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Superradiance on accelerated systems

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Title: Superradiance on accelerated systems

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Abstract:

In this work we will study the electromagnetic superradiance phenomenon on accelerated systems. We will briefly cover superradiance on a cylinder, which was thermodynamically proven by Zel'dovich. Then we will attempt to formulate the problem in accelerated coordinates, namely the flat-spacetime limit of the C-metric. Briefly introducing the C-metric, the Newman-Penrose formalism and the Geroch–Held–Penrose formalism along the way. Using the formalism of vector spherical harmonics, we point out the complications which arrise from not respecting the spherical symmetry of the spherical coordinate system.

Keywords: electromagnetism, relativity, superradiance, tetrad formalism

Contents

Introduction

Superradiance is generally understood as radiation amplification due to interaction with surrounding environment. Perhaps the most known form of supperadiance is the Cherenkov radiation, which is the radiation produced by a charges particle moving faster than the speed of light in medium. In the year 1958 Cherenkov received a nobel prize for describing this phenomenon. According to [Brito et al., 2020] "superradiance belongs to a wider class of classical problems displaying stimulated or spontaneous energy emission, such as Vavilov-Cherenkov effect, the anomalous Doppler effect, and other examples of "superluminal motion"".

One of the first macroscopic instances of superradiance were shown in Zel'dovich's work [Zel'dovich, 1972], where he showed that scattering of radiation on rotating conducting surfaces can result, under the right conditions, in reflected waves with larger power than the power of incoming waves. By Zel'dovich thermodynamic argument, this happens if

$$\omega < m\Omega \tag{1}$$

where ω is monochromatic frequency of incident radiation, m is the azimuthal number with respect to rotational axis and Ω the angular velocity of the body. Later the theory of superradiance was extended to black holes by Bekenstein [Bekenstein, 1973], where he again derived (1) through thermodynamic laws that hold for black holes.

In this work we will focus on electromagnetic superradiance in accelerated systems, as far as we know, such problem was never formulated or any attempt to discuss the potential difficulties was made. In Chapter 7 we briefly consider consider formulating the problems in spherical coordinates, in which [Bára, 2017] demonstrated the usefulness of employing the vector spherical harmonics while describing superradiance on a rotating spherical shell. We provide arguments against such approach for accelerated objects.

We then try to formulate and solve the problem in accelerated coordinates, this naturally leads to the C-metric, a solution of the Einstein fields equations representing two uniformly accelerated Kerr black holes, and its flat-spacetime limit. We provide the boundary problem formulation in Chapter 6. Since the C-metric is of special algebraic type (Petrov type D), the test electromagnetic fields can assume a separable ansatz which leads to a Sturm-Liouville problem, we then attempt solve in Chapter 5.

To understand the derivation of this field we shortly summarize the Newman-Penrose formalism in Chapter 3, which is a special form of tetrad formalism and the Geroch-Help-Penrose formalism in Chapter 4, which is a special case of Newman-penrose formalism. Along with providing a basic introduction to the C-metric in Chapter 2 and summarizing the simplest form of electromagnetic of electromagnetic superradiance, the case of cylindrical shell in Chapter 1.

1. Superradiance on a rotating cylinder

In this chapter we will discuss the [Zel'dovich, 1972] model of superradiance on a rotating cylindrical shell. This is one of the simplest form of superradiance. We will formulate the problem in covariant form and solve Maxwell equations. For further details we recommend [Bára, 2014].

1.1 Maxwell's equations in cylindrical coordinates system

In the following chapters we will make use of a few different notations, namely the geometrical and tensorial. To avoid confusion differential forms will be written bold $\mathbf{F}$, tensor in coordinate basis will always be written with indices $F_\mu{}^\nu$.

Considering the sign convention $(-+++)$, utilizing the geometrized units $(c = 1, G = 1)$ we can write the Minkowski metric in cylindrical coordinates as

$$\mathbf{ds}^2 = -\mathbf{d}t^2 + \mathbf{d}r^2 + r^2\mathbf{d}\phi^2 + \mathbf{d}z^2. \tag{1.1}$$

Identifying the components of *EM* tensor [Kvasnica, 1985]

$$F^{ij} = \begin{pmatrix} 0 & E_r & \frac{1}{r}E_\phi & E_z \\ -F^{01} & 0 & \frac{1}{r}B_z & -B_\phi \\ -F^{02} & -F^{12} & 0 & \frac{1}{r}B_r \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix}, \tag{1.2}$$

with the four-current

$$j^a = (\rho, j_r, \frac{j_\phi}{r}, j_z). \tag{1.3}$$

These have been found by comparing the Maxwell equations for the Faraday tensor

$$\mathbf{dF} = 0, \quad \mathbf{d} \star \mathbf{F} = \mu_0 \mathbf{j}, \tag{1.4}$$

to the "standard" EM equations with the observer 4-velocity $u^a = (1, 0, 0, 0)$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mu\mathbf{j}, \tag{1.5}$$

where we denoted $\star\mathbf{F}$ which is the Hodge dual to the tensor F, in coordinate basis we can find the dual of the Faraday tensor as

$$\begin{aligned} \star F^{ab} &= \frac{1}{2}\varepsilon^{abcd}F_{cd}, \\ \varepsilon_{abcd} &= \sqrt{-g}\epsilon_{abcd}, \end{aligned} \tag{1.6}$$

where ϵ is the standard Levi-Civita symbol.

Relations between the electromagnetic tensor and electric and magnetic field for the stationary observer with the four-velocity u^a are

$$\begin{aligned} E^a &= F^{aj}u_j, \\ B^a &= \frac{1}{2}\star F^{aj}u_j, \quad \text{where} \quad a = 1, 2, 3. \end{aligned} \tag{1.7}$$

Consider a cylinder with surface conductivity σ, whose axis is identified with the z coordinate of (1.1) rotating with angular velocity Ω. A point on the surface of the cylinder associated with the four-velocity

$$u^a = \gamma(1, 0, \Omega, 0), \quad \gamma^2 = \frac{1}{1 - \Omega^2 r^2}. \tag{1.8}$$

The normal of the surface can clearly be written as

$$n^a = (0, 1, 0, 0), \tag{1.9}$$

All the possible currents will be tangential to the surface of the cylinder, therefore we have to project out the normal contribution

$$j^a = \sigma E^a + \rho u^a - n^a n_b E^b, \tag{1.10}$$

with ρ being the charge density of the conductor. According to [Itin, 2012] the Maxwell equations at the boundary are

$$\epsilon^{ijkl}\left[F_{jk}\right] n_i = 0, \quad \left[\star F^{ij}\right] n_j = \mu_0 j^a, \tag{1.11}$$

where we describe the discontinuity of a physical variable $G = G(x^a)$ at boundary $\varphi = \varphi(x^a)$

$$[G] \equiv \lim_{\varphi \to 0+} G(x^a) - \lim_{\varphi \to 0-} G(x^a). \tag{1.12}$$

Writing out explicitly (1.11)

$$\begin{aligned} [E_r] &= \gamma\left(\frac{\rho}{\epsilon_0} + \mu_0 \sigma \Omega r E_\phi\right), \\ [B_\phi] &= \gamma\sigma\mu_0(E_z - \Omega r B_r), \\ [B_z] &= -\gamma(\Omega r \mu \rho + \sigma E_\phi), \\ [E_z] &= [B_r] = [E_\phi] = 0. \end{aligned} \tag{1.13}$$

1.2 Solving the wave equation

Were interested in a special solution to Maxwell equation. The electric field E and magnetic field B can be associated with a *electromagnetic four-potential*

$$A^\mu = (\phi, A^r, A^\phi, A^z). \tag{1.14}$$

The electromagnetic tensor can then be described in flat-spacetime as

$$F^{ij} = \partial^j A^i - \partial^i A^j, \tag{1.15}$$

which is equivalent to writing

$$\begin{aligned} E_a &= -\nabla_a \varphi - \partial_t A_a, \quad a = 1,2,3, \\ B_a &= \epsilon^{abc} \nabla_b A_c, \quad (a,b,c) = 1,2,3 \end{aligned} \tag{1.16}$$

Considering a vacuum space with no sources (i.e $j^a = 0$) along wth the Lorenz gauge, EM equation are solved by potential satisfying the equation [Kvasnica, 1985]

$$\Box A^\mu = g^{\nu\gamma} \partial_\nu \partial_\gamma A^\mu = 0. \tag{1.17}$$

where $\Box$ is called the d'Alambert operator.

As shown in [Bára, 2017] the solution of (1.17) is

$$\phi = 0, \quad \mathbf{A} = \{B J_m(\omega r) + C Y_m(\omega r)\} \mathbf{e}_z, \tag{1.18}$$

where (B, C) are complex constants, J_m is the Bessel function of the first kind and Y_m is the Bessel function of the second kind [Kvasnica, 1985], both Bessel functions solving the equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr}\left(1 - \frac{\omega^2}{r^2}\right) u = 0. \tag{1.19}$$

Finally, by using the relations (1.16) we get

$$\begin{aligned} \mathbf{E} &= i\omega \left[A J_m(\omega r) + B Y_m(\omega r)\right] e^{i(m\phi - \omega t)} \mathbf{e}_z, \\ \mathbf{B} &= \frac{im}{r} \left(\mathbf{A} \cdot \mathbf{e}_z\right) \mathbf{e}_r - \left(A \frac{dJ_m(kr)}{dr} + B \frac{dY_m(kr)}{dr}\right) e^{i(m\phi - \omega t)} \mathbf{e}_\phi. \end{aligned} \tag{1.20}$$

1.3 Fields inside and outside the cylinder

By comparing asymptotic behavior of the Bessel functions we can identify incoming a outgoing wave.

Since outgoing wave behaves at infinity as $e^{i(kr - \omega t)}$ and incoming wave behaves as $e^{-i(kr - \omega t)}$, since the asymptotic behavior of Bessel functions is

$$\begin{aligned} J_m(z) &\overset{z \sim \infty}{=} \sqrt{\frac{2}{\pi z}} \left[\cos(z - \frac{m\pi}{2} - \frac{\pi}{4})\right], \\ Y_m(z) &\overset{z \sim \infty}{=} \sqrt{\frac{2}{\pi z}} \left[\sin(z - \frac{m\pi}{2} - \frac{\pi}{4})\right], \end{aligned} \tag{1.21}$$

Which can be found for example by using the *Asymptotic* function in *Wolfram Mathematica* or in [Olver et al., 2010].

By identifying the incoming field $\mathbf{E}_{inc}, \mathbf{B}_{inc}$ as incoming wave, scattered field $\mathbf{E}_{sc}, \mathbf{B}_{sc}$ as outgoing waves and fields inside the cavity $\mathbf{E}_{cav}, \mathbf{B}_{cav}$ as standing waves we get

$$
\begin{aligned}
\mathbf{E}_{inc} &= i\omega A_1 H_m^{(2)}(kr) e^{i(m\phi-\omega t)}\mathbf{e}_z, \\
\mathbf{B}_{inc} &= A_1 \left(\frac{im}{r} H_m^{(2)}(kr)\mathbf{e}_r - \frac{dH_m^{(2)}(kr)}{dr}\mathbf{e}_\phi \right) e^{i(m\phi-\omega t)}, \\
\mathbf{E}_{sc} &= i\xi\omega A_1 H_m^{(1)}(kr) e^{i(m\phi-\omega t)}\mathbf{e}_z, \\
\mathbf{B}_{sc} &= \xi A_1 \left(\frac{im}{r} H_m^{(1)}(kr)\mathbf{e}_r - \frac{dH_m^{(1)}(kr)}{dr}\mathbf{e}_\phi \right) e^{i(m\phi-\omega t)}, \\
\mathbf{E}_{cav} &= i\tau\omega A_1 J_m(kr) e^{i(m\phi-\omega t)}\mathbf{e}_z, \\
\mathbf{B}_{cav} &= \tau A_1 \left(\frac{im}{r} J_m(kr)\mathbf{e}_r - \frac{dJ_m(kr)}{dr}\mathbf{e}_\phi \right) e^{i(m\phi-\omega t)},
\end{aligned}
\tag{1.22}
$$

where we denoted ξ and τ as the change of coefficient A_1 and we used the definition of the Hankel functions of the first and second kind

$$
\begin{aligned}
H_m^{(1)} &= J_m + iY_m \\
H_m^{(2)} &= J_m - iY_m
\end{aligned}
\tag{1.23}
$$

1.4 Results

Considering the fields (1.22) with the boundary conditions (1.11), we get the equations

$$
\begin{aligned}
i\tau J_m(kr)\left(\gamma\mu m r^3\sigma\Omega + im - \gamma\mu r\sigma\omega\right) - k\xi r H_{m-1}^{(1)}(kr) + \\
m\xi H_m^{(1)}(kr) + kr\tau J_{m-1}(kr) - krH_{m-1}^{(2)}(kr) + mH_m^{(2)}(kr) = 0 \\
\xi H_m^{(1)}(kr) - \tau J_m(kr) + H_m^{(2)}(kr) = 0
\end{aligned}
\tag{1.24}
$$

Solving the equations for ξ,τ

$$
\begin{aligned}
\xi &= \frac{-\gamma k J_m(kr) H_{m-1}^{(2)}(kr) + H_m^{(2)}(kr)\left(\gamma k J_{m-1}(kr) - i\mu\sigma J_m(kr)\left(\omega - mr^2\Omega\right)\right)}{\gamma k J_m(kr) H_{m-1}^{(1)}(kr) - H_m^{(1)}(kr)\left(\gamma k J_{m-1}(kr) - i\mu\sigma J_m(kr)\left(\omega - mr^2\Omega\right)\right)}, \\
\tau &= -\frac{4\gamma}{-2\gamma + \pi\mu e^{-i\pi m} r\sigma J_m(kr) H_{-m}^{(1)}(kr)\left(mr^2\Omega - \omega\right)}.
\end{aligned}
\tag{1.25}
$$

As was shown by [Bára, 2014] the coefficient ξ and τ directly correspond to the power output of the waves

$$
\langle S \rangle = \frac{1}{\mu}\langle E \times B \rangle, \tag{1.26}
$$

where $\langle\rangle$ denotes the mean time value. Iff $|\xi|^2 > 1$ then the refracted wave has larger power output than the incident waves. This is clearly shown in Fig. 1.1.

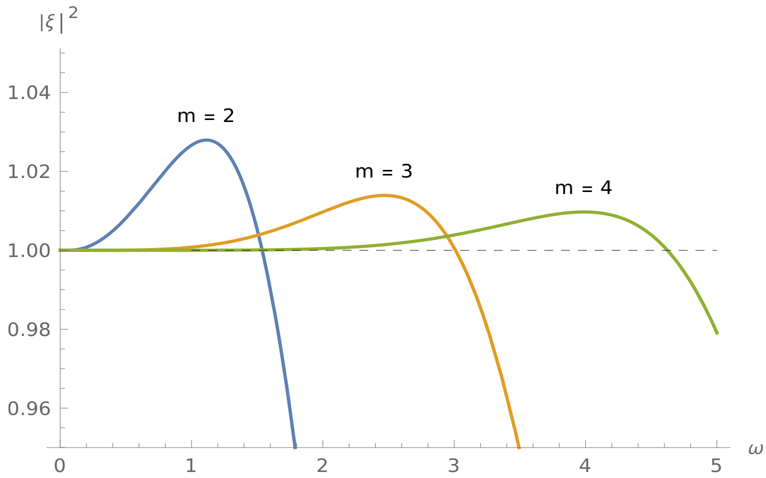


Figure 1.1: Superradiance of coefficient ξ for the numbers $m = 2, 3, 4$

1.5 Thermodynamical proof of rotational superradiance

For the sake of completeness let is include the thermodynamical proof of superradiance, which has been first provided in [Zel'dovich, 1972], further refined in [Bekenstein and Schiffer, 1998], a text which we will follow closely in this section.

Consider an axially symmetric body rotating around it's axial axis and radiation incident on this body in vacuum. We will consider only small frequency range $[\omega, \omega+\delta\omega]$, with the azimuthal number m along with denoting the intensity of the radiation as $I_m(\omega)$. The power of this radiation is then

$$P = I_m(\omega)\delta\omega, \tag{1.27}$$

note that according to [Griffiths et al., 2006] the mean power of radiation over surface Σ with the normal $\mathbf{n}$ can expressed as

$$\langle P \rangle = \int_\Sigma \langle \mathbf{S} \rangle \cdot d\mathbf{\Sigma} = \int_\Sigma \langle \mathbf{S} \rangle \cdot \mathbf{n}\, d\Sigma, \tag{1.28}$$

where $\mathbf{S}$ is the Poynting vector (1.26).

All incoming power P will be split into absorbed power P_{abs} by the object associated with the change of energy E_{abs} and reflected power $P_{ref} = P - P_{abs}$ associated with the change in energy E_{ref}, the absorption coefficient will be denoted $a_m(\omega)$. In [Bekenstein and Schiffer, 1998] another term W was added, which represent spontaneous emission

$$\begin{aligned} P_{abs} &= a_m(\omega)I_m(\omega)\delta\omega - W = \frac{dE_{abs}}{dt}, \\ P_{ref} &= (1 - a_m(\omega))\, I_m(\omega)\delta\omega - W = \frac{dE_{ref}}{dt}. \end{aligned} \tag{1.29}$$

Furtheremore the change of the the angular momentum along the axial axis is [Bára, 2017]

$$\frac{dJ_m}{dt} = \frac{m}{\omega} a_m(\omega)I_m(\omega)\delta\omega - J_e, \tag{1.30}$$

where J_e represent the change of angular momentum caused by spontaneous emission. From the definition of the absorbed heat of the system δQ as the difference between adiabatic and non adiabatic work [Luscombe, 2018] we have

$$\delta Q = dE_{abs} - \Omega \, dJ, \tag{1.31}$$

where Ω is the angular velocity of the rotating body. From the definition of entropy $dS = dQ/T$ divided by dt

$$\frac{dS}{dt} = \frac{1}{T}\frac{\delta Q}{dt} = \frac{1}{T}\left(1 - m\frac{\Omega}{\omega}\right) a_m(\omega) I_m(\omega) - \frac{W - \Omega J_e}{T}. \tag{1.32}$$

Since the second thermodynamic law takes the form $dS/dT > 0$, disregarding the subservient term $W - \Omega J_e$ we get the condition

$$\left(1 - m\frac{\Omega}{\omega}\right) a_m(\omega) > 0. \tag{1.33}$$

If we consider the Zel'dovič condition

$$\omega < m\Omega. \tag{1.34}$$

The absorption coefficient $a_m(\omega)$ must be less than zero, thus the object looses energy and since energy is conserved the wave must gain energy.

2. C-metric

The *C-metric* classified in [Witten, 1962] "describes a pair of casually separated black holes which accelerate in opposite directions under the action of forces represented by conical singularities". Our interested lies in the metric flat-spacetime limit, which has been extensively studied in [Bičák and Kofroň, 2009].

2.1 Introduction to C-metric

According to [Bičák and Kofroň, 2009] the *C-metric* reads

$$\begin{aligned}\mathbf{d}s^2 = &\frac{1}{A^2(x-y^2)}\{H(y,x)\left[(1+a^2A^2x^2)K\mathbf{d}t + aA(1-x^2)K\mathbf{d}\phi)\right]^2 \\ &+ H(x,y)\left[(1+a^2A^2y^2)K\mathbf{d}\varphi + aA(y^2-1)K\mathbf{d}t\right]^2 \\ &- \frac{1}{H(y,x)}\mathbf{d}y^2 + \frac{1}{H(x,y)}\mathbf{d}x^2\},\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}H(p,q) &= \frac{1}{1+(aApq)^2}(1-p^2)(1+r_+Ap)(1+r_-Ap), \\ r_\pm &= m \pm \sqrt{m^2-a^2-q^2},\end{aligned} \tag{2.2}$$

in SI units this would read

$$r_\pm = Gm \pm \sqrt{G^2m^2-a^2-Gq^2}, \tag{2.3}$$

with m being the mass parameter, q being the charge parameter, A being the acceleration parameter and a being the rotational parameter, K is a free constant. Numerator of (2.2) has four real roots

$$p_1 = -\frac{1}{r_-A}, \quad p_2 = -\frac{1}{r_+A}, \quad p_3 = -1, \quad p_4 = 1. \tag{2.4}$$

These root satisfy

$$p_1 < p_2 < p_3 < p_4. \tag{2.5}$$

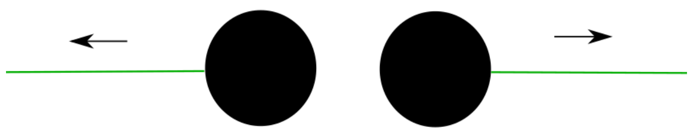


Figure 2.1: The interpretation of the C-metric as two black holes of equal mass and opposite charge, which are being pulled by a "cosmic string" in such a way, that they experience constant acceleration.

The variable is assumed to be periodical

$$\phi \sim \phi + 2\pi, \tag{2.6}$$

Furthermore the variable $x \in [-1, 1]$ and $y \in [p_2, p_3]$ so that the signature is conserved. At $x + y = 0$ we observe conformal infinity.

The black hole horizon is at $y = p_2$ and the acceleration horizon is at $y = p_1$. The metric along with the *electromagnetic 4 potential*

$$\mathbf{A} = \frac{qy}{1 + (aAxy)} \left[\mathbf{d}t + aA(1 - x^2)K\mathbf{d}\phi \right], \tag{2.7}$$

solve the Einstein-Maxwell equations.

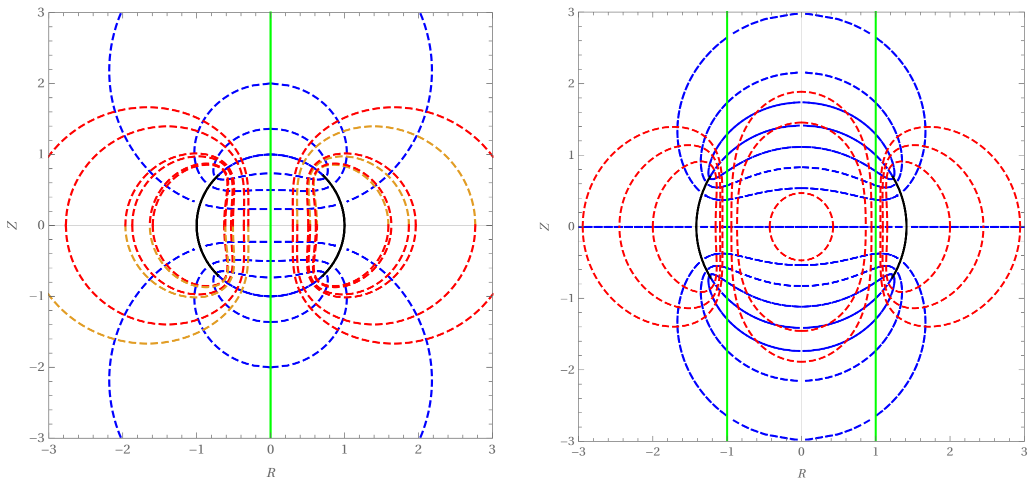
2.2 Flat-spacetime limit

As we've already said we are most interested in the flat-spacetime limit $G \to 0$, finding this limit in (2.1)

$$\mathbf{d}s^2 = \frac{1}{A^2(x-y)^2} \Bigg[-\frac{(y^2-1)(1+a^2A^2x^2)}{1+a^2A^2} dt^2 + \frac{1+(aAxy)^2}{(1-x^2)(1+a^2A^2x^2)} \mathbf{d}x^2 \tag{2.8}$$

$$\frac{1+(a^2A^2x^2)}{(1+a^2A^2)(y^2-1)} \mathbf{d}y^2 + \frac{(1-x^2)(1+a^2A^2y^2)}{1+a^2A^2} \mathbf{d}\phi^2 \Bigg],$$

where we also took $K = (1 + a^2A^2)^{-1}$ so that the axis is regular. This metric corresponds to uniformly accelerated frame. Lets take a closer look at the metric (2.8).



(a) $T = 0$ (b) $T = 1$

Figure 2.2: Coordinate curves of the *c-metric*, lines $y = const.$, $x = const.$ are red and blue respectively, the black lines are $y = \pm\infty$ and the green lines are the *acceleration horizons*

The metric along with the potential (2.7) describes an set of uniformly accelerated, rotating disk charged with a charge q these disk are bent "against" their direction of acceleration Fig. 2.2.

According to [Bičák and Kofroň, 2009] these transformation to Rindler coordinates reads

$$ds^2 = -\zeta^2 dt^2 + d\zeta^2 + d\rho^2 + \rho^2 d\phi^2, \tag{2.9}$$

as

$$\zeta = \frac{\sqrt{(y^2-1)(a^2A^2x^2+1)}}{A(x-y)\Gamma}, \qquad \rho = \frac{\sqrt{(1-x^2)(a^2A^2y^2+1)}}{A(x-y)\Gamma}, \tag{2.10}$$
$$t = t, \qquad \varphi = \varphi,$$

where $\Gamma = \sqrt{1+a^2A^2}$. The metric in the form The inverse transformation is also known [Bičák and Kofroň, 2009].

2.3 Magic Field

The term *magic-field* was first used in [Lynden-Bell, 2003] describing a electromagnetic field of the Kerr-Newman solution with gravitation constant $G = 0$. The field was obtained by considering harmonic complex potential

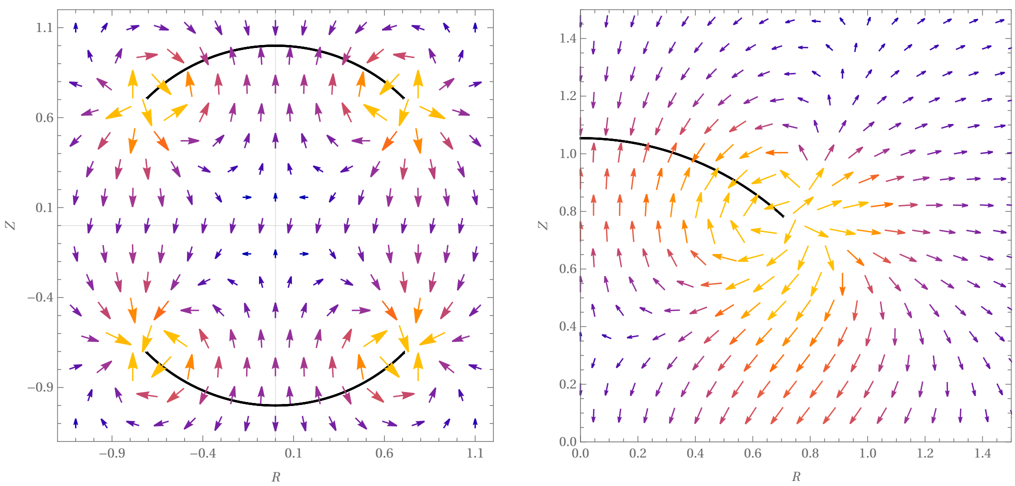
$$\psi = q/\sqrt{(\mathbf{r} - i\mathbf{a})^2}, \tag{2.11}$$

with $\mathbf{a}$ being a purely real vector. With special orientation of the z axis along $\mathbf{a}$ this can rewritten as

$$\psi = q/(R + (z - ia)^2)^{1/2}, \quad \text{where } R^2 = x^2 + y^2. \tag{2.12}$$

The field was obtained as

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = -\nabla\psi. \tag{2.13}$$



(a) $T = 0$ (b) $T = 1/3$

Figure 2.3: Vector plot of the *Eletric* field given by the potential (2.7), along with the coordinate curves $y = \pm\infty$

The source of the field lied at the disc $\rho < a$ and singular ring $r = a$, $z = 0$ in both cases. The singular ring had a divergent charge, opposite to the charge of the disk but the sum of these field resulted in a field, that had the finite charge q, magnetic dipole qa, electric quadrupole qa^2 and non vanishing even electric and odd magnetic multipole moments.

Our magic field is similar. Finding the electromagnetic tensor as

$$\mathbf{F} = \mathbf{dA}. \tag{2.14}$$

Writing this explicitly we get

$$\begin{aligned}\mathbf{F} = Bq\Big[&\frac{\mathrm{K}_\tau \left(a^2A^2x^2+1\right)\left(a^2A^2x^2y^2-1\right)}{\left(a^2A^2x^2y^2+1\right)^2}\mathbf{d}t\wedge\mathbf{d}y \\ &+\frac{2a^2A^2\mathrm{K}_\tau xy\left(y^2-1\right)}{\left(a^2A^2x^2y^2+1\right)^2}\mathbf{d}t\wedge\mathbf{d}x\frac{aA\mathrm{K}_\phi\left(x^2-1\right)\left(a^2A^2x^2y^2-1\right)}{\left(a^2A^2x^2y^2+1\right)^2}\mathbf{d}y\wedge\mathbf{d}\phi \\ &+-\frac{2\mathrm{K}_\phi x\left(a^3A^3y^3+aAy\right)}{\left(a^2A^2x^2y^2+1\right)^2}\mathbf{d}x\wedge\mathbf{d}\phi\Big].\end{aligned} \tag{2.15}$$

As is show graphically in 2.3 this corresponds to field a charged disk with total charge q, which has infinite charge at its rim.

3. Newman-Penrose formalism

Although Newman-Penrose formalism is in detail covered in many great textbooks and articles we still find it useful to include the basics here. We will generally follow [Chandrasekhar, 1998]. Since for our purposes we only need few basic conclusions we recommend [Penrose and Rindler, 1986], and for explicit calculations we also recommend [Miškovský, 2021]. We will also use the usual abbreviation of Newman-Penrose formalism as NP formalism.

Since NP formalism is a special case of more general tetrad formalism, we will first introduce the latter.

3.1 NP preliminaries - tetrad formalism

The tetrad formalism is a different way of tackling problems in, for example, General Relativity and classical electrodynamics. Instead of choosing a local coordinates basis the tetrad formalism considers a so called *tetrad basis*. *Tetrad basis* is composed of four linearly independent (possibly complex) vector-fields usually chosen to represent underlying symmetries of the space-time.

First we need to define the *tetrad basis*. Consider four linearly independent vectors

$$e_{(a)}{}^{j} \quad (a = 1, 2, 3, 4), \tag{3.1}$$

where we introduced new set of indices which are distinguished by being in round parentheses. The associated covariant vectors to (3.1) are

$$e_{(a)j} = e_{(a)}{}^{j} g_{ij}. \tag{3.2}$$

The tetrad indices behave as normal indices but they form a different basis

$$\eta_{(a)(b)} e^{(a)}{}_{j} = e_{(b)j}, \quad \eta^{(a)(b)} e_{(a)j} = e^{(a)}{}_{j}. \tag{3.3}$$

We assume that the matrix $\eta_{(a)(b)}$ is constant. We also define the inverse relation as

$$e^{i}_{(a)} e^{(b)}_{i} = \delta^{(b)}_{(a)}, \quad e^{i}_{(a)} e^{(a)}_{j} = \delta^{i}_{j}. \tag{3.4}$$

It is clear that any tensor can be projected onto the tetrad frame as

$$T_{(a)(b)(c)\ldots}{}^{(i)(j)(k)\ldots} = e_{(a)}{}^{l} e_{(b)}{}^{m} e_{(c)}{}^{n} \ldots e^{(i)}{}_{o} e^{(j)}{}_{p} e^{(k)}{}_{q} \ldots T_{lmn\ldots}{}^{opq\ldots}. \tag{3.5}$$

It is also useful to define directional derivatives

$$\mathbf{e}_{(a)} = e_{(a)}{}^{i} \frac{\partial}{\partial x^{i}}. \tag{3.6}$$

We also define

$$A_{(a),(b)} = e_{(a)}{}^{j} A_{j;l}\, e_{(b)}{}^{i} + e_{(c)}{}^{k} e_{(a)k;i}\, e_{(b)}{}^{i} A^{(c)}. \tag{3.7}$$

Rewriting this equation as

$$A_{(a),(b)} = e_{(a)}{}^{j} A_{j;l}\, e_{(b)}{}^{i} + \gamma_{(c)(a)(b)} A^{(c)}, \tag{3.8}$$

where the definition of $\gamma_{(a)(b)(c)}$ called *Ricci rotation-coefficients* is clear from comparison of equations (3.7) and (3.8).

3.2 Newman-Penrose formalism

As we already said the NP formalism is a special case of tetrad formalism: a *null* tetrad is chosen, two real vectors and a complex-conjugate pair denoted **l**, **n**,**m** and $\overline{\mathbf{m}}$. Which satisfy

$$l^a l_a = n^a n_a = m^a m_a = \overline{m}^a \overline{m}_a = l^a m_a = n^a m_a = l^a \overline{m}_a = n^a \overline{m}_a = 0, \tag{3.9}$$

and

$$l^a n_a = -m^a \overline{m}_a = 1 \tag{3.10}$$

Considering (3.6) we can identify

$$\mathbf{e}_1 = \mathbf{l}, \quad \mathbf{e}_2 = \mathbf{n}, \quad \mathbf{e}_3 = \mathbf{m}, \quad \mathbf{e}_4 = \overline{\mathbf{m}}. \tag{3.11}$$

It is also customary to denote directional derivatives as

$$l^a \frac{\partial}{\partial x^a} \equiv D, \quad n^a \frac{\partial}{\partial x^a} \equiv \Delta, \quad m^a \frac{\partial}{\partial x^a} \equiv \delta, \quad \overline{m}^a \frac{\partial}{\partial x^a} \equiv \overline{\delta}. \tag{3.12}$$

If the (3.9) (3.10) equations are to be satisfied the matrix $\eta_{(a)(b)}$ must take form

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.13}$$

Usually twelve special symbols are defined for Ricci rotation coefficients

$$\begin{aligned} \kappa &= \gamma_{(3)(1)(1)}, \quad \sigma = \gamma_{(3)(1)(3)}, \quad \lambda = \gamma_{(2)(4)(4)}, \quad \nu = \gamma_{(2)(4)(2)}, \\ \rho &= \gamma_{(3)(1)(4)} \quad \mu = \gamma_{(2)(4)(3)}, \quad \tau = \gamma_{(3)(1)(2)}, \quad \pi = \gamma_{(2)(4)(3)}, \\ \epsilon &= \frac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}), \quad \gamma = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}), \\ \alpha &= \frac{1}{2}(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}), \quad \beta = \frac{1}{2}(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}). \end{aligned} \tag{3.14}$$

Finally the so called *intrinsic derivative* is also defined

$$A_{(a)|(b)} = A_{(a);(b)} - \eta^{(n)(m)} \gamma_{(n)(a)(b)} A_{(m)}. \tag{3.15}$$

Among the most physically important tensor surely lie the Riemann curvature tensor, which "describes the failure of successive operations of differentiation to commute when applied to a dual vector field"[Wald, 1984], Weyl tensor, which measures the tidal forces on a object moving along a geodesic and the Ricci tensor, which describes the deformation of volume of an object along moving along a geodesic. The Weyl tensor is the trace free part of the Riemann tensor, defined in components as [Wald, 1984]

$$C_{abcd} = R_{abcd} - g_{a[a}R_{d]b} - \frac{1}{3} R g_{a[c} g_{d]b}, \tag{3.16}$$

where R is scalar curvature and $R_{ab} = R_{abc}{}^{b}$ is the Ricci tensor.

Projecting the Weyl tensor into tetrad basis we get [Chandrasekhar, 1998]

$$C_{(a)(b)(c)(d)} = R_{(a)(b)(c)(d)} - \eta_{(a)[(a)}R_{(d)](b)} - \frac{1}{3}R\eta_{(a)[(c)}\eta_{(d)](b)}, \tag{3.17}$$

with $R_{(a)(b)}$ denoting the tetrad components of Ricci tensor and R the tetrad components of the scalar curvature

$$R_{(a)(c)} = \eta^{(b)(d)}R_{(a)(b)(c)(d)}, \quad R = \eta^{(a)(b)}R_{(a)(b)}. \tag{3.18}$$

In NP formalism Weyl tensor is represented by 5 complex scalars

$$\begin{aligned}
\psi_0 &= -C_{(1)(3)(1)(3)}, \\
\psi_1 &= -C_{(1)(2)(1)(3)}, \\
\psi_2 &= -C_{(1)(3)(4)(2)}, \\
\psi_3 &= -C_{(1)(2)(4)(2)}, \\
\psi_4 &= -C_{(2)(4)(2)(4)},
\end{aligned} \tag{3.19}$$

and the 10 independent components of Ricci tensor are represented by four real scalars $(\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda)$ and three complex ones $\Psi_{10}, \Psi_{20}, \Psi_{21}$ along with their complex conjugates.

$$\begin{aligned}
&\Psi_{00} = \frac{1}{2}R_{(1)(1)}, \quad \Psi_{11} = \frac{1}{4}(R_{(1)(2)} - R_{(3)(4)}), \quad \Psi_{22} = \frac{1}{2}R_{(2)(2)}, \\
&\Lambda = \frac{R}{24}, \quad \Psi_{10} = \frac{1}{2}R_{(1)(4)}, \quad \Psi_{20} = \frac{1}{2}R_{(4)(4)}, \quad \Psi_{21} = \frac{1}{2}R_{(4)(3)}.
\end{aligned} \tag{3.20}$$

3.3 Maxwell Equations in NP formalism

For our purposes, Maxwell equations are most important. As with any tensor, the *EM* tensor can be projected onto the *null tetrad*, since the *EM* tensor is antisymmetric we get three complex scalars

$$\begin{aligned}
\phi_0 &= F_{ij}l^i m^j, \\
\phi_1 &= \frac{1}{2}F_{ij}(l^i n^j + \overline{m}^j m^j), \\
\phi_2 &= F_{ij}\overline{m}^i n^j.
\end{aligned} \tag{3.21}$$

This can of course be reconstructed as

$$F_{ij} = 2(\phi_1\ (n_{[i}l_{j]} + m_{[i}\overline{m}_{j]}) + \phi_0\ l_{[i}m_{j]} + \phi_2\ \overline{m}_{[i}n_{j]}) + c.c. \tag{3.22}$$

Where square brackets over indices indicate antisymmetrization and *c.c* is complex conjugate.

The vacuum Maxwell equations are

$$F_{[ij;k]} = 0, \quad g^{ij}F_{ik,j} = 0, \tag{3.23}$$

these take the form

$$\begin{aligned}
&\phi_{1|(1)} - \phi_{0|(4)} = 0, \quad \phi_{2|(1)} - \phi_{1|(4)} = 0, \\
&\phi_{1|(3)} - \phi_{0|(2)} = 0, \quad \phi_{2|(3)} - \phi_{1|(2)} = 0.
\end{aligned} \tag{3.24}$$

Which can be explicitly rewritten as

$$\begin{aligned}
D\phi_1 - \overline{\delta}\phi_0 &= (\kappa - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \\
D\phi_2 - \overline{\delta}\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2 * \epsilon)\phi_2, \\
\delta\phi_1 - \Delta\phi_0 &= (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \\
\delta\phi_2 - \Delta\phi_1 &= -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2,
\end{aligned} \tag{3.25}$$

which can be directly found using the definition (3.15).

3.4 Petrov classification

The complexity of many formulas in the NP formalism and especially in the GHP formalism, which is introduced latter, is greatly reduced if we limit ourself to algebraically special space-times. The classification of these space-times is based on the properties of the Weyl tensor, which we shall shortly discuss here. Weyl tensor has the same symmetries as the as the Riemann tensor

$$C_{abcd} = -C_{abdc} = -C_{bacd} = C_{cdab}, \tag{3.26}$$

and

$$C_{abcd} + C_{adbc} + C_{acdb} = 0. \tag{3.27}$$

But unlike the Riemann tensor the Weyl tensor is traceless

$$C^a{}_{bad} = 0. \tag{3.28}$$

These relations follow from the symmetry of the metric, the definition of Weyl tensor (3.16) and the definition of Riemann tensor

$$R^j_{bcd} g_{ja} = R_{abcd} = \frac{1}{2}(\partial_c \partial_b g_{ac} + \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ca} - \partial_d \partial_a g_{bc}). \tag{3.29}$$

Due to this fact the Weyl tensor can be represented by a 6×6 matrix [Duggal and Sharma, 1999], denoted as C_{AB} the matrix takes the form

$$C_{AB} = \begin{pmatrix} A & B \\ B^T & -A \end{pmatrix}, \tag{3.30}$$

both A and B are 3×3 are symmetric and traceless matrices. This matrix can be represented by a 3×3 complex matrix $A + iB$. Finally the space-times are classified based on these components

Petrov Type I

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \tag{3.31}$$

Petrov Type II

$$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}, \tag{3.32}$$

Petrov Type III

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.33}$$

Petrov Type D special case of Type I where $\alpha = \beta$,
Petrov Type O special case of Type I where $\alpha = \beta = \gamma = 0$,
Petrov Type D special case of Type II where $\alpha = 0$.

Note that since the Weyl tensor is traceless, the matrix eigen-values of the matrix C_{AB} must sum up to zero.

3.5 C-metric in NP formalism

Lets us now show the basic of NP formalism on the example of the C-metric (2.1). Considering the flat-spacetime limit ($G \to 0$) of the tetrad used in [Kofroň, 2016]

$$\mathbf{l} = \frac{(x-y)^2 A^2}{\sqrt{2} B^2} \Big[\frac{a^2 A^2 y^2 + 1}{\Gamma^2 \mathrm{K}_\tau G(y)} \boldsymbol{\partial}_t - \boldsymbol{\partial}_y + \frac{aA(1-y^2)}{K_\phi G(y) \Gamma^2} \boldsymbol{\partial}_\phi \Big], \tag{3.34}$$

$$\mathbf{n} = \frac{-1}{\sqrt{2}(1 + a^2 A^2 x^2 y^2)} \Big[\frac{a^2 A^2 y^2 + 1}{K_\tau \Gamma^2} \boldsymbol{\partial}_t + G(y) \boldsymbol{\partial}_y - \frac{aA\,(y^2 - 1)}{K_\phi \Gamma^2} \boldsymbol{\partial}_\phi \Big], \tag{3.35}$$

$$\mathbf{m} = \frac{A(x-y)\sqrt{G(x)}}{B\sqrt{2(a^2 A^2 x^2 y^2 + 1)}} \Big[-\frac{a^2 A^2 x^2 + 1}{\Gamma^2 K_\phi G(x)} \boldsymbol{\partial}_t - i \boldsymbol{\partial}_x - \frac{a^2 A^2 x^2 + 1}{\Gamma^2 K_\phi G(x)} \boldsymbol{\partial}_\phi \Big], \tag{3.36}$$

$$\overline{\mathbf{m}} = \frac{A(x-y)\sqrt{G(x)}}{B\sqrt{2(a^2 A^2 x^2 y^2 + 1)}} \Big[-\frac{a^2 A^2 x^2 + 1}{\Gamma^2 K_\phi G(x)} \boldsymbol{\partial}_t + i \boldsymbol{\partial}_x - \frac{a^2 A^2 x^2 + 1}{\Gamma^2 K_\phi G(x)} \boldsymbol{\partial}_\phi \Big], \tag{3.37}$$

where

$$\Gamma^2 = (1 + a^2 A^2). \tag{3.38}$$

Finding the Ricci rotation coefficients we get

$$\gamma = \frac{(x-y)G'(y)\left(a^2A^2x^2y^2+1\right)+G(y)\left(a^2A^2x^2y(4y-3x)+iaAx(x-y)+1\right)}{2\sqrt{2}(x-y)\left(a^2A^2x^2y^2+1\right)^2},$$
$$\beta = -\frac{iA(x-y)\sqrt{\frac{G(x)}{2a^2A^2x^2y^2+2}}\left(G'(x)\left(a^2A^2x^2y^2+1\right)+2aAyG(x)(-2aAxy+i)\right)}{4BG(x)\left(a^2A^2x^2y^2+1\right)},$$
$$\tau = \frac{A(-1-iaAy(2x-y))\sqrt{\frac{G(x)}{2a^2A^2x^2y^2+2}}}{B(aAxy-i)}, \quad \rho = \frac{A^2(x-y)\left(aAx^2-i\right)}{\sqrt{2}B^2(aAxy-i)},$$
$$\pi = -\frac{iA\left(aAy^2-i\right)\sqrt{\frac{G(x)}{2a^2A^2x^2y^2+2}}}{B(aAxy-i)}, \mu = \frac{iaAxG(y)}{\sqrt{2}\left(a^2A^2x^2y^2+1\right)^2},$$
$$\epsilon = -\frac{A^2(x-y)\left(aAx^2-i\right)}{2\sqrt{2}B^2(aAxy-i)}, \quad \alpha = \frac{A\left(1+iaAy^2\right)\sqrt{-\frac{G(x)}{2a^2A^2x^2y^2+2}}}{2B(aAxy-i)},$$
$$\kappa = \sigma = \nu = \lambda = 0. \tag{3.39}$$

All tensor Weyl tensor components and their projections (3.19) are zero since we are in flat-spacetime.

Althought the NP formalism has many uses, in searching of a EM field natural to the C-metric we will need to venture into a special case on the NP formalism, called the GHP formalism.

4. GHP formalism

In our search for a suitable EM field we first need to become acquainted with so called GHP (Geroch–Held–Penrose) formalism. The GHP is very useful, by noticing special tetrad set, it assign spin and boost weight to every NP quantity, then derives special operator, which are dependent on these two weights. This results in simplifying the otherwise cumbersome NP identities, mainly Bianchi identities but also Ricci and EM equations, are greatly reduced. The GHP formalism has been first introduced in [Geroch et al., 1973], since then it has become widely spread but under utilized form of spinor formalism. The GHP formalism along with the NP formalism is implemented in many computer algebra program, one example is the *xAct* [Gómez-Lobo and Martín-García, 2012] package for *Wolfram Mathematica*.

4.1 Introduction

Considers a tetrad $(l^a, n^a, m^a, \overline{m}^a)$ satisfying the equations (3.9) and (3.10). The transformation between another tetrad, denoted$(\hat{l}^a, \hat{n}^a, \hat{m}^a, \hat{\overline{m}}^a)$, which also fulfills (3.9) (3.10),if we fix the directions of l^μ and n^μ, can always be written in the form [Robin, 2014]

$$\begin{aligned} \hat{l}^a = r l^a, \quad n'^a = r^{-1} n^a, \\ \hat{m}^a = e^{i2\theta} m^a, \quad \hat{\overline{m}}^a = e^{-i2\theta} \overline{m}^a, \end{aligned} \tag{4.1}$$

where $(r, \theta) \in \mathbb{R}$. If we denote $\chi^2 = re^{i\theta}$ this takes symbolically the form a

$$(l^a, n^a, m^a, \overline{m}^a) \to (\chi^{-1}\overline{\chi}^{-1} l^a, \chi\overline{\chi} n^a, \chi\overline{\chi}^{-1} m^a, \chi^{-1}\overline{\chi}\, \overline{m}^a). \tag{4.2}$$

Since we project the tensors of general relativity onto the tetrad it is easy to see that every such tensor will, under the transformation (4.2) assume the form

$$T \to \chi^p \overline{\chi}^q T, \tag{4.3}$$

where $(p, q) \in \mathbb{Z}$. We say that T is of type p, q, while the number $\frac{1}{2}(p - q)$ is called the spin-weight and the number $\frac{1}{2}(p + q)$ is called the boost-weight. Let us illustrate this on a few examples.

$$\begin{aligned} \Psi_0 :&\{0, 4\}, & \Psi_1 &: \{0, 0\} & \Psi_2 &: \{0, 2\}, \\ \Psi_3 :&\{0, -2\}, & \Psi_4 &: \{0, -4\}, & \kappa &: \{3, 1\}, \\ \rho :&\{1, 1\}, & \pi &: \{-1, 1\}, & \nu &: \{-3, -1\}. \end{aligned} \tag{4.4}$$

Where ρ, κ, π, ν denotes the standard NP quantities (3.14) and Ψ_i the projections of the Weyl tensor (3.19).

In GHP formalism all NP quantities are defined it the same way as in NP formalism, we will thus assume knowledge of Chapter 3 *Newman penrose formalism*

One of the most important operations in the GHP formalism is the prime operator " ′ " which commutes with the complex conjugate operation. The prime operator is defined by it's action on the tetrad

$$(l^a)' = n^a, \quad (n^a)' = l^a, \quad (m^a)' = \overline{m}^a, \quad (\overline{m}^a)' = m^a. \tag{4.5}$$

Let us illustrate this operation on the quantities (4.4) along with the symbols 3.12

$$\begin{aligned} \Psi_0' = \Psi_4, \quad \Psi_1' = \Psi_3, \quad \Psi_2 = -\Psi_2 \\ \kappa' = \nu, \quad \tau' = -\pi, \quad \lambda' = -\sigma, \\ \gamma' = -\epsilon, \quad \alpha' = -\beta, \quad \mu' = -\rho \\ \delta' = \overline{\delta}, \quad D' = \delta. \end{aligned} \tag{4.6}$$

we thus formally reduced the number of independent scalars. Note also that under the operation " $'$ " the type $\{p, q\}$ changes to $\{-p, -q\}$.

Althought the operators (δ, Δ, D) are defined in the GHP formalism they do not behave "nicely", when the opretaros 3.12 act on a quantity fulfilling the transformation (4.3) the new quatnity doesn't satisfy the the transformation (4.3). Thus in GHP formalism new operators are defined

$$\begin{aligned} Þ &= l^a\nabla_a - p\epsilon - q\bar{\epsilon}, \\ ð &= \overline{m}^a\nabla_a - p\beta + q\overline{\beta}, \\ Þ' &= n^a\nabla_a + p\gamma + q\overline{\gamma}, \\ ð' &= m^a\nabla_a - p\alpha + q\overline{\alpha}, \end{aligned} \tag{4.7}$$

with the weights

$$Þ : \{1, 1\}, \; ð : \{1, -1\}. \tag{4.8}$$

4.2 GHP formalism in type D space-times

The following sections is limited to the Petrov type D space-times, not only because we are primarily interested in the C-metric (2.1), but also because the GHP equations and the Bianchi identities are greatly simplified in this algebraically special space-time [Wardell and Kavanagh, 2021].

According to the Golder-Sachs theorem [Goldberg and Sachs, 2009] by alignment of the tetrad components l^μ, n^μ to the null direction four spin coefficients and four Weyl scalars will vanish

$$\begin{aligned} \kappa = \kappa' = \sigma = \sigma' = 0, \\ \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0. \end{aligned} \tag{4.9}$$

In Chapter 3 we omnitted the so called NP equations, which describe the relation between the Ricci spin coefficients and the Weyl tensor scalars, there equations then read [Wardell and Kavanagh, 2021]

$$\begin{aligned} Þ\tau = \rho(\tau - \overline{\tau}'), \quad Þ\rho = \rho^2, \\ ð\rho = \tau(\rho - \overline{\rho}), \qquad ð\tau = \tau^2, \\ Þ'\rho = \rho\overline{\rho}' - \tau\overline{\tau} - \psi_2 + ð'\tau. \end{aligned} \tag{4.10}$$

Bianchi identities take the form

$$Þ\Psi_2 = 3\rho\Psi_2, \quad ð\Psi_2 = 3\tau\Psi_2, \tag{4.11}$$

One can also calculate the commutators of the GHP operators

$$\begin{aligned} [Þ, Þ'] = (\overline{\tau} - \tau')ð + (\tau - \overline{\tau}')ð' - p(\Psi_2\tau\tau') - q(\Psi_2 - \overline{\tau\tau}'), \\ [Þ, ð] = \overline{\rho}ð - \overline{\tau}'Þ + q\overline{\rho}\tau', \\ [ð, ð'] = (\overline{\rho}' - \rho')Þ + (\rho - \overline{\rho}Þ') + p(\Psi_2) + \rho\rho') - q(\overline{\rho\rho}' + \overline{\Psi}_2). \end{aligned} \tag{4.12}$$

and their primed, complex conjugate and complex cojugate prime versions.

4.3 EM field

The NP equations for scalar EM field can be rewritten in the GHP formalism as [Wardell and Kavanagh, 2021]

$$\begin{aligned}
\phi_0 &= F_{(1)(3)} = (\text{Þ} - \overline{\rho})A_{(3)} - (\eth - \overline{\tau}')A_{(1)}, \\
\phi_1 &= \frac{1}{2}(F_{(1)(2)} - F_{(3)(4)}) \\
&= \frac{1}{2}\Big[(\text{Þ} + \rho - \overline{\rho})A_{(2)} - (\text{Þ}' + \rho' - \overline{\rho}')A_{(1)}, \\
&\quad +(\eth' + \tau' - \overline{\tau})A_{(3)} - (\eth' + \tau - \overline{\tau}')A_{(4)}\Big] \\
\phi_2 &= F_{(4)(2)} = -(\text{Þ}' - \overline{\rho}')A_{(4)} + (\eth' - \overline{\tau})A_{(2)},
\end{aligned} \tag{4.13}$$

where A denotes the electromagnetic four-potential. The spin and boost wieght of the scalars are

$$\phi_0 : \{2, 0\} \quad \phi_1 : \{0, 0\}, \quad \phi_2 : \{-2, 0\}. \tag{4.14}$$

5. C-metric EM field

Although in Section 2.2 we have already provided a solution to the Einstein-Maxwell equations, the resulting field is suitable for the superradiance problem. We would like to model the situation such that, we have an incoming wave, an outgoing one and a wave which isn't divergent at the origin with analogy to Chapter 1. Finding such fields, not only EM ones but also gravitational ones, in solutions of the Einstein equations is generally complicated.

One of the most important ones was found by Teukolsky [Teukolsky, 1973], where he, by pertubating the gravitational field around Kerr black hole, generated a plethora of test fields (based on spin value of potential), which, surprisingly, was solved by assuming a separable ansantz. In Boyler-Lindquist coordinates [Boyer and Lindquist, 1967], in geometrized units ($c = 1$), while for the sake of uniformity we use the same coordinates as [Teukolsky, 1973] the metric takes the form

$$\begin{aligned} \mathbf{d}s^2 = &\left(1 - \frac{2Mr}{\Sigma}\right)\mathbf{d}t^2 + \frac{4Mar\sin^2(\theta)}{\Sigma}\mathbf{d}t\mathbf{d}\phi - \frac{\Sigma}{\Delta}\mathbf{d}r^2 - \Sigma\mathbf{d}\theta^2 \\ &- \frac{r^2 + a^2 + 2Ma^2 r\sin^2(\theta)}{\Sigma}\sin^2(\theta)\mathbf{d}\phi^2, \\ &\text{where} \quad \Sigma = r^2 + a^2\cos^2(\theta), \quad \Delta = r^2 - 2Mr + a^2, \end{aligned} \tag{5.1}$$

with M being the mass the black hole and aM it's angular momentum. For $a \to 0$ the metric reduces to Schwarzschild metric.

Choosing the tetrad

$$\begin{aligned} \mathbf{l} &= \frac{r^2 - a^2}{\Delta}\boldsymbol{\partial}t + \boldsymbol{\partial}r + \frac{a}{\Delta}\boldsymbol{\partial}\phi, \quad \mathbf{n} = \frac{1}{2\Sigma}\left[r^2 + a^2\boldsymbol{\partial}t - \Delta\boldsymbol{\partial}r + a\boldsymbol{\partial}\phi\right], \\ \mathbf{m} &= \frac{1}{\sqrt{2}(r + ia\cos(\theta))}\left[ia\sin(\theta)\boldsymbol{\partial}t + \boldsymbol{\partial}\theta + \frac{i}{\sin(\theta)}\boldsymbol{\partial}\phi\right], \end{aligned} \tag{5.2}$$

Teukolsky arrives at a separable ansantz for the unknown function ψ

$$\psi = e^{-i\omega t}e^{im\phi}S(\theta)R(r), \tag{5.3}$$

where R and S satisfy the equation

$$\begin{aligned} &\Delta^s\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{r}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0, \\ &\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dS}{d\theta}\right) + \\ &\qquad \left[\cos(\theta)\left(a^2\omega^2 - 2a\omega s - \frac{2ms}{\sin^2(\theta)}\right) - s^2\cot^2(\theta) + s + a\right]S = 0. \end{aligned}$$

These are now called the Teukolsky equations. These correspond to different field assigned to the number s, for example the field ϕ_0 is assigned to the number $s = 1$ and the field $\rho^{-2}\phi_2$ is assigned to the number $s = -1$.

By assuming regularity at $\phi = 0$ and π we get an Sturm-Liouville eingenvalue problem. The solution of ${}_s\Psi_l^m = {}_s S_l^m e^{im\phi}$ are the spin-weighted-spherodial harmonics, which, though not fully yet known, can we worked with and are implemented in certain programs, such as the *Black hole pertubation toolkit* for *Wolfram Mathematica*.

5.1 Generating the field

Many similar attempts have been made to find test fields in other metrics. Most notably for us, [Kofroň, 2016] 2015 introduces a set equations for perturbation in the C-metric, where, also surprisingly, the Debye potential assumes the separable ansantz which again leads to the Sturm-Liouville problem. Although the solution to this problem is not yet known, we will follow Kofroň's work in the flat-spacetime limit in attempt to reduce the problems complexity.

Consider the metric (2.1) alongside with a new tetrad

$$\begin{aligned}\mathbf{l} =& -\frac{\Omega_0}{\sqrt{2}}\Big[(1+a^2A^2x^2)\sqrt{-\varepsilon H(y)}\mathbf{d}\tau - \frac{1}{\sqrt{-\varepsilon H(y)}}K_\tau \mathbf{d}y \\ &+ aA(1-x^2)\sqrt{-\varepsilon H(y)}K\phi\mathbf{d}\phi\Big], \qquad (5.4)\\ \mathbf{n} =& -\frac{\Omega_0}{\sqrt{2}}\Big[(1+a^2A^2x^2)\sqrt{-\varepsilon H(y)}\mathbf{d}\tau + \frac{1}{\sqrt{-\varepsilon H(y)}}K_\tau \mathbf{d}y \\ &+ aA(1-x^2)\sqrt{-\varepsilon H(y)}K\phi\mathbf{d}\phi\Big], \qquad (5.5)\\ \mathbf{m} =& \frac{\Omega_0}{\sqrt{2}}\Big[-iaA(1-y^2)\sqrt{H(x)}\mathbf{d}\tau - \frac{1}{\sqrt{H(x)}}K_\tau \mathbf{d}y \\ &+ i(1-a^2A^2y^2)\sqrt{H(x)}K\phi\mathbf{d}\phi\Big], \qquad (5.6)\end{aligned}$$

where $\varepsilon = 1$ in regions $G(y) < 0$ and $\varepsilon = -1$ in regions $G(y) > 0$, in the flat-spacetime limit $G(y) < 0$ is always true, so the parameter ε will disappear in later discussion.

According to [Cohen and Kegeles, 1975] Maxwell equations can by generated by so called Debye potential ψ, which obeys the wave equation

$$\Big[\Delta + \overline{\gamma} + \overline{\mu})(D+\epsilon) - (\overline{\delta} + \overline{\beta} - \overline{\tau})(\delta + \beta)\Big]\,\psi = 0, \qquad (5.7)$$

the solution of the equation above then generates Maxwell fields as

$$\begin{aligned}\phi_0 &= \text{Þ}\text{Þ}\rho\overline{\psi},\\ \phi_1 &= \Big[-(D+\epsilon+\overline{\epsilon})(\overline{\delta}+2\overline{\beta}+\overline{\tau}) + (\pi+\overline{\tau})(D+2\overline{\epsilon}+\overline{\rho})\Big]\,\overline{\psi}, \qquad (5.8)\\ \phi_2 &= \eth'\eth'\rho\overline{\psi}.\end{aligned}$$

Now applying the separable asantz

$$\psi = e^{-i\omega\tau}e^{im\phi}(1+iaxy)Y^{(s)}_{lm}(y)X^{(s)}_{lm}(x), \qquad (5.9)$$

where $s = \pm 1$ and $(l, m) \in \mathbb{Z}$ are indices. The equations (5.8), (5.9) and (5.7) then lead to the equations [Kofroň, 2016]

$$
\begin{aligned}
&\frac{\frac{d}{dr}\left[G(x)X_{lm}^{(s)}(x)\right]}{X_{lm}^{(s)}(x)}+\frac{1}{2}\frac{d^2G(x)}{dx^2}+\Lambda_{lm}^{(s)}-\\
&-\frac{\left((1+a^2A^2x^2)\dot{m}-\frac{1}{2}\frac{dG(x)}{dx}+aA(1-x^2)\omega^2\right)^2}{G(x)}-4aA(aA\dot{m}-\dot{\omega}sx)=0,\\
&\frac{\frac{d}{dr}\left[G(y)Y_{lm}^{(s)}(y)\right]}{Y_{lm}^{(s)}(y)}+\frac{1}{2}\frac{d^2G(y)}{dy^2}+\Lambda_{lm}^{(s)}-\\
&-\frac{\left(i(1+a^2A^2y^2)\dot{m}-\frac{1}{2}\frac{dG(x)}{dx}-iaA(1-y^2)\omega^2\right)^2}{G(x)}+i4aA(aA\dot{m}-\dot{\omega}sx)=0.
\end{aligned}
\tag{5.10}
$$

with

$$
\dot{m}=\frac{m}{K_\phi\Gamma^2},\quad \dot{\omega}=\frac{\omega}{K_\tau\Gamma^2}. \tag{5.11}
$$

As already mentioned the equation (5.10) is a Sturm-Liouville problem, with the eigenvalues Λ yet to be determined. One of the standard method of solving such problems is the so called Spectral parameter power series method [Kravchenko and Porter, 2008], which finds the solution based on known solution of the equation with the eigenvalue $\Lambda = 0$, this is equivalent with having found any eingenvalue Λ. Sadly we were not able to guess no such eigenvalue, neither, as far as we know, has any such eigenvalue ever been found. Some computer systems, most notably the CAS software *Maple* can solve the equation (5.10). The results are too long to show here. The result given is in terms of the Heun functions, which are solution to the Heun equations [Maier, 2006]. Although we have a solution, this is sadly a stopping point for us, even if "an easy" solution was not expected the solution expression are too long and too complicated for further calculations even when using computer algebra software such as *Wolfram Mathematica* and *Maple*.

Even thought we have not found a suitable field, we still formulate the boundary problem, such that if the solutions of the equations (5.10) are ever found the basic groundwork will be laid.

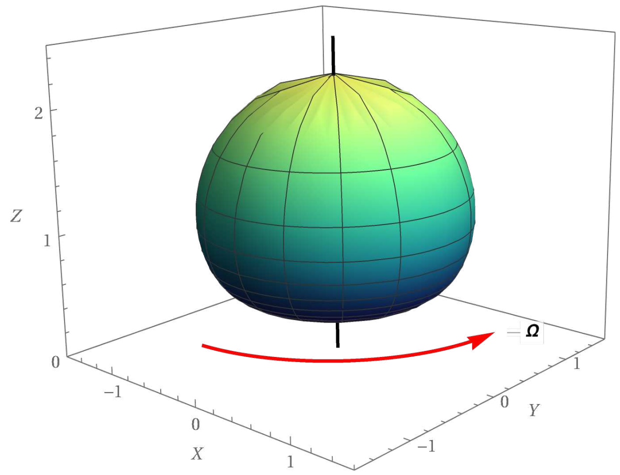


Figure 6.1: Surface of constant coordinate y rotating along the Z axis with angular velocity Ω

6. Superradiance in the C-metric

We would like to formulate the problem in the flat-spacetime C-metric (2.8). Since the formulation of the problems requires the normal of the object, it is sound to consider the object with the simplest normal, most natural of these are the surfaces of constant coordinates, for our purposes we will consider the $y = \alpha,\ \alpha < 1$ surface with the normal $\mathbf{n} = \mathbf{d}y$, normalized $n^\mu n_\mu = 1$. This object is shown in figure 6.1.

6.1 Jump conditions in electromagnetism

Although we already provided set of equations in Chapter 1 we would like to introduce the basic theory, about calculating these conditions. For further reference we refer the reader to [Itin, 2012]

Lets us have two tensors [Kulyabov and Korolkova, 2012]

$$F^{ab} = \begin{pmatrix} 0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\ \mathcal{E}^1 & 0 & \mathcal{B}_3 & -\mathcal{B}_2 \\ \mathcal{E}^2 & -\mathcal{B}_3 & 0 & \mathcal{B}_1 \\ \mathcal{E}^3 & \mathcal{B}_2 & -\mathcal{B}_1 & 0 \end{pmatrix}, \quad G^{ab} = \begin{pmatrix} 0 & -\mathcal{D}^1 & -\mathcal{D}^2 & -\mathcal{D}^3 \\ \mathcal{D}^1 & 0 & \mathcal{H}_3 & -\mathcal{H}_2 \\ \mathcal{D}^2 & -\mathcal{H}_3 & 0 & \mathcal{H}_1 \\ \mathcal{D}^3 & \mathcal{H}_2 & -\mathcal{H}_1 & 0 \end{pmatrix}. \tag{6.1}$$

Where $\mathcal{E}^\mu, \mathcal{H}_\mu, \mathcal{B}_\mu$ and $\mathcal{D}^\mu$ are the tetrad components of the electric field strength, magnetic field strength, magnetic flux density and electric displacement field components respectively for $\mu = 1, 2, 3$.

According to [Itin, 2012] the boundary conditions of EM field with a static boundary surface, described by the normal n^μ are described as

$$\begin{aligned} \varepsilon^{\alpha\beta\gamma} n_\alpha \left[F_{\beta\gamma}\right] = 0, \quad & \varepsilon^{\alpha\beta\gamma} n_\beta \left[F_{0\gamma}\right] = 0, \\ \left[G^{\alpha,0}\right] n_\alpha = j^0, \quad & \left[H^{\alpha\beta}\right] n_\beta = j^\alpha, \end{aligned} \tag{6.2}$$

where greek indices go over $1, 2, 3$, and $j^i, i = 0, 1, 2, 3$ is the four-current of the boundary.

These can be written in standard three-dimensional form as

$$\begin{aligned} \mathbf{n} \cdot [\mathbf{B}] = 0, \quad \mathbf{n} \times [\mathbf{B}] = 0, \\ \mathbf{n} \cdot [\mathbf{D}] = \sigma, \quad \mathbf{n} \cdot [\mathbf{H}] = \mathbf{j}. \end{aligned} \tag{6.3}$$

6.2 EM jump in C-metric

Having the boundary rotating with the four-speed

$$u^\mu = \gamma(1, 0, 0, \Omega), \tag{6.4}$$

where γ is the normalization coefficient $u^\mu u_\mu = -1$. Defining the four-current as to not have a component in the normal direction

$$j^\mu = \sigma\left(F^{\mu\nu}u_\nu - n^\mu n_\nu F^{\nu\gamma}u_\gamma\right) + \rho u^\mu, \tag{6.5}$$

where σ is the conductivity and ρ is the charge density.

Having the conditions (6.2) along with the definitions (6.4) and (6.5) we get

$$\begin{gathered} [\mathcal{B}_2] = \frac{B\gamma\mu\sqrt{\frac{a^2\alpha^2A^2x^2+1}{G(x)}}}{A^3(x-\alpha^3)}\left\{-A^2\mathcal{E}^3(x-\alpha)^2\sigma + (1+a^2A^2)B^2K_\phi^2(x^2-1)\Gamma^2\rho\Omega\right\}, \\ \left[\mathcal{E}^1\right] = -\frac{\gamma\sqrt{-\frac{B^2(\alpha^2(\Gamma^2-1)x^2+1)}{A^2G(x)(x-\alpha)^2}}, \left(\frac{A^2\mathcal{E}^3\sigma\Omega(x-\alpha)^2}{(\alpha^2-1)B^2\Gamma^2K\tau^2((\Gamma^2-1)x^2+1)} + \rho\right)}{\epsilon}, \\ [\mathcal{B}_3] = \frac{B\gamma\mu\sigma(\mathcal{B}_1\Omega - \mathcal{E}^2\sqrt{\frac{a^2\alpha^2A^2x^2+1}{G(x)}}}{A(x-\alpha)}, \\ [\mathcal{B}_1] = \left[\mathcal{E}^2\right] = \left[\mathcal{E}^3\right] = 0. \end{gathered} \tag{6.6}$$

The lattermost equation in (6.6) being the know equations.

$$\begin{aligned} \mathbf{n} \cdot \mathbf{B} = 0, \\ \mathbf{n} \times \mathbf{E} = 0. \end{aligned} \tag{6.7}$$

As we can see these equations are fairly simple, this is of course thanks to the fact that we chose a natural normal. We can also explicitly identify the components of the tensor F_{ab} from equation (3.22)

7. Coordinate transform approach

In this section we will consider a new approach towards electromagnetic superradiance in the C-metric. Instead of trying to find a radiation field in C-metric coordinates, we will attempt to describe the problem in Minkowski coordinates, thus we will have an uniformly accelerated rotating body, onto which EM waves will be incident. Main usefulness of such approach is that we can utilize the vector spherical harmonics (VSH). Since according to [Caltenco et al., 2002] the VSH form a complete set of orthonormal functions, every field can be decomposed into such functions. The efficiency of VSH has been shown in [Bára, 2017], where superradiance on a rotating spherical shell was described. We will first summaries the VSH and basic results from [Bára, 2017]. Then we will attempt to generalize this by reformulating the problem on a ellipsoid, here, since we do not respect the azimuthal symmetry, technical difficulties will surely arise.

7.1 Vector spherical harmonics

We will provide a short summary of the VSH [Carrascal et al., 2000], while also following the notation of [Bára, 2017]. The VSH have been known for some time, in canonical literature Jackson [1999] used a form of these to show how a standing wave interacts with a perfectly conducting sphere. Consider a radial vector $\tilde{\mathbf{r}}$ of unit length and a radial vector $\mathbf{r} = r\,\tilde{\mathbf{r}}$, then the VSH are defined as [Carrascal et al., 2000]

$$\begin{aligned} \mathbf{Y}^{(mag)}_{lm} &= \mathbf{r} \times \nabla Y_{lm}, \\ \mathbf{Y}^{(el)}_{lm} &= r\nabla Y_{lm}, \\ \mathbf{Y}^{(rad)}_{lm} &= \tilde{\mathbf{r}}\nabla Y_{lm}, \end{aligned} \tag{7.1}$$

with $Y_{lm} = Y_{lm}(\theta, \phi)$ being the spherical harmonics. With the orthogonality relations

$$\begin{aligned} \int d\Omega \mathbf{Y}^{(mag)}_{lm} \cdot \overline{\mathbf{Y}}^{(mag)}_{lm} = \int_0^{2\pi}\int_0^{\pi} r^4 \sin(\theta) \nabla Y_{lm} \nabla \overline{Y}_{l'm'} = \\ \int_0^{2\pi}\int_0^{\pi} r^4 \sin(\theta)\left[\nabla \cdot (Y_{lm}\nabla \overline{Y}_{l'm'}) - Y_{lm}\nabla^2 \overline{Y}_{l'm'}\right] = \\ l'(l'+1)\int d\Omega Y_{lm}\overline{Y_{lm}} = l(l+1)\delta_{ll}\delta_{mm}, \\ \int d\Omega \mathbf{Y}^{(el)}_{lm} \cdot \overline{\mathbf{Y}}^{(el)}_{lm} = l(l+1)\delta_{ll'}\delta_{mm'}, \\ \int d\Omega \mathbf{Y}^{(rad)}_{lm} \cdot \overline{\mathbf{Y}}^{(rad)}_{lm} = \delta_{ll'}\delta_{mm'}. \end{aligned} \tag{7.2}$$

Where we used the same "trick" as in [Carrascal et al., 2000], by noticing that the integral

$$\int d^3x \nabla \cdot [\Theta(r - |x|)\mathbf{A}] = 0, \tag{7.3}$$

where Θ is the heaviside distribution, writing out the integral

$$\int d^3x \Theta(r - |x|)\nabla \cdot \mathbf{A} - \int d^3x \delta(r - |x|)\mathbf{A}.\mathbf{n} = 0. \tag{7.4}$$

Now if we consider $\mathbf{A} = Y_{lm}\nabla\overline{Y_{l'm'}}$ we get $\mathbf{A}.\mathbf{n} = 0$ finally

$$\int_0^a r^2 dr \int d\Omega \nabla \cdot A = 0, \tag{7.5}$$

implies that

$$\int d\Omega \nabla \left(\cdot Y_{lm} \nabla \overline{Y}_{l'm'} \right) = 0. \tag{7.6}$$

Now according to [Carrascal et al., 2000] any EM field which satisfies the Maxwell equations can be decomposed as

$$\begin{aligned}
\mathbf{E} &= \sum_{lm} \left(f_l(\hat{r}) \mathbf{Y}_{lm}^{(mag)} - ic\frac{l(l+1)}{\hat{r}} g_l(\hat{r}) \mathbf{Y}_{lm}^{(rad)} - ic \left(g_l'(\hat{r}) + \frac{g_l(\hat{r})}{\hat{r}} \right) \mathbf{Y}_{lm}^{(el)} \right) e^{-i\omega t}, \\
\mathbf{B} &= \sum_{lm} \left(g_l(\hat{r}) \mathbf{Y}_{lm}^{(mag)} + \frac{i}{c}\frac{l(l+1)}{\hat{r}} f_l(\hat{r}) \mathbf{Y}_{lm}^{(rad)} + \frac{i}{c} \left(g_l'(\hat{r}) + \frac{g_l(\hat{r})}{\hat{r}} \right) \mathbf{Y}_{lm}^{(el)} \right) e^{-i\omega t},
\end{aligned} \tag{7.7}$$

where $\hat{r} = kr = \omega r/c$. The radial functions $f_l(r), g_l(r)$ are to be determined, the basic wave solutions are [Bára, 2017]

$$f_l(r) = \begin{cases} A(l)\ h_l^{(2)}(r) & \text{incoming wave} \\ B(l)\ h_l^{(1)}(r) & \text{outgoing wave} \\ C(l)\ j_l(r) & \text{standing wave} \end{cases}, \quad g_l(r) = \begin{cases} \alpha(l)\ h_l^{(2)}(r) & \text{incoming wave} \\ \beta(l)\ h_l^{(1)}(r) & \text{outgoing wave} \\ \gamma(l)\ j_l(r) & \text{standing wave} \end{cases} \tag{7.8}$$

where $\{A(l), B(l), C(l), \alpha(l), \beta(l), \gamma(l)\} \in \mathbb{C}$ are the coefficients.

7.2 Supperradiance on a ellipsoid

In this section we will formulate and attempt to solve the problem of superradiance on a ellipsoid in spherical coordinates. There of course exist more natural coordinate system, in which these objects are easily described and in which similar functions to the VSH also exist, but we are attempting to model the situation of the "C-metric object (coordinate surface $y = const.$)" interacting with waves in spherical coordinates.

We will consider a special type of ellipsoid, called the spheroid, which is just a ellipsoid with two identical axes, these quadratic surfaces are described in the cartesian coordinates by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{7.9}$$

where a and b are real parameters usually we recognize two types of spheroids, a prolate spheroid $a < c$ and and oblate spheroid $a > c$ with the degenerate case $a = b$ being a simple sphere. As a normal we can take the gradient of

$$\mathbf{n} = 2\left(\frac{x}{a^2}, \frac{y}{a^2}, \frac{z}{c^2} \right). \tag{7.10}$$

transforming this into the spherical coordinates (r,θ,ϕ) and denoting the normalization coefficient γ of $\mathbf{n}.\mathbf{n}=1$ we get

$$\mathbf{n}=\gamma\left(\frac{r\left(\left(a^2-\mathrm{c}^2\right)\cos(2\theta)+a^2+\mathrm{c}^2\right)}{2a^2\mathrm{c}^2},\frac{r^2\left(\mathrm{c}^2-a^2\right)\sin(\theta)\cos(\theta)}{a^2\mathrm{c}^2},0\right), \tag{7.11}$$

the normal component in the direction of ϕ is zero, this is expected since the spheroid follows the axial symmetry.

The rotation of the surface is described by the four velocity

$$u^\mu=\Gamma\left(1,0,0,\Omega\right), \tag{7.12}$$

where Γ is the normalization coefficient $u^\mu u_\mu=-1$ with respect to Minkowski metric

$$\mathbf{d}s^2=-\mathbf{d}t^2+\mathbf{d}r^2+r^2\left(\mathbf{d}\theta^2+\sin^2(\theta)\,\mathbf{d}\phi\right). \tag{7.13}$$

For the degenerate case $a=c$ the boundary equations (6.2) will become the boundary equations for a sphere, we found it useful to define the EM tensor with respects to orthogonal basis, so that we can check our result with the results provided in [Bára, 2017]. Thus identify the Faraday tensor and the 4-current as

$$F^{ab}=\begin{pmatrix} 0 & E_r & \frac{E_\theta}{r} & \frac{E_\phi}{r\sin(\theta)} \\ -E_r & 0 & \frac{B_\phi}{r} & -\frac{B_\theta}{r\sin(\theta)} \\ -\frac{E_\theta}{r} & -\frac{B_\phi}{r} & 0 & \frac{B_r}{r^2\sin(\theta)} \\ -\frac{E_\phi}{r\sin(\theta)} & \frac{B_\theta}{r\sin(\theta)} & -\frac{B_r}{r^2\sin(\theta)} & 0 \end{pmatrix}, \tag{7.14}$$

$$j^\mu=\sigma\left(F^{\mu\nu}u_\nu-u^\mu F^{\gamma\nu}u_\gamma u_\nu\right)-\rho u^\mu. \tag{7.15}$$

Now if we consider the equations (6.2) we get six linearly independent equations

$$\begin{aligned}
[B_\phi]=&-\frac{2\mathfrak{AC}a^2c^4e^2\mu\sigma\left(c^2+e^2\right)\csc^2(\theta)\sec(\theta)\left(\mathrm{E}_r\sqrt{\frac{1}{a^2}+\frac{1}{c^2}}-\mathrm{B}_\theta\Omega\right)}{\left(4\left(2c^2+e^2\right)^2\sec^2(\theta)\left(c^2+e^2\cos^2(\theta)\right)^2+4c^4e^4\left(c^2+e^2\right)^2\csc^2(\theta)\right)},\\
&-\frac{2\mathfrak{AC}a^2c^2\mu\sigma\left(2c^2+e^2\right)\csc(\theta)\sec^2(\theta)\left(c^2+e^2\cos^2(\theta)\right)\left(\mathrm{E}_\theta\sqrt{\frac{1}{a^2}+\frac{1}{c^2}}+\mathrm{B}_r\Omega\right)}{\left(4\left(2c^2+e^2\right)^2\sec^2(\theta)\left(c^2+e^2\cos^2(\theta)\right)^2+4c^4e^4\left(c^2+e^2\right)^2\csc^2(\theta)\right)}\\
[E_r]=&\frac{\mathfrak{AC}}{\epsilon}\left(a^2+\mathrm{cc}^2+e^2\cos(2\theta)\right)\left(\frac{\mathrm{E}_\phi\sigma\Omega}{\sqrt{\frac{1}{a^2}+\frac{1}{\mathrm{c}^2}}}+\rho\right),\\
[B_\theta]=&\mathfrak{AC}\mu\left(a^2+\mathrm{cc}^2+e^2\cos(2\theta)\right)\left(\frac{\rho\Omega}{\sqrt{\frac{1}{a^2}+\frac{1}{\mathrm{cc}^2}}}+\mathrm{E}\phi\sigma\right),\\
[E_\theta]=&-\mathfrak{AC}\frac{e^2\sin(2\theta)\csc^2(\theta)\left(\frac{\mathrm{E}_\phi\sigma\Omega}{\sqrt{\frac{1}{a^2}+\frac{1}{\mathrm{cc}^2}}}+\rho\right)}{\epsilon\left(\frac{1}{a^2}+\frac{1}{\mathrm{cc}^2}\right)},\\
[B_r]=&\mathfrak{AC}\frac{e^2\mu\sin(2\theta)\csc^2(\theta)\left(\frac{\rho\Omega}{\sqrt{\frac{1}{a^2}+\frac{1}{\mathrm{cc}^2}}}+\mathrm{E}_\phi\sigma\right)}{\left(\frac{1}{a^2}+\frac{1}{\mathrm{cc}^2}\right)},\quad [E_\phi]=0,
\end{aligned} \tag{7.16}$$

where we introduced

$$
\begin{aligned}
&\mathfrak{A} = \frac{1}{\sqrt{2\left(a^4 - \mathrm{c}^4\right)\cos(2\theta) + \left(a^2 + \mathrm{c}^2\right)^2 + \frac{4\mathrm{c}^4 e^4 (\mathrm{c}^2 + e^2)^2 \cot^2(\theta)}{\left(a^2 + \mathrm{c}^2\right)^2} + e^4 \cos^2(2\theta)}}, \\
&\mathfrak{C} = \frac{1}{\sqrt{1 - \frac{\Omega^2}{\frac{1}{a^2} + \frac{1}{c^2}}}}, \\
&e^2 = a^2 - c^2.
\end{aligned}
\tag{7.17}
$$

The parameter e represent the linear eccentricity of the spheroid. Let us take a closer look at the equations (7.16), the writing out the lattermost equation

$$
[E_\phi] = -ic\left(\frac{[g_l(kr)]}{kr} + \frac{d[g_l(kr)]}{dr}\right)\frac{\partial \mathrm{Y}_{lm}}{d\phi} + [f_l(kr)]\frac{\partial \mathrm{Y}_{lm}}{\partial \theta} = 0 \tag{7.18}
$$

where g_l, f_l are the functions (7.8). In order to get individual coefficients we would like to employ the orthogonality of spherical harmonics.

$$
\begin{gathered}
\int E_\phi \overline{Y_{l'm'}} d\Omega = \int_0^\pi \int_0^{2\pi} r^2 \sin(\theta) E_\phi \overline{Y_{l'm'}} d\phi d\theta, \\
\text{where} \quad r = \frac{\sqrt{2}ac}{\sqrt{2c^2 + \epsilon\cos(2\theta) + \epsilon}}.
\end{gathered}
\tag{7.19}
$$

This is the easiest of the equations (7.16) yet we weren't able to analytically solve such equations. Furthermore in [Alkhoori et al., 2018], where the case $\Omega = 0$ is considered, it is said that such integral only has analytical solution for the degenerate case $a = c$. Since we know not of any procedure for solving the equations (7.16) for the individual coefficients we can only state that we have found no solution. Of course the equations can be solved numerically, or should be solvable by considering small eccentricity parameter and then expanding it up to the n-th order.

Similar problems would surely arise in we were to formulate the problem on the "C-metric object" in spherical coordinates, we can only state that attempt to solve such problem analytically is not viable in our current understanding of the problem.

Conclusion

Having acquainted ourselves with the problem of superradiance on a sphere and on a cylinder, we provided argument against the utilizing non-accelerated spherical coordinates to describe the problem. Even the simplest problem that does not follow the azimuthal symmetry of spherical coordinates leads to equations with no analytical solution. This has been demonstrated by examining is the problem of superradiance on a ellipsoid.

We have provided a the (modified) Zel'dovich thermodynamical arugment of radiation superradiance for an axially symmetric conductor.

We further consider accelerated coordinates which could describe axially symmetric rotating objects on which the problem could be formulated, this lead to the C-metric. Which was introduced and the problem was formulated on constant coordinate surfaces of the C-metric. We shortly summarized the C-metric along with a electromagnetic "magic field" which together solve the Einstein-Maxwell equations.

In order to find a suitable electromagnetic field we provided a brief introduction the Newman-Penrose formalism and the Geroch–Held–Penrose formalism, both special cases of the tetrad formalisms. In these two formalism we attempted to find a suitable electromagnetic field. By generating the field from the Debye potential we arrived at two equations which consisted of the Sturm–Liouville problem. We were not able to solve these equations.

No analytical solution to the superradiance on accelerated systems was found, but we have augmented against the possibility of such an solution existing in non-accelerated coordinates and we laid the ground-work for further calculations if the already mentioned Sturm–Liouville was ever solved.

Bibliography

Hamad Alkhoori, Akhlesh Lakhtakia, James Breakall, and Craig Bohren. Plane-wave scattering by an ellipsoid composed of an orthorhombic dielectric–magnetic material. *Journal of the Optical Society of America A*, 35:1549, 09 2018. doi: 10.1364/JOSAA.35.001549.

Jacob D. Bekenstein. Extraction of Energy and Charge from a Black Hole. *Phys. Rev. D*, 7:949–953, Feb 1973. doi: 10.1103/PhysRevD.7.949. URL `https://link.aps.org/doi/10.1103/PhysRevD.7.949`.

Jacob D. Bekenstein and Marcelo Schiffer. The many faces of superradiance. *Phys. Rev. D*, 58:064014, Aug 1998. doi: 10.1103/PhysRevD.58.064014. URL `https://link.aps.org/doi/10.1103/PhysRevD.58.064014`.

Jiří Bičák and David Kofroň. Accelerating electromagnetic magic field from the C-metric. *General Relativity and Gravitation*, 41(9):1981–2001, may 2009. doi: 10.1007/s10714-009-0816-8. URL `https://doi.org/10.1007%2Fs10714-009-0816-8`.

Robert H. Boyer and Richard W. Lindquist. Maximal analytic extension of the kerr metric. *Journal of Mathematical Physics*, 8(2):265–281, 1967. doi: 10.1063/1.1705193. URL `https://doi.org/10.1063/1.1705193`.

Richard Brito, Vitor Cardoso, and Paolo Pani. *Superradiance*. Springer International Publishing, 2020. doi: 10.1007/978-3-030-46622-0. URL `https://doi.org/10.1007%2F978-3-030-46622-0`.

Václav Bára. Řešení polních rovnic se superradiací, bakalářská práce, 2014.

Václav Bára. Elektromagnetická rotační superradiace, diplomová práce, 2017.

Jose Hector Caltenco, Jon Bonilla, and Arturo Zúñiga-Segundo. Lanczos Spintensor and GHP formalism. *Czechoslovak Journal of Physics - CZECH J PHYS*, 52:901–909, 08 2002. doi: 10.1023/A:1019803803431.

B Carrascal, G Estevez, Peilian Lee, and V Lorenzo. Vector spherical harmonics and their application to classical electrodynamics. *European Journal of Physics*, 12:184, 07 2000. doi: 10.1088/0143-0807/12/4/007.

S. Chandrasekhar. *The Mathematical Theory of Black Holes*. International series of monographs on physics. Clarendon Press, 1998. ISBN 9780198503705. URL `https://books.google.cz/books?id=LBOVcrzFfhsC`.

J.M. Cohen and L.S. Kegeles. Space-time perturbations. *Physics Letters A*, 54 (1):5–7, 1975. ISSN 0375-9601. doi: https://doi.org/10.1016/0375-9601(75)90583-6. URL `https://www.sciencedirect.com/science/article/pii/0375960175905836`.

Krishan Duggal and Ramesh Sharma. *Symmetries of Spacetimes and Riemannian Manifolds*. 01 1999. ISBN 978-1-4613-7425-1. doi: 10.1007/978-1-4615-5315-1.

Robert P. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. *J. Math. Phys.*, 14:874–881, 1973. doi: 10.1063/1.1666410.

J. Goldberg and Rainer Sachs. Republication of: A theorem on Petrov types. *General Relativity and Gravitation*, 41:433–444, 02 2009. doi: 10.1007/s10714-008-0722-5.

J. Griffiths, P. Krtous, and J. Podolsky. Interpreting the C-metric. *Classical and Quantum Gravity*, 23, 09 2006. doi: 10.1088/0264-9381/23/23/008.

Alfonso García-Parrado Gómez-Lobo and José M. Martín-García. Spinors: A Mathematica package for doing spinor calculus in General Relativity. *Computer Physics Communications*, 183(10):2214–2225, 2012. ISSN 0010-4655. doi: https://doi.org/10.1016/j.cpc.2012.04.024. URL `https://www.sciencedirect.com/science/article/pii/S0010465512001634`.

Yakov Itin. Covariant jump conditions in electromagnetism. *Annals of Physics*, 327(2):359–375, feb 2012. doi: 10.1016/j.aop.2011.09.005. URL `https://doi.org/10.1016%2Fj.aop.2011.09.005`.

John David Jackson. *Classical electrodynamics*. Wiley, New York, NY, 3rd ed. edition, 1999. ISBN 9780471309321. URL `http://cdsweb.cern.ch/record/490457`.

David Kofron. Separability of test fields equations on the C-metric background. *Physical Review D*, 92(12), dec 2015. doi: 10.1103/physrevd.92.124064. URL `https://doi.org/10.1103%2Fphysrevd.92.124064`.

David Kofroň. Separability of test fields equations on the *C*-metric background. II. Rotating case and the Meissner effec. *Phys Rev D*, 93(10), may 2016. doi: 10.1103/physrevd.93.104012. URL `https://doi.org/10.1103%2Fphysrevd.93.104012`.

Vladislav V. Kravchenko and R. Michael Porter. Spectral parameter power series for sturm-liouville problems. 2008. doi: 10.48550/ARXIV.0811.4488. URL `https://arxiv.org/abs/0811.4488`.

D. S. Kulyabov and A. V. Korolkova. Maxwell's equations in arbitrary coordinate system. 2012. doi: 10.48550/ARXIV.1211.6590. URL `https://arxiv.org/abs/1211.6590`.

Kvasnica. Teorie elektromagnetického pole. *Academia Praha*, 1985.

J. Luscombe. *Thermodynamics*. CRC Press, 2018. ISBN 9780429017872. URL `https://books.google.cz/books?id=vshUDwAAQBAJ`.

Donald Lynden-Bell. *A magic electromagnetic field*, page 369–376. Cambridge University Press, 2003. doi: 10.1017/CBO9780511536335.026.

Robert S. Maier. The 192 solutions of the Heun equation. *Mathematics of Computation*, 76(258):811–843, nov 2006. doi: 10.1090/s0025-5718-06-01939-9. URL `https://doi.org/10.1090%2Fs0025-5718-06-01939-9`.

David Miškovský. Mathematical methods and exact spacetimes in quadratic gravity, 2021.

Frank Olver, Daniel Lozier, Ronald Boisvert, and Charles Clark. *The NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010-05-12 00:05:00 2010.

Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time*, volume 2 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 1986. doi: 10.1017/CBO9780511524486.

Ekman Robin. *The GHP formalism, with applications to Petrov type III spacetimes*. PhD thesis, 2014. URL `http://urn.kb.se/resolve?urn=urn:nbn:se:umu:diva-88352`.

Saul A Teukolsky. Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations. *The Astrophysical Journal*, 185:635–648, 1973.

Robert M. Wald. *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984. doi: 10.7208/chicago/9780226870373.001.0001.

Barry Wardell and Chris Kavanagh. Separable electromagnetic perturbations of rotating black holes. *Physical Review D*, 103(10), may 2021. doi: 10.1103/physrevd.103.104049. URL `https://doi.org/10.1103%2Fphysrevd.103.104049`.

L. Witten. *Gravitation: An Introduction to Current Research*. Wiley, 1962. URL `https://books.google.cz/books?id=3llAAAAAIAAJ`.

Ya. B. Zel'dovich. Amplification of cylindrical electromagnetic waves reflected from a rotating body. 1972.

List of Figures