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Filip Strakoš

Legendrian submanifolds in high-dimensional contact topology

Mathematical Institute of Charles University

Supervisor of the master thesis: Roman Golovko, Ph.D.
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Title: Legendrian submanifolds in high-dimensional contact topology

Author: Filip Strakoš

Institute: Mathematical Institute of Charles University

Supervisor: Roman Golovko, Ph.D., Mathematical Institute of Charles University

Abstract: This thesis deals with results concerning both flexible and rigid parts of contact topology. Basic notions of contact topology and constructions of higher-dimensional Legendrian submanifolds are stated. There is proved the existence of infinite family of pair-wise Legendrian non-isotopic loose Legendrian embeddings of 3-torus so that each embedding is not a Legendrian product of lower-dimensional tori. In the rest of the text, the bilinearized Legendrian contact homology invariant is described and the criterion for DGA-homotopy of augmentation of Chekanov-Eliashberg algebra for disconnected Legendrian submanifolds is proved.

Keywords: contact topology, Legendrian submanifolds, Legendrian product, augmentation of Chekanov-Eliashberg algebra

Contents

Introduction

Consider $\mathbb{R}^3$ with standard coordinates (x, y, z) and hyperplane distribution ξ spanned at each point by the tangent vectors

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \ \frac{\partial}{\partial y},$$

that is ξ is a kernel of 1-form $\alpha = dz - ydx$.

The Legendrian knot Λ is an image of an embedding $f : \mathbf{S}^1 \to \mathbb{R}^3$ so that $f^*(\alpha) = 0$, in other words, the tangent space of Λ is a sub-bundle of ξ restricted to Λ. Two Legendrian knots are said to be Legendrian isotopic if they are isotopic through Legendrian knots. The classification of Legendrian knots up to Legendrian isotopy has been the subject of intensive research in recent years. Therefore, the class of Legendrian knots is a reasonably well-understood and rich subclass of smooth knots.

This thesis will focus on the n-dimensional equivalent of Legendrian knots, so-called Legendrian submanifolds, in contact manifolds of dimension $2n+1$ for n any natural number greater than 1. We will be again interested in the classification of Legendrian submanifolds up to Legendrian isotopy, which has not been as intensively studied as its low-dimensional counterpart.

We will see that the family of all Legendrian submanifolds of a contact manifold is a rich class of submanifolds whose properties sometimes have serious implications on the ambient contact structure of the contact manifold.

Moreover, as it is quite symptomatic for the field, there is a difference between two important subfamilies of Legendrian submanifolds; one family performs some form of flexibility, in contrast, the other family consists of Legendrian submanifolds which are considered to be in some sense rigid. This dichotomy is essential, and it is reflected in the nature of the invariants used to study the corresponding classes.

The flexible family is studied using invariants coming from homotopy properties with respect to the Legendrian condition. Those are so-called classical invariants. Nevertheless, there are infinite families of rigid Legendrian submanifolds that have the same classical invariants and belong to distinct Legendrian isotopy classes, and so the classical invariants are insufficient.

The other, rigid class, can be studied using the homological invariants coming from the frame of the symplectic field theory. The vanishing of those invariants presents the necessary condition for being from the flexible family though the other implication might not hold in general.

We will see some basic constructions of higher-dimensional Legendrians from the lower ones, particularly various product constructions. We will also mention a particular kind of surgery that will enable us to attach handles to our original Legendrian submanifold.

One of the main results of this thesis is to prove a limitation of the Legendrian product construction using the classical invariants. The other original contribution lies on the rigid side and describes the DGA-homotopy criterion for augmentations of Chekanov-Eliashberg algebra of disconnected Legendrian submanifolds. To prove the criterion, we need results about the effect of a particular kind of surgery on the homological invariant mentioned above.

As far as the organisation of this thesis is considered, in Chapter 1, basic definitions and examples are introduced. In particular, the class of Whitney spheres and the class of loose Legendrian submanifolds.

In Chapter 2, the constructions of Legendrian submanifolds are introduced. With a general overview of recent developments in this direction.

In Chapter 3, the first invariants aiming to distinguish Legendrian isotopy classes are introduced in the setting of odd-dimensional Euclidean space.

One of the original contributions can be found in Chapter 4, where we use the rotation class to prove that a particular family of loose Legendrian embeddings of 3-torus is not a Legendrian product of lower-dimensional Legendrian tori.

In the last Chapter 5, the Chekanov-Eliashberg algebra and the notion of its augmentation are introduced, and classical technical tools and theory concerning this invariant is outlined. In addition, the second original contribution can be found there. More specifically, it is the criterion of differential graded algebra homotopy of augmentations of the Chekanov-Eliashberg algebra for disconnected Legendrian submanifolds. At the end of Chapter 5, a brief discussion on various Künneth formulas for the Legendrian contact homology is located.

1. Basic notions

In this whole thesis by a manifold we mean a smooth differentiable manifold, if not stated otherwise. The tangent bundle of the manifold M is to be denoted TM and for a smooth map of manifolds $f : M \to N$ we denote by $Tf : TM \to TN$ the tangent map. Moreover, in what follows n is an arbitrary natural number if not stated otherwise.

1.1 Contact and symplectic manifolds

Definition 1.1.1. *Let W be a manifold of dimension $2n$. A **symplectic form** is a globally defined 2-form ω so that*

i) $d\omega = 0$ (that is ω is a closed form),
ii) for each point $q \in W$ the bilinear form ω_q is non-degenerate.

*A **symplectic manifold** is the pair (W, ω). The symplectic manifold (P, η) is called an **exact symplectic manifold** if there is a globally defined 1-form λ so that $d\lambda = \eta$.*

Here note that every exact symplectic manifold P is not a closed manifold due to topological reasons.

Example 1.1.2. (The standard symplectic structure on $\mathbb{R}^{2n}$) Consider the Euclidean coordinates on $\mathbb{R}^{2n}$ denoted by

$$(x_1, y_1, \ldots, x_n, y_n,).$$

The Euclidean space $\mathbb{R}^{2n}$ equipped with the **standard symplectic form** given by the global non-degenerate closed 2-form

$$\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

is a basic example of a symplectic manifold.

Definition 1.1.3. *Let M be a manifold of dimension $2n+1$. A **contact structure** is a non-integrable subbundle ξ of TM of the maximal rank that is given locally by kernel of a global 1-form α, that is*

$$\alpha \wedge (d\alpha)^{\wedge n} \neq 0 \tag{1.1}$$

*where the inequality means that the wedge product is a non-vanishing form. Such a 1-form is called a **contact form** and the pair (M, ξ) is called a **contact manifold**.*

Remark. In this thesis, we will consider only the co-oriented contact structures. More specifically, we require that the contact form α is defined globally on M.

Let us remark that the contact structure ξ depends only on the equivalence class for the form α, where α and α' are equivalent if and only if there exists a non-zero real function $f : M \to \mathbb{R}$ so that $\alpha = f\alpha'$. Moreover, every contact manifold is orientable because $\alpha \wedge (d\alpha)^{\wedge n}$ is a globally non-vanishing $2n+1$ form. When we want to refer to a contact manifold with a particular choice of the contact form, we write $(M, \xi = \ker \alpha)$.

Example 1.1.4. (The standard contact structure on $\mathbb{R}^{2n+1}$) Let us consider the Euclidean coordinates

$$(x_1, y_1, \dots, x_n, y_n, z).$$

Moreover, let us have the global 1-form

$$\alpha_{st} = dz - \sum_{j=1}^{n} y_j dx_j.$$

First let us compute the exterior differential of α_{st}:

$$d\alpha_{st} = d\Big(dz - \sum_{j=1}^{n} y_j dx_j\Big) = ddz - \sum_{j=1}^{n}(dy_j \wedge dx_j + y_j ddx_j) = \sum_{j=1}^{n} dx_j \wedge dy_j = \omega_{st},$$

where ω_{st} is a standard symplectic structure on $\mathbb{R}^{2n}$. Hence,

$$(d\alpha_{st})^n = \omega_{st}^n = n!\; dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n = n!\; \mathrm{dvol}_{\mathbb{R}^{2n}}$$

and so $\alpha_{st} \wedge (d\alpha_{st})^n = (-1)^{2n} n!\; dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \wedge dz = n!\; \mathrm{dvol}_{\mathbb{R}^{2n+1}}$ which implies that the product is non-vanishing since it is equal to a positive multiple of the canonical volume form on $\mathbb{R}^{2n+1}$. We have just shown that α_{st} is a contact form. This is the so-called **standard contact form**. The **standard contact structure** ξ_{st} is then given by the kernel of α_{st}, that is it is trivial bundle with a trivialization induced by smooth vector fields

$$\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z}, \dots, \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial z}.$$

The vector field $\frac{\partial}{\partial z}$ is a uniquely defined unit vector field transverse to the contact structure later defined to be the Reeb vector field $R_{\alpha_{st}}$ associated with α_{st}.

▲

The contact geometry is considered to be an odd-dimensional sibling of the symplectic geometry. In effect, there are several construction to relate one to the other.

Definition 1.1.5. *Let $(P, d\eta)$ be an exact symplectic manifold. The* ***contactization*** *of P is a contact manifold $(\mathbb{R} \times M, dz + \eta)$.*

Let us note that there is equivalent definition, where the contact form differs in sign, that is $dz - \eta$. We will use both of those definitions.

Example 1.1.6. (Cotangent bundle T^*M and one-jet space $J^1(M)$) Let M be any manifold of dimension n and let $q_1, \dots, q_n$ be any local coordinates on M, then there are defined coordinates $p_1, \dots, p_n$ on the fibre of T^*M over this coordinate neighbourhood as the coordinates with respect to the dual basis to $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ the basis of the fibre of TM. It is straightforward to show that $d\eta$, where

$$\eta = \sum_{j=1}^{n} p_j dq_j,$$

is gives rise to a symplectic structure on T^*M which is by definition exact.

It is a standard fact that $J^1(M)$ is diffeomorphic to $\mathbb{R} \times T^*M$. If we denote by z the coordinate on the $\mathbb{R}$ factor, then it is clear that $(J^1(M), dz + \eta)$ is a contactization of T^*M. ▲

There is also a construction going in the opposite direction.

Definition 1.1.7. *A* ***symplectization*** *of a contact manifold* $(M, \xi = \ker \alpha)$ *is a manifold* $(\mathbb{R} \times M, d(e^t p^*(\alpha)))$*, where* $p : \mathbb{R} \times M \to M$ *is the canonical projection and* t *is a global coordinate on the* $\mathbb{R}$ *factor.*

Let us consider (R^{2n+1}, α_2) with the same coordinates as above but with a different contact form given by

$$\alpha_2 = dz + \sum_{j=1}^{n} (x_j dy_j - y_j dx_j).$$

The contact structure $\xi_2 = \ker \alpha_2$ is given by the span of vector fields

$$y_j \frac{\partial}{\partial y_j} + x_j \frac{\partial}{\partial x_j}, x_j y_j \frac{\partial}{\partial z} - y_j \frac{\partial}{\partial y_j} \text{ for } j = 1, \ldots, n.$$

One might get the impression that ξ_{st} and ξ_2 are different, however, they are basically the same in the following sense.

Definition 1.1.8. *Let* (M_i, ξ_i) *be contact manifolds for* $i = 1, 2$ *and* $\varphi : M_1 \to M_2$ *be a diffeomorphism. The map* φ *is called a* ***contactomorphism*** *if* $T\varphi(\xi_1) = \xi_2$*. If there is such a map, then* (M_i, ξ_i) *are said to be* ***contactomorphic****.*

Suppose $\xi_j = \ker \alpha_j$ for $j = 1, 2$. If $\mathbb{X} \in \xi_1$ is a vector field, in effect, $\alpha_1(\mathbb{X}) = 0$, then $\varphi^*\alpha(\mathbb{X}) = \alpha_2(\mathrm{T}\,\varphi(\mathbb{X})) = 0$. Then the condition of φ being a contactomorphism translates to $\ker \varphi^*\alpha_2 = \ker \alpha_1$ and so forms $\varphi^*\alpha_2$ and α_1 should induce the same contact structure ξ_1. One easily proves that there exists a smooth function $f : M \to \mathbb{R} \setminus \{0\}$ so that $\varphi^*\alpha_2 = f\alpha_1$.

Let us return to the Euclidean space $\mathbb{R}^{2n+1}$ equipped with forms α_{st} and α_2 as above. Observe that the map $f : (\mathbb{R}^{2n+1}, \alpha_{st}) \to (\mathbb{R}^{2n+1}, \alpha_2)$ given by

$$x_j \mapsto -\frac{x_j + y_j}{2}, \qquad y_j \mapsto \frac{x_j - y_j}{2}, \qquad z \mapsto z - \frac{1}{2} \sum_{j=1}^{n} x_j y_j, \text{ for } j = 1, \ldots, n,$$

is a contactomorphism of ξ_{st} and ξ_2.

Now, let us give an example of a contact structure on the Euclidean space which is not contactomorphic to the standard one.

Example 1.1.9. (Standard overtwisted contact structure) Consider the Euclidean space $\mathbb{R}^3$ with equipped with cylindrical coordinates (r, φ, z) and define

$$\alpha_{ot} = \cos r dz + r \sin r d\varphi.$$

Let us take for granted that α_{ot} is a contact form. The contact structure ξ_{ot} given by the kernel of α_0 is called the **standard overtwisted contact structure** on $\mathbb{R}^3$. Let us point out that $(\mathbb{R}^3, \xi_{st})$ and $(\mathbb{R}^3, \xi_{ot})$ are not contactomorphic.

▲

To distinguish between two symplectic structures we define similarly the following class of maps.

Definition 1.1.10. *Let (W_i, ω_i) be symplectic manifolds for $i = 1, 2$ and $\psi : W_1 \to W_2$ be a diffeomorphism. The map ψ is called a **symplectomorphism** if $\psi^* \omega_2 = \omega_1$. If there is such a map, then (W_i, ω_i) are said to be **symplectomorphic**.*

Example 1.1.11. Consider $\mathbf{S}^{2n+1} \subset \mathbb{R}^{2n+2}$ with the Euclidean coordinates

$$(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$$

and the contact form

$$\alpha_0 = \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j).$$

To prove the contact condition for α_0, observe that if we denote

$$\mathbb{Y} = \sum_{j=1}^{n+1} (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}),$$

then $\alpha_0 = \iota_{\mathbb{Y}} \omega_{st}$, where $\omega_{st} = \sum_{j=1}^{n+1} dx_j \wedge dy_j$ is the standard symplectic form on $\mathbb{R}^{2n+2}$. Moreover, let $r^2 = \sum_{j=1}^{n+1} (x_j^2 + y_j^2)$ so that $r = \sqrt{r^2}$ is the radial coordinate on $\mathbb{R}^{2n+2}$, then

$$\frac{\partial}{\partial r} = \frac{1}{2r} \sum_{j=1}^{n+1} 2(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j})$$

and so $\mathbb{Y} = r \frac{\partial}{\partial r}$. Now, realise that $\mathbf{S}^{2n+1}$ is the level hypersurface given by $r = 1$ and thus the vector field $\frac{\partial}{\partial r}$ is transversal to $T\mathbf{S}^{2n+1}$. In addition,

$$d(\iota_{\mathbb{Y}} \omega_{st}) = d\alpha_0 = \sum_{j=1}^{n+1} (dx_j \wedge dy_j - dy_j \wedge dx_j) = 2\omega_{st}.$$

Hence we can finally compute that

$$\alpha_0 \wedge (d\alpha_0)^n = \iota_{\mathbb{Y}} \omega_{st} \wedge (d(\iota_{\mathbb{Y}} \omega_{st}))^n = 2^n \iota_{\mathbb{Y}} \omega_{st} \wedge \omega_{st}^n = \frac{2^n}{n+1} \iota_{\mathbb{Y}} (\omega_{st}^{n+1}), \tag{1.2}$$

where in the last equality we have used the following formula

$$\iota_{\mathbb{Y}} (\omega_{st}^{n+1}) = (n+1) \iota_{\mathbb{Y}} \omega_{st} \wedge \omega_{st}^n. \tag{1.3}$$

Formula (1.3) can be easily proven by induction on $0 \leq l \leq n$ the l-th iteration of application the Leibniz rule for interior product $\iota_{\mathbb{Y}}$, where for l-th iteration we obtain that

$$\iota_{\mathbb{Y}}(\omega_{st}^{n+1}) = l(\iota_{\mathbb{Y}}\omega_{st} \wedge \omega_{st}^{n}) + \omega_{st}^{l} \wedge \iota_{\mathbb{Y}}(\omega_{st}^{n-l+1}).$$

Then the equation (1.2) implies that the form $\alpha_0 \wedge (d\alpha_0)^n$ restricted to the tangent bundle of $\mathbf{S}^{2n+1}$ is indeed a volume form, therefore, the contact condition for α_0 is satisfied.

▲

One can show that $(\mathbf{S}^{2n+1} \setminus \{p\}, \ker \alpha_0)$ where $p \in \mathbf{S}^{2n+1}$ is any point and $(\mathbb{R}^{2n+1}, \xi_{st})$ are also contactomorphic. This fact might raise a natural question whether any contact manifold is locally contactomorphic to the Euclidean space with the standard contact structure. The following Darboux's theorem affirms our suspicion in both contact and symplectic setting.

Theorem 1.1.12 (Darboux's Theorem, Theorem 2.5.1 in [29] and Theorem 3.2.2 in [38])**.** *Let $(M, \xi = \ker \alpha), (W, \omega)$ be a contact and a symplectic manifolds of dimensions $2n+1$ and $2n$, respectively. Then it holds that;*

i) for each $p \in M$ there is a neighbourhood $U \subset M$ of p and a choice of coordinates $\varphi : U \to \mathbb{R}^{2n+1}$ so that $\varphi(p) = (0, \ldots, 0)$ and

$$(\varphi^{-1})^*(\alpha|_U) = dz - \sum_{j=1}^{n} y_j dx_j = \alpha_{st},$$

ii) for each $q \in W$ there is a neighbouhood $q \subset W$ of q and a choice of coordinates $\psi : U \to \mathbb{R}^{2n}$ so that $\psi(p) = (0, \ldots, 0)$ and

$$(\psi^{-1})^*(\omega) = \sum_{j=1}^{n} dx_j \wedge dy_j = \omega_{st},$$

where in both cases we consider the standard Euclidean coordinates on the Euclidean spaces.

The important implication of the previous theorem is that there is a model for local behaviour of both contact and symplectic manifolds. Therefore, when equipped just with the ambient structures those families of manifolds do not posses any local invariants. Consequently, the study of corresponding structures must be done at the global level.

1.2 Reeb vector field dynamics and the space of contact structures

We defined the contact structure ξ to be given globally as $\xi = \ker\alpha$ where α is a one-form. In is also the complement of ξ in TM that plays major role in the contact topology.

Definition 1.2.1. *Let $(M, \xi = \ker\alpha)$ be a contact manifold. The unique vector field R_α defined by the equations*

i) $d\alpha(R_\alpha, \cdot) \equiv 0,$
ii) $\alpha(R_\alpha) \equiv 1.$

is the so-called ***Reeb vector field*** *associated with a contact form α,*

The existence is clear since the defining form α induces a global section of the dual bundle to TM/ξ. The second normalisation condition implies the uniqueness since both the dimension of TM/ξ and the dimension of its dual is equal to 1.

Definition 1.2.2. *(Isotopy) Let M, N be manifolds and $f_0, f_1 : M \to N$ be two embeddings. The smooth map $f : [0,1] \times M \to N$, equivalently, the smooth family $(f_t)_{t\in[0,1]}$, is said to be an* ***isotopy*** *of f_0 and f_1, if it holds that*

i) $\forall t \in [0,1]$ *the map* $f(t, -) : M \to N$ *is an embedding,*
ii) if $t = 0$*, then* $f(0, -) = f_0$*,*
iii) if $t = 1$*, then* $f(1, -) = f_1$*.*

If $M = N$, then we say that $(f_t)_{t\in[0,1]}$ is an ambient isotopy of M.

It is a well known fact than on a compact manifold the flow of a time-dependent vector field induces an ambient isotopy of the manifold (for example see [34], Chapter 8, Theorem 1.2). Thus it is a natural to ask whether a smooth family of Reeb vector fields induces an ambient isotopy going through contactomorphisms. This property is called stability and arbitrary smooth family of neither Reeb vector fields nor a family of contacts forms satisfy it (see [29], Chapter 2, Example 2.2.5). However, the following theorem due to Gray says that the stability property holds for contact structures.

Theorem 1.2.3. *(Gray stability theorem, Theorem 2.2.2 in [29]) Let M be a closed manifold. Consider $(\xi_t)_{t\in[0,1]}$ a smooth family of contact structures on M. Then there is $(f_t)_{t\in[0,1]}$ an isotopy of M such that for each $t \in [0,1]$*

$$Tf_t(\xi_0) = \xi_t.$$

Let $\Xi(M)$ be the set of all contact structures of a closed manifold M. Consider the Grassmannian manifold of codimension one subbundles of TM, which endowed with a C^k topology for $k \geq 1$ is a Fréchet manifold, and $\Xi(M)$ is its open subset (see Section 2.4 of [33] for more details). Then the Gray stability theorem in other words says that elements of the same path component in $\Xi(M)$ are contactomorphic and that it makes sense to study the topology $\Xi(M, \xi_0)$ the path component of a contact structure ξ_0.

Example *1.2.4.* Consider a torus $\mathbf{T}^3$ with coordinates (θ, x, y) and a contact structure ξ_n given as a kernel of $\alpha_n = \cos(n\theta)dx + \sin(n\theta)dy$. It is due to Geiges and Gonzalo (see [30]) that $\pi_1(\Xi(\mathbf{T}^3, \xi_n))$ contains a non-contractible circle for any $n \geq 1$.

On the other hand, let us have $\mathbf{S}^3$ with the contact structure ξ_0 defined above, then the space $\Xi(\mathbf{S}^3, p) \subseteq \Xi(\mathbf{S}^3, \xi_0)$ of contact structures ξ so that $\xi_p = (\xi_0)_p$ was shown to be contractible by Eliashberg (see [21]). It is a recent result of Eliashberg and Mishachev (see [24]) that $\Xi(\mathbf{S}^3, \xi_0)$ and even $\Xi(\mathbb{R}^3, \xi_{st})$ are contractible.

▲

To conclude, let us remark that the condition of M being closed is important, since Eliashberg proved in [20] that the stability property does not hold for contact structures on $\mathbf{S}^1 \times \mathbb{R}^2$.

1.3 Symplectic bundles, isotropic submanifolds and compatible structures

Definition 1.3.1. *Let $\eta = E \to B$ be any bundle,* ***symplectic bundle structure*** $\omega \in \Gamma(B, \bigwedge^2 \operatorname{Hom}(E, \mathbb{R}))$ *on η is the smooth section ω of the bundle*

$$\bigwedge_{j=1}^{2} (\operatorname{Hom}(E, \mathbb{R})) \to B$$

so that it is a symplectic form on each fibre.

Let $(M, \xi = \ker \alpha)$ be a $2n+1$ dimensional contact manifold. Now, let us digress on the equivalent formulation of the contact condition (1.1) using symplectic structure on ξ. For every $p \in M$ the kernel of α_p defines a $2n$-dimensional subspace ξ_p of $\mathrm{T}_p M$. Consider X a subset of $\mathrm{T}_p M$ which has $2n+1$ elements. For $\alpha \wedge (d\alpha)^n \neq 0$ to be true when we evaluate the product on the elements of the set X it must hold that all the elements of X must be linearly independent. Now if $\mathbb{X}_0$ is in the span of R_α and if we pick $\mathbb{X}_1, \dots, \mathbb{X}_{2n} \in \xi_p$ and $\pi \in \mathrm{S}_{2n+1}$ a permutation of $2n+1$ letters, then

$$\alpha \wedge (d\alpha)^{\wedge n}(\mathbb{X}_{\pi(0)}, \mathbb{X}_{\pi(1)}, \dots, \mathbb{X}_{\pi(2n)}) \neq 0$$

if and only if 0 is a fix point of π and the elements from ξ_p are linearly independent. Therefore, the contact condition (1.1) is equivalent to the requirement that $(d\alpha)^n|_\xi$ is non-vanishing. When we interpret this locally, it means that the 2-form $d\alpha_p|_{\xi_p}$ is skew-symmetric and non-degenerate and so $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle.

Definition 1.3.2. *We say that a bundle map $(\mathrm{id}_M, J) : \xi \to \xi$ is a* ***complex bundle structure*** *on ξ* ***compatible with*** *$d\alpha|_\xi$ if the following holds*

i) $J^2 = -\mathrm{id}_\xi$,
ii) for all $\mathbb{X}, \mathbb{Y} \in \xi$ we have that $d\alpha(J\mathbb{X}, J\mathbb{Y}) = d\alpha(\mathbb{X}, \mathbb{Y})$,
iii) for each $\mathbb{X} \in \xi$ non-zero we have that $d\alpha(\mathbb{X}, J\mathbb{X}) > 0$.

Propositon 1.3.3. *There exists J a complex bundle structure of ξ compatible with the symplectic form $d\alpha \mid_\xi$ on ξ for any contact structure ξ.*

For the proof see Proposition 2.5.4 in [38].

Definition 1.3.4. *Let (W, ω) be a $2n$-dimensional symplectic manifold, $p \in W$, and $U^\perp = \{v \in T_pW :$ for all $w \in T_pW$ $\omega_p(v, w) = 0\}$ is the symplectic complement of $U \subset T_pW$. Then a k-dimensional submanifold L of W is*

i) ***isotropic*** *if for every $p \in L$ it holds that $T_pL \subset T_pL^\perp$,*
ii) ***Lagrangian*** *if for every $p \in L$ it holds that $T_pL = T_pL^\perp$.*

Note that the Lagrangian submanifolds exist in abundance. For example, consider $(W, \omega) = (\mathbb{R}^2, \omega_{st})$ and let L be any 1-dimensional submanifold of $\mathbb{R}^2$, then because T_pL is one dimensional and ω_{st} is skew-symmetric we obtain that $\omega_{st}(v, w) = 0$, where $p \in L$ and $v, w \in T_pL$ are tangent vectors. Thus L is a Lagrangian submanifold of $\mathbb{R}^2$.

In the contact case, we have a similar notion.

Definition 1.3.5. *Let Λ be a k-dimensional submanifold of a contact manifold $(M, \ker\alpha)$ of dimension $2n+1$. Denote $V^\perp$ the symplectic complement of a subspace of ξ_p with respect to $d\alpha_p$ where $p \in M$. The submanifold Λ is*

i) ***isotropic*** *if for every $p \in L$ it holds that $T_p\Lambda \subset \xi$ and $T_p\Lambda \subset T_p\Lambda^\perp$,*
ii) ***Legendrian*** *if for every $p \in L$ it holds that $T_p\Lambda \subset \xi$ and $T_p\Lambda = T_p\Lambda^\perp$.*

Definition 1.3.6. *Let (M, ξ) be a contact manifold, $f : \Lambda \to M$ be an embedding. We say that f is a* ***Legendrian embedding*** *if the image of f is a Legendrian submanifolds of M. If $f : \Lambda \looparrowright (M, \xi)$ is an immersion so that $f(T_p\Lambda)$ is an isotropic subspace of ξ with respect to the symplectic structure on ξ for each $p \in \Lambda$, then we say that f is a* ***Legendrian immersion****.*

Again in the lowest dimensional case the Legendrian submanifolds form a wide class of submanifolds.

Theorem 1.3.7 (Theorem 3.3.1 in [29])**.** *Let $\gamma : \boldsymbol{S}^1 \to (M, \xi)$ be a knot in a contact 3-manifold. Then γ can be C^0-approximated by a Legendrian knot isotopic to γ.*

In higher dimensions the situation is much more difficult. In the Chapter 2 we will see how to obtain higher-dimensional Legendrian submanifolds from the lower-dimensional ones.

Let us remark that for the sake of brevity the notions of Legendrian submanifolds and Lagrangian submanifolds are usually shortened to Legendrians and Lagrangians respectively.

1.4 Canonical projections

Definition 1.4.1. *Let $(M, \xi = \ker \alpha)$ be a contact manifold and $\Lambda \subset M$ be a Legendrian submanifold. Let R_α be the Reeb vector field associated with the contact form α. Then we say that a curve $c : [a,b] \to M$ is a Reeb chord if $c(a), c(b) \in \Lambda$ and $Tc(\frac{d}{dt}) = R_\alpha$.*

In other words, the Reeb chord is a segment of the one dimensional submanifold given by integrating the vector field R_α.

Definition 1.4.2. *Let $(P, d\eta)$ be an exact symplectic manifold and $(P \times \mathbb{R}, dz - \eta)$ be its contactization. The **Lagrangian projection** is defined as the canonical projection $\Pi_P : P \times \mathbb{R} \to P$.*

Moreover, consider a Legendrian submanifold $L \subset P \times \mathbb{R}$. Then $\Pi_P(L)$, its Lagrangian projection to P, is a Lagrangian immersion, because

$$(\Pi_P \circ f)^*(d\eta) = f^*(d\eta(T\Pi_P \cdot, T\Pi_P \cdot)) = -f^*(ddz(T\Pi_P \cdot, T\Pi_P \cdot)) = 0.$$

Now let Λ be a Legendrian submanifold. Here we can easily observe that we can recover a Legendrian which projects to Π_P up to a translation. More specifically, choose any point $p \in \Pi_P(\Lambda)$ and also an arbitrary $z(p)$ coordinate of this point. Then the z coordinate of any other point p' is given by

$$z(p') - z(p) = \int_\gamma \eta,$$

where γ is the lift of a curve Γ connecting p with p' with respect to Π_P. That easily follows from the fact that $dz = \eta$ on Λ and Stokes' Theorem.

In particular, if $(P \times \mathbb{R}, dz + \eta)$ coincides with $(\mathbb{R}^{2n+1}, \alpha_{st})$, then with the choice of the standard coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ on $\mathbb{R}^{2n+1}$ we obtain that

$$\Pi_{\mathbb{R}^{2n}} : (x_1, y_1, \ldots, x_n, y_n, z) \mapsto (x_1, y_1, \ldots, x_n, y_n)$$

and so the Lagrangian projection projects out the z coordinate. In this case, we know that $R_{\alpha_{st}} = \frac{\partial}{\partial z}$ and so for any Legendrian submanifold Λ of $\mathbb{R}^{2n+1}$ the Reeb chords c are line segments parallel to the z axis. Therefore, $\Pi_{\mathbb{R}^{2n}}(c(a)) = \Pi_{\mathbb{R}^{2n}}(c(b))$ is a double point in the projection $\Pi_{\mathbb{R}^{2n}}$.

Definition 1.4.3. *Let $\Lambda \subset (P \times \mathbb{R}, dz + \eta)$ be a closed Legendrian submanifold of a contactization of an exact symplectic manifold. We say that Λ is **chord generic** if*

- *the image of Λ under the Lagrangian projection Π_P is a Lagrangian embedding except finitely many double points,*
- *and at every double point $c^* \in \Pi_P(\Lambda)$ so that $p, q \in \Lambda$ get mapped to c^* via Π_P we have that $\Pi_P(T_p\Lambda) \oplus \Pi_P(T_q\Lambda) = T_{c^*}P$.*

Let us remark that for a fixed manifold Λ the set of Legendrian embeddings with the chord generic image is open and dense in the space of all Legendrian embeddings of Λ to $P \times \mathbb{R}$ with C^∞-topology.

Similarly, for any double point of the projection $\Pi_{\mathbb{R}^{2n}}$ there is a Reeb chord projecting to this point. And thus we have showed the following correspondence.

Propositon 1.4.4. *Let $(P, d\eta)$ be an exact symplectic manifold and $(P\times\mathbb{R}, dz-\eta)$ be its contactization. Then for a chord generic Legendrian submanifold Λ in $P\times\mathbb{R}$ there is a one to one correspondence between the double points of Π_P and the Reeb chords of L.*

There is an important subclass of chord generic Legendrians.

Definition 1.4.5. *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ be a Legendrian submanifold. We say that Λ is an **admissible Legendrian submanifold** if*

- *Λ is chord generic,*
- *and the embedding $\iota : \Lambda \to \mathbb{R}^{2n+1}$ is real analytic in a neighbourhood of all Reeb chord endpoints.*

If we again fix the manifold Λ. We obtain that the set of all admissible Legendrian submanifolds is an open subset of the space of all chord generic Legendrian embeddings and is also dense in the C^∞-topology on the space of all Legendrian embeddings of Λ into $(\mathbb{R}^{2n+1}, \alpha_{st})$ for the proof see [15], Lemma 5.6.

Definition 1.4.6. *Let $(J^1(M), dz + \eta)$ be the one jet space of any manifold M. We define the **front projection** as the canonical projection*

$$\Pi_F : J^1(M) \to M \times \mathbb{R}.$$

*Moreover, if $\Lambda \subset J^1(M)$ is a Legendrian submanifold, then the set $\Pi_F(\Lambda)$ is called its **front**.*

Note that in the case of the one-jet space $J^1(M)$ the Lagrangian projection is the canonical projection

$$\Pi_{T^*M} : J^1(M) \to T^*M.$$

Example 1.4.7. Let $(\mathbb{R}^{2n+1}, \alpha_{st})$ the standard contact Euclidean space with standard coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$. The front projection $\Pi_F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is given by

$$(x_1, y_1, \ldots, x_n, y_n, z) \mapsto (x_1, x_2, \ldots, x_n, z).$$

▲

Thanks to the classification of the singularities in [1] one knows that the set $\Pi_F(\Lambda)$ contains more structure for a generic Legendrian submanifold Λ.

Definition 1.4.8. *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ be a Legendrian submanifold. We say that Λ is **front generic** if $\Pi_F(\Lambda)$ is an algebraic variety of codimension one and Σ the set of its singular points is a hypersurface that is smooth outside a set of codimension 3 in Λ. There is Σ' a codimension 2 subset in Σ so that we can describe the singularity for any $p \in \Sigma \setminus \Sigma'$ using some coordinates $(x_1, \ldots, x_n)$ around p as*

$$\Pi_F(x_1, \ldots, x_n) = (x_1^2, x_2, \ldots, x_n, \delta x_1^3 + \beta x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n),$$

for some constants $\delta = \pm 1$ and $\beta, \alpha_2, \ldots, \alpha_n$. The set $\Sigma \setminus \Sigma'$ is called a cusp edge.

It is a natural question, whether for a fixed manifold Λ the set of all front generic Legendrian embeddings is dense in the set of all chord generic or even admissible Legendrian embeddings. The answer to this question is expected to be positive, however, the details have not been fully dealt even in the 3-dimensional case.

Similarly to the case of the Lagrangian projection we can recover the front generic Legendrian Λ from its front projection, however, it this case the lift is uniquely determined. Moreover, we can also detect Reeb chords using the front projection.

Propositon 1.4.9 (Section 3.2 in [15])**.** *For L a front generic Legendrian submanifold of $(\mathbb{R}^{2n+1}, \alpha_{st})$ we have a one-to-one correspondence between the Reeb chords of L and a pair of points in the front projection so that they lie on a line parallel to the z axis and the tangent planes to L at those points are parallel to each other.*

1.5 Whitney spheres

Let a be a non-zero real number and denote $b = \sqrt{\frac{3a}{2}}$. Consider the standard Euclidean contact manifold $(\mathbb{R}^{2n+1}, \xi_{st})$ equipped with the standard coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ and $\mathbf{S}^n = \{u \in \mathbb{R}^n : u_1^2 + \cdots + u_{n+1}^2 = 1\}$ the unit sphere.

Define $\bar{w}_b : \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$ as

$$(u_1, \dots, u_n, u_{n+1}) \mapsto (au_1, au_{n+1}u_1, \dots, au_n, au_{n+1}u_n).$$

And compute its Jacobi matrix.

$$T\bar{w}_b = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ au_{n+1} & 0 & 0 & \cdots & 0 & au_1 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & au_{n+1} & 0 & \cdots & 0 & au_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & au_{n+1} & au_n \end{pmatrix} : T_u\mathbb{R}^{n+1} \to T_{\bar{g}_a(u)}\mathbb{R}^{2n}.$$

We observe that $T\bar{w}_b$ is injective for every $\mathbb{R}^{n+1}$ except the point $(0, \dots, 0, 0)$ because it has linearly independent columns otherwise. Now denote by $\iota : \mathbf{S}^n \to \mathbb{R}^{n+1}$ the canonical embedding and by $w_b : \mathbf{S}^n \looparrowright \mathbb{R}^{2n}$ the composition $w_b = \bar{w}_b \circ \iota$. We see that w_b is an immersion. From the form of w_a (injectivity of odd coordinates) one observes that the only double points of w_b are $(0, \dots, 0, 1)$ and $(0, \dots, 0, -1)$.

Define $\theta = \sum_{j=1}^n y_j dx_j$ so that $\omega_{st} = -d\theta$. Differentiating the defining equation of $\mathbf{S}^n$ we get that

$$u_1 du_1 + \ldots + u_n du_n + u_{n+1} du_{n+1} = 0.$$

Therefore,

$$u_1 du_1 \wedge du_{n+1} + \ldots + u_n du_n \wedge du_{n+1} = -u_{n+1} du_{n+1} \wedge du_{n+1} = 0.$$

Let us compute

$$w_b^*\theta = \sum_{j=1}^{n} a^2 u_{n+1} u_j du_j = -a^2 u_{n+1}^2 du_{n+1} = d(-a^2 \frac{u_{n+1}^3}{3})$$

$$\begin{aligned} w_b^*\omega_{st} &= \sum_{j=1}^{n} a^2 du_j \wedge (u_{n+1} du_j + u_j du_{n+1}) \\ &= \sum_{j=1}^{n} a^2 u_j du_j \wedge du_{n+1} \\ &= 0. \end{aligned}$$

This means that $w_b : \mathbf{S}^n \to \mathbb{R}^{2n}$ is a Lagrangian immersion so that $w_b^*\theta$ is an exact form and it has only one double point.

Definition 1.5.1. *Let $(P, -d\theta)$ be an exact symplectic manifold of dimension $2n$ and $f : L \looparrowright P$ be an immersion of a manifold L of dimension n. We say that L is an* ***exact Lagrangian immersion*** *if $f^*(-d\theta) = 0$ (that is f is a Lagrangian immersion) and $f^*\theta$ is exact.*

And so w_b is a particular case of exact Lagrangian immersion. Existence of this immersion is a characteristic of higher-dimensional spheres of even dimension. More specifically, we have the following theorem.

Theorem 1.5.2 (Ekholm, Smith, [19])**.** *Let L be a closed orientable manifold of dimension $2k$, where $k > 2$, and so that its Euler characteristic $\chi(L)$ is different from -2. If there exists $f : L \looparrowright (\mathbb{R}^{2n}, \omega_{st})$ an exact Lagrangian immersion with precisely one double point and no other self-intersections, then L is diffeomorphic to $\boldsymbol{S}^{2k}$.*

Define $f_b : \mathbf{S}^n \to \mathbb{R}^{2n+1}$ a Legendrian embedding as a Legendrian lift of the exact Lagrangian immersion w_b with respect to the Lagrangian projection. Note that this determines f_b up to translation in the z-direction.

Definition 1.5.3. *Define $W_b^n \subset \mathbb{R}^{2n+1}$ as the Legendrian submanifold that is given by the image of $f_b : \boldsymbol{S}^n \to \mathbb{R}^{2n+1}$. We will call this Legendrian the* ***Whitney sphere****.*

Since $\Pi_{\mathbb{R}^{2n}}(W_b^n) = w_b(\mathbf{S}^n)$, then W_b^n is a chord generic Legendrian submanifold of $\mathbb{R}^{2n+1}$ with precisely one Reeb chord c that has endpoints given by $p = f_b(0, \ldots, 0, 1)$ and $q = f_b(0, \ldots, 0, -1)$. If $\gamma : [0, 1] \to W_a^n$ is a path connecting p and q, so that its Lagrangian projection is given by $\Pi_{\mathbb{R}^{2n}}\gamma = w_b \circ c : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbf{S}^n$,

$$c : t \mapsto (\cos t, 0, \ldots, 0, \sin t).$$

. That is,

$$\Pi_{\mathbb{R}^{n_2}}(\gamma)(t) = (a \cos t, a \cos t \sin t, 0, \ldots, 0).$$

We can compute the length of c as

$$\begin{aligned} z(p) - z(q) &= \int_c dz = -\int_\gamma \theta = -\int_{\Pi_{\mathbb{R}^{2n}}(\gamma)} \theta \\ &= a^2 \int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \cos t \sin^2 t dt = \frac{2}{3} a^2 = b, \end{aligned}$$

where we used the Stokes' Theorem, the fact that W_b^n is a Legendrian submanifold and the form of the contact form on $\mathbb{R}^{2n+1}$, that is $dz - \theta$.

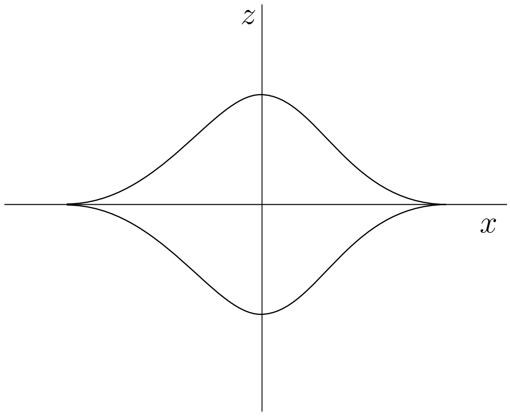

Figure 1.1: Front projection of the curve $W_1^1 \subset \mathbb{R}^3$.

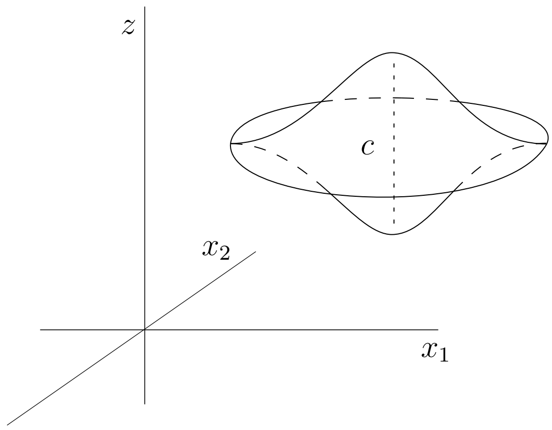

Figure 1.2: Front projection of $W_1^2 \subset \mathbb{R}^5$ with the unique Reeb chord marked by the dashed line. The set $\Pi_F(W_1^2)$ is called the flying saucer.

1.6 Different notions of isotopy for Legendrian embeddings

One of the main questions in contact topology is the classification of Legendrian submanifolds up to Legendrian isotopy.

Definition 1.6.1 (Legendrian isotopy)**.** *Let (M, ξ) be a contact manifold and $f_i : \Lambda \to M$ for $i = 1, 2$ be two Legendrian embeddings. We say that f_1 and f_2 are **Legendrian isotopic** if there exists a smooth homotopy $\{f_t\}_{t \in [0,1]}$ of f_1 and f_2 passing through Legendrian embeddings.*

We know that every Legendrian embedding can be approximated by a admissible Legendrian embedding that is arbitrary close in the C^∞-topology on the space of all Legendrian embeddings. The same can be done with a chord generic Legendrian isotopies.

Theorem 1.6.2 (Ekholm, Etnyre, Sullivan, Theorem 5.6 in [15])**.** *Consider a family of Legendrian embeddings* $\{\Lambda_t\}_{t\in[0,1]}$ *inducing a Legendrian isotopy of two admissible Legendrian submanifolds* Λ_0, Λ_1 *going through chord generic Legendrian embeddings, then there is a Legendrian isotopy through admissible Legendrian embeddings that is arbitrarily close to the original isotopy* $\{\Lambda_t\}_{t\in[0,1]}$ *in the* C^∞*-topology.*

Definition 1.6.3 (Formal Legendrian Embedding)**.** *Let* (M, ξ) *be a contact manifold of dimension* $2n+1$ *and* Λ *be a smooth manifold of dimension* n*. A* ***formal Legendrian embedding*** *is a pair* (f, F_s)*, where* $f : \Lambda \to M$ *is a smooth embedding and* F_s *is a homotopy of bundle maps covering* f*, so that:*

- $F_0 = Tf$*,*
- F_s *is fibrewise injective for all* $s \in [0, 1]$*, and*
- *the image of* F_1 *is contained in* ξ*, and the image is Lagrangian with respect to the linear conformal symplectic structure on* ξ*.*

Let us note that it is immediate that every Legendrian embedding f is a formal Legendrian embedding taking the constant homotopy covering f, that is $F_s = Tf$ for all $s \in [0, 1]$.

Definition 1.6.4. *Let* (M, ξ) *be a contact manifold and* $f_i : \Lambda \to M$ *for* $i = 1, 2$ *be two Legendrian embeddings. We say that* f_1 *and* f_2 *are* ***formally Legendrian isotopic*** *if there exists a path of formal Legendrian embeddings joining* (f_1, Tf_1) *and* (f_2, Tf_2)*.*

Again every Legendrian isotopy $\{f_s\}_{s\in[0,1]}$ is formal Legendrian isotopy taking the path to be $\{(f_s, Tf_s)\}_{s\in[0,1]}$.

Definition 1.6.5 (k-stable formal isotopy, Definition 1.11, [11])**.** *Let* $(M, \ker\alpha)$ *be a contact manifold and* $\Lambda_i \subset M$ *for* $i = 1, 2$ *be two Legendrian submanifolds. We say that* Λ_1 *and* Λ_2 *are* k*-stably formally Legendrian isotopic is their stabilisations defined as*

$$\{0\} \times \Lambda_1, \{0\} \times \Lambda_2 \subset (\mathbb{R}^{2k} \times M, -\sum_{j=1}^{k} y_j dx_j + \alpha)$$

are formally isotopic as subcritical manifolds.

1.7 Loose Legendrians and Homotopy principle

Before we turn our attention to the tools developed to study the Legendrian isotopy classes, let us digress a bit on one important class of Legendrian submanifolds called loose Legendrian submanifolds. First loose Legendrians were found by Sauvaget in [43] and later the whole class was introduced by Murphy in [40]. This class exhibit the flexibility phenomenon advertised in the introduction. More specifically, in dimension strictly bigger than 3 loose Legendrians are completely classified by its Legendrian isotopy classes.

Definition 1.7.1 (standard loose chart)**.** *Consider the standard neighbourhood $R_{abc} \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ defined for real numbers a, b, c satisfying $a < bc$ as*

$$R_{abc} = \{(x, y, x_1, y_1, \dots, x_{n-1}, y_{n-1}, z); |x|, |y| < 1, \|(x_1, \dots, x_{n-1})\| \leq b, \\ \|(y_1, \dots, y_{n-1})\| \leq c, |z| \leq a\}.$$

There denote by Λ_ℓ the product of

$$D_b = \{(x_1, y_1, \dots, x_{n-1}, y_{n-1}); y_1 = \dots = y_{n-1} = 0, \|(x_1, \dots, x_{n-1})\| \leq b\}$$

and a Legendrian curve $\gamma \subset \mathbb{R}^3$ with coordinates (x, y, z).

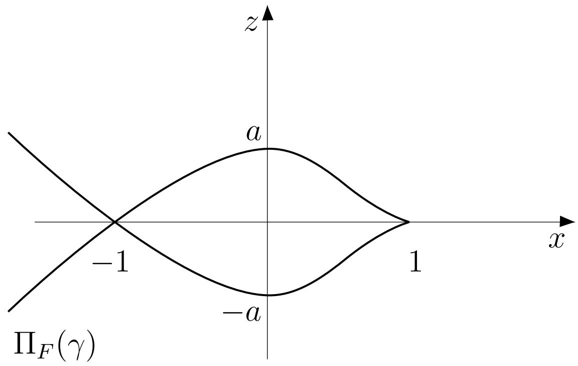

Figure 1.3: Front projection of the curve γ.

Definition 1.7.2. *Let (M, ξ) be a contact manifold and $\Lambda \subset M$ a connected Legendrian submanifold. We say that Λ is loose if there exists an open set U of M so that the pair $(U, U \cap \Lambda)$ is contactomorphic to the pair (R_{abc}, Λ_ℓ).*

Theorem 1.7.3 (h-principle for loose Legendrian embeddings, Murphy, Theorem 1.2. in [40])**.** *Let (M, ξ) be a $2n+1$-dimensional contact manifold and $f_0, f_1 : \Lambda \to (M, \xi)$ be two Legendrian embeddings, which are formally isotopic. If $n \geq 2$, then f_0 and f_1 are also Legendrian isotopic.*

Note that since looseness is a local property, loose Legendrian embeddings exist in abundance. For any Legendrian embedding it is enough to choose a small neighbourhood and modify the Legendrian so that we have the standard model of the loose chart. This yields that if we fix any formal Legendrian isotopy class, the space of loose Legendrian embeddings is its C^0-dense subset.

2. Constructions of Legendrians

We have seen some basic examples of Legendrian submanifolds. In this chapter, we will describe the standard constructions which from lower dimensional Legendrian submanifolds create higher dimensional ones by imitating product with a sphere, and moreover how to perform a surgery of Legendrian submanifolds.

2.1 Product constructions

2.1.1 Front spinning

The first product-like construction appeared in [15] and is due to Ekholm, Etnyre, and Sullivan. Consider $\mathbb{R}^{2n+1}$ with standard contact structure and $L \subset \mathbb{R}^{2n+1}$ a Legendrian submanifold which has parametrization $f : L \to \mathbb{R}^{2n+1}$ that is given by

$$f(p) = (x_1(p), y_1(p), \ldots, x_n(p), y_n(p), z(p)),$$

for each $p \in L$, moreover assume that $x_1(p) > 0$ for all $p \in L$. Then its front is parametrized as

$$\Pi_F \circ f(p) = (x_1(p), \ldots, x_n(p), z(p)).$$

Now fix an embedding of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ defined as

$$(x_1, \ldots, x_n, z) \mapsto (x_0, x_1, \ldots, x_n, z).$$

Consider the set $\Pi_F(\Sigma L)$ parametrized by $\theta \in \mathbf{S}^1$ as

$$(\sin\theta\, x_1(p), \cos\theta\, x_1(p), x_2(p), \ldots, x_n(p)).$$

This amounts to taking the set $\Pi_F(L)$ and rotating it around the z axis with respect to the x_0x_1-plane in $\mathbb{R}^{n+2}$. For L generic $\Pi_F(\Sigma L)$ lifts uniquely to ΣL a Legendrian submanifold of $\mathbb{R}^{2n+3}$ called suspension. Sometimes we also say that ΣL is a $\mathbf{S}^1$-spinning of L or $\mathbf{S}^1$-spun.

Propositon 2.1.1 (Lemma 4.16 in [15]). *Topological types of ΣL and $L \times \boldsymbol{S}^1$ coincide.*

Propositon 2.1.2 (Section 4.4 in [15]). *The $\boldsymbol{S}^1$-spun is invariant under Legendrian isotopy of L.*

This approach was generalized by Golovko in [32] to spinning with respect to $\mathbf{S}^m$ for arbitrary $m \in \mathbb{N}$. There we speak about $\mathbf{S}^m$-spuns $\Sigma_{\mathbf{S}^m}\Lambda$. In particular, having the parametrization of Λ above we obtain $g : \Lambda \times \mathbf{S}^m \to \mathbb{R}^{2(n+m)+1}$ a parametrization with the parametrization of its front given by

$$(\tilde{x}_{-m+1}(p, \theta, \bar{\phi}), \ldots, \tilde{x}_1(p, \theta, \bar{\phi}), x_2(p), \ldots, x_n(p), z(p))$$

where $\theta \in [0, 2\pi)$ and $\bar{\phi} = (\phi_1, \ldots, \phi_{m-1}) \in [0, \pi]^{m-1}$ and

$$\begin{aligned}
\tilde{x}_{-m+1}(p, \theta, \bar{\phi}) &= x_1(p) \sin\theta \sin\phi_1 \ldots \sin\phi_{m-1},\\
\tilde{x}_{-m+2}(p, \theta, \bar{\phi}) &= x_1(p) \cos\theta \sin\phi_1 \ldots \sin\phi_{m-1},\\
\tilde{x}_{-m+3}(p, \theta, \bar{\phi}) &= x_1(p) \cos\phi_1 \sin\phi_2 \ldots \sin\phi_{m-1},\\
&\vdots\\
\tilde{x}_1(p, \theta, \bar{\phi}) &= x_1(p) \cos\phi_{m-1},
\end{aligned}$$

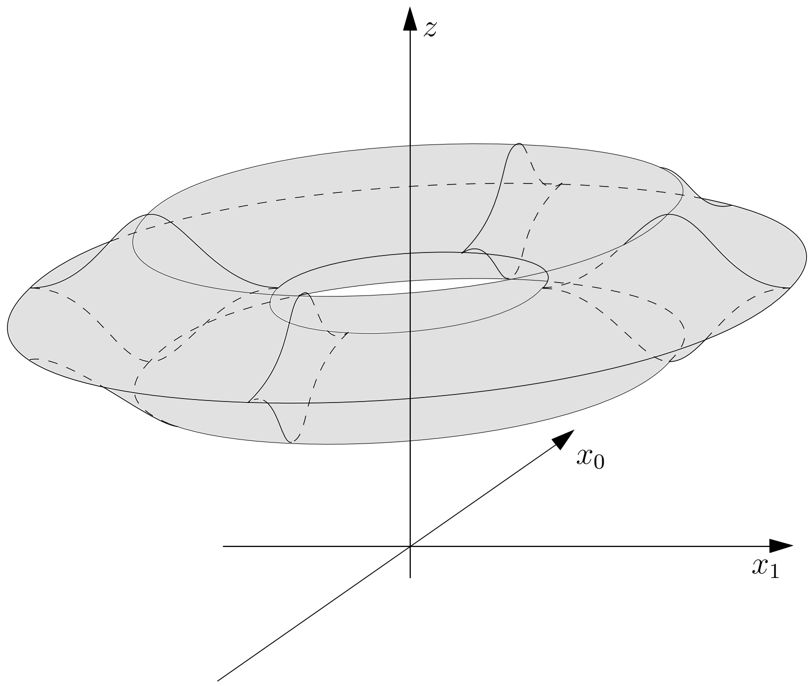

Figure 2.1: Front projection of $\Sigma_{\mathbf{S}^1}\Lambda$, where Λ is Legendrian isotopic to the Legendrian unknot W^1 .

where the trigonometric coordinates are just the usual spherical coordinates on $\mathbf{S}^m$.

Propositon 2.1.3 (Lemma 5.1 in [32])**.** *Manifolds $\Sigma_{\boldsymbol{S}^m}\Lambda$ and $\Lambda \times \boldsymbol{S}^m$ have the same topological type. In addition, $\Sigma_{\boldsymbol{S}^m}\Lambda$ is invariant of Legendrian isotopy class of Λ.*

2.1.2 Twist spun

The next construction is due to Ekholm and Kálmán who introduced it in [18]. It proceeds as follows. Let us have $(P \times \mathbb{R}, dz + \eta)$ a contactization of an exact symplectic manifold $(P, d\eta)$ a manifold Λ and $\{\Lambda_\theta\}$ for $\theta \in \mathbf{S}^1$ a loop of Legendrian embeddings $f_\theta : \Lambda \to P \times \mathbb{R}$. Then the map $F : \Lambda \times \mathbf{S}^1 \to \mathbb{R}^2 \times P \times \mathbb{R}$ defined by $(p, \theta) \mapsto (\theta, f_\theta(p))$ is a Legendrian embedding. Let us denote $\Sigma_{\mathbf{S}^1}\{\Lambda_\theta\}$ its image and call it a twist spun.

Observe that if the family of Legendrian embeddings is constant, we obtain precisely the $\mathbf{S}^1$-spun of the initial embedding of Λ.

We can again think what happens when we work with $\mathbf{S}^k$-families for $k \geq 1$. And that is precisely how this approach was generalized by Dimitroglou Rizell and Golovko in [11].

More specifically, with the set up above, where $P = Q \times T^*\mathbb{R}$ as symplectic manifolds (see [38], Chapter 2), consider $\{\Lambda_\theta\}_{\theta \in \mathbf{S}^k}$ a $\mathbf{S}^k$-family of Legendrian embeddings $\Lambda_\theta \subset P \times \mathbb{R}$.

Definition 2.1.4. *The **suspension** of the family* $\{\Lambda_\theta\}_{\theta\in \boldsymbol{S}^k}$ *is the unique Legendrian*

$$\Sigma_{\boldsymbol{S}^k}\{\Lambda_\theta\} \subset (P \times T^*\boldsymbol{S}^k \times \mathbb{R}, dz - ydx + \eta)$$

determined by the property that its image under the canonical projection

$$\Pi : P \times T^*\boldsymbol{S}^k \times \mathbb{R} \to P \times \boldsymbol{S}^k \times \mathbb{R},$$

induced by the canonical projection $\pi : T^*\boldsymbol{S}^k \to \boldsymbol{S}^k$ *is equal to*

$$\{(x, \theta, z) \in P \times \boldsymbol{S}^k \times \mathbb{R}; (x, z) \in \Lambda_\theta\},$$

which is an embedded submanifold.

The canonical identification and inclusion $\mathbb{R}\times\mathbf{S}^k \cong \mathbb{R}^{k+1}\setminus\{0\} \subset \mathbb{R}^{k+1}$ induces an exact symplectic inclusion $T^*(\mathbb{R} \times \mathbf{S}^k) \subset T^*\mathbb{R}^{k+1}$. And so using canonical product identifications we get an embedding ι of contact manifolds

$$\begin{array}{ccc} Q \times T^*(\mathbb{R} \times \mathbf{S}^k) \times \mathbb{R} & \hookrightarrow & Q \times T^*(\mathbb{R}^{k+1}) \times \mathbb{R} \\ \big\downarrow = & & \big\downarrow = \\ P \times T^*\mathbf{S}^k \times \mathbb{R} & & (P \times \mathbb{R}^{2k} \times \mathbb{R}, dz + \eta - \sum_{j=1}^k y_j dx_j), \end{array}$$

so that the pull-back of the contact form on the target is the contact form on the source.

Definition 2.1.5. *The image of the Legendrian suspension* $\Sigma_{\boldsymbol{S}^k}\{\Lambda_\theta\} \subset P\times\mathbb{R}^{2k+1}$ *under the canonical embedding* ι *is called the* $\boldsymbol{S}^k$*-twist spun of the family* $\{\Lambda_\theta\}_{\theta\in\boldsymbol{S}^k}$.

2.1.3 Legendrian product

Definition 2.1.6. *Let* Λ_1, Λ_2 *be two Legendrian submanifolds of a contact manifold* $(M, \ker\alpha)$. *Denote by* $\mathcal{R}(\Lambda_j)$ *the set of chords of corresponding Legendrian and for* $j = 1, 2$ *by*

$$\ell : \mathcal{R}(\Lambda_j) \to \mathbb{R},\ c \mapsto \int_c \alpha$$

define the length of Reeb chords. Then we say that

1. Λ_1 *is **smaller** then* Λ_2, $\Lambda_1 < \Lambda_2$ *if*

$$\max\{\ell(c) : c \in \mathcal{R}(\Lambda_1)\} < \min\{\ell(c) : c \in \mathcal{R}(\Lambda_2)\},$$

2. Λ_1 *and* Λ_2 *have **distinct Reeb chords lengths** if*

$$\{\ell(c) : c \in \mathcal{R}(\Lambda_1)\} \cap \{\ell(c) : c \in \mathcal{R}(\Lambda_2)\} = \emptyset.$$

Definition 2.1.7. *Let* $\iota_j : \Lambda_j \to (P_j \times\mathbb{R}, dz_j - \eta_j)$ *be two Legendrian embeddings, where* $(P_j, d\eta)$ *is an exact symplectic manifold. The **Legendrian product** is the Legendrian immersion*

$$\iota_1 \boxtimes \iota_2 : \Lambda_1 \times \Lambda_2 \to (P_1 \times P_2 \times \mathbb{R}, dz - \eta_1 - \eta_2),$$

defined by

$$\iota_1 \boxtimes \iota_2(u_1, u_2) = (\Pi_{P_1}(\iota_1(u_1)), \Pi_{P_2}(\iota_2(u_2)), z_1(\iota_1(u_1)) + z_2(\iota_2(u_2)))$$

The definition of the Legendrian product first appeared in the paper [37] by Lambert-Cole.

Lemma 2.1.8. *The Legendrian product $\Lambda_1 \boxtimes \Lambda_2$ is an embedded submanifold if and only if Λ_1 and Λ_2 have distinct Reeb chord lengths.*

Proof. Recall the genericity condition on Λ_j, in particular, they are both chord generic. The map $P_1 \times \mathbb{R} \times P_2 \times \mathbb{R} \to P_1 \times P_2 \times \mathbb{R}$, $(q_1, z_1, q_2, z_2) \mapsto (q_1, q_2, z_1 + z_2)$ is clearly smooth and so is $\iota_1 \boxtimes \iota_2$.

Denote by $\Phi = \iota_1 \boxtimes \iota_2$. If there would be two Reeb chods $c_j \in \Pi_{P_j}(\Lambda_j)$ starting at p_j^1 and ending at p_j^2 of the same length then the pairs (p_1^1, p_2^2) and (p_1^2, p_2^1) are mapped to the same point by Φ and so Φ is not injective, in particular it can not be an embedding. Otherwise, suppose that Λ_1 and Λ_2 have distinct Reeb chord lengths. We already know that Φ is smooth. To finish the proof we need to show that it is a homeomorphism onto its image. Choose (q_1, q_2, z) coordinates on $(P_1 \times P_2 \times \mathbb{R}, dz - \eta_1 - \eta_2)$. Now let $\tilde{p} \in \Phi(\Lambda_1 \times \Lambda_2)$ an arbitrary point. If $q_1(\tilde{p})$ and $q_2(\tilde{p})$ are both double points, that is $q_1(\tilde{p}) = \Pi_{P_1}(c_j)$ where c_j as above, and assume that $\ell(c_1) \neq \ell(c_2)$. Let z^{kl} be defined as $z_1(p_1^k) + z_2(p_2^l)$ for $k, l = 1, 2$. In particular, observe that $z^{kl} = z^{k'l'}$ if and only if $k = k'$ and $l = l'$. Then the points (p_1^k, p_2^l) have different images with respect to Φ and so Φ is a continuous bijection between compacts thus a homeomorphism. $\square$

We have seen that the lengths of the Reeb chords of Λ_1 and Λ_2 play a major role in behaviour of the Legendrian product. This is a big obstacle and even prevents the Legendrian product from being a Legendrian isotopy invariant of Λ_1 and Λ_2. More specifically, if we take $\{\Lambda_\theta\}_{\theta \in I}$ a Legendrian isotopy, it can change the Reeb chord lengths or even it may cancel out some of the original Reeb chords.

For example, let Λ_o be a Legendrian that is Legendrian isotopic to the Whitney embedding of the sphere, however, we perform three Reidemeister moves (with a fixed height ε) on the upper sheet of the flying saucer in the front projection (see Figure 2.1.3). The second Legendrian Λ_f is to be the Whitney embedding of the sphere. We choose a Legendrian isotopy $\{\Lambda_\theta\}_{\theta \in I}$ from Λ_o to Λ_f that will be the mere three-fold repetition of the equivalent of the first Reidemeister move. After the first repetition we obtain Λ_r, $r \in I$, that has a front of flying saucer with two Reidemeister moves performed on the upper sheet. Now, if we consider $\Lambda_o \boxtimes \Lambda_r$ we can clearly see that $\Lambda_o > \Lambda_r$ and $\Lambda_r > \Lambda_f$, however exactly for $\theta = r$ the immersion $\Lambda_\theta \boxtimes \Lambda_r$ fails to be an embedding because of the length reasons.

This phenomenon has an interesting consequence , see [37] Theorem 1.2.

Theorem 2.1.9 (Lambert-Cole,Theorem 1.2 in [37])**.** *Let $K \in \mathbb{R}^{2n+1}$, $L \in \mathbb{R}^{2m+1}$ be chord generic Legendrians such that n, m have different parity. Then there exists an infinite family of Legendrians $\{K_i\}$ all Legendrian isotopic to K such that the family of Legendrian products $\{K_i \times L\}$ are pairwise non-Legendrian isotopic.*

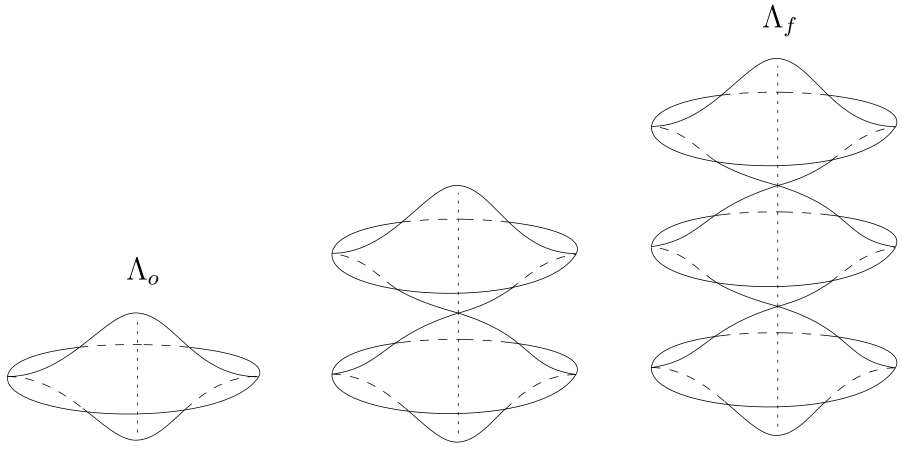

Figure 2.2: Front projection of the Legendrian isotopy of Λ_o to Λ_f given by higher-dimensional equivalent of the first Reidemeister move.

2.1.4 Spinning

This construction is also due to Lambert-Cole and it appeared in [37].

Let M, M' be contact manifolds, m a natural number, and $M \hookrightarrow M'$ a contact embedding of codimension $2m$ with trivial conformal symplectic normal bundle. Moreover, let $\iota_K : K \hookrightarrow (\mathbb{R}^{2m+1}, \alpha_{st})$ and $\iota_L : L \hookrightarrow (M, \alpha)$ be Legendrian embeddings. Now, identify a neighbourhood of L in M as a neighbourhood N of the 0-section in $J^1(M)$ and so using the contact embedding we obtain a identification of a neighbourhood of L in M' with

$$N \times \mathbb{R}^{2m} \subset T^*L \times \mathbb{R}^{2m} \times \mathbb{R}.$$

And thus $L \times K$ the product of manifolds embeds into $T^*L \times \mathbb{R}^{2m} \times \mathbb{R}$ which is identified with a neighbourhood of L in M'. There the first component embeds as γ a 0-section of T^*L and the second one is just ι_K.

Definition 2.1.10. *The Legendrian submanifold of M' given by the image of $L \times K$ under $\gamma \times \iota_K$ and the identifications above is said to be a **spinning** of K by L and is to be denoted by $K \times_M L$.*

Propositon 2.1.11 (Lambert-Cole, Section 1, [37])**.** *The spinning $K \times_M L$ is invariant under Legendrian isotopies of L in M, contact isotopies of M in M' and compactly supported isotopies of K.*

2.1.5 Interplay in between product constructions

It is due to Lambert-Cole (see [37], Theorem 1.4.) that if $\Lambda_1 \subset \mathbf{S}^{2k-1}$ (equivalently $\mathbb{R}^{2k-1}$) and $\Lambda_2 \subset \mathbb{R}^{2n-2k+1}$ Legendrian submanifolds so that $\Lambda_1 < \Lambda_2$, then $\Lambda_1 \times_{\mathbf{S}^{2k-1}} \Lambda_2$ and $\Lambda_1 \boxtimes \Lambda_2$ are Legendrian isotopic submanifolds of $\mathbb{R}^{2n+1}$. In particular, for the standard Whitney sphere W_1 and a Legendrian submanifold Λ so that $\Lambda < W$, we obtain that $\Sigma\Lambda$ is Legendrian isotopic to $W \boxtimes \Lambda$. And moreover, it is a corollary of Theorem 1.4. in [37] that the result extends to $\mathbf{S}^k$-spuns of arbitrary dimension by taking the standard Legendrian embedding of the Whitney sphere $W_k \subset \mathbb{R}^{2k+1}$ and a Legendrian submanifold Λ so that $\Lambda < W_k$. That is $\Sigma_{\mathbf{S}^k}\Lambda$ is Legendrian isotopic to $W_k \boxtimes \Lambda$.

In the work of Dimitroglou Rizell and Golovko, we can find the following generalization.

Theorem 2.1.12 (Dimitroglou Rizell, Golovko, Theorem 1.13, [11])**.** *Let $\Lambda_1 \subset (\mathbb{R}^{2n_1+1}, \alpha_{st})$ be a Legendrian submanifold and $\Lambda_2 \subset (\mathbb{R}^{2n_2+1,\alpha_{st}})$ a Legendrian sphere which is n_1-stably formally isotopic to the standard sphere W^{n_2}. If $\Lambda_1 < \Lambda_2$, then it follows that $\Lambda_1 \boxtimes \Lambda_2$ is Legendrian isotopic to a twist spun of an $\boldsymbol{S}^{n_2}$-family of Legendrians which is obtained from Λ_1 by a suitable family of rotations, each being a contact lift of a linear symplectic $U(n_1)$-action. Furthermore,*

- *In the case when Λ_2 is formally Legendrian isotopic to the standard sphere W^k, the product is Legendrian isotopic to the ordinary $\boldsymbol{S}^k$-spun of Λ_1.*
- *In the case when $n_2 = 1$ it follows that $\Lambda_1 \boxtimes \Lambda_2$ is Legendrian isotopic to the twist spun of the family of Legendrians that covers the image of $\Pi_{\mathbb{R}^{2n_1}}(\Lambda_1)$ under the family*

$$(z_1, z_2, \ldots, z_{n_1}) \mapsto (e^{i\,\mathrm{rot}(\Lambda_2)\theta} z_1, z_2, \ldots, z_n),\ \theta \in \boldsymbol{S}^1$$

of rotations.

2.2 Cobordisms, fillings, and surgery

Definition 2.2.1. *Let $(P, d\eta)$ be an exact symplectic manifold of dimension $2n$ and $(P\times\mathbb{R}, dz+\eta)$ be its contactization. An* ***exact Lagrangian cobordism*** *form Λ^- to Λ^+ Legendrian submanifolds of $P \times \mathbb{R}$ is a embedded $(n+1)$-dimensional submanifold*

$$V \subset (\mathbb{R} \times P^{2n} \times \mathbb{R}, d(e^t(dz + \eta)))$$

which for some real $T > 0$

- *$V \cap ([-T, T] \times P \times \mathbb{R})$ is a compact set,*
- *coincides with a cylinder over Λ^+ inside $[T, +\infty) \times P \times \mathbb{R}$,*
- *coincides with a cylinder over Λ^- inside $(-\infty, -T]$*
- *is exact Lagrangian in the sense that $e^t(dz + \eta)|_{TV}$ is exact with globally constant primitive on $(-\infty, T] \times \Lambda^- \subset V$.*

In particular, when $\Lambda^- = \emptyset$, we say that Λ^+ is ***(exact) fillable*** *and V is an* ***exact Lagrangian filling*** *of Λ^+.*

Remark. Using the invariants from Chapter 5 (Proposition 5.6.1, Theorem 5.1.12) one can prove that if $\Lambda \subset (P \times \mathbb{R}, dz + \eta)$ is a Legendrian submanifold, then it holds that:

- if Λ is fillable, then Λ is not loose,
- if Λ is loose, then Λ is not fillable.

And so the classes of loose and fillable Legendrian embeddings are disjoint.

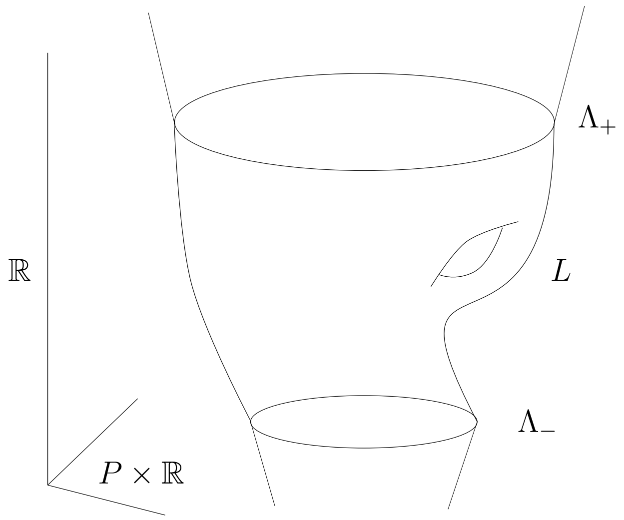

Figure 2.3: Example of an exact Lagrangian cobordism.

We can extend the Legendrian product construction, see Definition 2.1.7, to exact Lagrangian cobordisms as follows.

Let $(P_j, d\eta_j)$ be exact symplectic manifolds for $j = 1, 2$. Let $\iota_1 : V \to \mathbb{R} \times P_1$ be the Lagrangian embedding, where its image is the cobordism from Λ_1^- to Λ_1^+ both Legendrian submanifolds of the contactization of P_1. Let ι_2 be a Legendrian embedding of Λ_2 into the contactization of P_2. Define a Lagrangian immersion

$$V \boxtimes \Lambda_2 \looparrowright (\mathbb{R} \times P_1 \times \mathbb{R}, d(e^t(dz + a)))$$

by the assignment

$$\iota_1 \boxtimes \iota_2(u_1, u_2) = (\Pi_{\mathbb{R}\times P_1}(\iota_1(u_1)), \Pi_{P_2}(\iota_2(u_2)), z_1(\iota_1(u_1)) + z(\iota_2(u_2))).$$

In [11] Dimitroglou Rizell and Golovko proved the following.

Theorem 2.2.2 (Dimitroglou Rizell, Golovko, Theorem 1.14.,[11])**.**

- *The cobordism $V \boxtimes \Lambda_2$ is an immersed exact Lagrangian cobordism from $\Lambda_1^- \boxtimes \Lambda_2$ to $\Lambda_1^+ \boxtimes \Lambda_2$ which is embedded when the Reeb chords lengths of V are distinct from the Reeb chord lengths of Λ_2. Where the Reeb chord of V is defined as the length curve integral to the vector field $\frac{\partial}{\partial z}$ with endpoints on V and lying inside a slice $\{t\} \times P_1 \times \mathbb{R}$ for $t \in [-T, T]$.*
- *If $\Lambda_1 < \Lambda_2$ and Λ_1 is loose, then $\Lambda_1 \boxtimes \Lambda_2$ is loose.*

In particular,

Corollary 2.2.3. *If $\Lambda_1 < \Lambda_2$,*

- *Λ_1 is loose and Λ_2 is fillable, then $\Lambda_1 \boxtimes \Lambda_2$ is loose;*
- *Λ_1 is fillable and Λ_2 is loose, then $\Lambda_1 \boxtimes \Lambda_2$ is fillable.*

2.2.1 Cusp connected sum, Legendrian ambient surgery

In three dimensional case for Legendrian knots, or more generally one dimensional Legendrian manifolds in three dimensional tight contact manifolds we have a well-defined surgery producing a new Legendrian submanifold from former Legendrian submanifolds. This was proven in [27] by Etnyre and Honda.

The the question of defining a higher-dimensional surgery first appeared in [15]. There the construction proceed as follows.

Let Λ_1, Λ_2 be Legendrian submanifolds of $(\mathbb{R}^{2n+1}, \alpha_{st})$, and $K_1, K_2 \subset \mathbb{R}^{n+1}$ their fronts respectively. Moreover, assume that K_1 and K_2 can be divided by a hyperplane containing the z-direction. Consider $c : [-1, 1] \to \mathbb{R}^{n+1}$ a curve connecting two points on the cusp edges of K_1 and K_2. Choose T a tubular neighbourhood of c in $\mathbb{R}^{n+1}$ diffeomorphic to a subset of $[0, 1] \times \mathbb{R}^n$ so that there exists a function $r : [0, 1] \to (0, \infty)$ such that

- it has only one critical point which is the minimum attained for $s = 0$,
- and the vertical slice of the tubular neighbourhood, that is a subset of $\{c(s_0)\} \times \mathbb{R}^{n+1}$ is equal to $\{c(s_0)\} \times B_n(c(s_0), r(s_0))$ for some fixed $s_0 \in [-1, 1]$, and $B_n(c(s_0), r(s_0)))$ denotes a closed ball in $\mathbb{R}^n$ of radius $r(s_0)$ centred at $c(s_0)$.

We can define an edge as in the picture below so that the endpoints connect to the edges of corresponding front, denote this tube with edges by N.

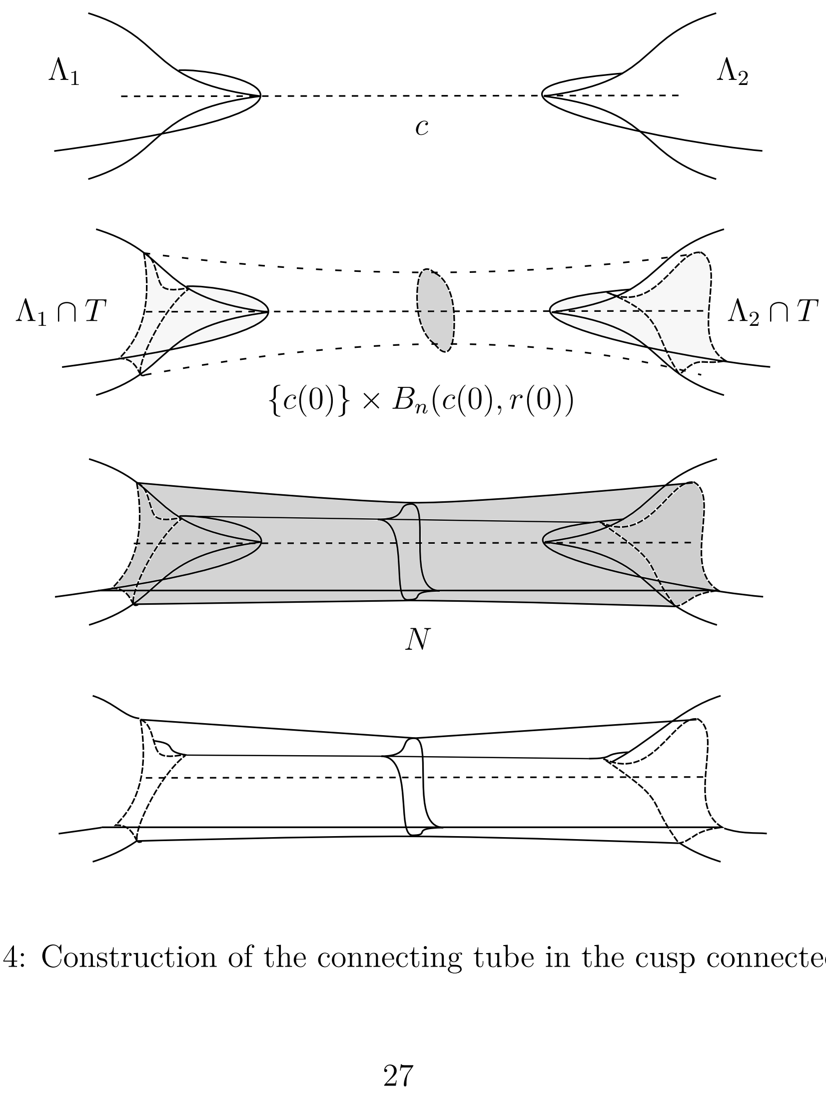

Figure 2.4: Construction of the connecting tube in the cusp connected sum.

Definition 2.2.4. *Let* Λ_1 *and* Λ_2 *be as above, then the* ***connected sum*** *is defined as the Legendrian submanifold obtained from the front given by the union and generic smoothening of* $\Lambda_1 \setminus (\Lambda_1 \cap N)$, $\Lambda_2 \setminus (\Lambda_2 \cap N)$ *and* ∂N.

Note that there has been made a lot of choices and it is not immediate whether the construction depends on them.

The answer to this question and much more is due to Dimitroglou Rizell and it can be found in [9], where the notion of Legendrian ambient surgery is introduced. There the income data for the construction are Λ a Legendrian submanifold of M a contact manifold (with additional assumption on the Reeb vector field dynamics), $\mathbf{S} \subset L$ a framed embedded sphere bounding $D_{\mathbf{S}}$ so-called isotropic surgery disk. As a result we obtain $\Lambda_{\mathbf{S}}$ a Legendrian submanifold obtained from Λ by a surgery on $\mathbf{S}$. In this thesis we will later use this construction for the case of the ambient contact manifold being $(J^1(M), dz + \eta)$, and when for $\mathbf{S} = \{-1, 1\}$.

Definition 2.2.5. *Let* $\Lambda \subset (J^1(M), dz + \eta)$ *be a Legendrian submanifold and let us have* $\boldsymbol{S} = \{-1, 1\} \to \Lambda$ *an embedding of* 0*-sphere and a choice of framing on* $T_{-1}\Lambda$ *and* $T_1\Lambda$*, where* $N\boldsymbol{S} = T_{-1}\Lambda \amalg T_1\Lambda$*. We say that* $D_{\boldsymbol{S}}$ *is an* ***isotropic surgery disk compatible with the framed sphere*** $\boldsymbol{S} \subset \Lambda$*, if there exists a choice of Lagrangian framing of* $(TD_{\boldsymbol{S}})^{\perp}/TD_{\boldsymbol{S}} \subset \xi = ker(dz + \eta)$ *so that:*

- $D_{\boldsymbol{S}} \subset J^1(M)$ *is an isotropic manifold,*
- $\partial D_{\boldsymbol{S}} = \boldsymbol{S}$ *and* $\operatorname{int} D_{\boldsymbol{S}} \cap \Lambda = \emptyset$,
- $(TD_{\boldsymbol{S}}^{\perp})|_{\boldsymbol{S}} \cap N\boldsymbol{S}$ *is an* $(n-1)$*-dimensional subspace.*

Remark. Let us remark that for a generic choice of two points of Σ the set of singular points of the front projection $\Pi_F(\Lambda)$ the isotropic surgery disk compatible with the framed sphere always exists as was observed in [10], Section 4.1. And so, we can always perform the connected sum for a particular choice of disconnected a Legendrian submanifold Λ.

Theorem 2.2.6 (Dimitroglou Rizell, Section 4.1, [9])**.** *Let* $\Lambda \subset (J^1(M), dz + \eta)$ *be a Legendrian submanifold* $\boldsymbol{S} \subset \Lambda$ *a framed embedded* 0*-sphere. Let* $D_{\boldsymbol{S}}$ *be a isotropic surgery disk compatible with* $\boldsymbol{S}$*, then there is a Legendrian submanifold* $\Lambda_{\boldsymbol{S}} \subset J^1(M)$ *and an exact Lagrangian cobordism* $V_{\boldsymbol{S}} \subset (\mathbb{R} \times J^1(M), d(e^t(dz + \eta)))$ *from* Λ *to* $\Lambda_{\boldsymbol{S}}$ *so that:*

- $V_{\boldsymbol{S}}$ *has the same diffeomorphism type as the one of a manifold that results from a handle-attachment on* $(-\infty, 0] \times \Lambda$ *along* $\boldsymbol{S} \subset \Lambda$ *that is identified with* $\partial((-\infty, 0] \times \Lambda)$.
- $\mathcal{R}(\Lambda_{\boldsymbol{S}}) = \mathcal{R}(\Lambda) \cup \{c_{\boldsymbol{S}}\}$*, where* $c_{\boldsymbol{S}}$ *is a Reeb chord of* $\Lambda_{\boldsymbol{S}}$.
- *Let us have two framed embeddings* S_i *of* $\boldsymbol{S}$ *in* Λ_i*, and* D_{S_i} *an isotropic surgery disk compatible with the framed sphere* S_i *for* $i = 1, 2$*. If there is an contact isotopy taking* $\Lambda_1 \cup D_{S_1}$ *to* $\Lambda_2 \cup D_{S_2}$*, and its tangent map maps the choice of Lagrangian framing for* D_{S_1} *onto the one of* D_{S_2}*, then* $(\Lambda_1)_{S_1}$ *is Legendrian isotopic to* $(\Lambda_2)_{S_2}$.

Definition 2.2.7. *We say that the Legendrian manifold* $\Lambda_{\boldsymbol{S}}$ *arises as the result of Legendrian ambient* 0*-surgery on* Λ *along* $\boldsymbol{S}$.

For illustration, consider a Legendrian link $L = \Lambda_1 \cup \Lambda_2$ in $(\mathbb{R}^3, \xi_{st})$, then denote $\pi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^2$ the projection given by $(t, x, y, z) \mapsto (t, x, z)$. See Figure 2.5.

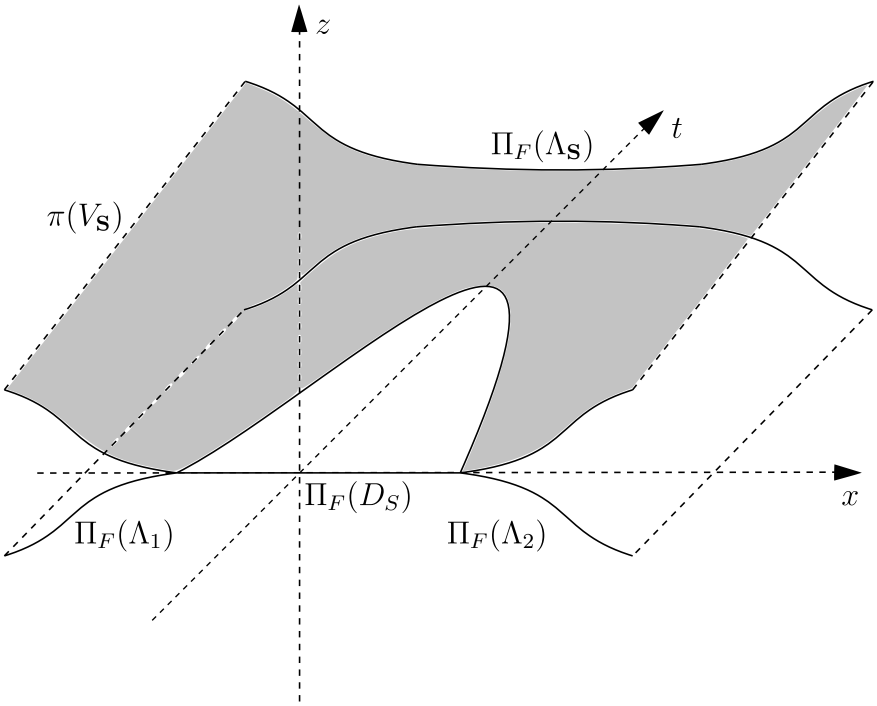

Figure 2.5: Front projection of the Legendrian ambient surgery cobordism for $n = 1$, where t is the coordinate of the $\mathbb{R}$-factor in the symplectization of $J^1(\mathbb{R}^2)$.

Remark. Let us note a that there is analogue of the previous construction of Lagrangian cobordisms comming from a particular handle attachment due to Bourgeois, Sabloff and Traynor. This can be found in [6]. In both [9] and [6], there is explained the effect on particular invariants. The difference of the constructions lies in the nature of those invariants. In [6] the invariant are so-called generating families, steming from Morse theory, while in [9] the invariant is the Chekanov-Eliashberg algebra that we describe in Chapter 5.

3. Classical invariants

3.1 Maslov and Conley-Zehnder index

There are multiple approaches how to define the Maslov index. The axiomatic one can be found in [38], page 48. The definition that we will adapt coincides with [15] and for proper treatment of this theory we refer to [42].

Let us consider the standard symplectic Euclidean space $(\mathbb{R}^{2n}, \omega_{st})$. Denote by $Gr_l(n)$ the Grassman manifold of Lagrangian subspaces in $\mathbb{R}^{2n}$ (recall that those are subspaces $V \leq \mathbb{R}^{2n}$ so that for each $v \in V$ we have $\omega(v, w) = 0$ for every $w \in V$). For a fixed Lagrangian subspace L of $\mathbb{R}^{2n}$ define the Maslov cycle as

$$\Sigma = \Sigma_1(L) \cup \Sigma_2(L) \cup \cdots \cup \Sigma_n(L),$$

where $\Sigma_k(L) \subset Gr_l(n)$ is a subset of all Lagrangian subspaces so that their intersection with L is a k-dimensional subspace. Note that Σ is an algebraic variety of codimension one in $Gr_l(n)$. Now consider a loop $\Gamma : [0, 1] \to Gr_l(n)$ that can be generically chosen to be transversal to Σ and so the intersection number in the sense of [34], page 131, is well-defined.

Definition 3.1.1. *For Γ a generic loop in $Gr_l(n)$ the* ***Maslov index*** *$\mu(\Gamma)$ is given as the intersection number of Γ and Σ. More specifically,*

$$\mu(\Gamma) = \sum_{t \in \Gamma \cap \Sigma} \operatorname{sgn}_t(\Gamma, \Sigma).$$

The Maslov index of Γ can be computed as follows. For a Lagrangian subspace L there is other Lagrangian subspace V of $\mathbb{R}^{2n}$ so that $L \oplus V = \mathbb{R}^{2n}$. Note that V is not uniquely determined, nevertheless, one can prove that the following construction is independent of the choice of V. The summand $\operatorname{sgn}_{t'}(\Gamma, \Sigma)$ for some t' is then computed as the signature of quadratic form Q defined on $\Gamma(t') \cap L$ as

$$Q(v) = \frac{d}{dt}\Big|_{t=t_0} \omega(v, w(t)),$$

where for t close to t' the vector $w(t) \in W$ is such that $v + w(t) \in \Gamma(t')$.

Now let $f : \Lambda \to \mathbb{R}^{2n}$ be a Lagrangian immersion, thus a generic loop γ in Λ induces Γ a generic loop in $Gr_l(n)$ via the pull-back of the tangent planes to $f(\Lambda)$ along $f(\gamma)$.

Definition 3.1.2. *The* ***Maslov index*** *$\mu(\gamma)$ of a loop γ in Λ for a fixed $f : \Lambda \to \mathbb{R}^{2n}$ Lagrangian immersion is given by $\mu(\gamma) = \mu(\Gamma)$. The* ***Maslov number*** *$m(f)$ is defined as*

$$m(f) = \min\{\mu(\gamma) \geq 0 : \gamma \textit{ is a generic non-trivial loop in } \Lambda \}.$$

If $g : \Lambda \to (\mathbb{R}^{2n+1}, \ker \alpha_{st})$ is a Legendrian embedding, then the ***Maslov number*** *$m(\Lambda)$ of Λ is given by $m(\Lambda) = m(\Pi_{\mathbb{R}^{2n}} \circ g)$.*

Definition 3.1.3. *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \ker \alpha_{st})$ be a chord generic Legendrian submanifold and let $c \in \mathcal{R}(\Lambda)$ be a Reeb chord with endpoints $a, b \in \Lambda$ so that $z(a) > z(b)$. The* ***capping path*** *γ is a path $\gamma : [0, 1] \to \Lambda$ with $\gamma(0) = a$ and $\gamma(1) = b$.*

Let γ be a capping from above, note that $\gamma' = \Pi_{\mathbb{R}^{2n}} \circ \gamma$ is a loop in $\Pi_{\mathbb{R}^{2n}}(\Lambda)$. Define $\Gamma' : [0,1] \to Gr_l(n)$ by

$$\Gamma'(t) = T\Pi_{\mathbb{R}^{2n}}(T_{\gamma(t)}\Lambda),$$

which is not a loop in $Gr_l(n)$ since Λ is assumed to be chord generic and so $\Gamma'(0) = \Pi_{\mathbb{R}^{2n}}(T_{\gamma(a)}\Lambda)$ and $\Gamma'(1) = \Pi_{\mathbb{R}^{2n}}(T_{\gamma(b)}\Lambda)$ are transverse, thus the endpoints of Γ' differ. We can choose J a complex structure $\mathbb{R}^{2n}$ such that

- J is compatible with ω (for all $z \in \mathbb{R}^{2n}$ it holds that $\omega(z, Jz) > 0$),
- $J(\Gamma(1)) = \Gamma(0)$.

Now consider a path $\lambda : [0, \frac{\pi}{2}] \to Gr_l(n)$ given by $\lambda(\Gamma(0), \Gamma(1))(t) = e^{tJ}(\Gamma(1))$. That clearly satisfies $\lambda(\Gamma(0), \Gamma(1))(0) = \Gamma(1)$ and $\lambda(\Gamma(0), \Gamma(1))(\frac{\pi}{2}) = \Gamma(0)$. By concatenating $\lambda(\Gamma(0), \Gamma(1))$ and Γ' we obtain a loop $\Gamma = \Gamma' * \lambda(\Gamma(0), \Gamma(1))$ in $Gr_l(n)$.

Definition 3.1.4. *Let $c \in \mathcal{R}(\Lambda)$ be a Reeb chord. Define for a capping path γ the **Conley-Zehnder index** $\nu_\gamma(c)$ as $\nu_\gamma(c) = \mu(\Gamma)$, where Γ is constructed from γ as above.*

Note that if we choose γ_1, γ_2 are two capping paths, then

$$\nu_{\gamma_1}(c) - \nu_{\gamma_2}(c) = \mu(\gamma_1 * (-\gamma_2)),$$

where $-\gamma_2$ is parametrized in the opposite direction as γ_2. And so if we choose γ_1, γ_2 so that $\mu(\gamma_1 * (-\gamma_2))$ is minimal in the homotopy class of $\gamma_1 * (-\gamma_2)$ we can define the Conley-Zehnder index modulo $m(\Lambda)$.

Definition 3.1.5. *Choose capping paths γ_j for each $c_j \in \mathcal{R}(\Lambda)$ and define $|c_j|$ a grading of Reeb chords by*

$$|c_j| = \nu_{\gamma_j}(c_j) - 1.$$

3.2 Rotation class

Let $(M, \xi = \ker\alpha)$ be a contact manifold and $(d\alpha|_\xi, J)$ be compatible (see Definition 1.3.2), we can view (ξ, J) as a complex bundle of the rank n where for each $\mathbb{X}, \mathbb{Y} \in \Gamma(\Lambda, \xi)$ smooth sections we consider the following identification

$$\mathbb{X} + i\mathbb{Y} \mapsto \mathbb{X} + J\mathbb{Y}.$$

Now let us complexify the bundle $T\Lambda$ that is a bundle $T\Lambda \otimes \mathbb{C} \to \Lambda$ where for $p \in \Lambda$ the the fiber of $T\Lambda \otimes \mathbb{C}$ is identified with the corresponding fibre of $(T\Lambda \oplus T\Lambda, J')$, where J' is defined by the assignment for all $v, w \in T_p\Lambda \oplus T_p\Lambda$: $J'(v, w) = (-w, v)$, by $v + iw \mapsto (v, w)$ for all $v, w \in T_p\Lambda$. The complex bundle $T\Lambda \otimes \mathbb{C}$ is of complex rank n. Denote by $Tf_{\mathbb{C}} : T\Lambda \otimes \mathbb{C} \to \xi$ its complexification, that is the map $Tf \oplus J \circ Tf : T\Lambda \oplus T\Lambda \to (\xi, J)$ sending $(\mathbb{X}, \mathbb{Y}) \mapsto Tf(\mathbb{X}) + J \circ Tf(\mathbb{Y})$. Locally, for $p \in \Lambda$ and $v, w \in T_p\Lambda$ it holds that

$$\begin{aligned} J \circ (Tf \oplus J \circ Tf)(v, w) &= (J \circ Tf(v) + J^2 \circ Tf(w)) \\ &= (J \circ Tf(v) - Tf(w)) \\ &= (Tf \oplus J \circ Tf)(-w, v) \\ &= (Tf \oplus J \circ Tf) \circ J'(v, w). \end{aligned}$$

And thus $Tf_{\mathbb{C}} \circ J' = J \circ Tf_{\mathbb{C}}$, that is $Tf_{\mathbb{C}}$ is fibre-wise a complex linear bundle map. Finally, because f is an immersion and the real rank of ξ is equal to twice the dimension of Λ, the map $Tf_{\mathbb{C}}$ is a complex fibre-wise bundle isomorphism.

Definition 3.2.1. *Homotopy class of $(f, Tf_{\mathbb{C}})$ in the space of complex fibre-wise isomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$ is called the* ***rotation class*** *of f and is denoted $r(f)$.*

$$\begin{array}{ccc} T\Lambda \otimes \mathbb{C} & \xrightarrow{Tf_{\mathbb{C}}} & (\xi, J) \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{\quad f \quad} & M \end{array}$$

If f is the inclusion embedding $\Lambda \subset M$, then we denote rotation class by $r(\Lambda)$.

Now we will use a form of the h-principle for Legendrian embeddings which directly follows from the h-principle for isotropic immersions.

Theorem 3.2.2 (Theorem 6.3.5,[29])**.** *Let (M, ξ) be a contact manifold and Λ be a n-dimensional manifold and $f_0 : \Lambda \to M$ be a map covered by a fiber-wise injective complex bundle map $F_0 : T\Lambda \otimes \mathbb{C} \to \xi$. Then f_0 is homotopic to an isotropic immersion $f_1 : \Lambda \to M$ so that $(f_1, (Tf_1)_{\mathbb{C}})$ is bundle homotopic to (f_0, F_0) through fiber-wise injective complex bundle maps. Moreover, the map f_1 and the homotopy between f_0 and f_1 can be chosen C^0-arbitrary close to f_0. If in addition is Λ assumed to be closed, then we can choose f_1 to be a Legendrian embedding.*

Therefore, as it was observed in [15],

Propositon 3.2.3 ([15], page 101)**.** *The rotation class $r(f)$ is complete invariant of f with respect to regular homotopy through Legendrian immersions. That means that for a fixed manifold Λ and two Legendrian embeddings $f, g : \Lambda \to M$ which are homotopic and $(f, Tf_{\mathbb{C}}), (g, Tg_{\mathbb{C}})$ are bundle homotopic through fiber-wise injective complex bundle maps we have that $r(f) = r(g)$.*

In the case $(M, \xi) = (\mathbb{R}^{2n+1}, \ker \alpha_{st})$, $\alpha_{st} = dz - \sum_{i=1}^{n} y_i dx_i$ in coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathbb{R}^{2n} \times \mathbb{R}$. Define a compatible complex structure J on ξ as follows

$$J(\partial_{x_j} + y_j \partial_z) = \partial_{y_j} \text{ and } J(\partial_{y_j}) = -\left(\partial_{x_j} + y_j \partial_z\right) \text{ for } j = 1, \dots, n.$$

Then the Lagrangian projection $\Pi_{\mathbb{C}} : \mathbb{R}^{2n+1} \to \mathbb{C}^n$ yields $(\mathrm{id}_M, T\Pi_{\mathbb{C}})$ a complex isomorphism from $(\xi, J) \to \mathbb{R}^{2n}$ to the trivial bundle $\mathbb{R}^{2n+1} \times \mathbb{C}^n \to \mathbb{R}^{2n+1}$, where we consider $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ equipped with the standard complex structure

$$J(\partial_{x_j}) = \partial_{y_j} \text{ and } J(\partial_{y_j}) = -\partial_{x_j} \text{ for } j = 1, \dots, n$$

with respect to the coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. The resulting situation is depicted by the following diagram.

$$\begin{array}{ccccc} T\Lambda \otimes \mathbb{C} & \xrightarrow{Tf_{\mathbb{C}}} & (\xi, J) & \xrightarrow{T\Pi_{\mathbb{C}}} & \mathbb{R}^{2n+1} \times \mathbb{C}^n \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda & \xrightarrow{\quad f \quad} & \mathbb{R}^{2n+1} & \xrightarrow{\mathrm{id}_{\mathbb{R}^{2n+1}}} & \mathbb{R}^{2n+1} \end{array}$$

Thus we may think of $Tf_{\mathbb{C}}$ as a trivialization $T\Lambda \otimes \mathbb{C} \to \mathbb{R}^{2n+1} \times \mathbb{C}^n$ in the following sense.

Definition 3.2.4. *If f is an embedding, then $F_{\mathbb{C}}(f)$ denotes the composition*

$$F_{\mathbb{C}}(f) = (f^{-1}\mid_\Lambda \times \mathrm{id}_{\mathbb{C}^n}) \circ T\Pi_{\mathbb{C}} \circ Tf_{\mathbb{C}}.$$

Then the complex bundle map $(\mathrm{id}_\Lambda, F_{\mathbb{C}})$ is a trivialization of the complex bundle $T\Lambda \otimes \mathbb{C} \to \Lambda$. See the diagram below.

$$\begin{array}{ccccc}
T\Lambda \otimes \mathbb{C} & \xrightarrow{T\Pi_{\mathbb{C}} \circ Tf_{\mathbb{C}}} & f(\Lambda) \times \mathbb{C}^n & \xrightarrow{f^{-1}|_\Lambda \times \mathrm{id}_{\mathbb{C}^n}} & \Lambda \times \mathbb{C}^n \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda & \xrightarrow{f} & f(\Lambda) & \xrightarrow{f^{-1}|_\Lambda} & \Lambda
\end{array}$$

Now from the proof of Lemma 2.5. in [12] it is immediate that if we choose a Hermitian metric μ on the bundle ξ, the map $Tf_{\mathbb{C}}$ will be a unitary map.

Lemma 3.2.5. *The group of continuous maps $\mathscr{C}(\Lambda, \mathrm{U}(n))$ acts freely and transitively on $\mathrm{U}(T\Lambda \otimes \mathbb{C}, \mathbb{R}^{2n+1} \times \mathbb{C}^n)$. Therefore, there is one-to-one correspondence of $\pi_0(\mathrm{U}(T\Lambda \otimes \mathbb{C}, \mathbb{R}^{2n+1} \times \mathbb{C}^n))$ and $[\Lambda, \mathrm{U}(n)]$.*

Proof. To make the notation less cumbersome G will represent the bundle map $(\mathrm{id}_\Lambda, G) \in \mathrm{U}_h(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$ in what follows.

Let us define the action

$$\Psi : \mathscr{C}(\Lambda, \mathrm{U}(n)) \times \mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n) \to \mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$$

by composition, that is $\Psi(\rho, G) = \rho \circ G$. Fix a point $p \in \Lambda$ and an element $\rho \in \mathscr{C}(\Lambda, \mathrm{U}(n))$ then it is clear that Ψ is free since the canonical action of $\mathrm{U}(n)$ on $\mathrm{U}(n)$ is a free action.

We know that $T\Lambda \otimes \mathbb{C}$ is trivial that is there are complex vector fields

$$\mathbb{X}_1, \ldots, \mathbb{X}_n \in \Gamma(\Lambda, T\Lambda \otimes \mathbb{C})$$

which are everywhere h-orthonormal, thus form an orthonormal basis B. In basis B any element G of $\mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$ gives rise to an element ΞG of $\mathscr{C}(\Lambda, \mathrm{U}(n))$ so that $p \mapsto [G_p]_{B_p}$. The correspondence $\Xi : \mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n) \to \mathscr{C}(\Lambda, \mathrm{U}(n))$ is obviously an bijection onto the set of smooth maps. Consider G, H two elements of $\mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$ and so we have the corresponding elements $\Xi G, \Xi H \in \mathscr{C}(\Lambda, \mathrm{U}(n))$ then define

$$\rho : p \mapsto \Xi H(p)(\Xi G(p))^{-1}$$

an element of $\mathscr{C}(\Lambda, \mathrm{U}(n))$. Now, at each point $p \in \Lambda$ we have that $\rho(p)\Xi G(p) = \Xi H(p)$ and by bijectivity of the correspondence Ξ we get that $\Psi(k, G) = \rho G = H$ and so the action is transitive.

If G, H are contained in the same path-component of $\mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$ then there is a continuous path $\gamma : [0, 1] \to \mathrm{U}(T\Lambda \otimes \mathbb{C}, M \times \mathbb{C}^n)$ such that $\gamma(0) = G$ and $\gamma(1) = H$. Then $\Gamma = \Xi \circ \gamma$ is the homotopy of ΞG and ΞH. And vice versa, if Γ is a homotopy of ΞG and ΞH, then $\gamma = \Xi^{-1} \circ \Gamma$ is the path from G to H.

□

Because of Lemma 3.2.5 we can think of $r(f)$ as an element in $[\Lambda, \mathrm{U}(n)]$.

For spheres then Bott periodicity (see [39], Theorem 23.5) yields the following.

Propositon 3.2.6. *If $\Lambda = \boldsymbol{S}^n$, then*

$$r(f) \in \pi_n(\mathrm{U}(n)) \cong \begin{cases} \mathbb{Z}\,; & n \text{ odd}, \\ 0\,; & n \text{ even}. \end{cases}$$

Thus for spheres we will refer to $r(f)$ as the **rotation number**.

3.3 Thurston-Bennequin invariant

Definition 3.3.1. *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ be a chord generic orientable Legendrian submanifold. The **sign** of a Reeb chord c of Λ from $b \in \Lambda$ to $a \in \Lambda$, $z(b) < z(a)$, is defined as follows.*

$$\operatorname{sgn}(c) = \begin{cases} 1 & \text{if the orientation of } V_a \oplus V_b \text{ agrees with} \\ & \text{the canonical orientation of } \mathbb{R}^{2n}, \\ -1 & \text{otherwise,} \end{cases}$$

where $V_a = T_a\Pi_{\mathbb{R}^{2n}}(T_a\Lambda)$ and $V_b = T_b\Pi_{\mathbb{R}^{2n}}(T_b\Lambda)$.

*The **Thurston-Bennequin invariant** $\operatorname{tb}(\Lambda)$ for Λ is then defined as*

$$\operatorname{tb}(\Lambda) = \sum_{c \in \mathcal{R}(\Lambda)} \operatorname{sgn}(c). \tag{3.1}$$

This is not the original higher-dimensional definition that is due to Tabachnikov, however, it is equivalent to it. For more details see Section 3.4. in [15].

Propositon 3.3.2. *The Thurston-Bennequin invariant $\operatorname{tb}(\Lambda)$ does not depend on the choice of the orientation of Λ.*

Theorem 3.3.3 (Proposition 3.2, [15])**.** *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ be an oriented Legendrian submanifold.*

- *The parity of $\operatorname{tb}(\Lambda)$ depends on $\mathrm{r}(\Lambda)$.*
- *When $n > 1$ is odd, then for any $k \in \mathbb{Z}$ we can find a Legendrian submanifold Λ' with $\operatorname{tb}(\Lambda) = 2k$ that is C^0 close to Λ and in addition both smoothly isotopic and Legendrian regular homotopic to Λ.*
- *In n is even, then*

  $$\operatorname{tb}(\Lambda) = \frac{(-1)^{\frac{n+2}{2}}}{2}\chi(\Lambda),$$

  where $\chi(\Lambda)$ is the Euler characteristic of Λ.

Propositon 3.3.4 (Proposition 3.3, [15])**.** *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ be an orientable chord generic Legendrian submanifold, then*

$$\operatorname{tb}(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \sum_{c \in \mathcal{R}(\Lambda)} (-1)^{|c|},$$

where $|c|$ is the grading defined using Conley-Zehnder index above.

3.4 Might of classical invariants

The rotation class and the Thurston-Bennequin invariant defined above are so-called classical invariants of Legendrian submanifolds. It is a classical result that those are enough to classify the Legendrian unknots up to Legendrian isotopy.

Theorem 3.4.1 (Eliashberg, Fraser, [22]). *Let $f_0, f_1 : \boldsymbol{S}^1 \to (\mathbb{R}^3, \xi_{st})$ be two Legendrian embeddings so that both are regularly homotopic to the embedding $f : \boldsymbol{S}^1 \to \mathbb{R}^3$, where $f(\boldsymbol{S}^1)$ is an unknot. Then f_0 and f_1 are Legendrian isotopic if and only if their classical invariants coincide.*

The Legendrian unknot is not the only case of knot family for which this phenomenon occurs.

Theorem 3.4.2 (Etnyre, Honda, [26]). *Both torus knots and figure eight Legendrian knots are classified by their smooth isotopy class, Thurston-Bennequin invariant, and rotation number.*

In particular, this means that the Legendrian condition is quite restrictive since the smooth isotopy class of a smooth knot might split into infinitely many Legendrian isotopy subclasses.

However, there are Legendrian knots that have the same classical invariants, even though they fail to be Legendrian isotopic.

Theorem 3.4.3 (Chekanov, [8]). *There are Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ with the same classical invariants that are not Legendrian isotopic.*

To prove this, Chekanov constructed combinatorially an invariant, which was later shown to fit into a bigger frame called Symplectic Field Theory (see [23]) that is due to Eliashberg, Givental, and Hofer, see Chapter 5. Along this line we have the following higher-dimensional result.

Theorem 3.4.4 (Ekholm, Etnyre, Sullivan, Theorem 1.1. in [15]). *For $n > 1$, there is an infinite class of n-dimensional Legendrian spheres in $(\mathbb{R}^{2n+1}, \xi_{st})$ so that they are not Legendrian isotopic, although their classical invariants coincide.*

Let us remark, that the Thurston-Bennequin invariant can be defined for any contact manifold (M, ξ) and not just for $(\mathbb{R}^{2n+1}, \xi_{st})$. In dimension three, we can combine both of classical invariants for a particular Legendrian submanifold to we obtain the information about the ambient contact structure of the 3-manifold.

Definition 3.4.5 (Definition 2.42, [31]). *Let (M, ξ) be a 3-dimensional contact manifold and $\Lambda \subset M$ is Legendrian submanifold that in a boundary of $\Sigma \subset M$ its Seifert surface. The following inequality is called the Thurston-Bennequin inequality.*

$$\mathrm{tb}(\Lambda) + |r(\Lambda)| \leq -\chi(\Sigma), \tag{3.2}$$

where χ is the Euler characteristic. Then we say that (M, ξ) is called:

- ***tight*** *if for any Λ as above, the inequality* (3.2) *holds,*
- ***overtwisted*** *otherwise.*

There is actually dichotomy between those two notions.

Theorem 3.4.6 (Eliashberg, [25])**.** *Let (M, ξ) be a 3-dimensional manifold, then ξ is either tight, or overtwisted. In particular, for (M, ξ_{ot}) for ξ_{ot} overtwisted and (M, ξ_{tg}) for ξ_{tg} tight contact structure on M are not contactomorphic.*

Eliashberg proved actually stronger result containing a version of homotopy principle for overtwisted contact structures and so the questions about them reduced to purely homotopical setting. This is why tight contact structures enjoy much more attention since they are in some sense rigid and so can tell us more about the ambient manifold. Using the Loose Legendrian embeddings in higher-dimensional submanifolds then in [2] Borman, Eliashberg and Murphy showed existence and classification of overtwisted contact structures in all dimensions. Note that our definition of overtwisted and tight contact structure applies just to 3-dimensional case.

Another application of the loose Legendrian embeddings for $n > 1$ is the following Lemma, which is a direct consequence of Remark A.5. in [40] due to Murphy.

Lemma 3.4.7. *If $n > 1$, $(\mathbb{R}^{2n+1}, \alpha_{st})$ is the standard contact manifold and Λ is a paralellizable n-dimensional manifold. Then for any homotopy class $\gamma \in [\Lambda \to U(n)]$ there is a loose Legendrian embedding $f : \Lambda \to U(n)$ so that $r(f) = \gamma$.*

4. Exceptional family of tori

In Chapter 2 we have seen several constructions of Legendrian submanifolds using various constructions imitating some form of rotation or product. It is a natural question, how those constructions differ with respect to Legendrian isotopy for a fixed diffeomorphism type of a manifold that is endowed with some product structure. The class of tori $\mathbf{T}^n = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ presents a natural choice. Considering the Legendrian product in comparison with the twist spun one can find that the following theorem holds.

Theorem 4.0.1 (Dimitroglou Rizell, Golovko, Theorem 1.16. in [11])**.** *There exist two Legendrian embeddings $\iota_i : \boldsymbol{S}^1 \to (\mathbb{R}^3, \xi_{st})$ so that the Legendrian product embedding $\iota_1 \boxtimes \iota_2 : \boldsymbol{T}^2 \to (\mathbb{R}^5, \xi_{st})$ is not Legendrian isotopic to any twist spun of a family of Legendrian knots.*

One can formulate a general question just for the Legendrian product construction:

Question 1. *Can be any Legendrian n-torus in $(\mathbb{R}^{2n+1}, \xi_{st})$ attained as the Legendrian product of Legendrian tori of lower-dimension?*

When $n = 1$ question degenerates to tautology. For $n = 2$ the question was originally posed in [11]. In this thesis Question 1 is answered in negative for $n = 3$.

Theorem (4.3.3)**.** *There exists an infinite family $T = (T_d)_{d \in \mathbb{Z}\setminus\{0\}}$ of loose Legendrian embeddings of 3-torus to $(\mathbb{R}^7, \xi_{st})$ so that*

- *for each $d \in \mathbb{Z} \setminus \{0\}$ the Legendrian embedding T_d is not Legendrian isotopic to a Legendrian product of embeddings of lower-dimensional tori,*
- *and if $d \neq d'$, then T_d and $T_{d'}$ are not Legendrian isotopic.*

Therefore, for $n \neq 0, 3$ this Question 1 remains open.

4.1 Splitting of rotation class

From now on Λ_1 and Λ_2 have distinct Reeb chord lengths. And $P_j = \mathbb{R}^{2n_j}$ with standard Liouville forms, where $n_j = \dim(\Lambda_j)$.

Let us for a moment imagine, that we have computed the rotation classes $r(\Lambda_j) \in [\Lambda_j, \mathrm{U}(n_j)]$ for $j = 1, 2$. Then Lemma 2.1.8 yields that $[\Lambda_1 \boxtimes \Lambda_2, \mathrm{U}(n_1+n_2)]$ is in bijection with $[\Lambda_1 \times \Lambda_2, \mathrm{U}(n_1 + n_2)]$. There is a canonical identification of $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$ with a subgroup of $\mathrm{U}(n_1 + n_2)$ given by

$$(V_1, V_2) \mapsto \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}.$$

And so we obtain that

$$\begin{aligned}[\Lambda_1, \mathrm{U}(n_1)] \times [\Lambda_2, \mathrm{U}(n_2)] &= [\Lambda_1 \times \Lambda_2, \mathrm{U}(n_1) \times \mathrm{U}(n_2)] \\ &\subseteq [\Lambda_1 \times \Lambda_2, \mathrm{U}(n_1 + n_2)] \\ &\cong [\Lambda_1 \boxtimes \Lambda_2, \mathrm{U}(n_1 + n_2)].\end{aligned}$$

Definition 4.1.1. *We say that $r(\Lambda_1 \boxtimes \Lambda_2) \in [\Lambda_1 \boxtimes \Lambda_2, \mathrm{U}(n_1+n_2)]$ **splits** if it can be represented by the pair $(r(\Lambda_1), r(\Lambda_2)) \in [\Lambda_1, \mathrm{U}(n_1)] \times [\Lambda_2, \mathrm{U}(n_2)]$.*

In what follows, we assume that both Λ_1 and Λ_2 are parallelizable, that is their tangent bundles are trivial, which in other words means that we have $\mathbb{X}_1^j, \ldots, \mathbb{X}_{n_j}^j$ for $j = 1, 2$ two sets of globally defined and everywhere linearly independent vector fields in $T\Lambda_j$ that give rise to corresponding global trivializations.

Let us start with a lemma with a quite long statement and short proof that we will need to prove the consecutive Theorem 4.1.3.

Lemma 4.1.2. *Denote by J_{st}^m the standard complex structure on $\mathbb{R}^{2m}$ for $m \in \mathbb{N}$. Let $n_1, n_2 \in \mathbb{N}$ and J_j be some complex structures on $\mathbb{R}^{n_j}$ and $\bar{J}_j$ be the matrix*

$$\bar{J}_j = \begin{pmatrix} J_j & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, denote by $J_\oplus$ the complex structure on $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ given by

$$J_\oplus = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

define $\bar{J}_\oplus$ similarly as $\bar{J}_j$ above. Let Q_k for $k = n_1, n_2, n_1 + n_2$ be a complex isomorphism of corresponding $(\mathbb{R}^{2n_j}, J_j) \to \mathbb{C}^{n_j}$ and $(\mathbb{R}^{n_1+n_2}, J_\oplus) \to \mathbb{C}^{n_1+n_2}$ and $\bar{Q}_k : \mathbb{R}^{2k+1} \to \mathbb{C}^k$ is the map that coinsides with Q_k on first $2k$ canonical basis vectors and the image of $(2k+1)$-th canonical basis vector vanishes. If we have real matrices A_j of type $(2n_j + 1) \times n_j$ so that their image lies in the span of the first $2n_j$ canonical vectors. Then for matrices

$$[G_1 \oplus G_2]_B = \begin{pmatrix} \bar{Q}_{n_1} A_1 & \bar{Q}_{n_1} \bar{J}_1 A_1 & 0 & 0 \\ 0 & 0 & \bar{Q}_{n_2} A_2 & \bar{Q}_{n_2} \bar{J}_2 A_2 \end{pmatrix}$$

and

$$[G]_C = \begin{pmatrix} \bar{Q}_{n_1} A_1 & 0 & \bar{Q}_{n_1} \bar{J}_1 & 0 \\ 0 & \bar{Q}_{n_2} A_2 & 0 & \bar{Q}_{n_2} \bar{J}_2 A_2 \end{pmatrix}$$

there are bases B and C of $\mathbb{R}^{2n_1+2n_2}$ so that $[\mathrm{id}]_C^B$ the transition matrix from basis B to basis C satisfy that

$$[G]_C = [G_1 \oplus G_2]_B [\mathrm{id}]_B^C \text{ and } \begin{pmatrix} J_{st}^{n_1} & 0 \\ 0 & J_{st}^{n_2} \end{pmatrix} [\mathrm{id}]_B^C = [\mathrm{id}]_B^C J_{st}^{n_1+n_2}. \tag{4.1}$$

Proof. Let

$$B = (v_1^1, \ldots, v_{n_1}^1, w_1^1 \ldots, w_{n_1}^1, v_1^2, \ldots, v_{n_2}^2, w_1^2, \ldots, w_{n_2}^2)$$

and

$$C = (v_1^1, \ldots, v_{n_1}^1, v_1^2, \ldots, v_{n_2}^2, w_1^1, \ldots, w_{n_1}^1, w_1^2, \ldots, w_{n_2}^2)$$

then the transition matrix is of the form

$$[\mathrm{id}]_B^C = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{pmatrix}$$

the rest of proof is obvious computation. $\square$

Theorem 4.1.3. *If Λ_j is parallelizable for both $j = 1, 2$ and they have distinct Reeb chord lengths, then the rotation class of their Legendrian product splits.*

Proof. We have two Legendrian embeddings $\iota_j : \Lambda_j \to \mathbb{R}^{2n+1}$ for $j = 1, 2$. By Lemma 2.1.8 we have that $\iota_1 \boxtimes \iota_2$ is also an embedding and so a diffeomorphism onto its image. Considering the set up above denote by $F_j = F_{\mathbb{C}}(\iota_j)$ the bundle map induced by ι_j. Then we have for $j = 1, 2$ that

$$\begin{array}{ccc} T\Lambda_j \otimes \mathbb{C} & \xrightarrow{F_j} & \Lambda_j \times \mathbb{C}^{n_j} \\ \downarrow & & \downarrow \\ \Lambda_j & \xrightarrow{\mathrm{id}_{\Lambda_j}} & \Lambda_j \end{array}$$

Then simply by product of those two diagrams we obtain the following product bundle map

$$\begin{array}{ccc} (T\Lambda_1 \otimes \mathbb{C}) \oplus (T\Lambda_2 \otimes \mathbb{C}) & \xrightarrow{F_1 \oplus F_2} & (\Lambda_1 \boxtimes \Lambda_2) \times \mathbb{C}^{n_1+n_2} \\ \downarrow & & \downarrow \\ \Lambda_1 \boxtimes \Lambda_2 & \xrightarrow{\mathrm{id}} & \Lambda_1 \boxtimes \Lambda_2 \end{array}$$

where we are using diffeomorphism $\Lambda_1 \times \Lambda_2 \simeq \Lambda_1 \boxtimes \Lambda_2$ from Lemma 2.1.8.

On the other hand, following the construction of $F = F_{\mathbb{C}}(\iota_1 \boxtimes \iota_2)$ we obtain the bundle map $(\mathrm{id}_{\Lambda_1 \boxtimes \Lambda_2}, F)$.

$$\begin{array}{ccc} T(\Lambda_1 \boxtimes \Lambda_2) \otimes \mathbb{C} & \xrightarrow{F} & (\Lambda_1 \boxtimes \Lambda_2) \times \mathbb{C}^{n_1+n_2} \\ \downarrow & & \downarrow \\ \Lambda_1 \boxtimes \Lambda_2 & \xrightarrow{\mathrm{id}} & \Lambda_1 \boxtimes \Lambda_2 \end{array}$$

We will show that there is a complex isomorphism S of the source bundles of our bundle maps F and $F_1 \oplus F_2$ so that after precomposition $(F_1 \oplus F_2) \circ S$ and F will coincide. In particular, the last two bundle maps will be bundle homotopic.

Let $\tilde{B}_j = (\mathbb{X}^j_1, \ldots, \mathbb{X}^j_{n_j})$ are smooth vector fields making $T\Lambda_j$ into a trivial bundle for $j = 1, 2$, and for $B_j = (\tilde{B}_j, i\tilde{B}_j)$ let $[-]_{B_j} : T\Lambda_j \otimes \mathbb{C} \to \Lambda_j \times \mathbb{C}^{n_j}$ be the coordinates bundle isomorphism acting as follows

$$\sum_{k=1}^{n_j}(v_k + iv_{n_j+1})\mathbb{X}^j_k(u_j) \mapsto (u_j, \sum_{k=1}^{n_j} v_k \oplus v_{n_j+k})$$

that is the bundle isomorphism to $\Lambda_j \times \mathbb{R}^{2n_j}$ where we view the fiber as $\mathbb{C}^{n_j}$ with the standard complex structure $J^{n_j}_{st}$.

And so we have a commutative diagram of bundle maps

$$\begin{array}{ccccc} & & T\Lambda_j \otimes \mathbb{C} & & \\ & \swarrow^{[-]_{B_j}} & & \searrow^{F_j} & \\ \Lambda_j \times \mathbb{C}^{n_j} & & \xrightarrow{[F_j]_{B_j}} & & \Lambda_j \times \mathbb{C}^{n_j} \end{array}$$

where $[F_j]_{B_j}$ denotes the composition $F_j \circ ([-]_{B_j})^{-1}$. This means that we can work with the bundle maps as with smooth families of matrices and vice versa, any matrix (a constant family of matrices) will give us a bundle map between $T\Lambda_j$ and $\Lambda_j \times \mathbb{C}^{n_j}$.

Since $\Lambda_1 \times \Lambda_2$ is diffeomorphic to $\Lambda_1 \boxtimes \Lambda_2$ by Lemma 2.1.8, they have isomorphic tangent bundles and so $T\Lambda_1 \oplus T\Lambda_2 = T(\Lambda_1 \times \Lambda_2) \cong T(\Lambda_1 \boxtimes \Lambda_2)$. Therefore, $\Lambda_1 \boxtimes \Lambda_2$ is also parallelizable. The complex bundle $T(\Lambda_1 \boxtimes \Lambda_2) \otimes \mathbb{C}$ then arises as a complexification with respect to the complex bundle structure $J_{st}^{n_1+n_2}$ the standard complex structure on $\mathbb{R}^{2(n_1+n_2)}$. While the complex bundle structure $J_\oplus$ on $(T\Lambda_1 \otimes \mathbb{C}) \oplus (T\Lambda_2 \otimes \mathbb{C})$ is given by the following matrix

$$\begin{pmatrix} J_{st}^{n_1} & 0 \\ 0 & J_{st}^{n_2} \end{pmatrix}$$

Denote by J_j the canonical complex structure on the contact structure of $\mathbb{R}^{2n_j+1}$ and observe that the standard contact structure on $\mathbb{R}^{2(n_1+n_2)+1}$ has the canonical complex structure equal to

$$J_\oplus = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

Racall the bar notation form Lemma 4.1.2. It is immediate from the construction that $[F]_C$ is of the form

$$\begin{pmatrix} \Pi_{\mathbb{C}^{n_1}} \circ T\iota_1 & 0 & \Pi_{\mathbb{C}^{n_1}} \circ \bar{J}_1 T\iota_1 & 0 \\ 0 & \Pi_{\mathbb{C}^{n_2}} \circ T\iota_2 & 0 & \Pi_{\mathbb{C}^{n_2}} \circ \bar{J}_2 T\iota_2 \end{pmatrix}$$

in the basis $C = (\tilde{B}_1, \tilde{B}_2, i\tilde{B}_1, i\tilde{B}_2)$ which is the real basis of complexification of $T\Lambda_1 \oplus T\Lambda_2$. This is because

$$\Pi_{\mathbb{C}^{n_1+n_2}} \circ T(\iota_1 \boxtimes \iota_2) = \begin{pmatrix} \Pi_{\mathbb{C}^{n_1}} \circ T\iota_1 & 0 \\ 0 & \Pi_{\mathbb{C}^{n_2}} \circ T\iota_2 \end{pmatrix}$$

$$\Pi_{\mathbb{C}^{n_1+n_2}} \circ \bar{J}_\oplus T(\iota_1 \boxtimes \iota_2) = \begin{pmatrix} \Pi_{\mathbb{C}^{n_1}} \circ \bar{J}_1 T\iota_1 & 0 \\ 0 & \Pi_{\mathbb{C}^{n_2}} \circ \bar{J}_2 T\iota_2 \end{pmatrix}$$

While the maps $[F_j]_{B_j}$ are of the form

$$\begin{pmatrix} \Pi_{\mathbb{C}^{n_j}} \circ T\iota_j & \Pi_{\mathbb{C}^{n_j}} \circ J_j T\iota_j \end{pmatrix}$$

with respect to (B_j, iB_j) the real basis of complexification of $T\Lambda_j$. And so we obtain that $[F_1 \oplus F_2]_C$ is of the form

$$\begin{pmatrix} \Pi_{\mathbb{C}^{n_1}} \circ T\iota_1 & \Pi_{\mathbb{C}^{n_1}} \circ \bar{J}_1 T\iota_1 & 0 & 0 \\ 0 & 0 & \Pi_{\mathbb{C}^{n_2}} \circ T\iota_2 & \Pi_{\mathbb{C}^{n_2}} \circ \bar{J}_2 T\iota_2 \end{pmatrix}$$

with respect to $C = (B_1, B_2)$ the real basis of the sum of complexifications of $T\Lambda_1$ and $T\Lambda_2$.

Now we use Lemma 4.1.2 and so we see that that the maps $F_1 \oplus F_2$ and F do not differ since one is equal to the other if we change the basis on the domain. The transition matrix induces the complex isomorphism S.

□

4.2 Degree map

Let us have an n-torus $\mathbf{T}^n$ and an n-sphere $\mathbf{S}^n$.

Lemma 4.2.1. *For any non-zero $d \in \mathbb{Z}$ there is a map $f_d : \boldsymbol{S}^n \to \boldsymbol{S}^n$ so that $\deg f_d = d$.*

Proof. Define map f_d using spherical coordinates on $\mathbf{S}^n$

$$\begin{aligned} x_1 &= \cos\phi_1 \\ x_2 &= \sin\phi_1 \cos\phi_2 \\ x_3 &= \sin\phi_1 \sin\phi_2 \cos\phi_3 \\ &\vdots \\ x_{n-1} &= \sin\phi_1 \dots \sin\phi_{n-2} \cos\phi_{n-1} \\ x_n &= \sin\phi_1 \dots \sin\phi_{n-1} \cos\phi_n \\ x_{n+1} &= \sin\phi_1 \dots \sin\phi_{n-1} \sin\phi_n, \end{aligned}$$

for $\phi_j \in (0, \pi)$ where $j = 1, \dots, n$ and $\phi_n \in \mathbb{R}/2\pi\mathbb{Z}$ by the assignment

$$(\phi_1, \dots, \phi_{n-1}, \phi_n) \mapsto (\phi_1, \dots, \phi_{n-1}, d\phi_n).$$

By straightforward computation one obtains that the Jacobian of f_d is equal to

$$d \prod_{k=1}^{n-1} \sin^{n-k} \phi_k.$$

We observe that the point $p = (0, \dots, 0, 1, 0)$ given by $(\phi_1, \dots, \phi_{n-1}, \phi_n) = (\frac{\pi}{2}, \dots, \frac{\pi}{2}, 0)$ has preimage consisting of d points and that it is a regular point of f_d. Therefore,

$$\deg(f_d) = \sum_{q \in f_d^{-1}(p)} \operatorname{sgn}(\det(d_q f_d)) = \sum_{q \in f_d^{-1}(p)} \operatorname{sgn}(d \prod_{k=1}^{n-1} 1^{n-k}) = \sum_{q \in f_d^{-1}(p)} \operatorname{sgn}(d) = d.$$

□

Lemma 4.2.2. *There is a map $F_d : \boldsymbol{T}^n \to \boldsymbol{S}^n$ which has a non-zero degree d.*

Proof. Denote $N = (1, 0, \dots, 0)$ the north pole of $\mathbf{S}^n$. Consider $\mathbf{S}^n$ as a submanifold of $\mathbb{R}^{n+1}$ and a smooth retraction $r : \mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbf{S}^n$ defined as $\mathbf{x} \mapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}$ for each $\mathbf{x} \in \mathbb{R}^{n+1}$. Using σ the stereographic projection from the north pole one can easily prove that $\mathbf{S}^n \setminus \{N\}$ is diffeomorphic to $\mathbb{R}^n$. Pick an arbitrary point $p \in \mathbf{T}^n$, there exists $\mathcal{U}$ an open neighbourhood of p in $\mathbf{T}^n$ and a diffeomorphism $\varphi' : \mathcal{U} \to \mathbb{R}^n$ so that $\varphi'(p) = (0, \dots, 0)$. Denote $\varphi = \sigma^{-1} \circ \varphi' : \mathcal{U} \to \mathbf{S}^n \setminus \{N\}$ and note that it is a diffeomorphism mapping p onto $S = (-1, 0, \dots, 0)$ the south pole in $\mathbf{S}^n$.

Let $\mathfrak{U} = (\mathcal{U}_\alpha)_{\alpha \in A}$ be any open covering of $\mathbf{T}^n$ satisfying:

- there is $\alpha_0 \in A$ so that $\mathcal{U} = \mathcal{U}_{\alpha_0}$,
- there is $\mathcal{V} \subset \mathcal{U}$ an open neighbourhood of p so that for every $\alpha \in A$ distinct from α_0 we have that $\mathcal{V} \cap \mathcal{U}_\alpha = \emptyset$.

Moreover, consider $(\rho_\alpha)_{\alpha\in A}$ a partition of unity subordinate to $\mathfrak{U}$. Define the map $h : \mathbf{T}^n \to \mathbb{R}^{n+1}$ by the following sum

$$h = \rho_{\alpha_0}\varphi + \sum_{\alpha\in A\setminus\{\alpha_0\}} \rho_\alpha N. \tag{4.2}$$

This yields that h is smooth. In addition, since we assumed the existence of $\mathcal{V}$ above, observe that $h : \mathbf{T}^n \to \mathbb{R}^{n+1} \setminus \{(0,\dots,0)\}$, which yields that $g = r\circ h : \mathbf{T}^n \to \mathbf{S}^n$ is a smooth map.

Consider $p : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection onto the first coordinate. Denote by B the compact $\mathbf{T}^n \setminus \mathcal{V}$, then $p\circ g|_B$ attains a minimum m on B which clearly satisfies $m \geq 1$. Since for every $s \in B\cap\mathcal{U}$ it holds that $p\circ h(s) > \varphi(s)$, then $m > 1$.

It is a classical fact that $H^n_{DR}(\mathbf{S}^n)$ the de Rham cohomology of $\mathbf{S}^n$ is generated by bump forms β whose support can be made arbitrarily small and $\int_{\mathbf{S}^n}\beta = 1$. Moreover, it can be shown that the notion of degree defined using simplicial homology coincides with the notion of degree defined as follows:

$$\deg(g) = \int_{\mathbf{T}^n} g^*\beta = \sum_{q\in g^{-1}(P)} \operatorname{sign}(\det(d_q g)) \text{ where } P\in\mathbf{T}^n \text{ is a regular point of } g. \tag{4.3}$$

For details on those classical results see Bott and Tu [3].

Let us set $\mathcal{W} = p^{-1}((-\infty,m))\cap g(\mathcal{V})$ and note that $\mathcal{W}\cap g(\mathcal{U}_\alpha) = \emptyset$ for every $\alpha\in A$ distinct from α and that $\mathcal{W}$ is an open neighbourhood of $S\in\mathbf{S}^n$. Denote $(g|_\mathcal{V})^{-1}(\mathcal{W})$ by $\mathcal{W}'$ where $\mathcal{W}'\subset\mathcal{V}$. This means that $g|_{\mathcal{W}'} : \mathcal{W}'\to\mathcal{W}$ is a diffeomorphism which coinsides with $\varphi|_{\mathcal{W}'}$, therefore,

$$\operatorname{sign}(\det(d_p g)) \neq 0.$$

Choose a bump form β as above so that the support of β is a subset of $\mathcal{W}$, this implies that the support of $g^*\beta$ is a subset of $\mathcal{W}'$. Consequently, $g^*\beta = (\varphi|_{\mathcal{W}'})^*\beta$ on $\mathcal{W}'$ and $g^*\beta = 0$ otherwise. Let us compute the degree of g:

$$\deg(g) = \int_{\mathbf{T}^n} g^*\beta = \int_\mathcal{V} (\varphi|_{\mathcal{W}'})^*\beta = \sum_{q\in g^{-1}(S)} \operatorname{sign}(\det(d_q g)) = \operatorname{sign}(\det(d_p g)) \neq 0.$$

Finally, thanks to Lemma 4.2.1 there are smooth maps $f_d : \mathbf{S}^n \to \mathbf{S}^n$ which are of non-zero degree $d\in\mathbb{Z}$. And so it suffices to set $f = f_d\circ g$ to obtain a smooth map with non-zero degree since the composition of maps acts as multiplication of their degrees.

□

4.3 Non-splittablity

Lemma 4.3.1. *Let X, Y be path-connected topological manifolds and $f, g : X \to Y$ to homotopy equivalent continuous maps, where $H : I \times X \to Y$ is the homotopy witnessing $f \sim g$. Let Z be another path-connected topological manifold and $\pi : Y \to Z$ be a continuous map. Then $\pi \circ H$ is the homotopy of the maps $\pi \circ f$ and $\pi \circ g$.*

Proof. By definition of a homotopy the continuous map $H : I \times X \to Y$ satisfies $H(0, x) = g(x)$ and $H(1, x) = f(x)$. Then as a composition of two continuous maps $\pi \circ H$ is a continuous map. Moreover, it is obvious that $\pi \circ H(0, x) = \pi \circ g(x)$ and $\pi \circ H(1, x) = \pi \circ f(x)$. □

Thanks to Lemma 4.2.2 we can consider a map $f : \mathbf{T}^3 \to \mathbf{S}^3$ of a non-zero degree $d \in \mathbb{Z}$. Before moving to the next result, let us fix some notation and identifications. Let $i : \mathbf{S}^3 \to U(2)$ be a canonical inclusion and $p : U(2) \to \mathbf{S}^3$ the canonical projection which are induced by the identification $U(2) = \mathbf{S}^3 \times \mathbf{S}^1$. Denote by $j : U(2) \to U(3)$ the smooth injective map defined by the assignment $A \mapsto \operatorname{diag}(1, A)$ and by $q : U(3) \to U(2)$ the projection defined as $(a_{ij})_{i,j=1,2,3} \mapsto (a_{ij})_{i,j=2,3}$. for all $A \in U(2)$.

Let us recall some well-known facts. The Lie group $U(2)$ acts on $U(3)$ by right multiplication by $j(A)$ for all $A \in U(2)$ and this action is transitive. The stabilizer of the matrix $B \in U(3)$ with the first column equal to the first canonical vector is the whole $U(2)$. Then the quotient of the Lie group $U(3)$ by the stabilizer of B, that is $U(3)/U(2)$, has a structure of a smooth manifold. In addition, if we identify (it is even a diffeomorphism) the sphere $\mathbf{S}^5$ with the set $\{(b_{11}, b_{21}, b_{31}) \in \mathbb{C}^3 : \sum_{i=1}^{3} |b_{i1}|^2 = 1\}$ then we obtain the diffeomorphism

$$U(3)/U(2) \cong \mathbf{S}^5 \tag{4.4}$$

and the canonical quotient map $\pi : U(3) \to \mathbf{S}^5$ wich amounts to the projection onto the first column.

Lemma 4.3.2. *For each $d \in \mathbb{Z}$ the map*

$$\boldsymbol{F}_d = j \circ i \circ F_d : \boldsymbol{T}^3 \to U(3)$$

defined as a composition of the maps above does not split.

Proof. For the sake of contradiction, let us suppose that $\mathbf{F}_d$ is homotopy equivalent to some map

$$G : \mathbf{S}^1 \times \mathbf{T}^2 \to U(3),$$

which is given by $G(\theta) = \operatorname{diag}(g_1(\theta_1), g_2(\theta_2, \theta_3))$, where $\theta = (\theta_1, \theta_2, \theta_3)$. This, in particular, means that there is homotopy $H : I \times \mathbf{T}^3 \to U(3)$ so that $H(0, \theta) = G(\theta)$ and $H(1, \theta) = \mathbf{F}_d(\theta)$. By the previous Lemma 4.3.1 and the identifications above we obtain that $\widetilde{H} = \pi \circ H$ is the homotopy of $\pi \circ G$ and $\pi \circ \mathbf{F}_d$. Observe that

$$\pi \circ G(\theta) = \begin{pmatrix} g_1(\theta_1) \\ 0 \\ 0 \end{pmatrix} \text{ and } \pi \circ \mathbf{F}_d(\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, considering $\pi' : \mathbf{S}^5 \to \mathbf{S}^1 \simeq U(1)$ the projection onto the first coordinate and again using Lemma 4.3.1 we obtain that $g_1 : \mathbf{S}^1 \to U(1)$ is homotopy equivalent to a constant map $1 : \mathbf{S}^1 \to \{1\} \subseteq \mathbf{S}^1$. That yields that G is homotopy equivalent to $G' : \mathbf{T}^3 \to U(3), \theta \mapsto \mathrm{diag}(1, g_2(\theta_2, \theta_3))$ and so is $\mathbf{F}_d$. Define the projection $P = p \circ q : U(3) \to \mathbf{S}^3$. Again we use Lemma 4.3.1 considering the maps $\mathbf{F}_d, G'$ and the projection P and so we obtain that the two maps $\widetilde{F}, \widetilde{G} : \mathbf{T}^3 \to \mathbf{S}^3$ are homotopic, where $\widetilde{F} = P \circ \mathbf{F}_d$ and $\widetilde{G} = P \circ G'$. Note that $q \circ j = \mathrm{id}_{U(2)}$ and $p \circ i = \mathrm{id}_{\mathbf{S}^3}$, therefore $\widetilde{F} = P \circ \mathbf{F}_d = p \circ q \circ \mathbf{F}_d = p \circ q \circ j \circ i \circ F_d = F_d$. In other words, F_d is homotopic to $\widetilde{G} = p \circ g_2$.

By the construction of $\widetilde{G}$ it is obvious that $\widetilde{G}$ factors through $\mathbf{T}^2$ as follows

$$\begin{array}{ccc} \mathbf{T}^3 & \xrightarrow{\widetilde{G}} & \mathbf{S}^3 \\ {\scriptstyle p_{23}}\downarrow & \nearrow {\scriptstyle g_2'} & \\ \mathbf{T}^2 & & \end{array},$$

where $p_{23} : \mathbf{T}^3 \to \mathbf{T}^2$ is defined as $(\theta_1, \theta_2, \theta_3) \mapsto (\theta_2, \theta_3)$. And so in homology we have that

$$\begin{array}{ccc} \mathbb{Z} \cong H_3(\mathbf{T}^3) & \xrightarrow{\widetilde{G}_*} & H_3(\mathbf{S}^3) \cong \mathbb{Z} \\ {\scriptstyle (p_{23})_*}\downarrow & \nearrow {\scriptstyle (g_2')_*} & \\ 0 \cong H_3(\mathbf{T}^2) & & \end{array},$$

which yields that $(\widetilde{G})_*$ act as the multiplication by zero on homology groups of degree and so $deg(\widetilde{G}) = 0$. However, this is a contradiction since we assumed that $deg(f) = d \neq 0$ and homotopic maps must induce the same map in homology, in particular they must have the same degree. □

Theorem 4.3.3. *There exists an infinite family $T = (T_d)_{d \in \mathbb{Z} \setminus \{0\}}$ of loose Legendrian embeddings of 3-torus to $(\mathbb{R}^7, \xi_{st})$ so that*

- *for each $d \in \mathbb{Z} \setminus \{0\}$ the Legendrian embedding T_d is not Legendrian isotopic to a Legendrian product of embeddings of lower-dimensional tori,*
- *and if $d \neq d'$, then T_d and $T_{d'}$ are not Legendrian isotopic.*

Proof. For the first part, let $\mathbf{F}_d : \mathbf{T}^3 \to U(3)$ be the map from Lemma 4.3.2 and $[\mathbf{F}_d]$ be its homotopy class. From Lemma 3.4.7 we know that there is an loose Legendrian embedding $T_d : \mathbf{T}^3 \to \mathbb{R}^7$ so that $r(T_d) = [\mathbf{F}_d]$.

Note that $\mathbf{T}^3$ is parallelizable since $\mathbf{S}^1$ is and $\mathbf{T}^3 = \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, similarly for $\mathbf{T}^2$. If T_d was a Legendrian product of lower-dimensional tori, then we would have two Legendrian embeddings $\iota_1 : \mathbf{S}^1 \to U(1)$ and $\iota_2 : \mathbf{T}^2 \to U(2)$ so that $r(T_d) = r(\iota^1) \times r(\iota^2)$ by Theorem 4.1.3, which is in contradiction with the statement of Lemma 4.3.2.

For the second part, if $T_{d'}$ and T_d were Legendrian isotopic, then $r(T_d) = r(T_{d'})$ and so we would have a homotopy of $\mathbf{F}_d$ and $\mathbf{F}_{d'}$. Let $P : U(3) \to \mathbf{S}^3$ be the projection form Lemma 4.3.2, then using Lemma 4.3.1 we have that $F_d = P \circ \mathbf{F}_d$ and $F_{d'} = P \circ \mathbf{F}_{d'}$ are homotopic. That is in contradiction with the well-known fact that the degree is an invariant under homotopy. □

5. Legendrian contact homology

5.1 Chekanov-Eliashberg algebra

Let M be a smooth manifold of dimension n and $J^1(M) = T^*M \times \mathbb{R}$ be the one-jet space of M. Recall that we have a canonical contact structure on $J^1(M)$ given by $\xi = \ker(\alpha)$, $\alpha = dz - \eta$, where z is the coordinate of the vertical fibre $\mathbb{R}$ over $T*M$ and η is the canonical one form so that its differential is the canonical exact symplectic form $d\eta$ on T^*M. Recall that we have two canonical projections $\Pi_{T^*M} : J^1(M) \to T^*M$ the Lagrangian projection and $\Pi_F : J^1(M) \to J^0(M) = M \times \mathbb{R}$ the front projection. Let Λ be a compact chord generic Legendrian submanifold of $J^1(M)$. Then recall that the Reeb chords associated with α above are exactly the segments of integrated flow of the vector field given by $\frac{\partial}{\partial z}$ starting and ending on Λ, that is vertical segments c and parametrized for some $T > 0$ as $c : [0, T] \to J^1(M)$ so that $c(0), c(T) \in \Lambda$ and

$$\frac{d}{dt}\Big|_{t=t_0} c(t) = \left(\frac{\partial}{\partial z}\right)_{c(t_0)}.$$

Moreover, we can easily observe that in this case $\ell(c)$ the length of c is equal to T. In Definition 3.1.5 we have defined

$$|\cdot| : \mathcal{R}(\Lambda) \to \mathbb{Z}_{m(\Lambda)}$$

the grading on $\mathcal{R}(\Lambda)$ Reeb chords of Λ. Note that the assumptions on Λ yield that $\mathcal{R}(\Lambda)$ is a finite set, notation $\#\mathcal{R}(\Lambda) < \infty$.

Definition 5.1.1. *Let $\Lambda \subset (J^1(M), \xi)$ be a compact chord generic Legendrian submanifold and $C(\mathcal{R}(\Lambda))$ is the vector space over $\mathbb{Z}_2$ generated by $\mathcal{R}(\Lambda)$. Then the tensor algebra $T(C(\mathcal{R}(\Lambda)))$ of $C(\mathcal{R}(\Lambda))$ is called the **Chekanov-Eliashberg algebra** and is denoted by $\mathcal{A}(\Lambda)$.*

The Chekanov-Eliashberg algebra can be endowed with the structure of differential graded algebra.

Definition 5.1.2. *Let for $I = (i_1, \ldots, i_k) \in \{1, \ldots, \#\mathcal{R}(\Lambda)\}^k$ and $c_I = c_{i_1} \otimes c_{i_2} \otimes \ldots \otimes c_{i_k} \in T^k(C(\mathcal{R}(\Lambda)))$ define the grading $|c_I|$ as*

$$|c_I| = \sum_{j=1}^{k} |c_{i_j}|.$$

In particular, if $k = 0$, then $c_\emptyset \in \mathbb{Z}_2$ and $|c_\emptyset| = 0$. Now extend linearly the grading to $T^k(C(\mathcal{R}(\Lambda)))$ and then define the grading for the product of $c \in T^k(C(\mathcal{R}(\Lambda)))$ and $c' \in T^l(C(\mathcal{R}(\Lambda)))$ as

$$|c \otimes c'| = |c| + |c'|.$$

Consider $(\mathbb{R} \times J^1(M), \omega = d(e^t p^*(\alpha)))$ the symplectization of $J^1(M)$ (see Definition 1.1.7), then the complex structure $J : \xi \to \xi$ compatible with $d\alpha$ can be extended to a almost complex structure on $\mathbb{R} \times J^1(M)$ compatible with ω by setting $J(\frac{\partial}{\partial t}) = \frac{\partial}{\partial z}$.

Definition 5.1.3. *Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$, where we can view $\mathbb{C}$ as $\mathbb{R}^2$ with a complex structure j. Let us have $z_1^-, \dots, z_k^-, z^+ \in \partial\mathbb{D}$. Define $\hat{\mathbb{D}}$ as $\mathbb{D} \setminus \{z_1^-, \dots, z_k^-, z^+\}$. Then a* ***punctured pseudo-holomorphic disk*** *is a map $u : \mathbb{D} \to \mathbb{R} \times J^1(M)$ so that we have parametrized Reeb chords $c_1^-, \dots, c_k^-, c^+ \in \mathcal{R}(\Lambda)$ of length $T_1^-, \dots, T_k^-, T^+$, $u((\partial\mathbb{D}) \setminus \{z_1^-, \dots, z_k^-, z^+\}) \subset \mathbb{R} \times \Lambda$ and u maps the ends of segments of the boundary of $\hat{\mathbb{D}}$ to the endpoints of Reeb chords so that denoting $u = (a, f)$*

- $J \circ Tu = Tu \circ j$,
- $\lim_{z \to z_j^-} a(z) = -\infty$ *and* $\lim_{z \to z^+} a(z) = +\infty$,
- $\lim_{z \to z_j^-} f(z) = c_j^-(\frac{T_j^-}{\pi} \arg(z - z_j^-))$ *and* $\lim_{z \to z^+} f(z) = c^+(\frac{T^+}{\pi} \arg(z - z^+))$.

We say that u has a ***positive puncture*** *at c^+ and* ***negative punctures*** *at c_j^-.*

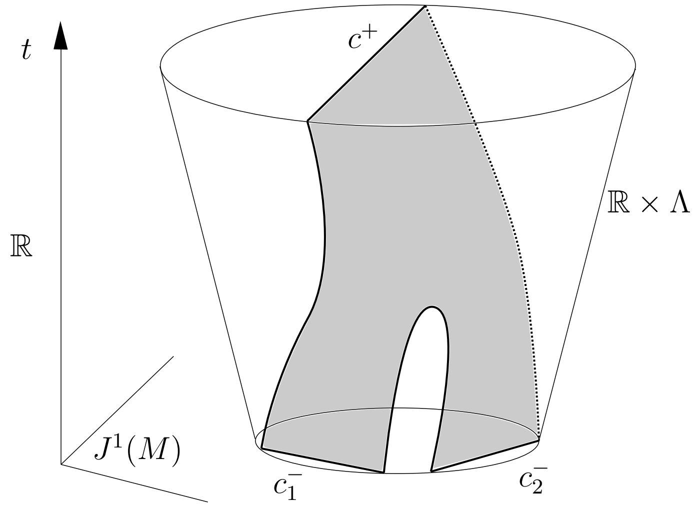

Figure 5.1: Example of a punctured pseudo-holomorphic disk with one positive puncture c^+ and two negative punctures c_1^- and c_2^+.

Definition 5.1.4. *Let $k \geq 0$ and $c_1^-, \dots, c_k^-, c^+ \in \mathcal{R}(\Lambda)$. Choose capping paths $\gamma_1^-, \dots, \gamma_k, \gamma^+$ for the Reeb chords respectively and $A = u(\partial\mathbb{D}) \cup (\bigcup_{j=1}^k \gamma_j) \in H_1(\Lambda)$. Denote by*

$$\mathcal{M}_A(c^+; c_1^-, \dots, c_k^-)$$

the set of all punctured pseudo-holomorphic disks with positive puncture at c^+ and negative punctures at c_j^- modulo biholomorphisms of the domain.

Note that there is a canonical $\mathbb{R}$ action on $\mathcal{M}_A(c^+; c_1^-, \dots, c_k^-)$ above given by the translation in the t-direction. And so we have well-defined quotient $\mathcal{M}_A(c^+; c_1^-, \dots, c_k^-)/\mathbb{R}$.

In the space of all Legendrian submanifolds there is a dense subset of so-called admissible Legendrian submanifolds for which the sets $\mathcal{M}_A(c^+; c_1^-, \dots, c_k^-)/\mathbb{R}$ for suitable choice of chords and their capping paths have the structure of smooth manifold (see Proposition 2.2 in [15]).

Theorem 5.1.5. *For generic admissible Legendrian submanifold $\Lambda \subset J^1(M)$ we have that if*

$$1 \geq d = \mu(A) + |c^+| + \sum_{j=1}^{k} |c_j^-| - 1,$$

then $\mathcal{M}_A(c^+; c_1^-, \ldots, c_k^-)/\mathbb{R}$ has a structure of d-dimensional manifold. In particular, if $d < 0$, then $\mathcal{M}_A(c^+; c_1^-, \ldots, c_k^-)/\mathbb{R}$ is empty.

Propositon 5.1.6. *If $u \in \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R}$ then*

$$\ell(c) - \sum_{j=1}^{k} \ell(c_j^-) > 0, \tag{5.1}$$

in particular, any u must have at a positive puncture.

Definition 5.1.7. *For $u \in \mathcal{M}_A(c^+; c_1^-, \ldots, c_k^-)/\mathbb{R}$ the number d is called the **formal dimension** of u. Punctured pseudo-holomorphic disks of formal dimension 0 are said to be **rigid pseudo-holomorphic disks**.*

In what follows we assume that the generic Legendrian submanifold Λ is in addition such that $m(\Lambda) = 0$ and so the grading of Reeb chords is a map $|\cdot| : \mathcal{R} \to \mathbb{Z}$. Consider arbitrary choice of indexing of the set $\mathcal{R}(\Lambda) = \{c_1, \ldots, c_{\#\mathcal{R}(\Lambda)}\}$. For index $I = (i_1, \ldots, i_k) \in \{1, \ldots, \mathcal{R}(\Lambda)\}^k$. We will denote $\mathbf{c}_I^- = (c_{i_1}^-, \ldots, c_{i_k}^-) \in \mathcal{R}(\Lambda)^k$. Moreover, considering the assumptions of the dimension formula for $\mathcal{M}_A(c^+; c_{i_1}^-, \ldots, c_{i_k}^-)$ from Theorem 5.1.5 is reduced to

$$d = |c^+| + \sum_{j=1}^{k} |c_{i_j}^-| - 1.$$

in addition denote the respective moduli space as $\mathcal{M}_A(c^+; \mathbf{c}_I^-)$ and its dimension as $d = |c^+| - |\mathbf{c}_I^-| - 1$.

Definition 5.1.8. *For generic admissible Legendrian submanifold $\Lambda \subset J^1(M)$, $m(A) = 0$, we define the differential on $\mathcal{A}(\Lambda)$ so that for $c \in \mathcal{R}(\Lambda)$*

$$\partial c = \sum_{\substack{I, \boldsymbol{c}_I^- \in \mathcal{R}(\Lambda)^k, \\ |c| - |\boldsymbol{c}^-| = 1}} \#_2 \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R} \; c_{i_1} \otimes \ldots \otimes c_{i_k}, \tag{5.2}$$

where for any $k \geq 0$ we have $I = (i_1, \ldots, i_k) \in \{1, \ldots, \#\mathcal{R}(\Lambda)\}^k$, a k-tuple of Reeb chords

$$\boldsymbol{c}^- = (c_{i_1}^-, \ldots, c_{i_k}^-),$$

and $\#_2 \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R}$ is modulo 2 count of the elements of the respective set. This differential is then extended to the whole $\mathcal{A}(\Lambda)$ by the Leibniz rule and linearity.

Theorem 5.1.9 (Lemma 2.5 in [15]). *For generic admissible Legendrian submanifold $\Lambda \subset J^1(M)$ the differential $\partial : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda)$ it holds that $\partial \circ \partial = 0$ and $|\partial(a)| = |a| - 1$ for any $a \in \mathcal{A}(\Lambda)$, in other words $\partial : \mathcal{A}_\bullet \to \mathcal{A}_{\bullet - 1}$.*

Definition 5.1.10. *For generic admissible Legendrian submanifold $\Lambda \subset J^1(M)$ define Legendrian contact homology by*

$$LCH(\Lambda) = \ker(\partial) / \operatorname{im}(\partial).$$

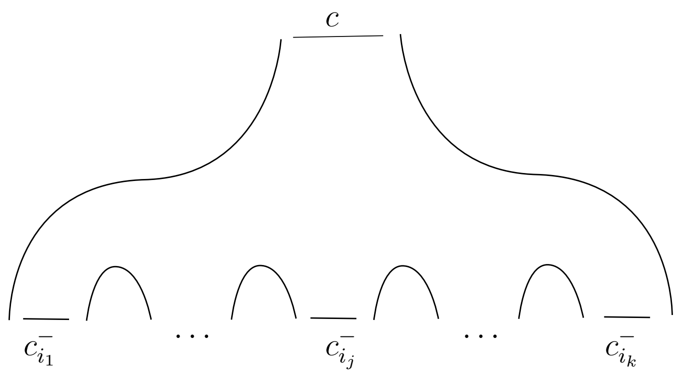

Figure 5.2: Example of a punctured pseudo-holomorphic disk with one positive puncture c and k negative punctures contributing to the differential of c.

Note that Legendrian contact homology $LCH(\Lambda)$ may be infinite dimensional and so hard to work with.

Theorem 5.1.11 (Proposition 2.6. in [15])**.** *Let $\{\Lambda_t\}_{t\in[0,1]}$ be a Legendrian isotopy so that Λ_0 and Λ_1 are generic admissible Legendrian submanifolds of $J^1(M)$, then $LCH(\Lambda_0) \cong LCH(\Lambda_1)$.*

Unfortunately, this invariant can not detect distinct Legendrian isotopy classes of loose Legendrians since the following holds.

Theorem 5.1.12 (Murphy, see Section 5 of [40])**.** *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$ be a loose Legendrian submanifold so that $(\mathcal{A}(\Lambda), \partial)$ is defined, then $\partial(a) = 1$ for some $a \in \mathcal{A}(\Lambda)$. Consequently, $LCH(\Lambda) = 0$.*

Corollary 5.1.13. *Let $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$ be a Legendrian submanifold so that $(\mathcal{A}(\Lambda), \partial)$ is defined, then if $LCH(\Lambda) \neq 0$, then Λ is not loose.*

Therefore, we can say that the class of Whitney spheres is disjoint from the loose Legendrians since for $n > 1$ (see Example 4.2. in [15])

$$LCH_k(W^n) = \begin{cases} 0, & k \not\equiv 0 \pmod{n}, \text{ or } k < 0, \\ \mathbb{Z}_2, & \text{otherwise.} \end{cases}$$

As far as the knowledge of the author is concerned, the opposite implication of Theorem 5.1.12 presents an open question.

Remark. The computation of the differential reduces to combinatorics when we consider Legendrian knots. Let us remark that this was the original approach of Chekanov (see [8]).

It is highly non-trivial problem to compute the differential ∂ of $\mathcal{A}(\Lambda)$ for generic Λ of dimension higher than 1. The only technique known to the author is the technique of Morse flow trees that is due to Ekholm (see [13]) and it considers the Legendrians embedded in $J^1(M)$ of a n-dimensional manifold M. There the holomorphic curves translate to gradients flow trees where the edges are (negative) gradient flow lines of difference of functions locally parametrizing the Legendrian by the one jet of them and the vertices are the critical points of those differences and generic points on the singular boundary Σ of the front projection.

5.2 Chekanov linearization

When $(\mathcal{A}(\Lambda), \partial)$ when is defined, the algebra $\mathcal{A}(\Lambda)$ can be endowed with a function $f : \mathcal{A}(\Lambda) \to \mathbb{Z}$ so that if $a = \sum_I a_I c_I \in \mathcal{A}(\Lambda)$, then $f(a) = \max_I |I|$, where $|I| = k$ is the length of the index $I = (i_1, \ldots, i_k)$. Consider $\mathcal{A}_j$ the ideal of $\mathcal{A}(\Lambda)$ generated by elements $a \in \mathcal{A}(\Lambda)$ so that $f(a) \geq j$.

Definition 5.2.1. *We say that any differential $\partial : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda)$ lowering the grading by one is* ***augmented*** *if $\partial(\mathcal{A}_1) \subset \mathcal{A}_1$. And the differential graded algebra $(\mathcal{A}(\Lambda), \partial)$ is called* ***good*** *if ∂ is augmented.*

Definition 5.2.2. *Let $(\mathcal{A}(\Lambda), \partial)$, where ∂ is any differential, be a well-defined good differential graded algebra. Denote by $V = \mathcal{A}_1/\mathcal{A}_2$ and $\partial^l : V \to V$ the induced map. Then the* ***linearized Legendrian contact homology*** *is defined as*

$$LCH^l(\Lambda) = \ker(\partial^l)/\operatorname{im}(\partial^l).$$

Because $\#\mathcal{R}(\Lambda) < \infty$ we obtain that V is a finite dimensional vector space over $\mathbb{Z}_2$ and $\partial^l : V \to V$ is a $\mathbb{Z}_2$-linear map of graded vector spaces.

Theorem 5.2.3 (Ekholm, Etnyre, Sullivan, Lemma 4.1, [15])**.** *Let $\{\Lambda_t\}_{t\in[0,1]}$ be a Legendrian isotopy so that Λ_0 and Λ_1 are generic admissible Legendrian submanifolds of $J^1(M)$ so that both $(\mathcal{A}(\Lambda_0), \partial)$ and $(\mathcal{A}(\Lambda_1), \partial)$ is good, then $LCH^l(\Lambda_0) \cong LCH^l(\Lambda_1)$.*

And so we obtained an invariant of Legendrian isotopy that is much easier to work with. On the other hand, the requirement of $(\mathcal{A}(\Lambda), \partial)$ being a good is very restrictive. Fortunately, there exist a construction that from a differential ∂ which is not necessarily augmented creates a differential ∂^ε using $\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$ unital homomorphism of DG-algebras.

Remark. The structure of differential graded algebra $(\mathbb{Z}_2, 0)$ on $\mathbb{Z}_2$ is defined such that every element of $\mathbb{Z}_2$ has grading zero and the differential is the multiplication by zero.

Definition 5.2.4. *Let $\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$ be a unital homomorphism of DG-algebras and $(\mathcal{A}(\Lambda), \partial)$ be the Chekanov-Eliashberg algebra of Λ. Define the* ***linearized differential*** *∂^ε by assignment*

$$\partial^\varepsilon(c) = \sum_{\substack{I, \boldsymbol{c}_I^- \in \mathcal{R}(\Lambda)^k, \\ |c| - |\boldsymbol{c}^-| = 1}} \#_2 \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R} \sum_{j=1}^{k} \varepsilon(c_{i_1}) \cdots \varepsilon(c_{i_{j-1}}) c_{i_j} \varepsilon(c_{i_{j+1}}) \cdots \varepsilon(c_{i_k}) \quad (5.3)$$

for $c \in \mathcal{R}(\Lambda)$. Using Leibniz rule and linearity extend this assignment to the whole algebra $\mathcal{A}(\Lambda)$.

Propositon 5.2.5. $(\mathcal{A}(\Lambda), \partial^{\varepsilon})$ *is a good differential graded algebra whenever it is defined.*

Proof. Note that ∂^{ε} lowers the degree, because the original differential of the Chekanov-Eliashberg algebra ∂ does.

Precise proof of the vanishing of $\partial^{\varepsilon} \circ \partial^{\varepsilon}$ is beyond the scope of this thesis. Nevertheless, the idea of the proof (even for the original ∂) is as follows.

Choose any Reeb chord and imagine that we have iterated the differential twice. Now pick a punctured disk from the first iteration, then one can glue to it to one of its lower ends a disk from the second iteration. We call this glued disk a pseudo-holomorphic building.

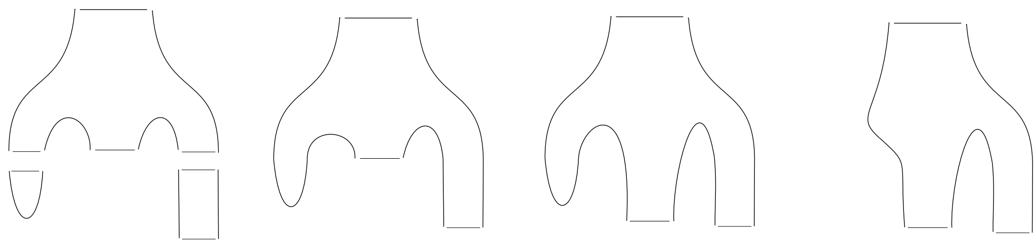

Figure 5.3: Example of a pseudo-holomorphic building.

It is non-trivial to show that those pseudo-holomorphic buildings are elements of the boundary of a 1-dimensional compact manifold. Note that any 1-dimensional compact manifold has boundary consisting of even number of points. Twice iterated differential counts precisely those pseudo-holomorphic buildings and thus vanishes. We refer to Lemma 2.5 in [14] for details in this setting.

Let $r = \#\mathcal{R}(\Lambda)$. Let $a \in \mathcal{A}_1(\Lambda)$, that is $a = a_{const} + \sum_{t=1}^{r} a_{i_t} c_{i_t}$, for $a_{i_t} \in \mathbb{Z}_2$ and $a_{const} \in \mathbb{Z}_2$, then

$$\begin{aligned}\partial^{\varepsilon}(a) &= \sum_{t=1}^{r} a_{i_t} \partial^{\varepsilon}(c_{i_t}) \\ &= \sum_{t=1}^{r} a_{i_t} \sum_{\substack{I, \mathbf{c}_I^- \in \mathcal{R}(\Lambda)^k, \\ |c_{i_t}| - |\mathbf{c}^-| = 1}} \#_2 \mathcal{M}_A(c_{i_t}, \mathbf{c}^-)/\mathbb{R} \sum_{j=1}^{k} \varepsilon(c_{i_1}) \cdots \varepsilon(c_{i_{j-1}}) c_{i_j} \varepsilon(c_{i_{j+1}}) \cdots \varepsilon(c_{i_k}).\end{aligned}$$

We used $\partial^{\varepsilon}(a_{const}) = 0$, because from the Leibniz rule we have that

$$\partial(1) = \partial(1 \cdot 1) = \partial(1) \cdot 1 + 1 \cdot \partial(1) = 0$$

and so this extends to the whole $\mathbb{Z}_2$ by linearity. ☐

Remark. The use of pseudo-holomorphic curves in symplectic setting is due to Gromov. The details of the Gromov's compactness result of moduli spaces of pseudo-holomorphic curves are explained in [35].

Definition 5.2.6. *Let* $\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$ *be a unital homomorphism of DG-algebras and* $(\mathcal{A}(\Lambda), \partial)$ *be the Chekanov-Eliashberg algebra of* Λ. *We say that* ε *is an* ***augmentation*** *of* $\mathcal{A}(\Lambda)$ *and we will denote the linearized Legendrian contact homology* $LCH^l(\Lambda)$ *with respect to* $V = C(\mathcal{R}(\Lambda))$ *and* ∂^{ε} *by* $LCH^{\varepsilon}(\Lambda)$. *The cohomology of the dual* V^* *with respect to dualized differential* μ_{ε}^1 *is called* ***linearized Legendrian contact cohomology*** *is is denoted by* $LCH_{\varepsilon}(\Lambda)$.

Definition 5.2.7. *Let $\varepsilon_1, \varepsilon_2$ be augmentations of $\mathcal{A}(\Lambda)$. A linear map $K : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ satisfying $K(ab) = \varepsilon_1(a)K(b) + K(a)\varepsilon_2(b)$ for all $a, b \in \mathcal{A}(\Lambda)$ is called a $(\varepsilon_1, \varepsilon_2)$-antiderivation. If it exists and $\varepsilon_1 - \varepsilon_2 = K \circ \partial$, then the augmentations are said to be* ***DGA-homotopic****, notation $\varepsilon_1 \sim \varepsilon_2$.*

It is standard fact (see [28], Lemma 26.3) that DGA-homotopy is an equivalence relation.

Note that each augmentation is uniquely determined by its values on Reeb chords of Λ and so for compact Legendrian Λ we have a finite set

$$\mathcal{E}(\Lambda) = \{[\varepsilon]_\sim \mid \varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0) \text{ is an augmentation of } \mathcal{A}(\Lambda)\}.$$

Theorem 5.2.8 (Bourgeois,Chantraine, Theorem 1.3. in [4])**.** *Let $\{\Lambda_t\}_{t\in[0,1]}$ is a Legendrian isotopy, then we have a bijection of $\mathcal{E}(\Lambda_0)$ and $\mathcal{E}(\Lambda_1)$.*

Therefore, the cardinality of the set $\mathcal{E}(\Lambda)$ is a Legendrian isotopy invariant. However, for $\{\Lambda_t\}_{t\in[0,1]}$ a Legendrian isotopy and $f : \mathcal{E}(\Lambda_0) \to \mathcal{E}(\Lambda_1)$ be the bijection from Theorem 5.2.8, then it is not necessarily true that $LCH^{\varepsilon}(\Lambda_0) \simeq LCH^{f(\varepsilon)}(\Lambda_1)$.

Theorem 5.2.9 (Bourgeois,Chantraine, Theorem 1.4, [4])**.** *Let Λ be a compact generic Legendrian submanifold of $J^1(M)$ with vanishing Maslov number, then if $\varepsilon_1, \varepsilon_2$ are two DGA-homotopic augmentations of $\mathcal{A}(\Lambda)$, then $LCH^{\varepsilon_1}(\Lambda) \cong LCH^{\varepsilon_2}(\Lambda)$.*

Theorem 5.2.10 (Bourgeois,Chantraine, Theorem 1.2, [4])**.** *Let Λ be a compact generic Legendrian submanifold of $J^1(M)$. Consider the set*

$$\mathcal{H}(\Lambda) = \bigcup_{[\varepsilon]_\sim \in \mathcal{E}(\Lambda)} \{LCH^{\varepsilon}(\Lambda)\}.$$

Let $\{\Lambda_t\}_{t\in[0,1]}$ is a Legendrian isotopy, then the sets $\mathcal{H}(\Lambda_0)$ and $\mathcal{H}(\Lambda_1)$ coincide.

5.2.1 Geometric origin of augmentations

In [36] Karlsson observed that the following theorem originally proved for $n = 1$ holds for any natural number n.

Theorem 5.2.11 (Ekholm, Honda, Kálmán, [17])**.** *Let $L \subset \mathbb{R}^{2n+2}$ be an exact Lagrangian cobordism from Λ_- to Λ_+ compact Legendrian submanifolds of $\mathbb{R}^{2n+1}$, then we have DGA-homomorphism*

$$\phi_L : (\mathcal{A}(\Lambda_-), \partial_-) \to (\mathcal{A}(\Lambda_+), \partial_+), \tag{5.4}$$

where on Reeb chords the map ϕ_L is given by

$$\phi_L(a) = \sum_{\dim \mathcal{M}_L(a,\boldsymbol{b})=0} \#_2\mathcal{M}_L(a, \boldsymbol{b})\boldsymbol{b}, \tag{5.5}$$

for $a \in \mathcal{R}(\Lambda_+)$, $\boldsymbol{b} = b_1, \dots, b_k \in \mathcal{R}(\Lambda_-)$ and $\mathcal{M}_L(a, \boldsymbol{b})$ is the moduli space of punctured pseudo-holomorphic disks $u : \hat{\mathbb{D}} \to \mathbb{R}^{2n+2}$ such that the boundary of u lies on L and the positive puncture is mapped to the strip over the Reeb chord a and negative punctures are mapped to strips over corresponding b_j. We extend this assignment linearly and multiplicatively onto the whole $\mathcal{A}(\Lambda_+)$.

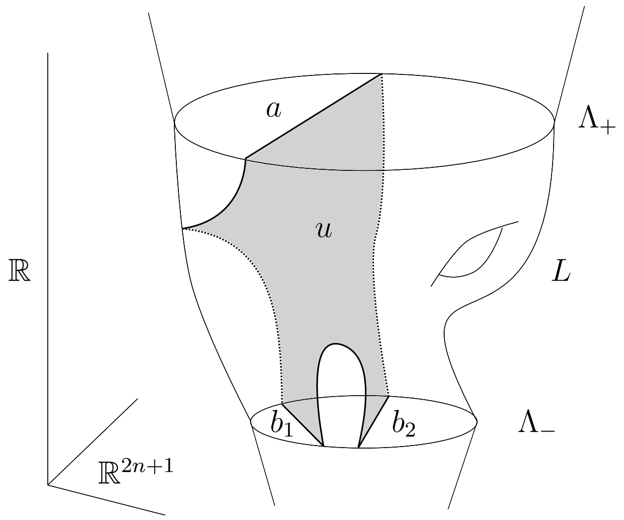

Figure 5.4: Example of u an element of $\mathcal{M}_L(a, \mathbf{b})$, where $\mathbf{b} = b_1, b_2$.

In particular, when $\Lambda_- = \emptyset$, so L is exact Lagrangian filling of Λ_+. And the obtained DGA-homomorphism ϕ_L is an augmentation that we denote by ε_L.

Theorem 5.2.12 (Ekholm, Honda, Kálmán, [17])**.** *Let Λ be a compact orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$ so that $m(\Lambda) = 0$. If L_0, L_1 are two exact Lagrangian fillings so that there is $\{L_t\}_{t\in[0,1]}$ isotopy through exact Lagrangian fillings. Then $\varepsilon_{L_0} \sim \varepsilon_{L_1}$.*

It is an interesting open question, whether there is an augmentation that is not of this geometric origin. The answer is likely to be negative when considering just the embedded fillings. However, the answer might be positive, when we would consider also immersed fillings.

Using the topology of the filling one might even control the Legendrian contact cohomology as we can see below.

Theorem 5.2.13 (Dimitroglou Rizell, [10])**.** *Let $\Lambda \subset (P \times \mathbb{R}, dz - \eta)$ be a Legendrian submanifold and L an exact Lagrangian filling of Λ. Then there is an isomorphism*

$$H_i(L; \mathbb{Z}_2) \simeq LCH^{n-i}_{\varepsilon_L}(\Lambda),$$

considering the grading i of H_i modulo $m(L)$.

5.2.2 Bilinearized Legendrian contact homology

Formula (5.3) in the definition of linearized differential in other words means that we decorate the negative punctures by numbers corresponding to the image under the augmentation.

In [4] Bourgeois and Chantraine took this idea further and decorated negative punctures with multiple augmentations that ultimately lead to $\mathcal{A}_\infty$ category whose objects are augmentations and morphisms are certain co-complexes, however, this is beyond the scope of this thesis. We will restrict ourselves to the case of two augmentations.

Definition 5.2.14. *Let $\varepsilon_1, \varepsilon_2 : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$ two augmentations of the Chekanov-Eliashberg algebra $(\mathcal{A}(\Lambda), \partial)$ of Λ. Define the **bilinearized differential** $\partial^{\varepsilon_1,\varepsilon_2}$ by assignment*

$$\partial^{\varepsilon}(c) = \sum_{\substack{I, \boldsymbol{c}_I^-, \\ \dim \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R}=0}} \#_2 \mathcal{M}_A(c, \boldsymbol{c}^-)/\mathbb{R} \sum_{j=1}^{k} \varepsilon_1(c_{i_1}) \cdots \varepsilon_1(c_{i_{j-1}}) c_{i_j} \varepsilon_2(c_{i_{j+1}}) \cdots \varepsilon_2(c_{i_k}) \tag{5.6}$$

for $c \in \mathcal{R}(\Lambda)$. Using Leibniz rule and linearity extend this assignment to the whole $\mathcal{A}(\Lambda)$.

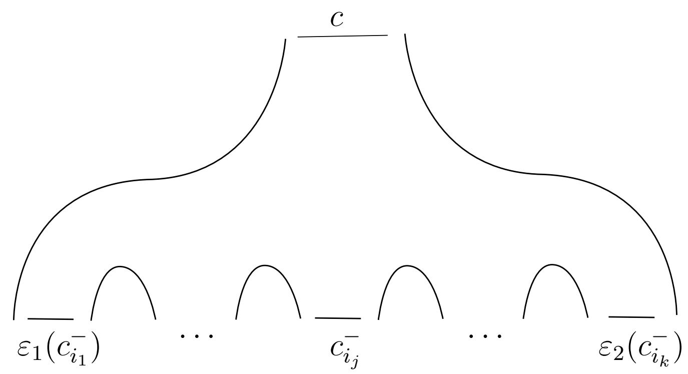

Figure 5.5: Example of a decorated punctured pseudo-holomorphic disk with one positive puncture c^+ and k negative punctures, where all negative punctures are decorated with respective augmentation except the j-th puncture.

Propositon 5.2.15. $(\mathcal{A}(\Lambda), \partial^{\varepsilon_1,\varepsilon_2})$ *is a good differential graded algebra.*

Proof. Analogous to the proof Proposition 5.2.5. □

Definition 5.2.16. *Let $\varepsilon_1, \varepsilon_2$ be two augmentations of $\mathcal{A}(\Lambda)$. We define **bilinearized Legendrian contact homology** $LCH^{\varepsilon_1,\varepsilon_2}(\Lambda)$ so-called as the homology of $C(\mathcal{R}(\Lambda))$ the graded vector space over $\mathbb{Z}_2$ with respect to the differential $\partial^{\varepsilon_1,\varepsilon_2}$. The cohomology of the dual V^* with respect to dualized differential $\mu_{\varepsilon_2,\varepsilon_1}$ is called **bilinearized Legendrian contact cohomology** is is denoted by $LCH_{\varepsilon_2,\varepsilon_1}(\Lambda)$.*

Analogously to linearized Legendrian contact homology we have the following.

Theorem 5.2.17 (Bourgeois,Chantraine, Theorem 1.2, [4])**.** *Let Λ be a compact generic Legendrian submanifold of $J^1(M)$ with vanishing Maslov number, then if $\varepsilon_1, \varepsilon_2, \varepsilon$ are augmentations of $\mathcal{A}(\Lambda)$, where $\varepsilon_1 \sim \varepsilon_2$, then $LCH^{\varepsilon_1,\varepsilon}(\Lambda) \cong LCH^{\varepsilon_2,\varepsilon}(\Lambda)$ and $LCH^{\varepsilon,\varepsilon_1}(\Lambda) \cong LCH^{\varepsilon,\varepsilon_2}(\Lambda)$.*

In particular,

Corollary 5.2.18. *Let Λ be a compact generic Legendrian submanifold of $J^1(M)$ with vanishing Maslov number, then if $\varepsilon_1, \varepsilon_2$ are augmentations of $\mathcal{A}(\Lambda)$, where $\varepsilon_1 \sim \varepsilon_2$, then $LCH^{\varepsilon_1,\varepsilon_2}(\Lambda) \cong LCH^{\varepsilon_2,\varepsilon_2}(\Lambda) = LCH^{\varepsilon_2}(\Lambda)$.*

Theorem 5.2.19 (Bourgeois,Chantraine, Theorem 1.2, [4])**.** *Let Λ be a compact generic Legendrian submanifold of $J^1(M)$. Consider the set*

$$\mathcal{H}^b(\Lambda) = \bigcup_{([\varepsilon_1]_\sim,[\varepsilon_2]_\sim)\in\mathcal{E}(\Lambda)\times\mathcal{E}(\Lambda)} \{LCH^{\varepsilon_1,\varepsilon_2}(\Lambda)\}.$$

Let $\{\Lambda_t\}_{t\in[0,1]}$ is a Legendrian isotopy, then the sets $\mathcal{H}^b(\Lambda_0)$ and $\mathcal{H}^b(\Lambda_1)$ coincide.

In view of Theorem 5.2.10 and Corollary 5.2.18, we see that $\mathcal{H}(\Lambda) \subset \mathcal{H}^b(\Lambda)$. And so bilinearized Legendrian contact homology is a stronger invariant of Legendrian isotopy which encodes the non-commutativity of the Chekanov-Eliashberg algebra that is lost in the process of linearization.

In general, it is not an easy task to determine the DGA-homotopy class of a augmentation of $\mathcal{A}(\Lambda)$ computationally and thus determining the cardinality of $\mathcal{E}(\Lambda)$ is an interesting problem. In the standard 3-dimensional space the cardinality of $\mathcal{E}(\Lambda)$ was studied by Ng, Rutheford, Shende, Sivek in [41].

It is quite surprising that there is a criterion for DGA-homotopy of augmentations in any dimension. This first appeared in work of Bourgeois and Galant (see [5]).

Theorem 5.2.20 (Bourgeois, Galant, Proposition 3.3. in [5])**.** *Let Λ be a connected compact generic Legendrian submanifold of $J^1(M)$ has dimension with vanishing Maslov number, where M has dimension n. Then if $\varepsilon_1, \varepsilon_2$ are augmentations of $\mathcal{A}(\Lambda)$ it holds that*

$$LCH_n^{\varepsilon_2,\varepsilon_1}(\Lambda) - LCH_{-1}^{\varepsilon_1,\varepsilon_2}(\Lambda) = \begin{cases} 0, & \varepsilon_1 \not\sim \varepsilon_2, \\ 1, & \varepsilon_1 \sim \varepsilon_2. \end{cases}$$

This criterion is an important step in proving so-called geography result for bilinearized Legendrian contact homology. More specifically, one is interested in polynomials arising as Poicaré polynomials of bilinearized Legendrian contact homology for some Legendrian Λ:

$$P^{\Lambda}_{\varepsilon_1,\varepsilon_2}(t) = \sum_{k\in\mathbb{Z}} \dim_{\mathbb{Z}_2}(LCH_k^{\varepsilon_1,\varepsilon_2}(\Lambda))\, t^k.$$

The question is whether the there is a so-called admissible class of polynomial so that each member of it can be attained as a Poincaré polynomial for some Legendrian Λ and DGA-non-homotopic augmentations $\varepsilon_1, \varepsilon_2$. For details on the geography of bilinearized Legendrian contact homology for connected Legendrians see [5].

In this thesis, we will generalize this criterion for the disconnected case. And so, if not stated otherwise. We will denote by Λ a disconnected Legendrian submanifold, that is $\Lambda = \coprod_{j=1}^{r} \Lambda_j$ for some $r \in \mathbb{N}$. Moreover, we suppose that Λ has vanishing Maslov number.

Regarding the geography question, the author of this thesis is preparing an article, where the obtained criterion for disconnected Legendrian then helps to extend the results of [5] to the disconnected case.

5.3 Duality long exact sequence

Theorem 5.3.1 below directly follows from Theorem 1.1 in [16] that was originally proven for linearized Legendrian contact homology. Nevertheless, both the statement and the proof translates to the bilinearized setting as was observed in [4].

Theorem 5.3.1 (Ekholm, Etnyre, Sabloff, Theorem 1.1 in [16])**.** *Let M be a smooth manifold of dimension n and $\Lambda \subset J^1(M)$ be a closed chord generic Legendrian submanifold such that we can completely displace the Lagrangian projection $\Pi_{T^*M}(\Lambda)$ from itself using a Hamiltonian isotopy, that is, if ϕ_t is a Hamiltonian isotopy of T^*M for $t \in [0,1]$, then $\phi_1(\Pi_{T^*M}(\Lambda)) \cap \Pi_{T^*M}(\Lambda) = \emptyset$. Moreover, we assume that we have $\varepsilon_1, \varepsilon_2$ two augmentations of $\mathcal{A}(\Lambda)$ over $\mathbb{Z}_2$, then we obtain a long exact sequence*

$$\cdots \to LCH^{n-k-1}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to LCH^{\varepsilon_1,\varepsilon_2}_k(\Lambda) \xrightarrow{\tau_k} H_k(\Lambda) \xrightarrow{\sigma_{n-k}} LCH^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to \cdots, \tag{5.7}$$

where $H_\bullet(\Lambda)$ is the Morse (or equivalently singular) homology of Λ with coefficients in $\mathbb{Z}_2$. This long exact sequence is to be called the duality long exact sequence of Λ.

Since the detailed description of this sequence is beyond the scope of this thesis, we present rather informal description below. We refer the reader to [16] for details.

The map τ_k above counts so called generalized lifted disks (u, γ), where u is a punctured pseudo-holomorphic as in the Definition 5.1.8 and γ is a negative gradient flow-line of some fixed perturbing Morse function $f : L \to \mathbb{R}$ which connects a critical point q of f with index k to some generic point p on the boundary of u and the orientation of the flow is oriented towards c (that is $f(p) > f(q)$).

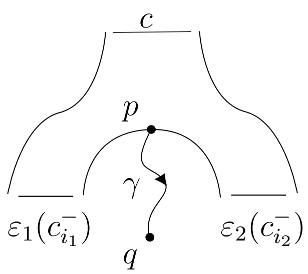

Figure 5.6: Example of a lifted generalized disk contributing to τ_k.

Denote the moduli space of suitable rigid (u, γ) where the boundary point is in between negative punctures $c_{i_{l-1}}$ and c_{i_l} as $\mathcal{M}_{c^+,\mathbf{c},q}$. Then we have

$$\tau_k(c) = \sum_{\dim \mathcal{M}(c;\mathbf{c}^-)/\mathbb{R} = \mathrm{Index}_f(q)-1} \#_2 \mathcal{M}_{c,\mathbf{c},q} \sum_{j=1}^{k} \varepsilon_1(c_{i_1}) \ldots \varepsilon_1(c_{i_{l-1}})\, q\, \varepsilon_2(c_{i_l}) \ldots \varepsilon_2(c_{i_k}).$$

On the other hand, the map σ_k counts lifted generalized disks as above, however, the flow-line heads in the opposite direction (that is $f(p) < f(q)$).

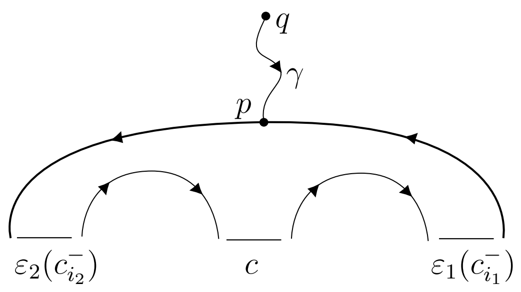

Figure 5.7: Example of a lifted generalized disk contributing to σ_k.

And so

$$\sigma_k(q) = \sum_{\dim \mathcal{M}(c;\mathbf{c}^-)/\mathbb{R}=\mathrm{Index}_f(q)-1} \#_2 \mathcal{M}_{c,\mathbf{c},q} \sum_{j=1}^{k} \varepsilon_1(c_{i_1})\dots\varepsilon_1(c_{i_{l-1}})\, c\, \varepsilon_2(c_{i_l})\dots\varepsilon_2(c_{i_k}).$$

The main difference between the linearized and bilinearized case is that the one has to pay attention to the ordering of augmentations in the duality formula from which the name of the sequence stems. More precisely, consider $\varepsilon_1, \varepsilon_2$ two augmentations of $\mathcal{A}(\Lambda)$ of some $\Lambda = \coprod_{j=1}^{r} \Lambda_j$ as above. We have two duality sequences: fist for the ordering $(\varepsilon_1, \varepsilon_2)$, that we call positive:

$$\dots \to LCH^{n-k-1}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to LCH^{\varepsilon_1,\varepsilon_2}_k(\Lambda) \xrightarrow{\tau_{+,k}} H_k(\Lambda) \xrightarrow{\sigma_{+,n-k}} LCH^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to \cdots \tag{5.8}$$

second for the ordering $(\varepsilon_2, \varepsilon_1)$, that we call negative:

$$\dots \to LCH^{n-k-1}_{\varepsilon_1,\varepsilon_2}(\Lambda) \to LCH^{\varepsilon_2,\varepsilon_1}_k(\Lambda) \xrightarrow{\tau_{-,k}} H_k(\Lambda) \xrightarrow{\sigma_{-,n-k}} LCH^{n-k}_{\varepsilon_1,\varepsilon_2}(\Lambda) \to \cdots \tag{5.9}$$

Recall that since all components Λ_j for $j = 1, \dots, r$ of our Legendrian Λ are closed we have an intersection pairing $\bullet : H_k(\Lambda_j) \otimes H_{n-k}(\Lambda_j) \to \mathbb{Z}_2$ for each $k \in \mathbb{Z}$. Now define the intersection pairing on Λ for $k \in \mathbb{Z}$ and for $c = (c_1, \dots, c_r) \in H_k(\Lambda) = H_k(\Lambda_1) \oplus \dots \oplus H_k(\Lambda_r)$ and $d = (d_1, \dots, d_r) \in H_{n-k}(\Lambda) = H_{n-k}(\Lambda_1) \oplus \dots \oplus H_{n-k}(\Lambda_r)$ to be

$$c \bullet d = \sum_{j=1}^{r} c_j \bullet d_j. \tag{5.10}$$

For precise definition of the pairing $\bullet$ on the components of the disconnected Legendrian submanifold Λ see Section 3.3.3. of [16].

Define a pairing $\langle \cdot, \cdot \rangle : C(\mathcal{R}(\Lambda))^* \otimes_{\mathbb{Z}_2} C(\mathcal{R}(\Lambda)) \to \mathbb{Z}_2$ on generators $\{c_i : i \in \{1, \dots, \#\mathcal{R}(\Lambda)\}\}$ as follows

$$\langle c_i^*, c_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this precisely corresponds to evaluation $\langle c_i^*, c_j \rangle = c_j^*(c_i)$. In particular, if $|c_i^*| \neq |c_j|$, then $\langle c_i^*, c_j \rangle = 0$.

Consider $\partial^{\varepsilon_1,\varepsilon_2}$ the differential on $C(\mathcal{R}(\Lambda))$ and $\mu^1_{\varepsilon_2,\varepsilon_1}$ on $C(\mathcal{R}(\Lambda))^*$ the dual differential to $\partial^{\varepsilon_1,\varepsilon_2}$, that is

$$\langle \mu^1_{\varepsilon_2,\varepsilon_1}(a), b\rangle = \langle a, \partial(b)\rangle,$$

for $a \in C(\mathcal{R}(\Lambda))$ and $b \in \mu^1_{\varepsilon_2,\varepsilon_1}$. Moreover, consider the tensor product $W = C(\mathcal{R}(\Lambda))^* \otimes_{\mathbb{Z}_2} C(\mathcal{R}(\Lambda))$ endowed with the standard differential

$$d_W(c^* \otimes d) = \mu^1_{\varepsilon_2,\varepsilon_1}(c^*) \otimes d + c^* \otimes \partial^{\varepsilon_2,\varepsilon_1}(d)$$

for $c^* \in C(\mathcal{R}(\Lambda))^*$ and $d \in C(\mathcal{R}(\Lambda))$. Using the generalized Künneth formula we obtain that

$$H_\bullet(W, d_W) \cong \bigoplus_{k+l=\bullet} H_k(C(\mathcal{R}(\Lambda))^*, \mu^1_{\varepsilon_2,\varepsilon_1}) \otimes_{\mathbb{Z}_2} H_l(C(\mathcal{R}(\Lambda)), \partial^{\varepsilon_2,\varepsilon_1}),$$

where torsion does not occur for we work over the field $\mathbb{Z}_2$.

Define $F : W \to \mathbb{Z}_2$ by $F(a) = \sum_i \langle c_i, d_i\rangle$ for $a = \sum_i c_i \otimes_{\mathbb{Z}_2} d_i \in W$, then $F \circ d_W = 0$ since the differentials are dual to each other. Therefore F descends to homology of (W, d_W) and so does the pairing.

We will denote the pairing that we have just constructed as

$$\langle \cdot, \cdot \rangle_+ : LCH_{\varepsilon_2,\varepsilon_1}(\Lambda) \otimes_{\mathbb{Z}_2} LCH^{\varepsilon_1,\varepsilon_2}(\Lambda) \to \mathbb{Z}_2,$$

and for the opposite order of augmentations we will write

$$\langle \cdot, \cdot \rangle_- : LCH_{\varepsilon_1,\varepsilon_2}(\Lambda) \otimes_{\mathbb{Z}_2} LCH^{\varepsilon_2,\varepsilon_1}(\Lambda) \to \mathbb{Z}_2.$$

Propositon 5.3.2. *Then for any non-zero class $[a] \in LCH^k_{\varepsilon_1,\varepsilon_2}(\Lambda)$, there is a Reeb chord $c \in \mathcal{R}(\Lambda)$ so that $\langle [a], [c]\rangle_- \neq 0$, where the pairing*

$$\langle \cdot, \cdot \rangle_- : LCH^k_{\varepsilon_1,\varepsilon_2}(\Lambda) \otimes_{\mathbb{Z}_2} LCH^{\varepsilon_2,\varepsilon_1}_k(\Lambda) \to \mathbb{Z}_2$$

is as above.

Proof. Let $a = \sum_{i=1}^m c_i^* + db$ be a representative of the class $[a]$, where $c_i \in \mathcal{R}(\Lambda)$ are all distinct, non-exact, and $|c_i^*| = k$, and moreover, $b \in C^*_{k+1}(\Lambda)$. Now because the class $[a]$ is non-zero then $m > 0$. If we had that $\langle [a], [c]\rangle_- = 0$ for all $c \in \mathcal{R}(\Lambda)$ of grading k, then

$$0 = \langle [a], [c]\rangle_- = \langle a, c\rangle = \langle \sum_{i=1}^m c_i^* + db, c\rangle = \langle \sum_{i=1}^m c_i^*, c\rangle = \sum_{i=1}^m c_i^*(c),$$

therefore, $m = 0$ which is a contradiction. □

We have an analogue of Proposition 3.9 in [16].

Propositon 5.3.3. *The pairs of maps $\tau_{+,k}$ and $\sigma_{-,k}$, and $\tau_{-,k}$ and $\sigma_{+,k}$ are adjoint in the following sense:*

Let us have $c \in H_k(\Lambda)$ and a chord q of grading $n-k$

$$\langle \sigma_{-,n-k}(c), [q]\rangle_- = c \bullet \tau_{+,n-k}([q]),$$
$$\langle \sigma_{+,n-k}(c), [q]\rangle_+ = c \bullet \tau_{-,n-k}([q]),$$

where $\bullet : H_k(\Lambda) \otimes H_{n-k}(L) \to \mathbb{Z}_2$ is the intersection pairing, and pairings

$$\langle \cdot, \cdot \rangle_- : LCH^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \otimes_{\mathbb{Z}_2} LCH^{\varepsilon_2,\varepsilon_1}_{n-k}(\Lambda) \to \mathbb{Z}_2,$$
$$\langle \cdot, \cdot \rangle_+ : LCH^{n-k}_{\varepsilon_1,\varepsilon_2}(\Lambda) \otimes_{\mathbb{Z}_2} LCH^{\varepsilon_1,\varepsilon_2}_{n-k}(\Lambda) \to \mathbb{Z}_2$$

are as above.

Proof. Let us prove the first equation the other case is analogous.

The right handed side counts holomorphic disks with q mixed positive puncture and c negative Morse puncture and with possibly other augmented negative punctures between q and c with ε_1 and between c and q with ε_2. Now the bijective correspondence from Theorem 3.6 form [16] implies that this disks corresponds to the lifted generalized disk with γ a negative gradient flow line of the perturbing function f ending at c and connecting it to the boundary of the disk.

To pass to the right side that is from homology to cohomology we change the sign of the perturbing function, that is f is $-f$ on the left side. Now the orientation of γ is reversed and so c is a positive Morse puncture and q is a negative mixed puncture. Since the order of augmentations is reversed , the disk contributes to $\sigma_{-,n-k}$. □

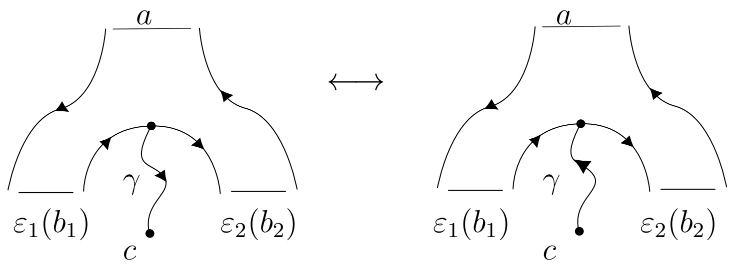

Figure 5.8: Effect of passing from homology to cohomology on a generalized lifted disk.

5.3.1 The map τ_n

Let us focus on the n-th level of the duality long exact sequence. Consider a Reeb chord a with grading n, than the differential ∂ counts (u, γ) lifted generalized disks where u is a pseudo-holomorphic curve and γ is the gradient flow line from a generic point of the boundary to m_j a maximum of the Morse function f on Λ_j the corresponding component of Λ, this in particular implies that the beginning point of γ on the boundary must be the maximum m_j. For $l \in \{1, \ldots, m+1\}$ denote this moduli space by

$$\mathcal{M}_{a;\mathbf{b},\mathbf{p}}(a; b_1, \ldots, b_{l-1}, p_l, b_l, \ldots, b_m),$$

where p_l lies on the boundary component of the punctured disk which was mapped to Λ_j where $j = j_l$ assuming b_l is a Reeb chord from the connected component Λ_{i_l} to the connected component Λ_{j_l} of the disconnected Legendrian submanifold L, for all $l \in \{1, \ldots, m\}$ if $l - 1 = m$ then $j = i_m$. We denote by $[\Lambda_j]$ the homology class which m_j represents. And so by the dimension formula ([16], Section 3.3.1)

$$0 = \dim(u, \gamma) = \dim \mathcal{M}(a; b_1, \ldots, b_m) + 1 - \text{Index}_f(p_l) = |a| - |\mathbf{b}| - 1 + 1 - n$$

and the fact that $|a| = n$ we get that $|\mathbf{b}| = 0$.

This gives us the description of action the map τ_n on a that is

$$\tau_n(a) = \sum_{|\mathbf{b}|=0} \#_2 \mathcal{M}_{a;\mathbf{b},\mathbf{p}} \sum_{j=1}^{m} \varepsilon_1(b_1) \ldots \varepsilon_1(b_{l-1})[\Lambda_{i_{l-1}}]\varepsilon_2(b_l) \ldots \varepsilon_2(b_m) \quad (5.11)$$

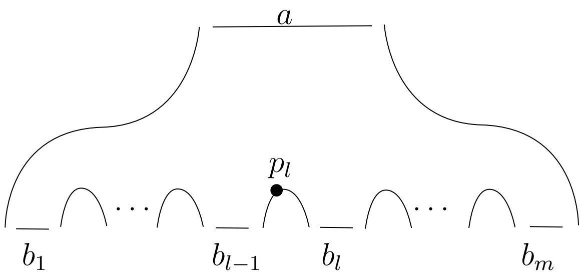

Figure 5.9: Pointed disc.

5.4 Effect of Legendrian ambient 0-surgery

Let $r \in \mathbb{N}$. Consider disconnected Legendrian submanifold $\Lambda = \coprod_{j=1}^{r} \Lambda_j$. Denote by $\Lambda_{S,1}$ the submanifold resulting from the Legendrian ambient 0-surgery by connecting Λ_1 and Λ_2. Now inductively $\Lambda_{S,k}$ denotes the submanifold resulting from the Legendrian ambient 0-surgery by connecting $\Lambda_{S,k-1}$ and Λ_k for $k = 3, \dots, r$.

Now we will restrict the setting to the first iteration of the 0-surgery for simplicity. The effect of the ambient Legendrian 0-surgery using the embedded sphere $\mathbf{S}^0$ into Λ_i and Λ_j for some $i, j \in \{1, \dots, r\}$ on the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda)$ was described by Dimitroglou Rizell in Section 1 of [9], more specifically, for embedded spheres of all dimensions from 0 to $n-1$. Denote by Λ_S the Legendrian submanifold obtained by performing the surgery. The situation is as follows:

The algebra $\mathcal{A}(\Lambda_S)$ is isomorphic to the algebra $\mathcal{A}(\Lambda, S)$ defined as the free product of the algebra $\mathcal{A}(\Lambda)$ and the line $\mathbb{Z}_2\langle s\rangle$ of a formal generator s which corresponds to c_S a new Reeb chord of Λ_S that is of degree $|c_S| = n-1$. Note that this means that for any ε augmentation of $\mathcal{A}(\Lambda_S)$ we have that $\varepsilon(c_S) = 0$. The differential on $\mathcal{A}(\Lambda, S)$ is to be denoted by ∂_S and it decomposes into $\partial_S = \partial + h$ on generators. Here ∂ is the differential of $\mathcal{A}(\Lambda)$ and

$$h(a) = \sum_{|a|-|\mathbf{b}|-|\mathbf{s}|=1} |\mathcal{M}_{a;\mathbf{b},\mathbf{w}}(a; b_1, \dots, b_m)| s^{w_1} b_1 \dots b_m s^{w_{m+1}} \tag{5.12}$$

counts number of holomorphic disks with boundary on L and with w_i marked points on the corresponding part of boundary of the disk that is mapped to one of the points that are in the image of $\mathbf{S}^0$. Here $|\mathbf{s}| = (w_1 + \dots + w_{m+1})(n-1)$. For more details see ([9], Section 6).

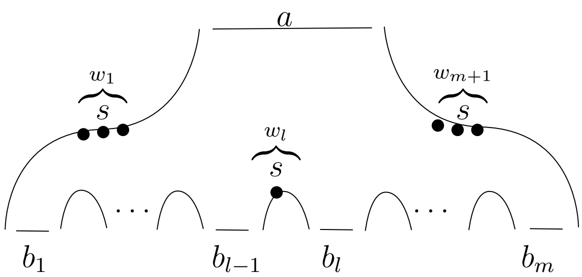

Figure 5.10: Twisted disc.

Consider $\varepsilon_1, \varepsilon_2$ two augmentations of $\mathcal{A}(\Lambda)$ then $\varepsilon_1', \varepsilon_2'$ are two augmentations of $\mathcal{A}(\Lambda, S)$ induced in the as the pull-back. More specifically, they both vanish on the element s and coincide with the original augmentations of original Reeb chords. Now $\partial_S^{\varepsilon_1,\varepsilon_2}$ the bilinearized differential decomposes $\partial_S^{\varepsilon_1,\varepsilon_2} = \partial^{\varepsilon_1,\varepsilon_2} + h^{\varepsilon_1,\varepsilon_2}$ on generators.

If one of $w_j > 1$ or if there are two $j \neq j'$, then the corresponding disk contributes by zero to the bilinearized differential. It must hold that there is exactly one $j \in \{1 \ldots, m+1\}$ such that $w_j = 1$ for a disk to contribute to $h^{\varepsilon_1,\varepsilon_2}$. Otherwise, it has already contributed to the usual bilinearized differential $\partial^{\varepsilon_1,\varepsilon_2}$. Let us denote by $\rho_\bullet = h_\bullet^{\varepsilon_1,\varepsilon_2}$

$$\rho_\bullet = \sum_{|a|-|\mathbf{b}|=n} |\mathcal{M}_{a;\mathbf{b},\mathbf{w}}(a; b_1, \ldots, b_m)| \sum_{l=1}^{m} \varepsilon_1(b_1)\ldots\varepsilon_1(b_{l-1})\, s\, \varepsilon_2(b_l)\ldots\varepsilon_2(b_m).$$

Let $\varepsilon_1, \varepsilon_2$ be two augmentations of $\mathcal{A}(\Lambda)$ over $\mathbb{Z}_2$. Denote by $\varepsilon_1^S, \varepsilon_2^S$ the induced augmentations of $\mathcal{A}(\Lambda, S)$.

The inclusion of the line $\mathbb{Z}_2\langle s\rangle_{n-1}$ into $C_\bullet(\Lambda, S)$ makes it into a subcomplex and the fact that $C_\bullet(\Lambda, S)/\mathbb{Z}_2\langle s\rangle \cong C_\bullet(\Lambda)$ now yields a short exact sequence of complexes

$$0 \to (\mathbb{Z}_2\langle s\rangle_\bullet, \partial_S^{\varepsilon_1^S,\varepsilon_2^S}) \to (C_\bullet(\Lambda, S), \partial_S^{\varepsilon_1^S,\varepsilon_2^S}) \xrightarrow{\pi} (C_\bullet(\Lambda), \partial^{\varepsilon_1,\varepsilon_2}) \to 0$$

this induces the following long exact sequence in homology

$$\cdots \to LCH_\bullet^{\varepsilon_1^S,\varepsilon_2^S}(\Lambda_S) \xrightarrow{\pi_\bullet} LCH_\bullet^{\varepsilon_1,\varepsilon_2}(\Lambda) \xrightarrow{\rho_\bullet} \mathbb{Z}_2\langle s\rangle_{\bullet-1} \to LCH_{\bullet-1}^{\varepsilon_1^S,\varepsilon_2^S}(L_S) \to \ldots.$$

Since $\mathbb{Z}_2\langle s\rangle_{\bullet-1} = 0$ if $\bullet \neq n$ we obtain the isomorphism:

$$0 \to LCH_\bullet^{\varepsilon_1^S,\varepsilon_2^S}(\Lambda_S) \xrightarrow{\pi_\bullet} LCH_\bullet^{\varepsilon_1,\varepsilon_2}(\Lambda) \xrightarrow{\rho_\bullet} 0.$$

5.5 DGA-homotopy criterion

Theorem 5.5.1. *Let $\Lambda = \Lambda_1 \sqcup \ldots \sqcup \Lambda_r$ be a disconnected Legendrian submanifold for $r > 1$ and denote by $[\Lambda]$ its fundamental class. Let $\varepsilon_1, \varepsilon_2$ be two augmentations of the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda)$ over $\mathbb{Z}_2$. Then the following holds:*

$$\varepsilon_1 \text{ and } \varepsilon_2 \text{ are DGA homotopic } \Leftrightarrow [\Lambda] \text{ is an element of the image of } \tau_{-,n}. \tag{5.13}$$

Note that since the relation of DGA-homotopy is symmetric we can substitute $\tau_{-,n}$ by $\tau_{+,n}$ and so the order of augmentations in the duality long exact sequence does not matter.

In this section let us consider disconnected Legendrian submanifold Λ with r components and Λ_S a connected Legendrian submanifold obtained by performing $r-1$ Legendrian ambient surgeries on Λ. And by $\varepsilon_1^S, \varepsilon_2^S$ denote the augmentations induced by this surgery so that they vanish on the formal generators added to the Chekanov-Eliashberg algebra and coincide with ε_1 and ε_2 otherwise. This implies that on chords of Λ are decorated with the same number by both ε_i^S and ε_i for both $i = 1, 2$. Considering Lemma 3.1 from [5] we obtain the following proposition

Propositon 5.5.2. $\varepsilon_1 \sim \varepsilon_2$ *if and only if* $\varepsilon_1^S \sim \varepsilon_2^S$

Lemma 5.5.3. *Consider the following diagram:*

$$\begin{array}{ccccc} LCH_0^{\varepsilon_1^S,\varepsilon_2^S}(\Lambda_S) & \xrightarrow{\pi^+} & & & LCH_0^{\varepsilon_1,\varepsilon_2}(\Lambda) \\ \downarrow \tau_{+,0}^S & & & & \downarrow \tau_{+,0} \\ H_0(\Lambda_S) & & \xleftarrow{\widetilde{\alpha}} & & H_0(\Lambda) \\ & \searrow^{\gamma} & & \swarrow^{\alpha} & \\ & & \mathbb{Z}_2 & & \end{array} \tag{5.14}$$

where $\widetilde{\alpha} : H_0(\Lambda) \to H_0(\Lambda)$ is defined as follows. Denote by $[*_{\Lambda_1}], [*_{\Lambda_2}]$ classes of points in $H_0(\Lambda)$ that represent distinct components Λ_1 and Λ_2 of Λ, and $[*_{\Lambda_S}] \in H_0(\Lambda_S)$ a class of point in Λ_S, that is connected. Moreover, consider the map $\gamma : H_0(\Lambda_S) \to \mathbb{Z}_2$ defined as $a[*_{\Lambda_S}] \to a$ for any $a \in \mathbb{Z}_2$, and let us denote by $\alpha : H_0(\Lambda) \to \mathbb{Z}_2$ the composition $\gamma \circ \widetilde{\alpha}$. Then $\widetilde{\alpha}(a_1[*_{\Lambda_1}] \oplus a_2[*_{\Lambda_2}]) \mapsto (a_1 + a_2)[*_{\Lambda_S}]$ for any $a_1, a_2 \in \mathbb{Z}_2$. The map π^+ is a composition of isomorphism level zero in the surgery long exact sequence. Then the diagram above commutes.

Proof. Let m_j be minimum of the perturbing Morse function on the component Λ_j if q is a chord of degree 0 that starts on Λ_i and ends on Λ_j, thanks to the rigidity of the formal disk the starting point of a generalized lifted disk must map either to the starting point of q or to the ending point of it. In the first case, the disk contributes with $\varepsilon_2(q)[*_{\Lambda_i}]$ to $\tau_{+,0}(q)$. In the second case, the disk contributes with $\varepsilon_1(q)[*_{\Lambda_j}]$ to $\tau_{+,0}(q)$. And so $\tau_{+,0}(q) = \varepsilon_1(q)[*_{\Lambda_j}] + \varepsilon_2(q)[*_{\Lambda_i}]$. By the proof of Proposition 3.2 in [5] we have that $\tau_{+,0}^S = \varepsilon_1 + \varepsilon_2$. Now it is clear that for each q the diagram commutes and so it commutes. □

Lemma 5.5.4. *For each $c \in H_0(\Lambda)$ and the map $\alpha : H_0(\Lambda) \to \mathbb{Z}_2$ from the statement of Lemma 5.5.3 it holds that*

$$\alpha(c) = c \bullet [\Lambda]. \tag{5.15}$$

Proof. The map α is an element of $(H_0(\Lambda))^*$ of the dual of $H_0(\Lambda)$. The intersection pairing $\bullet : H_0(\Lambda) \otimes_{\mathbb{Z}_2} H_n(\Lambda) \to \mathbb{Z}_2$ defines $\Theta : (H_0(\Lambda))^* \to H_n(\Lambda)$ an isomorphism that sends $\delta \in (H_0(\Lambda))^*$ to a class $\Theta(\delta) \in H_n(\Lambda)$ so that $p \bullet \Theta(\delta) = \delta(p)$ for each $p \in H_0(\Lambda)$.

Let us claim that $\Theta(\alpha) = [\Lambda]$. Then our claim is equivalent to the statement that for every $p \in H_0(\Lambda)$ it holds that $\alpha(p) = p \bullet [\Lambda]$. Since elements $e_j \in H_0(\Lambda)$ with only non-zero component equal to $[*_{\Lambda_j}]$ generate the space $H_0(\Lambda)$ and clearly $\alpha(e_j) = 1$. By the definition of the pairing $\bullet$ for the disconnected Legendrian submanifold Λ we have that

$$e_j \bullet [\Lambda] = [*_{\Lambda_j}] \bullet [\Lambda] + \sum_{i=1, i \neq j}^{r} 0 \bullet [\Lambda_1] = [*_{\Lambda_j}] \bullet [\Lambda] = 1 \tag{5.16}$$

where the last component is by the Poincaré duality for closed component Λ_j and so $([*_{\Lambda_1}] \oplus 0) \bullet ([\Lambda_1] \oplus [\Lambda_2]) = 1$. The reasoning for the other generating classes e_j is analogous. □

Proof of Theorem 5.5.1. First, consider $\varepsilon_1 \not\sim \varepsilon_2$. Then $\varepsilon_1^S \not\sim \varepsilon_2^S$ by Proposition 5.5.2. Thanks to Proposition 3.2 in [5] we have that $\tau_{+,0}^S \neq 0$. For the sake of contradiction suppose that $[\Lambda] \in \operatorname{im}_{\mathbb{Z}_2} \tau_{-,n}$, then by the exactness of the duality sequence for the negative order of augmentations we obtain that $\sigma_{-,0}([\Lambda]) = 0$. And so by Proposition 5.3.3 we obtain that

$$0 = \langle \sigma_{-,0}([\Lambda]), q \rangle_- = \tau_{+,0}(q) \bullet [\Lambda] \tag{5.17}$$

for every q Reeb chord that gives rise to a generator of $LCH_n^{\varepsilon_2,\varepsilon_1}(\Lambda)$ and a generator of $LCH_n^{\varepsilon_1,\varepsilon_2}(\Lambda)$. However, using the commutativity from Lemma 5.5.3 and the form of the map α from Lemma 5.5.4 we obtain that for every Reeb chord q

$$\gamma \circ \tau_{+,0}^S \circ (\pi_0^+)^{-1}(q) = \alpha \circ \tau_{+,0}(q) = \tau_{+,0}(q) \bullet [\Lambda] = 0 \tag{5.18}$$

which yields that $\tau_{+,0}^S = 0$ because both γ and π_n^+ are isomorphisms. This is the contradiction with $\tau_{+,0}^S \neq 0$.

On the other hand, assume that $\varepsilon_1 \sim \varepsilon_2$, then by Proposition 5.5.2 we have that $\varepsilon_1^S \sim \varepsilon_2^S$, which yields $\tau_{+,0}^S = 0$ as above. Suppose that $[\Lambda] \notin \operatorname{im}_{\mathbb{Z}_2} \tau_{-,n}$, then $\sigma_{-,0}([\Lambda]) \neq 0$, and thanks to Proposition 5.3.2, there exists a chord q so that $\langle \sigma_{-,0}([\Lambda]), q \rangle_- \neq 0$.

If $\tau_{+,0} = 0$, then $0 \neq \langle \sigma_{-,0}([\Lambda]), q \rangle_- = \tau_{+,0}(q) \bullet [\Lambda] = 0$ which is a contradiction.

If $\tau_{+,0} \neq 0$, then the commutativity of the diagram in Lemma 5.5.3, Lemma 5.5.4, and the fact that $\tau_{+,0}^S = 0$ imply that

$$0 = \gamma \circ \tau_{+,0}^S \circ (\pi_0^+)^{-1}(q) = \alpha \circ \tau_{+,0}(q) = \tau_{+,0}(q) \bullet [\Lambda] = \sigma_{-,0}([\Lambda]), q \rangle_- \neq 0 \tag{5.19}$$

which is a contradiction. This completes the proof.

□

Consider a disconnected Legendrian submanifold Λ as above. We want to show that there are no further obstructions regarding the dimension of the image of the map $\tau_{-,n}$ or map $\tau_{+,n}$.

Let us remark that for $r = 1$ the Legendrian submanifold Λ is a connected Legendrian and so the problem of the DGA-homotopy reduces to vanishing of the $\tau_{+,0}$ map in the duality sequence. More precisely, from Theorem 1.1 in [5] it follows that

$$\varepsilon_1 \text{ and } \varepsilon_2 \text{ are DGA-homotopic if and only if } \tau_{+,n} \text{ does not vanish.} \tag{5.20}$$

And so the non-existence of other obstruction with respect to the image of $\tau_{+,n}$ for $r \geq 2$ amounts to proving the following Proposition 5.5.5.

Propositon 5.5.5. *For any integer $r \geq 2$ and any non-negative integer $m < r$ there exists a disconnected Legendrian submanifold Λ^r and augmentations $\varepsilon_L^m, \varepsilon_R^m$ of its Chekanov-Eliashberg algebra so that* $\operatorname{im}_{\mathbb{Z}_2} \tau_{+,n} = m$.

Proof. As in [5] by $\Lambda^{(2)}$ let us denote the standard n-dimensional Legendrian Hopf link in $J^1(\mathbb{R}^n)$ so that the Maslov potential on the upper component is the Maslov potential of the lower component enlarged by 1.

The Corollary 4.6 in [5] then implies that there are two augmentations ε_L and ε_R of Chekanov-Eliashberg algebra of $\Lambda^{(2)}$ so that $\varepsilon_L(m_{12}) = 1$ and $\varepsilon_R(m_{12}) = 0$

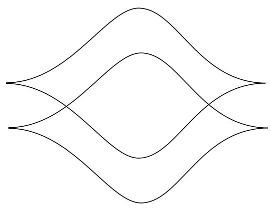

Figure 5.11: Front projection of the Hopf link $\Lambda^{(2)}$ when $n = 1$.

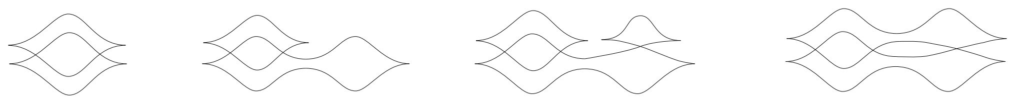

Figure 5.12: The construction of the Legendrian submanifold Λ' when $n = 1$.

for a chord m_{12} and they vanish otherwise. Perform a Legendrian ambient surgery on those two components of $\Lambda^{(2)}$ producing a Legendrian Λ'.

Pull-back the augmentations ε_L and ε_R onto the algebra $\mathcal{A}(\Lambda')$. Consequently Proposition 3.2 from [5] implies that those two pull-backed augmentations $\widetilde{\varepsilon_L}$ and $\widetilde{\varepsilon_R}$ are not DGA-homotopic since $\partial(m_{12}) = 0$ by Proposition 4.5 and $\tau_{+,0}(m_{12}) = \widetilde{\varepsilon_L}(m_{12}) - \widetilde{\varepsilon_R}(m_{12}) = 1 \neq 0$. The fact 5.20 yields that $\dim_{\mathbb{Z}_2} \operatorname{im} \tau_{+,n} = 0$.

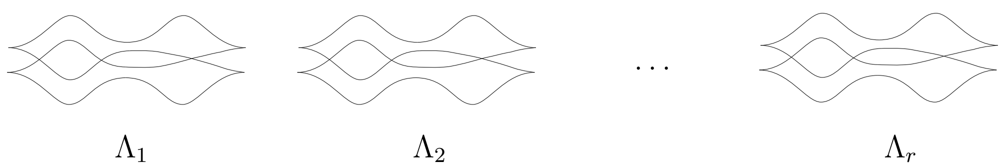

Figure 5.13: Front projection of the Legendrian Λ^r when $n = 1$.

The disconnected Legendrian submanifold $\Lambda^r = \coprod_{j=1}^r \Lambda_j$ is defined as r unlinked horizontally displaced copies Λ_j of Λ'. That means that the bilinearized complex splits into r copies because there are no Reeb chords among distinct components. In particular,

$$\tau_{+,n}^{\Lambda_p} = \bigoplus_{j=1}^{r} \tau_{+,n}^{\Lambda_j}$$

that is for the following augmentations the rank of the resulting map $\tau_{+,n}^{\Lambda^r} : LCH_n^{\varepsilon_L^m, \varepsilon_R^m}(\Lambda^r) \to H_n(\Lambda^r) = \bigoplus_{j=1}^r H_n(\Lambda_j)$ is the sum of the ranks of the maps $\tau_{+,n}^{\Lambda_j} : LCH_n^{\widetilde{\varepsilon_L}, \widetilde{\varepsilon_R}}(\Lambda_j) \to H_n(\Lambda_j)$ playing the same role as $\tau_{+,n}^{\Lambda^r}$ but in the duality exact sequence of the corresponding component. Fix some $m \in \{0, \dots, r-1\}$. Now define the augmentations by the assignment for any c a chord of Λ_j $\varepsilon_L^m(c) = \widetilde{\varepsilon_L}(c)$ and if $m = 0$, then $\varepsilon_R^m(c) = \widetilde{\varepsilon_R}(c)$, otherwise

$$\varepsilon_R^m(c) = \begin{cases} \widetilde{\varepsilon_L}(c); & 1 \leq j \leq m, \\ \widetilde{\varepsilon_R}(c); & m < j \leq r. \end{cases}$$

By the construction of ε_L^m and ε_R^m it is clear that since $m \neq r$ they are not DGA-homotopic because otherwise we could factor the corresponding $(\varepsilon_L^m, \varepsilon_R^m)$-derivative through the chords of r-th component Λ^r, which is impossible, and thus

$$\dim_{\mathbb{Z}_2} \operatorname{im} \tau_{+,n}^{\Lambda^r} = \sum_{j=1}^{r} \dim_{\mathbb{Z}_2} \operatorname{im} \tau_{+,n}^{\Lambda_j} = m \cdot 1 + (r-m) \cdot 0 = m$$

as we desired.

□

5.6 Künneth-like formulas for LCH

Let us return back to the product constructions. We have defined invariants $LCH(\Lambda)$ and $\mathcal{A}(\Lambda)$ and so regarding the standard (co-)homology theories that are appearing in topology, one may ask, whether there is some isomorphism of the algebraic invariant of the Legendrian submanifold stemming from some product construction and the tensor product of the original invariant with the space that we use as the second factor in the product, or that we use to rotate our Legendrian around.

If $\Lambda \subset J^1(M)$ Legendrian embedding for which $\mathcal{A}(\Lambda)$ is well-defined and moreover, we have an augmentation of $\mathcal{A}(\Lambda)$ over $\mathbb{Z}_2$. Then in [15] there first appeared the effect of the $\mathbf{S}^1$-spun on $\mathcal{A}(\Lambda)$ (for more details see Proposition 4.17. in [15]), in particular, we have an augmentation $\varepsilon_\Sigma : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ induced by ε. From this effect the first Künneth-like formula follows (Claim 5.4. in [16]).

$$LCH_\bullet^{\varepsilon_\Sigma}(\Sigma\Lambda) \cong \bigoplus_{k+l=\bullet} LCH_k^{\varepsilon}(\Lambda) \otimes_{\mathbb{Z}_2} H_l(\mathbf{S}^1),$$

where $H_\bullet(\Lambda)$ is the Morse (equivalently singular) homology of Λ over $\mathbb{Z}_2$. Therefore, since $H_\bullet(\mathbf{S}^1) = 0$ for $\bullet \neq 0, 1$ and $H_\bullet(\mathbf{S}^1) \cong \mathbb{Z}_2$ for $\bullet = 0, 1$ we obtain that

$$\begin{aligned} LCH_\bullet^{\varepsilon_\Sigma}(\Sigma\Lambda) &\cong LCH_\bullet^{\varepsilon}(\Lambda) \otimes_{\mathbb{Z}_2} H_0(\mathbf{S}^1) \oplus LCH_{\bullet-1}^{\varepsilon}(\Lambda) \otimes_{\mathbb{Z}_2} H_1(\mathbf{S}^1) \\ &\cong LCH_\bullet^{\varepsilon}(\Lambda) \oplus LCH_{\bullet-1}^{\varepsilon}(\Lambda). \end{aligned}$$

This was generalized for $\mathbf{S}^m$-spuns in [7] using Floer homology techniques for $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$ so that it admits L_i exact Lagrangian fillings for $i = 1, 2$, and so augmentations ε_{L_i}. Let us remark that then $\Sigma_{\mathbf{S}^m} L_i$ is an exact Lagrangian filling of $\Sigma_{\mathbf{S}^m}\Lambda$. Thus we have an augmentation $\varepsilon_{\Sigma L_i} : \mathcal{A}(\Sigma_{\mathbf{S}^m}\Lambda) \to \mathbb{Z}_2$. It holds that (see Theorem 2.11. in [7])

$$\begin{aligned} LCH_\bullet^{\varepsilon_{\Sigma L_1}, \varepsilon_{\Sigma L_2}}(\Sigma_{\mathbf{S}^m}\Lambda) &\cong \bigoplus_{k+l=\bullet} LCH^k_{\varepsilon_{L_1}, \varepsilon_{L_2}}(\Lambda) \otimes_{\mathbb{Z}_2} H_l(\mathbf{S}^m) \\ &\cong LCH^\bullet_{\varepsilon_{L_1}, \varepsilon_{L_2}}(\Lambda) \oplus LCH^{\bullet-m}_{\varepsilon_{L_1}, \varepsilon_{L_2}}(\Lambda). \end{aligned}$$

Nevertheless, when we consider the Legendrian product, we immediately run into troubles. First, let us note that

Propositon 5.6.1. *If* $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$ *be a fillable Legendrian submanifold so that* $(\mathcal{A}(\Lambda), \partial)$ *is defined, then* $\partial(a) \neq 1$ *for every* $a \in \mathcal{A}(\Lambda)$*.*

Proof. The exact Lagrangian filling induces the augmentation $\varepsilon_L : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ which is a unital differential algebra homomorphism, that is $\varepsilon_L(1) = 1$ and $\varepsilon_L \circ \partial(a) = 0$ for all $a \in \mathcal{A}(\Lambda)$. For the sake of contradiction, if there was $a \in \mathcal{A}(\Lambda)$ so that $\partial(a) = 1$, then it would hold that

$$1 = \varepsilon_L(1) = \varepsilon_L(\partial(a)) = 0.$$

□

Now, choose $\Lambda_1 < \Lambda_2$ Legendrians so that Λ_1 is fillable and Λ_2 is loose, then $\Lambda_1 \boxtimes \Lambda_2$ is fillable by Corollary 2.2.3. And so we have example of Λ_1, Λ_2, and $\Lambda_1 \boxtimes \Lambda_2$ so that $\mathcal{A}(\Lambda_2)$ is acyclic, however, $\mathcal{A}(\Lambda_1 \boxtimes \Lambda_2)$ is not acyclic. Therefore, if we had some tensor product formula some alleged tensor product with zero might yield a non-zero vector space (or module over more sophisticated coordinate rings). That is clearly troubling and so some kind of Künneth formula for Legendrian product is hard to expect.

On the other hand, it is an interesting question since in [37] Lambert-Cole asks whether for some particular choice of Λ_2, $LCH(\Lambda_1 \boxtimes \Lambda_2)$ is a stronger invariant of Λ_1 than $LCH(\Lambda_1)$ regarding Legendrian isotopies with a suitable behaviour on Reeb chord lengths.

Remark. Note that there are parallel developments regarding generating families and generating family homology. For example, in [6] the effect of front spinning construction on generating families of a Legendrian was described and used to prove geography-type result for the generating family homology.

Conclusion

In this thesis, we have seen two original contributions on both flexible and rigid side of contact topology.

On the flexible side, the existence of particular family of loose Legendrian embeddings of $\mathbf{T}^3$ into $(\mathbb{R}^7, \xi_{st})$ was showed in Theorem 4.3.3 that answers a question analogous to Question 1.4. in [11] asked by Dimitroglou Rizell and Golovko for $\mathbf{T}^2$ and $(\mathbb{R}^5, \xi_{st})$, however, this question remains open. To prove Theorem 4.3.3 we studied the rotation class in the case of parallelizable Legendrian submanifolds and then we constructed a particular family of maps from $\mathbf{T}^3$ to $\mathbf{S}^3$ of non-zero degree. That induced a map $\mathbf{T}^3$ to $U(3)$ whose homotopy class provided us with a non-splittable class. Therefore, using Theorem 4.1.3 we obtained the class of embeddings with desired properties.

On the rigid side, we may find a generalization of DGA-homotopy criterion for augmentations of Chekanov-Eliashberg algebra for disconnected Legendrian submanifolds (see Theorem 5.5.1) that was proved for connected Legendrian submanifolds in [5] by Bourgeois and Galant. To obtain this result, we studied the duality long exact sequence for bilinearized Legendrian contact homology (see [4]) that generalizes the linearized equivalent that is due to Ekholm, Etnyre, and Sabloff. From this long exact sequence, the topological obstruction for DGA-homotopy arises. There, it is helpful to work with two long exact sequences each for one order of augmentations because both of those sequences appear in the duality formula in the bilinearized setting (see Proposition 5.3.3). Finally, we use the effect of ambient Legendrian surgery on Chekanov-Eliashberg algebra due to Dimitroglou Rizell (see [9]). In particular, considering the bilinearized Legendrian contact homology, one obtains a surgery long exact sequence that enables us to reduce the question to the connected case.

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