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Ultrafilters and their monads

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I would like to thank my parents for their unconditional support during my studies.

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Abstract: Generalising the notion of an ultrafilter to structured sets, we construct the ultrafilter monad in the categories of partially ordered sets and finitely colourable graphs. This is done similarly to codensity monads, knowing that the codensity monad of the inclusion of finite sets into sets is the ultrafilter monad. We derive an equivalent definition of an ultrafilter on an object applicable for general graphs, also giving rise to a monad. We show that ultrafilters on a poset can be completely characterised in terms of suprema or infima of directed subsets when the poset has only finite antichains. We attempt to classify algebras over the poset ultrafilter monad; our results completely classify the algebras with all antichains finite as posets with a particular compact Hausdorff topology.

Keywords: ultrafilter, monad, partial order, topology

Contents

Introduction

Ultrafilters originate in set theory and have found many uses elsewhere, including but not limited to topology. It is straightforward to generalise the definition of ultrafilter for use in an arbitrary partial order instead of the power set of some set. Both topological uses and general ultrafilters in posets appear in this thesis.

Intuitively, one can think of ultrafilters as strategies in the following number guessing game. Player one fixes a secret natural number n. Player two wants to find out the value of n, they ask questions of the form "Is $n \in A$?", for some subset $A \subseteq \mathbb{N}$. Player one always answers yes or no. For their answers to be consistent, the following have to hold:

- If the answer was yes to '$n \in A$' and '$n \in B$', then surely '$n \in A \cap B$' has to be answered positively.
- Question '$n \in \emptyset$' has to be answered negatively. On the other hand, '$n \in \mathbb{N}$' has to be true.
- If we have $A \subseteq B$ and the answer was yes to '$n \in A$', then the answer to '$n \in B$' must also be yes.
- If the question '$n \in A$' was answered negatively, then '$n \in (\mathbb{N} \setminus A)$' has to be true.

We can see that these conditions match exactly the definition of ultrafilters on the set $\mathbb{N}$. Exploiting the fact that at any point in time player two has so far asked only finitely many questions, player one can cheat by picking a nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and answering yes to $n \in A$ iff $A \in \mathscr{U}$.

Important property of set ultrafilters is that they pick from finite partitions of the underlying set. That is if

$$A_1 \amalg A_2 \amalg \ldots \amalg A_n = X,$$

any ultrafilter $\mathscr{U}$ on X contains the set A_i for one and only one $i \in \{1, \ldots, n\}$. Such a partition is essentially a function

$$f : X \to \{1, \ldots, n\}, \qquad f(x) = i \iff x \in A_i.$$

Then the condition can be rephrased as $\mathscr{U}$ contains the fibre $f^{-1}(i)$ for exactly one $i \in \{1, \ldots, n\}$. This property can be used to define a peculiar integration operator assigning to f the element whose f-fibre belongs to $\mathscr{U}$. Properties of such integration operators are what we use to derive a definition of ultrafilter in other categories. From our results on partial orders it turns out that arbitrary partitions are in general not the same as partitions into f-fibres, but both of these partition conditions of ultrafilters still hold.

In chapter 1 we examine ultrafilters on sets using the language of integration operators as mentioned above. Similar concept is shown for elements of double dual of a vector space. In chapter 2 we define ultrafilters in categories of partial orders and graphs, using the properties of integration operators from chapter 1. The third chapter first connects these newly defined ultrafilters with ultrafilters in general partial orders. Then we define order or graph structure on the set of all ultrafilters, which arises from a categorical limit. Here we define the ultrafilter functor. Unlike for sets, we show it is sometimes possible to explicitly describe nontrivial ultrafilters on partial orders. The fourth chapter contains construction of the ultrafilter monad and incomplete classification of its algebras in the category of posets.

Previous work In 1969 Manes has shown in [7] that algebras over the set ultrafilter monad are precisely the compact Hausdorff spaces. A 2013 article [5] of Tom Leinster has connected this and other previously known results, showing the common category-theoretical properties of ultrafilters on sets and double dualisation of vector spaces. In 2016 Devlin, supervised by Leinster, has explored ultrafilters in general algebraic theories in his PhD thesis [2]. A 2020 article [1] by Adámek and Sousa shows general construction of the ultrafilter monad using concepts similar to double dualisation of vector spaces.

Ideas appearing in this thesis are more reminiscent of those in Devlin's work, but are applied in different context. Our approach naturally leads to a different, but equivalent, definition of ultrafilters on partial orders than the one in Adámek and Sousa [1]. Our definition of graph differs, as we do not allow loops. With this definition one cannot apply their general result about D-ultrafilters, as the resulting category of loopless (finitely colourable) graphs has no finite cogenerator.

1. First results

Here we briefly examine the better known cases of sets and vector spaces. We define integration against an ultrafilter and observe which properties characterise such an operator. Similar integral is defined for the vector space double dual. Proof of the fact that this defines a functor and a monad induced by it is omitted, as the proof for partial orders and graphs presented later is analogous.

1.1 Ultrafilters on sets

1.1.1 Definition

Given a set X, a *filter on* X is a nonempty collection $\mathscr{U} \subseteq \mathscr{P}(X)$ of its subsets, satisfying

- $\emptyset \notin \mathscr{U}$,
- $(\forall A, B \in \mathscr{U})\ A \cap B \in \mathscr{U}$,
- $(\forall A, B \subseteq X)(A \subseteq B \,\&\, A \in \mathscr{U}) \implies B \in \mathscr{U}$.

An *ultrafilter on a set* X is a maximal such collection. Important characterisation says a filter $\mathscr{U}$ is maximal if and only if for every finite partition

$$A_1 \amalg A_2 \amalg \ldots \amalg A_n = X,$$

exactly one A_i is an element of $\mathscr{U}$. By partition we mean a collection of pairwise disjoint, possibly empty, subsets, whose union is the whole set. We use the symbol $\amalg$ in place of the set union $\cup$, when the operands are disjoint sets. In particular,

$$A \in \mathscr{U} \quad \text{or} \quad X \setminus A \in \mathscr{U},$$

but not both. Proposition 1.5 in Leinster [5] tells us that any collection $\mathscr{U} \subseteq \mathscr{P}(X)$ satisfying this partition property for any $n \geq 3$ is an ultrafilter. A single axiom characterisation of an ultrafilter $\mathscr{U} \subseteq \mathscr{P}(X)$ is

$$(\forall A \subseteq X)(A \in \mathscr{U} \iff (\forall n \in \mathbb{N}\ \forall B_1, \ldots, B_n \in \mathscr{U})\ A \cap B_1 \cap \ldots \cap B_n \neq \emptyset).$$

The set of all ultrafilters on a set X is denoted βX.

1.1.2 Integration

Here we use the partition property of ultrafilters to define a special integration operator. Consider a function $f : X \to A$ of sets, where A is finite. Label $A = \{a_1, \ldots, a_n\}$, then

$$f^{-1}(a_1) \cup \ldots \cup f^{-1}(a_n)$$

is a partition of the set X. Thus for any ultrafilter $\mathscr{U} \in \beta X$ we have a unique $a \in A$, for which $f^{-1}(a) \in \mathscr{U}$.

For single points we denote the preimage (fibre) by $f^{-1}(a)$, whereas for a set A we denote the preimage by $f^{-1}[A]$. Throughout the text we use functions with finite codomain, we call such functions *simple*. We also use the word simple for morphisms with finite codomain in concrete categories.

Definition. Let $\mathscr{U}$ be an ultrafilter on a set X. If f is a function from X into a finite set, we define the integral of f with respect to $\mathscr{U}$, denoted

$$\int_X f \,\mathrm{d}\mathscr{U},$$

to be the unique element for which

$$f^{-1}\Big(\int f \,\mathrm{d}\mathscr{U}\Big) \in \mathscr{U}.$$

Proposition 1. *Let X be a set, A, B finite sets and $f : X \to A$, $g : A \to B$ functions. Then*

$$g\Big(\int_X f \,\mathrm{d}\mathscr{U}\Big) = \int_X g \circ f \,\mathrm{d}\mathscr{U}.$$

Proof. If $a = \int f \,\mathrm{d}\mathscr{U}$ and $b = g(a)$, then $(g \circ f)^{-1}(b) = f^{-1}[g^{-1}(b)] \supseteq f^{-1}(a)$. But $f^{-1}(a) \in \mathscr{U}$, thus the superset $(g \circ f)^{-1}(b)$ is also an element of $\mathscr{U}$. By definition $b = \int g \circ f \,\mathrm{d}\mathscr{U}$. □

Consider a map $\int \_ \,\mathrm{d}x$ assigning to a function $f : X \to A$ of sets with finite codomain A an element $\int f \,\mathrm{d}x \in A$. We will call this map a *simple integration operator* if it satisfies the condition

$$g\Big(\int f \,\mathrm{d}x\Big) = \int g \circ f \,\mathrm{d}x, \tag{1.1}$$

for any $f : X \to A$, $g : A \to B$ with A, B finite. Here the $\mathrm{d}x$ serves no other purpose than to distinguish the symbol from $\mathrm{d}\mathscr{U}$.

Lemma 2. *For any simple integration operator $\int f \,\mathrm{d}x \in \operatorname{rng} f$.*

Proof. Consider the inclusion $i : \operatorname{rng} f \hookrightarrow A$ and corestriction $\tilde{f} : X \twoheadrightarrow \operatorname{rng} f$ of f onto its range. Then $i \circ \tilde{f} = f$. By (1.1) we have $i(\int \tilde{f} \,\mathrm{d}x) = \int f \,\mathrm{d}x$, thus $\int f \,\mathrm{d}x \in \operatorname{rng} i = \operatorname{rng} f$. □

We showed that any ultrafilter induces a simple integration operator. Now from any such operator, we will recover the ultrafilter defining it in the above sense. From this we conclude that this correspondence is bijective. Later, this correspondence will be used to pick a proper definition of an ultrafilter in other categories.

For a fixed simple integration operator on X define

$$\mathscr{U} \coloneqq \Big\{ f^{-1}\Big(\int f \,\mathrm{d}x\Big) : f \colon X \to A, A \text{ finite} \Big\}.$$

This is a subset of the power set of X. We want to show it is an ultrafilter, and that integration against it is exactly $\int \_ \,\mathrm{d}x$. We will show for any partition $A_1, \ldots, A_n$ of X that exactly one $A_i \in \mathscr{U}$. Then by Leinster [5], Proposition 1.5, $\mathscr{U}$ is an ultrafilter on X.

Existence: Let $A_1, \ldots, A_n$ be a finite partition of X. Then $f : X \to \{1, \ldots, n\}$ sending $x \in A_i$ to i is a function with finite codomain. By definition $f^{-1}(j) \in \mathscr{U}$ for $j = \int f \,\mathrm{d}x$. But $f^{-1}(j) = A_j$, so $A_j \in \mathscr{U}$.

Uniqueness: First of all, if $S \in \mathscr{U}$, we will show that there exists an f, for which $f^{-1}(\int f \,\mathrm{d}x) = S$, its codomain is equal to $\{1,2\}$ and $\int f \,\mathrm{d}x = 1$. Let $h : \operatorname{cod} f \to \{1,2\}$ be defined as

$$h(a) \coloneqq \begin{cases} 1, & a = \int f \,\mathrm{d}x, \\ 2, & \text{otherwise.} \end{cases}$$

Then $\int h \circ f \,\mathrm{d}x = h(\int f \,\mathrm{d}x) = 1$ and $(h \circ f)^{-1}(1) = f^{-1}(\int f \,\mathrm{d}x) = S$. Thus $h \circ f : X \to \{1,2\}$ also witnesses $S \in \mathscr{U}$.

Now assume for a contradiction $F, G \in \mathscr{U}$ and $F \cap G = \emptyset$, with witnesses $f, g : X \to \{1,2\}$ satisfying $\int f \,\mathrm{d}x = \int g \,\mathrm{d}x = 1$. Define $h : X \to \{1,2,3\}$ by

$$h(x) \coloneqq \begin{cases} 1, & x \in F, \\ 2, & x \in G, \\ 3, & \text{otherwise.} \end{cases}$$

Then $f = \varphi \circ h$, $g = \gamma \circ h$, where

$$\varphi(k) \coloneqq \begin{cases} 1, & k = 1, \\ 2, & k = 2,3, \end{cases} \qquad \gamma(k) \coloneqq \begin{cases} 1, & k = 2, \\ 2, & k = 1,3, \end{cases} \qquad k \in \{1,2,3\}.$$

Thus $1 = \int f \,\mathrm{d}x = \varphi(\int h \,\mathrm{d}x)$, hence $1 = \int h \,\mathrm{d}x$. But then $\int g \,\mathrm{d}x = \gamma(\int h \,\mathrm{d}x) = \gamma(1) = 2$, a contradiction. Hence $\mathscr{U}$ does not contain disjoint sets and the choice from partition is unique.

Now we know that $\mathscr{U}$ is indeed an ultrafilter. The equality $\int \_ \,\mathrm{d}\mathscr{U} = \int \_ \,\mathrm{d}x$ is easy to see from definition of $\mathscr{U}$.

1.2 Vector spaces and their duals

Let us recall some basic notions from linear algebra. We will assume all vector spaces to be over a fixed field k. For a vector space V a *linear form* on V is a linear map $V \to k$. The space of all linear forms is also a vector space, denoted by V^* and called the *dual* of V. We can iterate this construction to arrive at the double dual $V^{**} = (V^*)^*$. For a finite dimensional vector space V with a basis $(e_1, \ldots, e_n)$ there is a *dual basis* of V^* $(\varepsilon^1, \ldots, \varepsilon^n)$, where ε^i is the form assigning to $x \in V$ the i-th coordinate of x w.r.t. the basis $(e_1, \ldots, e_n)$. For any V and $x \in V$ we can define an element $\mathscr{E}_x \in V^{**}$, $\mathscr{E}_x(f) \coloneqq f(x), f \in V^*$. This element is called *evaluation at x*. For finite dimensional vector spaces the evaluation map $\mathscr{E} : V \to V^{**}, x \mapsto \mathscr{E}_x$ is an isomorphism.

1.2.1 Integration

Let V be a k-vector space, $v \in V^{**}$. For a linear map $f : V \to N$, where N is finite dimensional and has a basis $(e_1, \ldots, e_n)$, denote the dual basis of N^* by $(\varepsilon^1, \ldots, \varepsilon^n)$ and define

$$\int_V f \,\mathrm{d}v \coloneqq \sum_{i=1}^{n} v(\varepsilon^i \circ f) e_i.$$

Simple integration operators on sets returned element whose f-fibre was element of the ultrafilter. If we want to somehow test f against v, we need to provide a linear form, not a subset. Natural option is to use the components of f with respect to the chosen basis of N. We can express them as $\varepsilon^i \circ f$, $i = 1, \dots, n$. We get back scalars instead of a yes or no answer from an ultrafilter, those can be used as coefficients with respect to the same basis of N.

However the choice of basis of N was arbitrary, and has to be done for each finite dimensional vector space. We will show the value of integral is in fact independent on the choice of basis. Similarly to the case of sets, the following identity

$$\int_V g \circ f \,\mathrm{d}v = g\Big(\int_V f \,\mathrm{d}v\Big) \tag{1.2}$$

holds for every linear map g of finite dimensional spaces.

Let N be finite dimensional, $f : V \to N$ linear, $(e_1, \dots, e_n)$, $(x_1, \dots, x_n)$ two bases of N with respective dual bases $(\varepsilon^1, \dots, \varepsilon^n)$, $(\xi^1, \dots, \xi^n)$. Express the coordinates of each with respect to the other as $x_i = \sum_{k=1}^n a_k^i e_k$, $e_k = \sum_{i=1}^n \alpha_i^k x_i$. Then $\sum_{k=1}^n a_k^i \alpha_j^k = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

First we prove the following statement:

$$\sum_{i=1}^n a_k^i \xi^i = \varepsilon^k$$

Proof. It suffices to show this equality on basis elements

$$\begin{aligned}\Big(\sum_{i=1}^n a_k^i \xi^i\Big)(e_l) = \sum_{i=1}^n a_k^i \xi^i \Big(\sum_{j=1}^n \alpha_j^l x_j\Big) &= \sum_{i=1}^n \sum_{j=1}^n a_k^i \alpha_j^l \xi^i(x_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_k^i \alpha_j^l \delta_{ij} = \sum_{i=1}^n a_k^i \alpha_i^l = \delta_{kl} = \varepsilon^k(e_l).\end{aligned}$$

□

Then

$$\begin{aligned}\sum_{i=1}^n v(\xi^i \circ f) x_i &= \sum_{i=1}^n v(\xi^i \circ f) \sum_{k=1}^n a_k^i e_k \\ &= \sum_{k=1}^n v\Big(\Big(\sum_{i=1}^n a_k^i \xi^i\Big) \circ f\Big) e_k = \sum_{k=1}^n v(\varepsilon^k \circ f) e_k = \int_V f \,\mathrm{d}v,\end{aligned}$$

hence $\int f \,\mathrm{d}v$ is defined correctly, independent of the basis.

Additionally let M be finite dimensional, $g : N \to M$ and $(b_1, \dots, b_m)$ a basis of M with a dual basis $(\beta_1, \dots, \beta_m)$. Express $g(e_i) = \sum_{k=1}^m \gamma_k^i b_k$. To prove (1.2) we will use

$$\sum_{i=1}^n \gamma_k^i \varepsilon^i = \beta^k \circ g.$$

Proof.

$$(\beta^k \circ g)(e_j) = \beta^k \Big(\sum_{i=1}^m \gamma_i^j b_i\Big) = \sum_{i=1}^m \gamma_i^j \delta_{ki} = \gamma_k^j = \Big(\sum_{i=1}^n \gamma_k^i \varepsilon^i\Big)(e_j).$$

□

Then

$$g\Big(\int_V f\,\mathrm{d}v\Big) = g\Big(\sum_{i=1}^n v(\varepsilon^i \circ f)e_i\Big) = \sum_{i=1}^n v(\varepsilon^i \circ f)g(e_i)$$
$$= \sum_{k=1}^m v\Big(\Big(\sum_{i=1}^n \gamma_k^i \varepsilon^i\Big) \circ f\Big)b_k = \sum_{k=1}^m v(\beta^k \circ (g \circ f))b_k = \int_V g \circ f\,\mathrm{d}v,$$

which proves the desired identity (1.2).

Now similarly to the case of ultrafilters, call $\int_V \_\,\mathrm{d}x$ a simple integration operator if it assigns to linear maps $f : V \to N$ with N some finite dimensional vector space a value $\int_V f\,\mathrm{d}x \in N$, and satisfies the identity (1.2). From this operator we want to extract a corresponding element of the double dual.

Lemma 3. *For any simple integration operator* $\int f\,\mathrm{d}x \in \operatorname{rng} f$.

Proof. Let $f : V \to N$ be linear, N finite dimensional. $\operatorname{rng} f$ is a subspace of N, therefore it has finite dimension. The inclusion $i : \operatorname{rng} f \hookrightarrow N$ is linear, so is the corestriction $\tilde{f} : V \twoheadrightarrow \operatorname{rng} f$ of f onto its image. Then $i \circ \tilde{f} = f$. By (1.2) we have $i(\int \tilde{f}\,\mathrm{d}x) = \int f\,\mathrm{d}x$, thus $\int f\,\mathrm{d}x \in \operatorname{rng} i = \operatorname{rng} f$. □

The map

$$p : N^2 \to N, \qquad (x, y) \mapsto x + y,$$

the projections

$$\pi_i : N^n \to N, \qquad (x_1 \ldots, x_n) \mapsto x_i,$$

and scalar multiplication

$$s_t : N \to N, \qquad x \mapsto tx, \qquad t \in k,$$

are all linear. By applying the identity (1.2) on all π_i's we get

$$\int_V (f_1, \ldots, f_n)\,\mathrm{d}x = \Big(\int_V f_1\,\mathrm{d}x, \ldots, \int_V f_n\,\mathrm{d}x\Big), \tag{1.3}$$

where $f_i : V \to N$ are linear, $i = 1, \ldots, n$, and $(f_1, \ldots, f_n)$ is the product map $V \to N^n$, $v \mapsto (f_i(v))_{i=1}^n$. In other words the i-th component of an integral is integral of the i-th component of a map. Using this fact, further applying the identity on p and s_t, we get linearity of the operator $\int_V \_\,\mathrm{d}x$ on each of the spaces $\operatorname{Hom}(V, N) = \{f : V \to N \text{ linear}\}$. In particular, taking $N = k$, we get $w := \int_V \_\,\mathrm{d}x \restriction V^*$ (restriction of integral on the dual) is an element of V^{**}. Now using (1.3) for $N = k$ and $f : V \to k^n$ we get

$$\pi_j\Big(\int_V f\,\mathrm{d}x\Big) = \int_V \pi_j \circ f\,\mathrm{d}x = w(\pi_j \circ f) = \pi_j\Big(\int_V f\,\mathrm{d}w\Big),$$

thus $\int_V f\,\mathrm{d}x = \int_V f\,\mathrm{d}w$, as all their components are equal. Choosing a basis of M and using (1.2) this equality transfers to $f : V \to M$ with M being an arbitrary finite dimensional vector space.

We have proved that simple integration operators on a vector space V are in bijective correspondence with elements of the double dual V^{**}, similarly to ultrafilters on sets.

2. Integration of simples in structured sets

So far, we have seen a general construction of ultrafilters on sets, which have no particular structure, and dualisation of a vector space. A different notion of finiteness occurred there, finite dimension instead of cardinality of the underlying set. Functions on sets have no condition to satisfy, whereas linear maps need to preserve the vector space structure. Thus the resulting objects are very different and their similarities are not immediately obvious. In this chapter, we will derive a sensible definition of an ultrafilter on sets with a single binary relation, partially ordered sets and (undirected) graphs.

2.1 Partially ordered sets

A set P is *partially ordered* by a relation $\leq$ if $\leq$ is reflexive, transitive and weakly antisymmetric, i.e. for all $x, y, z \in P$ it holds

- $x \leq x$,
- $x \leq y \,\&\, y \leq z \implies x \leq z$,
- $x \leq y \,\&\, y \leq x \implies x = y$.

Ordered sets need not be *linear*, meaning some elements might be incomparable. With partial orders we consider order homomorphisms, or *monotone maps*, those being $f : (P, \leq_P) \to (Q, \leq_Q)$ satisfying

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

This is the standard choice of morphisms, the resulting category is denoted Poset. For partially ordered sets we also use the short word posets.

For vector spaces subsets closed under relations were vector subspaces. In ordered sets we substitute subspaces with intervals, with the same definition used to characterise intervals on the real line.

Definition. A set $A \subseteq P$ in an ordered set $(P, \leq)$ is an *interval*, if

$$(\forall x, z \in A\ \forall y \in P)\ x \leq y \leq z \implies y \in A.$$

We write $A \sqsubseteq P$, whenever A is an interval in P.

Lemma 4. *The preimage of a single point under a monotone map is an interval.*

Proof. Let $f : P \to Q$ be homomorphism of orders, $q \in Q$. Then for $x, z \in f^{-1}(q)$ and $y \in P$ satisfying $x \leq y \leq z$ we have $f(x) \leq f(y) \leq f(z)$ due to the monotony of f. But $f(x) = f(z) = q$, thus $q \leq f(y) \leq q$, so $q = f(y)$. Hence $y \in f^{-1}(q)$. $\square$

This lemma tells us the importance of intervals for our use. In our context, elements of ultrafilters were preimages of points under maps with finite codomain. The same will happen for posets and graphs. Lemma 4 can be immediately generalised:

Lemma 5. *The preimage of an interval under a monotone map is again an interval.*

Proof. Let $f : P \to Q$ be monotone and $A \sqsubseteq Q$ an interval. Consider $x, z \in f^{-1}[A]$ and $y \in P$ such that $x \leq y \leq z$. Then again $f(x) \leq f(y) \leq f(z)$. Because A is an interval and $f(x), f(z) \in A$, we get $f(y) \in A$. This gives $y \in f^{-1}[A]$. $\square$

However, unlike for vector subspaces, the image of an interval need not be an interval. For example consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Lemma 6. *Let A be an interval in $(P, \leq)$. Then $B \subseteq A$ is an interval in A iff it is an interval in P. Intersection of any system of intervals is an interval.*

Proof. Clear. $\square$

Before diving into integration, we examine products of partial orders.

Definition. Let $(P_i, \leq_{P_i}), i \in I$, be ordered sets. By the *product order* we mean a relation on the cartesian product $\bigtimes_{i \in I} P_i$ defined as

$$(a_i)_{i \in I} \leq (b_i)_{i \in I} \iff (\forall i \in I)\ a_i \leq_{P_i} b_i.$$

We denote this product of ordered sets by $\prod_{i \in I}(P_i, \leq_{P_i})$.

Reflexivity, transitivity and weak antisymmetry all follow from the same property of every $\leq_{P_i}$. This is actually the categorical product in Poset.

Lemma 7. *Let $f_i : X \to P_i$ be order homomorphisms, $i \in I$. Define*

$$(f_i)_{i \in I} : X \to \bigtimes_{i \in I} P_i, \quad x \mapsto (f_i(x))_{i \in I}, \qquad x \in X.$$

Then

(i) $(f_i)_{i \in I}$ *is monotone,*

(ii) the projections $\pi_j : \prod_{i \in I} P_i \to P_j$, $(p_i)_{i \in I} \mapsto p_j$ *are all monotone,*

(iii) $f_j = \pi_j \circ (f_i)_{i \in I}$, $(f_i)^{-1}_{i \in I}((p_i)_{i \in I}) = \bigcap_{i \in I} f_i^{-1}(p_i)$.

Proof. Clear from the definitions. $\square$

2.1.1 Integration and ultrafilters

Similarly as in Set, by a simple integration operator on a partial order $(X, \leq)$ we mean an operator $\int_X \_ \,\mathrm{d}x$ satisfying the following condition. For P, Q finite ordered sets, $f : X \to P$ and $g : P \to Q$ monotone maps, it holds that

$$g\Big(\int_X f \,\mathrm{d}x\Big) = \int_X g \circ f \,\mathrm{d}x. \tag{2.1}$$

Fix $(X, \leq)$ and a simple integration operator on it. Consider the following collection of intervals

$$\mathscr{U} := \Big\{ f^{-1}\Big(\int_X f \,\mathrm{d}x\Big) : f \text{ is homomorphism } X \to P,\ P \text{ finite ordered} \Big\}.$$

We will show that $\mathscr{U}$ is closed under finite intersections and upwards closed for intervals. Precisely, with every $A \in \mathscr{U}$, all intervals $B \sqsubseteq X$ satisfying $A \subseteq B$ are also in $\mathscr{U}$. Additionally, we will show a partition property – given a partition of X into intervals

$$A_1 \amalg \ldots \amalg A_n = X,$$

exactly one A_i belongs to $\mathscr{U}$.

Firstly, same as in Lemma 2, we get $\int f \, \mathrm{d}x \in \operatorname{rng} f$. The proof works because $\operatorname{rng} f$ is a (finite) poset and the inclusion is monotone. This ensures $\emptyset \notin \mathscr{U}$. We already know that preimages of points are intervals. Here we show the converse.

Proposition 8. *Let $A \sqsubseteq X$ be an interval. Then there exists a homomorphism $f : (X, \leq) \to (P, \leq)$ and $a \in P$ such that $A = f^{-1}(a)$ and P is finite.*

Proof. Consider the sets

$$\begin{aligned} U &:= \{x \in X \setminus A : (\exists a \in A)\ a \leq x\}, \\ L &:= \{x \in X \setminus A : (\exists a \in A)\ x \leq a\}, \\ N &:= \{x \in X \setminus A : (\forall a \in A)\ x \nleq a \,\&\, a \nleq x\}. \end{aligned}$$

The following hold:

(i) There do not exist $u \in U$ and $a \in A$, such that $u \leq a$. Otherwise we have $a' \in A$ s.t. $a' \leq u$. Because A is an interval, $u \in A$, a contradiction.

(ii) U is upwards closed, hence an interval. Let $u \in U$ and $x \in X, u \leq x$. $x \notin A$ from (i). Then we have $a \in A$ s.t. $a \leq u$. But then $a \leq x$, thus $x \in U$.

(iii) $U \cap L = \emptyset$. Follows from (i).

(iv) There do not exist $l \in L$ and $a \in A$, such that $l \geq a$. Similarly to (i).

(v) L is downwards closed, hence an interval. Similarly to (ii).

(vi) There do not exist $u \in U$ and $l \in L$ such that $u < l$. Otherwise we have $a_1, a_2 \in A$ for which $a_1 \leq u$ and $l \leq a_2$. But then $a_1 \leq u \leq a_2$ and $a_1 \leq l \leq a_2$, thus $u, l \in A$, a contradiction.

(vii) $N \cap A = N \cap U = N \cap L = \emptyset$. From the definition of N.

(viii) N is an interval. Let $n_1, n_2 \in N$ and $x \in X$ satisfy $n_1 < x < n_2$. Suppose there is $a \in A$ for which $a \leq x$. Then $a < n_2$ contradicts $n_2 \in N$. Similarly, $x \leq a$ would imply $n_1 < a$, again a contradiction.

Thus A, U, L, N are pairwise disjoint intervals and for $a \in A, u \in U, l \in L, n \in N$ the following *cannot* occur: $u < a$ from (i), $a < l$ from (iv), $u < n$ from (ii) and (vii), $n < l$ from (v) and (vii), $n < a$ and $a < n$ from the definition, $u < l$ (vi).

Define a four-element ordered set $P_4 = \{p_A, p_U, p_L, p_N\}$ with relations $p_L < p_A < p_U$, $p_L < p_N < p_U$.

$$\begin{tikzcd}
p_U & \\
p_A \arrow[u] & p_N \arrow[ul] \\
p_L \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Let $f : X \to P_4$ assign to $x \in X$ the point p_S if $x \in S, S \in \{A, U, L, N\}$. Then f is monotone, because the relations missing in P_4 which could contradict monotony are exactly those ruled out above. By definition we have $f^{-1}(p_A) = A$. $\square$

Corollary 9. *Intervals in ordered sets are precisely the preimages of single points under monotone maps (maps into finite posets suffice).*

The ordered set P_4 from proof of above proposition is important and will be commonly used later.

Lemma 10. *Let $A \in \mathscr{U}$. Then there exists a homomorphism $f : X \to P_4$ with $f^{-1}(\int f \,\mathrm{d}x) = A$ and $\int f \,\mathrm{d}x = p_A$.*

Proof. By definition of $\mathscr{U}$ there exists finite $(Q, \leq)$ and $g : X \to Q$ monotone with $g^{-1}(\int g \,\mathrm{d}x) = A$. Let $i = \int g \,\mathrm{d}x$. Then $\{i\}$ is an interval. Let $h : Q \to P_4$ be the homomorphism from Proposition 8 with $h^{-1}(p_A) = \{i\}$. Then $h \circ g : X \to P_4$ and

$$\int h \circ g \,\mathrm{d}x = h\Big(\int g \,\mathrm{d}x\Big) = h(i) = p_A.$$

The condition $(h \circ g)^{-1}(p_A) = A$ is also satisfied, thus $f = h \circ g$ is the map we were looking for. $\square$

Lemma 11. *For $f_1, \ldots, f_n$ homomorphisms $X \to P_i$, with P_i finite, it holds*

$$\int (f_1, \ldots, f_n) \,\mathrm{d}x = \Big(\int f_1 \,\mathrm{d}x, \ldots, \int f_n \,\mathrm{d}x\Big).$$

Proof. For each $i = 1, \ldots, n$ use (2.1) for π_i:

$$\pi_i\Big(\int (f_1, \ldots, f_n) \,\mathrm{d}x\Big) = \int \pi_i \circ (f_1, \ldots, f_n) \,\mathrm{d}x = \int f_i \,\mathrm{d}x. \qquad \square$$

Lemma 12. *$\mathscr{U}$ is closed under finite intersections.*

Proof. Let $A_1, A_2 \in \mathscr{U}$ and f_1, f_2 be the associated homomorphisms for which $f_i^{-1}(\int f_i \,\mathrm{d}x) = A_i$, $i = 1, 2$. Set $s = (f_1, f_2)$ to be the product homomorphism. Then $\int s \,\mathrm{d}x = (\int f_1 \,\mathrm{d}x, \int f_2 \,\mathrm{d}x)$ by Lemma 11. By definition $s^{-1}(\int s \,\mathrm{d}x) \in \mathscr{U}$, because the product of finite codomains of f_1, f_2 is finite. But Lemma 7 gives us

$$\begin{aligned} s^{-1}\Big(\int s \,\mathrm{d}x\Big) &= (f_1, f_2)^{-1}\Big(\int f_1 \,\mathrm{d}x, \int f_2 \,\mathrm{d}x\Big) \\ &= f_1^{-1}\Big(\int f_1 \,\mathrm{d}x\Big) \cap f_2^{-1}\Big(\int f_2 \,\mathrm{d}x\Big) = A_1 \cap A_2, \end{aligned}$$

hence $A_1 \cap A_2 \in \mathscr{U}$. $\square$

Lemma 13. *With every element, $\mathscr{U}$ contains all larger intervals.*

Proof. Let $A \in \mathscr{U}$ and $A' \supseteq A$ an interval in X. We will show that the usual homomorphism $f' : X \to P_4$ mapping A' to $p_A \in P_4$ satisfies $\int f' \,\mathrm{d}x = p_A$. Then we will get $A' = f'^{-1}(\int f' \,\mathrm{d}x) \in \mathscr{U}$.

We already know that $\mathscr{U}$ is closed under finite intersections. If it were $\int f' \,\mathrm{d}x \neq p_A$, we would get

$$\mathscr{U} \ni A \cap f'^{-1}\Big(\int f' \,\mathrm{d}x\Big) \subseteq A' \cap f'^{-1}\Big(\int f' \,\mathrm{d}x\Big) = f'^{-1}(p_A) \cap f'^{-1}\Big(\int f' \,\mathrm{d}x\Big) = \emptyset,$$

but $\emptyset \notin \mathscr{U}$, a contradiction. Hence $\int f' \,\mathrm{d}x = p_A$, which proves the initial assertion. $\square$

Lemma 14. *For every finite partition of X into intervals $A_1, \dots, A_n$, exactly one $A_i \in \mathscr{U}$.*

Proof. Let $f_i : X \to P_4$ be such that $f_i^{-1}(p_A) = A_i$, $i = 1, \dots, n$. By Lemmata 7 and 11

$$\emptyset \neq (f_1, \dots, f_n)^{-1}\Big(\int f_1 \,\mathrm{d}x, \dots, \int f_n \,\mathrm{d}x\Big) = \bigcap_{i=1}^{n} f_i^{-1}\Big(\int f_i \,\mathrm{d}x\Big).$$

For each $i = 1, \dots, n$, $f_i^{-1}(\int f_i \,\mathrm{d}x)$ is either equal to A_i or an interval contained in the set $X \setminus A_i$. Because A_i's cover X, we have $\bigcap_{i=1}^{n}(X \setminus A_i) = \emptyset$. Therefore the right hand side must be of the form

$$\bigcap_{i=1}^{n} f_i^{-1}\Big(\int f_i \,\mathrm{d}x\Big) \subseteq A_j \cap \bigcap_{i \neq j}(X \setminus A_i) = A_j,$$

for some j. By upwards closure $A_j \in \mathscr{U}$. Uniqueness follows from $\emptyset \notin \mathscr{U}$. $\square$

We will use these properties to define ultrafilters on ordered sets. Afterwards we will show these suffice for determining a simple integration operator. Note the terminology: we say ultrafilter *on* sets, similarly we are about to define ultrafilters *on* partial orders. The order-theoretic filters (upwards closed and downwards directed subsets) for a given partial order will be called (ultra)filters *in* the partially ordered set. This may not be conventional, but is used to distinguish between these terms. Later we will prove a connection between these.

Definition. Let $(X, \leq)$ be partially ordered. We call a collection $\mathscr{U}$ of intervals of X an *ultrafilter on X*, if it satisfies

(i) $(\forall A, B \in \mathscr{U})\ A \cap B \in \mathscr{U}$,

(ii) $(\forall A \in \mathscr{U}\ \forall B \sqsubseteq X \text{ interval})\ A \subseteq B \implies B \in \mathscr{U}$,

(iii) For every finite partition into intervals $X = A_1 \amalg \dots \amalg A_n$, there is exactly one $i \in \{1, \dots, n\}$ such that $A_i \in \mathscr{U}$.

Now let $\mathscr{U}$ be any ultrafilter on a partial order $(X, \leq)$. For a homomorphism $f : X \to P$ with $P = \{p_1, \dots, p_n\}$ finite,

$$f^{-1}(p_1), \dots, f^{-1}(p_n)$$

is a partition of X into intervals upon applying Lemma 4. Then by the third property there exists exactly one $i \in \{1, \dots, n\}$ such that $f^{-1}(p_i) \in \mathscr{U}$. Thus we can define $\int_X f \,\mathrm{d}\mathscr{U} = p_i$. Proof of (2.1) is identical to the case of ultrafilters on sets.

2.2 Graphs

2.2.1 Choice of definitions

Multiple different types of structures are commonly used under the name graph. By default we consider undirected graphs, but results should generally transfer to the directed ones too. In our case there is at most one edge between a given pair of vertices (or at most one arrow between an ordered pair). Next we have to decide whether or not we allow loops, that is edges or arrows on a single vertex. In many practical uses loops are undesirable, but by disallowing them we lose some nice properties of the resulting category. For reasons discussed below, our definition does not allows loops. Additionally, any graph admitting "well behaved" ultrafilters needs to be finitely colourable. For morphisms we consider edge preserving maps. If we were to additionally allow contracting ends of an edge into a single vertex, this would be in some sense equivalent to allowing loops in the codomains.

Definition. *Graph* $\mathbf{G}$ is a set G of vertices together with a set E of edges, where an edge is a two element subset of G. Equivalently, we could have provided a symmetric antireflexive binary relation on G. Homomorphism of graphs is an *edge preserving map*, that is $f : (G, E) \to (H, F)$ such that

$$(\forall x, y \in G)\ \{x, y\} \in E \implies \{f(x), f(y)\} \in F.$$

For a graph $\mathbf{G}$ we also write $V(\mathbf{G})$ for its set of vertices and $E(\mathbf{G})$ for the set of edges. The resulting category of all graphs is denoted Graph, its full subcategory of finitely colourable graphs is denoted $\mathsf{FCGraph}$.

In Adámek and Sousa [1] loops on graphs are allowed. In that case, their result shows that ultrafilters on a graph are exactly the ultrafilters on its set of vertices. We provide a sketch of a proof of this fact.

Consider the complete graph $\mathbf{D}$ on two vertices $0, 1$, including the loops. If (G, E) is a graph and $A \subseteq G$ a set of its vertices, then the function $\chi_A : G \to D$, sending elements of A to 1 and elements of the complement to 0, is a homomorphism. It satisfies $\chi_A^{-1}(1) = A$. Thus ultrafilters need to differentiate among all subsets. Similarly, a partition of G into n sets can be represented as a homomorphism into a complete graph on n vertices. Hence we can use the same proof as in the case of sets.

2.2.2 Preliminary results

It is common knowledge that graph homomorphisms are closely related to colourings. Colouring of a graph $\mathbf{G} = (G, E)$ is a function $c : G \to S$, where S is a set of colours, such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$. Usually we have $S = \{1, \dots, n\}$ for some $n \in \mathbb{N}$, then we speak about an n-colouring. This can be interpreted as an edge preserving map into the complete graph $\mathbf{K}_S$ with vertices S. Below the word colouring is loosely used for this homomorphism. Two connected vertices can only be assigned two different colours, because those are also connected in the codomain $\mathbf{K}_S$. General homomorphisms extend this behaviour, in a set of colours we connect via an edge those colours which are compatible, meaning connected vertices can have these two colours.

Proposition 15. *Let* $\mathbf{G} = (G, E)$ *be a graph. There exists a homomorphism* $f : \mathbf{G} \to \mathbf{H}$ *into a finite graph* $\mathbf{H}$ *iff* $\mathbf{G}$ *is finitely colourable.*

Proof. If there exists an n-colouring, by the introductory discussion we can interpret it as a homomorphism $\mathbf{G} \to \mathbf{K}_n$. Conversely, we can add all possible edges to a codomain of any homomorphism, to turn it into a complete graph. Then the map stays edge preserving, but now is also a colouring. □

Next we look which sets are the analogues of intervals in posets.

Definition. In a graph $\mathbf{G}$ a set $A \subseteq V(\mathbf{G})$ is called *independent*, if no two of its vertices are connected with an edge. When A is an independent set of $\mathbf{G}$, we write $A \sqsubseteq \mathbf{G}$.

Lemma 16. *The preimage of a single point under an edge preserving map is an independent set.*

Proof. Consequence of the absence of loops. □

Lemma 17. *The preimage of an independent set under an edge preserving map is again independent.*

Proof. Clear. □

Lemma 18. *Let* $\mathbf{G}$ *be finitely colourable and* $A \subseteq V(\mathbf{G})$ *independent. Then there exists a homomorphism* $f : \mathbf{G} \to \mathbf{H}$ *into* $\mathbf{H}$ *finite such that* $f^{-1}(a) = A$ *for some vertex* a *of* $\mathbf{H}$.

Proof. Let c be any n-colouring of $\mathbf{G}$. Define $\tilde{c}$ by

$$\tilde{c}(v) = \begin{cases} c(v), & v \notin A, \\ n+1, & v \in A. \end{cases}$$

Because A is independent, we never assigned the same colour to connected vertices. Thus $\tilde{c}$ is a colouring and can be interpreted as a homomorphism. It holds that $\tilde{c}^{-1}(n+1) = A$. □

Notice we only added a new point and connected it to every old point in the codomain. This can be applied to the graph itself and identical (edge preserving) map $\mathrm{id} : \mathbf{X} \to \mathbf{X}$. Thus we get

Corollary 19. *Independent sets in graphs are precisely preimages of single points under edge preserving maps. For finitely colourable graphs, independent sets are preimages of single points under homomorphisms into finite graphs (or in particular, colourings).*

Similarly to partial orders, we will use products to capture intersections of independent sets. This definition is in fact the category-theoretical product.

Definition. Let $\mathbf{G}_i, i \in I$ be graphs. Their *product graph* $\prod_{i \in I} \mathbf{G}_i$ has as vertices the cartesian product $G = \bigtimes_{i \in I} V(\mathbf{G}_i)$ and edges satisfying

$$\{(x_i)_{i\in I}, (y_i)_{i\in I}\} \in E\Big(\prod_{i\in I} \mathbf{G}_i\Big) \iff (\forall i \in I)\{x_i, y_i\} \in E(\mathbf{G}_i).$$

Not only does $\prod_{i\in I} \mathbf{G}_i$ have no loops, but if any i-th component of two vertices is the same, they are not connected.

Lemma 20. *Let $f_i : \mathbf{X} \to \mathbf{G}_i, i \in I$ be homomorphisms of graphs. Define $(f_i)_{i\in I} : \mathbf{X} \to \prod_{i\in I} \mathbf{G}_i$ by*

$$(f_i)_{i\in I}(x) = (f_i(x))_{i\in I}, \qquad x \in X.$$

Then

(i) $(f_i)_{i\in I}$ preserves edges,

(ii) the projections $\pi_j : \prod_{i\in I} \mathbf{G}_i \to \mathbf{G}_j, (x_i)_{i\in I} \mapsto x_j$ preserve edges,

(iii) $f_j = \pi_j \circ (f_i)_{i\in I}$ and $(f_i)_{i\in I}^{-1}((x_i)_{i\in I}) = \bigcap_{i\in I} f_i^{-1}(x_i)$,

(iv) if any $\mathbf{G}_j$ is finitely colourable, then $\prod_{i\in I} \mathbf{G}_i$ is too.

Proof. Identically to the version for posets, (i) to (iii) are clear from definition. For the last point, if $\mathbf{G}_j$ has n-colouring c, then $c \circ \pi_j$ is a colouring of the whole product. $\square$

2.2.3 Integration and ultrafilters

For sets and partial orders we studied ultrafilters via morphisms with finite codomains. In order to do the same for graphs, we need finite colourability. Otherwise, no such morphisms exist at all.

Fix a finitely colourable graph $\mathbf{X} = (X, E)$ and a simple integration operator on it. That is for all finite graphs $\mathbf{G}, \mathbf{H}$ and edge preserving maps $f : \mathbf{X} \to \mathbf{G}$, $g : \mathbf{G} \to \mathbf{H}$, the following holds:

$$g\Big(\int_{\mathbf{X}} f \,\mathrm{d}x\Big) = \int_{\mathbf{X}} g \circ f \,\mathrm{d}x. \tag{2.2}$$

Consider

$$\mathscr{U} := \Big\{f^{-1}\Big(\int_{\mathbf{X}} f \,\mathrm{d}x\Big) : \mathbf{G} \text{ finite graph, } f : \mathbf{X} \to \mathbf{G} \text{ homomorphism}\Big\},$$

same as for posets, we will prove it is closed under finite intersections, upwards (for larger independent sets) and has the partition property. Argument for $\emptyset \notin \mathscr{U}$ follows right below.

Lemma 21. *Adding edges or new vertices to the codomain of f does not change the integral $\int f \,\mathrm{d}x$.*

Proof. Apply (2.2) for g inclusion. $\square$

Applying this to the induced subgraph $\operatorname{rng} f$, we get $\int f \,\mathrm{d}x \in \operatorname{rng} f$ for any f into a finite graph. Which in turn implies $\emptyset \notin \mathscr{U}$. Compare this with Lemma 2.

Corollary 22. *If $A \in \mathscr{U}$, then there exists an n-colouring c of $\mathbf{X}$ such that $A = c^{-1}(1)$ and $\int_{\mathbf{X}} c \,\mathrm{d}x = 1$.*

Proof. $A \in \mathscr{U}$ requires a witness $f : \mathbf{X} \to \mathbf{H}$, $\mathbf{H}$ some finite graph. Suppose it has n vertices. We can label its vertices $1, \ldots, n$. By Lemma 21 we can add edges to turn $\mathbf{H}$ into the complete graph $\mathbf{K}_n$ and the integral $\int f \, \mathrm{d}x$ stays the same. Applying (2.2) to a permutation of vertices, we can ensure $\int f \, \mathrm{d}x = 1$. □

Lemma 23. *$\mathscr{U}$ is closed under finite intersections.*

Proof. Apply Lemma 20 in the same way as we did for partial orders. □

Lemma 24. *Let $A \in \mathscr{U}$ and $A' \sqsubseteq \mathbf{X}$ be independent such that $A \subseteq A'$. Then $A' \in \mathscr{U}$.*

Proof. Let $A \in \mathscr{U}$ and $A' \supseteq A$ be an independent set in $\mathbf{X}$. We will show that an n-colouring c of $\mathbf{X}$ with $c^{-1}(1) = A'$ (which exists by Corollary 19) satisfies $\int c \, \mathrm{d}x = 1$. Then we will get $A' = c^{-1}(\int c \, \mathrm{d}x) \in \mathscr{U}$.

From Lemma 23 and the definition we know that $\mathscr{U}$ contains the set

$$A \cap c^{-1}\Big(\int c \, \mathrm{d}x\Big).$$

Hence the set is nonempty. For any $i \neq 1$ we have

$$A \cap c^{-1}(i) \subseteq A' \cap c^{-1}(i) = c^{-1}(1) \cap c^{-1}(i) = \emptyset,$$

implying $\int c \, \mathrm{d}x = 1$, which proves our assertion. □

Lemma 25. *Let $A_1 \amalg \ldots \amalg A_n$ be a partition of $V(\mathbf{X})$ into independent sets. Then exactly one A_i belongs to $\mathscr{U}$.*

Proof. Such a partition naturally defines an n-colouring c. From this we get $c^{-1}(\int c \, \mathrm{d}x) = A_i$ for some i, hence $A_i \in \mathscr{U}$. Uniqueness follows from closure under finite intersections and the fact $\emptyset \notin \mathscr{U}$. □

Definition. Let $\mathbf{X}$ be a finitely colourable graph. We call a collection $\mathscr{U}$ of independent sets of vertices of $\mathbf{X}$ an *ultrafilter on X*, if it satisfies

(i) $(\forall A, B \in \mathscr{U})\ A \cap B \in \mathscr{U}$,

(ii) $(\forall A \in \mathscr{U}\ \forall B \subseteq V(\mathbf{X})$ independent$)\ A \subseteq B \implies B \in \mathscr{U}$,

(iii) For every finite partition into independent sets $V(\mathbf{X}) = A_1 \amalg \ldots \amalg A_n$, there is exactly one $i \in \{1, \ldots, n\}$ such that $A_i \in \mathscr{U}$.

Again, if $\mathscr{U}$ is an ultrafilter on a finitely colourable graph, we could use it to define a simple integration operator.

3. Ultrafilters on structured sets

3.1 General filters

Filters and ultrafilters on a set are special cases of filters in general partial orders, which we discuss now. We consider only proper filters.

Definition. A *filter* in a partially ordered set $(P, \leq)$ is a nonempty subset $F \subseteq P$ such that

(i) $F \neq P$ (proper),

(ii) $(\forall a \in F\ \forall b \in P)\ a \leq b \implies b \in F$ (upwards closed),

(iii) $(\forall a, b \in F\ \exists c \in F)\ c \leq a \,\&\, c \leq b$ (downwards directed),

If $P = \mathscr{P}(X)$ is the powerset of a set X, then filters in P are precisely filters on X. The only difference in definition is closure under finite intersections vs. being downwards directed. But those are equivalent in P because of upwards closure.

Definition. An *ultrafilter* in an ordered set $(P, \leq)$ is a $\subseteq$-maximal filter.

We will not prove any general facts about ultrafilters in posets, but we will restrict ourselves to a special poset P. Its properties will be those of the meet-semilattices consisting of intervals of a particular partial order, or of independent sets of a graph.

For the remainder of this section, let X be a fixed set, $P \subseteq \mathscr{P}(X)$ a collection of some subsets ordered by inclusion, which has at least two nonempty elements and satisfies

(i) $(\forall a, b \in P)\ a \cap b \in P$,

(ii) $(\forall a \in P\ \exists x \in P \text{ maximal})\ a \subseteq x$ (i.e. Zorn's lemma holds in P),

(iii) $(\forall a \in P\ \forall x \in P \text{ maximal}, a \subseteq x\ \exists a_1, \ldots, a_n \in P)\ a \amalg a_1 \amalg \ldots \amalg a_n = x$.

If X itself is an element of P, the second condition is trivial and the third condition reduces to the existence of a finite partition of X containing a. For partial orders and finitely colourable graphs, we have seen existence of finite partitions of X, even though generally $X \in P$ only for posets. This more general setting will allow us to study graphs which are not finitely colourable (thus have no finite partition of X).

Proposition 26. *Let $\mathscr{U} \subseteq P$ be a filter. Then the following are equivalent*

(a) $\mathscr{U}$ is maximal (hence an ultrafilter),

(b) $\mathscr{U}$ has the following partition property

$$(\forall n \geq 2\ \forall a_1, \ldots, a_n \in P\ \forall x \in \mathscr{U})\ a_1 \amalg \ldots \amalg a_n \supseteq x \implies (\exists! j \in \{1, \ldots, n\})\ a_j \in \mathscr{U}.$$

Proof.

(b) $\Rightarrow$ (a) Let $\mathscr{U}$ satisfy the partition property, but suppose for contradiction that $\mathscr{V} \supsetneq \mathscr{U}$ is a larger filter. Thus there exists $a \in \mathscr{V} \setminus \mathscr{U}$. Because $\emptyset \neq \mathscr{U}$, let $b \in \mathscr{U}$ be arbitrary. Then $b \in \mathscr{V}$, hence $c = a \cap b \in \mathscr{V}$. Let $x \in P$ be any maximal element larger than b. We get $x \in \mathscr{U}$. Let $c \amalg a_1 \amalg \ldots \amalg a_n$ be any finite partition of x into disjoint elements of P. As $a \notin \mathscr{U}$, we have $c \notin \mathscr{U}$. Then the unique element of $\mathscr{U}$ from the partition property must be one of the a_i's. But $a_i \in \mathscr{U}$ gives $a_i \in \mathscr{V}$, thus $\emptyset = c \cap a_i \in \mathscr{V}$, a contradiction.

(a) $\Rightarrow$ (b) Let $x \in \mathscr{U}$ and $a'_1, \ldots, a'_n \in P$ such that $x \subseteq a'_1 \amalg \ldots \amalg a'_n$. Because they are disjoint, two of them together can not be elements of $\mathscr{U}$. Consider $a_i = a'_i \cap x, i = 1, \ldots, n$; if we prove $a_j \in \mathscr{U}$, then also $a'_j \in \mathscr{U}$, as we want.

Let
$$\mathscr{V}_i := \{b \in P : (\exists u \in \mathscr{U})\ u \cap a_i \subseteq b\}, \qquad i = 1, \ldots, n.$$
Then we see

- $a_i \in \mathscr{V}_i$,
- $\mathscr{U} \subseteq \mathscr{V}_i$,
- $\mathscr{V}_i$ is upwards closed,
- $\mathscr{V}_i$ is closed under finite intersections; if $b_1, b_2 \in \mathscr{V}_i$, we have $u_1, u_2 \in \mathscr{U}$ such that $u_j \cap a_i \subseteq b_j, j = 1, 2$. Then $(u_1 \cap u_2) \cap a_i = (u_1 \cap a_i) \cap (u_2 \cap a_i) \subseteq b_1 \cap b_2$ and $u_1 \cap u_2 \in \mathscr{U}$.

There are two possibilities:

- For some $i \in \{1, \ldots, n\}$, $\mathscr{V}_i$ is a (proper) filter. This means $\emptyset \notin \mathscr{V}_i$. Because $\mathscr{U} \subseteq \mathscr{V}_i$ and $\mathscr{U}$ is maximal, $\mathscr{V}_i = \mathscr{U}$ and thus $a_i \in \mathscr{U}$.
- No $\mathscr{V}_i$ is a filter, $i = 1, \ldots, n$. Then $\emptyset \in \mathscr{V}_i$ for all i, so there is $u_i \in \mathscr{U}$ such that $u_i \cap a_i = \emptyset$. Then
$$\begin{aligned}\mathscr{U} \ni x \cap \bigcap_{i=1}^n u_i &= \Big(\bigcup_{j=1}^n a_j\Big) \cap \bigcap_{i=1}^n u_i \\ &= \bigcup_{j=1}^n \Big(a_j \cap \bigcap_{i=1}^n u_i\Big) \subseteq \bigcup_{j=1}^n (a_j \cap u_j) = \bigcup_{j=1}^n \emptyset = \emptyset.\end{aligned}$$
Hence $\emptyset \in \mathscr{U}$, a contradiction. □

The following gives us a way to obtain ultrafilters in P from ultrafilters on sets. The assumption of nonemptyness is always satisfied if we can partition X into finitely many elements of P, which is true for posets and finitely colourable graphs.

Lemma 27. *Let $\mathscr{F}$ be an ultrafilter on the set X. Then if $\mathscr{F} \cap P \neq \emptyset$, this intersection is an ultrafilter in P.*

Proof. It is a filter, because P contains intersections. It satisfies the partition property of Proposition 26, because $\mathscr{F}$ satisfies it. □

Lemma 28 ("the ultrafilter theorem")**.** *Any nonempty subsystem of P with the finite intersection property can be extended into an ultrafilter.*

Proof. Instead of proving this directly, we use the same fact for ultrafilters on a set. Then use the previous lemma. $\square$

Lemma 29. *Let $\mathscr{U}$ be an ultrafilter in P. For $A \in P$ it holds*

$$A \in \mathscr{U} \iff (\forall U \in \mathscr{U})\ A \cap U \neq \emptyset.$$

Proof. If $A \in \mathscr{U}$, we get $U \cap A \neq \emptyset$ for all $U \in \mathscr{U}$, because $\mathscr{U}$ is closed under finite intersections and $\emptyset \notin \mathscr{U}$. Assuming the right-hand side, we get that $\mathscr{U} \cup \{A\}$ has the finite intersection property, thus it extends into an ultrafilter $\mathscr{F}$ in P containing $\mathscr{U}$. From maximality $\mathscr{F} = \mathscr{U}$, hence $A \in \mathscr{U}$. $\square$

Note that ultrafilters on posets and finitely colourable graphs as defined in chapter 2 indeed coincide with ultrafilters in the semilattice of intervals or of independent sets. The partition property restricted to partitions of X is equivalent to the stronger partition property from 26, since in these special cases any partition $a_1 \amalg \ldots \amalg a_n$ of a subset $x \subseteq X$ can be completed into a finite partition of the whole set X.

3.2 Posets

For a partial order $(P, \leq)$ denote by $U(P)$ the set of ultrafilters on the order P. By βP we will mean the set of ultrafilters on the underlying set P. To turn U into a functor, we need to endow $U(P)$ with some order relation and also define it on morphisms. Next in this section, we will show how to construct ultrafilters in terms of the order structure of P and provide a classification of ultrafilters with some additional assumptions on P.

Definition. For an order $(P, \leq)$ and $\mathscr{U}, \mathscr{V} \in U(P)$, set $\mathscr{U} \leq \mathscr{V}$ if

$$(\forall (Q, \leq) \text{ finite order } \forall f : P \to Q \text{ monotone}) \int_P f \, \mathrm{d}\mathscr{U} \leq \int_P f \, \mathrm{d}\mathscr{V}.$$

It is easy to see that this is indeed a partial order. For $x, y \in P$ and the trivial ultrafilters $\mathscr{E}(x), \mathscr{E}(y) \in U(P)$, we have $\mathscr{E}(x) \leq \mathscr{E}(y)$ iff for all f holds $\int f \, \mathrm{d}\mathscr{E}(x) = f(x) \leq f(y) = \int f \, \mathrm{d}\mathscr{E}(y)$ iff $x \leq y$. Note that if x, y are incomparable, a suitable homomorphism $P \to P_4$ sends x and y to the incomparable elements. Thus we see that P embeds into $U(P)$.

We can immediately formulate an important equivalent definition of order on $U(P)$

Proposition 30. *Let $\mathscr{U}, \mathscr{V} \in U(P)$. Then $\mathscr{U} \leq \mathscr{V}$ iff*

$$(\forall U \in \mathscr{U}\ \forall V \in \mathscr{V}\ \exists u \in U\ \exists v \in V)\ u \leq v.$$

Proof. ($\Rightarrow$) By contradiction, fix $U \in \mathscr{U}$, $V \in \mathscr{V}$ such that for any $u \in U, v \in V$ we have $u \not\leq v$. Then for $f : P \to P_4$ from Proposition 8 such that $f^{-1}(p_A) = U$, we get $f(u) \not\leq f(v)$ for any $u \in U, v \in V$ ($f(u)$ is p_A by definition, v is either incomparable with anything in U and then $f(v)$ is incomparable with $f(u)$, or its smaller than something and then $f(u) > f(v)$). Because U is sent to one point, $f^{-1}(f(u)) = U$ for any $u \in U$. This gives $\int f \, \mathrm{d}\mathscr{U} = f(u)$ for any $u \in U$. Next $f^{-1}(\int f \, \mathrm{d}\mathscr{V}) \cap V \in \mathscr{V}$, hence is nonempty, thus contains some v. This v then satisfies $f(v) = \int f \, \mathrm{d}\mathscr{V}$. But then $\int f \, \mathrm{d}\mathscr{U} = f(u) \not\leq f(v) = \int f \, \mathrm{d}\mathscr{V}$ contradicts $\mathscr{U} \leq \mathscr{V}$.

($\Leftarrow$) Let $f : P \to Q$ be a homomorphism, Q finite. We want $\int f \, \mathrm{d}\mathscr{U} \leq \int f \, \mathrm{d}\mathscr{V}$. Set $U = f^{-1}(\int f \, \mathrm{d}\mathscr{U}) \in \mathscr{U}$, $V = f^{-1}(\int f \, \mathrm{d}\mathscr{V}) \in \mathscr{V}$. By assumption there exist $u \in U, v \in V$ such that $u \leq v$. But then $\int f \, \mathrm{d}\mathscr{U} = f(u) \leq f(v) = \int f \, \mathrm{d}\mathscr{V}$. $\square$

Definition. Let $f : (P, \leq) \to (Q, \leq)$ be monotone and $\mathscr{U} \in U(P)$. Define

$$f_{\#}\mathscr{U} \coloneqq \{A \sqsubseteq Q : f^{-1}[A] \in \mathscr{U}\}.$$

Let us show $f_{\#}\mathscr{U} \in U(Q)$.

Proof. Let $A, B, A_1, \ldots, A_n \sqsubseteq Q$ be intervals. We check the definition of an ultrafilter on Q. Note the use of Lemma 5.

- Suppose $A \subseteq B$, $A \in f_{\#}\mathscr{U}$. Then $f^{-1}[B] \supseteq f^{-1}[A]$ gives $f^{-1}[B] \in \mathscr{U}$, thus $B \in f_{\#}\mathscr{U}$.
- Suppose $A, B \in f_{\#}\mathscr{U}$. Then $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] \in \mathscr{U}$, giving $A \cap B \in \mathscr{U}$.
- Assume $A_1 \amalg \ldots \amalg A_n = Q$. Then $f^{-1}[A_1] \amalg \ldots \amalg f^{-1}[A_n] = P$, thus for one and only one j we have $f^{-1}[A_j] \in \mathscr{U}$. But then only for j we get $A_j \in f_{\#}\mathscr{U}$. $\square$

Lemma 31. *Let $f : P \to Q$, $g : Q \to R$ be monotone, $\mathscr{U} \in U(P)$. Then $(g \circ f)_{\#}\mathscr{U} = g_{\#}(f_{\#}\mathscr{U})$.*

Proof.

$$\begin{aligned}(g \circ f)_{\#}\mathscr{U} &= \{A \sqsubseteq R : (g \circ f)^{-1}[A] \in \mathscr{U}\} = \{A \sqsubseteq R : f^{-1}[g^{-1}[A]] \in \mathscr{U}\} \\ &= \{A \sqsubseteq R : g^{-1}[A] \in f_{\#}\mathscr{U}\} = g_{\#}(f_{\#}\mathscr{U})\end{aligned}$$

$\square$

Lemma 32. *For homomorphisms $f : P \to Q$, $g : Q \to R$, R finite, $\mathscr{U} \in U(P)$, it holds*

$$\int_P g \circ f \, \mathrm{d}\mathscr{U} = \int_Q g \, \mathrm{d}(f_{\#}\mathscr{U}).$$

Proof. Let $i = \int_Q g \, \mathrm{d}(f_{\#}\mathscr{U})$. Thus $g^{-1}(i) \in f_{\#}\mathscr{U}$, which means $(g \circ f)^{-1}(i) = f^{-1}[g^{-1}(i)] \in \mathscr{U}$. This gives $\int_P g \circ f \, \mathrm{d}\mathscr{U} = i$. $\square$

Proposition 33. *The map U is a functor* $\mathsf{Poset} \to \mathsf{Poset}$*, assigning to $(P, \leq)$ the order $(U(P), \leq)$ and to a morphism $f : P \to Q$ the function $\mathscr{U} \mapsto f_{\#}\mathscr{U}$.*

Proof. We know U preserves composition and that $U(P)$ is ordered. Clearly $\mathrm{id}_{\#}\,\mathscr{U} = \mathscr{U}$. It only remains to show that $f_{\#-}$ is a homomorphism for $f : P \to Q$. Given $\mathscr{U} \leq \mathscr{V}$ in $U(P)$ and a morphism $g : Q \to R$ the previous lemma gives

$$\int g \, \mathrm{d}f_{\#}\mathscr{U} = \int g \circ f \, \mathrm{d}\mathscr{U} \leq \int g \circ f \, \mathrm{d}\mathscr{V} = \int g \, \mathrm{d}f_{\#}\mathscr{V}.$$

Hence $f_{\#}\mathscr{U} \leq f_{\#}\mathscr{V}$. $\square$

We can construct $U(P)$ as a limit – this limit appears in section 2 of Leinster [5] as a possible definition of the codensity monad. Codensity monads provide another approach to this topic, which we do not focus on in this thesis. Related construction of βX in Set as a limit of partitions can be found in [6]. The proof gives a reason for the particular definition of order on $U(P)$, it is in fact the only possible one for this proposition to hold.

Proposition 34. *Call $\Delta = (Q_\Delta, f_\Delta)$ a functional partition of $(P, \leq)$ if Q_Δ is a finite ordered set and $f_\Delta : P \to Q_\Delta$ homomorphism. Define morphisms of partitions $\Delta' \to \Delta$ as morphisms $h : Q'_\Delta \to Q_\Delta$ such that $h \circ f_{\Delta'} = f_\Delta$. Write $h \in [\Delta', \Delta]$. Then $U(P)$ is a limit of the diagram with objects Q_Δ and morphisms $h \in [\Delta', \Delta]$ over all functional partitions Δ', Δ of P.*

Proof. Note that the diagram can be equivalently expressed in terms of a small category. The corresponding limit projection has components $\int_P f_\Delta \,\mathrm{d}\_$.

$U(P)$ is a cone:

$$\begin{tikzcd} & Q_\Delta \\ U(P) \arrow[ur, "\int_P f_\Delta \,\mathrm{d}\_"] \arrow[dr, "\int_P f_{\Delta'} \,\mathrm{d}\_"'] & \\ & Q_{\Delta'} \arrow[uu, "h"'] \end{tikzcd}$$

This holds from the property (2.1) of integration, and the fact that h satisfies $h \circ f_{\Delta'} = f_\Delta$. The maps $\int f_\Delta \,\mathrm{d}\_$ are monotone by definition of order on $U(P)$.

It is a terminal cone: suppose (K, π) is a cone

$$\begin{tikzcd} & Q_\Delta & \\ K \arrow[ur, "\pi_\Delta"] \arrow[dr, "\pi_{\Delta'}"'] \arrow[rr, bend right=60, "\exists! \mu"'] & & U(P) \arrow[ul, "\int_P f_\Delta \,\mathrm{d}\_"'] \arrow[dl, "\int_P f_{\Delta'} \,\mathrm{d}\_"] \\ & Q_{\Delta'} \arrow[uu, "h"'] & \end{tikzcd}$$

Define $\mu : K \to U(P)$ by specifying a simple integration operator for each $k \in K$. Let $f : P \to Q$ be a morphism, Q finite. Then $\Delta = (Q, f)$ is a functional partition. Set

$$\int f \,\mathrm{d}k := \pi_\Delta(k), \qquad k \in K.$$

If $h : Q \to R$ is a morphism of finite orders, we get a partition $\Delta_0 = (R, h \circ f)$, by definition $h \in [\Delta, \Delta_0]$. Because (K, π) is a cone, we get $h(\pi_\Delta(k)) = \pi_{\Delta_0}(k)$, or $h(\int f \,\mathrm{d}k) = \int h \circ d \,\mathrm{d}k$. Hence $\int \_ \mathrm{d}k$ is a simple integration operator, thus it determines a unique element of $U(P)$, call it $\mu(k)$.

Uniqueness of μ: Suppose (K, π) also factors through $U(P)$ via $\nu \neq \mu$. Then there is $k \in K$ for which $\nu(k) \neq \mu(k)$. Those are different ultrafilters, hence there is $V \in \nu(k)$, $V \notin \mu(k)$. Consider the map $f : P \to P_4$ satisfying $f^{-1}(p_A) = V$. $\Delta = (P_4, f)$ is a functional partition, hence part of the limit diagram. By $V \in \nu(k) \setminus \mu(k)$ we get

$$\int_P f \,\mathrm{d}\mu(k) \neq p_A = \int_P f \,\mathrm{d}\nu(k).$$

This contradicts the assumption on ν, since we must have

$$\int_P f \,\mathrm{d}\nu(k) = \pi_\Delta(k) = \int_P f \,\mathrm{d}\mu(k). \qquad \square$$

Lemma 35. *Let $I \subseteq P$ be an upwards directed set and $f : P \to Q$ monotone, Q finite. Then $\max_{i \in I} f(i)$ is well defined. Similarly for F a downwards directed set, $\min_{u \in F} f(u)$ is well defined.*

Proof. If I is upwards directed and f monotone, so is $f[I] \subseteq Q$. Because Q is finite and any upwards directed set has upper bounds of finite subsets, $f[I]$ has a greatest element. $\square$

Lemma 36. *If $I \subseteq P$ has a greatest element x and $f : P \to Q$ is monotone, we have $f(x) = \max_{i \in I} f(i)$.*

Proof. This is obvious. $\square$

Proposition 37. *Let $(P, \leq)$ be ordered, $I \subseteq P$ nonempty upwards directed set. Then there exists $\mathscr{H} \in U(P)$ such that $\mathscr{H} = \sup_{U(P)}\{\mathscr{E}(i) : i \in I\}$. Integration against $\mathscr{H}$ is given by $\int f \, \mathrm{d}\mathscr{H} = \max_{i \in I} f(i)$.*

Dually, for $\emptyset \neq F \subseteq P$ downwards directed we get $\mathscr{D} \in U(P)$ such that $\mathscr{D} = \inf_{U(P)}\{\mathscr{E}(u) : u \in F\}$. Integration against $\mathscr{D}$ is given by $\int f \, \mathrm{d}\mathscr{D} = \min_{u \in F} f(u)$.

Proof. Lemma 36 gives the identity (2.1) for maximum. Thus $\max_I \_$ is a simple integration operator and as such defines an element $\mathscr{H} \in U(P)$. For a homomorphism $f : P \to Q$ into finite Q we get $\int f \, \mathrm{d}\mathscr{E}(i) = f(i) \leq \max_{x \in I} f(x) = \int f \, \mathrm{d}\mathscr{H}$, for every $i \in I$, giving that $\mathscr{H}$ is an upper bound of $\{\mathscr{E}(i) : i \in I\}$. If $\mathscr{U}$ is another upper bound, then $\int f \, \mathrm{d}\mathscr{U} \geq \int f \, \mathrm{d}\mathscr{E}(i) = f(i)$ for every $i \in I$. Hence $\int f \, \mathrm{d}\mathscr{U} \geq \max_{x \in I} f(x) = \int f \, \mathrm{d}\mathscr{H}$, giving $\mathscr{H} \leq \mathscr{U}$. Thus $\mathscr{H}$ is indeed the supremum. $\square$

In the example below we use the following notation: if P is ordered, and $x \in P$, we write

$$(\leftarrow, x] \coloneqq \{a \in P : a \leq x\},$$
$$(\leftarrow, x) \coloneqq \{a \in P : a < x\},$$

and similarly $[x, \rightarrow)$ or $(x, \rightarrow)$ for the reverse relation. By an *ideal* in a poset we mean the notion dual to filter, i.e. an upwards directed downwards closed set.

Example 38.

1. Let L be linearly ordered and $p \in L$. Suppose $C = (\leftarrow, p)$ is nonempty and does not have a greatest element, i.e. $p = \sup C$. Let

   $$\mathscr{C} = \sup\{\mathscr{E}(x) : x \in C\}.$$

   Since $\mathscr{E}(p)$ is also an upper bound, we have $\mathscr{C} \leq \mathscr{E}(p)$. Moreover, the function $f : L \to \{0, 1\}$ such that $f(x) = 0$ iff $x < p$, otherwise $f(x) = 1$, is monotone for which $\int f \, \mathrm{d}\mathscr{C} = \max_{x<p} f(x) = 0 < 1 = f(p) = \int f \, \mathrm{d}\mathscr{E}(p)$. Hence $\mathscr{C} < \mathscr{E}(p)$.

2. Let P be order with an infinite antichain A of cardinality $\varkappa$. Then there exist $2^{2^\varkappa}$ set-ultrafilters on A. Those have the finite intersection property, and since antichains are intervals, these ultrafilters extend into (distinct) elements of $U(P)$.

3. Let A be infinite and P be the disjoint union of A copies of $(\mathbb{N}, \leq)$ (or any other infinite chain without a greatest element). Label those $\mathbb{N}_\alpha$ for $\alpha \in A$. For every $\mathscr{U} \in \beta A$ the system $S = \{\coprod_{\alpha \in U}(n_\alpha, \to) : n_\alpha \in \mathbb{N}_\alpha, U \in \mathscr{U}\}$ has the finite intersection property. Hence it extends into $\mathscr{S} \in U(P)$. If $\mathscr{U}$ is nontrivial, for each $\alpha \in A$ we have $U \in \mathscr{U}$ such that $\alpha \notin U$. This gives an interval $D \in \mathscr{S}$ disjoint with $\mathbb{N}_\alpha$. Hence for every $p \in \mathbb{N}_\alpha$, $\mathscr{S}$ is incomparable to $\mathscr{E}(p)$.

4. Noteworthy applications of Proposition 37 are when I is a chain or filter/ideal in P. In Proposition 43 we show the converse statement in a special case.

From the examples 2 and 3 we see that infinite antichains in P induce ultrafitlters with unclear structure.

Lemma 39. *If $\mathscr{U}_1, \mathscr{U}_2 \in U(P)$ are incomparable, then there exists a simple morphism h such that $\int h \,\mathrm{d}\mathscr{U}_1$ and $\int h \,\mathrm{d}\mathscr{U}_2$ are incomparable.*

Proof. Because neither $\mathscr{U}_1 \leq \mathscr{U}_2$ or $\mathscr{U}_2 \leq \mathscr{U}_1$, by the definition of order on $U(P)$ either the claim holds or there exist simple morphisms f, g such that $\int f \,\mathrm{d}\mathscr{U}_1 < \int f \,\mathrm{d}\mathscr{U}_2$ and $\int g \,\mathrm{d}\mathscr{U}_1 > \int g \,\mathrm{d}\mathscr{U}_2$. But then the codomain of (f, g) is also finite. From properties of product and its pojections we get

$$\int (f, g) \,\mathrm{d}\mathscr{U}_i = \Big(\int f \,\mathrm{d}\mathscr{U}_i, \int g \,\mathrm{d}\mathscr{U}_i\Big), \qquad i = 1, 2.$$

But these elements are incomparable. Hence we can use $h = (f, g)$. □

Proposition 40. *Let $(P, \leq)$ be ordered and $\mathscr{U}_1, \ldots, \mathscr{U}_n$ an antichain in $U(P)$. Choose arbitrary $A_i \in \mathscr{U}_i$, $i = 1, \ldots, n$. Then there exists an antichain $a_1, \ldots, a_n$ in P such that $a_i \in A_i$, $i = 1, \ldots, n$.*

Proof. Case $n = 1$ is trivial since $A_1 \neq \emptyset$. Otherwise let $n > 1$. For each i, j s.t. $1 \leq i < j \leq n$ use Lemma 39 and find a simple morphism h_{ij} such that $\int h_{ij} \,\mathrm{d}\mathscr{U}_i$ is incomparable with $\int h_{ij} \,\mathrm{d}\mathscr{U}_j$. Let $h = (h_{ij} : 1 \leq i < j \leq n)$, its codomain is a finite product, hence finite. Then all elements $\int h \,\mathrm{d}\mathscr{U}_i$, $i = 1, \ldots, n$ are incomparable in the product order. That is because for $i < j$ the i, j-th components of the integrals $\int h \,\mathrm{d}\mathscr{U}_i$ and $\int h \,\mathrm{d}\mathscr{U}_j$ are $\int h_{ij} \,\mathrm{d}\mathscr{U}_i$ and $\int h_{ij} \,\mathrm{d}\mathscr{U}_j$, respectively, and those are incomparable. Now

$$h^{-1}\Big(\int h \,\mathrm{d}\mathscr{U}_i\Big) \cap A_i \in \mathscr{U}_i, \qquad i = 1, \ldots, n,$$

hence these intersections are nonempty and we can choose $a_i \in A_i$ such that $h(a_i) = \int h \,\mathrm{d}\mathscr{U}_i$. Then the elements $h(a_i)$ are incomparable, from monotony also a_i, $i = 1, \ldots, n$ must have been incomparable in P. □

Lemma 41. *Suppose $(P, \leq)$ has only finite antichains. Then for every $\mathscr{U} \in U(P)$ we have*

$$\{x \in P : \mathscr{E}(x) \leq \mathscr{U}\} \in \mathscr{U} \vee \{x \in P : \mathscr{E}(x) \geq \mathscr{U}\} \in \mathscr{U}.$$

Proof. This is clear for $\mathscr{U}$ trivial. Assume $\mathscr{U}$ is nontrivial. Union of the intervals in question is $R = \{x \in P : \mathscr{E}(x) \leq \mathscr{U} \vee \mathscr{E}(x) \geq \mathscr{U}\}$. Let $p_1, \ldots, p_m$ be a maximal antichain in $P \setminus R$. Then $\mathscr{E}(p_1), \ldots, \mathscr{E}(p_m), \mathscr{U}$ is an antichain in $U(P)$. For every $A \in \mathscr{U}$ Proposition 40 gives $a \in A$ such that $p_1, \ldots, p_m, a$ is an antichain in P. By maximality $a \in R$. Hence $A \cap R \neq \emptyset$ for any $A \in \mathscr{U}$. We cannot simply say $R \in \mathscr{U}$, because R may not be an interval. But $P \setminus R$ is an interval. Now

$$(P \setminus R) \amalg \{x \in P : \mathscr{E}(x) \leq \mathscr{U}\} \amalg \{x \in P : \mathscr{E}(x) \geq \mathscr{U}\} = P$$

is a partition into intervals. Since $R \cap (P \setminus R) = \emptyset$ and we have proven that R has nonempty intersection with elements of $\mathscr{U}$, we have $P \setminus R \notin \mathscr{U}$. Thus the claim must hold. $\square$

Lemma 42. *Suppose $\mathscr{E}(m) \leq \mathscr{U}$ in $U(P)$ for some $m \in P$. Then $[m, \rightarrow) \in \mathscr{U}$.*

Proof. $M = [m, \rightarrow)$ is an interval, because it is upwards closed. Since $\mathscr{E}(m) \leq \mathscr{U}$, we must have witnesses of this relation in the sets $\{m\} \in \mathscr{E}(m)$ and any $U \in \mathscr{U}$. This gives $u \in U$ such that $m \leq u$. But then $u \in M$. Hence $M \cap U \neq \emptyset$ for any $U \in \mathscr{U}$, giving $M \in \mathscr{U}$ according to Lemma 29. $\square$

Proposition 43. *Suppose $(P, \leq)$ has only finite antichains. Then for any nontrivial $\mathscr{U}$ there are two possibilities*

- *there exists an infinite ideal $I \subseteq P$ without a greatest element, such that $\mathscr{U} = \sup\{\mathscr{E}(x) : x \in I\}$, or*
- *there exists an infinite filter $F \subseteq P$ without a least element, such that $\mathscr{U} = \inf\{\mathscr{E}(x) : x \in F\}$.*

Proof. Denote

$$H = \{x \in P : \mathscr{E}(x) \geq \mathscr{U}\}, \quad D = \{x \in P : \mathscr{E}(x) \leq \mathscr{U}\}.$$

Those are disjoint because $\mathscr{U}$ is nontrivial. By Lemma 41, either $H \in \mathscr{U}$, or $D \in \mathscr{U}$. We will assume $D \in \mathscr{U}$, the proof and the conclusion in the other case is dual. We want to show that D is the sought ideal. It is clearly downwards closed. Let $a, b \in D$. By Lemma 42 we get

$$A := [a, \rightarrow) \in \mathscr{U}, \quad B := [b, \rightarrow) \in \mathscr{U}.$$

Thus $A \cap B \cap D \in \mathscr{U}$, in particular it is nonempty. But

$$A \cap B \cap D = \{x \in D : x \geq a \,\&\, x \geq b\},$$

giving that D is directed upwards. Hence D is indeed an ideal. We claim

$$\mathscr{U} = \sup\{\mathscr{E}(x) : x \in D\}.$$

For a simple morphism f we have that $\max_{x \in D} f(x)$ exists because D is directed upwards. Thus there is $p \in D$ such that $f(p) = \max_{x \in D} f(x)$. Next $[p, \rightarrow) \in \mathscr{U}$, therefore $S = [p, \rightarrow) \cap D = \{x \in D : x \geq p\} \in \mathscr{U}$. For $x \in f^{-1}(\int f \,\mathrm{d}\mathscr{U}) \cap S$ we have

$$\max_{i \in D} f(i) \geq f(x) \geq f(p) = \max_{i \in D} f(i),$$

hence $\int f \,\mathrm{d}\mathscr{U} = \max_{i\in D} f(i)$. From this and Proposition 37 we know that $\mathscr{U}$ is the supremum of $\{\mathscr{E}(x) : x \in D\}$.

Furter we claim D does not contain a greatest element, in fact, D has no maximal element. Let $m \in D$. Then $[m, \to) \in \mathscr{U}$ and from nontriviality $(m, \to) = [m, \to) \setminus \{m\} \in \mathscr{U}$. Hence there must exist $l \in (m, \to) \cap D$, giving m is not maximal in D. $\square$

It is not the case, that every $\mathscr{U} \in U(P)$ would be "supremum" of a chain in P. Suppose the first option from previous proposition holds, then $I \in \mathscr{U}$. For any $a \in I$ we have $I \cap (a, \to) \in \mathscr{U}$. Let f be a homomorphism satisfying $I \cap (a, \to) = f^{-1}(\int f \,\mathrm{d}\mathscr{U}) = f^{-1}(\max_{i\in I} f(i))$. If $\mathscr{U}$ were a supremum of some chain C, there would have to be $\max_{i\in I} f(i) = f(c)$ for some $c \in C$. Therefore

$$(\forall a \in I\, \exists c \in C)\ c \geq a.$$

Some orders do not have this property, as the following example shows.

Example 44. Let $P = \omega \times \varkappa$ be the product order of these ordinals, where $\varkappa$ has uncountable cofinality. Then P is an ideal. Therefore we have $\mathscr{U} = \sup\{\mathscr{E}(x) : x \in P\}$ with integration given by calculating the maximum. Let us show there is no chain $C \subseteq P$ such that

$$(\forall a \in P\, \exists c \in C)\ c \geq a.$$

For contradiction suppose such a C exists. Define the following sequence $\beta_n, n < \omega$ of elements of $\varkappa$ by

$$\beta_n \coloneqq \min\{\beta : (\exists m \in \omega)\ (m, \beta) \in C \,\&\, m \geq n\}.$$

For any $\alpha \in \varkappa$ there exists $(k, \beta) \in C$ such that $(k, \beta) \geq (0, \alpha)$. In particular $\beta \geq \alpha$. Then $\beta_{k+1} \geq \beta$ because C is a chain. Thus $\beta_{k+1} \geq \alpha$. We proved $(\beta_n)_{n<\omega}$ is cofinal in $\varkappa$, but this is a contradiction.

Remark 45. In Adámek and Sousa [1], D-ultrafilters are prime collections of up-sets (upwards closed sets), whereas here they contain intervals. However, such a collection contains the same information, thus these notions are equivalent: Let $\mathscr{U} \in U(P)$ be an ultrafilter on a poset P, and $\mathscr{U}' \subseteq \mathscr{U}$ the collection of up-sets in $\mathscr{U}$. Then $\mathscr{U}'$ is prime ($X \cup Y \in \mathscr{U}'$ implies $X \in \mathscr{U}'$ or $Y \in \mathscr{U}'$), because $\mathscr{U}$ is (the partition property is stronger), hence it is a D-ultrafilter. Any D-ultrafilter can be extended into an element of $U(P)$. Moreover, $\mathscr{U}$ can be recovered from $\mathscr{U}'$: for an interval A set

$$A^+ \coloneqq \{x \in P : (\exists a \in A)\ a \leq x\}, \qquad A^- \coloneqq \{x \in P : (\exists a \in A)\ x \leq a\}.$$

Then $A = A^+ \cap A^-$ (because it is an interval), A^+ is an up-set, A^- is a down-set (that is, complement of an up-set). A down-set is in $\mathscr{U}$ iff its complement is not in $\mathscr{U}'$. Hence $A \in \mathscr{U}$ iff $A^+ \in \mathscr{U}'$ and $P \setminus A^- \notin \mathscr{U}'$.

3.3 Graphs

For a graph $\mathbf{G}$ denote $U(\mathbf{G})$ the set of ultrafilters on $\mathbf{G}$, i.e. ultrafilters in the semilattice of independent sets of $\mathbf{G}$. Let $\mathscr{E}(x)$ be the trivial ultrafilter for $x \in V(\mathbf{G})$.

Definition. For a finitely colourable graph $\mathbf{G}$ and $\mathscr{U}, \mathscr{V} \in U(\mathbf{G})$ let $\{\mathscr{U}, \mathscr{V}\}$ be an edge of $U(\mathbf{G})$ iff for every homomorphism $f : \mathbf{G} \to \mathbf{H}$, with $\mathbf{H}$ finite, it holds

$$\left\{\int f \,\mathrm{d}\mathscr{U}, \int f \,\mathrm{d}\mathscr{V}\right\} \in E(\mathbf{H}).$$

The resulting graph has no loops because the codomains had no loops. Finitely colourable $\mathbf{G}$ embeds into $U(\mathbf{G})$ via the map $x \mapsto \mathscr{E}(x)$: $\{x, y\} \in E(\mathbf{G})$ gives $\{f(x), f(y)\}$ is an edge for any edge preserving f, for f simple this means that

$$\left\{\int f \,\mathrm{d}\mathscr{E}(x), \int f \,\mathrm{d}\mathscr{E}(y)\right\}$$

is an edge. Conversely, if x, y do not form an edge, given any n-colouring c define $f : \mathbf{G} \to \{1, \dots, n+2\}$ as $f \restriction V(\mathbf{G}) \setminus \{x, y\} = c$ and $f(x) = n+1$, $f(y) = n+2$. Regard $\mathbf{K}_{n+2} - \{n+1, n+2\}$ complete graph without this one additional edge as the codomain of f. Then f is a homomorphism and $\int f \,\mathrm{d}\mathscr{E}(x), \int f \,\mathrm{d}\mathscr{E}(y)$ do not form an edge.

The following equivalent definition of edges in $U(\mathbf{G})$, analogous to Proposition 30 for posets, does not use simple morphisms and as such can be used for non-finitely colourable graphs.

Proposition 46. *Let $\mathbf{G}$ be a finitely colourable graph, $\mathscr{U}, \mathscr{V} \in U(G)$. Then $\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G}))$ iff*

$$(\forall U \in \mathscr{U} \ \forall V \in \mathscr{V} \ \exists u \in U \ \exists v \in V) \ \{u, v\} \in E(\mathbf{G}).$$

Proof. ($\Rightarrow$) For a contradiction assume $\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G}))$ and let $U \in \mathscr{U}$, $V \in \mathscr{V}$ be such that $\{u, v\} \notin E(\mathbf{G})$ for every $u \in U$, $v \in V$. Let c be an n-colouring of $\mathbf{G}$ such that $U = c^{-1}(1)$, using Lemma 18. Define

$$A \coloneqq \{x \in V(\mathbf{G}) : (\exists u \in U)\{u, x\} \in E(\mathbf{G})\}.$$

Then set

$$\begin{aligned} f(u) &= (0, c(u)) = (0, 1), && u \in U, \\ f(a) &= (0, c(a)), && a \in A, \\ f(x) &= (1, c(x)), && x \in V(\mathbf{G}) \setminus (U \cup A). \end{aligned}$$

Because $c(x) \neq c(y)$ implies $f(x) \neq f(y)$ for any pair of vertices, f does not contract any edge into a point. This means we can turn $\operatorname{im} f$ into a graph without loops such that f is a homomorphism. Consider only necessary edges, that is $\{a, b\} \in E(\operatorname{im} f)$ only if $a = f(x)$, $b = f(y)$ for some $\{x, y\} \in E(\mathbf{G})$. In particular, $(0, 1) \in \operatorname{im} f$ is connected only to $(0, c(a))$ for $a \in A$.

We claim that this f contradicts $\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G}))$ because $\int f \,\mathrm{d}\mathscr{U}$ and $\int f \,\mathrm{d}\mathscr{V}$ are not connected: The assumption $\forall u \in U \forall v \in V : \{u, v\} \notin E(\mathbf{G})$ implies $V \cap A = \emptyset$. For the projection $\pi : \operatorname{im} f \to \mathbf{K}_n, (i, k) \mapsto k$ we get $c = \pi \circ f$, implying $\int f \,\mathrm{d}\mathscr{U} = (0, 1)$. Then $f^{-1}(\int f \,\mathrm{d}\mathscr{U}) = U$. Next

$$f^{-1}\Big(\int f \,\mathrm{d}\mathscr{V}\Big) \cap V \in \mathscr{V},$$

giving that the intersection is nonempty. However, if $\int f \,\mathrm{d}\mathscr{V}$ were connected to $(0, 1) = \int f \,\mathrm{d}\mathscr{U}$, we would get $f^{-1}(\int f \,\mathrm{d}\mathscr{V}) \subseteq A$ from construction and the property of A discussed above. This contradicts $V \cap A = \emptyset$.

($\Leftarrow$) Let f be a simple homomorphism. For $U = f^{-1}(\int f \,\mathrm{d}\mathscr{U}) \in \mathscr{U}$ and $V = f^{-1}(\int f \,\mathrm{d}\mathscr{V}) \in \mathscr{V}$ we have some $u \in U$, $v \in V$ such that $\{u, v\} \in E(\mathbf{G})$. Then $\{\int f \,\mathrm{d}\mathscr{U}, \int f \,\mathrm{d}\mathscr{V}\} = \{f(u), f(v)\}$ is an edge of the codomain of f. The morphism f was arbitrary, giving $\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G}))$ by definition. $\square$

Lemma 47. *For finitely colourable graph $\mathbf{G}$ the graph $U(\mathbf{G})$ has the same colourability.*

Proof. Let c be an n-colouring of $\mathbf{G}$. Define $\hat{c} : U(G) \to \{1, \dots, n\}$ by $\mathscr{U} \mapsto \int c \,\mathrm{d}\mathscr{U}$. If $\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G}))$, by definition $\{\int c \,\mathrm{d}\mathscr{U}, \int c \,\mathrm{d}\mathscr{V}\}$ is an edge of $\mathbf{K}_n$, giving $\int c \,\mathrm{d}\mathscr{U} \neq \int c \,\mathrm{d}\mathscr{V}$. Hence $\hat{c}$ is an n-colouring of $U(\mathbf{G})$. $\square$

Definition. Let $f : \mathbf{G} \to \mathbf{H}$ be a graph homomorphism, $\mathscr{U} \in U(\mathbf{G})$. Define

$$f_\# \mathscr{U} := \{A \sqsubseteq \mathbf{H} : f^{-1}[A] \in \mathscr{U}\}.$$

We want to show $f_\# \mathscr{U} \in U(\mathbf{H})$. Here we will not assume finite colourability.

Proof. Let $A, B, A_1, \dots, A_n$ are independent sets of $\mathbf{H}$.

- upwards closure: Assume $A \in f_\# \mathscr{U}$ and $B \supseteq A$. Then $f^{-1}[B] \supseteq f^{-1}[A] \in \mathscr{U}$, giving $f^{-1}[B] \in \mathscr{U}$, hence $B \in f_\# \mathscr{U}$.
- intersections: Assume $A, B \in f_\# \mathscr{U}$. This means $f^{-1}[A], f^{-1}[B] \in \mathscr{U}$, giving $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] \in \mathscr{U}$. Thus $A \cap B \in f_\# \mathscr{U}$.
- nontriviality follows from $\emptyset = f^{-1}[\emptyset]$, as $\emptyset \notin \mathscr{U}$.
- partition property, as in Proposition 26. Suppose $A_1 \amalg \ldots \amalg A_n \supseteq A$ and $A \in f_\# \mathscr{U}$. We have $f^{-1}[A] \in \mathscr{U}$ and $f^{-1}[A_1] \amalg \ldots \amalg f^{-1}[A_n] \supseteq f^{-1}[A]$. Thus for one and only one j we get $f^{-1}[A_j] \in \mathscr{U}$, further giving $A_j \in f_\# \mathscr{U}$. $\square$

Lemma 48. *Let $f : \mathbf{G} \to \mathbf{H}$ and $g : \mathbf{H} \to \mathbf{F}$ be graph homomorphisms, $\mathscr{U} \in U(\mathbf{G})$. Then*

$$(g \circ f)_\# \mathscr{U} = g_\#(f_\# \mathscr{U}).$$

Proof is identical to that of Lemma 31.

Lemma 49. *Let $f : \mathbf{G} \to \mathbf{H}$ and $q : \mathbf{H} \to \mathbf{F}$ be graph homomorphisms, where $\mathbf{F}$ is finite. Let $\mathscr{U} \in U(\mathbf{G})$. Then*

$$\int_{\mathbf{H}} q \,\mathrm{d} f_\# \mathscr{U} = \int_{\mathbf{G}} q \circ f \,\mathrm{d}\mathscr{U}.$$

Proof is identical to that of Lemma 32.

Proposition 50. *Let U assign to graph $\mathbf{G}$ the graph $U(\mathbf{G})$. If f is a morphism, set $U(f) : \mathscr{U} \mapsto f_\# \mathscr{U}$. Then U is a functor $\mathsf{Graph} \to \mathsf{Graph}$ and restricts to a functor $\mathsf{FCGraph} \to \mathsf{FCGraph}$.*

Proof. First we need to show the definition is correct. $U(\mathbf{G})$ is a graph; for finitely colourable we defined edges via morphisms and came to equivalent formulation in Proposition 46, which we use as definition in the general case. $U(\mathbf{G})$ has no loops because every $U \in \mathscr{U}$ is independent, and as such no $u \in U$ is connected to a $v \in U$. $U(f) = f_\# -$ preserves edges; suppose $f \in \mathsf{Graph}(\mathbf{G}, \mathbf{H})$, then $U(f)$ is indeed a function $U(\mathbf{G}) \to U(\mathbf{H})$ – proven after definition of $f_\# -$. Let $\{\mathscr{U}, \mathscr{V}\}$ be an edge of $U(\mathbf{G})$; we want $\{f_\# \mathscr{U}, f_\# \mathscr{V}\}$ to be an edge of $U(\mathbf{H})$. Let $A \in f_\# \mathscr{U}, B \in f_\# \mathscr{V}$, then $f^{-1}[A] \in \mathscr{U}$, $f^{-1}[B] \in \mathscr{V}$. Because $\{\mathscr{U}, \mathscr{V}\}$ is an edge, by definition there exist $a \in f^{-1}[A]$, $b \in f^{-1}[B]$ such that $\{a, b\} \in E(\mathbf{G})$. Then $\{f(a), f(b)\} \in E(\mathbf{H})$ because f preserves edges and $f(a) \in A$, $f(b) \in B$. Hence $f_\# - \in \mathsf{Graph}(U(\mathbf{G}), U(\mathbf{H}))$.

It is clear from definition that $U(\mathrm{id}_\mathbf{G}) = \mathrm{id}_{U(\mathbf{G})}$. Lemma 48 states that U preserves composition. Hence U is a functor.

If $\mathbf{G}$ is finitely colourable, Lemma 47 gives us that $U(\mathbf{G})$ is also finitely colourable. $\mathsf{FCGraph}$ is a full subcategory of Graph, thus U restricts to a functor $\mathsf{FCGraph} \to \mathsf{FCGraph}$. □

For partial orders without infinite antichains, we have seen in Proposition 40 that any finite antichain in $U(P)$ was inherited from P. Here we show a similar type of statement about a first order property being preserved for finitely colourable graphs.

Definition. Let $\mathbf{G}$ be a graph. We say that $\mathbf{G}$ has finite *diameter* $d \in \mathbb{N}$, if for every $x, y \in V(\mathbf{G})$ there exists a path from x to y of edge length at most d, and d is the smallest such number.

Proposition 51. *Suppose $\mathbf{G}$ is a finitely colourable graph with finite diameter $\leq n$. Then $U(\mathbf{G})$ has diameter $\leq n$.*

Converse of this statement is not trivial, but still significantly simpler to prove.

Proof. We will use the following notation. A path P in $\mathbf{G}$ is a (finite) sequence of vertices $(p_0, p_1, \dots, p_m)$ such that $\{p_{i-1}, p_i\}$ is an edge of $\mathbf{G}$, $i = 1, \dots, m$. We also index the elements as $P(i) = p_i$. Length of P is m, i.e. the number of edges. Distance between two vertices is the minimal length of a path having them as endpoints. Denote the distance with $\mathrm{dist}(u, v)$.

For $\mathscr{U}, \mathscr{V} \in U(P)$ define the number

$$m = \min\{k \in \mathbb{N} : (\forall U \in \mathscr{U}\ \forall V \in \mathscr{V}\ \exists u \in U\ \exists v \in V)\ \mathrm{dist}(u, v) \leq k\}.$$

By the assumption $m \leq n$. We will show there is a path from $\mathscr{U}$ to $\mathscr{V}$ of length m. For each $U \in \mathscr{U}, V \in \mathscr{V}$ choose a path P^{UV} of length exactly m with $P^{UV}(0) \in U$, $P^{UV}(m) \in V$. Such a path must exist; suppose there are $U' \in \mathscr{U}$, $V' \in \mathscr{V}$ such that for all $u \in U'$, $v \in V'$ there are only paths of length $\leq m - 1$ from u to v. But then

$$(\forall U \in \mathscr{U}\ \forall V \in \mathscr{V}\ \exists u \in U\ \exists v \in V)\ u \in U \cap U' \ \&\ v \in V \cap V',$$

therefore for such u, v we have $\operatorname{dist}(u, v) \leq m-1$. Thus $k = m-1$ satisfies the condition in definition of m, contradicting minimality of m.

Our goal is to construct a path $(\mathscr{P}_0, \mathscr{P}_1, \dots, \mathscr{P}_m)$ in $U(\mathbf{G})$ with $\mathscr{P}_0 = \mathscr{U}$, $\mathscr{P}_m = \mathscr{V}$. Continue recursively, there are two nested inductions. For $i = 1, \dots, m-1$ consider the system of sets

$$\sigma^i_{UV} \coloneqq \{P^{U'V'}(i) : U' \in \mathscr{U}, V' \in \mathscr{V}, P^{U'V'}(i-1) \in U, P^{U'V'}(m) \in V\},$$
$$S_i \coloneqq \{\sigma^i_{UV} : U \in \mathscr{P}_{i-1}, V \in \mathscr{V}\}.$$

We will show that S_i's have the finite intersection property, hence we can extend them into set-ultrafilters $\mathscr{S}_i \in \beta V(\mathbf{G})$ and use these to construct $\mathscr{P}_i \in U(\mathbf{G})$; via $\mathscr{P}_i = \mathscr{S}_i \cap \{A \sqsubseteq \mathbf{G}\}$. Note that we use finite colourability for $\mathscr{P}_i \neq \emptyset$. Let us first prove by induction that σ^i_{UV} are nonempty. For $i = 1$ let $U \in \mathscr{U}, V \in \mathscr{V}$, then $P^{UV}(0) \in U$ by choice of P^{UV}, hence $P^{UV}(1) \in \sigma^1_{UV}$ by definition. For $i > 1$ let $U \in \mathscr{P}_{i-1}, V \in \mathscr{V}$, pick $\sigma^{i-1}_{U_0V} \in S_{i-1}$ with U_0 arbitrary. By the induction hypothesis $\mathscr{S}_{i-1}$ is an ultrafilter and contains the set $\sigma^{i-1}_{U_0V} \cap U$, hence $\sigma^{i-1}_{U_0V} \cap U \neq \emptyset$. Thus there exist $U' \in \mathscr{U}$ and $V' \in \mathscr{V}$ such that $P^{U'V'}(i-1) \in U \cap \sigma^{i-1}_{U_0V}$ and $P^{U'V'}(m) \in V$. But then $P^{U'V'}(i) \in \sigma^i_{UV}$ by definition.

For $U_0, U_1 \in \mathscr{P}_{i-1}$, $V_0, V_1 \in \mathscr{V}$ we have

$$\sigma^i_{(U_0 \cap U_1),(V_0 \cap V_1)} = \sigma^i_{U_0V_0} \cap \sigma^i_{U_1V_1},$$

hence S_i indeed has the finite intersection property, which completes the construction.

We want $(\mathscr{P}_0, \mathscr{P}_1, \dots, \mathscr{P}_m)$ to be a path in $U(\mathbf{G})$. Let $j \in \{0, \dots, m-2\}$, $A \in \mathscr{P}_j$, $B \in \mathscr{P}_{j+1}$. Let $V \in \mathscr{V}$ be arbitrary. Then because $\mathscr{S}_{j+1}$ is closed under finite intersections, $B \cap \sigma^{j+1}_{AV} \in \mathscr{S}_{j+1}$, so it is nonempty. It is even independent, because B is, hence $B \cap \sigma^{j+1}_{AV} \in \mathscr{P}_{j+1}$. Thus for some $U' \in \mathscr{U}, V' \in \mathscr{V}$ we have $P^{U'V'}(j+1) \in B \cap \sigma^{j+1}_{AV}$. By definition $P^{U'V'}(j) \in A$. Then there is an edge from A to B, namely $\{P^{U'V'}(j), P^{U'V'}(j+1)\}$. Thus $\{\mathscr{P}_j, \mathscr{P}_{j+1}\} \in E(U(\mathbf{G}))$. Lastly, for $j = m-1$ we want $\{\mathscr{P}_{m-1}, \mathscr{V}\}$ to be an edge: For $A \in \mathscr{P}_{m-1}$, $V \in \mathscr{V}$ we have $P^{U'V'}(m-1) \in A \cap \sigma^{m-1}_{UV}$ for some $U, U' \in \mathscr{U}$ and $V' \in \mathscr{V}$, but then $P^{U'V'}(m) \in V$ by the definition of σ^{m-1}_{UV}, hence $\{P^{U'V'}(m-1), P^{U'V'}(m)\}$ is an edge from A to V. □

Example 52. We examine properties of $U(\mathbf{G})$ for some choices of the graph $\mathbf{G}$:

1. Remarks on degree. Suppose v is a vertex of a finitely colourable graph $\mathbf{G}$ and $\deg v = \varkappa$. Let N be the set of neighbours of v. If $\varkappa$ is finite, then $\deg \mathscr{E}(v) = \varkappa$: consider an edge $\{\mathscr{U}, \mathscr{E}(v)\}$, then for any $U \in \mathscr{U}$ either $U \cap N$ or $U \setminus N$ is an element of $\mathscr{U}$. But $U \setminus N \in \mathscr{U}$ would contradict the edge $\{\mathscr{U}, \mathscr{E}(v)\}$, hence $U \cap N \in \mathscr{U}$. But $N \cap U$ is finite, because N is, thus $\mathscr{U}$ is trivial. For trivial ultrafilters $\mathscr{E}(x)$ is connected to $\mathscr{E}(v)$ iff x is connected to v, hence $\deg \mathscr{E}(v) = \varkappa$.

   On the other hand, if $\varkappa$ is infinite, N contains an infinite independent set N' of cardinality $\varkappa$ by finite colourability of $\mathbf{G}$ and the pigeonhole principle. Then any set-ulrafilter $\tilde{\mathscr{U}} \in \beta N'$ is a system of independent sets with the finite intersection property, hence it extends into an element $\mathscr{U} \in U(\mathbf{G})$ (and different elements of $\beta N'$ give different elements of $U(\mathbf{G})$). Clearly $\{\mathscr{U}, \mathscr{E}(v)\}$ is an edge. Thus $\deg \mathscr{E}(v) \geq 2^{2^\varkappa}$, as we have $2^{2^\varkappa}$ elements of $\beta N'$.

2. Consider a *star graph* $\mathbf{G}$. We have a "central" vertex c and an infinite set S; we define the edges by

$$E(\mathbf{G}) := \{(c,s) : s \in S\}.$$

Then S is an independent set. If A is an independent set containing more than one element, then $c \notin A$. We can see that $\beta S \cup \{\mathscr{E}(c)\} = U(\mathbf{G})$. For any $\mathscr{U} \in \beta S$ and $U \in \mathscr{U}$, we have edges from U to $\{c\}$ (all vertices are connected to c), hence $\mathscr{U}$ is connected to $\mathscr{E}(c)$. For different $\mathscr{U}, \mathscr{V} \in \beta S$ we have some $U \in \mathscr{U}$ and $V \in \mathscr{V}$ such that $U \notin \mathscr{V}$ and $V \notin \mathscr{U}$. Because $U, V \subseteq S$, there is no edge between them, giving that $\{\mathscr{U}, \mathscr{V}\}$ is not an edge. We can conclude that $U(\mathbf{G})$ is also a star graph, and has larger cardinality.

3. Let $\mathbf{G}$ have vertices $\mathbb{N} \times 2 = \{(n,i) : n \in \mathbb{N}, i = 0,1\}$ and edges only between vertices $(n,0)$ and $(n,1)$ for each $n \in \mathbb{N}$. Set $T = \{(n,1) : n \in \mathbb{N}\}$, $B = \{(n,0) : n \in \mathbb{N}\}$, then $T \amalg B$ is a partition of $V(\mathbf{G})$ into independent sets. For $\mathscr{U} \in U(\mathbf{G})$ define its mirror image $M(\mathscr{U})$ by

$$M(\mathscr{U}) := \{M(A) : A \in \mathscr{U}\},$$

where

$$M(A) = \{(n, 1-i) : (n,i) \in A\}, \quad A \subseteq V(\mathbf{G}).$$

Let $\{\mathscr{U}, \mathscr{V}\}$ be an edge in $U(\mathbf{G})$, and $U \in \mathscr{U}, V \in \mathscr{V}$. Set U' to be the set of those vertices of U, which are connected to a vertex of V, similarly define $V' \subseteq V$. Because $\{\mathscr{U}, \mathscr{V}\}$ is an edge, we must have $U' \in \mathscr{U}$, $V' \in \mathscr{V}$. By construction of $\mathbf{G}$ we get $M(V') = U'$. But then $V' = MM(V') = M(U') \in M(\mathscr{U})$, by upwards closure $V \in M(\mathscr{U})$, hence $\mathscr{V} \subseteq M(\mathscr{U})$. Since $M(\mathscr{U})$ is clearly an element of $U(\mathbf{G})$, this inclusion gives $\mathscr{V} = M(\mathscr{U})$. It is easy to see that $\mathscr{U}$ is connected to $M(\mathscr{U})$ for every $\mathscr{U} \in U(\mathbf{G})$. Therefore every element of $U(\mathbf{G})$ is connected to precisely one vertex, just as holds in $\mathbf{G}$.

4. Path infinite in one direction. Regard $\mathbb{N}$ as a graph with edges $\{n, n+1\}, n \in \mathbb{N}$. Consider an ultrafilter $\mathscr{U}$ on $\mathbb{N}$ different from $\mathscr{E}(1)$. For a set $A \subseteq \mathbb{N}$ set

$$A^+ := \{n+1 : n \in A\}, \qquad A^- := \{n-1 : n \in A \,\&\, n > 1\}.$$

Then denote

$$\mathscr{U}^+ := \{A^+ : A \in \mathscr{U}\} \cup \{A^+ \cup \{1\} : A \in \mathscr{U} \,\&\, 1 \notin A\},$$
$$\mathscr{U}^- := \{A^- : A \in \mathscr{U}\}$$

One can check that $\mathscr{U}^+, \mathscr{U}^- \in U(\mathbb{N})$. They are clearly both connected to $\mathscr{U}$. We can deduce $\mathscr{U}^+ \neq \mathscr{U}^-$ from the partition of $\mathbb{N}$ into independent sets

$$U_i := \{3n + i : n \in \mathbb{N}\}, \quad i = 0,1,2,$$

the fact that $U_i \in \mathscr{U}$ for only one i and that $U_i^+ \cap U_i^- = \emptyset$. Conversely, let $\mathscr{V}$ be connected to $\mathscr{U}$. For $V \in \mathscr{V}$ and $U \in \mathscr{U}$ we have either $V \cap U^- \in \mathscr{V}$ or $V \setminus U^- \in \mathscr{V}$. For other $V' \in \mathscr{V}$ the same must happen, otherwise

$$(V \cap U^-) \cap (V' \setminus U^-) = \emptyset \in \mathscr{V},$$

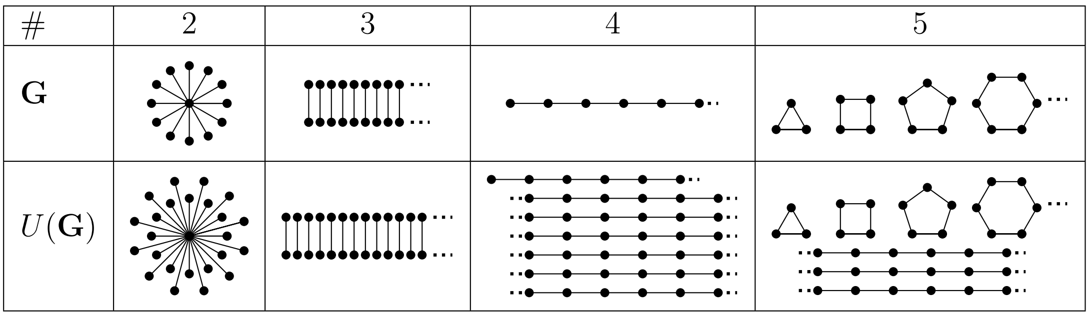
or the other way around, a contradiction. In the first case we get $U^- \supseteq V \cap U^- \in \mathscr{V}$, hence $\mathscr{U}^- = \mathscr{V}$. In the second case we must have had $V \cap U^+ \in \mathscr{V}$, then we conclude $U^+ \supseteq U^+ \cap V \in \mathscr{V}$, same for $U^+ \cup \{1\}$ if relevant, therefore $\mathscr{U}^+ = \mathscr{V}$. We now see that $\deg \mathscr{U} = 2$. Moreover, one of the neighbours of $\mathscr{U}$ must be $f_\# \mathscr{U}$ where $f(n) = n + 1$.

Because $\mathscr{E}(1)$ can only be connected to $\mathscr{E}(2)$, we can conclude that $\mathscr{U}(\mathbb{N})$ is composed from a copy of $\mathbb{N}$ and many copies of the double sided infinite path $\mathbb{Z}$ (we get one of these from any of the uncountably many ultrafilters on the set of even natural numbers).

5. Let $\mathbf{G} = \coprod_{n \geq 3} \mathbf{C}_n$, where $\mathbf{C}_n$ is the n-cycle with vertices $v_1^n, \dots, v_n^n$. We will show that $U(\mathbf{G})$ contains an infinite path. Put $A_k := \{v_k^n : n \geq k\}$ for $k \geq 3$. Choose a nontrivial ultrafilter $\mathscr{U}_3$ on the set A_3 arbitrarily. We will recursively construct nontrivial ultrafilters $\mathscr{U}_k \in \beta A_k$, $k \geq 3$. For $k \geq 3$ the ultrafilter $\mathscr{U}_k$ must contain the set $A_k \smallsetminus \{v_k^k\}$, hence $\mathscr{U}_k' = \mathscr{U}_k \cap \mathscr{P}(A_k \smallsetminus \{v_k^k\}) \in \beta(A_k \smallsetminus \{v_k^k\})$. Then set

$$\mathscr{U}_{k+1} := \{\{v_{k+1}^n : v_k^n \in A\} : A \in \mathscr{U}_k'\}.$$

We have pushed $\mathscr{U}_k'$ along a bijection $A_k \smallsetminus \{v_k^k\} \to A_{k+1}$, giving that $\mathscr{U}_{k+1} \in \beta A_{k+1}$. It is nontrivial, because it contains only infinite sets. By construction, for any $A \in \mathscr{U}_k$ and $B \in \mathscr{U}_{k+1}$ there is an edge from A to B. Now if we extend $\mathscr{U}_k$ into $\mathscr{V}_k \in U(\mathbf{G})$, we get that $\{\mathscr{V}_k, \mathscr{V}_{k+1}\}$ is an edge for any $k \geq 3$. With minor changes to the construction, we could extend the path infinitely in the other direction as well.



Illustrations of the graphs in Example 52

After seing these examples and a partial result leading to Proposition 51, we have conjectured the following:

> For a finitely colourable graph $\mathbf{G}$, the graph $U(\mathbf{G})$ is an elementary extension of $\mathbf{G}$.

To interpret graphs as first order logic structures, we use the language with one symmetric antireflexive binary relation describing the edges. This language is too weak to capture finite colourability, which is also preserved by Lemma 47, hence one could even strengthen the question. So far, neither one of us knows how to prove or disprove this statement.

4. Monads

Monad is a category-theoretical construction consisting of an endofunctor $F : \mathscr{C} \to \mathscr{C}$, and two natural transformations: multiplication $\mu : T^2 \to T$ and unit $\eta : I \to T$. Following axioms are required, expressed as commutative diagrams, referred to as associativity and unit law

$$\begin{tikzcd} T^3(X) \arrow[r, "T\mu_X"] \arrow[d, "\mu_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[d, "T\eta_X"'] \arrow[dr, equal] & TT(X) \arrow[d, "\mu_X"] \\ TT(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

Algebra over a monad (T, μ, η) is an object A of $\mathscr{C}$ together with a morphism $s \in \mathscr{C}(T(A), A)$ such that the following diagrams commute

$$\begin{tikzcd} T^2(A) \arrow[r, "Ts"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "s"] \\ T(A) \arrow[r, "s"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & T(A) \arrow[d, "s"] \\ & A \end{tikzcd}$$

Algebras over a monad form a category called the Eilenberg-Moore category of (T, μ, η). Morphisms of this category are homomorphisms of algebras, i.e. for algebras (A, s), (B, t) morphism $f \in \mathscr{C}(A, B)$ is a homomorphism if the following commutes

$$\begin{tikzcd} T(A) \arrow[r, "Tf"] \arrow[d, "s"'] & T(B) \arrow[d, "t"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

4.1 Partial orders

Ernst Manes proved in [7] that algebras over the ultrafilter monad on Set are the compact Hausdorff topological spaces. Here we provide a similar, yet incomplete classification of algebras over the ultrafilter monad on Poset. For posets without infinite antichains, this characterisation is complete. Overall structure of the proof was inspired by this original article of Manes.

4.1.1 Construction of ultrafilter monad

Lemma 53. *Let $(P, \leq)$ be ordered. For interval $A \sqsubseteq P$ denote*

$$\hat{A} := \{\mathscr{U} \in U(P) : A \in \mathscr{U}\}.$$

Then $\hat{A}$ is interval in $U(P)$.

Poset specific proof. If $\mathscr{U}, \mathscr{V} \in \hat{A}$ and $\mathscr{W}$ satisfy $\mathscr{U} \leq \mathscr{W} \leq \mathscr{V}$, let $f : P \to P_4$ be the morphism from Proposition 8 such that $A = f^{-1}(p_A)$. Because $A \in \mathscr{U}, \mathscr{V}$, we have $\int f \,\mathrm{d}\mathscr{U} = \int f \,\mathrm{d}\mathscr{V} = p_A$. But $\int f \,\mathrm{d}\mathscr{U} \leq \int f \,\mathrm{d}\mathscr{W} \leq \int f \,\mathrm{d}\mathscr{V}$ now implies $\int f \,\mathrm{d}\mathscr{W} = p_A$, hence $A \in \mathscr{W}$. This gives $\mathscr{W} \in \hat{A}$, hence $\hat{A}$ is indeed an interval. $\square$

Generalisable proof. Let $f : P \to P_4$ be a morphism from Proposition 8 such that $f^{-1}(p_A) = A$ (the fact that $\operatorname{cod} f = P_4$ is not important). Then $\hat{f} = \int f \,\mathrm{d}\_ :$ $U(P) \to P_4$ is a morphism for which $\hat{A} = \hat{f}^{-1}(p_A)$ because $\int f \,\mathrm{d}\mathscr{U} = p_A$ iff $A = f^{-1}(p_A) \in \mathscr{U}$. Recall that preimage of a point is an interval. $\square$

Proposition 33 tells us that U is an endofunctor on Poset. Our goal is to show, that U is monadic. Iterated applications of U are written as UU or U^n.

Lemma 54. *For $\mathfrak{U} \in UU(P)$ set*

$$\mu\mathfrak{U} := \{A \sqsubseteq P : \hat{A} \in \mathfrak{U}\}.$$

Then $\mu\mathfrak{U} \in U(P)$.

Proof. Let $A, B, A_1, \ldots, A_n$ be intervals of P. Then

- $\emptyset \notin \mu\mathfrak{U}$ because $\emptyset = \hat{\emptyset} \notin \mathfrak{U}$. $P \in \mu\mathfrak{U}$ because $U(P) = \hat{P} \in \mathfrak{U}$.
- Suppose $A \subseteq B$ and $A \in \mu\mathfrak{U}$. Then $\hat{A} \in \mathfrak{U}$; because $\hat{A} \subseteq \hat{B}$, we get $\hat{B} \in \mathfrak{U}$ and hence $B \in \mu\mathfrak{U}$.
- Suppose $A, B \in \mu\mathfrak{U}$. Then $\hat{A}, \hat{B} \in \mathfrak{U}$; because $(A \cap B)^\wedge = \hat{A} \cap \hat{B}$, we have $(A \cap B)^\wedge \in \mathfrak{U}$. Thus $A \cap B \in \mu\mathfrak{U}$.
- Assume $A_1 \amalg \ldots \amalg A_n = P$. Since every $\mathscr{U} \in U(P)$ contains precisely one A_i, we get
$$\hat{A}_1 \amalg \ldots \amalg \hat{A}_n = U(P).$$
Hence for only one i it holds that $\hat{A}_i \in \mathfrak{U}$, giving $A_i \in \mu\mathfrak{U}$. $\square$

Lemma 55. *$\mu : UU(P) \to U(P)$ is monotone.*

Proof. We will use Proposition 30. Let $\mathfrak{U} \leq \mathfrak{V}$ in $UU(P)$. Then consider $U \in \mu\mathfrak{U}$, $V \in \mu\mathfrak{V}$. We have $\hat{U} \in \mathfrak{U}$, $\hat{V} \in \mathfrak{V}$, from the relation we get $\mathscr{U} \in \hat{U}$ and $\mathscr{V} \in \hat{V}$ such that $\mathscr{U} \leq \mathscr{V}$. Then $U \in \mathscr{U}$ and $V \in \mathscr{V}$, hence for some $u \in U$, $v \in V$ we get $u \leq v$. It follows that $\mu\mathfrak{U} \leq \mu\mathfrak{V}$. $\square$

Proposition 56. *(U, μ, η) is a monad, where U is the ultrafilter functor on* Poset*, multiplication $\mu = (\mu_P; (P, \leq) \in \mathsf{Poset})$ is defined as*

$$\mu_P : UU(P) \to U(P), \quad \mathfrak{U} \mapsto \{A \sqsubseteq P : \hat{A} \in \mathfrak{U}\}.$$

Unit $\eta = (\eta_P : (P, \leq) \in \mathsf{Poset})$ is defined as

$$\eta_P : P \to U(P), \quad x \mapsto \mathscr{E}(x) = \{A \sqsubseteq P : x \in A\}.$$

Proof. First we show that μ, η are natural transformations, i.e. the following commute

$$\begin{array}{ccc} UU(P) & \xrightarrow{\mu_P} & U(P) \\ {\scriptstyle UUf}\downarrow & & \downarrow{\scriptstyle Uf} \\ UU(Q) & \xrightarrow{\mu_Q} & U(Q) \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\eta_P} & U(P) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle U(f)} \\ Q & \xrightarrow{\eta_Q} & U(Q) \end{array}$$

For the first diagram we want $\mu_Q f_{\#\#}\mathfrak{U} = f_\# \mu_P \mathfrak{U}$, $\mathfrak{U} \in UU(P)$.

$$\begin{aligned}
f_\#\mu_P\mathfrak{U} &= \{A \sqsubseteq Q : f^{-1}[A] \in \mu\mathfrak{U}\}\\
&= \{A \sqsubseteq Q : f^{-1}[A] \in \{B \sqsubseteq P : \{\mathscr{U} \in U(P) : \mathscr{U} \ni B\} \in \mathfrak{U}\}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{U} \in U(P) : \mathscr{U} \ni f^{-1}[A]\} \in \mathfrak{U}\}\\
\mu_Q f_{\#\#}\mathfrak{U} &= \mu_Q\{B \sqsubseteq U(Q) : (f_\#)^{-1}[B] \in \mathfrak{U}\}\\
&= \mu_Q\{B \sqsubseteq U(Q) : \{\mathscr{U} \in U(P) : f_\#\mathscr{U} \in B\} \in \mathfrak{U}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{V} \in U(Q) : \mathscr{V} \ni A\}\\
&\qquad\qquad \in \{B \sqsubseteq U(Q) : \{\mathscr{U} \in U(P) : f_\#\mathscr{U} \in B\} \in \mathfrak{U}\}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{U} \in U(P) : f_\#\mathscr{U} \in \{\mathscr{V} \in U(Q) : \mathscr{V} \ni A\}\} \in \mathfrak{U}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{U} \in U(P) : f_\#\mathscr{U} \ni A\} \in \mathfrak{U}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{U} \in U(P) : A \in \{B \sqsubseteq Q : f^{-1}[B] \in \mathscr{U}\}\} \in \mathfrak{U}\}\\
&= \{A \sqsubseteq Q : \{\mathscr{U} \in U(P) : f^{-1}[A] \in \mathscr{U}\} \in \mathfrak{U}\}
\end{aligned}$$

The second diagram says $f_\#\eta_P(x) = \eta_Q f(x)$ for all $x \in P$.

$$\begin{aligned}
f_\#\eta_P(x) = f_\#\mathscr{E}(x) &= \{A \sqsubseteq Q : f^{-1}[A] \in \mathscr{E}(x)\}\\
&= \{A \sqsubseteq Q : x \in f^{-1}[A]\} = \mathscr{E}(f(x)) = \eta_Q f(x).
\end{aligned}$$

Associativity and the unit law hold

$$\begin{tikzcd}
U^3(P) \arrow[r, "U\mu_P"] \arrow[d, "\mu_{U(P)}"'] & U^2(P) \arrow[d, "\mu_P"] \\
U^2(P) \arrow[r, "\mu_P"] & U(P)
\end{tikzcd}
\qquad
\begin{tikzcd}
U(P) \arrow[r, "\eta_{U(P)}"] \arrow[d, "U\eta_P"'] \arrow[dr, equal] & UU(P) \arrow[d, "\mu_P"] \\
UU(P) \arrow[r, "\mu_P"] & U(P)
\end{tikzcd}$$

Associativity requires $\mu_P(\mu_{P\#}u) = \mu_P(\mu_{U(p)}u)$ for $u \in U^3(P)$.

$$\begin{aligned}
&\mu_P(\mu_{P\#}u)\\
&\quad= \mu_P\{B \sqsubseteq U(P) : \mu_P^{-1}[B] \in u\}\\
&\quad= \{A \sqsubseteq P : \{\mathscr{U} \in U(P) : \mathscr{U} \ni A\} \in \{B \sqsubseteq U(P) : \mu_P^{-1}[B] \in u\}\}\\
&\quad= \{A \sqsubseteq P : \mu_P^{-1}[\{\mathscr{U} \in U(P) : \mathscr{U} \ni A\}] \in u\}\\
&\quad= \{A \sqsubseteq P : \{\mathfrak{U} \in U^2(P) : \mu_P(\mathfrak{U}) \ni A\} \in u\}\\
&\quad= \{A \sqsubseteq P : \{\mathfrak{U} \in U^2(P) : A \in \{B \sqsubseteq P : \{\mathscr{U} \in U(P) : B \in \mathscr{U}\} \in \mathfrak{U}\}\} \in u\}\\
&\quad= \{A \sqsubseteq P : \{\mathfrak{U} \in U^2(P) : \{\mathscr{U} \in U(P) : A \in \mathscr{U}\} \in \mathfrak{U}\} \in u\}\\
&\mu_P(\mu_{U(P)}u)\\
&\quad= \mu_P\{B \sqsubseteq U(P) : \{\mathfrak{U} \in U^2(P) : \mathfrak{U} \ni B\} \in u\}\\
&\quad= \{A \sqsubseteq P : \{\mathscr{U} \in U(P) : \mathscr{U} \ni A\}\\
&\qquad\qquad \in \{B \sqsubseteq U(P) : \{\mathfrak{U} \in U^2(P) : \mathfrak{U} \ni B\} \in u\}\}\\
&\quad= \{A \sqsubseteq P : \{\mathfrak{U} \in U^2(P) : \{\mathscr{U} \in U(P) : A \in \mathscr{U}\} \in \mathfrak{U}\} \in u\}
\end{aligned}$$

For the unit $\mu_P(\eta_{U(P)}\mathscr{U}) = \mu_P(\eta_{P\#}\mathscr{U}) = \mathscr{U}$ for all $\mathscr{U} \in U(P)$

$$\begin{aligned}
\mu_P\eta_{P\#}\mathscr{U} &= \mu_P\{B \sqsubseteq U(P) : \eta_P^{-1}[B] \in \mathscr{U}\} \\
&= \{A \sqsubseteq P : \{\mathscr{V} \in U(P) : A \in \mathscr{V}\} \in \{B \sqsubseteq U(P) : \eta_P^{-1}[B] \in \mathscr{U}\}\} \\
&= \{A \sqsubseteq P : \eta_P^{-1}[\{\mathscr{V} \in U(P) : A \in \mathscr{V}\}] \in \mathscr{U}\} \\
&= \{A \sqsubseteq P : \{x \in P : \mathscr{E}(x) \in \{\mathscr{V} \in U(P) : A \in \mathscr{V}\}\} \in \mathscr{U}\} \\
&= \{A \sqsubseteq P : \{x \in P : A \in \mathscr{E}(x)\} \in \mathscr{U}\} \\
&= \{A \sqsubseteq P : \{x \in P : x \in A\} \in \mathscr{U}\} = \{A \sqsubseteq P : A \in \mathscr{U}\} = \mathscr{U} \\
\mu_P\eta_{U(P)}\mathscr{U} &= \mu_p\mathscr{E}(\mathscr{U}) \\
&= \mu_P\{B \sqsubseteq U(P) : \mathscr{U} \in B\} \\
&= \{A \sqsubseteq P : \{\mathscr{V} \in U(P) : A \in \mathscr{V}\} \in \{B \sqsubseteq U(P) : \mathscr{U} \in B\}\} \\
&= \{A \sqsubseteq P : \mathscr{U} \in \{\mathscr{V} \in U(P) : A \in \mathscr{V}\}\} = \{A \sqsubseteq P : A \in \mathscr{U}\} = \mathscr{U}.
\end{aligned}$$

□

4.1.2 Topology and algebras

Well known theorems of topology used in this subsection are not cited directly. Good reference on the subject is Engelking [3].

Interval topology

Firstly, we define the interval topology for general partial orders, which extends the natural choice of topology for linear orders; the one generated by a basis of open-ended intervals $(a, b) = \{x \in L : a < x < b\}$, $a, b \in L$.

Definition. Let $(P, \leq)$ be ordered. By *interval topology* we mean the topology whose closed sets are generated by the subbasis consisting of intervals

$$\begin{aligned}
(\leftarrow, a] &= \{x \in P : x \leq a\}, \\
[a, \rightarrow) &= \{x \in P : a \leq x\}, \qquad a \in P
\end{aligned}$$

and $\emptyset$. Any closed set is thus an intersection of finite unions of these intervals.

It is easy to see that the open subbasic sets $P \setminus (\leftarrow, a]$, $P \setminus [a, \rightarrow)$ are intervals (they are even upwards or downwards closed). As their intersections are also intervals, we can conclude that interval topology has a basis from open intervals.

We will later use the following statement, which is a special case of Theorem 1 in Erné [4]

Proposition 57. *Let $(P, \leq)$ be ordered. If P does not contain any infinite antichain, then the interval topology on P is Hausdorff.*

It is not difficult to prove the compactness of the interval topology on $U(P)$ directly, but we will not need this fact.

Topology induced by algebra structure

For an algebra (A, s) over the set-ultrafilter monad β, we can define a topology by saying a set $U \subseteq A$ is open iff

$$(\forall \mathscr{U} \in \beta A)(s(\mathscr{U}) \in U \implies U \in \mathscr{U}).$$

This definition mostly only rephrases what we want from limits of ultrafilters; a set is open if it is contained in all ultrafilters which converge into it. This topology on A is compact Hausdorff. Checking Hausdorffness is undoubtedly the hardest part of the proof. The other way around, given a compact Hausdorff space A, we define the structure map s to assign to an ultrafilter its (unique) limit with respect to this topology.

Let (P, s) be an algebra over U. We want to define a topology in a similar way to the Set case. Ultrafilters on P only contain intervals and not every open set need be an interval, thus we need to adapt the formula. We want to keep the fact that an interval $A \sqsubseteq P$ is open iff

$$(\forall \mathscr{U} \in U(P))(s(\mathscr{U}) \in A \implies A \in \mathscr{U}).$$

In general, we define a set $A \subseteq P$ to be open iff

$$(\forall \mathscr{U} \in U(P))(s(U) \in A \implies (\exists A' \sqsubseteq P)\ A' \subseteq A \,\&\, A' \in \mathscr{U}). \tag{4.1}$$

Intuitively, this means A would have been an element of a set-ultrafilter extending $\mathscr{U}$.

Conversely, a set $F \subseteq P$ is closed iff

$$(\forall \mathscr{U} \in U(P))((\forall A' \in \mathscr{U})\ A' \cap F \neq \emptyset) \implies s(\mathscr{U}) \in F.$$

The name s-topology is taken from Devlin [2].

Proposition 58. *Open sets of an algebra (P, s) defined by the formula (4.1) indeed form a topology. We call it the s-topology.*

Proof. Empty set never satisfies the assumption, whole P always satisfies the conclusion, hence both are open. For open sets A_i, $i \in I$ if $s(\mathscr{U}) \in \bigcup_{i \in I} A_i$, there exists $j \in I$ such that $s(\mathscr{U}) \in A_j$; since A_j is open there is an interval $A \subseteq A_j$ such that $A \in \mathscr{U}$. But then $A \subseteq \bigcup_{i \in I} A_i$, hence the union is open. If A, B are open and $s(\mathscr{U}) \in A \cap B$, by the assumption we get intervals $A' \subseteq A$, $B' \subseteq B$ such that $A' \in \mathscr{U}$, $B' \in \mathscr{U}$. Then $A' \cap B' \in \mathscr{U}$ and it is a subset of $A \cap B$, giving $A \cap B$ is open. In conclusion, we have a topology. □

Lemma 59. *In the free algebra $(U(P), \mu_P)$ μ_P-topology is generated by the basis $\{\hat{A} : A \sqsubseteq P\}$. Hence for $T \subseteq U(P)$ open we have*

$$T = \bigcup \{\hat{A} : A \sqsubseteq P \,\&\, \hat{A} \subseteq T\}$$

Proof. For $A \sqsubseteq P$ we know that $\hat{A}$ is an interval in $U(P)$ (Lemma 53); openness follows from definitions as follows. $\mu_P \mathfrak{U} \in \hat{A}$ means $A \in \mu_P \mathfrak{U}$ by definition of $\hat{A}$, from this $\hat{A} \in \mathfrak{U}$ by definition of μ_P. Hence $\hat{A}$ is open.

Let $T \subseteq U(P)$ be open, we will prove the formula. The nontrivial inclusion is $\subseteq$. Thus we want

$$(\forall \mathscr{U} \in T\ \exists A \in \mathscr{U})\ \hat{A} \subseteq T.$$

For a contradiction, suppose there is $\mathscr{U} \in T$ such that for all $A \in \mathscr{U}$ it holds $\hat{A} \cap (U(P) \setminus T) \neq \emptyset$. Then

$$\{\hat{A} : A \in \mathscr{U}\} \cup \{U(P) \setminus T\}$$

has the finite intersection property, let $\tilde{\mathfrak{U}} \in \beta U(P)$ be an ultrafilter extending it. We get $\mathfrak{U} = \tilde{\mathfrak{U}} \cap \{A \sqsubseteq U(P)\} \in UU(P)$. Consider any interval $I \sqsubseteq U(P)$ such that $I \subseteq T$; then $I \notin \tilde{\mathfrak{U}}$ because $T \notin \tilde{\mathfrak{U}}$. Thus $I \notin \mathfrak{U}$. Hence $\mu_P\mathfrak{U} \notin T$ because T is open. But for $A \in \mathscr{U}$ we have $\hat{A} \in \mathfrak{U}$, giving $A \in \mu_P\mathfrak{U}$. We just proved $\mu_P\mathfrak{U} = \mathscr{U}$, but $\mathscr{U} \in T$ and $\mu_P\mathfrak{U} \notin T$, a contradiction. □

As a corollary we get that the μ_P-topology on $U(P)$ is generated by open intervals. Dually, a set $F \subseteq U(P)$ is closed iff

$$F = \bigcap\{U(P) \setminus \hat{A} : A \sqsubseteq P \,\&\, \hat{A} \cap F = \emptyset\}.$$

In Set, one can simplify this expression using complements, because there $U(P) \setminus \hat{A} = (P \setminus A)^\wedge$. However, complement of an interval need not be an interval, so we cannot use this.

Lemma 60. *Let (P, s) be an algebra over U. Then the structure map s is closed, i.e. the image of a closed set is closed.*

Proof. Let $F \subseteq U(P)$ be closed. From Lemma 59 we know $F = \bigcap_{i\in I}(U(P) \setminus \hat{M}_i)$ for some intervals $M_i \sqsubseteq P, i \in I$. We want $s[F]$ to be closed. This means

$$(\forall \mathscr{U} \in U(P))((\forall A \in \mathscr{U})\ A \cap s[F] \neq \emptyset) \implies s(\mathscr{U}) \in s[F].$$

Let $\mathscr{U} \in U(P)$ satisfy the assumption $(\forall A \in \mathscr{U})\ A \cap s[F] \neq \emptyset$, thus for each $A \in \mathscr{U}$ the set

$$\alpha_A \coloneqq \{\mathscr{V} \in U(P) : s(\mathscr{V}) \in A \,\&\, (\forall i \in I)\ M_i \notin \mathscr{V}\}$$

is nonempty. For $A, B \in \mathscr{U}$ it holds $\alpha_A \cap \alpha_B = \alpha_{A\cap B}$, therefore the system $\{\alpha_A : A \in \mathscr{U}\}$ has the finite intersection property. Extend it into an ultrafilter $\tilde{\mathfrak{U}} \in \beta U(P)$ and extract from it $\mathfrak{U} \in UU(P)$ (via $\mathfrak{U} = \tilde{\mathfrak{U}} \cap \{A \sqsubseteq U(P)\}$). Then

$$Us(\mathfrak{U}) = s_\#\mathfrak{U} = \{B \sqsubseteq P : s^{-1}[B] \in \mathfrak{U}\} = \mathscr{U},$$

because for $A \in \mathscr{U}$ we have $s^{-1}[A] \supseteq \alpha_A$ and $\alpha_A \in \tilde{\mathfrak{U}}$, hence also $s^{-1}[A] \in \tilde{\mathfrak{U}}$. This is an interval, thus $s^{-1}[A] \in \mathfrak{U}$. Thus $s_\#\mathfrak{U} = \mathscr{U}$. Next

$$\mu_P\mathfrak{U} = \{A \sqsubseteq P : \hat{A} \in \mathfrak{U}\}$$

by definition. For every $i \in I$ and any $A \in \mathscr{U}$ we have $\hat{M}_i \cap \alpha_A = \emptyset$. Therefore $\hat{M}_i \notin \mathfrak{U}$ and thus $M_i \notin \mu_P\mathfrak{U}$. From this $\mu_P\mathfrak{U} \notin \hat{M}_i$. In conclusion $\mu_P\mathfrak{U} \in \bigcap_{i\in I}(U(P) \setminus \hat{M}_i) = F$. By commutativity of the following algebra diagram

$$\begin{array}{ccc} U^2(P) & \xrightarrow{Us} & U(P) \\ {\scriptstyle \mu_P}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ U(P) & \xrightarrow{\quad s \quad} & P \end{array}$$

we have

$$s(\mathscr{U}) = s(s_\#\mathfrak{U}) = s(\mu_P\mathfrak{U}) \in s[F].$$

□

Definition. Let $(P, \leq)$ be ordered and $\mathscr{G}$ a topology on P. We call the order relation *continuous* with respect to $\mathscr{G}$, if

$$R_{(P,\leq)} \coloneqq \{(a,b) \in P^2 : a \leq b\}$$

is a closed set of P^2, i.e. the product of the space $(P, \mathscr{G})$ with itself.

Proposition 61. *Let $(P, \leq)$ be ordered. Then the order on $U(P)$ is continuous (w.r.t. the μ_P-topology).*

Proof. Let $(\mathscr{U}, \mathscr{V}) \notin R_{(U(P),\leq)}$, that is $\mathscr{U} \not\leq \mathscr{V}$. We want an open neighbourhood which does not intersect $R_{(U(P),\leq)}$. The lack of relation $\mathscr{U} \leq \mathscr{V}$ by Lemma 30 means there exist $A \in \mathscr{U}$, $B \in \mathscr{V}$ such that for all $a \in A, b \in B$ we have $a \not\leq b$. From this any $\mathscr{T}, \mathscr{S} \in U(P)$ such that $A \in \mathscr{T}, B \in \mathscr{S}$ satisfy $\mathscr{T} \not\leq \mathscr{S}$. Meaning $\hat{A} \times \hat{B} \subseteq U(P) \smallsetminus R_{(U(P),\leq)}$. But $\hat{A}, \hat{B}$ are open in P by Lemma 59, hence $\hat{A} \times \hat{B}$ is open in the product. $\square$

Lemma 62. *μ_P-topology on $U(P)$ is compact Hausdorff.*

Proof. We will show that every set-ultrafilter $\tilde{\mathfrak{U}} \in \beta U(P)$ has precisely one limit, namely $\mu_P(\mathfrak{U})$ for $\mathfrak{U} = \tilde{\mathfrak{U}} \cap \{A \sqsubseteq U(P)\}$. We know that the intervals $\hat{A}$, $A \sqsubseteq P$ are a basis for this topology. We can test limits of ultrafilters on basic sets. Assume $\hat{A} \sqsubseteq U(P)$ is basic open and $\mu_P \mathfrak{U} \in \hat{A}$. By definition of μ we have $\hat{A} \in \mathfrak{U}$, therefore $\hat{A} \in \tilde{\mathfrak{U}}$. Thus $\mu_P \mathfrak{U}$ is a limit of $\tilde{\mathfrak{U}}$.

Assume $\mathscr{U} \in U(P)$ is a limit of $\tilde{\mathfrak{U}}$. For $A \in \mathscr{U}$ we have $\mathscr{U} \in \hat{A}$. Since $\hat{A}$ is open and $\mathscr{U} \in \hat{A}$ a topological limit, we get $\hat{A} \in \tilde{\mathfrak{U}}$. But $\hat{A}$ is interval, hence $\hat{A} \in \mathfrak{U}$. But this gives $A \in \mu_P \mathfrak{U}$, concluding we have $\mathscr{U} = \mu_P \mathfrak{U}$. Hence the limit is unique. $\square$

Lemma 63. *Structure map s of a U-algebra (P, s) is continuous.*

Proof. We will show that s preserves limits of ultrafilters, this implies continuity. Let $\tilde{\mathfrak{U}} \in \beta U(P)$ a set-ultrafilter on $U(P)$ and $\mathfrak{U} = \tilde{\mathfrak{U}} \cap \{A \sqsubseteq U(P)\} \in UU(P)$. Suppose $\mathscr{U} \in U(P)$ is a topological limit of $\tilde{\mathfrak{U}}$, we want to show that $s(\mathscr{U})$ is a limit of $s_\# \tilde{\mathfrak{U}} \in \beta P$ (the definition of $s_\#$ seamlessly extends to set-ultrafilters). Because $U(P)$ is compact Hausdorff, the unique limit of $\tilde{\mathfrak{U}}$ is $\mu_P \mathfrak{U}$, therefore $\mathscr{U} = \mu_P \mathfrak{U}$. From the algebra axiom we get $s(\mathscr{U}) = s(\mu_P \mathfrak{U}) = s(s_\# \mathfrak{U})$. The point $s(s_\# \mathfrak{U})$ is a topological limit of $s_\# \tilde{\mathfrak{U}}$ from the definition of the topology. Hence $s_\# \tilde{\mathfrak{U}}$ converges to $s(\mathscr{U})$ in P. $\square$

Theorem 64. *Let (P, s) be an algebra over U. Then the s-topology on P is compact Hausdorff and order on P is continuous.*

Proof. We showed that s is continuous and closed. Its action on trivial ultrafilters implies surjectivity. Image of a compact Hausdorff space under a closed continuous map is itself compact Hausdorff, hence P is.

Because P is compact Hausdorff, $P \times P$ with the product topology is also compact Hausdorff. So is $U(P) \times U(P)$. The map $(s, s) : U(P) \times U(P) \to P \times P$ is continuous, because it has continuous components. Its domain is compact and codomain Hausdorff, implying (s, s) is a closed map. Since s is monotone, it maps the relation set $R_{(U(P),\leq)}$ into $R_{(P,\leq)}$. The existence of trivial ultrafilters implies this is onto $R_{(P,\leq)}$. By Proposition 61 $R_{(U(P),\leq)}$ is closed, therefore $R_{(P,\leq)}$ is also closed. Hence by definition the order on P is continuous. $\square$

Proposition 65. *Interval topology on an algebra (P, s) is coarser than s-topology.*

Proof. We will show that subbasic closed sets of the interval topology are closed in the s-topology. Let $x \in P$ and without loss of generality $F = [x, \rightarrow)$ be subbasic closed. Suppose $\mathscr{U} \in U(P)$ with $s(\mathscr{U}) \notin F$ and $F \in \mathscr{U}$. $F \in \mathscr{U}$ gives for any $M \in \mathscr{U}$ that $M \cap F \neq \emptyset$, i.e. there exists $m \in M$ such that $m \geq x$. By Proposition 30 we now have $\mathscr{E}(x) \leq \mathscr{U}$. But s is monotone, giving $x = s(\mathscr{E}(x)) \leq s(\mathscr{U})$, which contradicts $s(\mathscr{U}) \notin F = [x, \rightarrow)$. □

Theorem 66. *Let $(P, \leq)$ be ordered and have a compact Hausdorff topology generated by a basis of open intervals, with respect to which the order is continuous. Then P forms an algebra over U.*

Proof. If $\mathscr{U} \in U(P)$, let $s(\mathscr{U})$ denote the unique limit of any set-ultrafilter $\tilde{\mathscr{U}} \in \beta P$ extending $\mathscr{U}$. We will show that s is a well defined function $U(P) \to P$ and is the structure map of an algebra.

$s : U(P) \to P$ is a well defined function: Let $\tilde{\mathscr{U}}, \mathscr{U}'$ be two extensions of $\mathscr{U} \in U(P)$. Suppose $\lim \tilde{\mathscr{U}} \neq \lim \mathscr{U}'$; then because of Hausdorffness they can be separated by disjoint basic open sets A, B, we must have $A \in \tilde{\mathscr{U}}$ and $B \in \mathscr{U}'$. We have chosen A, B basic, i.e. open intervals, hence $A \in \mathscr{U}$ and $B \in \mathscr{U}$, then $\emptyset = A \cap B \in \mathscr{U}$, a contradiction.

s is monotone: let $\mathscr{U}, \mathscr{V} \in U(P)$ such that $\mathscr{U} \leq \mathscr{V}$. For a contradiction assume $s(\mathscr{U}) \nleq s(\mathscr{V})$. Then $(s(\mathscr{U}), s(\mathscr{V})) \notin R_{(P,\leq)}$. The set $P \times P \smallsetminus R_{(P,\leq)}$ is open by continuity of the order. Therefore the point $(s(\mathscr{U}), s(\mathscr{V})) \notin R_{(P,\leq)}$ has a basic open neighbourhood $A \times B \subseteq P \times P \smallsetminus R_{(P,\leq)}$, where A, B are open intervals, $s(\mathscr{U}) \in A$, $s(\mathscr{V}) \in B$. Because they are open and contain the limit of any extension of $\mathscr{U}$, $\mathscr{V}$, respectively, into a set-ultrafilter, we have $A \in \mathscr{U}$, $B \in \mathscr{V}$. By Proposition 30 and the assumption $\mathscr{U} \leq \mathscr{V}$, we get $a \in A$, $b \in B$ such that $a \leq b$. But then $(a, b) \in A \times B \subseteq P \times P \smallsetminus R_{(P,\leq)}$, meaning $a \nleq b$, a contradiction.

s satisfies the algebra axioms

$$
\begin{tikzcd}
U^2(P) \arrow[r, "Us"] \arrow[d, "\mu_P"'] & U(P) \arrow[d, "s"] \\
U(P) \arrow[r, "s"] & P
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[dr, equal] & U(P) \arrow[d, "s"] \\
& P
\end{tikzcd}
$$

The second diagram is clear; for any $a \in P$ let A be an open basic set (interval), which is a neighbourhood of a. By definition of $\eta_P(a) = \mathscr{E}(a)$ we have $A \in \mathscr{E}(a)$ from $a \in A$. Therefore a is a limit of any ultrafilter extending $\mathscr{E}(a)$, giving $s(\mathscr{E}(a)) = a$.

For the first diagram let $\mathfrak{U} \in UU(P)$. We can test limits of set-ultrafilters, by extension the values of s, via the basic open sets (intervals). By definitions

$$Us(\mathfrak{U}) = s_{\#}\mathfrak{U} = \{A \sqsubseteq P : s^{-1}[A] \in \mathfrak{U}\}, \qquad \mu_P(\mathfrak{U}) = \{A \sqsubseteq P : \hat{A} \in \mathfrak{U}\}.$$

Let $B \subseteq P$ be an open interval containing $s(s_{\#}\mathfrak{U})$. Since B is an open neighbourhood, we get $B \in s_{\#}\mathfrak{U}$. Then by the definition $s^{-1}[B] \in \mathfrak{U}$. Condition that B is open can be rewritten as $s^{-1}[B] \subseteq \hat{B}$ (all ultrafilters converging into B must contain B, s is the limit by definition). Therefore $\hat{B} \in \mathfrak{U}$ by upwards closure, hence $B \in \mu_P(\mathfrak{U})$. We proved that $\mu_P(\mathfrak{U})$ contains all basic neighbourhoods of $s(s_\#\mathfrak{U})$, so does any extension. Hence $s(s_\#\mathfrak{U})$ must be the limit, giving $s(\mu_P(\mathfrak{U})) = s(s_\#\mathfrak{U})$. □

In the following special case we can apply the preceding theorems to arrive at a characterisation of algebras. Note that we are using Proposition 57. In the general case we are left with the assumption of basis from open intervals, which we need in order to construct an algebra but have not proven for a given general algebra.

Corollary 67. *Suppose $(P, \leq)$ has only finite antichains, then P forms an algebra over U if and only if the interval topology on P is compact and the order continuous.*

Proof. Follows from the above classification if we use interval topology, see Proposition 65 and recall that bijective continuous map of compact Hausdorff spaces is a homeomorphism. □

4.2 Graphs

In this section we briefly discuss the ultrafilter monad in categories Graph and FCGraph. For general graphs we have to modify the definition compared to the one from chapter 2, because morphisms into finite graphs might not exist. We say $\mathscr{U}$ is an ultrafilter on $\mathbf{G}$ if it is an ultrafilter in the poset $(\{A \sqsubseteq \mathbf{G}\}, \subseteq)$. Finitely colourable graphs are very similar to posets, but general graphs have one major difference. Finitely colourable graphs have finite partitions on the vertex set, implying that the condition $\mathscr{F} \cap \{A \sqsubseteq \mathbf{G}\} \neq \emptyset$ in formulation of Lemma 27 is trivial. However, a general graph $\mathbf{G}$ with $\varkappa > \omega$ vertices might have all independent sets of cardinality $< \varkappa$. If $\mathscr{F}$ is a uniform ultrafilter on $V(\mathbf{G})$ (meaning every $A \in \mathscr{F}$ has cardinality $\varkappa$), then $\mathscr{F} \cap \{A \sqsubseteq \mathbf{G}\}$ is empty. Nevertheless, nontrivial ultrafilters still exist if $\mathbf{G}$ has some infinite independent sets. Many of the desired categorical properties will not work for general graphs ($U(\mathbf{G})$ is not a limit of finite graphs, concept of codensity, etc.).

In this section we take the formula

$$\{\mathscr{U}, \mathscr{V}\} \in E(U(\mathbf{G})) \iff (\forall U \in \mathscr{U}\, \forall V \in \mathscr{V}\, \exists u \in U\, \exists v \in V)\ \{u, v\} \in E(\mathbf{G})$$

from Proposition 46 as the definition of edges on $U(\mathbf{G})$, as it is applicable without finite colourability.

Analogue of Lemma 53:

Lemma 68. *Let $\mathbf{G}$ be a graph. For independent $A \sqsubseteq \mathbf{G}$ denote*

$$\hat{A} := \{\mathscr{U} \in U(\mathbf{G}) : A \in \mathscr{U}\}.$$

Then $\hat{A}$ is independent in $U(\mathbf{G})$.

Proof. Suppose that ultrafilters $\mathscr{U}, \mathscr{V}$ both contain A. There is no edge from A to A, because A is independent. By definition $\mathscr{U}, \mathscr{V}$ do not form an edge. □

Below we use the fact that $U(\mathbf{G})$ is finitely colourable if $\mathbf{G}$ was.

Proposition 69. *Let $\mathscr{G} = \mathsf{Graph}, \mathsf{FCGraph}$. (U, μ, η) is a monad, where U is the ultrafilter functor on $\mathscr{G}$, multiplication μ is defined as*

$$\mu_{\mathbf{G}} : UU(\mathbf{G}) \to U(\mathbf{G}), \quad \mathfrak{U} \mapsto \{A \sqsubseteq \mathbf{G} : \hat{A} \in \mathfrak{U}\}.$$

Unit $\eta = (\eta_{\mathbf{G}} : \mathbf{G} \in \mathscr{G})$ is defined as

$$\eta_{\mathbf{G}} : \mathbf{G} \to U(\mathbf{G}), \quad x \mapsto \mathscr{E}(x) = \{A \sqsubseteq \mathbf{G} : x \in A\}.$$

Proof is mostly identical to the case for Poset. We only have to check a different partition condition when showing that $\mu_{\mathbf{G}}\mathfrak{U} \in U(\mathbf{G})$ for $\mathfrak{U} \in UU(\mathbf{G})$, but the argument is the same.

Possible generalisation

The following are my (informal) thoughts on a possible generalisation of the entire construction. Recall that the Čech–Stone compactification βT of a Tichonov topological space T can be viewed as a quotient of the set of all set-ultrafilters on T, with respect to the equivalence:

$$\mathscr{U} \sim \mathscr{V} \iff (\forall f \in M)\ \lim f_\# \mathscr{U} = \lim f_\# \mathscr{V},$$

where M is the class of all continuous functions from T into any compact Hausdorff space (then the limits on the right exist and are unique). An extensive reference on the Čech–Stone compactification is Walker [8]. Since we are concerned about convergence of the pushforwards $f_\# \mathscr{U}$, the important information is which open (or closed) sets does $f_\# \mathscr{U}$ contain. The definition is

$$f_\# \mathscr{U} = \{A \subseteq \operatorname{cod} f : f^{-1}[A] \in \mathscr{U}\}.$$

Hence if A is an open neighbourhood of $\lim f_\# \mathscr{U}$, $\mathscr{U}$ must contain $f^{-1}[A]$, and so does any other equivalent ultrafilter. From this we see that the only subsets of T important for the definition of $\sim$ are preimages of open sets in compact Hausdorff spaces under continuous maps (for now call such sets *compact-open*, they are complements of the more commonly used zero-sets). Then

$$\bigcap_{\mathscr{V} \sim \mathscr{U}} \mathscr{V} \tag{*}$$

is a collection of subsets of T closed upwards and under finite intersections, which contains all of those compact-open sets, which are preimages of neighbourhoods of the limit. Important question is, when is the filter (*) generated by these compact-open sets – i.e. they form a filter basis. Then we may restrict our attention only to

$$C(\mathscr{U}) = \{A \in \mathscr{U} : A \text{ is compact-open}\}.$$

For our investigation of ultrafilters the codomains were finite sets. Finite set (with discrete topology) is a special case of compact Hausdorff space. We were concerned with those subsets, which were preimages of single points under appropriate morphisms. In discrete topology, singletons are open.

In light of these facts, the following setting is worth investigating: let $\mathcal{C}$ be a complete concrete category, $\mathcal{K}$ its subcategory, which is also a subcategory of CHaus, the category of compact Hausdorff spaces with continuous maps (plus some assumptions on limits in $\mathcal{K}$). To extend the construction from this thesis, if $\mathscr{U}$ is an "ultrafilter" (to be defined precisely) on an object X of $\mathcal{C}$, and f a morphism from X to Y an object of $\mathcal{K}$, let

$$\int_X f \,\mathrm{d}\mathscr{U} = \lim f_\# \mathscr{U}.$$

In the other direction, for such an integral/limit let

$$\mathscr{U} = \Big\{ f^{-1}[A] : f \in \mathscr{C}(X, Y), Y \in \operatorname{obj} \mathcal{K}, A \text{ open neighbourhood of } \int_X f \,\mathrm{d}x \Big\}.$$

For the analogy to hold, there would need to be a bijective correspondence between these. Whether or not this could be sufficient for calculating the algebras in general is unclear to me.

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