Selected problems in theoretical gravitational physics

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Contents

1 Introduction

My research focuses on theoretical gravitational physics, black holes, and mathematical relativity. I am a recognized expert on hidden symmetries, separability, and integrability of field equations in curved spacetime. I am one of the founders of a new subdiscipline of black hole thermodynamics known as black hole chemistry. In the last couple of years I have pushed forward our understanding of exotic black hole spacetimes, studied the interaction of ultralight bosons with rotating black holes, and proposed a new theory of higher curvature gravity. Most recently, I have become interested in relativistic quantum information and quantum detection of various spacetime features.

In this Habilitation I have gathered a number of selected publications that I have published in collaboration with my students and numerous researchers in the past 15 years or so. These papers have been split into four categories: i) hidden symmetries of rotating black holes ii) black hole thermodynamics iii) modified gravity theories and iv) miscellaneous results. A brief introduction to each of these topics is provided in Sec. 2. This section also sets the context for each of the attached papers.

The central topic of this Habilitation is that of *black holes*. Black holes are one of the most fascinating predictions of Einstein's general relativity. With a high resolution image of a supermassive black hole at the center of M87 by the Event Horizon Telescope[1] and recent gravitational wave observations of binary black hole collisions by LIGO[2] at hand, there are no longer any doubts that black holes exist in our Universe. However, the very existence of black holes still raises many fundamental theoretical questions. Those related to quantum processes, such as the mystery of black hole entropy, information loss, quantum evaporation and black hole thermodynamics, are expected to provide key insights towards understanding how to reconcile gravity with quantum theory. Other are purely classical and range from describing astrophysical processes to pure mathematical physics

One of the most exciting theoretical developments in classical black hole thermodynamics in the past couple of years is due to the *black hole chemistry* – a new discipline I helped to establish. My original papers on this topic[3,4] have generated a groundswell of activity, resulting in more than 700 papers written in this field. The subject has grown substantially and now spans many diverse directions[5]. While the mainstream focuses on studying black hole phase transitions and understanding the notion of black hole volume, more recently there have been attempts to understand the thermodynamics of "*exotic*" black hole spacetimes, for example those with acceleration[6,7] and Newman–Unti–Tamburino (NUT) parameters[8,9]. These works have helped to discover a more complete thermodynamic dictionary and raise new challenges for the interpretation of the black hole entropy and the AdS/CFT correspondence.

Perhaps my most ground breaking discovery in the past 6 years is the demonstration[10] of *separability of the massive vector perturbations* around rotating black holes in any number of dimensions – a completely unexpected result that was awaiting its discovery for almost 50 years. The demonstrated separability allowed us to study instability modes of such perturbations, providing constraints on various candidates for the dark matter[10]. The key tool towards this discovery was to exploit a *hidden symmetry* encoded in the principal Killing–Yano tensor that is present in these black hole spacetimes[11].

Another topic discussed in this Habilitation is that of *modified gravity theories.* It is very likely that Einstein's theory of gravity is only an approximation and will eventually have to be modified. At classical level, the departures are expected to be described by the higher curvature terms. Of special interest are the higher curvature theories that preserve some of the remarkable properties of Einstein's theory. In this spirit, we have proposed two novel modified higher curvature gravities. One that is obtained by taking a singular limit of the well known Gauss–Bonnet gravity to four dimensions[12] and another that generalizes the recently proposed quasi-topological gravities[13].

Finally, we shall discuss three '*miscellaneous results*', related to the calculation of gravitational wave production in black hole mergers[14], to quantum detection of inertial frame dragging[15], and cosmic string hair of black holes[16].

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2 The Hitchhiker's guide to the selected papers

2.1 Black hole thermodynamics

It has been nearly 50 years since Bardeen, Carter and Hawking first formulated the laws of *black hole mechanics*[17]. These laws are geometric in origin and are consequences of classical general relativity. In particular, the *first law* of black hole mechanics is a constraint enforced by the Einstein equation, relating physical perturbations of the black hole to variations in its area and conserved charges. That the laws of black hole mechanics are in fact the *laws of thermodynamics* (induced by quantum effects in the vicinity of a black hole horizon) was only firmly established somewhat later by Bekenstein and Hawking[18,19], yielding the famous relation for the black hole temperature and entropy:

$$\begin{aligned} \text{temperature} \quad T \quad &\leftrightarrow \quad \frac{\kappa}{2\pi} \quad \text{surface gravity} \\ \text{entropy} \quad S \quad &\leftrightarrow \quad \frac{A}{4} \quad \text{horizon area} \end{aligned} \tag{2.1}$$

in units where $G = c = \hbar = k_B = 1$, yielding a correspondence between the laws of thermodynamics and black hole mechanics:

$$\delta E = T\delta S - P\delta V + \text{work terms} \quad \leftrightarrow \quad \delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \Omega\delta J + \phi\delta Q + \dots\,, \tag{2.2}$$

where on the r.h.s. the ΩdJ and ΦdQ terms are the standard 'kinetic' and 'chemical potential' black hole work terms, written in terms of the asymptotic angular momentum J and electric charge Q (and their conjugate variables Ω and ϕ), and M is the mass of the black hole. The first law is standardly accompanied by its 'integral version', the so called *Smarr–Gibbs–Duhem relation*[20], which is an equality relating the finite thermodynamic charges:

$$\frac{d-3}{d-2}M = TS + \Omega J + \frac{d-3}{d-2}\phi Q\,, \tag{2.3}$$

valid for asymptotically flat black holes in d number of spacetime dimensions.

While the above laws are well established for simple black holes, two natural questions arise. 1) Is it possible to find, on the black hole side, the term that would correspond to the standard $P\delta V$ work term? 2) Can the laws of black hole thermodynamics be also formulated for more complicated black hole spacetimes, such as those including acceleration, Taub-NUT charges, and/or cosmic strings? Whereas the answer to the first question led to a new subdiscipline of black hole thermodynamics known as the *black hole chemistry*, the latter gave rise to an extended thermodynamic dictionary where new thermodynamic charges on the black hole side were properly identified. In what follows, we shall describe these recent developments.

2.1.1 Black hole chemistry

Black hole chemistry[5,21] is a new and fast developing subdiscipline of classical black hole thermodynamics. The original idea was to reconsider the behavior of Anti de Sitter (AdS) black holes, that is black holes in an asymptotically AdS space, in the context of a dynamical cosmological constant, which provided the basis for introducing the pressure/volume term into black hole thermodynamics[22]. This simple idea has far reaching consequences and leads to a radical new understanding of black holes from the "viewpoint of chemistry", in terms of concepts such as chemical enthalpy, Van der Waals fluids, and holographic heat engines.

Thermodynamics with variable Λ

Asymptotically AdS black holes feature many attractive properties that are absent for their asymptotically flat cousins. Such black holes provide a description of the dual conformal field theory (CFT) at finite temperature via the AdS/CFT correspondence[23]. Moreover, they can be in thermal equilibrium with their Hawking radiation and exhibit interesting thermodynamic phase transitions, such as the first order Hawking–Page phase transition[24], or the existence of a second order Van der Waals type phase transition for charged AdS black holes[4,25–27].

An asymptotically AdS black hole in d spacetime dimensions is a black hole solution to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}\,, \tag{2.4}$$

where the cosmological constant $\Lambda < 0$ is often parameterized by the *AdS radius* l

according to

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2} < 0\,, \tag{2.5}$$

and $T_{\mu\nu}$ is the matter stress-energy tensor (that vanishes sufficiently quickly as we approach the asymptotic region). The starting point of the black hole chemistry is to identify the *thermodynamic pressure* with the cosmological constant Λ according to

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}\,, \tag{2.6}$$

and allow it to vary in the first law of black hole thermodynamics.* The *thermodynamic volume* is then defined as a quantity that is thermodynamically conjugate to pressure:

$$V \equiv \left(\frac{\partial M}{\partial P}\right)_{S,Q,J}\,. \tag{2.7}$$

This means that the mass of the black hole M is now identified with *enthalpy* rather than energy† and the extended first law reads[22]:

$$\delta M = T\delta S + V\delta P + \sum_i \Omega^i \delta J^i + \sum_j \phi^j \delta Q^j\,, \tag{2.8}$$

where we allowed for a possibility of having multiple angular momenta J^i and multiple $U(1)$ charges Q^j. This law is now consistent (via the dimensional Euler argument) with the following extended Smarr relation[22,30]:

$$\frac{d-3}{d-2}M = TS + \sum_i \Omega^i J^i - \frac{2}{d-2}PV + \frac{d-3}{d-2}\sum_j \phi^j Q^j\,. \tag{2.9}$$

Obviously, without the $P-V$ term this equality would not be valid, giving a 'practical reason' as to why the variations of Λ have to be included in the first law of black hole thermodynamics.

*The idea that Λ might be a dynamical variable was first proposed by Teitelboim and Brown[28,29].

†In standard thermodynamics, enthalpy is energy to create the system and place it in an environment. In black hole physics this corresponds to 'creating' a black hole and 'placing' it in an environment of the negative cosmological constant.

Simple example

To illustrate the above concepts, let us consider the 'simplest possible' (vacuum) spherical AdS black hole in $d=4$ dimensions, the *Schwarzschild-AdS* spacetime:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}\,, \tag{2.10}$$

whose horizon is located at the radius r_+, determined as the largest positive root of $f(r_+) = 0$. This black hole can be assigned the following thermodynamic quantities:

$$M = \frac{r_+}{2}\Big(1 + \frac{r_+^2}{l^2}\Big)\,, \quad S = \frac{\text{Area}}{4} = \pi r_+^2\,, \quad T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\Big(1 + 3\frac{r_+^2}{l^2}\Big)\,. \tag{2.11}$$

Thence, upon identifying $P = 3/(8\pi l^2)$, we find an 'amusing formula'

$$V = \frac{4}{3}\pi r_+^3 \tag{2.12}$$

for the thermodynamic volume of this black hole, which is a volume of a sphere in a 3-dimensional Euclidean space. Such a volume can also be recovered by integrating the 'space' hidden behind the black hole horizon – a quantity know as the *geometric* black hole volume[31,32].

Thermodynamic volume

In general, there is no reason for the thermodynamic volume to have any relation to the volume in Euclidean space or to the black hole geometric volume. In fact, already for rotating black holes in four dimensions, the three concepts are different[3]. That thermodynamic definition yields a sensible definition for the black hole volume is indicated by the *reverse isoperimetric inequality.* Namely, the following conjecture has been put forward in[3]:

> *Reverse isoperimetric inequality.* For any AdS black hole solution in Einstein gravity the following ratio:
>
> $$\mathcal{R} = \Big(\frac{(d-1)V}{\omega_{d-2}}\Big)^{\frac{1}{d-1}} \Big(\frac{\omega_{d-2}}{A}\Big)^{\frac{1}{d-2}}\,, \tag{2.13}$$

where A is the black hole horizon area, V is its thermodynamic volume, and

$$\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,, \tag{2.14}$$

is the volume of the unit n-sphere, obeys the following inequality:

$$\mathcal{R} \geq 1\,, \tag{2.15}$$

with the bound being saturated for Schwarzschild-AdS black holes. In other words, for a fixed thermodynamic volume the entropy of the black hole is maximized for the Schwarzschild-AdS spacetime.

It turns out that many of the AdS black holes obey the conjectured inequality[3]. Those that do not, possess for a given thermodynamic volume more entropy than the Schwarzschild-AdS spacetime, and are known as the *'superentropic black holes'*[33].

Phase transitions

Interestingly, already the simplest AdS black hole (2.10) admits an interesting phase transition, known as the *Hawking–Page phase transition*[24]. To uncover this, one has to study the corresponding free energy of the system and seek its global minimum. If such a minimum is 'discontinuous', there will be the corresponding phase transition. In particular, according to Ehrenfest's classification the continuity in the minimum itself but discontinuity in its first derivatives implies a *first order phase transition.*

In our case, the total system consists of AdS black hole and its Hawking radiation. While the (Gibbs) free energy of the latter, sometimes referred to as the *thermal AdS*, is negligible[24], $G_{\mathrm{rad}} \approx 0$, the free energy of the black hole (and thence the total free energy) reads

$$G = G(T,P) = M - TS = \frac{r_+}{4}\left(1 - \frac{r_+^2}{l^2}\right). \tag{2.16}$$

For a given pressure, this is parametrically plotted in Fig. 2.1. We observe a minimum temperature $T_{\mathrm{min}} = 2\sqrt{3}/(4\pi l)$, corresponding to $r_{\mathrm{min}} = l/\sqrt{3}$, below which no black holes can exist. Above this temperature we have two branches of black holes that meet at a cusp. The upper branch describes small black holes with negative specific heat; these are thermodynamically unstable and cannot be in a thermal equilibrium with a thermal bath of radiation. The large ($r_+ > r_{\mathrm{min}}$) black holes at lower branch have positive specific heat and are locally thermodynamically stable. More importantly, the radiation with zero free energy represents a global minimum of the free energy for temperatures lower

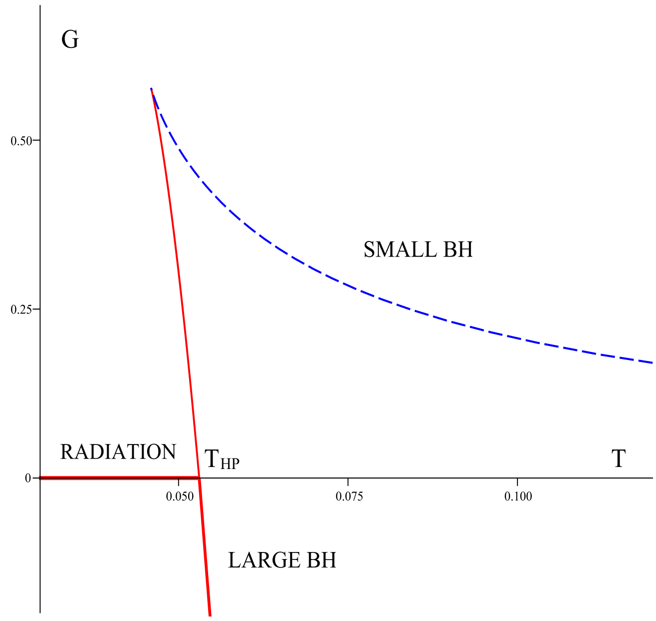

Figure 2.1: *Hawking–Page phase transition.* Above $T_{\rm min}$, the free energy of the Schwarzchild-AdS black hole displays two branches of black holes. The upper branch (small black holes) has negative specific heat and is thermodynamically unstable. The lower branch (large black holes) has positive specific heat. For $T > T_{\rm HP}$ this branch has negative free energy (lower than that of the Hawking radiation) and the corresponding black holes represent the globally thermodynamically preferred state. As the temperature increases, at $T_{\rm HP}$, the system undergoes a first order Hawking–Page phase transition from radiation to large black hole.

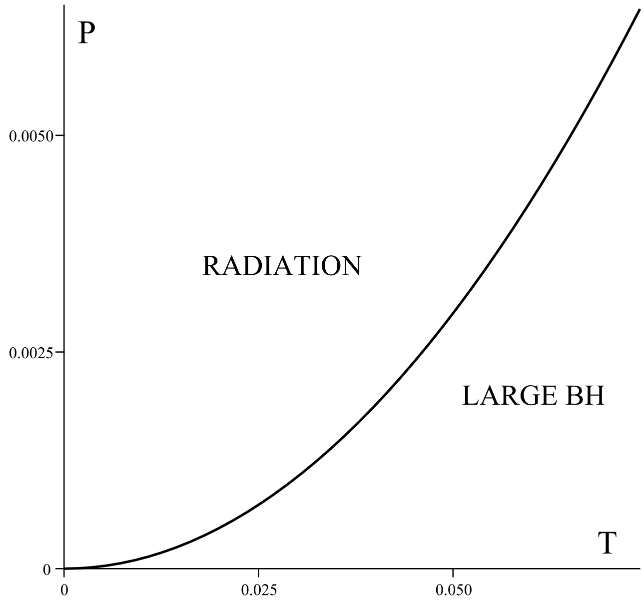

Figure 2.2: *Phase diagram: Schwarzschild-AdS.* We display a semi-infinite coexistence line between the radiation and large black hole phases, reminiscent of the solid-liquid phase transition.

than the Hawking–Page temperature:

$$T_{\mathrm{HP}} = \frac{1}{\pi l}\,, \tag{2.17}$$

determined from $G = 0$, above which the branch of large black holes thermodynamically dominates. In other words, as the system is heated up, it undergoes a first order *Hawking–Page* phase transition from thermal radiation to a large black hole[24].

This phase transition was later re-interpreted as a confinement/deconfinement phase transition in the dual quark gluon plasma[34]. Alternatively, it is obvious that as we vary the cosmological constant, the Hawking–Page critical temperature, (2.17), changes. This yields the following coexistence line between the two phases:

$$P|_{\text{coexistence}} = \frac{3\pi}{8}T^2\,. \tag{2.18}$$

The corresponding $P-T$ phase diagram is displayed in Fig. 2.2. It is reminiscent of a solid-liquid phase transition, yielding thus a familiar interpretation.

So far we have only dealt with the simplest AdS black hole (2.10). Considering more complicated AdS black hole spacetimes has revealed remarkable similarities between the phase behaviour of black holes and that of ordinary matter[5]. The paradigmatic example of this analogy is the Van der Waals-like phase transition of charged AdS black holes[4,25,26]. While the Van der Waals behavior is the most 'typical' for AdS black holes, more complicated phase diagrams can also be identified. For example, people observed the so called reentrant phase transitions[35], water-like tricritical points[36], an isolated critical point[37,38], superfluid-like behaviour[39], or the existence of 'snapping' swallow tails[40]. Most recently, these phase transitions were further investigated employing the Ruppener's thermodynamic geometry to reveal possible features of the underlying black hole microstructure[41].

Holographic interpretation

One of the main motivations to study AdS black holes is the AdS/CFT correspondence[23] where such black holes provide a dual description of the boundary CFT at finite temperature. This in particular regards the bulk black hole phase transitions which are dual to the phase transitions of the boundary CFT.

Surprisingly, the *holographic interpretation* of black hole chemistry has been elusive for many years[42–48]. The reason for this is that the extended first law (2.8) cannot be straightforwardly related to the corresponding thermodynamics of the holographic dual field theory[49,50], because variations of the bulk cosmological constant Λ correspond to changing both the central charge C (or the number of colours N) and the CFT volume $\mathcal{V}$. Namely, one has the following holographic first law[50]:

$$\delta E = T\delta S - pd\mathcal{V} + \tilde{\phi}\delta\tilde{Q} + \Omega\delta J + \tilde{\mu}\delta C\,, \tag{2.19}$$

where E is the CFT energy (not enthalpy), p and $\mathcal{V} = \mathcal{V}_0 l^{D-2}$ are the CFT pressure and volume, $\tilde{\mu}$ is the chemical potential for the central charge C, which is proportional to N to some power ($C \propto N^2$ for $SU(N)$ gauge theories with conformal symmetry), J and Ω are the angular momentum and conjugate angular velocity, and $\tilde{Q}, \tilde{\phi}$ are its respective holographic charge and conjugate potential.

When the holographic first law (2.19) is used a starting point, and upon employing the standard dictionary between boundary and bulk quantities (recovering the Newton's constant G for the moment):

$$E = M\,, \quad \tilde{Q} = \frac{Ql}{\sqrt{G}}\,, \quad \tilde{\phi} = \frac{\phi\sqrt{G}}{l}\,, \tag{2.20}$$

together with the two holographic Smarr relations $E=(D-2)p\mathcal{V}$, $E=TS+\tilde{\phi}\tilde{Q}+\Omega J+\tilde{\mu}C$, and the duality relation

$$C=k\frac{l^{D-2}}{16\pi G}\,, \tag{2.21}$$

where the numerical factor k depends on the details of the particular holographic system[49], one arrives at the following extended bulk first law:

$$\delta M=\frac{\kappa}{8\pi G}\delta A+\Omega\delta J+\phi\delta Q-\frac{V}{8\pi G}\delta\Lambda-\alpha\frac{\delta G}{G}\,. \tag{2.22}$$

Thus varying Λ is naturally (at least from the holographic perspective) accompanied by varying the Newton's constant G. (See[51] for additional reasons as to why G should be varied in the first law of black hole thermodynamics.)

Alternatively, one may rewrite the extended bulk first law (2.22) in the following *'mixed' form*[52]:

$$\delta M=T\delta S+\Omega\delta J+\phi\delta Q+V_C\delta P+\mu\delta C\,, \tag{2.23}$$

where V_C and μ are the new thermodynamic volume and chemical potential, respectively. In this form, it is possible to decouple the variations of the cosmological constant from the effect of changing the boundary theory – allowing one to study the black hole bulk behavior, holding the central charge C fixed. As shown in[52], this has the very interesting consequence that varying bulk pressure (as done in the black hole chemistry literature) no longer qualitatively changes phase diagrams. Their qualitative behavior depends entirely on the value of the central charge – in particular, phase transitions of charged AdS black holes only exist provided the dual CFT has a sufficient number of degrees of freedom. In this sense the work[52] marks "the fall" of black hole chemistry as traditionally understood, but opens up a new frontier for exploring its relationship with the AdS/CFT correspondence.

Comments on selected papers

As described above the black hole chemistry is a fast developing discipline. More than 700 papers have been written on the topic in past 10 years, with the author of the current manuscript playing a leading role in many of the developments.

Paper[3] is one of the first studies probing the notion of the black hole thermodynamic volume (2.7). Having calculated this volume for a large variety of black holes, the authors were led to the reverse isoperimetric inequality conjecture (2.15). This has influenced many following studies and led eventually to the concept of superentropic black holes, discussed in Paper[33].

One of the most interesting conclusions that come from the black hole chemistry is that black holes feature many interesting phase transitions, reminiscent of phase transitions of everyday substances. The Paper[4] represents the first study in this direction, showing that the charged AdS black holes demonstrate Van der Waals-like behavior, where the coexistence line of first order phase transition terminates at a second order critical point characterized by standard mean field theory critical exponents. While many other more complicated phase transitions were later identified, one of the most remarkable discoveries is the existence of an 'isolated critical point', studied in[38] for higher-curvature black holes, which is a special critical pint with polymer-like critical exponents.

Finally, the holographic interpretation of the black hole chemistry was recently clarified in[52].

2.1.2 Thermodynamics of exotic black hole spacetimes

Euclidean methods and regularity paradigm

While Hawking's original derivation of black hole radiation[19] was performed in the context of Lorentzian quantum field theory in curved spacetime, very shortly after this a number of works established relationships between properties of the complexified geometry and black hole thermodynamics[53–55]. These so-called *Euclidean methods* have since become common practice for deducing the thermodynamic properties of spacetimes containing horizons. A key point advocated in this approach is that the Euclidean sector be *regular*. For example, York writes[56]

> "This geometry must be topologically regular (no conical singularity at its axis or horizon), a geometrical condition equivalent to the physical requirement of thermal equilibrium."

For many examples of gravitating solutions regularity is indeed an essential ingredient for the validity of the first law — as is the case of the Schwarzschild solution, or for certain gravitational solitons[57]. In particular, regularity implies that the Euclidean time τ is periodic, $\tau \sim \tau + \beta$, and yields a finite temperature T, given by:

$$T = \frac{1}{\beta}\,, \quad \tau \sim \tau + \beta\,. \tag{2.24}$$

In what follows we are going to question the regularity paradigm, and probe whether or not the first law can also be formulated for more '*exotic black hole spacetimes*' for which regularity of the Euclidean solution cannot be fully achieved.

Lorentzian Taub-NUT solution

A prototypical example of a spacetime that has been puzzling physicists for the past 70 years is the Lorentzian Taub-NUT(-AdS) solution[58–60], whose metric reads

$$\begin{aligned} ds^2 &= -f(dt+2n\cos\theta d\phi)^2+\frac{dr^2}{f}+(r^2+n^2)(d\theta^2+\sin^2\theta d\phi^2)\,, \\ f &= \frac{r^2-2mr-n^2}{r^2+n^2}-\frac{3n^4-6n^2r^2-r^4}{l^2(r^2+n^2)}\,. \end{aligned} \tag{2.25}$$

Here, m stands for the gravitational mass, l is the AdS radius, and n represents the so called NUT charge (a gravitational analogue of the magnetic monopole). The solution is plagued by the existence of a *Misner string singularity*[61], located along the axis $\theta=0$, and the associated with it closed timelike curves in its vicinity. The Misner string, which is the gravitational analog of the Dirac string, can be eliminated by requiring the Lorentzian time t to be periodic[61]

$$t\sim t+8\pi n\,, \tag{2.26}$$

which, however, has fatal consequences for the spacetime causality and geodesic completeness[62–65]. A recently preferred alternative[66,67] is to preserve the Misner string and interpret it as a (rotating string) source of angular momentum[68–70].

Turning to the Euclidean sector, which is obtained by Wick rotating $t\to i\tau$ and $n\to i\nu$, we observe several singularities. Namely, similar to Schwarzschild, the Euclidean metric possesses a conical singularity at the root of $f(r_+)=0$ that can be eliminated by choosing the Euclidean time τ to have period β, given by

$$\beta=\frac{4\pi}{f'(r_+)}\,. \tag{2.27}$$

In addition to the conical singularity at the horizon $f(r_+)=0$, there is the Euclidean version of the Misner string singularity along the axis $\theta=0$, which can be eliminated by requiring that $\tau\sim\tau+\beta$, where[60,61,71]

$$\beta=8\pi\nu\,, \tag{2.28}$$

in accord with the Lorentzian condition (2.26). To proceed further, we calculate the

Euclidean action

$$I=\frac{1}{16\pi}\int_M d^4x\sqrt{g}\Big(R+\frac{6}{\ell^2}\Big)+\frac{1}{8\pi}\int_{\partial M}d^3x\sqrt{h}\Big[\mathcal{K}-\frac{2}{\ell}-\frac{\ell}{2}\mathcal{R}(h)\Big]\,,\tag{2.29}$$

where $\mathcal{K}$ and $\mathcal{R}(h)$ are respectively the extrinsic curvature and Ricci scalar of the boundary. In this expression we have included, apart from the Einstein–Hilbert and York–Gibbons–Hawking terms, also the standard AdS counter-terms[72]. Assuming general periodicity $\tau\sim\tau+\beta$, we obtain the following simple result for the free energy of the Euclidean solution[8,72]:

$$F=\frac{I}{\beta}=\frac{m}{2}-\frac{1}{2l^2}\Big(r_+^3-3\nu^2r_+\Big)\,.\tag{2.30}$$

So what do we infer from here?

The *conventional approach* is to concentrate on the thermodynamics of the Euclidean solution and require its *full regularity.* This means that both conditions (2.27) and (2.28) have to be simultaneously satisfied, which implies

$$T=\frac{1}{\beta}=\frac{f'(r_+)}{4\pi}=\frac{1}{8\pi\nu}\quad\Leftrightarrow\quad r_+=r_+(\nu,l)\tag{2.31}$$

for the temperature and reduces the number of free parameters of the solution. At the same time, the free energy is now understood as a function of two variables, $F=F(T,P)$, which yields the following entropy:

$$S=-\frac{\partial F}{\partial T}\Big|_P=\frac{\pi(3r_+^4-12\nu^2r_+^2+r_+^2l^2+\nu^2l^2-3\nu^4)}{l^2+3r_+^2-3\nu^2}\,,\tag{2.32}$$

providing thus the only example of entropy in Einstein gravity that is not equal to Area/4, a result that is interpreted by assigning the Misner strings themselves an entropy[60,73].[‡]

Despite this being the conventional approach, it is possible to obtain fully consistent thermodynamics *without* imposing the absence of Misner strings – abandoning the regularity paradigm. Let us concentrate on the Lorentzian setting, where the Misner strings are themselves Killing horizons associated with the Killing vector $\xi=\partial_t-\partial_\phi/(2n)$, and can be assigned the following *Misner potential*[8,9,74]:

$$\psi=\frac{1}{8\pi n}\,,\tag{2.33}$$

[‡] The author finds this conclusion very counter-intuitive, as the Misner string has already been removed by imposing the Misner condition (2.28).

where ψ has been identified with either the Misner string surface gravity $\kappa_{\text{MS}}/(4\pi)$[74], or alternatively, with its angular velocity $\Omega_{\text{MS}}/(4\pi)$[75]. The Lorentzian versions of (2.27) and (2.30) are given by

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1 + \frac{3(n^2 + r_+^2)}{l^2}\right), \quad F = \frac{m}{2} - \frac{r_+(r_+^2 + 3n^2)}{2l^2}. \tag{2.34}$$

We stress that, since the absence of Misner strings is no longer required, all of r_+, n, l remain independent, and the free energy is now a function of three variables, $F = F(T, \psi, P)$. The conjugate potentials are then computed in the standard way:

$$S = -\frac{\partial F}{\partial T}\Big|_{\psi,P} = \pi(r_+^2 + n^2), \quad N = -\frac{\partial F}{\partial \psi}\Big|_{T,P} = -\frac{4\pi n^3}{r_+}\Big(1 + \frac{3(n^2 - r_+^2)}{l^2}\Big), \tag{2.35}$$

while the energy $E = m$ is deduced from the relation $F = E - TS - \psi N$. The obtained quantities can easily be verified to satisfy the first law

$$\delta E = T\delta S + \delta W, \quad \delta W = \psi \delta N, \tag{2.36}$$

where the new work term δW is associated with either the surface gravity or the rotational energy of Misner strings. The entropy S is now Area/4, while N can be shown to have the geometric interpretation as the integral of $*d\xi$ over the Misner string[9].

We thus managed to formulate a 'meaningful' first law for the Taub-NUT spacetime, without demanding the full regularity of its Euclidean sector. The same considerations apply also to other spacetimes containing NUT charge. For example, demanding regularity for rotating NUT solutions eliminates all free parameters of the solution and leads to a 'discrete first law'[76]. However, it is entirely possible to obtain a 'normal' full cohomogeneity first law when regularity is not demanded[77].

Accelerating black holes and other examples

There are many other examples of black hole solutions for which a sensible first law of black hole mechanics can be formulated – even using Euclidean methods – without demanding regularity of the Euclidean solution.

A very interesting (and only recently understood) example is that of *accelerated black holes* with[78,79] or without[6,7,80,81] accelerated horizons. The Euclidean action is not regular due to the presence of conical singularities, or *cosmic strings*, that extend along the north pole/south pole symmetry axes. It is precisely the tension μ of these strings that causes the acceleration of the black hole. As understood in the above studies, the variation of this tension needs to be included in the first law, leading to a new black hole work term

$$\delta W = \lambda d\mu\,, \tag{2.37}$$

with a new conjugate variable λ, known as the thermodynamic length[80,82]. Similar work term also appears for the 'balanced' multi black hole solutions, e.g.[82–86].

Perhaps the most famous example of spacetimes where full regularity cannot be achieved is that of *de Sitter black holes*[87,88] whose Euclidean sector has necessarily conical singularity on at least one of the horizons, cosmological or black hole. Although, such spacetimes are in general out of thermodynamic equilibrium, the corresponding laws of black hole mechanics can be formulated for each horizon separately and consequently combined to a 'mixed' black hole/cosmological horizon first law, e.g.[88].

We may also consider an example of a black hole with *magnetic charge* Q_m. Here the Euclidean geometry is regular, but the $U(1)$ gauge potential

$$A \propto dt + Q_m \cos\theta d\phi \tag{2.38}$$

is not, unless the Dirac quantization condition is imposed. Nevertheless, the first law of black hole mechanics can still be formulated without imposing regularity, picking up an additional work term

$$\delta W = \phi_m \delta Q_m\,, \tag{2.39}$$

where ϕ_m is the conjugate 'magnetostatic' potential, e.g.[30].

To summarize, we have illustrated here through a number of examples, that the potential singularity of the Euclidean solution need not be fatal and does not necessarily prevent one from formulating the first law of black hole mechanics. Instead, one typically picks up additional work terms which come with a well-motivated physical interpretation such as string tension μ in the case of accelerating black holes, or Misner charge N in case of the Taub-NUT solutions. These quantities can be interpreted as the physical sources responsible for the loss of regularity in the Euclidean solution. Their inclusion extends the black hole *thermodynamic dictionary* and poses a challenge for the potential holographic interpretation. While some of the obtained first laws obviously cannot be associated with equilibrium thermodynamics in the spirit of the identification (2.1), as is the case of de Sitter black holes, for others this is not so obvious – for example the Taub-NUT solutions. Therefore, it seems that the precise role played by regularity of the Euclidean solution remains an open question.

Comments on selected papers

The author has devoted a number of papers towards understanding thermodynamics of exotic objects, and in particular the thermodynamics of accelerated black holes and Lorentzian Taub-NUT solutions.

Paper[6] marks the first study of the author on thermodynamics of the accelerated black holes (before the necessity of including the $\lambda d\mu$ term, (2.37), was first realized). The work on this subject culminated in[81] where the full thermodynamics of accelerated black holes with charges and rotations was finally understood.

As discussed above, the thermodynamics of the Lorentzian Taub-NUT solutions finds natural description in terms of the Misner potential ψ, (2.33), and its conjugate quantity N. In[9] the geometric interpretation of these quantities has been first proposed, filling an important gap in understanding these spacetimes.

2.2 Hidden symmetries of rotating black holes

With motivations coming from string theory and braneworld scenarios, higher-dimensional black holes have recently attracted a lot of attention. One of the most surprising discoveries of recent years is a realization that the properties of higher-dimensional rotating black holes are very similar to those of the simple four-dimensional Kerr metric.[§] This remarkable result stems from the existence of a single object called the *principal Killing–Yano tensor*. The very existence of this tensor determines uniquely the Kerr-NUT-AdS family of metrics in all dimensions. It also generates a tower of explicit and hidden symmetries that stand behind complete integrability of geodesic motion and separability of the Hamilton–Jacobi equation, as well guarantee separability of the scalar, spinor, and vector perturbations in these spacetimes. In this section we shall review some of these remarkable results.

Principal Killing–Yano tensor

The Principal Killing–Yano tensor[11,92] is a special object which obeys a number of algebraic and differential restrictions. More specifically, it is a non-degenerate closed conformal Killing–Yano 2-form h obeying the following equation:

$$\nabla_\gamma h_{\alpha\beta} = g_{\alpha\gamma}\xi_\beta - g_{\beta\gamma}\xi_\alpha\,, \tag{2.40}$$

where

$$\xi_\alpha = \frac{1}{d-1}\nabla_\beta h^\beta{}_\alpha \tag{2.41}$$

is the corresponding characteristic 1-form, the primary Killing vector[93]. The non-degeneracy means that h has a maximal matrix rank and that its eigenvalues are functionally independent.[¶]

Contrary to *explicit symmetries* of Killing vectors, whose action is evident on the configuration space (manifold itself), the principal Killing–Yano tensor is a dynamical symmetry of the phase space, and its action on configuration space remains 'hidden'. For this reason we call the corresponding symmetry a *hidden symmetry*. In what follows, we shall assume the existence of this tensor and derive some of the implications.

[§]Here we refer to the black holes with spherical horizon topology that belong to the 'Myers–Perry class'[89]. This excludes both the non-trivial topology generalizations, such as black rings[90], as well as 'bumpy' black holes[91].

[¶]When the algebraic conditions on h are not imposed, the existence of h is far less restrictive and many of the results discussed below become weaker. For example, we have to deal with the corresponding generalized Kerr-NUT-AdS spacetimes which contain unspecified Kähler manifolds[94].

Towers of symmetries

Starting with a single principal Killing–Yano tensor h, one can generate the whole towers of explicit and hidden symmetries – the '*symmetry descendants*' of h[95]. Namely, we can construct the following tower of closed conformal Killing–Yano tensors:

$$h^{(j)} = \frac{1}{j!}\underbrace{h\wedge\cdots\wedge h}_{j\ \text{times}}\,. \tag{2.42}$$

Their Hodge duals $f^{(j)} = \star h^{(j)}$ are Killing–Yano tensors, and their square gives rise to a tower of rank-2 Killing tensors:

$$k^{\alpha\beta}_{(j)} = \frac{1}{(d-2j-1)!}f^{(j)\alpha}{}_{\gamma_1\ldots\gamma_{d-2j-1}}f^{(j)\beta\gamma_1\ldots\gamma_{d-2j-1}}\,. \tag{2.43}$$

In turn, these tensors give rise to the tower of Killing vectors:

$$l^{\alpha}_{(j)} = k^{\alpha\beta}_{(j)}\xi_\beta\,. \tag{2.44}$$

Note that the $j=0$ Killing tensor is just the inverse metric and the zeroth Killing vector is the primary Killing vector, $l_{(0)} = \xi$.

Since the principal Killing–Yano tensor is non-degenerate – it admits n independent eigenvalues in

$$d = 2n+\epsilon \tag{2.45}$$

number of dimensions, where $\epsilon = 0,1$ in even, odd dimensions – the above construction yields n independent Killing tensors, and $(n+\epsilon)$ independent Killing vectors, with the last $l_{(n)}$ in odd dimensions given by $l_{(n)} = f_{(n)}$. Moreover, being constructed from a single object, all these symmetries mutually Schouten–Nijenhuis commute

$$\begin{aligned}\Big[l_{(i)},l_{(j)}\Big]^{\alpha}_{\text{SN}} &= \mathcal{L}_{l_{(i)}}l^{\alpha}_{(j)} = 0\,, \quad \Big[l_{(i)},k_{(j)}\Big]^{\alpha\beta}_{\text{SN}} = \mathcal{L}_{l_{(i)}}k^{\alpha\beta}_{(j)} = 0\,,\\ \Big[k_{(i)},k_{(j)}\Big]^{\alpha\beta\gamma}_{\text{SN}} &= k^{\delta(\alpha}_{(i)}\nabla_\delta k^{\beta\gamma)}_{(j)} - k^{\delta(\alpha}_{(j)}\nabla_\delta k^{\beta\gamma)}_{(i)} = 0\;,\end{aligned} \tag{2.46}$$

and the Killing tensors commute as matrices: $k^{\alpha}_{(i)\,\beta}k^{\beta}_{(j)\,\gamma} - k^{\alpha}_{(j)\,\beta}k^{\beta}_{(i)\,\gamma} = 0$, see[11] for all the details and proofs of the above statements.

Remarkable properties of rotating black holes

It is not surprising that the above towers of symmetries are very special and determine many remarkable properties of the corresponding spacetime. Namely, the following *uniqueness theorem* was proved in[93,96]:

> *Uniqueness theorem:* The most general solution of the Einstein equations with the cosmological constant which admits the principal Killing–Yano tensor is the Kerr-NUT-(A)dS black hole spacetime[97]. Even when the Einstein equations are not imposed, any spacetime admitting such a hidden symmetry can be written in the off-shell Kerr-NUT-AdS form which guarantees the following properties: it is of the special algebraic type D, it allows the separation of variables for the Hamilton–Jacobi, Klein–Gordon, and Dirac equations, the geodesic motion in such a spacetime is completely integrable.

We are already equipped to intuitively understand *complete integrability* of geodesic motion. To this purpose we would need to find d number of integrals of motion that are all independent and mutually Poisson commute[98]. These integrals are simply given by the $(n+\epsilon)$ linear and n quadratic in particle's momenta integrals of motion that are generated from Killing vectors and Killing tensors of the above constructed tower:

$$L_j = p_\alpha l^\alpha_{(j)}\,, \quad K_k = p_\alpha k^{\alpha\beta}_{(k)} p_\beta\,. \tag{2.47}$$

Note that K_0 is related to the Hamiltonian of the geodesic motion, $H = \frac{1}{2}K_0 = \frac{1}{2}p_\alpha g^\beta p_\beta$. The Schouten–Nijenhuis commutation relations between the tensors (2.46) then precisely translate into the mutual Poisson commutation of these integrals:

$$\{L_j, L_k\} = 0\,, \quad \{L_k, K_k\} = 0\,, \quad \{K_j, K_k\} = 0\,. \tag{2.48}$$

The final step in the proof of integrability is to show that all such integrals are independent. This fact was shown in[99–101] and is intimately related to the fact that h possesses independent eigenvalues.

The proof of geodesic integrability[99–101] was shortly followed by the demonstration that the Hamilton–Jacobi[102], the Klein–Gordon[102], and the Dirac[103] equations all *separate* in Kerr-NUT-AdS spacetimes. The geodesic integrability result was later extended to integrability of the bosonic sector of the Grassmann spinning particle model[104].

Perhaps the most remarkable is a more recent result of *separability of vector perturbations* in these spacetimes[10,105,106]. The key ingredience for this development is to consider a novel ansatz for the vector field

$$A^\mu = B^{\mu\nu}\nabla_\nu Z\,, \tag{2.49}$$

where the 'polarization tensor $B^{\mu\nu}$ is the following 'inverse' of the principal Killing–Yano tensor: $B^{\mu\nu} = \left[(g+i\mu h)^{-1}\right]^{\mu\nu}$, and μ is an arbitrary constant. Imposing the *Lorentz 'gauge' condition*,

$$\nabla_\mu A^\mu = 0\,, \tag{2.50}$$

yields a set of differential equations for the separated scalar Z. Interestingly, the ansatz is applicable to both the Maxwell ($m=0$) and massive perturbations:

$$\nabla_\nu F^{\mu\nu} + m^2 A^\mu = 0\,, \tag{2.51}$$

whose equations simply impose an additional algebraic constraint on the separated solution Z. This method therefore provides an alternative to the famous Teukolsky approach[107,108] over which it has many advantages: it is i) covariant ii) applicable in all dimensions, and iii) applies to massive vector fields as well. In particular, the novel ansatz allowed one to study the instability modes of the *ultralight massive vector fields* in the vicinity of rotating black holes, providing thus a characteristic feature for possible observations of axionic-type dark matter candidates[10,109].

Generalized Killing–Yano tensors

The above uniqueness theorem essentially restricts the applications of the principal Killling–Yano tensor to 'vacuum spacetimes' of Einstein gravity – we are inevitably led to the Kerr-NUT-AdS family of metrics. To go beyond, one has to relax some of the assumptions imposed on the principal Killing–Yano tensor. One such generalization, especially useful for supergravity and string theory solutions, is that of *Killing–Yano tensors with torsion*, where the torsion is identified with the 3-form flux naturally present in these theories[110–114].

In particular, we define a generalized principal Killing–Yano tensor with torsion by the following equation generalizing (2.40):

$$\nabla^T_\gamma h_{\alpha\beta} = g_{\alpha\gamma}\xi_\beta - g_{\beta\gamma}\xi_\alpha\,, \tag{2.52}$$

where ∇^T is a covariant derivative with (totally antisymmetric) torsion T, properly identified with the 3-form flux of the given solution. This simple generalization turned out to be very fruitful for example for black holes of $d=5$ minimal gauged supergravity[110] (with T identified with the Maxwell flux, $T = *F/\sqrt{3}$), or the Kerr–Sen solution of the string theory[113] (with T given by the Kalb–Ramond field, $T=H$). At the moment, there is no uniqueness theorem for such a generalized principal Killing–Yano tensor, though a partial classification was achieved in[114].

Other natural generalizations of Killing–Yano tensors, that can be obtained by studying the symmetry operators of the Dirac equation in the presence of general fluxes, were studied in[112].

Comments on selected papers

The author of this Habilitation has played a central role in many of the above developments. The following publications provide a selected overview of some of these works.

Paper[92] presents a discovery of the principal Killing–Yano tensor in higher-dimensional rotating black hole spacetimes. Before its publication it was strongly believed, e.g.[115], that such symmetries are very unlikely to play any role for black holes in higher dimensions. Paper[99] demonstrates complete integrability of geodesic motion in rotating black hole spacetimes, while paper[104] presents a generalization of thereof to the case of a spinning particle described by the Grassmann variables. The separability of the Klein–Gordon and Hamilton–Jacobi equations has been demonstrated in[102]. In[93] the Uniqueness theorem for the principal Killing–Yano tensor was formulated. Paper[110] presents the torsion generalization of the Killing–Yano tensors and demonstrates their relevance for black holes of $d = 5$ minimal gauged supergravity. Finally, paper[10] shows separability of massive vector perturbations in general Kerr-NUT-AdS spacetimes and provides its application to calculating the black hole instability modes due to ultralight massive vector bosons.

2.3 Modified gravity theories

Although incredibly successful and in full agreement with all its experimental verifications, *Einstein's general relativity* cannot be the final theory of gravity. First, it is intrinsically incompatible with quantum theory – the smooth spacetime on which the theory is founded likely emerges only in the low energy limit. Even at the classical level one has to deal with the problem of spacetime singularities, which are a generic prediction of the theory. Furthermore, it is an open question whether the Einstein's relativity provides the correct description at cosmological scales. For all these reasons, it is important to seek new modified theories of gravity. In particular, at the classical level it is natural to study *higher curvature corrections* to the Einstein–Hilbert action, as the latter is only expected to provide effective description for weak gravitational fields. In this section, we shall describe two attempts of the author at formulating new (classical) *higher curvature theories of gravity.*

Lovelock gravities

The gravitational action of a theory that possesses the *diffeomorphism invariance* as a fundamental symmetry is determined by specifying a scalar Lagrangian density $\mathcal{L}$, a function of the metric tensor g and its derivatives,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\mathcal{L}(g,\partial g,\partial^2 g,\dots)\,. \tag{2.53}$$

In order to avoid the so called 'runaway solutions' (or ghosts at the quantum level) an additional assumption, that the resulting equations of motion for the metric should be at most second order in metric derivatives, is often imposed. The simplest solution to these requirements is provided by the *Einstein–Hilbert* action where $\mathcal{L}$ is identified with the Ricci scalar:

$$\mathcal{L}_{\text{EH}} = R\,. \tag{2.54}$$

Is this really just the simplest solution, or can we include some other terms like:

$$R^2\,,\quad R_{\alpha\beta}R^{\alpha\beta}\,,\quad R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,,\quad R^3\,,\quad \nabla_\alpha R\nabla^\alpha R,\quad \dots\,? \tag{2.55}$$

The answer is provided by the following theorem[116]:

> *Lovelock theorem (1971)*: In four dimensions, the Einstein–Hilbert action is the only local action, apart from the cosmological constant and topological

terms (total derivatives), that leads to the second-order partial differential equations for the metric.

An example of an interesting topological term in four dimensions is the following *Gauss–Bonnet term*:

$$\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,. \tag{2.56}$$

Interestingly, this term is topological only in four dimensions. It identically vanishes in $d < 4$, and yields a non-trivial equations of motion in $d = 5$ and higher dimensions, governed by the corresponding Gauss–Bonnet tensor:

$$H_{\alpha\beta} = -\frac{1}{2}g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_{\beta}{}^{\gamma} + 4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_{\alpha}{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa}\,. \tag{2.57}$$

By inspection, this tensor leads to second-order equations of motion for the metric, and thence a generalization of the Einstein gravity in $d \geq 5$ dimensions.

More generally, one may consider the *Lovelock gravity*[116]. This is a unique higher-curvature (with local action) gravity in d dimensions that yields second order partial differential equations for the metric. The corresponding Lagrangian density is given by

$$\mathcal{L} = \frac{1}{16\pi G}\sum_{k=0}^{K}\alpha_k\mathcal{L}^{(k)}\,, \tag{2.58}$$

where $\mathcal{L}^{(k)}$ are the $2k$-dimensional Euler densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k}\delta^{\alpha_1\beta_1\ldots\alpha_k\beta_k}_{\gamma_1\delta_1\ldots\gamma_k\delta_k}R_{\alpha_1\beta_1}{}^{\gamma_1\delta_1}\ldots R_{\alpha_k\beta_k}{}^{\gamma_k\delta k}\,, \tag{2.59}$$

and α_k are the associated coupling constants. In particular, $k = 0$ is the cosmological constant term, $\Lambda = -\alpha_0/2$, $k = 1$ corresponds to the Einstein–Hilbert action, $k = 2$ recovers the Gauss–Bonnet term, and so on. Interestingly, the basic property of the Euler densities is that they are topological in $d = 2k$ dimension, trivial below, and contribute to equations of motion for $d > 2k$. This is why the sum terminates at $K = \left[\frac{d-1}{2}\right]$. Many people consider the Lovelock gravity to be a natural generalization of Einstein's gravity to higher dimensions, though the problem of well-posedness is a non-trivial one, see e.g.[117,118].

Beyond Lovelock

While Lovelock gravities are very natural geometric generalizations of the Einstein gravity, they are only non-trivial in higher dimensions. If one wants to restrict to (what seems to be a physical dimension) $d = 4$, one must therefore seek other alternatives.

In a purely geometric framework, where gravity is described by the metric only, it follows from the Lovelock theorem that one must consider theories that lead to higher-order equations of motion. A famous example of one such theory is the so called *quasi-topological gravity*[119]. This is a higher curvature theory (a generalization of the Lovelock gravity) whose equations of motion become second-order on spherically symmetric spacetimes. Another very specific theory which is non-trivial even in four dimensions is known as the *Einsteinian cubic gravity*[120]. Guided by the importance of spherically symmetric solutions, a *generalized quasi-topological theory* was proposed by the author and his collaborators[13], see also[121,122] for its later generalizations. These new theories play, for example, a natural role in 'geometric inflation', e.g.[123].

Going beyond the purely geometric framework, one may consider theories with extra gravitational degrees of freedom. Among such theories, the forefront position belongs to the *Horndeski gravity*[124], which is the most general scalar-tensor theory that yields second order equations of motion for both the metric and the scalar field. As shown by the author[12], a special subclass of Horndeski theories is obtained by taking a (singular) limit of the Gauss–Bonnet gravity to four dimensions.

Comments on selected papers

In paper[13] the author constructed the most general cubic in curvature theory with the following remarkable properties: i) It has a well-defined Einstein gravity limit, ii) it admits "Schwarzschild-like" solutions characterized by a single metric function, iii) on maximally symmetric backgrounds it propagates the same degrees of freedom as Einstein's gravity, and iv) all of the following: Lovelock[116], quasitopological[119], and Einsteinian cubic[120] gravities are recovered as special cases.

Recently, there has the been a lot of interest in whether or not there exists a meaningful theory of gravity obtained by taking a singular limit of the Gauss–Bonnet gravity to $d = 4$ dimensions[125–129]. In paper[12] we have shown that a well-defined $d \to 4$ limit of the Gauss–Bonnet Gravity is obtained by generalizing a method of Mann and Ross[130], used many years ago to obtain a limit of the Einstein gravity in $d = 2$ dimensions. The resulting theory is a scalar-tensor theory of the Horndeski type, obtained alternatively by a special dimensional reduction[127].

2.4 Miscellaneous results

Apart from hidden symmetries, black hole thermodynamics, and modified gravity theories, the author of this Habilitation has also been interested in many other problems of gravitational physics. In this section, we shall present three different results on: gravitational waves, quantum detection of inertial frame dragging, and black hole cosmic string hair, that provide a glimpse into the author's other interests.

Gravitational waves

Gravitational waves provide a new window into our Universe. The great discovery made by LIGO on September 14, 2015[2] provided the first direct confirmation that strong gravitational waves are produced in the violent process of the coalescence of two black holes. Other such observations shortly followed. The effort culminated in an observation of a neutron star-neutron star collision[131], marking the beginning of *multi messenger astronomy.*

The precise modelling of the merger and of the corresponding gravitational wave production are complicated and have to often be simulated numerically or by using various approximations. In[14] we have employed the so called *moduli space approximation* to study the collision of charged black holes surrounded by a dilatonic field, picking up the threads on the work[132] where no dilatonic field was considered. The special feature of such an approximation is that it remains valid in strong gravitational regime (as opposed to the post-Newtonian expansion). The price to pay is that the approximation is only valid for extrenal black holes that are supported by (most likely unphysical) strong electromagnetic fields. The obtained results clearly illustrate the effect of the dilatonic field on the gravitational wave production, and in turn impose some restrictions on the corresponding theories with dilatons, string theory for example.

Quantum detection of inertial frame dragging

Frame dragging (or gravitomagnetism) is a general-relativistic effect induced by the motion (and in particular rotation) of matter and gravitational waves, that is in many ways analogous to electromagnetic induction. Already in the early days of general relativity Lens and Thiring observed[133,134] that in the vicinity of a rotating body an infalling geodesic observer experiences a Coriolis-type force, and that gyroscopes are subject to precession, which can be thought of as a Larmor precession induced by the gravitomagnetic fields. The extreme frame dragging manifest itself by the existence of an ergosphere in the vicinity of rotating black holes. Frame-dragging is also behind various astrophysical phenomena such as relativistic jets and the Bardeen-Petterson effect[135], which aligns accretion disks perpendicular to the axis of a rotating black hole. The frame dragging by the planet Earth has been measured recently by the Gravity Probe B satellite mission[136,137].

In[15] we have proposed a toy model for quantum measurement of the frame dragging effect. Namely, we have positioned an Unruh–deWitt detector[138,139] inside a rotating shell and shown that it can pick up the information about the frame dragging with respect to distant stars. This happens despite that the space inside the shell is flat and the detector remains inertial for all times. Perhaps even more interestingly, it happens even when the detector is switched on for a finite time interval within which a light signal cannot travel to the shell and back as required by a classical measurement. In principle, our results open a possibility of measuring the frame dragging effect in the laboratory settings and analogue systems.

Cosmic string hair on rotating black holes

Cosmic strings are examples of field theory topological defects that could have been created by phase transitions in the early Universe. An interaction of such strings with a black hole might result in a string capture, and black hole featuring a novel type of cosmic string hair – providing thus another counter-example to black hole *no hair theorems*[140,141].

In[16] we have shown that rotating black holes can indeed sport cosmic string hair. Interestingly, it was found that, contrary to the common wisdom, the backreaction of the string can no longer be described by a simple conical deficit, as is the case for non-rotating black holes[142]. Moreover, it is well known, e.g.[143–146], that extremal black holes feature a flux expulsion for simple electromagnetic fields, also known as the black hole *Meissner effect*. In the case of cosmic strings, however, the situation is much more subtle[16,147]. Namely, small black holes demonstrate flux expulsion, while large ones are pierced by the cosmic string. The phase transition between the two situations is of the second order for charged black holes[147], and of the first order for the rotating ones[16], demonstrating that interesting critical phenomena also occur in classical black hole physics.

3 References

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4 List of author's publications

Research Papers

[1] F. Gray, D. Kubiznak, T. May, S. Timmerman and E. Tjoa, *Quantum imprints of gravitational shockwaves*, `arXiv:2105.09337`.

[2] W. Cong, D. Kubiznak and R. B. Mann, *Thermodynamics of AdS Black Holes: Critical Behavior of the Central Charge*, Physical Review Letters **127** (2021) 091301, `doi:10.1103/PhysRevLett.127.091301` [`arXiv:2105.02223`].*

[3] W. Cong, J. Bicak, D. Kubiznak and R. B. Mann, *Quantum detection of conicity*, Physics Letters B **820** (2021) 136482, `doi:10.1016/j.physletb.2021.136482` [`arXiv:2103.05802`].

[4] F. Gray, T. Houri, D. Kubiznak and Y. Yasui, *Symmetry operators for the conformal wave equation in rotating black hole spacetimes*, `arXiv:2101.06700`.

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These study texts are available upon request. The corresponding lectures are available online on `PIRSA`.