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H-compactifications of Topological Spaces

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Abstract: H-compactifications are compactifications that continuously extend all automorphisms of a given topological space over its compactifciation. Van Douwen proved there are only three H-compactifications of $\mathbb{R}$ and only one of $\mathbb{Q}$. Vejnar proved that there are precisely two H-compactifications of higher dimensional Euclidean spaces.

We prove that there is only one H-compactification of $\mathbb{Q}^\omega$, which is precisely the Stone-Čech compactification. For the proof, we use the properties related to (strong) zero-dimensionality, which shows to play important role in topological characterization of other spaces and their H-compactificaions.

We ask a question about all H-compactifications of l^2 and provide some tools that may be helpful to answer this question.

Finally, we describe all H-compactifications of a space as a category and study properties of such category.

Keywords: Compactification, Tychonoff space, H-compactification, Homeomorphism, Category theory

Contents

List of Notation

$\mathcal{C}$	1) Cantor set 2) Arbitrary category
$C^{\star}(X)$	Set of bounded real valued continuous functions from X to $\mathbb{R}$
$cHaus$	Category of compact Hausdorff spaces
F_{σ}	F-sigma set
$F_{\sigma\delta}$	F-sigma-delta set
$\mathfrak{F}$	Z-filter
G_{δ}	G-delta set
$G_{\delta\sigma}$	G-delta-sigma set
grf	Graph of f
$\mathbb{H}$	Halfline
$\mathcal{H}$	Set of all automorphisms on X
$\mathcal{H}^{+}(X)$	Space of increasing homeomorphisms of X
$Hom(X, Y)$	Morphisms from X to Y
$K(X)$	Set of all compactifications of X
$\mathcal{K}(X)$	Hyperspace of X (equivalently denoted 2^{X})
l^2	Space of all square summable real sequences
$\mathbb{N}$	Natural numbers (equivalently denoted ω)
$Ob(X)$	Objects of X
P	1) Irrational numbers 2) Partially ordered set
P_X	Lattice of all H-compactifiactions of X
$\mathcal{P}_{\mathcal{X}}$	Category of all H-compactifiactions of X
$\mathbb{Q}$	Rational numbers
$\mathbb{Q}^{\omega}$	Set of all rational sequences (also denoted $\mathbb{Q}^{\mathbb{N}}$ or $\mathbb{Q}^{\infty}$)
Q_X	Lattice of all compactifiactions of X
$\mathcal{Q}_{\mathcal{X}}$	Category of all compactifiactions of X
$\mathbb{R}$	Real numbers
$\mathbb{R}^X$	Set of all functions from X in to $\mathbb{R}$
S^n	n-dimensional sphere
Top	Category of topological spaces

X_1	Class of all zero-dimensional absolute $F_{\sigma\delta}$-spaces which are nowhere σ-complete and of first category
$\mathbb{Z}$	Integers
αX	Alexandroff one-point compactification of X
βX	Stone-Čech compactification of X
γX	Arbitrary compactification of X
Δ^0_α	Class of sets that is both in Σ^0_α and Π^0_α
Π^0_α	Class of sets whose complement is in Σ^0_α
ρ_H	Hausdorff metric
Σ^0_α	Class of sets closed under countable unions and finite intersections
ω_1	Set of All Countable Ordinals
$[0,1]^{(0,1)}$	Tychonoff cube
$\{0,1\}^X$ or 2^X	Cantor cube
$\{0,1\}^{\mathbb{N}}$	Set of n-tuples consisting of 0s and 1s
∞	Point at infinity
2^ω	Cantor space

Introduction and Preliminaries

Let us start the thesis by recalling some basic facts from general topology and mathematical analysis that will make it easier to provide some intuition behind the notion of H-compactifications and its importance.

Definition 1. *A topological space X is Hausdorff if and only if for every two points x and y, there exist disjoint open sets U and V with $x \in U$ and $y \in V$.*

Hausdorff spaces represent a nice world for proving theorems. They formalize the idea of separating points from each other, they allow limits of sequences to be unique, they have many more of the properties one would intuitively associate with a space.

Metric spaces, manifolds, topological groups and many other objects of interest are Hausdorff spaces and in many mathematical texts, all spaces are automatically assumed Hausdorff.

Compact Hausdorff spaces that are compact are even more favorable to work with.

Definition 2. *A compact space is a space in which every open covering of X has a finite subcovering, in other words, if for any collection $\{U_\alpha\}_{\alpha \in A}$ of open sets with $X \subset \cup_{\alpha \in A} U_\alpha$ there exists a finite set of indices $\{\alpha_1, ...\alpha_n\}$ such that $X \subset U_{\alpha_1} \cup ... \cup U_{\alpha_n}$.*

In layman terms,

"If a city is compact, it can be guarded by a finite number of arbitrarily near-sighted policemen."

This paraphrase of the finite subcover definition of compactness is attributed to Hermann Weyl. However, compact spaces can be also described using other properties, for example the following statements define compactness for metric spaces:

1. *All continuous functions are bounded*,
2. *All continuous functions attain a maximum*,
3. *Every sequence has a convergent subsequence.*

Note that all these characteristics are deducible from each other. However, without more specific assumptions, only the first and most general definition via open covers can be used in all cases.

Proposition 3. *Compactness is a topological property.*

Proof. Let X be a compact space. The claim follows from the fact that if a function $f : X \to Y$ is continuous, then $f(X)$ is compact. To see this, let $f : X \to Y$ be a continuous function. If $\{U_\alpha \mid \alpha \in A\}$ is an open cover of $f(X)$, then $\{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open cover of X, since the inverse image of an open set is open. By the assumption, X is compact, so it has a finite subcover $\{f^{-1}(U_{\alpha_i}) \mid i = 1, 2, 3, ...\}$. Then $\{U_{\alpha_i} \mid i = 1, 2, 3, ...\}$ makes a finite subcover of $f(X)$, which proves that $f(X)$ is compact. □

Theorem 4. *(Chandler, 1976, Theorem 1.11)*

(i) Closed subsets of compact spaces are compact.

(ii) Compact subsets of Hausdorff spaces are closed.

(iii) If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

(iv) If $f : X \to Y$ is one-to-one and continuous, X is compact, and Y is Hausdorff then f is a homeomorphism onto $f(X)$.

Proof. (i) Choose a compact space X and its closed subset F. Let $\{U_\alpha \mid \alpha \in A\}$ be an open covering of F. We take the open covering $\{U_\alpha \mid \alpha \in A\}$ and add it to the set $X \setminus F$ which is open. $\{U_\alpha \mid \alpha \in A\} \cup \{X \setminus F\}$ is still an open covering of F, in fact, it is also an open covering of X, since $(X \setminus F) \cup \bigcup_{\alpha \in A} U_\alpha \supset (X \setminus F) \cup F = X$. In conclusion, there exists a finite subcovering of X and it already contains the finite subcovering of the set F as well.

(ii) For a fixed point $x \in X \setminus F$ and each point y of F select disjoint open sets U_y, V_y containing x, y. $\{V_y \mid y \in F\}$ is an open covering of F. Take a finite subcovering (which we can do by assupmtion) and so the intersection of the corresponding $U's$ will be a neighborhood of x in $X \setminus F$.

(iii) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open covering of $f(X)$. Then $\{f^{-1}(\mathcal{U})\}$ is an open covering of X. The U_α corresponding to the finite subcovering of X provide a finite subcovering of $f(X)$.

(iv) For an open set $U \subset X$ we have $X \setminus U$ is closed. Then, of course, it has to be compact, from which $f(X \setminus U)$ is also compact. Hence $f(X \setminus U)$ is closed; hence $f(U)$ is open. Thus, $f : X \to f(X) \subset Y$ is an open mapping . □

Theorem 5 (Tychonoff)**.** *The product topological space (endowed with product topology) of an arbitrary set of compact topological spaces is also compact.*

Several ways of proving this very famous theorem are available in the literature, three of them given by a relatively new Matheron´s paper (see Matheron (2020)). The original Tychonoff's proof proof from 1930 used the concept of a complete accumulation point, the Cartan's proof using ultrafilters or Chernoff's proof using nets. The theorem also comes as a corollary of the Alexander subbase theorem saying that if (X, τ) is a topological space and X has a subbasis S such that every cover of X by elements from S has a finite subcover, then X is compact.

Example 5.1. $[0,1]^{\mathbb{N}}$ *is compact, being a product of closed intervals* $[0,1]$.

Compact spaces indeed carry many advantageous properties, however, compactness is not hereditary (unlike other properties like metrizability, Hausdorffness, regularity etc.). This is actually the reason why "putting a non-compact space into something compact" can broaden our knowledge about that space. In particular, we can construct a dense *embedding* of a Hausdorff space into a compact space which brings us to the concept of compactifications.

Remark. *Recall that we say that a topological space X embeds into a topological space Y if X is homeomorphic to a subspace of Y. If we take a closure of the embedding, the name for this process, or equivalently for the resulting compact Hausdorff space, is compactification.*

Definition 6. *A (Hausdorff) compactification γX of a topological space X is a compact (Hausdorff) space γX together with an embedding*

$$\gamma : X \to \gamma X$$

so that $\gamma(X)$ is dense in γX.

Some authors instead use a pair (X, γ) as a notation for compactification, where γ is a homeomorphism between X and the compact Hausdorff space γX and γ is a dense embedding. The word compactification is sometimes used when referring only to *Hausdorff compactification*.

Definition 7. *A topological space X is called completely regular if given any point $x \in X$ and any closed $S \subset X$ such that $x \notin S$, there is a continuous map $f : X \to [0, 1]$ such $f(x) = 0$ and $f(S) = 1$.*

Loosely speaking, points of a complete regular space can be separated from closed sets via (bounded) continuous real-valued functions.

Definition 8. *X is called a Tychonoff space (alternatively a $T_{3\frac{1}{2}}$ space) if it is a completely regular Hausdorff space.*

Proposition 9. *A topological space has a Hausdorff compactification if and only if it is Tychonoff.*

Proof. One direction is a not difficult - note that every compact Hausdorff space is automatically a Tychonoff space. Since every subspace of a Tychonoff space is Tychonoff, we conclude that any space possessing a Hausdorff compactification must be a Tychonoff space. The converse, i.e. that every Tychonoff space has a Hausdorff compactification was proved in the famous Tychonoff´s 1930 article. where he also defined completely regular spaces and proved Tychonoff's theorem (see Theorem 5). □

Remark. *Every Tychonoff space embeds into a product of type $[0, 1]^I$, whence it always admit a compactification (take the closure of the embedded copy of X).*

Regarding the facts above, we will automatically assume that all topological spaces in this text are Tychonoff.

Constructing compactifications

There are various ways to go about compactifying a space. Intuitively, each such construction somehow controls points from going off to infinity by adding "points at infinity" or preventing such an "escape" in whatever other way.

Example 9.1. *We will show on one the simplest spaces that embedding a space in numerous different compact Hausdorff spaces is not that difficult to construct, but neither that insightful in all cases.*

Take the interval $(0, 1)$ *with its usual topology. The most natural way to compactify the* $(0, 1)$ *is embedding as a subspace of* $[0, 1]$ *via the natural inclusion map*

$$f_1 : (0, 1) \to [0, 1].$$

Here, the compactifying process adds just the two endpoints $\{0\}, \{1\}$*. Another possibility is embedding the space* $(0, 1)$ *as a subspace of a 1-sphere* S^1 *via the map*

$$f_2(x) := (cos(2\pi x), sin(2\pi x)).$$

One easily recognizes that f_2 *behaves the most "efficiently" - it only "misses" the point* $(1, 0)$ *of the original space in* S^1*, while* f_1 *"misses" two points.*

There are other ways of compactifying $(0, 1)$*, e.g. embedding into* $[0, 1]^{\mathbb{N}}$ *which is compact (as we have seen in the example), via the map*

$$f_3(x) := (x, 1, 1, 1, ...)$$

or embedding as a subspace of the Topologist's Sine Curve via

$$f_4 : x \to (x, sin(\tfrac{1}{x})).$$

Note that apart from f_1, f_2, f_3, f_4, we can have other possible ways how to embed $(0, 1)$ in something compact.

Introducing H-compactifications

We have seen that sometimes, given a space X, we can go on and generate as many compactifications as we like by choosing different representations of X. We would, however, soon recognize that many of such compactifications may be rather bizarre and not telling much about X.

This is why it is useful to introduce a special type of compactifications - such that they do not depend on a specific "representation" of a given space, but exclusively on its topological properties.

Definition 10. *Let* $h : X \to Y$ *be a continuous mapping and* γX *a compactification on* X*. We say that* h *has an extension to* γX*, denoted by* γh*, such that* $\gamma h : \gamma X \to Y$ *if* $\gamma h \circ \gamma = h$ *where*
γh *is continuous.*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & \gamma X \\ & \searrow^{h} & \downarrow{\scriptstyle \gamma h} \\ & & Y \end{array}$$

Alternatively, $\gamma h \restriction_X = h$*.*

Definition 11. *Given a topological space X, a compactification γX of X is said to be an H-compactification if each automorphism of X can be continuously extended to an automorphism (a mapping of γX into γX).*

Remark. *For a space X, if an automorphism*

$$f : X \to X$$

extends continuously to

$$\gamma f : \gamma X \to \gamma X,$$

then also γf^{-1} exists. Therefore, γf is an automorphism of γX too.

The inverses of automorphisms are a hint that all automorphisms on a given space naturally form a group. Moreover, if the group of automorphisms admits a natural topology, we get a topological group.

Many common topological spaces like $\mathbb{R}, \mathbb{Q}$, the set of irrational numbers or the Cantor set, carry "rich" automorphism groups, which manifests that a lot of their points behave topologically the same. Spaces with large number of automorphisms have their special name - *homogeneous spaces*.

Definition 12. *Let $\mathcal{H}(X)$ be the group of all automorphisms of a space X. We say that a continuous mapping $f : X \to Y$ is homogeneous if for every $h \in \mathcal{H}(X)$ there exists $g \in \mathcal{H}(X)$ such that*

$$f \circ h = g \circ f.$$

In some literature, a homogeneous space X is also defined as a space X such that for every $x, y \in X$ there is an automorphism of X that maps x to y. It is intuitive that more automorphisms a space admits, less H-compactifications of that space exist.

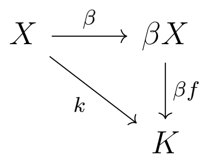
Example 12.1. *Rigid space is a space on which the only homeomorphisms are the trivial ones (i.e. the identity homeomorphisms). It is not difficult to see that all compactifications of rigid spaces are automatically H-compactifications.*

The most common H-compactifications

The Stone-Čech H-compactification

Embedding a space into a compactification that continuously extends all its automorphisms is not always that straightforward. However, for a Tychonoff space X, the most famous H-compactification of any space βX called the Stone-Čech compactification always exists.

Definition 13. *The Stone–Čech compactification of the topological space X is a compact Hausdorff space βX together with a continuous map β with the universal property, which means that any continuous map $k : X \to K$, where K is a compact Hausdorff space, extends uniquely to a continuous map $\beta f : \beta X \to K$, i.e. $\beta f \circ \beta = k$.*



The universal property, expressed by the commutative diagram, will get more attention in the last chapter of this thesis, where functorial properties of βX are studied.

βX is the most "general" compactification in the sense that it is characterized by the universal property. Any continuous function from X to a compact Hausdorff space K can be extended to a continuous function from βX to K in a unique way.

Theorem 14. *Tychonoff (1930) Let X be a Tychonoff space. Then its Stone-Čech compactification exists and it is unique (up to homeomorphism).*

Proof. We will provide the main steps of the proof, which originates in the Tychonoff´s paper from 1930.

For uniqueness, we first prove that if a compactification of a Tychonoff space X satisfying the given universal property exists, then it is unique up to equivalence. Suppose we have two compactifications of X satisfying the property, namely $\gamma_1 : X \to \gamma_1 X$ and $\gamma_2 : X \to \gamma_2 X$. Then γ_1 and γ_2 have to be continuous functions from X into compact Hausdorff spaces, and so by the universal property we can find other continuous functions $f : \gamma_1 X \to \gamma_2 X$ and $g : \gamma_2 X \to \gamma_1 X$ such that $\gamma_2 = f \circ \gamma_1$ and $\gamma_1 = g \circ \gamma_2$. From this it immediately follows that $g \circ f$ is the identity map on $\gamma_1 X$, and that $f \circ g$ is the identity map on $\gamma_2 X$. Therefore f is a continuous function with a continuous inverse, making it the homeomorphism we require.

To show existence, one has to construct c compactification satisfying the required property. Let C be the collection of all continuous functions $X \to [0, 1]$. Because X is completely regular, C is certainly nonempty. By Tychonoff´s theorem (see Theorem 5), the Hausdorff product space $[0, 1]^C$ is compact. Now define an evaluation map $i : X \to [0, 1]^C$ such that each $x \in X$ is sent by i to the evaluation at x. That is, given a $x \in X$, define

$$i(x) = e_x : C \to [0, 1] \text{ given by}$$
$$e_x(f) = f(x) \text{ for each } f \in C.$$

Finally, define $\beta X = \overline{i(X)}$, with its subspace topology inherited from $[0, 1]^C$.

Finally, it needs to be checked that βX is a compactification and that it satisfied the defined universal property which we skip in this chapter. □

There are several ways to construct the Stone-Čech compactification - some of them are presented in the Chapter 3 and for at least three different detailed construction, see (Chandler, 1976, Chapters 2, 3).

The Alexandroff one-point H-compactification

The other very typical H-compactification is the Alexandroff one-point compactification αX.

Definition 15. *A Hausdorff compactification αX of a locally compact (i.e. such that each point of it has a compact neighborhood), non-compact, Hausdorff space X, obtained by adding a single point ∞ to X is called the Alexandroff one-point compactification.*

The extra point ∞ is usually called "a point at infinity".

Definition 16. *For a locally compact X, denote the topology of X by τ_X and define the topology $\tau_{\alpha X}$ on its one-point compactification αX the following way:*

$$\tau_{\alpha X} := \tau_X \cup \{U \subseteq \alpha X \mid \infty \in U \text{ and } X \setminus U \text{ is a compact subset of } X\}.$$

Remark. *In the situation where X is not locally compact, we can still construct the one-point compactification, however, this case is usually called Alexandroff extension.*

1. Known sets of all H-compactifications

This chapter synthesizes all that is known about sets of H-compactifications of more or less common topologcial spaces. What interests us is description of the set of *all* H-compactifications of a given space.

Remark. *Different sources use a lot of different names for H-compactification, e.g. "G-compactification", where G is a subgroup of the group of all automorphisms of a space X, in de Groot und McDowell (1959/60) or "equivariant extension" in Smirnov (1994). We find the most suitable for this notion the term topological compactification" from van Douwen (1979) and "H-compactification" used in Vejnar (2011), which we will keep throughout this text.*

For a space X, the βX usually does not look that nice - on the other hand, constructing αX is often more convenient. Hence, for some spaces we will look closely at their one-point compactifications.

1.1 Halfline

Let $\mathbb{H}$ be the halfline $[0, \infty)$. Van Douwen´s observation about $\mathbb{H}$ leads to the description of H-compactifications of the real line.

Proposition 17. *(van Douwen, 1979, Proposition 4) The set of all H-compactifications of $\mathbb{H}$ consists of just two elements: $\alpha\mathbb{H}$ and $\beta\mathbb{H}$.*

Proof. Let $\lambda\mathbb{H}$ be any H-compactification of $\mathbb{H}$ with $|\lambda\mathbb{H} \setminus \mathbb{H}| > 1$ (so it is distinct from $\alpha\mathbb{H}$). We show that $\lambda\mathbb{H} = \beta\mathbb{H}$ by showing that disjoint closed subsets of $\mathbb{H}$ have disjoint closures in $\lambda\mathbb{H}$. (A part of Chapter 3 is devoted to explaining how this property characterizes the Stone-Čech compactification). So let F and G be disjoint closed subsets of $\mathbb{H}$. Without loss of generality, assume that $0 \in F$. We can define discrete family $\mathcal{A}$ in a way that each of them is a family of closed subsets of $\mathbb{H}$ - i.e. of closed intervals such that every point of $\mathbb{H}$ has a neighbourhood intersecting at most one element of $\mathcal{A}$. Analogously, define another discrete family $\mathcal{B}$. One can construct $\mathcal{A}$ and $\mathcal{B}$ such that

$$F \subseteq \cup\,\mathcal{A},$$
$$G \subseteq \cup\,\mathcal{B} \text{ and}$$
$$(\cup\,\mathcal{A}) \cap (\cup\,\mathcal{B}) = 0.$$

Let p and q be any two distinct points of $\lambda\mathbb{H} \setminus \mathbb{H}$. Let U and V be neighborhoods of p and q in $\lambda\mathbb{H}$ with $0 \in U$ and $\overline{U} \cap \overline{V} = \emptyset$. Without difficulty one can construct an autohomeomorphism h of $\mathbb{H}$ such that $h(A) \subseteq U$ for $A \in \mathcal{A}$ and $h(B) \subseteq V$ for $B \in \mathcal{B}$. If λh is the extension of h then

$$(\lambda h)(\overline{F}) \cap (\lambda h)(\overline{G}) \subseteq \overline{U} \cap \overline{V} = \emptyset,$$

hence $\overline{U} \cap \overline{V} = \emptyset$. □

1.2 Real Line

Proposition 18. *(van Douwen, 1979, Proposition 2) $\alpha\mathbb{R}$, the two-point compactification of $\mathbb{R}$ and $\beta\mathbb{R}$ are the only H-compactifications of $\mathbb{R}$.*

Proof. This follows directly from the result about the halfline. For any $x \in \mathbb{R}$, the map

$$f : x \to -x$$

induces an autohomeomorphism of $\beta\mathbb{R}$ which implies that $\beta[0, \infty)$ is identical with $\beta(-\infty, 0]$. □

$\beta\mathbb{H}$ resembles $\beta\mathbb{N}$ in the sense that it is a "thin" locally compact space with a large compact lump at its end.

Proposition 19. *The one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.*

Proof. The construction can be given explicitly as an inverse stereographic projection. Consider the map

$$s : \mathbb{R} \to S^1 \text{ given by}$$
$$x \to (\tfrac{1-x^2}{1+x^2}, \tfrac{2x}{1+x^2})$$

Then s is a homeomorphism between $\mathbb{R}$ and $S^1 \setminus \{(-1, 0)\}$. It is because for any real x, taking $f(x) = 2arctan(x)$ is in the interval $(-\pi, \pi)$ and $s(x) = (cos(f(x)), sin(f(x)))$ is a continuous bijection. That s is an open map follows from the observation that s maps any bounded open real interval (x, y) to $\{(cos(u), sin(u)) : f(x) < u < f(y)\}$, which is an open set in $S^1 \setminus \{(-1, 0)\}$. So, since $S^1 \setminus \{(-1, 0)\}$ is dense in S^1 and $S^1 \setminus (S^1 \setminus \{(-1, 0)\})$ consists of a single point, S^1 is the one-point compactification of $\mathbb{R}$. □

1.3 Higher dimensional Euclidean spaces $\mathbb{R}^n$

Obviously, any space $\mathbb{R}^n, n \geq 2$ is still locally compact, however, the previous argument using the half-line H-compactifications fails and a more sophisticated proof is needed.

Proposition 20. *(Vejnar, 2011, Corollary 28) For $\mathbb{R}^n$ for $n \geq 2$, there are exactly two H-compactifications, namely $\alpha\mathbb{R}$ and $\beta\mathbb{R}$.*

We will rephrase the proof of this proposition in the second part of the Chapter 3 which explores compactifications of l^2.

Example 20.1. *We have already constructed $\alpha\mathbb{R}$, so it is not surprising that $\alpha\mathbb{R}^n$ is (homeomorphic to) the n-sphere S^n. What is impressive is that this allows proving things about S^n by passing to $\mathbb{R}^n$ and vice versa. The embedding of $\mathbb{R}^n$ in S^n is the inverse of the stereographic projection.*

One example is the fact that S^n is simply connected (it is path-connected and any loop can be contracted to a point). We can see this if we show that a path in S^n can be deformed so that it misses one point. From here, removing the missed point gives a path in $\mathbb{R}^n$, which can be deformed into a constant path using linear functions.

1.4 $\mathbb{N}$

It is easy to embed $\mathbb{N}$ (with the usual discrete topology) into a Hausdorff space. It only admits two H-compactifications - α and β. For the precise description the one-point compactification $\alpha\mathbb{N}_\infty$, we direct a reader to (Escardó, 2013, Parts 5-9) where it is obtained from the discrete space $\mathbb{N}$ as

$$\alpha\mathbb{N} = \{x \in \{0,1\}^{\mathbb{N}} \mid \forall i \in \mathbb{N}(x_i \geq x_{i+1})\},$$

also known as the generic convergent sequence. The $\{0,1\}^{\mathbb{N}}$ refers to a set of (ordered) n-tuples consisting of 0´s and 1´s.

Regarding the construction of $\beta\mathbb{N}$, one example can be found in a paper by Tychonoff (1935) where it is built as a closure of a countable set

$$A = \{a_n(x) : n \in \mathbb{N}\}$$

of points in the Tychonoff cube $[0,1]^{(0,1)}$. (The $a_n(x)$ refers to a dyadic expansion of every $x \in (0,1)$). One can also identify $\beta\mathbb{N}$ with the set of ultrafilters on $\mathbb{N}$, with the topology generated by sets of the form $\{F : U \in F\}$ where U is a subset of $\mathbb{N}$. (We define filters and ultrafilters in chapter 3).

Overall, $\beta\mathbb{N}$ is notoriously elusive and highly sensitive to various set-theoretic axioms - a lot of the research on $\beta\mathbb{N}$ instead concentrates on its remainder - $\beta\mathbb{N} \setminus \mathbb{N}$, see for example the chapter Dow und Hart (2003) in the Encyclopedia of General Topology.

Remark. *The set of all H-compactifications of $\mathbb{Z}$ is the same as for $\mathbb{N}$ (since these are homeomorphic).*

1.5 $\mathbb{Q}$

Proposition 21. *(van Douwen, 1979, Proposition 1) The Stone-Čech compactification $\beta\mathbb{Q}$ is the only H-compactification of $\mathbb{Q}$.*

Proof. This is proven as an immediate consequence of the following proposition which we will also find very useful in Chapter 2. □

Proposition 22. *(van Douwen, 1979, Proposition 3) If X is a non-compact strongly zero-dimensional space in which every nonempty clopen subspace is homeomorphic to X, then the only H-compactification of X is the Stone-Čech compactification.*

Proof. Let γX be an arbitrary H-compactification of X. In order to show that γX is the same as βX it is enough to prove that every clopen subset of X has an open closure in γX. Denote by $\bar{\ }$ the closure operator in γX.

If U is a clopen subset of X, we can assume $\emptyset \neq U \neq X$. Then we can find a nonempty clopen subset V of X such that $\overline{U} \cap \overline{V} = \emptyset$. U and V are indeed homeomorphic. Then there exists an automorphism h of X such that $h(U) = V$ and h sends any $x \notin U \cup V$ to itself. By the assumption, γX is a H-compactification, so we can define an extension of h over γX - denote such extension by γh. Observe that γh satisfies

$$\overline{U} \cap (\gamma h)(\overline{U}) = \overline{U} \cap \overline{V} = \emptyset.$$

But since $\gamma h(x) = x$ for each $x \in [X - (U \cup V)]^{-}$, we have

$$\overline{U} \cap (X - U)^{-} = (\overline{U} \cap (\gamma h)(\overline{U})) \cup (\overline{U} \cap [X - (U \cup V)]^{-}) = \emptyset.$$

Hence the closure $\overline{U}$ is open in γX. □

1.6 Other spaces

Van Douwen noted that even for many other zero-dimensional spaces, such as irrationals P or Sorgenfrey line, the above mentioned conclusion about H-compactifications still works and hence such the set of all H-compactifications of each such space consists only of the Stone-Čech compactification. Some modifications of these spaces, such as the product of $\mathbb{Q}$ and P also only admit the Stone-Čech compactification.

There are other spaces studied by several authors, often as "variations" of already analysed spaces. For example Vejnar described 26 H-compactifications of the space $\omega \times \mathbb{R}$ in (Vejnar, 2011, Theorem 33) and proved that the spaces $\omega \times S_n, n > 1$ have only three H-compactifications in (Vejnar, 2011, Corollary 32) and from the same paper, the space $\omega \times S$ has exactly four H-compactifications. Recall that spaces that are topologically the same (homeomorphic) have the same set of H-compactifications. Therefore, we can use topological characterization of the spaces introduced here to find similar conclusion on other spaces.

2. H-compactifications of $\mathbb{Q}^\omega$

In this section, we find the set of all H-compactifications of the space $\mathbb{Q}^\omega$ - the set of all rational sequences. Our result will be shown using characteristics from (van Douwen, 1979, Proposition 3). To achieve that, we have to first analyze properties of $\mathbb{Q}^\omega$ and its nonempty clopen subsets to conclude that the only H-compactification on $\mathbb{Q}^\omega$ is $\beta\mathbb{Q}^\omega$.

Definition 23. *$\mathbb{Q}^\omega$ is defined as a set of all rational sequences.*

Throughout some literature, $\mathbb{Q}^\omega$ is denoted as $\mathbb{Q}^\infty$ or $\mathbb{Q}^\mathbb{N}$ which emphasizes the fact that we can practically interpret $\mathbb{Q}^\omega$ as the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$.

Definition 24. *Let I be a non-empty index set and for each $i \in I$, let X_i be a topological space. If $\prod_{i\in I} X_i$ is a Cartesian product and $\pi_i : \prod_{i\in I} X_i \to X_i$ are canonical projections, then the product topology on $\prod_{i\in I} X_i$ is defined as the coarsest topology (i.e. the topology with the fewest open sets) for which every π_i is continuous.*

Throughout this text, we will always consider $\mathbb{Q}^\omega$ endowed with the standard product topology. It is important to have in mind the definition of this topology and that $\mathbb{Q}^\omega$ is built as a countable product of $\mathbb{Q} \subset \mathbb{R}$, where rational numbers have their standard topology inherited from real numbers (with the standard metric).

$\mathbb{Q}^\omega$ is non-compact, metrizable and separable (note that it is a countable product of $\mathbb{Q}$). Sine the H-compactifications of the space $\mathbb{Q}$ have been described in van Douwen (1979), we now want to study the same problem in the more complicated case of $\mathbb{Q}^\omega$.

Question 1. *What is the set of all H-compactifications of $\mathbb{Q}^\omega$?*

2.1 Characterization and properties of $\mathbb{Q}^\omega$

Before answering the question, we will present all the necessary properties of $\mathbb{Q}^\omega$ needed for analysing its H-compactifications.

2.1.1 (Strong) zero-dimensionality

Further on, we will refer to spaces that are open and closed at the same time as *clopen*.

Definition 25. *A Hausdorff topological space X is zero dimensional if for every point x of X and every neighborhood U of x in X, there exists a nonempty clopen subset V of X such that $x \in V \subset U$. The clopen basis of any zero-dimensional space is a collection of clopen sets that is closed under complements and finite intersections.*

Thus, a space is zero dimensional if and only if it has a basis of clopen sets.

Proposition 26. *The space $\mathbb{Q}$ endowed with the topology inherited from $\mathbb{R}$ is a zero-dimensional space.*

Proof. To show zero dimensionality of $\mathbb{Q}$ (regarded as a subspace of $\mathbb{R}$, which itself is not zero-dimensional), pick $q \in \mathbb{Q}$ and let V be a neighborhood of q in $\mathbb{Q}$. Then there exists an open subset U of $\mathbb{R}$ such that $U \cap \mathbb{Q} = V$. We can find irrational numbers $x < y$ such that $q \in (x, y)$ and $(x, y) \subset U$. Hence, applying set intersection, also $q \in (x, y) \cap \mathbb{Q} \subset U \cap \mathbb{Q} = V$. Moreover, $(x, y) \cap \mathbb{Q} = [x, y] \cap \mathbb{Q}$. But that means exactly that $(x, y) \cap \mathbb{Q}$ is a clopen subset of $\mathbb{Q}$. □

It is a well-known fact that any product of zero-dimensional spaces is zero-dimensional, whence $\mathbb{Q}^\omega$ has zero-dimensionality guaranteed, being a product of zero-dimensional spaces $\mathbb{Q}$.

Definition 27. *A Hausdorff topological space X is said to be strongly zero-dimensional whenever for every closed subset A of X and every open subset U of X such that $A \subseteq U$, there exists a clopen subset V of X such that $A \subseteq V \subseteq U$.*

Remark. *Zero-dimensionality and strong zero-dimensionality are equivalent for all separable metrisable spaces.*

This guarantees strong zero-dimensionality of $\mathbb{Q}^\omega$ which we will utilize at the end of this chapter. we need to introduce some more concepts.

2.1.2 Borel hierarchy

Let us recall some basic definitions and facts concerning the Borel hierarchy - a classical and widely studied topic.

Definition 28. *Given a set S, a σ-algebra over S is a family of subsets of S closed under countable union, countable intersection and complement. The Borel algebra over $\mathbb{R}$ is the smallest σ-algebra containing the open sets of $\mathbb{R}$. A Borel set of real numbers is an element of the Borel algebra over $\mathbb{R}$.*

Informally, a Borel set is any set in X that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and complement.

Definition 29. *Choose a separable metric space X and $\alpha \in [1, \omega_1]$ where ω_1 means the set of all countable ordinals. We will define the Borel classes Σ^0_α and Π^0_β. First let Σ^0_1 be the class of open sets. For each $\alpha > 1$ let Σ^0_α be the class of countable unions of elements of*

$$\cup\{\Pi^0_\beta \mid \beta < \alpha\}$$

where

$$\Pi^0_\beta = \{X - A \mid A \in \Sigma^0_\beta\}.$$

We will not deep dive into the Borel sets and hierarchy, so for more information, see Miller (1979) or, within the framework of separable metrizable spaces, Kuratowski (1966) and Hausdorff (1962)).

Definition 30. *Each Borel set is assigned a unique countable ordinal number called the rank of the Borel set. A Borel set is said to have a finite rank if it belongs in Σ^0_α for a finite ordinal α - else it is said to have an infinite rank.*

The following theorem summarizes the properties of Borel classes in a more intuitive manner.

Theorem 31. *(Miller, 1995, Theorem 2.1) Σ^0_α is closed under countable unions and finite intersections, Π^0_α is closed under countable intersections and finite unions.*

Proof. From the definition of Σ^0_α, it is clearly closed under countable unions. Now denote by $A_n, B_m, n, m \in \mathbb{N}$ arbitrary families of sets. Since

$$(\bigcup_{n\in\mathbb{N}} A_n) \cap (\bigcup_{n\in\mathbb{N}} B_n) = \bigcup_{m,n\in\mathbb{N}} (A_n \cap B_m),$$

Σ^0_α is closed under finite intersections. It also follows from the De Morgan's laws

$$(A \cup B)' = A' \cap B' \text{ and}$$
$$(A \cap B)' = A' \cup B' \text{ and}$$

that Π^0_α is closed under finite unions and countable intersections (for the latter, take complements). □

Definition 32. *Some authors add also the ambiguous classes Δ^0_α to the definition of Borel hierarchy. A set is in Δ^0_α if and only if it is in both Σ^0_α and Π^0_α, whence it is closed under finite intersections, finite unions and complements.*

Example 32.1. *On $\mathbb{R}$, the first level of Borel hierarchy consists of all open and closed subsets of On $\mathbb{R}$, and upon having defined levels $2, 3, 4, ..., n-1$, level n is obtained by taking countable unions and intersections of the previous level.*

When working on higher levels of Borel hierarchy, the classical notation introduced above comes handy. We will, however, work with just the first two levels, hence, further on, we will stick to an alternative notation of Borel hierarchy which uses letters F and G.

Definition 33. *A G_δ set (said "G-delta set") is a a subset of a topological space X that is a countable intersection of open sets.*

A dual notion to the G_δ set is a $F_\sigma$set

Definition 34. *An F_σ set (said "F-sigma set") is a subset of a topological space X which is the countable union of closed sets.*

Remark. *Note that G_δ is synonymous with Π^0_2 and F_σ is synonymous with Σ^0_2 from the abstract definition above. We can apply this correspondence over all Borel classes and obtain*

$$\Sigma^0_1 = \textit{open set} = G$$
$$\Pi^0_1 = \textit{closed set} = F$$
$$\Sigma^0_2 = F_\sigma$$
$$\Pi^0_2 = G_\delta$$
$$\Sigma^0_3 = G_{\delta\sigma}$$
$$\Pi^0_3 = F_{\sigma\delta}$$

and so on.

Analogously to the F_σ and G_δ, we can define the Borel sets of rank three.

Definition 35. *An $F_{\sigma\delta}$ set is a set that can be expressed as a countable intersection of F_σ-sets. A $G_{\delta\sigma}$ set is a set that can be expressed as a union of countably many G_δ-sets.*

Remark. $\mathbb{Q}^\omega$ *is not a $G_{\delta\sigma}$-subset in the completely metrizable space $\mathbb{R}^\omega$. (See van Engelen (1984)). It is, however, $F_{\sigma\delta}$, which can be rewritten as $\bigcap_{n\in\mathbb{N}} \bigcup_{m\in N_n} F_m$ where each N_n is a countable set and all $F_m, m \in N_n, n \in \mathbb{N}$ are closed.*

Definition 36. *A metrizable space X is an absolute $F_{\sigma\delta}$-set (or an absolute $F_{\sigma\delta}$-space) provided that X is an $F_{\sigma\delta}$-subset in every metrizable space in which X is embedded.*

It follows from Engelen´s characterization of $\mathbb{Q}^\omega$ in (van Engelen, 1985, Part 2) that it is absolute $F_{\sigma\delta}$. In the following section we will show this fact, besides other findings.

2.1.3 Sets of first and second category

For making a conclusion about the H-compactifications of $\mathbb{Q}^\omega$, we have to demand $\mathbb{Q}^\omega$ is of first category (note that this terminology should not be confused with category theory concepts).

Informally, a set of first category is a set whose elements are not tightly clustered together anywhere in the space.

Definition 37. *A subset $Y \subseteq X$ of a topological space X is called nowhere dense (or rare) in X if its closure has empty interior. Equivalently, Y is nowhere dense in X if we cannot find any nonempty open subset of X which would be contained in Y*

Definition 38. *A subset of a topological space X is said to be of first category in X if it is a countable union of nowhere dense subsets of X.*

The sets of first category are also called meagre sets or meager sets. Informally, one thinks of a first category subset as a "small" subset of the host space. A subset that is not of first category in X is said to be of the second category in X

Proposition 39. $\mathbb{Q}$ *with the usual topology (as the countable union of one-point subsets of $\mathbb{R}$) is of first category in itself.*

Proof. We need to write $\mathbb{Q}$ as a countable union of nowhere dense subsets of $\mathbb{Q}$. Choose an enumeration $\{q_1\}, \{q_2\}, ...$ of $\mathbb{Q}$. Then $\mathbb{Q} = \bigcup_{i=1,...}\{q_1\}$ and $\{q_i\}$ is nowhere dense for each i. Equivalently, each $\mathbb{Q} - \{q_i\}$ is an open subset of $\mathbb{Q}$ and observe that $\bigcap_{i=1,...}(\mathbb{Q} - \{q_i\}) = \emptyset$, which means that the empty set can be expressed as a countable intersection of open dense subsets of $\mathbb{Q}$. ☐

The case of $\mathbb{Q}$ is not that difficult to visualize. The question is, however, whether the same holds for the infinite product of $\mathbb{Q}$.

Throughout this sub-section, we will, for the sake of convenience, assume all spaces embedded in the Cantor space 2^ω.

Definition 40. *Define and denote the following:*

1. $\Gamma =$ *an arbitrary class of spaces*
2. $\Gamma' = \{X \mid 2^\omega - X \in \Gamma\}$ *(the dual class of* Γ*).*
3. $Q_0 = \{x \in 2^\omega \mid \exists m : \forall n \geq m : x_n = 0\}$
4. $Q_1 = \{x \in 2^\omega \mid \exists m : \forall n \geq m : x_n = 1\}$
5. $P = 2^\omega - (Q_0 \cup Q_1)$
6. *Mapping* $\phi : P \to 2^\omega$ *by* $\phi(x)_n = 0$ *iff the* $n^t h$ *block of zeros in* x *has even length.*

Definition 41. *Let* Γ *be a class of topological spaces and everything as defined above. We say that* Γ *has a property* $(\star)$ *if* Γ *is continuously closed (meaning closed under continuous preimage), and for each* $X \in \Gamma$, $\phi^{-1}(X) \cup Q_0 \in \Gamma$.

Definition 42. *Let* Γ *be a class of sets. We say that a space* X *has a property* $(\star\star)$ *if for each non-empty clopen subset* U *of* 2^ω, $U \cap X \in \Gamma \setminus \Gamma'$.

Theorem 43. *If a class of Borel sets* Γ *has the property* $(\star)$*, and* A *and* B *both have the property* $(\star\star)$*, and are either both of first category or both Baire, then* $A \simeq B$.

For the proof of this theorem, which is rather technical and three pages long, see (Steel, 1980, Theorem 2). What is of higher interest here is the implication for $\mathbb{Q}^\omega$.

Corollary 43.1. $\mathbb{Q}^\omega$ *densely embedded in* 2^ω *is of first category.*

Proof. If we set $\Gamma = \Pi^0_3 = F_{\sigma\delta}$ or $\Gamma = \Sigma^0_3 = G_{\delta\sigma}$, then either of them have the property $(\star)$. Assume $\mathbb{Q}^\omega$ is densely embedded in 2^ω. For $\Gamma = \Pi^0_3$, $\mathbb{Q}^\omega$ has the property $(\star\star)$ and for $\Gamma = \Sigma^0_3$, $2^\omega - \mathbb{Q}^\omega$ has the property $(\star\star)$. Then it is concluded in (van Engelen, 1996, Part 2) that the theorem above implies exactly that $\mathbb{Q}^\omega$ is of first category. □

We will further comment the fact that $\mathbb{Q}^\omega$ is of first category (in itself) in the proof of the Theorem 55.

2.1.4 Characterization of $\mathbb{Q}^\omega$

The notions from the previous sub-chapters will be beneficial for describing $\mathbb{Q}$ in terms of a special class of spaces denoted $\mathcal{X}$, which is inspired by (van Engelen, 1985, Part 3).

Definition 44. *A space* X *is said to be* σ*-complete if* $X = \cup_{i=1}^{\infty} X_i$ *where each* X_i *is complete (i.e. an absolute* G_δ *space), Equivalently, this definition says that* X *is an absolute* $G_{\delta\sigma}$.

Definition 45. *A local ring is a ring (with unit, usually also assumed commutative) such that:* $0 \neq 1$*; and whenever* $a + b = 1$*,* a *or* b *is invertible.*

Definition 46. *A presheaf* $\mathcal{F}$ *of sets on* X *is defined as a rule assigning to each open* $U \subset X$ *a set* $\mathcal{F}(U)$ *and to each inclusion* $V \subset U$ *a map*

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V) \text{ such that}$$
$$\rho_U^U = id_{\mathcal{F}(U)}$$

and whenever $W \subset V \subset U$ *we have* $\rho_W^U = \rho_U^V \circ \rho_V^U$*.*

Presheaves have more ways to be defined, for example the category-theoretic definition can be found in (Leinster, 2014, Definition 1.2.15). However, we only introduce presheaves and sheaves shortly in order to explain properly all the notions used for three consequent lemmas in this subsection.

Definition 47. *A sheaf* $\mathcal{F}$ *of sets on* X *is a presheaf of sets which satisfies the following additional property: Given any open covering* $\mathcal{U} = \cup_{i \in I} U_i$ *and any collection of elements* $s_i \in \mathcal{F}(U_i)$ *called sections such that* $\forall i, j \in I$

$$s_i \restriction_{U_i \cap U_j} = s_j \restriction_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ *such that* $s_i = s \restriction_{U_i}$ *for all* $i \in I$*.*

Informally, a sheaf is the best way of pack a "local" data together on a topological space.

Definition 48. *A ringed space is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of unital rings (that is rings with an identity element for multiplication). The sheaf* $\mathcal{O}_X$ *is called the structure sheaf of the ringed space* $(X, \mathcal{O}_X)$*.*

Definition 49. *Let* X *be a topological space. Let* $x \in X$ *be a point. Let* $\mathcal{F}$ *be a presheaf of sets on* X*. The stalk of* F *at* x *is the set* $\mathcal{F}_x = colim_{x \in U} F(U)$ *where the colimit is over the set of open neighbourhoods* U *of* x *in* X*.*

Definition 50. *A locally ringed space is a ringed space* $(X, \mathcal{O}_X)$ *such that the stalks of the structure sheaf* $\mathcal{O}$ *are local rings.*

Definition 51. *An analytic space is a locally ringed space* $(X, \mathcal{O}_X)$ *such that around every point* x *of* X*, there exists an open neighborhood* U *such that* $(U, \mathcal{O}_U)$ *is isomorphic (as locally ringed spaces) to an analytic variety with its structure sheaf, where an analytic variety is defined locally as the set of common solutions of several equations involving analytic functions.*

Notation. *Let us introduce an important notation that will help us prove the Theorem 55.*

1. *Denote by* $\mathcal{X}$ *the class of all zero-dimensional absolute* $F_{\sigma\delta}$*-spaces which are nowhere* σ*-complete and of first category.*

2. *Denote by* $\{X_i : i \in \mathbb{N}\}$ *a family of non-empty non-complete absolute* $F_{\sigma\delta}$ *spaces.*

Prior to the main theorem of this subsection, we provide three lemmas that lead to the result that $\mathbb{Q}^\omega \in \mathcal{X}$. Proofs of the lemmas are omitted here and can be found in van Engelen (1985). Recall that Y is nowhere dense in X if its closure in X has empty interior.

Lemma 52. *If X is an analytic space which is not σ-complete then X contains a closed nowhere σ-complete subspace Y which is nowhere dense in X.*

Definition 53. *A topological space is said to be σ-compact if it is the union of countably many compact sets.*

Example 53.1. *The set of rational numbers $\mathbb{Q}$ with its usual topology is countable (rational numbers are countable infinite), which implies that $\mathbb{Q}$ is σ-compact. In fact, any countable Space is σ-compact.*

Lemma 54. *Let A be a Borel set in $\mathcal{C}$ which is not σ-complete, and let F be a σ-compact space such that $A \subset F \subset \mathcal{C}$. Then A contains a closed nowhere dense subset Y which is nowhere σ-complete and first category, such that $Cl_{\mathcal{C}}Y \subset F$.*

This lemma helps prove the following lemma which is crucial for the characterization of $\mathbb{Q}^\omega$ provided in the subsequent theorem. We need more definition to proceed.

Definition 55. *Let X_i be a subset of a metrizable space X. Define the diameter of (X_i) as*

$$diam(X_i) = \sup\{\rho(x_1, x_2) \mid x_1, x_2 \in X_i\}$$

where ρ is a metric on X.

Lemma 56. *Let $X \in \mathcal{X}$, let F be a σ-compact space such that $X \subset F \subset \mathcal{C}$ and let $\epsilon > 0$. Then there exist closed nowhere dense subsets X_i of X such that*

- *(i)* $X = \cup_{i=1}^{\infty} X_i$
- *(ii)* $X_i \in X$ *for each* $i \in \mathbb{N}$
- *(iii)* $Cl_C(X_i) \subset F$
- *(iv)* $diam(X_i) < \epsilon$

Definition 57. *A topological space is called a Baire space if the countable intersection of open dense subsets is also dense. Equivalently, it is a space such that a countable union of closed sets each with empty interior also has empty interior.*

Theorem 58. *(van Engelen, 1985, Theorem 3.4) Up to homeomorphism, $\mathbb{Q}^\omega$ is the only element of $\mathcal{X}$.*

Proof. $\mathbb{Q}^\omega$ is a product of σ-compacta, hence an absolute $F_{\sigma\delta}$. It is also of first category - consider the following definition for finite sequences of rational numbers $(q_0, \ldots, q_n)$, $n \in \mathbb{N}$:

$$[(q_0, \ldots, q_n)] = \{x \in \mathbb{Q}^\omega : x_i = q_i, 0 \leq i \leq n, n \in \mathbb{N}\}.$$

Clearly $\mathbb{Q}^\omega$ is the countable union of all the sets of the form $[(q_0, \ldots, q_n)]$ and such sets are closed in $\mathbb{Q}^\omega$ and have empty interiors.

To show that $\mathbb{Q}^\omega$ is nowhere σ-complete, we take inspiration in (van Engelen, 1984, Lemma 2.1 (b)). Suppose $\{A_i \mid i \in \mathbb{N}\}$ is a countable family of subsets of $\mathbb{Q}^\omega$ that are complete. $\mathbb{Q}^\omega$ is not Baire space, hence $\overline{A_1} \neq \mathbb{Q}^\omega$, which implies there is a basic non-empty open subset U of $\mathbb{Q}^\omega$ such that $U \cap A_1 = \emptyset$. Recall the notation of $\{X_i : i \in \mathbb{N}\}$ introduced before Lemma 52 and let $n_1 \in \mathbb{N}$ and $(q_1, ..., q_{n_1}) \in \mathbb{Q}^{n_1}$ be such that

$$X_1 = (q_1, ..., q_n) \times \mathbb{Q} \times \mathbb{Q} \times ... \subset U.$$

Observe that $X_1 \simeq \mathbb{Q}^\omega$. (Each X_i contains a closed copy of $\mathbb{Q}^\omega$ by a result of Hurewicz from 1928.) Since $A_2 \cap X_1$ is closed in A_2, it is complete, and since X_1 is not Baire, there exist $n_1, n_2 \in \mathbb{N}$ such that $n_1 < n_2$ and $(q_{n_1+1}, ..., q_{n_2}) \in \mathbb{Q}^{n_2-n_1}$ such that

$$X_2 = (q_1, ..., q_{n_2}) \times \mathbb{Q} \times \mathbb{Q} \times ... \subset \mathbb{Q}^\omega \setminus (A_1 \cup A_2).$$

Proceeding in this way, we discover a point $(q_i)_{i\in\mathbb{N}} \in \mathbb{Q}^\omega \setminus (\cup_{i=1}^\infty A_i)$. Hence $\mathbb{Q}^\omega$ is not σ-complete, and because it is strongly homogeneous (see the next part of this chapter), it is nowhere σ-complete.

Now, we need to see that $\mathbb{Q}^\omega$ is the *only* element of $\mathcal{X}$. Further on, denote by M the set of all finite sequences of natural numbers, including the empty sequence $\emptyset$. Take an arbitrary X such that $X \in \mathcal{X}$ which we embed in the Cantor set $\mathcal{C}$. Let $\{F_n \mid n \in \mathbb{N}\}$ be a family of σ-compact subsets of $\mathcal{C}$ such that $\cap_{n=1}^\infty F_n$ and put $F_0 = \mathcal{C}$. We will construct closed subspaces X_s of X, for each $s \in M$, satisfying the following conditions:

1. $X = X_\emptyset$ and $X_s = \cup_{i=1}^\infty$ for each $s \in M$
2. For each $i \in \mathbb{N}$ and each $s \in M, X_{s,i}$ is nowhere dense in X_s
3. for each $s \in M$, $X_s \in \mathcal{X}$
4. for each $s \in M$, $diam(X_s) < (|s| + 1)^{-1}$
5. for each $s \in M$, $\overline{X}_s \subset F_{|s|}$ (closure in $\mathcal{C}$).

To construct such X_s, put $X_\emptyset = X$, and if X_s has been defined for all $s \in M$ with $abs\mathbf{s} \leq n$, then we obtain the sets $X_{s,i}$. by applying Lemma 56 of this subsection to $X_s \subset F_{|s|+1} \subset \mathcal{C}$ where $\epsilon = (|s| + 2)^{-1}$. Now, we claim that the sets X_s satisfy the following condition: If $\sigma \in \mathbb{N}^\omega$ and $p_n \in X_{\sigma|n}$ for each $n \in \mathbb{N}$, then the sequence $(p_n)_{n\in\mathbb{N}}$ converges. For that, let $\sigma \in \mathbb{N}^\omega$ and since $\overline{X}_{\sigma|1} \supset \overline{X}_{\sigma|2} \supset ...$ is a decreasing sequence of compacta, $\cap_{n=1}^\infty \overline{X}_{\sigma|n} = \emptyset$, let $x \in \cap_{n=1}^\infty \overline{X}_{\sigma|n}$ By the condition 5, $x \in \cap_{n=1}^\infty F_n = X$. Thus, $x \in \cap_{n=1}^\infty \overline{X}_{\sigma|n}$ and if U is any open neighborhood of x in X, then by the condition 4, $X_{\sigma|k} \subset U$ for some $k \in \mathbb{N}$. Hence, if $p_n \in X_{\sigma|n}$ for each $n \in \mathbb{N}$, then $p_n \in U$ for $n \geq k$, which implies that $(p_n)_n$. converges to X as we wanted to prove. □

2.1.5 Strong homogeneity

Once we have described all relevant properties of $\mathbb{Q}^\omega$, we will proceed to the conclusion about the set of all H-compactifications, using the notion of strong homogeneity.

Definition 59. *A space X is homogeneous if for each $x, y \in X$, there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.*

Example. $\mathbb{Q}^\omega$ is homogeneous. This is overt, considering the homogeneity of $\mathbb{Q}$, since any product of homogeneous spaces is homogeneous. This implies $\mathbb{Q}^\omega$, similarly as $\mathbb{Q}$ has a rich group of automorphisms and hence we can come up with a hypothesis that the set of all H-compactifications would be quite narrow.

Definition 60. *A separable metrizable topological space X is strongly homogeneous if for each non-empty clopen subset U of X, we have $U \simeq X$.*

Lemma 61. *(van Engelen, 1984, Lemma 2.1) $\mathbb{Q}^\omega$ is strongly homogeneous.*

Proof. Choose a nonempty clopen subset U of $\mathbb{Q}^\omega$ for which we want to show that $U \simeq \mathbb{Q}^\omega$. Observe that any non-empty open subspace U can be written as an infinite disjoint union of non-empty basic clopen subsets; hence, $U \simeq \mathbb{N} \times \mathbb{Q}^\omega \simeq \mathbb{Q}^\omega$. □

The strong homogeneity of $\mathbb{Q}^\omega$ implies the following fact.

Corollary 61.1. *Let V be a nonempty clopen subset of $\mathbb{Q}^\omega$. Then $V \simeq \mathbb{Q}^\omega$.*

2.2 Conclusion

The previous analysis of $\mathbb{Q}^\omega$ and its nonempty clopen subsets above allows us to answer the Question 1.

Theorem 62 (Van Douwen)**.** *If X is a non-compact strongly zero-dimensional space in which every nonempty clopen subspace is homeomorphic to X, then βX is the only H-compactification of X.*

Proof. See the proof of the Proposition 22 in the first chapter of this work. □

We have seen that $\mathbb{Q}^\omega$ complies with all assumptions of this theorem and that every non-empty clopen subset of $\mathbb{Q}^\omega$ is homeomorphic to $\mathbb{Q}^\omega$. This answers our question.

Corollary 62.1. *The only H-compactification of $\mathbb{Q}^\omega$ is precisely the Stone-Čech compactification $\beta\mathbb{Q}^\omega$.*

2.3 Spaces homeomorphic to $\mathbb{Q}^\omega$

Homeomorphic spaces generally have identical sets of H-compactifications. Many authors studied spaces homeomorphic to $\mathbb{Q}^\omega$. For instance,

$$Y = \{(y_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega : lim_{i\to\infty} y_i = \infty\} \simeq \mathbb{Q}^\omega$$

(see van Engelen (1985)), or

$$X \times \mathbb{Q}^\omega \simeq \mathbb{Q}^\omega \text{ for every zero-dimensional } F_{\sigma\delta}\text{-space } X$$

(see van Engelen (1984)).

3. H-compactifications of l^2

The purpose of this chapter is to speculate about H-compactifications of the space l^2. Our, maybe too ambitious, hypothesis that this set only has one element - βl^2 has been neither successfully proved nor disproved, so the question stays unanswered.
However, we analyze several topics that could help understanding behaviour of l^2 and its compactifications and motivate future aims in this problem.
Throughout this chapter, l^2 will be considered as the Hilbert space of all square summable real sequences.

Definition 63. *ℓ^2 is the set of sequences $x = (x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$, such that $\sum_{n\in\mathbb{N}} |x_n|^2 < \infty$.*

l^2 can be regarded as a topological vector space and a topological group at the same time. It is endowed with the classical norm topology.

Question 2. *What H-compactifications does the space l^2 admit?*

With a bit of imagination, we can envision l^2 like a generalized version of Euclidean spaces, where each "vector" has infinite number of coordinates. (Here, the "vectors" are represented by the infinite sequences). Higher dimensional Euclidean spaces are therefore something we can begin with when exploring the H-compactification of l^2.
Recall that $\mathbb{R}^n, n \geq 2$ only admit $\beta\mathbb{R}^n$ and $\alpha\mathbb{R}^n$ as H-compactifications, the latter due to local compactness of $\mathbb{R}^n$. Seeing that l^2 and $\mathbb{R}^n, n \geq 2$ have some things in common (i.e. both admit rich group of automorphisms), it is natural to expect that the set of all H-compactifications of l^2 will only contain βl^2 - l^2 is not locally compact, so we have to immediately exclude αl^2.

3.1 Characterization of βX

In response to the hypothesis above, we will present one of the interesting characterizations of the Stone-Čech compactification. We will prove that if an arbitrary H-compactification meets certain criteria, it has to be homeomorphic to the Stone-Čech compactification.

Definition 64. *For an arbitrary topological space X, we introduce the following notation and definitions:*

1. *Denote by $C(X)$ a ring of all real-valued continuous functions on an arbitrary topological space X*
2. *Denote by $C^\star(X) =$ subring of $C(X)$ of bounded functions*
3. *We say that X is $C^\star-$embedded as a subspace in its Stone-Čech compactification βX if every function in $C^\star(X)$ can be extended to a function in $C^\star(\beta X)$.*
4. *Let $f \in C(X)$. A subset Z of X which is of the form $Z = \{x \in X \mid f(x) = 0\}$ is called the zero-set of f. Clearly, such sets are closed.*

5. *Denote by $Z(X)$ the family of all zero-sets in X.*

The notion of zero sets is useful for introducing z-filters - special cases of filters that play an important role giving a connection between βX and $C(X)$.

Definition 65. *A nonempty subfamily $\mathfrak{F}$ is called a z-filter on X provided that*

1. $0 \notin \mathfrak{F}$
2. *if $Z_1, Z_2 \in \mathfrak{F}$, then $Z_l \cap Z_2 \in \mathfrak{F}$ and*
3. *if $Z_1 \in \mathfrak{F}, Z_2 \in Z(X)$, and $Z_1 \subset Z_2$ then $Z_2 \in \mathfrak{F}$.*

Definition 66. *We call a z-filter on X a z-ultrafilter on X if it is a maximal z-filter, i.e., one not contained in any other z-filter. We call a z-filter free if the intersection of all its members is empty.*

Definition 67. *The z-filter $\mathfrak{F}$ is said to converge to the limit p if every neighborhood of p contains a member of $\mathfrak{F}$.*

The Stone-Čech compactification βX of X can be constructed by adjoining to X one new point for each free z-ultrafilter and it is essentially unique. For βX, distinct free z-ultrafilters on X converge to distinct points of βX. These properties are captured by the next two theorems which demonstrate equivalent characterizations of βX and methods of construction.

Theorem 68. *(Gillman und Jerison, 1960, Theorem 6.4, (III)) Let X be dense in γX. The following statements are equivalent.*

1. *Every continuous mapping γ from X into any compact space K has an extension to a continuous mapping from γX into K.*
2. *X is $C^\star$-embedded in γX.*
3. *Any two disjoint zero-sets in X have disjoint closures in γX.*
4. *For any two zero-sets Z_l and Z_2 in X, $cl_{\gamma X}(Z_l \cap Z_2) = cl_{\gamma X} Z_l \cap cl_{\gamma X} Z_2$*
5. *Every point of γX is the limit of a unique z-ultrafilter on X.*

Proof. We will prove the most relevant implications for this chapter, which are 2. $\implies$ 3. and 3. $\implies$ 4.. The complete proof can be found in (Gillman und Jerison, 1960, Theorem 6.4, (III))
The former is due to the Urysohn´s theorem saying that any subspace of X is $C^\star$-embedded in X if and only if any two completely separated sets in that subspace are completely separated in X. For the latter implication, we will show $clZ_l \cap clZ_2$ is contained in $cl(Z_l \cap Z_2)$ (the reverse inclusion is done trivially). Take a point $x \in clZ_l \cap clZ_2$. Then if V is an arbitrary zero-set neighborhood (in γX) of x, we have $x \in cl(V \cap Z_l)$ and $x \in cl(V \cap Z_2)$. Hence, by 3., $V \cap Z_l$ intersects $V \cap Z_2$ i.e., V intersects $Z_l \cap Z_2$ Therefore $p \in cl(Z_l \cap Z_2)$ and hence, $clZ_l \cap clZ_2$ is contained in $cl(Z_l \cap Z_2)$. □

Definition 69. *Take a completely regular space X, a compact space K and a mapping h such that $h : X \to K$. Define the Stone extension as the extension βh of h into K such that $\beta h : X \to \beta X$.*

Theorem 70. *(Gillman und Jerison, 1960, Theorem 6.5, (III)) Every (completely regular) space X has a unique compactification βX such that any two disjoint zero-sets in X have disjoint closures in βX. If another compactification γX of X satisfies the said condition, then there exists a homeomorphism of βX onto γX that leaves X point-wise fixed.*

Proof. Uniqueness: By the previous theorem, if γX satisfies one of the conditions $(i), ...(iv)$, it satisfies all of them. From (i), the identity mapping on X, which is a continuous mapping into the compact space γX, has a Stone extension from all of βX into γX; similarly, it has a Stone extension from γX into βX. A discussion regarding uniqueness of βX has also been provided within the proof of Theorem 14 (see the introductory chapter of this work).

Construction of βX: This part is more technical and lengthy, so we omit it here and refer to (Gillman und Jerison, 1960, Theorem 6.5, (III)) for the complete proof. The idea behind constructing βX is using a one-one correspondence between the z-ultrafilters on X and the points of βX, each z-ultrafilter converging to its corresponding point. We will define the topology on βX such that each point corresponds to the limit of one z-ultrafilter. A base for the closed sets of βX will be created from the family of all zero-sets corresponding to ultrafilters with limits $p \in X$. □

3.2 H-compactifications of $\mathbb{R}^2$

The goal of this chapter is to describe the set of all H-compactifications of $\mathbb{R}^2$. We will rephrase findings from (Vejnar, 2011, Part 3.3) who proved the same for the general case $\mathbb{R}^n$, $n = 2, 3, 4, ...$

Definition 71. *Let $\mathcal{U}$ be a collection of subsets of a metric space X. Recall that $diam(X) = \sup\{\rho(x_1, x_2) \mid x_1, x_2 \in X\}$ where ρ is a metric on X and define the mesh of $\mathcal{U}$ such that:*

$$mesh\,\mathcal{U} = sup\{diam(U) : U \in \mathcal{U}\} \in [0, +\infty].$$

Definition 72. *We say that an open subset U of a space X has property $(\star)$ if for every $E \subseteq U$ closed in X and for every $F \subseteq U$ which is open and non-empty, we can find an automorphism h of X such that $h(E) \subseteq F$ and $h(x) = x$ on $X \setminus U$.*

Definition 73. *We say that a space X has property $(\star\star)$ if there exists a number $N \in \mathbb{N}$ such that for arbitrary small $\epsilon > 0$ there is an open covering $\mathcal{U}$ which can be expressed as a union of N discrete sub-collections $U \in \mathcal{U}$ with mesh less than ϵ, where each U has property $(\star)$, i.e. as in the previous definition, we can find an automorphism h that sends closed subsets of U to open subsets of U and is an identity on $X \setminus U$.*

Lemma 74. *(Vejnar, 2011, Lemma 22) Let X be a separable locally compact metric space with property $(\star\star)$. Let $M = 2N$. Then:*

1. *Let F be a closed set contained in an open set U. Then, for F there exist a closed discrete set $C \subseteq F$ and closed sets $F_0, ..., F_{M-1}$ such that $F = \cup F_i$ and for every $i < M$ and each open neighbourhood G of C there there exists an automorphism h of X such that $h(F_i) \subseteq G$ and h is the identity on the complement of U.*

2. *For every pair of closed sets $F,\ F^{'} \subseteq X$ such that $F \cap F^{'} = \emptyset$, we can find closed discrete sets $C, C^{'} \subseteq X$, $C \cap C^{'} = \emptyset$ and closed sets $F_0, ..., F_{M-1}, F_0^{'}, ...F_{M-1}^{'}$ such that*

$$F = \cup F_i,$$
$$F^{'} = \cup F_j^{'}$$

and for any $i, j < M$ and neighbourhoods G and $G^{'}$ of C and $C^{'}$ respectively there is an automorphism h of X such that $h(F_i) \subseteq G$ and $h(F_j^{'}) \subseteq G^{'}$.

For will skip the proof of this lemma, which is quite lengthy and technical and refer to (Vejnar, 2011, Lemma 22). More important for this chapter will be Theorem 78 and its corollary.

Definition 75. *A space X is called strongly locally homogeneous if for every $x \in X$ and each neighbourhood U of x there exists a neighbourhood V of x in U such that for every $y \in V$ we can find a homeomorphism of X which sends x to y and is the identity on the complement of V.*

Proposition 76. *(Vejnar, 2011, Proposition 26) Every bijection of closed discrete subsets of $\mathbb{R}^2$ can be extended to a homeomorphism of the whole space.*

We will also omit conducting a proof this proposition, which, as was remarked by Vejnar, is more convenient to prove for Euclidean spaces of higher dimension than 2, else it is rather complicated.

Definition 77. *Let X be a topological space and $\mathcal{P}$ a collection of subspaces of X. Then X is called 2-homogeneous with respect to $\mathcal{P}$ if for any sets $C_1, C_2, D_1, D_2 \in \mathcal{P}$ such that $C_1 \simeq D_1$, $C_2 \simeq D_2$ and $C_1 \cap C_2 = \emptyset = D_1 \cap D_2$ there exists an automorphism h of X such that $h(C_1) = D_1$ and $h(C_2) = D_2$.*

Theorem 78. *(Vejnar, 2011, Theorem 24) Let X be a separable locally compact but non-compact metric space with property $(\star\star)$ and 2-homogeneous with respect to closed discrete sets. Then αX and αX are the only H-compactifications of X.*

Proof. First, let γX be an H-compactification of X such that γX is distinct from αX. Take $F, G \subset X$ such that both F and G are closed and $F \cap G = \emptyset$. Our goal is to show that for their closures in γX, $clF \cap clG = \emptyset$ because then, from findings in the previous section, we can tell that γX is equivalent to βX. Note that in metric spaces, zero sets and closed sets are the same.

Lemma 74 provides that there are two closed discrete sets C_1, C_2 and families of closed sets $F_0, ...F_M$ and $G_0, ...G_M$ having the properties mentioned in the lemma. Since $F = \cup F_i$ and $G = \cup G_i$, it suffices to prove that for any $i, j < M$ the $clF_i \cap clG_j = \emptyset$ (we take the closures in γX).

We can find two countable infinite closed discrete sets D_1 and D_2 of X such that clD_1 and clD_2 in γX are disjoint, since γX is not a one-point compactification by assumption, whence $\gamma X \setminus X$ contains at least two points. By Proposition 76, X is 2-homogeneous with respect to closed discrete sets and since γX is an H-compactification, we infer that $clC_1 \cap clC_2 = \emptyset$ in γX. Hence we are able to separate them by open sets U_1 and U_2 in X with disjoint closures in γX.

By Lemma 74 we can find an automorphism h of X with $h(F_i) \subseteq U_1$ and $h(G_j) \subseteq U_2$. Consequently, the closures of $h(F_i)$ and $h(G_j)$ in γX are disjoint and since γX is an H-compactification, indeed the closures of F_i and G_j are also disjoint, hence, from $F = \cup F_i$ and $G = \cup G_i$, $clF \cap clG = \emptyset$, as we wanted.

Thus we have proved that $\gamma X \simeq \beta X$. □

Corollary 78.1. *(Vejnar, 2011, Corollary 28) The set of all H-compactifications of* $\mathbb{R}^2$ *has exactly two elements -* $\alpha\mathbb{R}^2$ *and* $\beta\mathbb{R}^2$*.*

Proof. The corollary follows from the previous theorem - to see that, we are going to play with the premises and outcomes of Lemma 74 and Proposition 76 introduced in this section.

(i) Lemma 74: Consider the maximum metric ρ on $\mathbb{R}^2$, defined, for each $x, y \in \mathbb{R}^2$, $x = (x_1, x_2), y = (y_1, y_2)$ by

$$\rho(x, y) = max(|x_1 - y_1|\,; |x_2 - y_2|)$$

and denote by $B_\rho(x, r)$ the open ball with centre x and diameter r. We need to verify the property $(\star\star)$ - choose $N \in \mathbb{N}$ equal to 4 so it corresponds to the number of sequences of length 2 formed by zeros and ones. Now we have to find for arbitrary $\epsilon > 0$ an open cover which can be written as the $\mathcal{U}$ in Definition 70. Pick an $\epsilon > 0$ and put

$$\mathcal{U}_j = \{B_\rho(\tfrac{\epsilon i}{3}, \tfrac{\epsilon}{4}) : i \in \mathbb{Z}^2, i_k \equiv j_k(mod2)\}.$$

.

Then, define the required open cover $\mathcal{U}$ as

$$\mathcal{U} = \cup\{\mathcal{U}_j \mid j \in 4\} \text{ of } \mathbb{R}^2.$$

Observe that the sets $B_\rho(\frac{\epsilon i}{3}, \frac{\epsilon}{4})$ from the definition of $\mathcal{U}_j$ obey the property $(\star)$ because every ball $B_\rho(x, r)$ in $\mathbb{R}^2$ has property $(\star)$. In conclusion, since all collections U_j are discrete and made of sets with property $(\star)$, we obtain that $\mathbb{R}^2$ has property $(\star\star)$.

(ii) Proposition 76: Let $\mathcal{P}$ be a collection of closed discrete subsets of $\mathbb{R}^2$. From Proposition 76, we know that every bijection of any $P \in \mathcal{P}$ can be extended to a homeomorphism of $\mathbb{R}^2$. This implies that $\mathbb{R}^2$ is 2-homogeneous with respect to closed discrete sets, which is the last assumption that we needed for the Theorem 78. □

Remark. *The two H-compactifications of* $\mathbb{R}^2$ *are distinct from one another. This is an immediate consequence of the fact that there exists a continuous bounded function with no limit at infinity.*

3.3 Compactifying the space $\mathcal{H}^+([0, 1])$

This section provides a different way of investigating the H-compactifications of l^2. We will study the space $\mathcal{H}^+([0, 1])$ which is topologically the same as l^2 and hence can provide a new intriguing view on the compactifications of l^2. Inspired by Kennedy (1988), we will construct an example of a compactification of $\mathcal{H}^+([0, 1])$, using a hyperspace.

Definition 79. $\mathcal{H}^+([0,1])$ *is the space of increasing homeomorphisms of the closed interval* $[0,1]$*, endowed with the supremum metric. Analogously we could define the space* $\mathcal{H}^-([0,1])$ *of decreasing homeomorphisms of* $[0,1]$.

The topology on $\mathcal{H}^+([0,1])$ is the topology of uniform convergence, induced by the supremum metric. The $\mathcal{H}^+([0,1])$ also possesses a group structure, where the group multiplication operation is the composition of homeomorphisms.

Perhaps not so intuitively, l^2 is homeomorphic to $\mathcal{H}^+([0,1])$. We will prove this for

Theorem 80. *(Keesling, 1971, Theorem III.1)* $\mathcal{H}_0([0,1]) \simeq l^2$.

Proof. In this proof, we consider $\mathcal{H}_0([0,1]) \simeq l^2$ as a set of all functions on a unit interval $[0,1]$ that are monotone, increasing and onto. We endow $\mathcal{H}_0([0,1])$ with the compact open topology.

Here, we will just sketch the proof - verifying the details is a routine procedure. What we will show is that $\mathcal{H}_0([0,1]) \simeq \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$. Then, the homeomorphism $\mathcal{H}_0([0,1]) \simeq l^2$, is guaranteed by Anderson und Bing (1968) where the the authors prove the result that l^2 is homeomorphic to the space of all sequences $\{x_i\}_{i>1}$ of real numbers (endowed with the product topology.)

Let $\{x_{n,i}\} \in \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$. Define an orientation preserving homeomorphism h of $[0,1]$ associated with $\{x_{n,i}\}$. Suppose that we chose an arbitrary $n \in \mathbb{N}$ and that we have defined points

$$A_n = \{0 = \alpha_0^n < \alpha_1^n < ...\alpha_{2^n}^n = 1 \text{ and}$$
$$B_n = \{0 = \beta_0^n < \beta_1^n < ...\beta_{2^n}^n = 1$$

that we defined h such that $h(\alpha_i^n) = \beta_i^n$ for $i = 0, 1, ..., 2^n$. Now we have to extend the definition of h to a set of points $A_{n+1} \supset A_n$ onto $B_{n+1} \supset B_n$ where each A_{n+1} and B_{n+1} contain exactly $2^{n+1} + 1$ points. We investigate the two cases of n:

(i) n is odd: Let z_i be the midpoint of the interval $[\alpha_{i-1}^n, \alpha_i^n]$ for $i = 0, 1, ..., 2^n$ and let $y_i = h(z_i) = x_{n,i}(\beta_i^n - \beta_{i-1}^n) + \beta_{i-1}^n$.

(ii) n is even: Let y_i be the midpoint of the interval $[\beta_{i-1}^n, \beta_i^n]$ and let

$$z_i = h^{-1}(y_i) = x_{n,i}(\alpha_i^n - \alpha_{i-1}^n) + \alpha_{i-1}^n.$$

Then let $A_{n+1} = A_n \cup \{z_i\}^2 N_{i=1}$ and $\beta_{n+1} = \beta_n \cup \{y_i\}^2 N_{i=1}$. Then we will obtain $A_{n+1} = \{0 = \alpha_0^{n+1} < ... < \alpha_{2^n+1}^{n+1} = 1\}$ and $\{0 = \beta_0^{n+1} < ... < \beta_{2^n+1}^{n+1} = 1\}$ with $h(\alpha_i^{n+1}) = \beta_i^{n+1}$. Proceeding in this way and put $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$. Then A and B will be dense in $[0,1]$ and $h[A] = B$ will be order preserving. Hence h admits a continuous extension to an orientation preserving homeomorphism of $[0,1]$ onto itself which will be also denoted by h.

Finally, define

$$F : \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i} \to \mathcal{H}_0([0,1]) \text{ by}$$
$$F(\{x_{n,i}\}) = h.$$

Then F is the desired homeomorphism of $\prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$, onto $\mathcal{H}_0([0,1])$. □

Definition 81. *A hyperspace* $\mathcal{K}(X)$ *of a topological space* X *is defined as*

$$\mathcal{K}(X) := \{K \subseteq X, K \text{ compact, } K \neq \emptyset\}.$$

The metric on $\mathcal{K}(X)$ is the Hausdorff metric

$$\rho_H(K, L) := max\{sup_{x\in K}\rho(x, L), sup_{y\in L}\rho(y, K)\}.$$

The hyperspace is sometimes denoted 2^X, which can refer to hyperspace defined with closed, not compact sets. The Hausdorff metric induces the so-called Vietoris topology. It is a well known fact that if the space X is compact, then the hyperspace $\mathcal{K}(X)$ is compact as well.

Note that the subsets of the X in the definition of hyperspace have to be compact, while for the whole X, this condition is not necessary. However, when X happens to be compact too, one interesting fact about the hyperspace arises.

Notation. *Denote by d a metric on X, compatible with its topology and by grf a graph of f.*

If $X = [0, 1]$, we can think of any $f \in \mathcal{H}^+([0, 1])$ as a closed collection of ordered pairs in $[0, 1] \times [0, 1] = [0, 1]^2$. Hence we can associate any such f with its graph, grf. Therefore, if we take $[0, 1]^2$, which is compact, then we can use a hyperspace to construct a nice compactification of $\mathcal{H}^+([0, 1]) \simeq l^2$.

Proposition 82. *(Kennedy, 1988, Observation 1) If $X = ([0, 1], d)$ is a compact metric space, then the space $\mathcal{H}([0, 1])$ can be embedded in its hyperspace $\mathcal{K}([0, 1])$.*

Proof. Define the function

$$\phi : \mathcal{H}([0, 1]) \to \{grf \mid f \in \mathcal{H}([0, 1])\} \subseteq \mathcal{K}([0, 1]^2) \text{ such that}$$
$$\phi(f) = grf.$$

Observe that ϕ is a one-one function from $\mathcal{H}([0, 1])$ onto $\{grf \mid f \in \mathcal{H}([0, 1])\}$. Then, we need to show that ϕ is a homeomorphism, which is proven in (Kennedy, 1988, Observation 1). Using relationship between metrics on $[0, 1]$ and $[0, 1]^2$, it is shown that ϕ is continuous and then, by contradiction, that ϕ^{-1} is continuous as well). □

3.4 Conclusion

There are more ways to go about the analysis of l^2. If we stick to the hypothesis that there is no other H-compactification except βl^2, one can find useful the characterization via disjoint zero-sets or any of the equivalent conditions in Theorem 68.
Another possibility is recreating the process that Vejnar used on the case of $\mathbb{R}^n$, $n \geq 2$ and compensate the local compactness used in Lemma 74 and Theorem 78 with something else.
Finally, there exist spaces homeomorphic to l^2, which admit the same set of H-compactifications and can serve as an intermediary in this problem.

4. H-compactifications from a Category-theoretic View

In this final chapter we will study various categorical aspects of the theory of compactifications. It has been shown that H-compactifications certainly form a mathematical structure, for instance they have been analyzed from lattice-theoretic perspective by Lubben (1941), Chandler (1976) or Vejnar (2011).

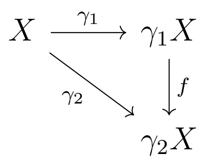
4.1 Preliminaries

First, we introduce a reasonable ordering, which, for H-compactifications is inherited from the natural ordering of compactifications.

Definition 83. *We say that, given a topological space X, a H-compactification γ_1 is finer than a H-compactification γ_2 if for the embeddings*

$$\gamma_1 : X \to \gamma_1 X \text{ and } \gamma_2 : X \to \gamma_2 X$$

there exists a continuous map $f : \gamma_1 X \to \gamma_2 X$ such that the diagram



commutes.

Further on, we will use the notation $\gamma_2 X \leq \gamma_1 X$ for the fact that $\gamma_1 X$ is finer than $\gamma_2 X$. The complementary notion is a coarser H-compactification.

An equivalent definition states that $\gamma_2 X \leq \gamma_1 X$ if there exists a continuous $f : \gamma_1 X \to \gamma_2 X$ that *fixes all points of* X, which means the points stay unaffected by applying f.

The f is a morphism and if it exists, it is automatically surjective and unique. Recall that in the definition of Hausdorff compactifications, $\gamma_i(X)$ is dense in $\gamma_i X$.

Definition 84. *Recall that a lattice L is a partially ordered set where any pair of its elements has both a greatest lower bound (called meet and denoted $\wedge$) and a least upper bound (called join and denoted $\vee$) in L (which, by induction, implies this for all finite subsets).*

We will show how the lattice structure of H-compactifications corresponds to category-theoretic notions.

Definition 85. *A complete lattice L is a lattice where all subsets have a greatest lower bound and a least upper bound in L.*

Definition 86. *A complete sub-lattice L of a (complete) lattice M is a lattice where the meet and join operations in L agree with the meet and join operations in M.*

Definition 87. *An upper semi-lattice L is a partially ordered set that has a join operation $\vee$ for any nonempty finite subset. Analogously, a lower semi-lattice has $\wedge$ operation defined for each such subset.*

4.2 Category of H-compactifications

The ordering of H-compactifications brings for each topological space X a partially ordered set (shortly called poset) P_X endowed with a binary relation which is precisely the above defined ordering $\leq$. We will preserve the notation P_X for such poset (or, with more properties, even a lattice or semi-lattice). We will show how P_X can turn into a category.

Definition 88. *Given a Hausdorff topological space X, a partially ordered set of all H-compactifications of X is defined as*

$$P_X = \{\gamma X, \leq \mid \gamma X \text{ is a H-compactification of } X\},$$

where $\leq$ is the natural ordering of H-compactifications.

Notation. *To avoid confusion, we will denote lattices (or posets) by capital letters and categories by calligraphic font.*

1. *P_X = (Semi-)lattice of all H-compactifiactions of X*
2. *$\mathcal{P}_X$ = Category of all H-compactifiactions of X*
3. *Q_X = (Semi-)l of all compactifiactions of X*
4. *$\mathcal{Q}_X$ = Category of all compactifiactions of X*

Definition 89. *Given a Hausdorff topological space X, define a category $\mathcal{P}_X$ of all H-compactifications of X consisting of*

- *A collection $Ob(\mathcal{P}_X)$ of objects which are H-compactifications $\gamma_i X$ of X and the elements of the poset P_X introduced above*
- *A collection $Hom(\gamma_1 X, \gamma_2 X)$ of morphisms which has exactly one element if $\gamma_1 X \leq \gamma_2 X$ and is an empty set otherwise. The morphisms are given by the relation $\leq$ on P_X.*

The morphisms in this category are the arrows pointing from γ_1 to γ_2 when $\gamma_1 X \leq \gamma_2 X$.

It is straightforward to see that $\mathcal{P}_X$ is a well-defined category. For any object γX, the identity morphism exists trivially and composition is provided by the transitivity of the partial order $\leq$. A composition of morphisms of H-compactifications results in another morphism of H-compactifications.

Definition 90. *Categories defined as $\mathcal{P}_X$ above are sometimes called posetal or thin.*

$\mathcal{P}_X$ is also a *small category* - a category such that all H-compactifications and all collections of arrows between them are sets and every subcategory of $\mathcal{P}_X$ is isomorphism-closed. Indeed, the structure of H-compactifications depends on the nature of the space X. Therefore we will look separately at properties of $\mathcal{P}_X$ in the case when X is locally compact and then in the general case.

Properties of $\mathcal{P}_X$ when X is locally compact

Throughout this subsection, we will assume X to be an arbitrary space. Before getting to the interesting part, we need to introduce more definitions.

Definition 91. *Define* $C^\star(X) = \{f : X \to \mathbb{R} \mid f \textit{ is continuous and bounded}\}$

Recall the notion of *extension*, as we defined it in the introductory chapter of this work.

Definition 92. *For any compactification γX of X, let*

$$C_\gamma = \{h \in C^\star(X) \mid h \textit{ has an extension } \gamma h : \gamma X \to \mathbb{R}\}.$$

Theorem 93. *(Chandler, 1976, Theorem 2.19), based on Lubben (1941) The set of all compactifications of X forms a complete lattice if and only if X is a locally compact topological space.*

Proof. We provide an idea and main steps for proving the theorem, detailed proof is conducted for example in (Chandler, 1976, Chapter 2). Denote the set of all compactifications of X by $K(X)$.
(i) First, we show by contradiction that if $K(X)$ (as a lattice) is complete, then X is locally compact. Suppose X is chosen such that it is not locally compact. Then for any $\gamma X \in K(X)$, $\gamma X \setminus X$ consists of more than one point. If X is open in γX, then $\forall x \in X$, there is a continuous map $h : \gamma X \to [0, 1]$ such that

$$h(x) = 0$$
$$h(\gamma X \setminus X) = \{1\}.$$

Now, we shrink the $[0, 1]$ to $[0, \frac{1}{2}]$. Then, $h^{-1}([0, \frac{1}{2}])$ is a closed neighborhood of x which is still contained in X and the closedness implies compactness of $h^{-1}([0, \frac{1}{2}])$.

We then proceed by determining for each compactification $\gamma X \in K(X)$ another element $\gamma' X$ of $K(X)$ such that $\gamma X > \gamma' X$ and hence conclude that $K(X)$ has no greatest lower bound, hence it cannot be complete. This way the contradiction is reached.
(ii) Conversely, let X be locally compact and define

$$C = \{f \in C^\star(X) \mid \forall \epsilon > 0 \,\exists \text{ a compact } K_\epsilon \subset X \text{ and } |f(x)| < \epsilon \,\forall x \in X \setminus K_\epsilon\}.$$

Chandler proceeds in the proof with observing that C separates points from closed sets and $C \subset C_\gamma$ for any $\gamma X \in K(X)$. From that, we conclude that for any $\gamma X \in K(X)$, $C \subset \cap_{i \in I} C_{\gamma_i}$ for an arbitrary subset $\{\gamma_i X\}_{i \in I} \subset K(X)$. Finally, we will use the fact from (Chandler, 1976, Theorem 2.18) stating that the greatest lower bound of any set of compactifications $\{a_i X\}_{i \in I} \subset K(X)$ exists if and only if $\cap_{i \in I} C_{\alpha_i}$ separates points and separates points from closed sets. From this, $K(X)$ is complete. ☐

The following proposition describes H-compactifications of a locally compact space in terms of lattices and sublattices.

Proposition 94. *The set of all H-compactifications of a locally compact space X is a complete sub-lattice of the (complete) lattice of all compactifications of X.*

Proof. Let X be locally compact and denote by $\mathcal{H}(X)$ the group of all automorphisms of X.
For a given family of H-compactifications $\mathcal{K}_i = \{\gamma_i X \mid i \in I\}$ denote by C_i the set of all continuous functions from $\mathcal{C}^{\star}(X)$ that admit an extension over $\mathcal{K}_i$. Let $C = \bigcap_{i \in I} \mathcal{K}_i$. The greatest lower bound of the elements of $\mathcal{K}_i$ is given by $\overline{\gamma(X)}$ where γ is a map

$$\gamma : X \to \mathbb{R}^C$$
$$f(x)_e = e(x).$$

We now choose any $f \in \mathcal{C}^{\star}(X)$ such that it is continuously extendable over $\overline{\gamma X}$ and an arbitrary $h \in \mathcal{H}(X)$. We will verify that $f \circ h$ can be continuously extended over $\overline{\gamma X}$, which is equivalent with $\overline{\gamma X}$ being H-compactification, following (Vejnar, 2011, Proposition 5) which is used in the same paper for proving this proposition. Let $h \in \mathcal{H}(X)$ and let $f \in \mathcal{C}^{\star}(X)$ be continuously extendable over $\overline{\gamma(X)}$. Then $f \in C$. Consequently, $f \circ h \in \mathcal{K}_i$ for every $i \in I$ since $\gamma_i X$ is an H-compactification. Since $f \circ h$ is in each $\mathcal{K}_i$, it follows that $f \circ h \in \bigcap \mathcal{K}_i = C$ can be continuously extended over $\overline{\gamma X}$. We have shown that $\mathcal{K}_i$ is closed under the join operation, which is enough to see that the set of all H-compactifications as a lattice is complete. □

The category $\mathcal{P}_X$ is determined by the lattice of all H-compactifications P_X. Analogously, the lattice of all compactifications, which we will denote Q_X also determines a category - to be precise, we provide its definition too.

Notation. *To differentiate between P_X and Q_X, denote by $\leq'$ the natural ordering on the poset Q_X of all compactifications of X, defined as before.*

Definition 95. *Given a Hausdorff topological space X, a partially ordered set Q_X determines a category $\mathcal{Q}_{\mathcal{X}}$ of all compactifications of X, consisting of*

- *A collection $Ob(Q_X)$ of objects, which are compactifications $\gamma_i X$ of X*
- *A collection $Hom(\gamma_1 X, \gamma_2 X)$ of morphisms has exactly one element whenever $\gamma_1 X \leq' \gamma_2 X$ and is an empty set otherwise.*

Definition 96. *A category $\mathcal{P}_X$ is a full subcategory of a category $\mathcal{Q}_X$ if and only if the ordering $\leq$ defined on $\mathcal{P}_X$ is a restriction of the ordering $\leq'$ defined on $\mathcal{Q}_X$.*

In the following proposition, which is more an exercise, we summarize the category-theoretic properties of $\mathcal{P}_X$.

Proposition 97. *Let X be a locally compact topological space and the categories $\mathcal{P}_X$ and $\mathcal{Q}_X$ defined as above. Then $\mathcal{P}_X$ is closed in $\mathcal{Q}_X$ under all products and all co-products of arbitrary subsets of P_X. Moreover, $\mathcal{P}_X$ is a full subcategory of $\mathcal{Q}_X$.*

Proof. Trivially, $\mathcal{P}_X$ is a full subcategory of $\mathcal{Q}_X$, since the collection of all H-compactifications of X is by definition a sub-collection of all compactifications of X and for each pair $\gamma_1 X, \gamma_2 X \in Ob(\mathcal{P}_X)$, we have $Hom_{P_X}(\gamma_1 X, \gamma_2 X) = Hom_{Q_X}(\gamma_1 X, \gamma_2 X)$.

Let $Ob(\mathcal{P}_X)$ be objects of $\mathcal{P}_X$. Let $\gamma_1 X$ and $\gamma_2 X \in Ob(\mathcal{P}_X)$. Observe that the co-product of $\gamma_1 X$ and $\gamma_2 X$ corresponds to the supremum of $\gamma_1 X$ and $\gamma_2 X$ in P_X in the sense that if either of them exists, so does the other. Symmetrically, the products corresponds to infima.

Since X is locally compact, by the previous proposition, the lattice P_X is complete, hence the category $\mathcal{P}_X$ contains all products and co-products. By the same logic, the category $\mathcal{Q}_X$ contains all products and co-products. □

Remark. *The completeness of Q_X is crucial. If Q_X was not a complete lattice, but just a lattice, or even just a partially ordered set from which P_X inherits its natural ordering, then the supremum and infimum of any subset A of the lattice P_X in P_X would not necessarily correspond to the supremum and infimum of A in Q_X.*

Properties of $\mathcal{P}_X$ when X is not necessarily locally compact

In the general case when X is not required to be locally compact, it still possesses a semi-lattice structure.

Proposition 98. *Lubben (1941) The set of all compactifications of a space X with natural order is a complete upper semi-lattice.*

Proof. Let X be an arbitrary space and let $K(X)$ be the set of all compactifications of X. For a subset $\{\gamma_i X \mid i \in I\}$ of $K(X)$, denote for each $i \in I$ by γ_i the map $X \to \gamma_i X$. Define a map e such that

$$e : X \to \textstyle\prod_{i \in I} \gamma_i X$$
$$e(x)(i) = \gamma_i(x).$$

and denote by π_i the projections for each $i \in I$. The e can be seen as a function evaluating the functions γ_i and is determined by $\{\gamma_i X \mid i \in I\}$, hence from (Chandler, 1976, Theorem 1.24), since each γ_i is a homeomorphism, e is also a homeomorphism. This means that $eX = \overline{e(X)}$ is a compactification of X. For each $i \in I$, $f_i : eX \to \gamma_i X$ is the projection map restricted to eX. Then $(f_i \circ e)(x) = e(x)(i) = \gamma_i X$ so that $f_1 \circ e = \gamma_i$ which implies $eX \geq \gamma_i X \; \forall i \in I$. Now suppose for all $i \in I$ $e_1 X \geq \gamma_i X$ and define g_i such that

$$g_i : e_1 X \to \gamma_i X$$
$$g_i \circ e_1 = \gamma_i.$$

Define

$$f : e_1 X \to \textstyle\prod_{i \in I} \gamma_i X$$
$$f(y)(i) = g_i(y).$$

Then $\pi_i \circ f = g_i$, hence f is continuous and moreover $f(e_1(x)(i) = g_i(e_1(x)) = \gamma_i(x) = e(x)(i)$. In conclusion, $f \circ e_1 = e$ so that $f(e_1 X) = eX$ and $e_1 X \geq eX$ which implies that eX is the desired least upper bound of $\{\gamma_i X \mid i \in I\}$ with respect to the binary relation $\geq$. □

Once we have established the properties for a locally compact space, we can use similar approach in the general case.

Proposition 99. *The set of all H-compactifications of a space X is a complete (upper) sub-semi-lattice of the complete (upper) semi-lattice of all compactifications of X.*

Proof. Let $\mathcal{K}_i = \{\gamma_i X \mid i \in I\}$ be a set of H-compactifications of X. For each $i \in I$ denote the corresponding inclusion mapping by $\gamma_i : X \to \gamma_i X$. Define a diagonal mapping $\Delta\gamma_i$ by $\Delta\gamma_i(x) = \prod_{i\in I} \gamma_i X$. Observe that the least upper bound of $\mathcal{K}_i$ is given by $\overline{\gamma(X)}$ where $\gamma = \Delta\gamma_i$.
The final step is to show that $\overline{\gamma(X)}$ is also a H-compactification, whence a part of $\mathcal{K}_i$. This is proven as a part of (Vejnar, 2011, Proposition 4) by showing that for any autohomeomorphism h of X there exists an autohomeomorphism g of $\overline{\gamma(X)}$ such that $\gamma \circ h = g \circ \gamma$. Analogously we can find such h and g for γ. Two lemmas are used, saying that if this property holds for a mapping $X \to Y_i, i \in I$, then it can be extended to mappings $X \to \prod_{i\in I} Y_i$ and $X \to \overline{f(X)}$. □

Again, we will transfer the semi-lattices P_X and Q_X to the category-theoretic world.

Proposition 100. *Let X be a topological space and the categories $\mathcal{P}_X$ and $\mathcal{Q}_X$ defined as before. Then $\mathcal{P}_X$ is closed in $\mathcal{Q}_X$ under all co-products of arbitrary subsets of P_X. Moreover, $\mathcal{P}_X$ is a full subcategory of $\mathcal{Q}_X$.*

Proof. The proof is analogous to the case of local compactness. □

4.2.1 Initial and terminal objects

If we stay on the general (not necessarily locally compact) space X, we can explore other properties of the category $\mathcal{P}_X$, utilizing the natural ordering of H-compactifications. We have introduced the notions of finer and coarser H-compactifications and defined the finest ("most general") H-compactification. Similarly we can define a coarsest H-compactification.

Definition 101. *The coarsest H-compactification is defined as a compactification $(\gamma_1 X, \gamma_1)$ with the property that for any H-compactification $(\gamma_2 X, \gamma_2)$ there is a unique extension*

$$h : \gamma_2 X \to \gamma_1 X$$
$$\text{of } \gamma_1 : X \to \gamma_1 X.$$

Definition 102. *An initial object of a category $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for every object X in $\mathcal{C}$, there exists precisely one morphism $I \to X$. The dual notion is terminal object - it is an object T such that for every object X in $\mathcal{C}$, there exists exactly one morphism $X \to T$.*

Example 102.1. *In a poset, an object is apparently initial iff it is the greatest element, and terminal iff it is the least element.*

Consequently, in the lattice P_X, the largest element, provided it exists, is precisely the initial object in the corresponding category $\mathcal{P}_X$, and the smallest element, provided it exists, is the terminal object.

Remark. *If an initial (or terminal) object exists, it is always unique. Hence, the finest and coarsest H-compactification of P_X is equivalent to the initial and terminal object in the category of H-compactifications (each of this objects is unique, provided it exists).*

Proposition 103. *For a topological (not necessarily Tychonoff) space X, the one point H-compactification αX is the coarsest H-compactification of X.*

Proof. Recall that for a locally compact space X, αX is defined as $X \cup \{\infty\}$. Let $(\gamma X, \gamma)$ be another H-compactification of X. Then, define the desired morphism h from γX to αX by

$$h(x) = \begin{cases} \alpha(\gamma^{-1}(x)) & \text{if } x \in \gamma(X), \\ \infty & \text{if } x \notin \gamma(X), \end{cases}$$

where α means the inclusion $X \to \alpha X$. □

Recall that for Tychonoff spaces that are not locally compact, the Alexandroff one-point compactification does not even exist, so $\mathcal{P}_X$ does not always have the terminal object.

The opposite case is the initial object, which corresponds exactly to the Stone-Čech compactification.

Recall that for βX of a topological space X, if we take an arbitrary compact Hausdorff space K and k such that $k : X \to K$, there always exists a morphism $f : \beta X \to K$ such that $f \circ \beta = k$, where β is the embedding $\beta : X \to \beta X$. The f is always unique (we say that the k extends uniquely to f.)

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \beta X \\ & \searrow_{k} & \downarrow f \\ & & K \end{array}$$

This universal property of the Stone-Čech compactification makes it the *initial object* in the category $\mathcal{P}_X$. Informally, the Stone-Čech compactification is the initial object, because all the arrows "start there".

4.2.2 Functorial properties

Given a topological space X, the αX (if exists) and βX certainly play a special role, not only as the "biggest" and "smallest" object, but also as functors between different categories.

The (Hausdorff) Stone-Čech compactification can be seen as a functor

$$\beta : Top \to cHaus$$

where Top refers to the category of topological spaces and $cHaus$ to the category of compact Hausdorff spaces.

Definition 104. *Let*

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}.$$

be categories and functors. Let $A \in Ob(\mathcal{A})$ and $B \in Ob(\mathcal{B})$. We say that F is left adjoint to G (and G is right adjoint to F) if

$$\mathcal{B}(F(A), B) \simeq \mathcal{A}(A, G(B))$$

naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

"Naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$" means that there is a specified bijection satisfying a naturality axiom. For more details, see (Leinster, 2014, Chapter 2).

Adjoints can be viewed intuitively as expressing conceptual duality between mathematical notions.

Remark. *The functor β is a left adjoint functor to the inclusion functor*

$$\mathcal{I} : cHaus \to Top.$$

Corollary 104.1. *The adjunction between β and $\mathcal{I}$ exhibits cHaus as a reflective subcategory of all of Top. (Informally, a full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is said to be reflective in $\mathcal{B}$ when the inclusion functor from $\mathcal{A}$ to $\mathcal{B}$ has a left adjoint.)*

For αX, the situation is more complicated. The one-point compactification does not generally extend to a functor on the whole category of topological spaces. However, it does extend to a functor on locally compact Hausdorff spaces with proper maps between them.

Definition 105. *Let X and Y be two topological spaces. A continuous map $f : X \to Y$ is called proper if $f^{-1}(K)$ is compact for every $K \subset Y$ compact.*

Definition 106. *A based topological space is a pair $(X, \star)$, where X is a nonempty space and $\star \in X$ is a chosen basepoint. We will normally suppress basepoints from notation and write a based space simply X, denoting all basepoints ambiguously by $\star$.*

If X, Y are based spaces, then a continuous map $f : X \to Y$ is said to be based if $f(\star) = \star$.

Based spaces are also called *pointed.*

Remark. *The collection of all based spaces and maps forms a category $Top_\star$.*

We will now look at the Alexandroff one-point compactification as a based space where the basepoint is disjoint. This will shed a light on functorial property of α.

Definition 107. *Given a map $f : X \to Y$ we can define a function of sets*

$$\alpha f : \alpha X \to \alpha Y$$

in the obvious way by letting $\alpha f \restriction X = f$ *and* $\alpha f(\infty) = \infty$ *where* ∞ *denotes the one point added to* X*.*

This definition ensures that αf is a pointed map. If we somehow manage that it is continuous, then we will be able to operate with α as a functor.

Proposition 108. *(Cutler, 2020, Proposition 1.5) If* $f : X \to Y$ *is a map between unpointed locally compact Hausdorff spaces, then* $\alpha f : \alpha X \to \alpha Y$ *is continuous if and only if* f *is proper.*

Proof. First, suppose that $f : X \to X$ is proper. Then, from the definition, for any compact subspace K of Y, $f^{-1}(K) \subset X$ is also compact and $\alpha f^{-1}(\alpha Y - X) = \alpha X - f^{-1}(K)$ is open in αX. Moreover, for each open subset U of Y, $\alpha f^{-1}(U) = f^{-1}(U)$ is open, but this means that αf, as a map between topological spaces, is continuous.

For the other implication, suppose that $\alpha f : \alpha X \to \alpha Y$ is continuous. Then, for each compact $K \subset Y$, $\alpha f^{-1}(\alpha Y - K) = \alpha X - f^{-1}(K)$ which is open in αX by continuity of αf. Therefore $f^{-1}(K)$ is compact in X. Because we work with locally compact Hausdorff topological spaces, this is enough for concluding that f is proper. □

Denote by $\mathcal{H}$ the category of locally compact Hausdorff spaces where morphisms are proper maps and by $Top_\star$ the category of pointed topological spaces.

Corollary 108.1. $X \to \alpha X$ *defines a faithful functor*

$$\mathcal{H} \to Top_\star.$$

Conclusion

The overall purpose of this thesis is to synthesise all the scattered findings about H-compactifications into a complex and structured overview. We have summarized the main findings about H-compactifications of some well-known spaces in the Chapter 1.

We devoted the Chapter 2 to taking a closer look at the space $\mathbb{Q}^\omega$ for which the set of H-compactifications has not been shown before. We have proven that the only H-compactification of $\mathbb{Q}^\omega$ is $\beta\mathbb{Q}^\omega$, which is naturally true for all spaces homeomorphic to $\mathbb{Q}^\omega$ as well.

In the Chapter 3, we have shown several methods possible to solve the problem of describing the set of all H-compactifications of l^2 where the question stays unanswered.

We closed the thesis by analzsing the structure of sets of H-compactifications and made a point about category-theoretic properties of the Alexandroff and Stone-Čech compactifications.

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