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REPORT OF THE PHD THESIS

Structural and categorical description of classes of abelian groups

by Josef Dvořák

When we try to understand a category (sets, abelian groups, modules etc.) a quite naturally strategy is to start with objects satisfying various finiteness assumptions and to see how can we built more complicated objects from these small pieces. One of the most useful finiteness assumption is the one called smallness: Roughly speaking an object S in a category with coproducts $\mathcal{A}$ is called small, if the covariant functor $\mathrm{Hom}_{\mathcal{A}}(S,-)$ preserves coproducts. Here the target category of this functor is Set for general categories and Ab for (pre)additive ones (sure one can consider categories enriched over many other monoidal categories, but this is beyond the purposes of this report). The thesis under review considers a variant of this notion: An object S in a category $\mathcal{A}$ is called *compact* if the canonical morphism from $\coprod \mathrm{Hom}_{\mathcal{A}}(S, X_i)$ to $\mathrm{Hom}_{\mathcal{A}}(S, \coprod X_i)$ is surjective for any family of objects $(X_i)_{i\in I}$. In the important case in which $\mathcal{A}$ is abelian Ab5 category, e. g. $\mathcal{A}$ is the category of modules over a unitary associative ring, this morphism is automatically injective, therefore the definition above says that the canonical morphism is actually bijective and we recover the preservation of coproducts we spoke above. The thesis studies various relative versions of this notion in various settings: abelian groups, modules over unital associative rings, acts over monoids etc. From this point of view I think the title of the thesis does not cover all cases considered within, but this could be explained by the fact that the author started the research having in the mind some directions, and in time many another directions came into consideration.

The thesis is based on five papers written by the author one of them alone and four in collaboration. The papers are presented in distinct Chapters, each of which having its own Bibliography.

Chapter 1 is actually *Introduction* where the author lists the papers on which the thesis is based and highlights the main results.

A module A over a unitary associative ring R is called self-small if the functor $\mathrm{Hom}_R(A,-)$ commutes with arbitrary coproducts (i. e. direct sums) of copies of A itself. In Chapter 2, *On products of self-small abelian groups*, it is constructed an elementary counterexample showing that for a family of self-small abelian groups (i. e. $\mathbb{Z}$-modules) $\{A_i \mid i \in I\}$, satisfying $\mathrm{Hom}_{\mathbb{Z}}(A_i, A_j) = 0$ for $i \neq j$, the direct product $A = \prod_{i \in I} A_i$ needs not to be self-small. The counterexample shows that Corollary 1.3 from the seminal paper by Arnold and Murley (reference [4] in the Bibliography of this Chapter) is false. However in Corollary 2.6 it is shown that if we assume additionally that I is finite, then the conclusion of [4, Corollary 1.3] holds true.

Chapter 3, titled *Self-small products of abelian groups*, starts by considering a relative version of smallness, more precisely an abelian group A is called small relative to a family of abelian groups $\mathcal{N}$, provided that the natural morphism

$$\bigoplus_{N \in \mathcal{N}} \mathrm{Hom}_{\mathbb{Z}}(A, N) \to \mathrm{Hom}_{\mathbb{Z}}(A, \bigoplus_{N \in \mathcal{N}} N)$$

is bijective. The study of closure properties, under various categorical constructions as (co)products, extensions etc., of this relative notion enabled the author to characterize and even to give a structure theorem for self-small products of finitely generated abelian groups (see Theorems 3.23, respectively 3.27). In particular it follows that $\mathbb{Z}^{\kappa}$ is self-small for any cardinal κ (Corollary 3.26).

Chapter 4, *Autocompact objects of Ab5 categories*, contains a generalization of the study of self–small abelian groups and modules, in the sense that there are studied autocompact objects in an arbitary abelian Ab5 category $\mathcal{A}$. By the way, the usage of a double terminology, that is "small", "relative small", or "self-small" when one speaks about modules and "compact", "relative compact", respectively "autocompact" when one speaks about objects in another categories, could be a little confusing. From the results contained in this Chapter, I would mention Lemma 4.21 and Theorem 4.22, where one can find a categorical version of a useful negative criterion for self-smallness (i. e. autocompactness in the terms of this Chapter) proved for the first time in [3] (from the Biblography for this Chapter; this choice to list a Biblography for every Chapter could also lead to misunderstandings). This criterion is used in order to study in Section 4.5 autocompact products in a category $\mathcal{A}$ as before, generalizing some results from the previous Chapter. A world about Theorem 4.34: If $\mathcal{A}$ fulfills the hypothesis of this theorem, namely it is abelian Ab5 with a compact projective generator, then it is equivalent to a module category, by a celebrated theorem by Freyd and Mitchell (see for example P. Freyd, *Abelian Categories*, Harper & Row, 1964, p. 120, H. Categories representable as functor categories). Thus, perhaps a more direct approach should work in order to prove the conclusion of this theorem (see [21, Proposition 1.6]).

The fifth chapter is titled *Compact objects in categories of S-acts.* Here S is a monoid, and a (left) S-act is a set A together with an action $S \times A \to A$, satisfying two compatibility conditions. The analogy with R-modules is transparent, but unlike the case of R-modules, the category obtained here is not more enriched over Ab (preadditive). After introducing an axiomatic description of categories of acts, allowing the identification of indecomposable and projective objects, it is shown that the classes of compact, autocompact and indecomposable acts coincide (Theorem 5.45). Provided that S have a zero element, one can consider the category of acts whose morphisms are compatible with zero. Here an act A is compact if and only if it is not a union of a pair of proper subacts (Proposition 5.40), whereas the class of autocompact acts in this new category is a larger.

The last chapter, *Perfect monoids with zero and categories of S-acts*, studies the relationship between the existence of projective covers in both categories of acts considered above. We say that the monoid with zero S is left perfect, respectively left 0-perfect, provided that projective covers do exist for all objects in the category of acts, respectively in the category of acts with zero-preserving morphisms. Then in Theorem 6.18 it is shown that a monoid is left perfect if and only if it is left 0-perfect.

With very few exceptions mentioned above, the thesis is carefully written. I also want to point out that the author used many modern and powerful techniques. The proofs are correct, and many of them are non-trivial. Some results of the thesis are already published in peer-reviewed journals. All these convinced me that the author acquired very good research skills, permitting him to continue his independent research work.

In conclusion, taking into account the observations above but also the significance of the mathematical results, **I strongly recommend Mr. Josef Dvořák to be awarded a Ph.D. in Mathematics**.

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July 28, 2021