take an arbitrary right R-module M. Tensoring φ by M from the left and by S from the right produces homomorphism $M \otimes_R R \otimes_R S \to M \otimes_R S \otimes_R S$, which is a split monomorphism, admitting a section $m \otimes s \otimes s' \mapsto m \otimes 1 \otimes ss'$. As φ reflects monomorphisms, $M \otimes_R R \to M \otimes_R S$ is a monomorphism and $R \to S$ is left pure. $\square$

Lemma 6.3. *The homomorphism φ is right object faithful if and only if it reflects epimorphisms from the right.*

Proof. For the if direction let φ reflect epimorphisms from the right, and let M be such that $M \otimes_R S = 0$. Then the zero map $0 \to M$ produces after tensoring a map $0 \to 0$ which is epimorphic. So $0 \to M$ is itself epimorphic rendering M zero.

For the only if part let φ be object faithful and let $\pi : A \to B$ be such that $\pi \otimes_R \mathrm{id}_S$ is an epimorphisms. Take cokernel $Q = \mathrm{Coker}(\pi)$. Then tensoring it with S produces $0 = \mathrm{Coker}(\pi \otimes_R \mathrm{id}_S) = Q \otimes_R S$. As φ is object faithful, also $Q = 0$ and π is epimorphic. $\square$

Lemma 6.4. *The homomorphism φ is right faithful if and only if it reflects both epimorphisms and monomorphisms from the right.*

Proof. For the only if direction, let φ be right faithful. Let $\nu : A \to B$ be an R-module homomorphism such that $\nu \otimes_R \mathrm{id}_S$ is a monomorphism. Let us take cokernel

$$A \to B \to Q \to 0,$$

then tensoring produces exact sequence

$$0 \to A \otimes_R S \to B \otimes_R S \to Q \otimes_R S \to 0.$$

By faithfulness of φ $0 \to A \to B \to Q \to 0$ is also exact, rendering $A \to B$ a monomorphism.

To show that φ reflects epimorphisms, we show that it is object faithful. Let M be such a right R-module, that $R \otimes_R S = 0$. Then the sequence $0 \to M \to 0$

when tensored with S produces an exact sequence $0 \to M \otimes_R S \to 0$ and hence is itself exact. So, $M = 0$.

For the if direction, let $A \to B \to C$ be a sequence of right R modules, such that

$$0 \to A \otimes_R S \to B \otimes_R S \to C \otimes_R S \to 0$$

is an exact sequence. As φ reflects both mono- and epimorphisms, $A \to B$ is already a monomorphism and $B \to C$ is an epimorphism. It remains to show, that the sequence is exact in the center. First let us show, that it is at least a complex. As φ is left pure, according to Lemma 6.2, tensoring φ with the sequence $A \to B \to C$ produces diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "h"] & B \arrow[r, "g"] \arrow[d, hook] & C \arrow[d, hook, "i"] \\ A \otimes_R S \arrow[r, "j"] & B \otimes_R S \arrow[r, "k"] & C \otimes_R S \end{tikzcd},$$

in which we seek to show, that $gf = 0$. For any $a \in A$ we have $igf(a) = kjh(a) = 0$ and as i is a monomorphism, $gf(a) = 0$, so $gf = 0$.

We'll show now that in fact $C \cong \mathrm{Coker}(A \to B)$. Take a cokernel Q of $A \to B$. The map $B \to C$ factorizes through it, we obtain diagram

$$\begin{tikzcd} & & & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[ur] & C \arrow[r] & 0 \end{tikzcd}.$$

Tensoring it with S produces

$$\begin{tikzcd} & & & Q \otimes_R S \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & A \otimes_R S \arrow[r] & B \otimes_R S \arrow[r] \arrow[ur] & C \otimes_R S \arrow[r] & 0 \end{tikzcd},$$

where $Q \otimes_R S \to C \otimes_R S$ is an isomorphism from the universal property of cokernel. As φ reflects both monomorphisms and epimorphisms, the homomorphism $Q \to C$ must be a bijection an as such an isomorphism. Thus, C is in fact cokernel of $A \to B$ and the sequence

$$0 \to A \to B \to C \to 0$$

is exact. $\square$

Lemma 6.5. *If the homomorphism φ reflects monomorphisms from the right, then it is right object faithful. If φ is left flat, then also if it is right object faithful, it reflects monomorphisms.*

Proof. Let φ reflect monomorphisms and let M be a right R-module such that $M \otimes_R S = 0$. The zero map $M \to 0$, when tensored with S becomes a monomorphism $0 \to 0$. It must therefore be itself a monomorphism, rendering $M = 0$.

Let now be φ left flat and right object faithful. Let $A \to B$ be a homomorphism of right R-modules such that $A \otimes_R S \to B \otimes_R S$ is a monomorphism of right S-modules. Consider kernel K of the map $A \to B$. As S is flat, this means that $K \otimes_R S = \operatorname{Ker}(A \otimes_R S \to B \otimes_R S) = 0$. As φ is object faithful, also $K = 0$ and $A \to B$ is monic. $\square$

Lemma 6.6. *If φ reflects pure monomorphisms from the right, then it is both left and right pure.*

Proof. Tensoring φ from the right with S produces split and hence also pure monomorphism $R \otimes_R S \to S \otimes_R S$ with a section $s \otimes s' \mapsto 1 \otimes ss'$. As φ reflect pure monomorphisms from the right, it must be a right pure ring homomorphism.

To see that φ is left pure, we utilize again the proof of Lemma 6.2, this time noting that the monomorphism $M \otimes_R R \otimes_R S \to M \otimes_R S \otimes_R S$ is pure. $\square$

Lemma 6.7. *If φ reflects pure epimorphisms from the right, then it is right object faithful and reflects epimorphisms from the right.*

Proof. We will prove that φ is right object faithful. Let M be a right R-module such that $M \otimes_R S = 0$. Consider a presentation $R^{(J)} \to R^{(I)} \to M \to 0$. Tensoring it with S produces $S^{(J)} \to S^{(I)} \to 0$ with the map $S^{(J)} \to S^{(I)}$ being an epimorphisms ending in a flat module and therefore a pure epimorphism (cf Corollary 4.21). Then however $R^{(J)} \to R^{(I)}$ must also be a pure epimorphism and its cokernel M is zero. $\square$

Remark. Reflection of pure epimorphisms is stronger than reflection of epimorphisms. Ring homomorphism which reflects epimorphisms might not pick the pure among them.

Lemma 6.8. *If φ reflects pure epimorphisms from the right, then it also reflects pure monomorphisms from the right.*

Proof. Let φ reflect pure epimorphisms. Let $\nu : A \to B$ be such that $\nu \otimes_R \mathrm{id}_S$ is a pure monomorphism. Now we take cokernel

$$A \to B \to Q \to 0.$$

Tensoring with S produces pure cokernel

$$0 \to A \otimes_R S \xrightarrow{*} B \otimes_R S \xrightarrow{*} Q \otimes_R S \to 0$$

and as φ reflects pure epimorphisms,

$$A \to B \xrightarrow{*} Q \to 0$$

is pure.

We will show, that A is in fact kernel of the pure epimorphism $B \to Q$. To this end, construct its kernel K. The homomorphism $A \to B$ factorizes through it. Let us also take the kernel L of the map $A \to K$.

$$\begin{tikzcd}
K \arrow[dr, "*"] & & & \\
A \arrow[u, two heads] \arrow[r] & B \arrow[r, "*"] & Q \arrow[r] & 0 \\
L \arrow[u, hook] & & &
\end{tikzcd}$$

Tensoring the whole diagram with S produces

$$\begin{tikzcd}
0 \arrow[r] & K \otimes_R S \arrow[dr, hook, "*"] & & & \\
0 \arrow[r] & A \otimes_R S \arrow[u, "\wr"'] \arrow[r] & B \otimes_R S \arrow[r, "*"] & Q \otimes_R S \arrow[r] & 0 \\
& L \otimes_R S \arrow[u] & & &
\end{tikzcd}$$

where $A \otimes_R S \to K \otimes_R S$ is an isomorphism, by the universal property of kernel. But that is also a pure epimorphism, so $A \to K$ is pure. Thus, $L \to A$ is a pure monomorphism, being a kernel of pure epimorphism. But then $L \otimes_R S = \mathrm{Ker}(A \otimes_R S \to L \otimes_R S) = 0$ and as φ is also object faithful (Lemma 6.7), $L = 0$, and $A \to K$ is an isomorphism. So, A is in fact the kernel of $B \to Q$ and $A \to B$ is a pure monomorphism. $\square$

Lemma 6.9. *If φ is left flat and reflects pure monomorphisms from the right, then it also reflects pure epimorphisms from the right.*

Proof. Let $A \to B$ be a right R-module homomorphism such that $A \otimes_R S \to B \otimes_R S$ is a pure epimorphism. Let K be kernel of $A \to B$. As S is flat

$$0 \to K \otimes_R S \to A \otimes_R S \to B \otimes_R S \to 0$$

is an exact sequence of right S-modules, as $A \otimes_R S \to B \otimes_R S$ is pure it is pure exact. By combination of previous Lemmas φ is faithful, so the sequence

$$0 \to K \to A \to B \to 0$$

is exact. As φ reflects pure monomorphisms, it is pure exact and $A \to B$ is a pure epimorphism. $\square$

From previous lemmas it follows, that the strongest considered property of pure ring homomorphisms is the reflection of pure epimorphisms. This is the property which we will end up needing for descent of pure projectivity.

Following are several examples and non-examples.

Example. Homomorphism $k[x]/(x^2) \to \mathrm{End}_k(k[x]/(x^2))$ from section 5.2 is an example of both left and right faithfully flat homomorphism, which doesn't reflect pure epimorphisms from the right. If it did so, it would, as will be proven later, descend flatness.

Example. Any commutative ring homomorphisms which is split as a module homomorphism will be pure and also descend pure epimorphism. Homomorphism $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_p$ for any prime number p will be an example of such homomorphism, which is not flat.

Example. Ring homomorphism $\mathbb{Z} \to \prod_{p \in \mathbb{P}, n \in N} \mathbb{Z}_{p^n}$ is not flat, but it is a non-splitting example of a pure homomorphism of commutative rings. Such homomorphisms in fact possess all of the properties from Definition 6.1, except flatness. This shows that, in general, flatness of ring homomorphisms is not needed for descent.

More examples of non-commutative ring homomorphisms possessing these properties will be generated in Chapter 8.

For the descent of relative properties, preserving relative purity will be important. Unlike the usual purity, it is not preserved by arbitrary homomorphisms.

Definition 6.10. Take a subcategory $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$, then we say that φ

(a) *preserves $\mathcal{C}$-pure epimorphisms*, if whenever $\pi : A \to B$ is a $\mathcal{C}$-pure epimorphism of right R-modules, then $\pi \otimes_R \mathrm{id}_S$ is a $\mathcal{C}_S$-pure epimorphism of right S-modules,

(b) *preserves $\mathcal{C}$-pure monomorphisms*, if whenever $\nu : A \to B$ is a $\mathcal{C}$-pure monomorphism of right R-modules, then $\nu \otimes_R \mathrm{id}_S$ is a $\mathcal{C}_S$-pure monomorphism of right S-modules.

Lemma 6.11. *Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$. If φ preserves $\mathcal{C}$-pure monomorphisms, then it also preserves $\mathcal{C}$-pure epimorphisms. If it is flat, then also the other direction holds.*

Proof. Let first φ preserve $\mathcal{C}$-pure monomorphisms. Let $M \to N$ be a $\mathcal{C}$-pure epimorphism. Taking the kernel we get a $\mathcal{C}$-pure exact sequence

$$0 \to K \to M \to N \to 0.$$

As φ preserves $\mathcal{C}$-pure monomorphisms, the sequence

$$0 \to K \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$$

is $\mathcal{C}_S$-pure exact, specially $M \otimes_R S \to N \otimes_R S$ is a $\mathcal{C}_S$-pure epimorphism.

Let now φ be flat and preserve $\mathcal{C}$-pure epimorphisms. Let $M \to N$ be a $\mathcal{C}$-pure monomorphism. Taking the cokernel we get a $\mathcal{C}$-pure exact sequence

$$0 \to M \to N \to Q \to 0.$$

As φ is flat and preserves $\mathcal{C}$-pure epimorphisms, the sequence

$$0 \to M \otimes_R S \to N \otimes_R S \to Q \otimes_R S \to 0$$

is exact, epimorphism $N \otimes_R S \to Q \otimes_R S$ is $\mathcal{C}$-pure and the sequence is $\mathcal{C}$-pure exact. Specially, monomorphism $M \otimes_R S \to N \otimes_R S$ is $\mathcal{C}$-pure. $\square$

When relating the purity over different rings, we can check that the forgetful functor in fact preserves purity. For the usual purity it is an easy tensor product computation, this however holds for relative purity as well.

Lemma 6.12. *Let $R \to S$ be an arbitrary ring homomorphism and let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$. Let $M \to N$ be a $\mathcal{C}_S$-pure epimorphism (monomorphism) of right S-modules. Then it is also $\mathcal{C}$-pure as a homomorphism of R-modules.*

Proof. Let $\pi : M \to N$ be a $\mathcal{C}_S$-pure epimorphism of right S-modules. Take $Q \in \mathcal{C}$ and an R-module homomorphism $f : Q \to N$. By properties of functor $- \otimes_R S$, this homomorphism factorizes uniquely through R-module homomorphism $Q \to Q \otimes_R S$ and S-module homomorphism $f' : Q \otimes_R S \to N$. As $Q \otimes_R S \in \mathcal{C}_S$, there is a homomorphism $g' : Q \otimes_R S \to M$ such that $\pi g' = f'$. Then $\pi g' f'' = f' f'' = f$, so the R-module homomorphism $g' f''$ witnesses that $M \to N$ is a $\mathcal{C}$-pure epimorphism. If $M \to N$ is a $\mathcal{C}_S$-pure monomorphism, then its cokernel is a $\mathcal{C}$-pure epimorphisms, making the monomorphism $M \to N$ also $\mathcal{C}$-pure. $\square$

6.2 The case of pure and flat homomorphisms of commutative rings

In the classical setting of pure homomorphisms of commutative rings (though in fact it is enough for the domain to be commutative, as long as the homomorphism is central) the entirety of previous section collapses into a simple case, as pure homomorphisms of commutative rings posses all the properties of Definition 6.1 apart from being flat. It is enough to show that they reflect pure epimorphisms. The following proof is an adaptation of Proposition 2.3 in [Mesablishvili, 2002].

Lemma 6.13. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R*-modules, then* $0 \to A \to B \to C \to 0$ *is pure exact if and only if the sequence* $0 \to C^* \to B^* \to A^* \to 0$ *splits.*

Proof. Omitted, [Göbel and Trlifaj, 2006, Lemma 2.19]. □

Proposition 6.14. *Let* $\varphi : R \to S$ *be a pure homomorphism of commutative rings. Let* $\pi : M \to N$ *be such homomorphism of* R*-modules, that* $\pi \otimes_R S : M \otimes_R S \to N \otimes_R S$ *is a pure epimorphism of (right)* S*-modules. Then* π *is a pure epimorphism.*

Proof. First, $R \to S$ is object faithful, so π is an epimorphism. We will determine that it is pure by showing, that $\pi^* : N^* \to M^*$ splits. First, as $\varphi : R \to S$ is a pure monomorphism, the epimorphism $\varphi^* : S^* \to R^*$ splits. Let us fix a section $\tau : R^* \to S^*$. Now for any module M we denote $\tau_M = \mathrm{Hom}_R(M, \tau) : \mathrm{Hom}_R(M, R^*) \to \mathrm{Hom}_R(M, S^*)$. By tensor-hom adjunction and the fact that R and S are commutative this is in fact a homomorphism of R-modules

$$\tau_M : M^* \to (M \otimes_R S)^*.$$

Furthermore, it constitutes a section of epimorphism $\sigma_M = (\mathrm{id}_M \otimes_R \varphi)^* \cong \mathrm{Hom}_R(M, \varphi^*) : (M \otimes_R S)^* \to M^*$, as we have

$$\mathrm{Hom}_R(M, \varphi^*) \circ \mathrm{Hom}_R(M, \tau) = \mathrm{Hom}_R(M, \varphi^* \circ \tau) = \mathrm{Hom}_R(M, \mathrm{id}_{R^*}) = \mathrm{id}_{M^*}$$

Now, as $\pi \otimes_R \mathrm{id}_S : M \otimes_R S \to N \otimes_R S$ is a pure epimorphism of S-modules, the monomorphism $(\pi \otimes_R \mathrm{id}_S)^* : (N \otimes_R S)^* \to (M \otimes_R S)^*$ admits a section ψ. The square

$$\begin{tikzcd}
N^* \arrow[r, "\pi^*"] \arrow[d, "\tau_N"] & M^* \arrow[d, "\tau_M"] \\
(N \otimes_R S)^* \arrow[r, "(\pi \otimes_R \mathrm{id}_S)^*"] & (M \otimes_R S)^*
\end{tikzcd}$$

commutes, as maps τ_M are constructed functorially. Finally, the map $\sigma_N \psi \tau_M$ is a section of π^*, as

$$\sigma_N \psi \tau_M \pi^* = \sigma_N \psi (\pi \otimes_R \mathrm{id}_S)^* \tau_N = \mathrm{id}_{N^*}.$$

Hence, π is a pure epimorphism. □

The setting of commutative rings also simplifies the situation for relative properties

Proposition 6.15. *Let $R \to S$ be a flat homomorphism of commutative rings and $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. Then $R \to S$ preserves $\mathcal{C}$-pure monomorphisms and epimorphisms.*

Proof. Let $M \to N$ be a $\mathcal{C}$-pure monomorphism of R-modules. We want to check, that $M \otimes_R S \to N \otimes_R S$ is $\mathcal{C}_S$-pure. Take a module $Q \in \mathcal{C}^\top$. Then $M \otimes_R Q \to N \otimes_R Q$ is a monomorphism. As S is a flat R-module, also $M \otimes_R Q \otimes_R S \to N \otimes_R Q \otimes_R S$ is a monomorphism and thanks to natural isomorphisms

$$- \otimes_R Q \otimes_R S \cong - \otimes_R S \otimes_R Q \cong (- \otimes_R S) \otimes_S (S \otimes_R Q)$$

also $(M \otimes_R S) \otimes_S (S \otimes_R Q) \to (N \otimes_R S) \otimes_S (S \otimes_R Q)$ is a monomorphism. This is enough to check, that the monomorphism $M \otimes_R S \to N \otimes_R S$ is $\mathcal{C}_S$-pure. Pure epimorphisms are then preserved by Lemma 6.11. □

This lets us give a first minor result on descent.

Proposition 6.16. *Let $R \to S$ be a faithfully flat homomorphism of commutative rings. Then relative flatness descends through $R \to S$.*

Proof. Take $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$ and let M be an R-module such that $S \otimes_R M$ is $\mathcal{C}_S$-flat. Let $A \to B$ be a $\mathcal{C}$-pure monomorphism. Then $A \otimes_R S \to B \otimes_R S$ is a $\mathcal{C}_S$ pure monomorphism by Proposition 6.15. Therefore, $(A \otimes_R S) \otimes_S (S \otimes_R M) \to (B \otimes_R S) \otimes_S (S \otimes_R M)$ is a monomorphism. By the same chain of natural isomorphisms used in the previous proof it is isomorphic to monomorphism $(A \otimes_R M) \otimes_R S \to (B \otimes_R M) \otimes_R S$ and as $R \to S$ is faithful, the homomorphism $A \otimes_R M \to B \otimes_R M$ is a monomorphism. So, M is a $\mathcal{C}$-flat module. □

This shows, that in the case of faithfully flat homomorphisms of commutative rings, descent of relative properties is sensible.

Corollary 6.17. *Let $R \to S$ be faithfully flat homomorphism of commutative rings. Then the map $\mathcal{C} \mapsto \mathcal{C}_S$ is injective.*

Proof. Let us take $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. We need to show, that for any finitely presented R-module M such that $M \otimes_R S \in \mathcal{C}_S$, already $M \in \mathcal{C}$. But clearly $M \otimes_R S \in \mathcal{F}_{\mathcal{C}_S{}^\top}$, by Proposition 6.16 $M \in \mathcal{F}_{\mathcal{C}^\top}$ and, as M is finitely presented, by Proposition 4.19 already $M \in \mathcal{C}$. □

Remark. This proof will be almost verbatim repeated in the next section, when we get different criteria for descent of relative flatness in a non-commutative setting.

Remark. For wider generality, in this section it suffices for R to be commutative and for S to be a central algebra. Important property coming from commutativity is that all R-modules posses (R, R)-bimodule structure (from commutativity) and that the left and right module structure on S is the same (centrality).

7. Ascent and descent of pertinent properties

This chapter shows proofs of ascent and descent of properties discussed and defined in previous parts of the text. Importantly, the proofs are done over not necessarily commutative rings, trying to pose as little hypotheses on the morphisms as possible in each case. In the next chapter some special non-commutative morphisms will be proven to possess the relevant properties.

7.1 Ascent

It is quite easy to show that all discussed properties ascend, in fact through arbitrary ring homomorphisms. This follows from the fact, that each of discussed properties admits some characterization by a construction commuting with the tensor product.

Proposition 7.1. *Let $R \to S$ be a ring homomorphism, let M be a right R-module, and let us fix $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$. Then if M is*

(A) *$< \kappa$-generated*

(B) *$< \kappa$-presented*

(C) *a direct sum of countably generated submodules*

(D) *Mittag-Leffler*

(E) *$\mathcal{C}^\top$-flat*

(F) *$\mathcal{C}$-projective*

(G) *flat*

(H) *projective*

(I) *pure projective*

then so is $M \otimes_R S$ as an S-module (relative to subcategory $\mathcal{C}_S$, where applicable).

Proof. In parts (A), (B) it is enough to take a tensor of a free precover or presentation. Part (C) follows from (A) (with $\kappa = \omega^+$) and the fact, that tensor product commutes with direct sums.

For part (D) we express the module M as direct limit of a Mittag-Leffler system $(M_i, f_{ji})_{i \leq j \in I}$. As tensor product commutes with taking direct limits, $M \otimes_R S$ is the limit of system $(M_i \otimes_R S, f_{ji} \otimes_R \mathrm{id}_S)_{i \leq j \in I}$, which can be shown to be Mittag-Leffler as it clearly satisfies condition (ii) of Theorem 2.3.

For part (E) we express M as direct limit of a system $(C_i, f_{i,j})_{i \leq j \in I}$ where $C_i \in \mathcal{C}$. Then $M \otimes_R S$ is the limit of system $(C_i \otimes_R S, f_{ji} \otimes_R \mathrm{id}_S)_{i \leq j \in I}$ where, by definition, $C_i \otimes_R S \in \mathcal{C}_S$.

Part (F) follows similarly from expressing M as a direct summand in a direct sum $\bigoplus_{i\in I} C_i$ where $C_i \in \mathcal{C}$ and noting that, as tensor product commutes with direct sums, $M \otimes_R S$ is a direct summand in $\bigoplus_{i\in I}(C_i \otimes_R S)$.

Parts (G) and (H) follow from parts (E) and (F) respectively, taking $\mathcal{C} = \mathcal{P}_0^{<\omega}$.

Part (I) follows from (F) taking $\mathcal{C} = \text{mod}-R$. Note that in this case $\mathcal{C}_S$ projectivity is not the same as pure projectivity over S, but in fact more special. $\square$

7.2 Generatedness and presentedness

Descent of upper limits on amount of generators and relations plays important role in the proof of descent for projectivity and pure projectivity. Countable generation lies in the heart of Kaplansky decomposition. While a limit on generation descends already with an object faithful morphism, we use left purity for the presentedness.

Proposition 7.2. *Let $R \to S$ be a right object faithful ring homomorphism, let κ be an infinite cardinal and M be such a right R-module, that $M \otimes_R S$ is $< \kappa$-generated. Then M is $< \kappa$-generated.*

Proof. Take a set $\{\sum_{i=1}^{n_\alpha} m_{\alpha i} \otimes s_{\alpha i} \mid \alpha < \lambda\}$ of generators of $M \otimes_R S$ for some cardinal $\lambda < \kappa$. Then we construct map $p : R^{\left(\bigsqcup_{\alpha<\lambda} n_\alpha\right)} \to M$ given on the basis elements $e_{(\alpha,i)}$, $\alpha < \lambda$, $0 < i \leq n_\alpha$ as $p(e_{(\alpha,i)}) = m_{\alpha i}$. Then the map $p \otimes_R \mathrm{id}_S : S^{\left(\bigsqcup_{\alpha<\lambda} n_\alpha\right)} \to M \otimes_R S$ is onto, as for each $\alpha < \lambda$ we have

$$(p \otimes_R \mathrm{id}_S)\left(\sum_{i=1}^{n_\alpha} e_{(\alpha,i)} \otimes s_{\alpha i}\right) = \sum_{i=1}^{n_\alpha} m_{\alpha i} \otimes s_{\alpha i}.$$

Because $R \to S$ is object faithful and so reflects epimorphisms, map p is also onto. As $\lambda < \kappa$ and each of the n_α are finite, the disjoint union $\bigsqcup_{\alpha<\lambda} n_\alpha$ is of cardinality less than κ. So, map p witnesses that M is $< \kappa$ generated. $\square$

Proposition 7.3. *Let $R \to S$ be a left pure ring homomorphism, then if M is such right R-module that $M \otimes_R S$ is an at $< \kappa$-presented S-module, then also M is a $< \kappa$-presented R-module.*

Proof. By Proposition 7.2 M is $< \kappa$-generated, as $R \to S$ is object faithful. Consider some exact sequence

$$0 \to K \xrightarrow{\nu} R^{(\lambda)} \xrightarrow{\pi} M \to 0$$

for a cardinal $\lambda < \kappa$. Tensoring with S produces exact sequence

$$K \otimes_R S \to S^{(\lambda)} \xrightarrow{\pi \otimes_R \mathrm{id}_S} M \otimes_R S \to 0.$$

As $M \otimes_R S$ is $< \kappa$ presented, there is some cardinal $\mu < \kappa$ such that $\mathrm{Ker}(\pi \otimes_R \mathrm{id}_S)$ is μ generated and therefore there is a set of elements $\{k_\alpha \in K\}_{\alpha<\mu}$ such that elements $\{(\nu \otimes_R \mathrm{id}_S)(k_\alpha \otimes 1)\}$ generate $\mathrm{Ker}(\pi \otimes_R \mathrm{id}_S)$. We will show, that elements $\{k_\alpha \in K\}_{\alpha<\mu}$ generate the kernel K.

Let $r = (r_\beta)_{\beta<\lambda}$ be an element of $R^{(\mu)}$ such that $\pi(r) = 0$. Then $(\pi \otimes_R \mathrm{id}_S)(r \otimes 1) = 0$ and therefore there is an element $x = \sum_{i=1}^n (k_{\alpha_i} \otimes s_i) \in K \otimes_R S$ such that $\sum_{i=1}^n (\nu(k_{\alpha_i}) \otimes s_i) = (\nu \otimes_R \mathrm{id}_S)(x) = r \otimes 1$. As all of $\nu(k_{\alpha_i}) = (k_{i\beta})_{\beta<\lambda}$ are elements of $R^{(\lambda)} \subseteq S^{(\lambda)}$, we can rewrite this identity by components as a finite system of identities

$$\sum_{i=1}^n k_{i\beta_j} \cdot s_i = r_{\beta_j},\ \ j = 1 \dots m,$$

where $\{\beta_j\}_{1\le j\le m}$ are the indices at which at least one of $\nu(k_{\alpha_i})$ or r is nonzero. As $R \to S$ is left pure, there is a set $\{r'_i\}_{1\le i\le n}$ such that

$$\sum_{i=1}^n k_{i\beta_j} \cdot r'_i = r_{\beta_j},\ \ j = 1 \dots m.$$

Then however $\sum_{i=1}^n k_{\alpha_i} \cdot r'_i = r$. So, the kernel K is generated by the elements $\{k_\alpha \in K\}_{\alpha<\mu}$, and M is $< \kappa$ presented.

□

7.3 Flatness and relative flatness

Descent of flatness and its relative versions is, where the reflection of pure epimorphism first shines. It suffices for descent of flatness. Relative version requires that the ring homomorphism further preserves relatively pure epimorphisms.

Proposition 7.4. *Let $R \to S$ be a ring homomorphism reflecting pure epimorphisms from the right. Take $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$ and let $R \to S$ preserve $\mathcal{C}$-purity. Then $\mathcal{C}^\top$-flatness of right R-modules descends through $R \to S$.*

Proof. Let M be such right R-module, that $M \otimes_R S$ is a $\mathcal{C}_S^\top$-flat. Lemma 4.6 yields a $\mathcal{C}$-pure epimorphism $P \to M$ for some $\mathcal{C}$-projective module P. Tensoring this map with S produces a $\mathcal{C}_S$-pure epimorphism $P \otimes_R S \to M \otimes_R S$, with $M \otimes_R S$ being $\mathcal{C}_S$-flat. By Proposition 4.20 this epimorphism is already pure. By hypothesis on the homomorphism $R \to S$ the epimorphism $P \to M$ is pure and by Proposition 4.22 the module M is $\mathcal{C}^\top$-flat. □

This has immediate corollary for the usual flatness.

Corollary 7.5. *Flatness of right modules descends through ring homomorphisms reflecting pure epimorphisms from the right.*

Proof. Taking $\mathcal{C} = \mathcal{P}_0^{<\omega}$, this follows immediately. In this case $\mathcal{C}$-pure epimorphism reduces to just an epimorphism, which is preserved by any ring homomorphism (as tensor product is right exact). □

Thanks to the Proposition 6.14 we have a good result for pure homomorphisms of commutative rings.

Corollary 7.6. *Let R be a commutative ring, S a central R-algebra and let the corresponding ring homomorphism $R \to S$ be pure. Then flatness descends through $R \to S$.*

Remark. There are multiple proofs of pure descent for flatness, using commutativity of R in different ways. One computational is recounted in [Angermüller, 2015], the one in [Stacks project authors, 2022] is a version of the one used here, adapted also from [Mesablishvili, 2002].

Now we're finally able to name some non-commutative conditions, under which the relative properties ascend distinctly, and therefore descent for them makes sense.

Corollary 7.7. *Let $R \to S$ be ring homomorphism reflecting pure epimorphisms from the right, such that for any $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$, homomorphism $R \to S$ preserves $\mathcal{C}$-pure epimorphisms. Then the map $\mathcal{C} \mapsto \mathcal{C}_S$ is injective (and hence strictly monotonous).*

Proof. Let us take $\mathcal{C} \in \mathfrak{A}(\text{mod}-R)$. We need to show, that for any finitely presented R-module M such that $M \otimes_R S \in \mathcal{C}_S$, already $M \in \mathcal{C}$. But clearly $M \otimes_R S \in \mathcal{F}_{\mathcal{C}_S{}^\top}$, by Proposition 7.4 $M \in \mathcal{F}_{\mathcal{C}^\top}$ and, as M is finitely presented, by Proposition 4.19 already $M \in \mathcal{C}$. □

Remark. As any ring homomorphism reflecting pure epimorphisms from the right is already pure on both sides, we don't even need to take M finitely presented to begin with, as it will be finitely presented by Proposition 7.3.

Corollary 7.8. *Let R be a commutative ring and S a central R-algebra, let the corresponding ring homomorphism $R \to S$ be faithfully flat. Then the map $\mathfrak{A}(\text{mod}-R) \to \mathfrak{A}(\text{mod}-S) : \mathcal{C} \mapsto \mathcal{C}_S$ is strictly monotonous.*

7.4 Mittag-Leffler modules

Descent of Mittag-Leffler property another key part of descent of projectivity and pure projectivity. Here the reflection of pure epimorphisms can be weakened in two different ways. The next section will make use of the second one, however in the spirit of trying to use as little hypotheses as possible, the first version is also included.

The first weakening of this condition comes from only requiring reflection of pure monomorphisms, rather than pure epimorphisms.

Theorem 7.9. *Let $f : R \to S$ be a ring homomorphism reflecting pure monomorphisms. Then Mittag-Leffler property descends through f, that is if M is a right R-module such that $M \otimes_R S$ is a Mittag-Leffler module as a right S-module then also M is Mittag-Leffler as a right R-module.*

Proof. The ideas of this proof come from [Stacks project authors, 2022, Lemma 10.95.1], care has been taken to respect the non-commutative setting.

Write M as a direct limit of a directed system of finitely presented modules $(M_i, f_{ji})_{i \leq j \in I}$. We will prove that this system is a Mittag-Leffler direct system by verifying the condition (iv) of Theorem 2.3. Pick arbitrary $i \in I$. As tensor product commutes with taking colimits, we have $M \otimes_R S \cong \varinjlim (M_i \otimes_S S)$. Because

$M \otimes_R S$ is a Mittag-Leffler module the system $(M_i \otimes_R S, f_{ji} \otimes_R \mathrm{id}_S)_{i \leq j \in I}$ is a Mittag-Leffler system and for chosen i there exists $i \leq j \in I$ satisfying the condition 2.3 (iv). Tensoring the pushout diagram

$$\begin{tikzcd} M_i \arrow[r, "f_{ji}"] \arrow[d, "f_i"] & M_j \arrow[d, "h"] \\ M \arrow[r, "g"] & N \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

with S produces pushout diagram (as tensor commutes with pushouts)

$$\begin{tikzcd} M_i \otimes_R S \arrow[r, "f_{ji} \otimes_R \mathrm{id}_S"] \arrow[d, "f_i \otimes_R \mathrm{id}_S"] & M_j \otimes_R S \arrow[d, "h \otimes_R \mathrm{id}_S"] \\ M \otimes_R S \arrow[r, "g \otimes_R \mathrm{id}_S"] & N \otimes_R S \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Here by the choice of j the map $h \otimes_R \mathrm{id}_S$ is a pure monomorphism. Map f reflects pure monomorphisms, hence also h is a pure monomorphism. This verifies the condition and M is indeed a Mittag-Leffler module. □

The second way to weaken the hypothesis is to require descent for flatness and purity from the right. Again, a ring homomorphism which reflects pure epimorphisms possesses both of these properties.

In the *a priori* presence of flatness it is enough for the ring homomorphism to be right pure.

Theorem 7.10. *Let $R \to S$ be a right pure ring homomorphism. If M is a flat right R-module such that $M \otimes_R S$ is Mittag-Leffler as a right S-module, then M is Mittag-Leffler as a right R-module.*

Proof. The ideas of this proof are taken from [Angermüller, 2015, Lemma 5], changes are made for the non-commutative setting, especially in considering the particular type of purity.

We will check that M satisfies the condition (iii) of Theorem 2.5 Let $\{Q_k\}_{k \in K}$ be a system of left R-modules. In the diagram

$$\begin{tikzcd} M \otimes_R \prod_{k \in K} Q_k \arrow[r, hook] \arrow[d] & M \otimes_R \prod_{k \in K} (S \otimes_R Q_k) \arrow[r, "\sim"] & (M \otimes_R S) \otimes_S \prod_{k \in K} (S \otimes_R Q_k) \arrow[d, hook] \\ \prod_{k \in K} (M \otimes_R Q_k) \arrow[r, hook] & \prod_{k \in K} (M \otimes_R (S \otimes_R Q_k)) \arrow[r, "\sim"] & \prod_{k \in K} ((M \otimes_R S) \otimes_S (S \otimes_R Q_k)) \end{tikzcd}$$

homomorphisms $Q_k \to S \otimes_R Q_k$ are monomorphisms, as $R \to S$ is right pure, maps $M \otimes_R Q_k \to M \otimes_R S \otimes_R Q_k$ are monic because M is right flat, $\prod_{k \in K}(M \otimes_R Q_k) \to \prod_{k \in K}(M \otimes_R (S \otimes_R Q_k))$ is monic as a product of monomorphisms, and similarly $M \otimes_R \prod_{k \in K} Q_k \to M \otimes_R \prod_{k \in K}(S \otimes_R Q_k)$ is monic, as it is a product of monomorphisms tensored with a flat module. The isomorphisms arise from the fact that $S \otimes_R Q_k$ and $\prod_{k\in K} S \otimes_R Q_k$ are left S-modules.

Finally, the homomorphism

$$(M \otimes_R S) \otimes_S \prod_{k\in K} (S \otimes_R Q_k) \to \prod_{k\in K} ((M \otimes_R S) \otimes_S (S \otimes_R Q_k))$$

is monic, as $M \otimes_R S$ is Mittag-Leffler. This makes the composition along the top and right side of the diagram monic, and hence also the homomorphisms

$$M \otimes_R \prod_{k\in K} (Q_k) \to \prod_{k\in K} (M \otimes_R Q_k)$$

must be monic and the module M Mittag-Leffler.

$\square$

This immediately leads to following corollary, using condition weaker than reflection of pure epimorphisms.

Corollary 7.11. *Let $f : R \to S$ be a ring homomorphism, which is pure as a right R-module homomorphism and flatness of right modules descends through it. Then the property of being flat Mittag-Leffler module descends through f.*

7.5 Projectivity

With most of the parts of Drinfeld characterization of projectivity and pure projectivity descending, it remains to show, that also the Kaplansky decomposition descends, in presence of projectivity or pure projectivity. The following proof is done in two different, ways. Branch (A) deals with descent of projectivity and uses the weaker condition of purity (here on both sides) and descent of flatness. This is not an unexpected result, it is mentioned (though without proper proof and specification of whether left or right purity is needed) in [Osofsky, 1979, page 233].

Branch (B) proves descent of pure projectivity in the case, that the ring homomorphism reflects pure epimorphisms. Corollary of this is pure descent for pure projectivity (over commutative rings). This is again an expected result. As communicated orally to the advisor result on pure descent for pure projectivity will also appear in the independent work [Herbera et al.] by Herbera, Příhoda, and Wiegand.

The proof is done by devissage à la Kaplansky, each step will make use of the following lemma.

Lemma 7.12. *Let $R \to S$ be a ring homomorphism, let M be a right R-module such that $M \otimes_R S = \bigoplus_{i\in I} Q_i$ is a direct sum of countably generated right S-modules. Let $N \subseteq M$ be a countably generated submodule. Then there is a countably generated submodule N' such that $N \subseteq N' \subseteq M$ and $(\subseteq \otimes_R \mathrm{id}_S)(N' \otimes_R S) = \bigoplus_{i\in I'} Q_i$ for some countable subset $I' \subseteq I$.*

Proof. We construct a chain of countably generated submodules $(N_n)_{n<\omega}$ and a chain of countable subsets $(I_n)_{n<\omega}$ as follows. Let $N_0 = N$. If for $n < \omega$ the module N_n is already constructed, we put $I_n = \{i \in I \mid \pi_i((\subseteq \otimes_R \mathrm{id}_S)(N_n \otimes_R S)) \neq 0\}$, where $\pi_i : N \otimes_R S \to Q_i$, $i \in I$ are the canonical projections. The set I_n is countable, as the module N_n is countably generated, therefore also $N_n \otimes_R S$ is countably generated and for each its generator x the set $\{i \in I \mid \pi_i((\subseteq \otimes_R \mathrm{id}_S)(x)) \neq 0\}$ is finite. Clearly, $(\subseteq \otimes_R \mathrm{id}_S)(N_n \otimes_R S) \subseteq \bigoplus_{i \in I_n} Q_i$.

If the set I_n is already constructed, construct the submodule $N_{n+1} \subseteq M$ as follows. The submodule $\bigoplus_{i \in I_n} Q_i \subseteq M \otimes_R S$ is a countable direct sum of countably generated modules and is therefore countably generated. Let $\left\{\sum_j^{k_\alpha} m_{\alpha j} \otimes s_{\alpha j}\right\}_{\alpha<\omega}$ a set of its generators. Then we construct module N_n' as a submodule of M generated by the countable set $\{m_{\alpha j}\}_{\alpha<\omega, 1 \leq j \leq k_\alpha}$ and module $N_{n+1} = N_n + N_n'$. Clearly, $\bigoplus_{i \in I_n} Q_i \subseteq (\subseteq \otimes_R \mathrm{id}_S)(N_n \otimes_R S)$ and $N_n \subseteq N_{n+1}$.

As for each $n < \omega$ we have $\bigoplus_{i \in I_n} Q_i \subseteq (\subseteq \otimes_R \mathrm{id}_S)(N_n \otimes_R S)$, by construction of I_{n+1} also $I_n \subseteq I_{n+1}$.

Having constructed the chains, we put $I' = \bigcup_{n<\omega} I_n$ and $N' = \bigcup_{n<\omega} N_n$. As a union of a countable chain of countable sets, I' is countable and as a union of a countable chain of countably generated modules N' is countably generated. It is easily checked that $(\subseteq \otimes_R \mathrm{id}_S)(N' \otimes_R S) = \bigoplus_{i \in I'} Q_i$, as $(\subseteq \otimes_R \mathrm{id}_S)(N' \otimes_R S) = \bigcup_{n<\omega} (\subseteq \otimes_R \mathrm{id}_S)(N_n \otimes_R S)$ and $\bigoplus_{i \in I'} Q_i = \bigcup_{n<\omega} \bigoplus_{i \in I_n} Q_i$.

□

Theorem 7.13. *Let $R \to S$ be a ring homomorphism, which is pure as a left R-module homomorphism. Let M be a right R-module. Then:*

(A) *If $R \to S$ is further also right pure and flatness descends through it, then projectivity descends through $R \to S$.*

(B) *If $R \to S$ further recognizes pure epimorphisms, then pure projectivity descends through $R \to S$ In this case $R \to S$ also descends flatness and projectivity.*

Proof. The proofs of both parts are similar, we'll do them at the same time, noting where the proof differs.

Let M be such that $M \otimes_R S$ is (pure) projective. To finish the proof it is enough to show that $M = \bigoplus_{j \in J} M_j'$ for some system of countably generated modules, because

(A) by lemma 7.10 M is already Mittag-Leffler and by the hypothesis on $R \to S$ it is flat. Hence, M is projective by Theorem 4.25 for usual projectivity.

(B) by lemma 7.9 M is already Mittag-Leffler and hence it is already pure projective by Theorem 4.25 for pure projectivity.

By Kaplansky's theorem (Theorem 4.10) we can decompose $M \otimes_R S = \bigoplus_{i \in I} Q_i$ as a direct sum of countably generated (pure) projective modules.

Fix on M a well ordering by a cardinal κ and denote $M = \{m_\alpha\}_{\alpha<\kappa}$. We'll construct a chain $(M_\alpha)_{\alpha<\kappa}$ of submodules of M and chain $(I_\alpha)_{\alpha<\kappa}$ of subsets of I with following properties.

1. $M_0 = 0$, $I_0 = \emptyset$

2. $M_\alpha \subseteq M_\beta$, $I_\alpha \subseteq I_\beta$ whenever $\alpha < \beta < \kappa$,

3. for each limit ordinal $\beta < \kappa$ we have $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ and $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$, and

4. for each $\alpha < \kappa$:

   (a) $m_\alpha \in M_{\alpha+1}$

   (b) $0 \to M_\alpha \to M \to M/M_\alpha \to 0$ is pure exact and $M_\alpha \otimes_R S = \bigoplus_{i\in I_\alpha} Q_i$,

   (c) $0 \to M_\alpha \to M_{\alpha+1} \to M'_\alpha \to 0$ splits and M'_α is countably generated.

We claim that such chain $(M_\alpha)_{\alpha<\kappa}$ already produces direct decomposition $M = \bigoplus_{\alpha\in\kappa} M'_\alpha$.

By induction on κ we see that for each $\alpha < \kappa$ already $M_\alpha = \bigoplus_{\gamma<\alpha} M'_\gamma$. For $\alpha = 0$ we have $M_0 = 0$. If for some $\alpha < \kappa$ already $M_\alpha \cong \bigoplus_{\gamma<\alpha} M'_\gamma$, then by condition 4.(c)

$$M_{\alpha+1} \cong M_\alpha \oplus M'_\alpha \cong \bigoplus_{\gamma<\alpha+1} M'_\gamma.$$

For $\beta < \kappa$ a limit ordinal we have by condition 3.

$$M_\beta = \bigcup_{\alpha<\beta} M_\alpha = \bigcup_{\alpha<\beta} \bigoplus_{\gamma<\alpha} M'_\gamma = \bigoplus_{\gamma<\beta} M'_\gamma,$$

viewing M'_γ as submodules in M. Finally, by condition 4.(a)

$$M = \bigcup_{\alpha<\kappa} M_\alpha = \bigcup_{\alpha<\kappa} \bigoplus_{\gamma<\alpha} M'_\gamma = \bigoplus_{\gamma<\kappa} M'_\gamma.$$

The chains shall be constructed by induction on the cardinal κ. Put $M_0 = 0$, $I_0 = \emptyset$. Now, if for a limit cardinal $\beta < \kappa$ there already are M_α, I_α for all $\alpha < \beta$, we put $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ and $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$. Then all required conditions except 4.(b) are trivially satisfied. To check the condition 4.(b) express the short exact sequence

$$0 \to M_\beta \to M \to M/M_\beta \to 0$$

as a direct limit of the system of pure exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_\alpha \arrow[r] \arrow[d, "\subseteq_{\alpha'\alpha}"] & M \arrow[r] \arrow[d, "\wr"] & M/M_\alpha \arrow[r] \arrow[d, "\varphi_{\alpha'\alpha}"] & 0 \\
0 \arrow[r] & M_{\alpha'} \arrow[r] & M \arrow[r] & M/M_{\alpha'} \arrow[r] & 0
\end{tikzcd}$$

with the map on the left being the inclusion for $\alpha \le \alpha'$ and the map on the right being the projection $M/M_\alpha \to (M/M_\alpha)/(M/M_{\alpha'}) \cong M/M_{\alpha'}$. Furthermore, because union of a chain is a type of direct limit and tensor commutes with direct limits we have

$$M_\beta \otimes_R S = \varinjlim_{\alpha<\beta}(M_\alpha \otimes_R S) = \bigcup_{\alpha<\beta} \bigoplus_{i\in I_\alpha} Q_i = \bigoplus_{i\in I_\beta} Q_i.$$

If for $\alpha < \kappa$ we already have M_α, we construct $M_{\alpha+1}$ as follows. Tensoring the pure exact sequence

$$0 \to M_\alpha \to M \to M/M_\alpha \to 0$$

with S produces a sequence

$$0 \to M_\alpha \otimes_R S \to M \otimes_R S \to M/M_\alpha \otimes_R S \to 0$$

isomorphic to

$$0 \to \bigoplus_{i \in I_\alpha} Q_i \to \bigoplus_{i \in I} Q_i \to \bigoplus_{i \in I \setminus I_\alpha} Q_i \to 0$$

by the condition 4.(b) for α.

Put $N_\alpha \subseteq M/M_\alpha$ as the submodule generated by element $m_\alpha + M_\alpha$. Then lemma 7.12 yields a countable generated submodule $M'_\alpha \subseteq M/M_\alpha$ and the set $I'_\alpha \subseteq I$ such that $\operatorname{Im}(\subseteq \otimes_R \mathrm{id}_S) = \bigoplus_{i \in I'_\alpha} Q_i$. Put $I_{\alpha+1} = I_\alpha \cup I'_\alpha$ and take $M_{\alpha+1}$ as the pullback in the diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & M/M_{\alpha+1} \arrow[u] \arrow[r, phantom, "\cong"] & M/M_{\alpha+1} \arrow[u] & \\
0 \arrow[r] & M_\alpha \arrow[r] & M \arrow[r] \arrow[u] & M/M_\alpha \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M_\alpha \arrow[r] \arrow[u, "\wr"'] & M_{\alpha+1} \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner"] & M'_\alpha \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{7.1}$$

where all rows and columns are exact, and the isomorphisms arise from properties of pullback of an epimorphism along a monomorphism. Let us prove, that the sequence

$$0 \to M'_\alpha \to M/M_\alpha \to M/M_{\alpha+1} \to 0$$

is in fact pure. Tensoring it with S produces a right exact sequence

$$M'_\alpha \otimes_R S \to (M/M_\alpha) \otimes_R S \to (M/M_\alpha \otimes_R S)/\operatorname{Im}(\subseteq \otimes_R \mathrm{id}_S) \to 0$$

which is in turn isomorphic to

$$M'_\alpha \otimes_R S \to \bigoplus_{i \in I \setminus I_\alpha} Q_i \to \bigoplus_{i \in I \setminus I_{\alpha+1}} Q_i \to 0$$

which splits.

(A) Module $(M/M_{\alpha+1}) \otimes_R S \cong \bigoplus_{i \in I \setminus I_{\alpha+1}} Q_i$ is therefore a direct summand in a flat module $M \otimes_R S$ and as such it is itself flat. By the hypothesis on homomorphism $R \to S$ so is the module $M/M_{\alpha+1}$. Then however the exact sequence

$$0 \to M'_\alpha \to M/M_\alpha \to M/M_{\alpha+1} \to 0$$

ends in a flat module and is pure.

(B) The epimorphism $M/M_\alpha \otimes_R S \to M/M_{\alpha+1} \otimes_R S$ is split and therefore pure. By the hypothesis on $R \to S$, also $M/M_\alpha \to M/M_{\alpha+1}$ is a pure epimorphism and hence the exact sequence

$$0 \to M'_\alpha \to M/M_\alpha \to M/M_{\alpha+1} \to 0$$

is pure.

With $M'_\alpha \subseteq M/M_\alpha$ being a pure inclusion we have already $M'_\alpha \otimes_R S = \operatorname{Im}(\subseteq \otimes_R \mathrm{id}_S) = \bigoplus_{i \in I'_\alpha} Q_i$. As a direct summand in a (pure) projective module $M \otimes_R S$ the module $M'_\alpha \otimes_R S$ is itself (pure) projective. As it is countably generated, as a pure projective module it is already countably presented, and as $R \to S$ is pure as a left module morphism, by Proposition 7.2 the module M'_α is already countable generated.

(A) By hypothesis on $R \to S$ the module M'_α is flat, by Proposition 7.10 it is Mittag-Leffler and by Proposition 4.24 it is projective. Hence, the sequence

$$0 \to M_\alpha \to M_{\alpha+1} \to M'_\alpha \to 0$$

splits.

(B) By Proposition 7.9 the module M'_α is Mittag-Leffler and by Proposition 4.24 it is pure projective. The sequence

$$0 \to M_\alpha \to M_{\alpha+1} \to M'_\alpha \to 0$$

is pure as the epimorphism $M_{\alpha+1} \to M'_\alpha$ is a pullback of a pure epimorphism. Therefore, it splits.

Finally, the homomorphism $M_{\alpha+1} \to M$ is a pure extension of an isomorphism and a pure monomorphism and is therefore itself a pure monomorphism. So the sequence

$$0 \to M_{\alpha+1} \to M \to M/M_{\alpha+1} \to 0$$

is pure. This concludes construction of desired chains and as such the whole proof. In part (B) further by Corollary 7.5 the homomorphism $R \to S$ already descends flatness and thanks to characterization of a projective module as a flat pure projective module, it already descends projectivity. $\square$

Corollary 7.14. *Let $R \to S$ be a pure homomorphism of commutative rings. Then pure projectivity and projectivity descend through $R \to S$.*

Corollary 7.15. *Let $R \to S$ be a ring homomorphism recognizing pure epimorphisms from the right. Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}{-}R)$ be a class and let $R \to S$ further preserve $\mathcal{C}$-pure epimorphisms. Then $\mathcal{C}$-projectivity descends through $R \to S$.*

Proof. Let M be such right R-module, that $M \otimes_R S$ is $\mathcal{C}_S$-projective. By version (B) of Theorem 7.13, the homomorphism $R \to S$ descends pure projectivity and so M is pure projective. By Proposition 7.4 it also descends $\mathcal{C}$-flatness and by characterization in Corollary 4.27 it descends $\mathcal{C}$-projectivity. $\square$

Corollary 7.16. *Let $R \to S$ be a faithfully flat homomorphisms of commutative rings, then relative projectivity descends through $R \to S$.*

8. Application on central extensions of central pure homomorphisms

Having proved the descent of projectivity and pure projectivity through ring homomorphisms under some conditions, we will show descend through a concrete type of homomorphisms of central algebras over a commutative ring – those, which arise as extensions of pure (faithfully flat) central homomorphisms starting in R, by a central R-algebra. Note that the non-example $k[x]/(x^2) \to \operatorname{End}_k k[x]/(x^2)$ of section 5.2 is not central and is therefore a non-example for this setting as well.

8.1 Descent through central extensions of central homomorphisms

For this section, we fix the following situation. Let R be a commutative ring and let S and A be central R-algebras with $\varphi : R \to S$ and $\psi : R \to A$ being the corresponding ring homomorphisms. The module $A \otimes_R S$ is imbued with a central R algebra structure via

$$(a \otimes s) \cdot (a' \otimes s') = aa' \otimes ss'.$$

Now the R bimodule isomorphism $A \otimes_R S \to S \otimes_R A : (a \otimes s \mapsto s \otimes a)$ is in fact an R algebra isomorphism. We denote $A \otimes_R S \cong S \otimes_R A \cong B$. This data makes a commutative square

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ A & \xrightarrow{\varphi_A} & B \end{array}$$

where $\varphi_A = \mathrm{id}_A \otimes_R \varphi$ and $\psi' = \psi \otimes_R \mathrm{id}_S$.

It is the bottom homomorphism in this square that is of interest. Algebra homomorphisms constructed in this way offer a generalization of centrality. While B is certainly not a central A-algebra, images of elements from A nonetheless commute with anything in B that "does not come from A", that is for any $a \in A$ and $s \in S$

$$(a \otimes 1) \cdot (1 \otimes s) = a \otimes s = (1 \otimes s) \cdot (a \otimes 1).$$

Importantly, ring homomorphism φ_A inherits important properties of the homomorphism φ. This will secure descent of (pure) projectivity when φ is pure and relative projectivity when φ is faithfully flat. This inheritance of properties is facilitated by a handy natural isomorphism concerning extension of A-modules by B.

Lemma 8.1. *There are natural isomorphisms*

$$(- \otimes_A B) \cong (- \otimes_R S)|_{\mathrm{Mod}-A} \ : \ \mathrm{Mod}-A \to \mathrm{Mod}-B$$

and

$$(B \otimes_A -) \cong (S \otimes_R -)|_{A-\mathrm{Mod}} \ : \ A-\mathrm{Mod} \to B-\mathrm{Mod}$$

Proof. Let M be a right A-module. Let us check that $M \otimes_R S$ is in fact a right B-module. The B-module structure is given by

$$(m \otimes s) \cdot b = (m \otimes s) \cdot \sum_{i=1}^{n} (a_i \otimes s_i) = \sum_{i=1}^{n} (ma_i \otimes ss_i),$$

where $\sum_{i=1}^{n} (a_i \otimes s_i)$ is the interpretation of b in $A \otimes_R S \cong B$.

Then the natural isomorphism of right S-modules

$$M \otimes_A B \cong M \otimes_A (A \otimes_R S) \cong M \otimes_R S$$

is also an isomorphism of B-modules. The other isomorphism is constructed similarly. □

Another important observation is the fact, that the tensor product over R is commutative, without jeopardizing the A-module structure.

Lemma 8.2. *Let M be a right A module. Then the $M \otimes_R S \cong S \otimes_R M : m \otimes s \mapsto s \otimes m$ is an isomorphism of right A-modules.*

Proof. The A-module on $M \otimes_R S$ is given as

$$(m \otimes_R s) \cdot a = (ma \otimes s).$$

From this the R-module isomorphism $M \otimes_R S \cong S \otimes_R M : m \otimes s \mapsto s \otimes m$ is clearly also a bijective A-module homomorphism. □

These simple lemmas let us prove that φ_A has some useful properties. Immediately we get flatness and preservation of relatively pure morphisms.

Proposition 8.3. *Let in the situation above further $R \to S$ be flat, then $A \to B$ is both right and left flat. Let $\mathcal{C} \in \mathfrak{A}(\mathrm{mod}-A)$. Then $A \to B$ preserves $\mathcal{C}$-pure monomorphisms and epimorphisms.*

Proof. Let $M \to N$ be a right module monomorphism. Then $M \otimes_A B \to N \otimes_A B$ is isomorphic to $M \otimes_R S \to N \otimes_R S$, which is monic. So, $A \to B$ is left flat. Similarly we show, that $A \to B$ is right flat.

Let now $M \to N$ be a $\mathcal{C}$-pure monomorphism of right A-modules. We would like to show, that $M \otimes_A B \to N \otimes_A B$ is a $\mathcal{C}_B$-pure monomorphism of B-modules. Take a module $Q \in \mathcal{C}^\top$. Then $M \otimes_A Q \to N \otimes_A Q$ is a monomorphism. As S is flat as a right R-module, also $S \otimes_R M \otimes_A Q \to S \otimes_R N \otimes_A Q$ is monic and thanks to natural isomorphisms

$$S \otimes_R - \otimes_A Q \cong (- \otimes_R S) \otimes_A Q \cong - \otimes_A B \otimes_A Q \cong (- \otimes_A B) \otimes (B \otimes_A Q)$$

so is $(M \otimes_A B) \otimes (B \otimes_A Q) \to (N \otimes_A B) \otimes (B \otimes_A Q)$. So, $M \otimes_A B \to N \otimes_A B$ is in fact $\mathcal{C}_B$-pure. By Lemma 6.11 $A \to B$ also preserves $\mathcal{C}$-pure epimorphisms. □

More importantly, if $R \to S$ is pure, then $A \to B$ inherits all the important properties of purity.

Proposition 8.4. *Let in the situation above further $R \to S$ be a pure central ring homomorphism, then $A \to B$ reflects pure epimorphisms.*

Proof. Let $R \to S$ be pure. First we show that $A \to B$ reflects pure monomorphisms from the right. Let $K \to M$ be such homomorphism of right A-modules, that $K \otimes_A B \to M \otimes_A B$ is a pure homomorphism of right B-modules. Let Q be an arbitrary left A-module. Then $(K \otimes_A B) \otimes_B (B \otimes_A Q) \to (M \otimes_A B) \otimes_B (B \otimes_A Q)$ is a monomorphism. Thanks to natural isomorphism

$$(- \otimes_A B) \otimes_B (B \otimes_A Q) \cong (- \otimes_A B) \otimes_A Q \cong \\ (- \otimes_R S) \otimes_A Q \cong (S \otimes_R -) \otimes_A Q \cong S \otimes_R (- \otimes_A Q)$$

also $S \otimes_R (K \otimes_A Q) \to S \otimes_R (M \otimes_A Q)$ is monic and as $R \to S$ is pure and therefore faithful, $K \otimes_A Q \to M \otimes_A Q$ is monic. So, $K \to M$ is a pure monomorphism of right A-modules.

Let now $M \to N$ be a homomorphism of right A-modules such that $M \otimes_A B \to M \otimes_A B$ is a pure epimorphism of B-modules. Homomorphism $M \otimes_A B \to M \otimes_A B$ is also a pure epimorphism of S-modules (Lemma 6.12), and it is isomorphic to $M \otimes_R S \to M \otimes_R S$. As $R \to S$ recognizes pure epimorphisms (Proposition 6.14, using centrality), $M \to N$ is a pure epimorphism of R-modules. Considering its kernel K, we obtain pure exact sequence of R-modules

$$0 \to K \to M \to N \to 0.$$

Tensoring it with S produces exact sequence

$$0 \to K \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0.$$

Modules M and N are right A-modules and as $M \to N$ is a right A-module homomorphism, so is K. Therefore, the sequence is isomorphic to

$$0 \to K \otimes_A B \to M \otimes_A B \to N \otimes_A B \to 0,$$

which is a pure exact sequence of B-modules, specially $K \otimes_A B \to M \otimes_A B$ is a pure monomorphism of B-modules. We already know, that $A \to B$ reflects pure monomorphisms, so $K \to M$ is a pure monomorphism of right A-modules and its cokernel $M \to N$ is a pure epimorphism of right A-modules. □

With all these properties in place, projectivity, pure projectivity and relative projectivity descend through homomorphism $A \to B$.

Theorem 8.5. *Let R be a commutative ring, $\phi : R \to S$ be a central pure ring homomorphism, let A be a central R-algebra. Then (pure) projectivity of both left and right modules descends through the R-algebra homomorphism $\varphi_A = \mathrm{id}_A \otimes_R \varphi :$ $A \to B \cong A \otimes_R S$.*

Proof. By Proposition 8.4 homomorphism φ_A reflects pure epimorphisms from the right and from the left. By 7.13 pure projectivity and projectivity descend through φ_A. □

Theorem 8.6. *Let R be a commutative ring, $\varphi : R \to S$ be a central faithfully flat ring homomorphism, let A be a central R-algebra. Then relative projectivity of left and right modules descends through* $\mathrm{id}_A \otimes_R \varphi : A \to B \cong A \otimes_R S$

Proof. By Proposition 8.4 homomorphism φ_A reflects pure epimorphisms from the right and from the left, and by Proposition 8.3 it preserves relatively pure epimorphisms. By 7.15 relative projectivity descends through φ_A. □

Type of algebra homomorphisms described in this section offers a wide selection of homomorphisms of non-commutative algebras. One such case is when $R \to S$ is a $\mathfrak{p}$-adic completion of a noetherian ring and A a Noether R-algebra, as in [Kanda and Nakamura, 2021, Appendix A]. Homomorphisms of this type are also used in [Breaz et al., 2022, Section 4, Section 6] in the context of silting complexes. Finally, we might simply take $R \to S$ to be a Zariski covering, then discussed properties are local properties of sheaves of modules over a sheaf of algebras over a scheme.

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